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Let T be a complete theory of fields, possibly with extra structure. Suppose that model-theoretic algebraic closure agrees with field-theoretic algebraic closure, or more generally that model-theoretic algebraic closure has the exchange property. Then T has uniform finiteness, or equivalently, it eliminates the quantifier $\exists^\infty$. It follows that very slim fields in the sense of Junker and Koenigsmann are the same thing as geometric fields in the sense of Hrushovski and Pillay. Modulo some fine print, these two concepts are also equivalent to algebraically bounded fields in the sense of van den Dries.

From the proof, one gets a one-cardinal theorem for geometric theories of fields: any infinite definable set has the same cardinality as the field. We investigate whether this extends to interpretable sets. We show that positive dimensional interpretable sets must have the same cardinality as the field, but zero-dimensional interpretable sets can have smaller cardinality. As an application, we show that any geometric theory of fields has an uncountable model with only countably many finite algebraic extensions.

1. Introduction

Throughout the paper, T denotes a complete theory. We use $\mathrm{acl}(-)$ to denote the model-theoretic algebraic closure. When T expands the theory of fields, we use $(-)^{\mathrm{alg}}$ to denote the field-theoretic algebraic closure. Following [Hrushovski and Pillay 1994; Gagelman 2005], we say that T is *geometric* if (1)–(2) hold and *pregeometric* if (1) holds:

(1) $\mathrm{acl}(-)$ satisfies the exchange property.

(2) T eliminates $\exists^\infty$, or equivalently, T has uniform finiteness.

A *(pre)geometric structure* is a structure M whose complete theory is (pre)geometric.

Using a simple argument, we show that pregeometric fields are geometric (Theorem 2.5). This seems to not be well-known. For example, it is implicitly unknown in [Hrushovski and Pillay 1994, Remark 2.12], and a special case of

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this implication is asked as an open problem in [Junker and Koenigsmann 2010, Question 1].

As a consequence of Theorem 2.5, several concepts in the literature are equivalent, namely the *very slim* fields of Junker and Koenigsmann [2010, Definition 1.1], the *geometric fields* of Hrushovski and Pillay [1994, Remark 2.10], and (modulo some fine print) the *algebraically bounded fields* of van den Dries [1989].

Our method also shows that in a geometric field, any infinite definable set has the same cardinality as the field (Proposition 3.1), which may be of independent interest. It is natural to ask whether this extends to interpretable sets. In Proposition 3.6, we show that if X is an interpretable set of positive dimension, then $|X| = |K|$, but there are models where all the zero-dimensional interpretable sets satisfy $|X| < |K|$. As an application, there is an uncountable model K with only countably many finite algebraic extensions (Corollary 4.3), which may be of interest to field theorists. In the special case of ω-free perfect PAC fields, this recovers examples such as [Bary-Soroker and Paran 2013, Example 2.2].

2. Uniform finiteness from the exchange property

If M is a geometric structure, there is a well-established dimension theory on M defined as follows. If $A \subseteq M$ and a is a tuple, we define $\dim(a/A)$ to be the length of the maximal acl_A-independent subtuple of a. This is well-defined by the exchange property. If X is an A-definable subset of M^n, we define

$$\dim(X) = \max\{\dim(x/A) : x \in X(\mathbb{U})\}$$

for some monster model $\mathbb{U} \succeq M$.

Fact 2.1. *Let M be a pregeometric structure, and X, Y be definable sets.*

(1) $\dim(X)$ *is well-defined, independent of the choice of A.*

(2) $\dim(X) > 0 \iff |X| = \infty$.

(3) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

(4) $\dim(M^n) = n$ *unless M is finite.*

(5) *If $f : X \to Y$ is a definable injection or surjection, then* $\dim(X) \leq \dim(Y)$ *or* $\dim(X) \geq \dim(Y)$, *respectively.*

Gagelman observed that the dimension theory can also be extended to M^{eq}. If $a \in M^{\mathrm{eq}}$, then $\dim(a/A)$ is defined to be $\dim(b/Aa) - \dim(b/A)$ for any tuple b with $a \in \mathrm{acl}^{\mathrm{eq}}(Ab)$. If X is an interpretable set, then $\dim(X)$ is defined as for definable sets. By [Gagelman 2005, Lemma 3.3], these definitions are well-defined, and all of Fact 2.1 holds *except for* (4) and the $\Leftarrow$ direction of (2) [Gagelman 2005, Proposition 3.4 and p. 321].

Lemma 2.2. *Let K be a pregeometric field. Then K is perfect.*

Proof. Otherwise, K is an infinite field of characteristic p. Take $a \in K \setminus K^p$. Then the map

$$f(x, y) \mapsto x^p + ay^p$$

is a definable injection $f : K \times K \to K$. But $\dim(K^2) = 2 > \dim(K) = 1$, as K is infinite. □

Lemma 2.3. *Let K be a field and X be a subset. Then one of the following holds*:

(1) *The set $S = \{(y - y')/(x' - x) : x, y, x', y' \in X, x \neq x'\}$ equals K.*

(2) *There is an injection $f : X^2 \to K$ of the form $f(x, y) = ax + y$.*

Proof. If $a \in K \setminus S$, then the function $f(x, y) = ax + y$ is injective on X^2. □

Lemma 2.4. *Let K be an infinite pregeometric field*, *and let $X \subseteq K$ be definable. Then X is infinite if and only if $K = \{(y - y')/(x' - x) : x, y, x', y' \in X,\ x \neq x'\}$.*

Proof. If X is infinite, then $\dim(X) > 0$, so $\dim(X^2) \geq 2 > \dim(K) = 1$ and there are no definable injections $f : X^2 \to K$. Therefore, case (2) of Lemma 2.3 cannot hold, so (1) holds. Conversely, if X is finite, then case (1) of Lemma 2.3 cannot hold, as the set S would be finite. □

Theorem 2.5. *Let T be a complete theory of fields*, *possibly with extra structure. If T is pregeometric*, *then T is geometric.*

Proof. It suffices to eliminate $\exists^\infty$. If models of T are finite, then $\exists^\infty$ is trivially eliminated. If the models of T are infinite, then Lemma 2.4 gives a first-order criterion for telling whether a definable set $X \subseteq K$ is infinite. □

The proof of Theorem 2.5 is the same argument used in [Johnson 2018, Observation 3.1].

Remark 2.6. Let $(K, +, \cdot, \dots)$ be a field, possibly with extra structure. If $\operatorname{acl}(-)$ agrees with field-theoretic algebraic closure, then $\operatorname{acl}(-)$ satisfies exchange.

Definition 2.7 [Junker and Koenigsmann 2010, Definition 1.1]. A field K is *very slim* if $\operatorname{acl}(-)$ agrees with field-theoretic algebraic closure in any elementary extension of K.

Definition 2.8 [Hrushovski and Pillay 1994, Remark 2.10]. A *strongly geometric field* is a perfect field $(K, +, \cdot)$ with a geometric theory that is very slim.

Hrushovski and Pillay call these *geometric fields*, but we prefer the term *strongly geometric* to distinguish strongly geometric fields from the more general case of fields that are geometric (as structures). There are geometric fields that are not strongly geometric (Remark 2.16).

Corollary 2.9. *A field K is very slim if and only if it is strongly geometric.*

Proof. If K is very slim, then K is pregeometric, hence geometric by Theorem 2.5 and perfect by Lemma 2.2 (or [Junker and Koenigsmann 2010, Proposition 4.1]). □

Definition 2.10 [van den Dries 1989]. Let $(K, +, \cdot, \dots)$ be an expansion of a field, and F be a subfield. Then K is *algebraically bounded over* F if for any formula $\varphi(\bar{x}, y)$, there are finitely many polynomials $P_1, \dots, P_m \in F[\bar{x}, y]$ such that for any $\bar{a}$, *if* $\varphi(\bar{a}, K)$ is finite, then $\varphi(\bar{a}, K)$ is contained in the zero set of $P_i(\bar{a}, y)$ for some i such that $P_i(\bar{a}, y)$ does not vanish. Following the convention in [van den Dries 1989; Junker and Koenigsmann 2010] we say that K is *algebraically bounded* if it is algebraically bounded over K.

Lemma 2.11. *Suppose K is algebraically bounded over some subfield F.*

(1) *If A is a subset of K, then* $\mathrm{acl}(AF)$ *is the field-theoretic relative algebraic closure* $(AF)^{\mathrm{alg}} \cap K$.

(2) *K has uniform finiteness.*

(3) *If K^* is an elementary extension of K, then K^* is algebraically bounded over F.*

Proof. (1) Clearly $(AF)^{\mathrm{alg}} \cap K \subseteq \mathrm{acl}(AF)$. Conversely if $b \in \mathrm{acl}(AF)$, then b is in a finite set $\varphi(\bar{a}, K)$ for some tuple $\bar{a}$ from AF. By algebraic boundedness, $\varphi(\bar{a}, K)$ is contained in a finite zero set of some polynomial $P(\bar{a}, y)$, where $P(\bar{x}, y) \in F[\bar{x}, y]$. Therefore b is field-theoretically algebraic over AF.

(2) For a fixed formula $\varphi(\bar{x}, y)$, let $P_1, \dots, P_m \in F[\bar{x}, y]$ be polynomials as in Definition 2.10. Then the cardinality of finite sets of the form $\varphi(\bar{a}, K)$ is bounded by the maximum of the degrees of the P_i's.

(3) By (2), the theory of K has uniform finiteness, and so $\exists^\infty$ is uniformly eliminated across elementary extension of K. It follows that for fixed $\varphi, P_1, \dots, P_m$, the following condition is preserved in elementary extensions:

> For any $\bar{a}$, if $\varphi(\bar{a}, y)$ defines a finite set, then there is $i \in \{1, \dots, m\}$ such that $P_i(\bar{a}, y)$ has finitely many zeros and $\varphi(\bar{a}, y) \to P_i(\bar{a}, y) = 0$.

Thus, algebraic boundedness transfers from K to any elementary extension K^*. □

Lemma 2.12. *Let $K = (K, +, \cdot, \dots)$ be a field or an expansion of a field. Let F be a subfield. The following are equivalent*:

(1) *In any elementary extension $K^* \succeq K$, field-theoretic algebraic closure over F agrees with model-theoretic algebraic closure over F: if $A \subseteq K^*$ and $b \in \mathrm{acl}(AF)$, then $b \in (AF)^{\mathrm{alg}}$.*

(2) *Condition* (1) *holds and K has uniform finiteness, or equivalently, $\exists^\infty$ is eliminated in elementary extensions of K.*

(3) *K is algebraically bounded over F.*

Proof. (1)$\Longrightarrow$(2). Field-theoretic algebraic closure satisfies the exchange property. Therefore $\mathrm{acl}(-)$ satisfies exchange (after naming the elements of F as parameters). By Theorem 2.5, elementary extensions of K eliminate $\exists^\infty$.

(2)$\Longrightarrow$(3). Suppose (2) holds but K fails to be algebraically bounded over F, witnessed by some formula $\varphi(\bar{x}, y)$. Using elimination of $\exists^\infty$, we may assume that there is $n \in \mathbb{N}$ such that $|\varphi(\bar{a}, K)| \le n$ for every $\bar{a}$. For any finite set of polynomials $P_1, \dots, P_m \in F[\bar{x}, y]$, there is $\bar{a} \in K$ such that $\varphi(\bar{a}, K)$ is finite, but is not contained in the zero set of $P_i(\bar{a}, y)$ unless $P_i(\bar{a}, y) \equiv 0$. By compactness, there is an elementary extension $K^* \succeq K$ and a tuple $\bar{a} \in K^*$ such that $\varphi(\bar{a}, K^*)$ is finite, but is not contained in the zero set of $P(\bar{a}, y)$ for *any* $P \in F[\bar{x}, y]$ except those with $P(\bar{a}, y) \equiv 0$. Then $\varphi(\bar{a}, K^*)$ contains a point not in $F(\bar{a})^{\mathrm{alg}}$, contradicting (2).

(3)$\Longrightarrow$(1). Lemma 2.11 shows that if K^* is an elementary extension of K, then

- K^* is algebraically bounded over F, and
- field-theoretic and model-theoretic algebraic agree over F.

Therefore (1) holds. □

Specializing to the case where K is a pure field and F is the prime field, we get the following:

Theorem 2.13. *Let K be a pure field. Then K is algebraically bounded over the prime field if and only if K is very slim.*

We have thus answered [Junker and Koenigsmann 2010, Question 1] positively.

Remark 2.14. If $(K, +, \cdot, \dots)$ is an *expansion* of a field, and $(K, +, \cdot, \dots)$ is algebraically bounded over the prime field, then the reduct $(K, +, \cdot)$ is also algebraically bounded over the prime field, and so the underlying field $(K, +, \cdot)$ is very slim.

Remark 2.15. Reducts of geometric structures are geometric structures. This is folklore, but we include a proof for completeness. Let N be a geometric structure and M be a reduct. Without loss of generality, N and M are highly saturated. Uniform finiteness transfers from N to M in a trivial way: if M fails uniform finiteness, the same definable family fails uniform finiteness in N. Suppose $\mathrm{acl}^M(-)$ does not satisfy exchange. Then there are some $a, b \in M$ and $C \subseteq M$ such that $a \notin \mathrm{acl}^M(C)$, $b \notin \mathrm{acl}^M(Ca)$, but $a \in \mathrm{acl}^M(Cb)$. Let $p(x)$ and $q(x, y)$ be $\mathrm{tp}^M(a/C)$ and $\mathrm{tp}^M(a, b/C)$. The number of realizations of $p(x)$ is large, so we may find $a' \models p$ with $a' \notin \mathrm{acl}^N(C)$. Similarly, the number of realizations of $q(a', y)$ is large, so we may find a realization $b' \notin \mathrm{acl}^N(Ca')$. Then $a'b' \equiv_C ab$ in M, so $a' \in \mathrm{acl}^M(Cb') \subseteq \mathrm{acl}^N(Cb')$. Then a' and b' contradict the exchange property in N.

Therefore, any reduct of an algebraically bounded field is geometric, though not necessarily strongly geometric.

Remark 2.16. In future work, we will give an example of a pure field $(K, +, \cdot)$ of characteristic 0 with a subfield K_0 such that

(1) Field-theoretic algebraic closure and model-theoretic algebraic closure agree over K_0, and this remains true in elementary extensions.

(2) $\mathrm{acl}(\varnothing)$ contains elements of K_0 that are transcendental over $\mathbb{Q}$. In particular, field-theoretic algebraic closure and model-theoretic algebraic closure do not agree over $\mathbb{Q}$.

It follows that this field K is algebraically bounded over K_0, but not over $\mathbb{Q}$. In particular, K is algebraically bounded (over K) but not very slim and not a strongly geometric field.[1] Additionally, the field K is geometric (by Remark 2.15) but not strongly geometric.

Failure of $\mathrm{acl}(\varnothing)$ to be algebraic over the prime field is the *only* way an algebraically bounded field can fail to be very slim:

Proposition 2.17. *If $K = (K, +, \cdot, \ldots)$ is algebraically bounded, then K is algebraically bounded over the subfield $F = \mathrm{dcl}(\varnothing)$.*

Proof. We use the criterion of Lemma 2.12(1) to show that K is algebraically bounded over F. Embed K into a monster model $\mathbb{K}$. Suppose $b, \bar{c} \in \mathbb{K}$ and $b \in \mathrm{acl}(F\bar{c})$. We must show $b \in F(\bar{c})^{\mathrm{alg}}$. By Remark 2.15, $\mathrm{Th}(K)$ is geometric, because it is geometric after naming parameters from F. Replacing $\bar{c}$ by a basis of $\bar{c}$ in the acl-pregeometry, we may assume that the tuple $\bar{c}$ is field-theoretically algebraically independent over F. Now suppose for the sake of contradiction that $b \notin F(\bar{c})^{\mathrm{alg}}$. Then $(\bar{c}, b)$ is also field-theoretically algebraically independent over F.

As $b \in \mathrm{acl}(F\bar{c}) = \mathrm{acl}(\bar{c})$, there is a 0-definable set $D \subseteq \mathbb{K}^n$ with $(b, \bar{c}) \in D$ and $D_{\bar{c}'}$ finite for each $\bar{c}'$. By algebraic boundedness over K, there are finitely many nonzero polynomials $P_1, \ldots, P_i \in K[x, \bar{y}]$ such that D is contained in the union of the zero-sets of the P_i.

Let $M = \mathbb{K}^{\mathrm{alg}}$ and let V be the Zariski closure of D in M^{n+1}. The polynomials P_i show that $V \subsetneq M^{n+1}$, and so $\dim(V) < n+1$. By elimination of imaginaries in ACF, there is a finite tuple e in M which codes V. Recall that $\mathbb{K}$ is perfect by Lemma 2.2. If $\sigma \in \mathrm{Aut}(M/\mathbb{K}) = \mathrm{Gal}(\mathbb{K})$, then $\sigma(D) = D$, $\sigma(V) = V$, and $\sigma(e) = e$. As the tuple e is fixed by $\mathrm{Gal}(M/\mathbb{K})$, it must be in the perfect field $\mathbb{K}$.

If σ is any automorphism of $\mathbb{K}$, then σ can be extended to an automorphism σ' of M. The fact that D is 0-definable implies that $\sigma'(D) = \sigma(D) = D$, which then implies $\sigma'(V) = V$ and $\sigma(e) = \sigma'(e) = e$. Thus e is $\mathrm{Aut}(\mathbb{K})$-invariant, which implies that e is in $F = \mathrm{dcl}(\varnothing)$. Therefore, in the structure M, the e-definable set V is in fact F-definable. However, the tuple $(b, \bar{c}) \in D \subseteq V$ is algebraically

[1] This contradicts the claim in [Junker and Koenigsmann 2010, p. 482] that algebraically bounded fields are very slim.

independent over F, so this implies $\dim(V) = n+1$, contradicting the earlier fact that $\dim(V) < n+1$. □

Remark 2.18. Algebraically bounded fields are closely related to fields of size at most S in the sense of [Junker and Koenigsmann 2010]. For fields of size at most S, [Junker and Koenigsmann 2010, Proposition 3.4] and Lemma 2.12 show that they are algebraically bounded over $\operatorname{dcl}(\varnothing)$. On the other hand, by Lemma 2.12 and Proposition 2.17, any algebraically bounded field with $\operatorname{tr.deg}(\operatorname{dcl}(\varnothing)) \in \mathbb{N}$ is of size at most S.

Question 2.19. Is there a pure field K that is geometric, but not algebraically bounded?

3. Cardinalities

Fix a complete geometric theory T expanding the theory of fields, not necessarily algebraically bounded.

Proposition 3.1. *If $K \models T$ and $X \subseteq K^n$ is an infinite definable set, then $|X| = |K|$.*

Proof. Clearly $|X| \le |K|$. We must show $|X| \ge |K|$. Replacing X with a projection onto one of the coordinate axes, we may assume $X \subseteq K^1$. By Lemma 2.4,

$$K = \{(y-y')/(x'-x) : x, y, x', y' \in X,\ x \neq x'\}.$$

Therefore $|K| \le |X|^4 = |X|$. □

Proposition 3.1 does not generalize to interpretable sets, as exhibited by the example of the value group $\mathbb{Z}$ in the geometric field $\mathbb{Q}_p$. In Proposition 3.6 below, we will see that the obstruction is precisely the zero-dimensional interpretable sets. Before proving this, we need a few general lemmas on geometric structures.

Lemma 3.2. *Suppose M is a geometric structure and X is an interpretable set in M with* $\dim(X) = d > 0$.

(1) *There is an interpretable set Y in M and finite-to-one interpretable functions $f : Y \to X$ and $g : Y \to M^d$ such that the image $g(Y)$ has dimension d.*

(2) *There is an infinite definable set D with $|X| \ge |D|$.*

Proof. Note that (1) implies (2), by taking $D = g(Y)$. We prove (1). Embed M into a monster model $\mathbb{M} \succeq M$. Take $e \in X$ with $\dim(e/M) = d$. Then $e \in \mathbb{M}^{\mathrm{eq}}$. Every imaginary is definable from a real tuple, so there is a real tuple $\bar{a} \in \mathbb{M}^m$ with $e \in \operatorname{acl}^{\mathrm{eq}}(M\bar{a})$. Replacing $\bar{a}$ with a subtuple, we may assume that $\bar{a}$ is independent over M. By [Gagelman 2005, Lemma 3.1], $\operatorname{acl}(-)$ continues to satisfy exchange after naming the parameter e. Therefore, we can meaningfully talk about real tuples being independent over eM. Write $\bar{a}$ as $\bar{b}\bar{c}$, where $\bar{b}$ is a maximal subtuple that is independent over eM. Then $\bar{c} \in \operatorname{acl}(\bar{b}eM)$. At the same time, $e \in \operatorname{acl}(\bar{b}\bar{c}M)$, so e is interalgebraic with $\bar{c}$ over $\bar{b}M$.

Meanwhile, $\dim(\bar{b}/eM) = \dim(\bar{b}/M) = |\bar{b}|$, because $\bar{b}$ is an independent tuple over eM. Then $\bar{b}$ and e are independent from each other over M, implying $\dim(e/\bar{b}M) = \dim(e/M) = d$ by symmetry. As e and $\bar{c}$ are interalgebraic over $\bar{b}M$, we have $\dim(\bar{c}/\bar{b}M) = d$. But $\bar{b}\bar{c}$ is an independent tuple over M, so $\dim(\bar{c}/\bar{b}M)$ is the length of $\bar{c}$. Thus $\bar{c} \in M^d$.

As e and $\bar{c}$ are interalgebraic over $\bar{b}M$, there is a $\bar{b}M$-interpretable set $Y_0 \subseteq X \times \mathbb{M}^d$ such that $(e, \bar{c}) \in Y_0$, and the projections $f_0 : Y_0 \to X$ and $g_0 : Y_0 \to \mathbb{M}^d$ have finite fibers. By saturation, there is a uniform bound N on the fiber size. The image $g_0(Y_0)$ is $\bar{b}M$-definable and contains the point $\bar{c}$ with $\dim(\bar{c}/\bar{b}M) = d$. Therefore $g_0(Y_0)$ has dimension d.

Now we have the desired configuration (Y_0, f_0, g_0), but defined over the parameter $\bar{b}$ outside M. Because $M \preceq \mathbb{M}$ and dimension is definable in families [Gagelman 2005, Fact 2.4], we can replace the parameter $\bar{b}$ with something in M, getting an M-definable configuration (Y, f, g) in which the fibers of f and g are still bounded in size by N. □

Definition 3.3. Let M be a structure. A *definable notion of largeness*[2] on M is a partition of the M-definable sets into two classes — large and small — such that the following axioms hold:

(1) Any finite set is small.

(2) Any definable subset of a small set is small.

(3) If Y is small and $\{X_a\}_{a\in Y}$ is a definable family of small sets, then the union $\bigcup_{a\in Y} X_a$ is small.

(4) Smallness is definable in families: if $\{X_a\}_{a\in Y}$ is a definable family, then the set $\{a \in Y : X_a \text{ is small}\}$ is definable.

If $N \succeq M$, then any definable notion of largeness on M extends in a canonical way to a definable notion of largeness on N by extending the definition according to (4) above.

Fact 3.4. *Let M be a countable structure in a countable language. Fix a definable notion of largeness on M. Then there is an elementary extension $N \succeq M$ such that if X is definable in N, then X is uncountable if and only if X is large.*

Fact 3.4 is essentially Keisler's completeness theorem for $\mathcal{L}(Q)$ [Keisler 1970, Section 2], but the translation between these settings is sufficiently confusing that we give the details.

[2] This is a purely model-theoretic notion and should not be confused with the notion of large fields.

Proof of Fact 3.4. Let T be the elementary diagram of M. Let $\mathcal{L}$ be the language of T, and let $\mathcal{L}(Q)$ be the language obtained by adding a new quantifier (Qx). Let $\psi \mapsto \psi^*$ be the map from $\mathcal{L}(Q)$-formulas to $\mathcal{L}$-formulas interpreting $(Qx)\varphi(x, \bar{y})$ as "the set of x such that $\varphi(x, \bar{y})$ holds is large." More precisely,

- $\varphi^* = \varphi$ if φ doesn't involve the quantifier Q.
- $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, and similarly for the other logical operators including $\exists$ and $\forall$.
- $((Qx)\varphi(x, \bar{y}))^*$ is the formula $\psi(\bar{y})$ such that in models $N \models T$,

$$N \models \psi(\bar{b}) \iff (\varphi^*(N, \bar{b}) \text{ is large}).$$

Let T' be the set of $\mathcal{L}(Q)$-sentences φ such that $T \vdash \varphi^*$. It is straightforward to verify that T' is closed under the rules of inference on pages 6–7 of [Keisler 1970]. For example, the "axioms of $\mathcal{L}(Q)$" [Keisler 1970, p. 6] correspond to the axioms in Definition 3.3. By the completeness theorem for $\mathcal{L}(Q)$ [Keisler 1970, Section 2], there is an $\mathcal{L}$-structure N which satisfies the sentences T', when (Qx) is interpreted as

"there are uncountably many x such that…".

Ignoring the sentences involving Q, we see that $N \models T$, and so $N \succeq M$. Finally, suppose $X = \varphi(N, \bar{b})$ is definable in N. Let $\psi(\bar{y})$ be the $\mathcal{L}$-formula such that $\psi(\bar{b})$ holds iff $\varphi(N, \bar{b})$ is large. Then T' contains the sentence

$$(\forall \bar{y})[\psi(\bar{y}) \leftrightarrow (Qx)\varphi(x, \bar{y})],$$

because its image under $(-)^*$ is the tautology

$$(\forall \bar{y})[\psi(\bar{y}) \leftrightarrow \psi(\bar{y})].$$

Therefore,

$$\varphi(N, \bar{b}) \text{ is uncountable} \iff N \models \psi(\bar{b}) \iff \varphi(N, \bar{b}) \text{ is large.} \qquad \square$$

Lemma 3.5. *Let T be a complete geometric theory in a countable language. Then there is a model $N \models T$ such that for any interpretable set X,*

$$\dim(X) > 0 \iff |X| > \aleph_0.$$

Proof. Take a model $M \models T$. There is a definable notion of largeness on M^{eq} in which X is large iff $\dim(X) > 0$. The requirements of Definition 3.3 hold by properties of dimension in M^{eq} [Johnson 2022, Propositions 2.8, 2.9, and 2.12]. Applying Fact 3.4, we get an elementary extension $N^{\text{eq}} \succeq M^{\text{eq}}$ such that if X is definable in N^{eq}, then X is uncountable iff $\dim(X) > 0$. $\square$

Proposition 3.6. *Let T be a complete, geometric theory of infinite fields, possibly with extra structure.*

1. *If $K \models T$ and X is an interpretable set of positive dimension, then $|X| = |K|$.*
2. *If the language is countable, there is a model $K \models T$ of cardinality $\aleph_1$ such that every zero-dimensional interpretable set is countable.*

Proof. (1) Suppose X has positive dimension. Lemma 3.2 shows that $|X| \geq |D|$ for some infinite definable set D. Then $|D| \geq |X| \geq |K|$ by Proposition 3.1. Finally, $|K| \geq |X|$ is clear.

(2) Lemma 3.5 gives an uncountable model K in which every zero-dimensional interpretable set is countable. By downward Löwenheim–Skolem, we can replace K with an elementary substructure of cardinality $\aleph_1$. □

4. Finite extensions

Let T be a complete, geometric theory of fields, possibly with extra structure, not necessarily algebraically bounded.

Proposition 4.1. *If $K \models T$ and $n \in \mathbb{N}_{>0}$, then the (interpretable) set of degree n finite extensions has dimension zero.*

Proof. By Lemma 2.2, K is perfect. Hence any finite extension of K is a simple extension. Let X be the set of irreducible monic polynomials of degree n. We can regard X as a definable subset of K^n by identifying a polynomial

$$P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$$

with the n-tuple $(c_0, c_1, \ldots, c_{n-1})$. Let Y be the interpretable set of degree n finite extensions. Let $f : X \to Y$ be the map sending $P(x)$ to the extension $K[x]/(P(x))$.

Note that $\dim(X) \leq \dim(K^n) = n$. By the definition of dimension for interpretable sets, it suffices to show that each fiber of f has dimension at least n. Fix some $b \in Y$ corresponding to a degree n extension L/K. We claim $f^{-1}(b)$ has dimension n. By identifying L with K^n, we can regard L as a definable set with $\dim(L) = n$. Because K is perfect, there are only finitely many fields between K and L. Let U be the union of the intermediate fields. Then U has lower dimension than n, so $\dim(L \setminus U) = n$. The elements of $L \setminus U$ are generators of L. The fiber $f^{-1}(b)$ is the set of minimal polynomials of elements of $L \setminus U$. Let $\rho : (L \setminus U) \to f^{-1}(b)$ be the map sending $a \in L \setminus U$ to its minimal polynomial. Then ρ is finite-to-one, so $\dim(f^{-1}(b)) \geq \dim(L \setminus U) = n$. □

We recover the following corollary, which is presumably well-known (for example, it follows from [Gagelman 2005, Corollary 3.6] and [Pillay and Poizat 1995, Théorème]).

Corollary 4.2. *If T eliminates imaginaries, then models of T have bounded Galois group — there are only finitely many extensions of degree n for each n.*

Proof. Zero-dimensional definable sets are finite. If elimination of imaginaries holds, then zero-dimensional interpretable sets are finite. □

Combining Propositions 3.6 and 4.1, we have the following.

Corollary 4.3. *If T is a geometric theory of infinite fields in a countable language, then there is an uncountable model $K \models T$ with countably many finite extensions of degree n for each n.*

For example, since perfect PAC fields are geometric [Chatzidakis and Hrushovski 2004], Corollary 4.3 recovers results such as [Bary-Soroker and Paran 2013, Example 2.2]. The fact that all very slim fields satisfy this property might have some field-theoretic consequences.

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