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We define the notion of mock hyperbolic reflection spaces and use it to study Frobenius groups. These turn out to be particularly useful in the context of Frobenius groups of finite Morley rank including the so-called *bad groups*. We show that connected Frobenius groups of finite Morley rank and odd type with nilpotent complement split or interpret a bad field of characteristic zero. Furthermore, we show that mock hyperbolic reflection spaces of finite Morley rank satisfy certain rank inequalities, implying, in particular, that any connected Frobenius group of odd type and Morley rank at most ten either splits or is a simple nonsplit sharply 2-transitive group of characteristic $\neq 2$ of Morley rank 8 or 10.

1. Introduction

This paper contributes to the study of groups acting on geometries arising naturally from conjugacy classes of involutions. We define the notion of a mock hyperbolic reflection space and use it to study certain Frobenius groups. Such an approach to the classification of groups and their underlying geometries based on involutions was developed by Bachmann [1959]. Mock hyperbolic reflection spaces generalize real hyperbolic spaces and their definition is motivated by the geometry arising from the involutions in certain nonsplit sharply 2-transitive groups.

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The points of such a mock hyperbolic space are given by a conjugacy class of involutions, and we view the conjugation action by an involution in the space as a point-reflection. More precisely, a conjugacy class of involutions in a group forms a *mock hyperbolic reflection space* if it admits the structure of a linear space such that three axioms are satisfied: three points are collinear if and only if the product of their point-reflections is a point-reflection, for any two points there is a unique midpoint, i.e., a unique point reflecting one point to the other, and given two distinct lines there is at most one point reflecting one line to the other.

We will consider in particular mock hyperbolic reflection spaces arising from Frobenius groups of finite Morley rank. One of the main open problems about groups of finite Morley rank is the *algebraicity conjecture*, which states that any infinite simple group of finite Morley rank should be an algebraic group over an algebraically closed field. While the conjecture was proved by Altınel, Borovik, and Cherlin [Altınel et al. 2008] in the characteristic 2 setting, it is still wide open in general and in particular in the situation of small (Tits) rank. The conjecture would in fact imply that any sharply 2-transitive group of finite Morley rank and, more generally, any Frobenius group of finite Morley rank splits.

A Frobenius group is a group G together with a proper nontrivial malnormal subgroup H, i.e., a subgroup H such that $H \cap H^g = \{1\}$ for all $g \in G \setminus H$. (If G is a *bad group* of finite Morley rank with Borel subgroup B then $B < G$ is a Frobenius group.) A classical result due to Frobenius states that finite Frobenius groups split, i.e., they can be written as a semidirect product of a normal subgroup and the subgroup H. In the setting of finite groups the methods used by Frobenius play an important role in the classification of CA-groups, CN-groups, and groups of odd order. For groups of finite Morley rank, all the corresponding classification problems are still wide open.

Sharply 2-transitive groups of finite Morley rank came to renewed attention when recently the first sharply 2-transitive groups without nontrivial abelian normal subgroup were constructed in characteristic 2 in [Rips et al. 2017] (see also [Tent and Ziegler 2016]) and in characteristic 0 in [Rips and Tent 2019]. However, as we show below, these groups do not have finite Morley rank. We also show that specific nonsplit sharply 2-transitive groups of finite Morley rank would indeed be direct counterexamples to the algebraicity conjecture.

We prove the following splitting criteria for groups with an associated mock hyperbolic reflection space:

Theorem 1.1. *If G is a group with an associated mock hyperbolic reflection space J, then the following are equivalent*:

(a) $G \cong A \rtimes \mathrm{Cen}(q)$ *for some abelian normal subgroup A and any $q \in J$.*

(b) *J is a (possibly degenerate) projective plane.*

(c) *J consists of a single line.*

We show a rank inequality for mock hyperbolic reflection spaces in groups of finite Morley rank: if J is a mock hyperbolic reflection space of Morley rank n such that lines are infinite and of Morley rank k, then $n \leq 2k$ implies that J consists of a single line (and hence $n = k$). If $n = 2k + 1$, then there exists a normal subgroup similar to the one in the above theorem (see Theorem 6.11).

We then consider mock hyperbolic reflection spaces arising from Frobenius groups. A connected Frobenius group G of finite Morley rank with Frobenius complement H falls into one of three classes: it is either degenerate or of odd or of even type depending on whether or not G and H contain involutions (see Section 4). A connected Frobenius group is of odd type if and only if the Frobenius complement contains an involution. In particular, every sharply 2-transitive group of finite Morley rank and characteristic different from 2 is a Frobenius group of odd type. We show:

Theorem 1.2. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type.*

(a) *The involutions J in G form a mock hyperbolic reflection space and all lines are infinite.*

(b) *If a generic pair of involutions is contained in a line of Morley rank k and* $\mathrm{MR}(J) \leq 2k + 1$, *then $H < G$ splits.*

(c) *If G does not split and a generic pair of involutions is contained in a line of Morley rank* 1, *then G is a simple sharply* 2*-transitive group of characteristic $\neq 2$ and hence a direct counterexample to the algebraicity conjecture.*

(d) *If* $\mathrm{MR}(G) \leq 10$, *then either G splits or G is a simple nonsplit sharply* 2 *transitive group of characteristic $\neq 2$ and* $\mathrm{MR}(G)$ *is either* 8 *or* 10.

For nilpotent Frobenius complements we show the following splitting criteria:

Theorem 1.3. *If $H < G$ is a connected Frobenius group of finite Morley rank and odd type with nilpotent complement H, then any of the following conditions implies that $H < G$ splits*:

- *H is a minimal group.*
- *The lines in the associated mock hyperbolic reflection space have Morley rank* 1.
- *G does not interpret a bad field of characteristic* 0.

If G is a uniquely 2-divisible Frobenius group, then G does not contain involutions. However if the complement H is abelian, then we can use a construction from the theory of K-loops to extend G to a group containing involutions and if $H < G$ is full, i.e., if $G = \bigcup_{g \in G} H^g$, then the involutions in this extended group will again form a mock hyperbolic reflection space (see Section 4).

This construction allows us to use mock hyperbolic reflection spaces to study Frobenius groups of finite Morley rank and degenerate type. This class contains potential bad groups. Frécon [2018] showed that bad groups of Morley rank 3 do not exist. Subsequently, Wagner [2017] used Frécon's methods to show more generally that if $H < G$ is a simple *full* Frobenius group of Morley rank n with abelian Frobenius complement H of Morley rank k, then $n > 2k + 1$. Note that the existence of full Frobenius groups was claimed by Ivanov and Olshanski, but to the authors' best knowledge no published proof exists (see also [Jaligot 2001, Fact 3.1]).

If $H < G$ is a not necessarily full or simple Frobenius group of finite Morley rank and degenerate type, we obtain a weaker version of mock hyperbolic reflection spaces which still allows us to extend Frécon's and Wagner's results:

Theorem 1.4. *If $H < G$ is a connected Frobenius group of Morley rank n and degenerate type with abelian Frobenius complement H of Morley rank k, then $n \geq 2k + 1$.*

If $n = 2k + 1$, then G splits as $G = N \rtimes H$ for some definable connected normal subgroup N. Moreover, if N is solvable, then there is an interpretable field K of characteristic $\neq 2$ such that $G = K_+ \rtimes H$, $H \leq K^*$, and H acts on K_+ by multiplication.

2. Mock hyperbolic reflection spaces

We now introduce the notion of mock hyperbolic reflection spaces, which will be central to our work. The motivating example for our construction comes from sharply 2-transitive groups in characteristic different from 2 (see Section 3) in which the involutions have a rich geometric structure, which is reflected in the following definition.

Let G be a group and $J \subset G$ a conjugacy class of involutions in G, and let $\Lambda \subset \mathcal{P}(J)$ be a G-invariant family of subsets of J such that each $\lambda \in \Lambda$ contains at least two elements. We view involutions in J as points and elements of Λ as lines, so that the conjugation action of J on itself corresponds to point reflections.

For involutions $i \neq j \in J$, we write

$$\ell_{ij} = \{k \in J : ij \in kJ\},$$

and we say that the line ℓ_{ij} *exists in* Λ if $\ell_{ij} \in \Lambda$.

Definition 2.1. Let G be a group, let $J \subset G$ be a conjugacy class of involutions in G, and let $\Lambda \subset \mathcal{P}(J)$ be G-invariant and such that each $\lambda \in \Lambda$ contains at least two elements. The pair (J, Λ) is a *partial mock hyperbolic reflection space* if the following conditions are satisfied:

(a) For all $\lambda \in \Lambda$ and $i \neq j \in \lambda$, we have

$$\lambda = \ell_{ij} = \{k \in J : ij \in kJ\}.$$

In particular, if $i \neq j$ are contained in lines $\lambda, \delta \in \Lambda$, then $\lambda = \ell_{ij} = \delta$. Therefore any two points are contained in at most one line.

(b) Midpoints exist and are unique; i.e., given i, j in J there is a unique $k \in J$ such that $i^k = j$.

(c) Given two distinct lines there is at most one point reflecting one line to the other. In other words, if $\lambda^i = \lambda^j$ for $i \neq j$ in J, then $\lambda^i = \lambda = \lambda^j$.

We say that (J, Λ) is a *mock hyperbolic reflection space* if it satisfies (a)–(c) and furthermore $\ell_{ij} \in \Lambda$ for all $i \neq j \in J$.

Given a group G and a conjugacy class J of involutions in G, in light of Definition 2.1 and in a slight abuse of notation, we say that J forms a mock hyperbolic reflection space if $(J, \{\ell_{ij} : i \neq j \in J\})$ is a mock hyperbolic reflection space.

For a group G and subset $A \subset G$, we write

$$A^{\cdot n} = \{a_1 \cdots a_n \mid a_1, \ldots, a_n \in A\} \subseteq G.$$

Remark 2.2. Let (J, Λ) be a partial mock hyperbolic reflection space. We say that involutions $i, j, k \in J$ are *collinear* if there is some $\lambda \in \Lambda$ with $i, j, k \in \lambda$.

Furthermore, if J is a conjugacy class of involutions in G and $\Lambda \subset \mathcal{P}$ is such that every $\lambda \in \Lambda$ contains at least two elements, we will see below that if (a) and (b) hold, then (c) is equivalent to either of the following conditions:

(c′) If $\lambda^i = \lambda^j$ for $i \neq j$ in J and $\lambda \in \Lambda$, then $i, j \in \lambda$.

(c″) For every line $\lambda \in \Lambda$, we have $N_G(\lambda) \cap J^{\cdot 2} = \lambda^{\cdot 2}$.

If $\Lambda = \{\ell_{ij} : i \neq j \in J\}$, then (a) is equivalent to

$$i, j, k \in J \quad \text{are collinear if and only if} \quad ijk \in J.$$

Example 2.3. Let $\mathbb{H}^n$ be the n-dimensional real hyperbolic space. Then $\mathrm{Isom}(\mathbb{H}^n)$, the group of all isometries of $\mathbb{H}^n$, contains the point-reflections as a conjugacy class J of involutions. J can be identified with $\mathbb{H}^n$, and hence J forms a mock hyperbolic reflection space. In case $n = 2$ the simple group $\mathrm{PSL}_2(\mathbb{R})$ consists of all orientation-preserving isometries of $\mathbb{H}^2$. $\mathrm{PSL}_2(\mathbb{R})$ is generated by the point-reflections and the point-reflections are the only involutions.

Example 2.4. Let A be a uniquely 2-divisible abelian group, and let $\epsilon \in \mathrm{Aut}(A)$ be given by $\epsilon(x) = x^{-1}$. Put $G = A \rtimes \langle \epsilon \rangle$. Then the set of involutions in G is given by $J = A \times \{\epsilon\}$ and J forms a mock hyperbolic reflection space consisting of a single line.

Other examples arise from sharply 2-transitive groups (Section 3) or can be constructed from a class of uniquely 2-divisible Frobenius groups (Section 4).

Lemma 2.5. *Let G be a group and J a conjugacy class of involutions in G such that J acts regularly on itself by conjugation, i.e., J satisfies condition* (b) *in Definition 2.1. Then the following holds for any $i \in J$:*

(a) *iJ is uniquely 2-divisible.*

(b) $J^{\cdot 2} \cap \mathrm{Cen}(i) = \{1\}$.

(c) *$G = (iJ)\,\mathrm{Cen}(i)$ and every $g \in G$ can be written uniquely as $g = ijh$ with $j \in J$ and $h \in \mathrm{Cen}(i)$.*

Proof. (a) Fix $ia \in iJ$. We have to show that there is a unique $b \in J$ such that

$$ia = (ib)^2 = ibib = ii^b.$$

This is exactly condition (b) in Definition 2.1.

(b) Suppose a and b are involutions in J such that $i^{ab} = i$. Then $i^a = i^b$, and hence $a = b$ by the uniqueness in condition (b). Hence $ab = 1$.

(c) Let $g \in G$, and set $k = i^{g^{-1}}$. Then there is a unique $j \in J$ such that $k^{ij} = i$. Now put $h = jig$. Then $g = ijh$ and, we have

$$k^{ij} = i = k^g = k^{ijh} = i^h,$$

and therefore $h \in \mathrm{Cen}(i)$. This shows existence of such a decomposition, and uniqueness follows from part (b). □

In accordance with the terminology from real hyperbolic spaces or from sharply 2-transitive groups, we call elements of the set

$$S = \{\sigma \in J^{\cdot 2} \setminus \{1\} : \ell_\sigma \text{ exists in } \Lambda\} \cup \{1\}$$

translations. Then (b) of Lemma 2.5 implies that nontrivial translations have no fixed points (in their action on J).

B. H. Neumann [1940] showed that a uniquely 2-divisible group admitting a fixed-point-free involutionary automorphism must be abelian. More generally, uniquely 2-divisible groups with involutionary automorphisms can be decomposed as follows:

Proposition 2.6 [Borovik and Nesin 1994, Exercise 14 on p. 73]. *Let G be a uniquely 2-divisible group, and let $\alpha \in \mathrm{Aut}(G)$ be an involutionary automorphism. Define the sets* $\mathrm{Inv}(\alpha) = \{g \in G : g^\alpha = g^{-1}\}$ *and* $\mathrm{Cen}(\alpha) = \{g \in G : g^\alpha = g\}$.

Then $G = \mathrm{Inv}(\alpha)\,\mathrm{Cen}(\alpha)$, and for every $g \in G$ there are unique $a \in \mathrm{Inv}(\alpha)$ and $b \in \mathrm{Cen}(\alpha)$ such that $g = ab$. In particular, if α has no fixed points, then G is abelian and α acts by inversion.

Lemma 2.7. *Suppose* (J, Λ) *satisfies conditions* (a) *and* (b) *in Definition 2.1. Let* λ *be a line in* Λ.

(a) $N_G(\lambda) \cap J = \lambda$.

(b) *If* $i \in \lambda$, *then* $\lambda^{\cdot 2} = i\lambda$.

(c) *If* a *and* b *are distinct involutions in* J *such that* $ab \in \lambda^{\cdot 2}$, *then* $a, b \in \lambda$.

(d) $\lambda^{\cdot 2}$ *is a uniquely* 2*-divisible abelian group.*

(e) *If* $i, j, k \in J$ *are such that* ℓ_{ij} *and* ℓ_{jk} *exist in* Λ, *then* $\ell_{ij}^{\cdot 2} \cdot \ell_{jk}^{\cdot 2} = \ell_{ij} \cdot \ell_{jk} \subseteq J^{\cdot 2}$.

(f) $N_G(\lambda) = N_G(\lambda^{\cdot 2})$.

Proof. (a) We first show $\lambda \subseteq N_G(\lambda)$: If $\lambda = \ell_{ij}$, then $j^i = iji$, and hence $ij \in j^i J$. Therefore $j^i \in \lambda$.

Now assume $k \in N_G(\lambda) \cap J$ and $\lambda = \ell_{ij}$. We may assume $k \neq i$. Then $i \neq i^k \in \lambda$, and hence $\lambda = \ell_{i^k i}$. Now $i^k i = kk^i$, so $i^k i \in kJ$, and therefore $k \in \lambda$.

(b) Fix $a \neq b$ in λ. Then $ab \in iJ$, and hence $ab = ij$ for some $j \in J$. It remains to show $j \in \lambda$: we have $ab = ij \in Jj = jJ$, and hence $j \in \lambda$.

(c) Suppose $ab = ij$ and $\lambda = \ell_{ij}$. Then $(ij)^a = (ij)^{-1} = ji$, and therefore $\lambda^a = \lambda$, so $a \in N_G(\lambda) \cap J = \lambda$. Now $a\lambda = \lambda^{\cdot 2}$, and hence $b \in \lambda$.

(d) We first show that $\lambda^{\cdot 2} = i\lambda$ is uniquely 2-divisible. Since we know that iJ is uniquely 2-divisible, it remains to show that $i\lambda$ is 2-divisible. Fix $ia \in i\lambda$, say $ia = (ib)^2$ for some $b \in J$. Then $ia = ii^b$, so $a = i^b$, and thus $b \in N_G(\lambda) \cap J = \lambda$.

It remains to show that $\lambda^{\cdot 2} = i\lambda$ is an abelian group. Note that $i\lambda = \lambda i$, and hence $\lambda^{\cdot 2}$ is closed under multiplication and taking inverses. Therefore $\lambda^{\cdot 2}$ is a uniquely 2-divisible group. Moreover, i acts on $\lambda^{\cdot 2}$ as an involutionary automorphism without fixed points. Now Proposition 2.6 implies that $\lambda^{\cdot 2}$ is abelian.

(e) Since j normalizes ℓ_{ij}, we have $\ell_{ij}^{\cdot 2} = \ell_{ij} j$ by (b), and hence the claim follows.

(f) We only need to show that $N_G(\lambda^{\cdot 2}) \subseteq N_G(\lambda)$. Take $g \in N_G(\lambda^{\cdot 2}) \setminus \{1\}$ and fix $i \neq j \in \lambda$. Then $ij \in \lambda^{\cdot 2}$, and hence $i^g j^g \in \lambda^{\cdot 2}$. Therefore $i^g, j^g \in \lambda$ by (c), and thus $\lambda^g = \lambda$. □

Lemma 2.8. *Suppose* (J, Λ) *satisfies* (a) *and* (b) *in Definition 2.1. Then the following are equivalent*:

(a) (J, Λ) *is a partial mock hyperbolic reflection space.*

(b) *Every line* $\lambda \in \Lambda$ *satisfies* $N_G(\lambda) \cap J^{\cdot 2} = \lambda^{\cdot 2}$.

Proof. Suppose that (J, Λ) forms a partial mock hyperbolic reflection space and fix $ij \in N_G(\lambda) \cap J^{\cdot 2}$ and assume $i \neq j \in J$. Then $\lambda^i = \lambda = \lambda^j$, and therefore $i, j \in N_G(\lambda) \cap J = \lambda$. Thus $N_G(\lambda) \cap J^{\cdot 2} = \lambda^{\cdot 2}$.

Conversely, assume $N_G(\lambda) \cap J^{\cdot 2} = \lambda^{\cdot 2}$ and $\lambda^i = \lambda^j$ for $i \neq j \in J$. Then $ij \in \lambda^{\cdot 2}$, and hence $i, j \in \lambda$ by Lemma 2.7(c). This shows $\lambda^i = \lambda = \lambda^j$. □

Proposition 2.9. *Let G be a group and J a conjugacy class of involutions in G, and suppose (J, Λ) is a partial mock hyperbolic reflection space. Then the following holds*:

(a) *If $\lambda = \ell_{ij} \in \Lambda$, then $\lambda^{\cdot 2} = iJ \cap jJ = \mathrm{Cen}(ij) \cap J^{\cdot 2}$.*

(b) *The set $S \setminus \{1\} = \{ij \in J^{\cdot 2} \setminus \{1\} : \ell_{ij}$ exists in $\Lambda\}$ is partitioned by the family $\{\lambda^{\cdot 2} \setminus \{1\} : \lambda \in \Lambda\}$.*

Proof. By Lemma 2.7, (b) follows from (a). In order to prove (a), we first show $\lambda^{\cdot 2} = iJ \cap jJ$. Fix $ia = jb \in iJ \cap jJ$. Then $ab = ij \in \lambda^{\cdot 2}$, and hence $a, b \in \lambda$ by Lemma 2.7(c). This shows $iJ \cap jJ \subseteq \lambda^{\cdot 2}$. Moreover, we have $\lambda^{\cdot 2} = i\lambda = j\lambda$, and hence $\lambda^{\cdot 2} \subseteq iJ \cap jJ$. Thus $\lambda^{\cdot 2} = iJ \cap jJ$.

The group $\lambda^{\cdot 2}$ is abelian and contains ij. Hence $\lambda^{\cdot 2} \subseteq \mathrm{Cen}(ij) \cap J^{\cdot 2}$. Any element $g \in \mathrm{Cen}(ij)$ normalizes $\lambda = \ell_{ij} = \ell_{ji}$. Thus $\mathrm{Cen}(ij) \cap J^{\cdot 2} \subseteq N_G(\lambda) \cap J^{\cdot 2} = \lambda^{\cdot 2}$ (Lemma 2.8), and hence $\mathrm{Cen}(ij) \cap J^{\cdot 2} = \lambda^{\cdot 2}$. □

If i is an involution in J, then we define $\Lambda_i = \{\lambda \in \Lambda : i \in \lambda\}$ to be the set of all lines that contain i.

Proposition 2.10. *Suppose (J, Λ) forms a partial mock hyperbolic reflection space.*

(a) *Suppose $\lambda \cap \lambda^j \neq \varnothing$ for a line λ and an involution j in J. Then $j \in \lambda$, and therefore $\lambda = \lambda^j$.*

(b) *G acts transitively on Λ if and only if $\mathrm{Cen}(i)$ acts transitively on Λ_i for each $i \in J$.*

Proof. (a) Suppose $\{i\} = \lambda \cap \lambda^j$. Then $i = i^j$ and therefore $j = i \in \lambda$.

(b) If $\mathrm{Cen}(i)$ acts transitively on Λ_i, then G is transitive on Λ, because all involutions in J are conjugate.

Now assume G acts transitively on Λ and suppose $i \in \lambda \cap \lambda^g$ for some $g \in G$. By Lemma 2.5, g can be written as $g = ijh$ for some $j \in J$ and $h \in \mathrm{Cen}(i)$. Note that $\lambda^g = \lambda^{ijh} = \lambda^{jh}$, because i is contained in λ.

Since $h \in \mathrm{Cen}(i)$, this implies that i must be contained in λ^j, and hence

$$i \in \lambda \cap \lambda^j.$$

Therefore (a) implies that j must be contained in λ, and hence $\lambda = \lambda^j$. Hence $\lambda^g = \lambda^h$. Since g was arbitrary, this shows that $\mathrm{Cen}(i)$ acts transitively on Λ_i. □

The geometry of a mock hyperbolic reflection space. Recall that a mock hyperbolic reflection space is a partial hyperbolic space such that any two points are contained in a line. As a first step, we show that the geometry of a mock hyperbolic reflection space cannot contain a proper projective plane:

Lemma 2.11. *Suppose that (J, Λ) is a mock hyperbolic reflection space in a group G and that $X \subseteq J$ is a projective plane. That is, suppose*

(a) *for all* $i \neq j \in X$ *the line* ℓ_{ij} *is contained in* X, *and*

(b) *if* λ *and* δ *are lines contained in* X *then* $\lambda \cap \delta \neq \varnothing$.

Then $X^{\cdot 2}$ *is a uniquely* 2*-divisible subgroup of* G.

Proof. The set $X^{\cdot 2}$ is a group by Lemma 2.7(e). This group is uniquely 2-divisible by Lemma 2.7(d) and Proposition 2.9(b). □

Lemma 2.12. *Suppose* J *forms a mock hyperbolic reflection space in a group* G, *and let* $H \subseteq J^{\cdot 2}$ *be a subgroup of* G *which is uniquely* 2*-divisible and normalized by an involution* $i \in J$. *Then* $H \subseteq \mathrm{Cen}(\sigma)$ *for some* $\sigma \in J^{\cdot 2} \setminus \{1\}$.

Proof. Since H is uniquely 2-divisible and i acts as an involutionary automorphism without fixed points, Proposition 2.6 implies that H is abelian and hence must be contained in the centralizer of some translation. □

Proposition 2.13. *If* J *is a mock hyperbolic reflection space in a group* G, *then it does not contain a proper projective plane. That is*, *if* $X \subseteq J$ *is a projective plane*, *then* X *contains at most one line.*

Proof. By Lemma 2.11, the set $X^{\cdot 2}$ is a uniquely 2-divisible subgroup of G. By Lemma 2.5(b), each $j \in X$ acts on $X^{\cdot 2}$ as an involutionary automorphism without fixed points. By the previous lemma, $X^{\cdot 2} \leq \mathrm{Cen}(\sigma)$ for some $\sigma \in J^{\cdot 2} \setminus \{1\}$, and hence $X \subseteq \ell_\sigma$ by Lemma 2.7(c). □

Theorem 2.14. *Suppose* J *forms a mock hyperbolic reflection space in a group* G. *Then the following are equivalent*:

(a) Λ *consists of a single line.*

(b) J *is a projective plane.*

(c) G *has an abelian normal subgroup* $A \not\subseteq \bigcap_{i \in J} \mathrm{Cen}(i)$.

(d) $J^{\cdot 2} = iJ$ *for any involution* $i \in J$.

(e) iJ *is commutative for any involution* $i \in J$.

(f) iJ *is a subgroup of* G *for any involution* $i \in J$.

(g) $J^{\cdot 2}$ *is a subgroup of* G.

(h) iJ *is an abelian normal subgroup of* G, *and* G *splits as* $G = iJ \rtimes \mathrm{Cen}(i)$ *for any involution* $i \in J$.

Proof. We show the following implications:

$$\text{(d)} \iff \text{(a)} \implies \text{(b)} \implies \text{(g)} \implies \text{(e)} \implies \text{(f)} \implies \text{(h)} \implies \text{(c)} \implies \text{(a)}.$$

To show (a) $\iff$ (d), assume (d) and fix a line $\lambda = \ell_{ij}$. Then

$$i\lambda = \lambda^{\cdot 2} = iJ \cap jJ = J^{\cdot 2} = iJ,$$

and hence $\lambda = J$ is the only line. Conversely, assume (a) holds and $\lambda = J$ is the unique line. Then $J^{\cdot 2} = \lambda^{\cdot 2} = i\lambda = iJ$ by Lemma 2.7.

(a) $\Longrightarrow$ (b) is trivial.

(b) $\Longrightarrow$ (g) holds by Lemma 2.11.

Now assume (g) holds. By Proposition 2.9, $J^{\cdot 2} \setminus \{1\}$ is partitioned by the family $\{\lambda^{\cdot 2} \setminus \{1\} : \lambda \in \Lambda\}$. Each $\lambda^{\cdot 2}$ is a uniquely 2-divisible abelian group by Lemma 2.7. Therefore $J^{\cdot 2}$ is uniquely 2-divisible. If i is any involution, then i normalizes $J^{\cdot 2}$ and acts by conjugation as an involutionary automorphism without fixed points. Therefore $J^{\cdot 2}$ is an abelian group by Proposition 2.6. In particular, $iJ \subseteq J^{\cdot 2}$ is commutative. This shows (e).

Now assume (e). iJ is partitioned by $\{\lambda^{\cdot 2} : \lambda \in \Lambda, i \in \lambda\}$ and if $\lambda = \ell_{ij}$, then $\lambda^{\cdot 2} = iJ \cap jJ = \mathrm{Cen}(ij) \cap J^{\cdot 2}$ by Proposition 2.9(a). Since iJ is commutative, this implies $iJ = \lambda^{\cdot 2}$, and hence iJ is a subgroup of G by Lemma 2.7. This shows (f).

We next show (f) $\Longrightarrow$ (h): iJ is a uniquely 2-divisible group and i acts as an involutionary automorphism without fixed points. Therefore iJ is an abelian subgroup of G by Proposition 2.6. Note that $N_G(iJ)$ contains $\mathrm{Cen}(i)$ and iJ. Therefore $G = iJ\,\mathrm{Cen}(i) = N_G(iJ)$ by Lemma 2.5. Hence iJ is an abelian normal subgroup of G, and therefore $G = iJ \rtimes \mathrm{Cen}(i)$ by Lemma 2.5.

(h) $\Longrightarrow$ (c) is obvious.

To see that (c) implies (a), let $i \in J$ and $a \in A \setminus \mathrm{Cen}(i)$. Then

$$1 \neq a^{-1}a^i = i^a i \in A \cap iJ.$$

In particular, $A \cap iJ$ is nontrivial. Now fix $\sigma \in (A \cap iJ) \setminus \{1\}$, and set $\lambda = \ell_\sigma$. Then

$$\mathrm{Cen}(\sigma) \cap J^{\cdot 2} = \lambda^{\cdot 2},$$

so $A \cap J^{\cdot 2} \subseteq \lambda^{\cdot 2}$. This implies that $\lambda^{\cdot 2}$ is a normal subset of G, and hence λ is a normal subset of G by Lemma 2.7. Therefore $\lambda = J$ by Lemma 2.7. □

3. Sharply 2-transitive groups

In this section we consider a particular class of Frobenius groups: a permutation group G acting on a set X, where $|X| \geq 2$, is called *sharply 2-transitive* if it acts regularly on pairs of distinct points, or equivalently, if G acts transitively on X and for each $x \in X$ the point stabilizer G_x acts regularly on $X \setminus \{x\}$. Thus, a sharply 2-transitive group *splits* if it can be written as a semidirect product of a regular normal subgroup with a point-stabilizer. For two distinct elements $x, y \in X$ the unique $g \in G$ such that $(x, y)^g = (y, x)$ is an involution. Hence the set J of involutions in G is nonempty and forms a conjugacy class.

The (permutation) characteristic of a group G acting sharply 2-transitively on a set X is defined as follows: put $\mathrm{char}(G) = 2$ if and only if involutions have no fixed points. If involutions have a (necessarily unique) fixed point, the G-equivariant bijection $i \mapsto \mathrm{fix}(i)$ allows us to identify the given action of G on X with the conjugation action of G on J. Thus, in this case, the set $S \setminus \{1\}$ of nontrivial translations also forms a single conjugacy class. We put $\mathrm{char}(G) = p$ (or 0) if translations have order p (or infinite order, respectively). For the standard examples of sharply 2-transitive groups, namely $K \rtimes K^*$ for some field K, this definition of characteristic agrees with the characteristic of the field K.

Remark 3.1. Let G be a sharply 2-transitive group of characteristic $\mathrm{char}(G) \neq 2$. Since G acts sharply 2-transitively by conjugation on the set J of involutions in G, the following properties are easy to see:

(a) $\mathrm{Cen}(i)$ acts regularly on $J \setminus \{i\}$.

(b) The set J acts regularly on itself by conjugation, that is, condition (b) of Definition 2.1 holds.

(c) $J^{\cdot 2} \cap \mathrm{Cen}(i) = \{1\}$ for all $i \in J$.

In particular, a nontrivial translation does not have a fixed point.

In order to define the lines for a mock hyperbolic reflection space on J, we need the following equivalent conditions to be satisfied:

Proposition 3.2. *If G is a sharply 2-transitive group of characteristic different from 2, the following conditions are equivalent*:

(a) *Commuting is transitive on $J^{\cdot 2} \setminus \{1\}$.*

(b) *$iJ \cap kJ$ is uniquely 2-divisible for all involutions $i \neq k \in J$.*

(c) *$\mathrm{Cen}(ik) = iJ \cap kJ$ is abelian and is inverted by k for all $i \neq k \in J$.*

(d) *The set $\{\mathrm{Cen}(\sigma) \setminus \{1\} : \sigma \in J^{\cdot 2} \setminus \{1\}\}$ forms a partition of $J^{\cdot 2} \setminus \{1\}$.*

Note that these conditions are satisfied in split sharply 2-transitive groups by Theorem 3.5 whenever $\mathrm{char}(G) = p \neq 0, 2$ or if G satisfies the descending chain condition for centralizers, so in particular if G has finite Morley rank by [Borovik and Nesin 1994, Lemma 11.50].

Proof. For (a) $\Longrightarrow$ (b), note that since $(ij)^2 = ii^j \in iJ$ every element of iJ has a unique square root in iJ. Let $\tau \in iJ \cap kJ$. Since commuting is transitive, the group $A = \langle \mathrm{Cen}(\tau) \cap J^{\cdot 2} \rangle \leq \mathrm{Cen}(\tau)$ is abelian. Moreover, $A \cap J = \varnothing$ by Remark 3.1. Hence the square map is an injective group homomorphism from A to A.

There is $\sigma_i \in iJ$ such that $\sigma_i^2 = \tau$ and so $\sigma_i \in \mathrm{Cen}(\tau) \cap iJ$. Similarly we find $\sigma_k \in \mathrm{Cen}(\tau) \cap kJ$ such that $\sigma_k^2 = \tau$. Since the square map is injective, it follows that $\sigma_i = \sigma_k \in iJ \cap kJ$. Therefore $iJ \cap kJ$ is uniquely 2-divisible.

(b) $\Longrightarrow$ (c) is contained in [Borovik and Nesin 1994, Lemma 11.50(iv)].

(c) $\Longrightarrow$ (d) and (d) $\Longrightarrow$ (a) are obvious. □

The examples constructed in [Rips et al. 2017] (see also [Tent and Ziegler 2016]) show that in characteristic 2 these conditions need not be satisfied. The nonsplit examples in characteristic 0 constructed in [Rips and Tent 2019] satisfy the assumptions and it is an open question whether nonsplit sharply 2-transitive groups exist in characteristic 0 which fail to satisfy these conditions. Note that [Rips and Tent 2019, Lemmas 2.3 and 5.3] imply that the maximal near-field in these examples is planar.

Assume now that the conditions of Proposition 3.2 are satisfied. Then for $i \neq j \in J$ we put

$$\ell_{ij} = \{k \in J : ij \in kJ\} \quad \text{and} \quad \Lambda = \{\ell_{ij} : i \neq j \in J\}.$$

By Remark 3.1(b), (J, Λ) satisfies conditions (a) and (b) of Definition 2.1 and by Proposition 3.2, we have

$$\ell_{ij} = \{k \in J : ij \in kJ\} = i\,\mathrm{Cen}(ij) = \{k \in J : (ij)^k = ji\}.$$

The point-line geometry (J, Λ) is equivalent to the incidence geometry considered by Borovik and Nesin [1994, Section 11.4].

If $\lambda = \ell_{ij}$ is a line, then $N_G(\lambda) = N_G(\mathrm{Cen}(ij))$ is a split sharply 2-transitive group,

$$N_G(\lambda) = \mathrm{Cen}(ij) \rtimes N_{\mathrm{Cen}(i)}(\lambda),$$

and corresponds to the maximal near-field (see, e.g., [Kerby 1974] or [Borovik and Nesin 1994, Chapter 11]). The maximal near-field is called *planar* if

$$N_G(\lambda) = \mathrm{Cen}(ij) \cup \bigcup_{k \in \lambda} N_{\mathrm{Cen}(k)}(\lambda),$$

i.e., if $\mathrm{Cen}(ij)$ coincides with the set of fixed-point-free elements of $N_G(\lambda)$.

Lemma 3.3. *Assume that G is sharply 2-transitive and* $\mathrm{char}(G) \neq 2$, *and if* $\mathrm{char}(G) = 0$, *assume furthermore that G satisfies the descending chain condition on centralizers. Assume moreover that the maximal near-field is planar. If $\lambda \in \Lambda$ and $i \neq j \in J$ such that $\lambda^i = \lambda^j$, then $i, j \in \lambda$ and so $\lambda^i = \lambda = \lambda^j$, and thus condition* (c) *of Definition 2.1 holds.*

Proof. This is contained in the proof of [Borovik and Nesin 1994, Theorem 11.51]. Since our definition of lines is slightly different from the one given in that work, we include a proof. If $\lambda^i = \lambda^j$ then $ij \in N_G(\lambda)$, and hence $ij \in N_G(\lambda^{\cdot 2})$. By Propositions 2.9(a) and 3.2(c), we have $\lambda^{\cdot 2} = \mathrm{Cen}(\sigma)$ for some $\sigma \in J^{\cdot 2} \setminus \{1\}$ such that $\lambda = \ell_\sigma$. Fix $s \in \lambda$. The group $N_G(\mathrm{Cen}(\sigma)) = \mathrm{Cen}(\sigma) \rtimes N_{\mathrm{Cen}(s)}(\mathrm{Cen}(\sigma))$ is split sharply 2-transitive by [Borovik and Nesin 1994, Proposition 11.51]. Since the maximal near-field is planar, we have

$$ij \in N_G(\mathrm{Cen}(\sigma)) \cap J^{\cdot 2} = \mathrm{Cen}(\sigma),$$

and therefore $i, j \in \ell_\sigma = \lambda$. $\square$

Corollary 3.4. *Let G be a sharply 2-transitive group. Then the set of involutions $J \subset G$ forms a mock hyperbolic reflection space in any of the following cases*:

(a) *G is a split sharply 2-transitive group corresponding to a planar near-field of characteristic $\neq 2$*;

(b) $\mathrm{char}(G) = p \neq 0, 2$ *and the maximal near-field is planar*; *or*

(c) $\mathrm{char}(G) = 0$, *G satisfies the descending chain condition for centralizers*, *and the maximal near-field is planar.*

In particular, *if* $\mathrm{char}(G) \neq 2$ *and G is of finite Morley rank*, *then the involutions in G form a mock hyperbolic reflection space.*

In the case of sharply 2-transitive groups, Theorem 2.14 reduces to the following well-known result of Neumann [1940]:

Theorem 3.5. *A sharply 2-transitive group G splits if and only if the set of translations $J^{\cdot 2}$ is a subgroup of G (and in that case, $J^{\cdot 2}$ must in fact be abelian).*

4. Uniquely 2-divisible Frobenius groups

In this section we will construct (partial) mock hyperbolic reflection spaces from uniquely 2-divisible Frobenius groups with abelian Frobenius complement. This construction makes use of K-loops and quasidirect products.

K-loops and quasidirect products. K-loops are nonassociative generalizations of abelian groups. They are also known as Bruck loops and gyrocommutative gyrogroups. We mostly follow Kiechle's book [2002].

Definition 4.1. A groupoid $(L, \cdot, 1)$ is a *K-loop* if

(a) it is a loop, i.e., the equations
$$ax = b \quad \text{and} \quad xa = b$$
have unique solutions for all $a, b \in L$,

(b) it satisfies the Bol condition, i.e.,
$$a(b \cdot ac) = (a \cdot ba)c$$
for all $a, b, c \in L$, and

(c) it satisfies the automorphic inverse property, i.e., all elements of L have inverses, and we have
$$(ab)^{-1} = a^{-1}b^{-1}$$
for all $a, b \in L$.

Given $a \in L$, let $\lambda_a : L \to L$ be defined by $\lambda_a(x) = ax$. Given $a, b \in L$, we define the precession map
$$\delta_{a,b} = \lambda_{ab}^{-1}\lambda_a\lambda_b.$$

These maps are characterized by

$$a \cdot bx = ab \cdot \delta_{a,b}(x) \quad \text{for all } x \in L.$$

If L is a K-loop, then the precession maps are automorphisms, and we set

$$\mathcal{D} = \mathcal{D}(L) = \langle \delta_{a,b} : a, b \in L \rangle \leq \mathrm{Aut}(L).$$

The following identities will be used in this section:

Proposition 4.2. *Let L be a K-loop, $a, b \in L$, and $\alpha \in \mathrm{Aut}(L)$. Then the following identities hold*:

(a) [Kiechle 2002, 2.4(2)] $\alpha^{-1}\delta_{a,b}\alpha = \delta_{\alpha^{-1}(a),\alpha^{-1}(b)}$.

(b) [Kiechle 2002, 6.1(1)] $\delta_{a,a^{-1}} = \mathrm{id}$.

(c) [Kiechle 2002, Theorem 6.4(1)(VI)] $\delta_{a,ba} = \delta_{a,b}$.

(d) [Kiechle 2002, part of Theorem 3.7] $\delta_{a,b} = \delta_{b,a}^{-1} = \delta_{a^{-1},b^{-1}}$.

Definition 4.3. Let G be a group. A subset $L \subseteq G$ is a twisted subgroup of G if and only if $1 \in L$, $L^{-1} \subseteq L$, and $aLa \subseteq L$ for all $a \in L$.

Note that twisted subgroups are closed under the square map. A twisted subgroup is *uniquely* 2*-divisible* if the square map is bijective.

Proposition 4.4 [Kiechle 2002, Theorem 6.14]. *Let G be a group with a uniquely* 2*-divisible twisted subgroup $L \subseteq G$. Then*

$$a \otimes b = a^{1/2} b a^{1/2}$$

makes L into a K-loop $(L, \otimes, 1)$ and integer powers of elements in L agree in G and $(L, \otimes)$. Given $a, b \in L$, the precession map $\delta_{a,b}$ is given by conjugation with

$$d_{a,b} = b^{1/2} a^{1/2} (a^{1/2} b a^{1/2})^{-1/2}.$$

Proof. The formula for the precession maps follows from simple calculation. Everything else is contained in [Kiechle 2002, Theorem 6.14]. □

Proposition 4.5 [Kiechle 2002, Theorem 2.13]. *Let L be a K-loop, and let $\mathcal{A} \leq \mathrm{Aut}(L)$ be a group of automorphisms such that $\mathcal{D}(L) \subseteq \mathcal{A}$. Then*:

(a) *The quasidirect product $L \rtimes_Q \mathcal{A}$ given by the set $L \times \mathcal{A}$ together with the multiplication*

$$(a, \alpha)(b, \beta) = (a \cdot \alpha(b), \delta_{a,\alpha(b)}\alpha\beta)$$

forms a group with neutral element $(1, \mathrm{id})$. *Inverses are given by*

$$(a, \alpha)^{-1} = (\alpha^{-1}(a^{-1}), \alpha^{-1}).$$

(b) *$L \rtimes_Q \mathcal{A}$ acts faithfully and transitively on L by*

$$(a, \alpha)(x) = a\alpha(x) \quad \textit{for all } (a, \alpha) \in L \rtimes_Q \mathcal{A} \textit{ and } x \in L.$$

Mock hyperbolic reflection spaces from uniquely 2-divisible Frobenius groups. Let $H < G$ be a uniquely 2-divisible Frobenius group with abelian complement H.

We set L to be the K-loop $L = (G, \otimes)$, where $\otimes$ is defined by

$$a \otimes b = a^{1/2} b a^{1/2}.$$

Set $\mathcal{A} = G \times \langle \epsilon \rangle < \mathrm{Aut}(L)$, where ϵ inverts all elements of L. Put $\mathcal{G} = L \rtimes_Q \mathcal{A}$. Let J be the set of all involutions in $\mathcal{G}$, and put $\iota = (1, \epsilon) \in J$.

Lemma 4.6. (a) $J = L \times \{\epsilon\}$.

(b) $\mathrm{Cen}(\iota) = 1 \times \mathcal{A}$.

(c) *For all $i, j \in J$, there is a unique $k \in J$ such that $j = i^k$.*

Proof. L is a K-loop by Proposition 4.4.

(a) Fix $(a, \alpha) \in \mathcal{G}$ such that $(a, \alpha)^2 = (1, \mathrm{id})$. Note that

$$(a, \alpha)(a, \alpha) = (a \otimes \alpha(a), \delta_{a,\alpha(a)} \alpha^2).$$

Now $a \otimes \alpha(a) = 1$ implies $\alpha(a) = a^{-1}$, and therefore $\delta_{a,\alpha(a)} = \mathrm{id}$. Hence we must have $\alpha^2 = \mathrm{id}$.

If $\alpha = \mathrm{id}$, then $a \otimes \alpha(a) = a^2$, so $a^2 = 1$, and thus $a = 1$. In that case, $(a, \alpha) = (1, \mathrm{id})$ is the neutral element in $\mathcal{G}$.

This shows $J = L \times \{\epsilon\}$, because ϵ is the only involution in $\mathcal{A}$.

(b) Fix $(a, \alpha) \in \mathrm{Cen}(\iota)$. We have

$$(a, \alpha)(1, \epsilon) = (a, \alpha\epsilon) \quad \text{and} \quad (1, \epsilon)(a, \alpha) = (a^{-1}, \epsilon\alpha).$$

Hence $(a, \alpha) \in \mathrm{Cen}(\iota)$ if and only if $a = a^{-1}$ if and only if $a = 1$.

(c) Take involutions $(a, \epsilon), (b, \epsilon), (c, \epsilon) \in J = L \times \{\epsilon\}$. Then

$$\begin{aligned}(b, \epsilon)(a, \epsilon)(b, \epsilon) &= (b, \epsilon)(a \otimes b^{-1}, \delta_{a,b^{-1}}) \\ &= (b \otimes (a^{-1} \otimes b), \delta_{b,a^{-1}\otimes b} \delta_{a,b^{-1}} \epsilon) \\ &= ((b \otimes a^{-1/2})^2, \epsilon).\end{aligned}$$

Hence we have $(a, \epsilon)^{(b,\epsilon)} = (c, \epsilon)$ if and only if $b \otimes a^{-1/2} = c^{1/2}$. The loop conditions ensure that for all $a, c \in L$ there is a unique b satisfying this equation. $\square$

Now set $\lambda_0 = H \times \{\epsilon\} \subseteq J$, and put $\Lambda = \{\lambda_0^g : g \in \mathcal{G}\}$. We view elements of Λ as lines, and we view involutions as points. Note that Λ is $\mathcal{G}$-invariant and all lines are conjugate.

The following will be shown in this section:

Theorem 4.7. (a) (J, Λ) *is a partial mock hyperbolic reflection space in* $\mathcal{G}$.

(b) *If* G *is full, i.e., if* $G = \bigcup_{g\in G} H^g$, *then* (J, Λ) *is a mock hyperbolic reflection space.*

(c) *Suppose* $i, j, k \in J$ *are pairwise distinct such that the lines* ℓ_{ij} *and* ℓ_{ik} *exist, and assume that* i, j, k *are not collinear. Then* $\mathrm{Cen}_{\mathcal{G}}(i, j, k) = 1$. *In particular,* $\mathcal{G}$ *acts faithfully on* J.

Lemma 4.8. *Let* λ *be a line containing* ι. *Then* λ *is of the form*

$$\lambda = H^g \times \{\epsilon\}$$

for some $g \in G$.

Proof. We have $\lambda = \lambda_0^g$ for some $g = (a, \alpha) \in \mathcal{G}$. So elements of λ are of the form

$$\begin{aligned}(\alpha^{-1}(a^{-1}), \alpha^{-1})(c, \epsilon)(a, \alpha)&\\ &= (\alpha^{-1}(a^{-1}), \alpha^{-1})(c \otimes a^{-1}, \delta_{c,a^{-1}}\epsilon\alpha)\\ &= \big(\alpha^{-1}(a^{-1}) \otimes \alpha^{-1}(c \otimes a^{-1}), \delta_{\alpha^{-1}(a^{-1}),\alpha^{-1}(c\otimes a^{-1})}\alpha^{-1}\delta_{c,a^{-1}}\alpha\epsilon\big)\\ &= \big(\alpha^{-1}(a^{-1} \otimes (c \otimes a^{-1})), \alpha^{-1}\delta_{a^{-1},c\otimes a^{-1}}\delta_{c,a^{-1}}\alpha\epsilon\big)\end{aligned}$$

for some $c \in H$, where the last equality holds by Proposition 4.2(a).

Note that $a^{-1} \otimes (c \otimes a^{-1}) = (a^{-1} \otimes c^{1/2})^2$. We assume $\iota \in \lambda$. Hence

$$1 = a^{-1} \otimes c^{1/2}$$

for some $c \in H$, and thus $a = c^{1/2} \in H$. This implies $\lambda = (\alpha^{-1}(H), \epsilon) \subseteq J$. □

Corollary 4.9. *Any two distinct points are contained in at most one line.*

Lemma 4.10. *Fix distinct involutions* $i, j \in J$ *and suppose* ℓ_{ij} *exists in* Λ. *Then*

$$\begin{aligned}\ell_{ij} &= \{k \in J : ij \in kJ\}\\ &= \{k \in J : (ij)^k = (ij)^{-1}\}.\end{aligned}$$

Proof. We may assume that $\ell_{ij} = H \times \{\epsilon\}$ and $ij = (c, 1)$ for some $c \in H \setminus \{1\}$. The second equality is easy, and therefore we only show the first equality.

We first show $\ell_{ij} \subseteq \{k \in J : ij \in kJ\}$: Take $d \in H \setminus \{1\}$. Then

$$(d, \epsilon)(c, 1)(d, \epsilon) = (d, \epsilon)(c \otimes d, \delta_{c,d}\epsilon) = (d \otimes (c \otimes d)^{-1}, \delta_{d,(c\otimes d)^{-1}}\delta_{c,d}).$$

The Frobenius complement H is abelian, and therefore

$$(d \otimes (c \otimes d)^{-1}, \delta_{d,(c\otimes d)^{-1}}\delta_{c,d}) = (c^{-1}, 1).$$

This shows $(c, 1)^{(d,\epsilon)} = (c, 1)^{-1}$, and hence $\ell_{ij} \subseteq \{k \in J : ij \in kJ\}$.

We now show $\supseteq$ for the first equality: Suppose $(c, 1) = (a, \epsilon)(b, \epsilon)$. We have to show that a is an element of H. We have

$$(c, 1) = (a, \epsilon)(b, \epsilon) = (a \otimes b^{-1}, \delta_{a,b^{-1}}),$$

and hence $a^{1/2}b^{-1}a^{1/2} = a \otimes b^{-1} = c \in H$ and $\delta_{a,b^{-1}} = \mathrm{id}$. By Proposition 4.4, this implies

$$b^{-1/2}a^{1/2}(a \otimes b^{-1})^{-1/2} = 1.$$

So $b^{-1/2}a^{1/2} = c^{1/2}$, and since $c = a^{1/2}b^{-1}a^{1/2}$, this implies $a^{1/2}b^{-1/2} = c^{1/2}$. Hence

$$c^{1/2} = b^{-1/2}a^{1/2} = (a^{1/2}b^{-1/2})^{a^{1/2}} = (c^{1/2})^{a^{1/2}},$$

and therefore $a^{1/2} \in \mathrm{Cen}(c) = H$. □

Lemma 4.11. *Suppose $(a, \alpha) \in N_{\mathcal{G}}(\lambda_0)$. Then $a \in H$ and α normalizes H.*

Proof. Given $c \in H$, we have

$$\begin{aligned}(a, \alpha)^{-1}(c, \epsilon)(a, \alpha) &= (\alpha^{-1}(a^{-1}), \alpha^{-1})(c \otimes a^{-1}, \delta_{c,a^{-1}}\epsilon\alpha)\\ &= \big(\alpha^{-1}(a^{-1}) \otimes \alpha^{-1}(c \otimes a^{-1}), \delta_{\alpha^{-1}(a^{-1}),\alpha^{-1}(c\otimes a^{-1})}\alpha^{-1}\delta_{c,a^{-1}}\alpha\epsilon\big)\\ &= \big(\alpha^{-1}(a^{-1} \otimes (c \otimes a^{-1})), \alpha^{-1}\delta_{a^{-1},c\otimes a^{-1}}\delta_{c,a^{-1}}\alpha\epsilon\big)\\ &= \big(\alpha^{-1}(a^{-1} \otimes (c \otimes a^{-1})), \epsilon\big).\end{aligned}$$

We have $(1, \epsilon) \in \lambda_0$, and therefore

$$1 = a^{-1} \otimes (c_0 \otimes a^{-1})$$

for some $c_0 \in H$. Note that

$$a^{-1} \otimes (c_0 \otimes a^{-1}) = (a^{-1/2}c_0^{1/2}a^{-1/2})^2 = (a^{-1} \otimes c_0^{1/2})^2,$$

and therefore $1 = a^{-1} \otimes c_0^{1/2}$. This shows $a = c_0^{1/2} \in H$.

Moreover, $\alpha^{-1}(a^{-1} \otimes (c \otimes a^{-1})) \in H$ for all $c \in H$, and hence α normalizes H. □

Proposition 4.12. $$N_G(\lambda_0) \cap J^{\cdot 2} = \lambda_0^{\cdot 2}.$$

Proof. Fix $a \neq b$ in L such that $(a, \epsilon)(b, \epsilon) = (a \otimes b^{-1}, \delta_{a,b^{-1}}) \in N_{\mathcal{G}}(\lambda_0)$. By Lemma 4.11, we have $a \otimes b^{-1} \in H$ and $\delta_{a,b^{-1}}$ normalizes H.

By Proposition 4.4, the latter is equivalent to

$$b^{-1/2}a^{1/2}(a^{1/2}b^{-1}a^{1/2})^{-1/2} \in H.$$

Since $a^{1/2}b^{-1}a^{1/2} = a \otimes b^{-1} \in H$, this implies $b^{-1/2}a^{1/2} \in H$, and therefore $a^{1/2}b^{-1/2} = a^{1/2}b^{-1}a^{1/2}(b^{-1/2}a^{1/2})^{-1} \in H$.

This shows

$$b^{-1/2}a^{1/2} = (a^{1/2}b^{-1/2})^{b^{1/2}} \in H \cap H^{b^{1/2}}.$$

Thus $b^{1/2} \in h$ and $a^{1/2} = (a^{1/2}b^{-1/2})b^{1/2} \in H$, because H is malnormal in G. □

Proposition 4.13. *Suppose $i, j, k \in J$ are pairwise distinct such that the lines ℓ_{ij} and ℓ_{ik} exist in Λ, and assume that i, j, k are not collinear. Then* $\mathrm{Cen}(i, j, k) = 1$.

Proof. Let $i = (1, \epsilon)$ and fix $j = (a, \epsilon) \in J \setminus \{i\}$. We already know $\mathrm{Cen}(i) = 1 \times \mathcal{A}$. Now fix $(1, \beta) \in \mathrm{Cen}(i) \cap \mathrm{Cen}(j)$. Then

$$(\beta(a), \epsilon\beta) = (1, \beta)(a, \epsilon) = (a, \epsilon)(1, \beta) = (a, \epsilon\beta).$$

Therefore $\beta \in \mathrm{Cen}_{\mathcal{A}}(a) = 1 \times \mathrm{Cen}_G(a)$, and hence $\mathrm{Cen}(i, j) = 1 \times \mathrm{Cen}_G(a)$. This shows the claim, because G is a Frobenius group. □

Proof of Theorem 4.7. We start by checking conditions (a) and (b) of Definition 2.1. Condition (a) follows from Corollary 4.9 and Lemma 4.10. Condition (b) is part (c) of Lemma 4.6.

Now Proposition 4.12 and Lemma 2.8 imply that (J, Λ) is a partial mock hyperbolic reflection space.

If the Frobenius group is full, then it follows from Lemma 4.8 and from the definition of λ_0 that all lines exist and hence that J forms a mock hyperbolic reflection space.

The final statement is Proposition 4.13. □

5. Mock hyperbolic reflection spaces in groups of finite Morley rank

We now turn to the finite Morley rank setting. We refer the reader to [Borovik and Nesin 1994; Poizat 1987] for a general introduction to groups of finite Morley rank. If X is a definable set of finite Morley rank, then we denote its Morley rank by $\mathrm{MR}(X)$ and its Morley degree by $\mathrm{MD}(X)$.

Convention. In the context of finite Morley rank, we say that a definable property P holds for *Morley rank k many elements* if the set defined by P has Morley rank k. In a slight abuse, we may also say that P holds for *generically many* elements of a definable set X if the set of elements in S not satisfying P has smaller Morley rank than X.

We will repeatedly make use of the following:

Proposition 5.1 [Borovik and Nesin 1994, Exercises 11 and 12 on p. 72]. *If G is a group of finite Morley rank and G does not contain an involution, then G is uniquely* 2*-divisible.*

Now let G be a group of finite Morley rank, and let J be a conjugacy class of involutions such that $\mathrm{MD}(J) = 1$. Moreover, we assume that $\Lambda \subseteq \mathcal{P}(J)$ is a G-invariant definable family of subsets of J such that each $\lambda \in \Lambda$ is of the form

$$\lambda = \{k \in J : ij \in kJ\}$$

for any $i \neq j \in \lambda$.

Definition 5.2. We call (J, Λ) a *generic mock hyperbolic reflection space* if (J, Λ) is a partial mock hyperbolic reflection space and for each $i \in J$ the set

$$\{j \in J : \ell_{ij} \in \Lambda\}$$

is generic in J.

Remark 5.3. Let (J, Λ) be a generic mock hyperbolic reflection space.

(a) The condition in the above definition is equivalent to the statement that

$$\{(i, j) \in J^2 : i \neq j \text{ and } \ell_{ij} \text{ exists in } \Lambda\} \subseteq J \times J$$

is a generic subset of J^2.

(b) Write

$$\Lambda(k) = \{\lambda \in \Lambda : \mathrm{MR}(\lambda) = k\}.$$

Fix $i \in J$, and set $B_{(k)}(i) = \{j \in J \setminus \{i\} : \ell_{ij} \in \Lambda(k)\}$. Since $\mathrm{MD}(J) = 1$, there is exactly one $k \leq n$ such that $B_{(k)}(i)$ is a generic subset of J. In that case $(J, \Lambda(k))$ is a generic mock hyperbolic reflection space. Hence we may *assume from now on that all lines in* Λ *have the same Morley rank*.

(c) If (J, Λ) is a generic mock hyperbolic reflection space of finite Morley rank in which all lines have Morley rank k, then we have $\mathrm{MR}(\Lambda) = 2n - 2k$ and $\mathrm{MD}(\Lambda) = 1$ for $n = \mathrm{MR}(J)$. The set of translations

$$S = \{\sigma \in J^{\cdot 2} \setminus \{1\} : \ell_\sigma \text{ exists in } \Lambda\} \cup \{1\}$$

has Morley rank $2n - k$ and Morley degree 1.

If X and Y are definable sets, then we write $X \approx Y$ if X and Y coincide up to a set of smaller rank, i.e., if the sets X, Y, and $X \cap Y$ all have the same Morley rank and Morley degree. This defines an equivalence relation on the family of definable sets. One important property of this equivalence relation is the following:

Proposition 5.4 [Wagner 2017, Lemma 4.3]. *Let G be a group acting definably on a set X in an ω-stable structure. Let Y be a definable subset of X such that $gY \approx Y$ for all $g \in G$. Then there is a G-invariant set $Z \subseteq X$ such that $Z \approx Y$.*

By Theorem 2.14, a mock hyperbolic reflection space consists of one line if and only if the set of translations forms a normal subgroup. For generic mock hyperbolic reflection spaces the following will be shown in this section:

Theorem 5.5. *Suppose (J, Λ) is a generic mock hyperbolic reflection space such that J has Morley rank $\mathrm{MR}(J) = n$. Assume that Λ consists of more than one line and that all lines $\lambda \in \Lambda$ are infinite and of Morley rank $\mathrm{MR}(\lambda) = k$. Then $n \geq 2k + 1$.*

If $n = 2k + 1$, then the translations almost form a normal subgroup: G has a definable connected normal subgroup N of Morley rank $\mathrm{MR}(N) = 2n - k$ such that $N \approx S$. Moreover, $\mathrm{MR}(N \cap \mathrm{Cen}(i)) = n - k$ for any involution $i \in J$.

For the remainder of this section we assume that (J, Λ) is a generic mock hyperbolic reflection space in a group of finite Morley rank G such that (J, Λ) satisfies the assumptions in Theorem 5.5. In particular, $n > k \geq 1$.

Note that we do not state any assumption about the Morley degree of lines.

Generic projective planes.

Definition 5.6. A definable subset $X \subseteq J$ is a *generic projective plane* if

(a) $\mathrm{MR}(X) = 2k$ and $\mathrm{MD}(X) = 1$, and

(b) $\mathrm{MR}(\Lambda_X) = 2k$ and $\mathrm{MD}(\Lambda_X) = 1$,

where Λ_X is the set of all lines $\lambda \subseteq J$ such that $\mathrm{MR}(\lambda \cap X) = k$.

The next lemma follows from easy counting arguments.

Lemma 5.7. *Let $X \subseteq J$ be a definable set of Morley rank* $2k$ *and Morley degree* 1. *The following are equivalent*:

(a) *X is a generic projective plane.*

(b) $\mathrm{MR}(\Lambda_X) \geq 2k$.

(c) *The set of $x \in X$ such that* $\mathrm{MR}(\{\lambda \in \Lambda_X : x \in \lambda\}) = k$ *is generic in X.*

Proof. (a) $\Longrightarrow$ (b) This holds by definition.

(b) $\Longrightarrow$ (c) Given $x \in X$ consider $L_x = \{\lambda \in \Lambda_X : x \in \lambda\}$ and note that

$$\mathrm{MR}\Big(\bigcup L_x\Big) = \mathrm{MR}(L_x) + k$$

holds for each $x \in X$. In particular, $\mathrm{MR}(L_x) \leq k$, since $\mathrm{MR}(X) = 2k$. Moreover, $\mathrm{MR}(\Lambda_X) \geq 2k$ and each $\lambda \in \Lambda_X$ is contained in rank k many sets of the form L_x. Hence we must have $\mathrm{MR}(L_x) = k$ for generically many $x \in X$.

(c) $\Longrightarrow$ (b) We have $\mathrm{MR}(X) = 2k$ and $\mathrm{MR}(L_x) = k$ for generically many $x \in X$. Moreover, each $\lambda \in \Lambda_X$ contains rank k many points from X. Thus $\mathrm{MR}(\Lambda_X) \geq 2k$.

(b) $\Longrightarrow$ (a) Consider the set

$$P = \{(x, y) \in X \times X : x \neq y \text{ and } \ell_{xy} \in \Lambda_X\}.$$

Note that each $\lambda \in \Lambda_X$ has rank $2k$ many preimages in P. Since X has rank $2k$ and degree 1, this implies $\mathrm{MR}(\Lambda_X) = 2k$ and $\mathrm{MD}(\Lambda_X) = 1$. □

Lemma 5.8. *Suppose $X \subseteq J$ is a generic projective plane. Then set of $x \in X$ such that $X^x \approx X$ is generic in X.*

Proof. Let $\lambda \in \Lambda_X$ be a line. Recall that $\lambda^{\cdot 2}$ is a group by Lemma 2.7(d). For $i \in \lambda$, set

$$\lambda_i = \{j \in \lambda : ij \in (\lambda^{\cdot 2})^0\} = i(\lambda^{\cdot 2})^0.$$

Then $\{\lambda_i : i \in \lambda\}$ is a partition of λ into sets of rank k and degree 1. Moreover, we have $(\lambda_i)^i = \lambda_i$ for all $i \in \lambda$. In particular, if $\lambda_i \cap X \approx \lambda_i$, then $(\lambda_i \cap X)^i \cap X \approx \lambda_i$.

Hence for all $\lambda \in \Lambda_X$ the set

$$X_\lambda = \{x \in \lambda \cap X : \mathrm{MR}((\lambda \cap X)^x \cap X) = k\}$$

has Morley rank k. Moreover, each $x \in X$ is contained in at most rank k many lines in Λ_X and hence is contained in at most rank k many sets X_λ.

We have $\mathrm{MR}(\Lambda_X) = 2k$, and hence the set

$$\{(x, \lambda) \in X \times \Lambda_X : x \in X_\lambda\}$$

has Morley rank $3k$. Since $\mathrm{MR}(X) = 2k$, this implies that the set of $x \in X$ contained in rank k many sets X_λ is generic in X.

Now if $x \in X_\lambda$ for rank k many λ, then

$$X^x \cap X \supseteq \Bigg(\bigcup_{\lambda : x \in X_\lambda} \lambda \cap X \Bigg)^x \cap X = \bigcup_{\lambda : x \in X_\lambda} (\lambda \cap X)^x \cap X$$

must have Morley rank $2k$, and hence $X^x \approx X$. $\square$

Lemma 5.9. *If $X \subseteq J$ is a generic projective plane and $Z \subseteq J$ is a definable subset with $X \approx Z$, then Z is a generic projective plane.*

Proof. For $x \in X$ put $\Lambda_x = \{\lambda \in \Lambda_X : x \in \lambda\}$. If $\mathrm{MR}(\Lambda_x) = k$, then $B(x) = \bigcup \Lambda_x \approx X$. In particular, $B(x) \approx Z$ for a generic set of $x \in X \cap Z$. If $B(x) \approx Z$, then $\Lambda_x \cap \Lambda_Z$ must have Morley rank k. Hence it follows from Lemma 5.7 that Z must be a generic projective plane. $\square$

Lemma 5.10. *Let $H \leq G$ be a definable subgroup such that* $\mathrm{MR}(H \cap J) = 2k$ *and* $\mathrm{MD}(H \cap J) = 1$. *Then* $\mathrm{MR}(\Lambda_{H \cap J}) < 2k$, *i.e., $H \cap J$ does not form a generic projective plane.*

Proof. This is proved in the same way as [Borovik and Nesin 1994, Proposition 11.71]. Put $Z = H \cap J$.

Assume towards contradiction that $\mathrm{MR}(\Lambda_Z) \geq 2k$. Then Z is a generic projective plane, and hence $\mathrm{MR}(\Lambda_Z) = 2k$ and $\mathrm{MD}(\Lambda_Z) = 1$ (Lemma 5.7).

Let $\lambda \in \Lambda_Z$ be a line. By Proposition 2.10, the family $\{\lambda^i : i \in Z \setminus \lambda\}$ consists of Morley rank $2k$ many lines which do not intersect λ. Therefore the set $\{\delta \in \Lambda_Z : \lambda \cap \delta = \varnothing\} \subseteq \Lambda_Z$ is a generic subset of Λ_Z.

We aim to find a line which intersects Morley rank $2k$ many lines contradicting $\mathrm{MD}(\Lambda) = 1$. For $x \in Z$, set $\Lambda_x = \{\lambda \in \Lambda_Z : x \in \lambda\}$, and set $B(x) = \bigcup \Lambda_x \cap Z \subseteq Z$. Note that $\mathrm{MR}(B(x)) = \mathrm{MR}(\Lambda_x) + k$, and hence $\mathrm{MR}(\Lambda_x) \leq k$ for all $x \in Z$. Since each $\lambda \in \Lambda$ contains Morley rank k many points, we must have $\mathrm{MR}(\Lambda_x) = k$ for a generic set of $x \in Z$.

Fix $x_0 \in Z$ such that Λ_{x_0} has Morley rank $2k$. Then $B(x_0) \subseteq Z$ is generic, and hence $\mathrm{MR}(\Lambda_x) = k$ for a generic set of $x \in B(x_0)$. Since $B(x_0) = \bigcup \Lambda_{x_0}$, we can find a line $\lambda \in \Lambda_{x_0}$ such that $\mathrm{MR}(\Lambda_x) = k$ for a generic set of $x \in \lambda$. But then λ intersects Morley rank $2k$ many lines in Λ_Z. □

Proposition 5.11. *J does not contain a generic projective plane X.*

Proof. Assume $X \subseteq J$ is a generic projective plane, and put

$$H = N_G^{\approx}(X) = \{g \in G : X^g \approx X\}.$$

By Lemma 5.8, the set $X \cap H$ is generic in X. Hence $\mathrm{MR}(H \cap J) \geq 2k$.

Now consider the action of G on J by conjugation. Note that, by Proposition 5.4, there is a definable subset $Z \subseteq J$, $X \approx Z$, such that H normalizes Z. Since J forms a generic mock hyperbolic space, J acts regularly on itself, and hence $\mathrm{MR}(H \cap J) \leq \mathrm{MR}(Z) = 2k$. Therefore $\mathrm{MR}(H \cap J) = 2k$ and $\mathrm{MD}(H \cap J) = 1$ (since $\mathrm{MD}(Z) = 1$). This contradicts Lemma 5.10. □

A rank inequality and a normal subgroup. A line $\lambda \in \Lambda$ is called *complete for some $i \in J \setminus \lambda$* if the set $\{j \in \lambda : \ell_{ij} \in \Lambda\}$ is a generic subset of λ.

Definition 5.12. Let (i, j, p) be a triple of noncollinear involutions in J.

- (i, j, p) is *good* if ℓ_{ij}, ℓ_{jp} exist and ℓ_{ij} is complete for p.
- (i, j, p) is *perfect* if ℓ_{ij}, ℓ_{jp} exist and

  $$\{j' \in \ell_{jp} : \ell_{ij'} \in \Lambda \text{ is complete for } p' = j'jp\}$$

  is generic in ℓ_{jp}.

Lemma 5.13. *A generic triple $(i, j, p) \in J^3$ is good. In particular, for any $i \in J$ a generic element of $\{i\} \times J^2$ is good.*

Proof. Fix $i \in J$, and put $B(i) = \{j \in J : \ell_{ij} \in \Lambda\}$. Then $B(i)$ is a generic subset J. Now fix $p \in J \setminus \{i\}$. We aim to show that (i, j, p) must be good for generically many $j \in J \setminus \{i, p\}$.

Note that $B(i)$ and $B(p)$ are generic subsets of J. Therefore $B(i) \cap B(p)$ must be generic in $B(i)$ and $B(i) \setminus B(p)$ is not generic in $B(i)$. Note that

$$B(i) \cap B(p) = \bigcup_{\lambda \in \Lambda_i} (\lambda \cap B(p)) \quad \text{and} \quad B(i) \setminus B(p) = \bigcup_{\lambda \in \Lambda_i} (\lambda \setminus B(p)).$$

Since $\mathrm{MR}(J) = \mathrm{MR}(\Lambda_i) + p$, the set

$$\{\lambda \in \Lambda_i : \mathrm{MR}(\lambda \setminus B(p)) < p\}$$

must be generic in Λ_i.

Hence $\lambda \cap B(p) \approx \lambda$ for generically many $\lambda \in \Lambda_i$. Moreover, if $\lambda \cap B(p) \approx \lambda$ for some $\lambda \in \Lambda_i$ and j is contained in $\lambda \setminus \{i, p\}$, then (i, j, p) is good. The last sentence follows since all elements in J are conjugate. $\square$

Proposition 5.14. *A generic triple $(i, j, p) \in J^3$ is perfect, and for any $i \in J$ a generic element of $\{i\} \times J^2$ is perfect.*

Proof. Since J is a generic mock hyperbolic reflection space, the set $U = \{(j, p) : jp \in S \setminus \{1\}\} \subseteq J^2$ is generic in J^2. For $\sigma \in S$ put $U_\sigma = \{(j, p) : jp = \sigma\}$. Then each U_σ has Morley rank p and U is the disjoint union

$$U = \bigcup_{\sigma \in S} U_\sigma \subseteq J \times J.$$

Now fix $i \in J$. A generic triple in $\{i\} \times U$ is good, and we have $\mathrm{MD}(\{i\} \times U) = 1$. Since $\mathrm{MD}(S) = 1$, this implies that for generically many $\sigma \in S$ the set

$$\{(i, r, s) \in \{i\} \times U_\sigma : (i, r, s) \text{ is good}\}$$

is a generic subset of $\{i\} \times U_\sigma$.

Moreover, if a generic triple in $\{i\} \times U_\sigma$ is good, then a generic triple in $\{i\} \times U_\sigma$ must be perfect. This proves the lemma. $\square$

Now let $\mu : J^3 \to G$ be the multiplication map, and put

$$T = \{(i, j, p) \in J^3 : \ell_{jp} \text{ exists}\} \quad \text{and} \quad T_{\text{perf}} = \{(i, j, p) \in J^3 : (i, j, p) \text{ is perfect}\}.$$

Note that $T_{\text{perf}} \subseteq T$. If (J, Λ) is a mock hyperbolic reflection space, i.e., if all lines exist, then T_{perf} consists of all triples of noncollinear involutions in J.

Lemma 5.15. $\mathrm{MR}(\mu(T_{\text{perf}})) \geq 2n - k.$

Proof. For any $i \in J$ the set $\{(j, p) \in J^2 : (i, j, p) \text{ is perfect}\}$ has Morley rank $2n$ by Proposition 5.14. Clearly $ijp = ij'p'$ if and only if $jp = j'p'$. If ℓ_{jp} exists, the set $\{(j', p') \in J^2 : jp = j'p'\}$ has Morley rank k. Hence $\mu(T_{\text{perf}})$ has Morley rank at least $2n - k$. $\square$

Proposition 5.16. *Suppose* $\mathrm{MR}(\mu(T_{\text{perf}})) = 2n - k$. *Then G has a definable connected normal subgroup N of Morley rank* $\mathrm{MR}(N) = 2n - k$ *such that $N \approx S$. Moreover,* $\mathrm{MR}(N \cap \mathrm{Cen}(i)) = n - k$ *for any involution $i \in J$.*

Proof. Set $d = \mathrm{MD}(\mu(T_{\text{perf}}))$ and write $\mu(T_{\text{perf}})$ as a disjoint union

$$\mu(T_{\text{perf}}) = Y_1 \cup \cdots \cup Y_d,$$

where each Y_r has rank $2n - k$ and degree 1. Put $T_i = T_{\text{perf}} \cap (\{i\} \times J \times J)$. Then each T_i has rank $2n$ and degree 1 by Proposition 5.14. Moreover, $\mu(T_i)$ has rank $2n-k$ and degree 1. We can find $1 \leq f \leq d$ such that

$$\mu(T_i) \approx Y_f$$

for generically many $i \in J$. Put $Y = Y_f$, set $N = \mathrm{Stab}^{\approx}(Y) = \{g \in G : gY \approx Y\}$, and note that N must be a normal subgroup of G, because Y is G-normal up to $\approx$-equivalence.

Now, by Proposition 5.4, there is some $Z \approx Y$ such that $N \subseteq \mathrm{Stab}(Z)$. In particular, N has rank $\leq 2n-k$, since $\mathrm{MR}(Z) = 2n-k$. Moreover, if $\mathrm{MR}(N) = 2n-k$, then we must have $N = \mathrm{Stab}(Z)$, since $\mathrm{MD}(Z) = 1$.

Let $J_Y = \{i \in J : \mu(T_i) \approx Y\}$. Given $i \neq j \in J_Y$, we have

$$ij\mu(T_j) \approx \mu(T_i),$$

and hence $ij \in N$. Therefore $J_Y^{\cdot 2} \subseteq N$. Since J_Y is a generic subset of J, we have $\Lambda_{J_Y} \approx \Lambda$, and therefore $J_Y^{\cdot 2} \cap S \approx S$. Thus $\mathrm{MR}(N) = 2n-k$, and hence $N = \mathrm{Stab}(Z)$ is connected. In particular, $J_Y^{\cdot 2} \cap S$ is generic in both N and S, and hence $N \approx S$.

We now show $\mathrm{MR}(N \cap \mathrm{Cen}(i)) = n-k$ for any involution $i \in J$: Fix an involution $i \in J$. If $i \in N$, then $iJ \subseteq N$, and hence $N = iJ(N \cap \mathrm{Cen}(i))$ by Lemma 2.5, and therefore $\mathrm{MR}(N \cap \mathrm{Cen}(i)) = n-k$.

If $i \notin N$, then note that $iJ \cap N$ must be a generic subset of iJ, and therefore the conjugacy class i^N is generic in J. This implies that $N \rtimes \langle i \rangle$ must contain J, and hence is a normal subgroup of G. Now argue as in the first case. □

For $\alpha \in \mu(T)$, we set

$$X_\alpha = \{i \in J : \exists (j, p) \in J \times J \text{ such that } (i, j, p) \in T \text{ and } ijp = \alpha\}.$$

Note that $\mathrm{MR}(\mu^{-1}(\alpha) \cap T) = \mathrm{MR}(X_\alpha) + k$.

If A and B are definable sets, then we write $A \subsetapprox B$ if A is almost contained in B, i.e., if $A \cap B \approx A$.

Lemma 5.17. *Fix a triple $(i, j, p) \in T$.*

(a) *If (i, j, p) is good, then $\ell_{ij} \subsetapprox X_{ijp}$.*

(b) *If (i, j, p) is perfect, then $\ell_{it} \subsetapprox X_{ijp}$ for generically many $t \in \ell_{jp}$. In particular,* $\mathrm{MR}(X_{ijp}) \geq 2k$.

Proof. (a) Since (i, j, p) is good, the line ℓ_{ij} is p-complete. Hence $\ell_{j'p}$ exists for generically many $j' \in \ell_{ij}$. Fix such an j' and write $ij = i'j'$. Then (i', j', p) is good, and hence $i' \in X_{ijp}$.

(b) This follows immediately from (a). □

Lemma 5.18. *Set $l = \mathrm{MR}(\mu(T_{\mathrm{perf}})) - (2n-k)$. Then $2k \leq \mathrm{MR}(X_\alpha) \leq n-l$ for generically many $\alpha \in \mu(T_{\mathrm{perf}})$. In particular, $n \geq 2k+l$.*

Proof. We have that $\mathrm{MR}(\mu^{-1}(\alpha) \cap T) = \mathrm{MR}(X_\alpha) + k$ for each $\alpha \in \mu(T)$ and that T has rank $3n$, and we trivially have

$$\bigcup_{\alpha \in \mu(T_{\mathrm{perf}})} \mu^{-1}(\alpha) \cap T \subseteq T.$$

Therefore a generic $\alpha \in \mu(T_{\mathrm{perf}})$ must satisfy the inequality

$$\mathrm{MR}(\mu(T_{\mathrm{perf}})) + \mathrm{MR}(X_\alpha) + k \leq \mathrm{MR}(T) = 3n.$$

Moreover, we have $\mathrm{MR}(X_\alpha) \geq 2k$ by Lemma 5.17. Hence

$$2k \leq \mathrm{MR}(X_\alpha) \leq \mathrm{MR}(T) - k - \mathrm{MR}(\mu(T_{\mathrm{perf}})) = n - l$$

for generically many $\alpha \in \mu(T_{\mathrm{perf}})$. □

Proposition 5.19. *Set* $l = \mathrm{MR}(\mu(T_{\mathrm{perf}})) - (2n - k)$. *Then* $n > 2k + l$. *In particular*, $n > 2k$.

Proof. Assume not. Then $n = 2k + l$ and $\mathrm{MR}(X_\alpha) = 2k$ for generically many $\alpha \in \mu(T_{\mathrm{perf}})$. Set $M = \{\alpha \in \mu(T_{\mathrm{perf}}) : \mathrm{MR}(X_\alpha) = 2k\}$. This is a generic subset of $\mu(T_{\mathrm{perf}})$. We have

$$\mathrm{MR}\Big(\bigcup_{\alpha \in M} \mu^{-1}(\alpha) \cap T\Big) = (2n - k + l) + 3k = 6k + 3l = 3n.$$

So $\bigcup_{\alpha \in M} \mu^{-1}(\alpha) \cap T$ is a generic subset of $J \times J \times J$. Note $\mathrm{MR}(M) = 3k + 3l$. Therefore we can find $\alpha \in M$ such that $\mu^{-1}(\alpha) \cap T$ has rank $3k$ and contains rank $3k$ many perfect triples. Set $X = X_\alpha$ and $\Lambda_X = \{\lambda \in \Lambda : \lambda \precsim X\}$. Now Lemma 5.17 implies that for a generic $i \in X$ the set

$$\{\lambda \in \Lambda_X : i \in \lambda\}$$

has Morley rank k. Hence $\mathrm{MR}(\Lambda_X) = 2k$, and therefore a degree 1 component of X must be a generic projective plane. This contradicts Proposition 5.11. □

Proof of Theorem 5.5. Set $l = \mathrm{MR}(\mu(T_{\mathrm{perf}})) - (2n - k)$. By Proposition 5.19, we have $2k + 1 = n > 2k + l$, and hence $l = 0$. Now Proposition 5.16 implies the theorem. □

6. Frobenius groups of finite Morley rank

We now consider Frobenius groups of finite Morley rank. If G is a group of finite Morley rank and H is a Frobenius complement in G, then H is definable by [Borovik and Nesin 1994, Proposition 11.19]. If G splits as $G = N \rtimes H$, then N is also definable by [Borovik and Nesin 1994, Proposition 11.23].

Epstein and Nesin showed that if $H < G$ is a Frobenius group of finite Morley rank and H is finite, then $H < G$ splits [Borovik and Nesin 1994, Theorem 11.25]. As a consequence it suffices to consider connected Frobenius groups of finite Morley rank [Borovik and Nesin 1994, Corollary 11.27].

Solvable Frobenius groups of finite Morley rank split and their structure is well understood [Borovik and Nesin 1994, Theorem 11.32].

Lemma 6.1. *Let $H < G$ be a connected Frobenius group of finite Morley rank with Frobenius complement H, and let $X \subseteq H \setminus \{1\}$ be a definable H-normal subset such that* $\mathrm{MR}(X) = \mathrm{MR}(H)$. *Then $\bigcup_{b \in G} X^b \subseteq G$ is a generic subset of G.*

Proof. Set $n = \mathrm{MR}(G)$ and $k = \mathrm{MR}(H)$. Consider the map $\alpha : G \times X \to G$, $(b, x) \mapsto x^b$. If $x^b = y^c$ for $x, y \in X$ and $b, c \in G$, then bc^{-1} must be contained in $N_G(H) = H$. Therefore we have

$$\alpha^{-1}(x^b) = \{(c, x^{bc^{-1}}) \in G \times X : bc^{-1} \in H\}.$$

Hence all fibers of α have Morley rank k. This shows that $\alpha(G \times X) = \bigcup_{b \in G} X^b$ must have Morley rank n, and hence is a generic subset of G. □

Groups of finite Morley rank can be classified by the structure of their 2-Sylow subgroups. In case of Frobenius groups this classification is simpler:

Proposition 6.2. *Let G be a connected Frobenius group of finite Morley rank with Frobenius complement H. Then H is connected and G lies in one of the following mutually exclusive cases*:

(a) *H contains a unique involution, and G is of odd type*;

(b) *G does not contain any involutions, and in particular, G is of degenerate type*;

(c) *$G \setminus \left(\bigcup_{g \in G} H^g\right)$ contains involutions, and G is of even type.*

Proof. We first show that H must be connected: if H is not connected, then $\bigcup_{g \in G} (H^0 \setminus \{1\})^g$ and $\bigcup_{g \in G} (H \setminus H^0)^g$ would be two disjoint generic subsets of G. This is impossible, because G is connected.

If H contains an involution, then Delahan and Nesin showed this involution must be unique and moreover all involutions in G are conjugate, so $G \setminus \left(\bigcup_{g \in G} H^g\right)$ cannot contain any involution [Borovik and Nesin 1994, Lemma 11.20]. In particular, G is of odd type, because the connected subgroup H contains a unique involution.

If $G \setminus \left(\bigcup_{g \in G} C^g\right)$ contains an involution, then the proof of [Altınel et al. 2019, Theorem 2] shows that G is of even type. □

Remark 6.3. If $H < G$ is of even type, then Altınel, Berkman, and Wagner showed in [Altınel et al. 2019] that there is a definable normal subgroup N such that $N \cap H = 1$ and N contains all involutions of G. By [Borovik and Nesin 1994, Lemma 11.38], either $G = N \rtimes H$ splits or $HN/N < G/N$ is a Frobenius group of finite Morley rank. Now if $HN/N < G/N$ splits, then it is easy to see that $H < G$ must split. Hence a nonsplit Frobenius group of minimal Morley rank cannot be of even type. Therefore to show that all Frobenius groups of finite Morley rank split, it suffices to consider Frobenius groups of odd and degenerate type.

Frobenius groups of odd type. Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type. Note that G contains a single conjugacy class of involutions, which we denote by J. Moreover, J has Morley degree 1.

Proposition 6.4 [Borovik and Nesin 1994, Proposition 11.18]. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type and J its set of involutions. If $a \in J^{\cdot 2} \setminus \{1\}$ and $i \in J$, then* $\mathrm{Cen}(a) \cap \mathrm{Cen}(i) = \{1\}$.

Lemma 6.5. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type and J its set of involutions. Fix distinct involutions $i, j \in J$.*

(a) *If $a \in iJ \setminus \{1\}$, then $\mathrm{Cen}(a) \subseteq iJ$ is a uniquely 2-divisible abelian group.*

(b) *iJ is uniquely 2-divisible.*

(c) *J acts regularly on itself; i.e., given $i, j \in J$ there is a unique $p \in J$ such that $j = i^p$.*

(d) *$iJ \cap jJ$ is uniquely 2-divisible.*

(e) $\mathrm{Cen}(ij) = iJ \cap jJ$.

(f) *The family $\{\mathrm{Cen}(a) \setminus \{1\} : a \in J^{\cdot 2} \setminus \{1\}\}$ forms a partition of $J^{\cdot 2} \setminus \{1\}$.*

Proof. (a) By the previous proposition, we have $\mathrm{Cen}(a) \cap \mathrm{Cen}(k) = \{1\}$ for all involutions k. In particular, $\mathrm{Cen}(a)$ does not contain an involution and hence is uniquely 2-divisible by Proposition 5.1. Note that i acts on $\mathrm{Cen}(a)$ as a fixed-point-free involutionary automorphism. Hence, by Proposition 2.6, $\mathrm{Cen}(a)$ is abelian and inverted by i, therefore $i\,\mathrm{Cen}(a) \subseteq J$, and we have $\mathrm{Cen}(a) \subseteq iJ$.

(b) Fix $a = ip \in iJ \setminus \{1\}$. Since $\mathrm{Cen}(a) \subseteq iJ$ is uniquely 2-divisible, we have $a = b^2$ for some $b = iq \in iJ$. If $a = c^2$ for another element $c = ir \in iJ$, then $ii^q = ii^r$, and hence $qr \in \mathrm{Cen}(i) \cap J^{\cdot 2} = \{1\}$. Thus $b = c$.

(c) Note that $j = i^p$ if and only if $ij = ii^p = (ip)^2$. Since iJ is uniquely 2-divisible, p exists and is unique.

(d) It suffices to show that $iJ \cap jJ$ is 2-divisible. Given $a \in iJ \cap jJ$, we have $\mathrm{Cen}(a) \subseteq iJ \cap jJ$, and hence $a = b^2$ for some $b \in \mathrm{Cen}(a) \subseteq iJ \cap jJ$.

(e) By (a), we have $\mathrm{Cen}(ij) \subseteq iJ \cap jJ$. Hence it remains to show that $iJ \cap jJ \subseteq \mathrm{Cen}(ij)$. Given $a \in iJ \cap jJ$, a is inverted by i and j, and hence $a \in \mathrm{Cen}(ij)$.

(f) Suppose $c \in \mathrm{Cen}(a) \cap \mathrm{Cen}(b)$ for some $c \neq 1$. Then $a, b \in \mathrm{Cen}(c)$, and hence $a \in \mathrm{Cen}(b)$, because $\mathrm{Cen}(c)$ is abelian. This implies $\mathrm{Cen}(b) \subseteq \mathrm{Cen}(a)$, because $\mathrm{Cen}(b)$ is abelian. Hence $\mathrm{Cen}(a) = \mathrm{Cen}(b)$ by symmetry. This implies (f). □

Given two distinct involutions $i \neq j$ in J, we define the line

$$\ell_{ij} = \{p \in J : (ij)^p = (ij)^{-1}\}.$$

Lemma 6.6. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type and J its set of involutions. Let $i \neq j \in J$. Then $i\ell_{ij} = \mathrm{Cen}(ij)$.*

Proof. Clearly $i\ell_{ij} \subseteq \mathrm{Cen}(ij)$. On the other hand, we have $\mathrm{Cen}(ij) \subseteq iJ$ by Lemma 6.5(e). Given $\sigma = ip \in \mathrm{Cen}(ij)$, we have

$$(ji)^p = (ij)^{ip} = (ij)^\sigma = ij.$$

Therefore $(ij)^p = (ij)^{-1}$, and thus $p \in \ell_{ij}$. □

Lemma 6.7. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type and J its set of involutions. Fix $i \neq j \in J$, and let $p, q \in \ell_{ij}$ be distinct involutions. Then $\ell_{pq} = \ell_{ij}$.*

Proof. We have $pq \in \mathrm{Cen}(ij)$, and hence $\mathrm{Cen}(pq) = \mathrm{Cen}(ij)$. Moreover, $ip \in \mathrm{Cen}(ij) = \mathrm{Cen}(pq)$, and hence

$$\ell_{pq} = p\,\mathrm{Cen}(pq) = i\,\mathrm{Cen}(ij) = \ell_{ij}. \qquad \square$$

Hence the set J together with the above notion of lines satisfies conditions (a) and (b) of Definition 2.1.

Lemma 6.8. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type and J its set of involutions. Let $i, j \in J$ be distinct involutions, and let T be a subgroup of G such that $\mathrm{Cen}(ij) \leq T \leq N_G(\mathrm{Cen}(ij))$. Then T can be written as a semidirect product $T = \mathrm{Cen}(ij)(T \cap \mathrm{Cen}(i))$.*

Proof. Note G can be decomposed as $G = iJ\,\mathrm{Cen}(i)$, and put $\lambda = \ell_{ij}$. Given $t \in T$, we can write $t = ipg$ for (unique) elements $k \in J$ and $g \in \mathrm{Cen}(i)$. Then

$$\lambda = \lambda^t = \lambda^{pg}.$$

In particular, $i \in \lambda^p \cap \lambda$. If $\lambda^p \cap \lambda = \{i\}$, then $p = i \in \lambda$. If $\lambda^p = \lambda$, then $p \in \lambda$ by part (a) of Lemma 2.7. Therefore $t = ipg \in \mathrm{Cen}(ij)(T \cap \mathrm{Cen}(i))$. □

We will make use of the following result about conjugacy of complements:

Proposition 6.9 [Borovik and Nesin 1994, Theorem 9.11]. *Let G be a group of finite Morley rank and $H \lhd G$ be a definable normal nilpotent subgroup. Assume that G/H is abelian and $\mathrm{Cen}(g) = 1$ for some $g \in G$. Then $G = H \rtimes \mathrm{Cen}(g)$ and any two complements of H in G are H-conjugate. Furthermore, $[H, g] = H$.*

Proposition 6.10. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type and J its set of involutions. If $i \neq j$ are two distinct involutions in J, then*

$$N_G(\mathrm{Cen}(ij)) \cap J^{\cdot 2} = \mathrm{Cen}(ij).$$

Proof. Assume there is $a \in \left(N_G(\mathrm{Cen}(ij)) \cap J^{\cdot 2}\right) \setminus \mathrm{Cen}(ij)$ and consider the group $A = \mathrm{Cen}(a) \cap N_G(\mathrm{Cen}(ij))$. Note that A and $\mathrm{Cen}(ij)$ are abelian by Lemma 6.5, and by Lemma 6.5(f), we obtain a semidirect product $K = \mathrm{Cen}(ij) \rtimes A$. Moreover, K is a solvable subgroup of $N_G(\mathrm{Cen}(ij))$. By Lemma 6.8, we have $K = \mathrm{Cen}(ij) \rtimes (K \cap \mathrm{Cen}(i))$. Now Proposition 6.9 implies that A and $K \cap \mathrm{Cen}(i)$ are conjugate. This is impossible by Proposition 6.4. □

Theorem 6.11. *Let $H < G$ be a connected Frobenius group of Morley rank n and odd type, and let J be the set of all involutions in G.*

(a) *J forms a mock hyperbolic reflection space, and all lines in J are infinite.*

(b) *Choose Λ such that (J, Λ) is a generic mock hyperbolic reflection space such that all lines are of Morley rank k, and set $n = \mathrm{MR}(J)$. If $n \leq 2k + 1$, then G splits.*

Proof. (a) We first show that J forms a mock hyperbolic reflection space. We already know that conditions (a) and (b) of Definition 2.1 are satisfied. Fix a line $\lambda = \ell_{ij}$. Then $\lambda^{\cdot 2} = i\lambda$ by Lemma 2.7, and hence $\lambda^{\cdot 2} = \mathrm{Cen}(ij)$ by Lemma 6.6. Therefore J forms a mock hyperbolic reflection space by Lemma 2.8.

Moreover, by [Borovik et al. 2007, Proposition 1.1], the centralizer of any element in a connected nontrivial group of finite Morley rank is infinite. In particular, $\mathrm{Cen}(ij)$ is infinite, and therefore all lines in J must be infinite.

(b) Note that if the mock hyperbolic reflection space J consists of a single line, then $H < G$ splits by Theorem 2.14. Hence, by Theorem 5.5, we may assume $n = 2k + 1$. Then again by Theorem 5.5, B has a connected normal subgroup N of rank $2n - k$ such that $N \approx S$, where

$$S = \{\sigma \in J^{\cdot 2} \setminus \{1\} : \ell_\sigma \text{ exists}\} \cup \{1\}$$

is the set of translations. Recall that $\mathrm{MR}(S) = 2n - k$ and $\mathrm{MD}(S) = 1$.

On the other hand $N \cap \mathrm{Cen}(i) < N$ is a connected Frobenius group of finite Morley rank, and hence $\bigcup_{i \in J} N \cap \mathrm{Cen}(i) \subseteq N$ is a generic subset of N. This contradicts $N \approx S$. □

As a direct consequence, we get the following known corollary (which also follows from [Borovik and Nesin 1994, Lemma 11.21 and Theorem 11.32]).

Corollary 6.12. *Let $H < G$ be a connected Frobenius group of finite Morley rank of odd type. If G has a nontrivial abelian normal subgroup, then G splits.*

Proof. This follows directly from Theorem 2.14. □

Proposition 6.13. *Let $H < G$ be a connected nonsplit Frobenius group of finite Morley rank and odd type, and let (J, Λ) be the associated mock hyperbolic reflection space. If generic lines have Morley rank* 1, *then G is a nonsplit sharply 2-transitive group of characteristic $\neq 2$.*

Proof. Set $n = \mathrm{MR}(J)$. The set of translations has Morley rank $2n - 1$ and is not generic in G. On the other hand, $\mathrm{Cen}(i)$ acts on $J \setminus \{i\}$ without fixed points. Therefore $\mathrm{MR}(\mathrm{Cen}(i)) \leq 2n$. Hence $G = i\,J\,\mathrm{Cen}(i)$ must have Morley rank $2n$ and $\mathrm{Cen}(i)$ has Morley rank n. This implies that $\mathrm{Cen}(i)$ acts regularly on $J \setminus \{1\}$, and hence G is a sharply 2-transitive group. □

Remark 6.14. We will see in Corollary 7.5 that the group in the above proposition must in fact be simple.

Proposition 6.15. *Let $H < G$ be a connected Frobenius group of Morley rank at most* 10 *and odd type. Then either $H < G$ splits or G is a simple nonsplit sharply* 2*-transitive group of Morley rank* 8 *or* 10.

Proof. Assume G does not split. Suppose the set of involutions has Morley rank n and the lines in the associated generic mock hyperbolic reflection space have rank k. Since the set of translations is not generic in G, we have $\mathrm{MR}(G) > 2n-k$. Moreover, we know $n > 2k+1$ and $k \geq 1$. This shows $\mathrm{MR}(G) > 2(2k+2)-k = 3k+4$. Since $\mathrm{MR}(G) \leq 10$, we obtain $k = 1$ and $\mathrm{MR}(G) > 7$. The previous proposition and the remark show that G is a simple sharply 2-transitive group, and hence $\mathrm{MR}(G)$ must be an even number, so $\mathrm{MR}(G)$ is either 8 or 10. □

Frobenius groups of odd type with nilpotent complement. Delahan and Nesin showed that a sharply 2-transitive group of finite Morley rank of characteristic $\neq 2$ with nilpotent point stabilizer must split [Borovik and Nesin 1994, Theorem 11.73]. We will show that the same is true for a Frobenius group of odd type if the lines in the associated mock hyperbolic reflection geometry are strongly minimal or if there is no interpretable bad field of characteristic 0.

We fix a connected Frobenius group $H < G$ of finite Morley rank of odd type, and we denote the set of involutions by J. By Theorem 6.11, J forms a mock hyperbolic reflection space with infinite lines. Note that $H = \mathrm{Cen}(i)$ if i is the unique involution in H. If λ is a line containing i, then $N_G(\lambda) = \lambda^{\cdot 2} \rtimes N_H(\lambda)$ is a split Frobenius group by Theorem 2.14.

Lemma 6.16. *If i and j are involutions with $i \neq j$, then $\mathrm{Cen}(ij)$ has infinite index in $N_G(\mathrm{Cen}(ij))$. In particular, $N_{\mathrm{Cen}(i)}(\ell_{ij})$ is infinite.*

Proof. Otherwise $\bigcup_{g\in G} \mathrm{Cen}(ij)^g \subseteq J^{\cdot 2}$ by Lemma 6.5, and hence $J^{\cdot 2}$ would be generic in G. This is impossible, since $\bigcup_{i\in J} \mathrm{Cen}(i) = \bigcup_{g\in G} H^g$ is generic in G and the elements of $J^{\cdot 2}$ do not have fixed points by Lemma 2.5. Therefore $\mathrm{Cen}(ij)$ has infinite index in $N_G(\mathrm{Cen}(ij))$.

Now $N_G(\mathrm{Cen}(ij)) = N_G(\ell_{ij})$, and $N_G(\ell_{ij}) \cap J = \ell_{ij}$ forms a mock hyperbolic reflection space (consisting of one line). Therefore $N_G(\mathrm{Cen}(ij)) = i\ell_{ij} N_{\mathrm{Cen}(i)}(\ell_{ij})$, and thus $N_{\mathrm{Cen}(i)}(\ell_{ij})$ is infinite. □

If the point stabilizer in a sharply 2-transitive group of characteristic $\neq 2$ with planar maximal near-field contains an element $g \notin \{1, i\}$ such that g normalizes all lines containing i, then by [Sozutov et al. 2014] the sharply 2-transitive group splits. We are going to prove a similar result for Frobenius groups of finite Morley rank of odd type.

If A is a group, then we write $A^* = A \setminus \{1\}$.

Lemma 6.17. *Let λ be a line containing $i \in J$ and fix a definable solvable subgroup $A \leq N_{\mathrm{Cen}(i)}(\lambda)$. Then $A^* i \lambda \cup \{1\} = A^\lambda$.*

Proof. Note that $H = \lambda^{\cdot 2} \rtimes A$ is a solvable Frobenius group of finite Morley rank. By [Borovik and Nesin 1994, Theorem 11.32], we have $H = \lambda^{\cdot 2} \cup \bigcup_{j \in \lambda} A^j$. This proves the lemma. □

Proposition 6.18. *Let Λ be a set of lines on J such that (J, Λ) forms a generic mock hyperbolic reflection space. Suppose there exists a definable infinite solvable normal subgroup $A \trianglelefteq \mathrm{Cen}(i)$ such that $A \leq \bigcap_{\lambda \in \Lambda_i} N_{\mathrm{Cen}(i)}(\lambda)$. Then $H < G$ splits.*

Proof. We may assume that all lines in Λ have Morley rank k. Since i is central in $\mathrm{Cen}(i)$, we may also assume that $i \in A$.

Now set $J_i = \bigcup_{\lambda \in \Lambda_i} \lambda = \{j \in J \setminus \{i\} : \ell_{ij} \text{ exists}\} \cup \{i\}$. By the previous lemma, we have $A^* i \lambda \cup \{1\} = A^\lambda$ for all $\lambda \in \Lambda_i$. Hence we have

$$A^* i J_i \cup \{1\} = \bigcup_{\lambda \in \Lambda_i} A^* i \lambda \cup \{1\} = \bigcup_{\lambda \in \Lambda_i} A^\lambda = A^{J_i}.$$

We have $A^* i J_i \approx A^* i J$ as a consequence of Lemma 2.5 and $A^{J_i} \approx A^J$, since J acts regularly on the set of conjugates of H and hence also on the set of conjugates of A.

Therefore

$$A^* i J \cup \{1\} \approx A^* i J_i \cup \{1\} = A^{J_i} \approx A^J = A^{\mathrm{Cen}(i) J} = A^G.$$

Put $N = \mathrm{Stab}^{\approx}(A^G)$. Then $A \leq N$ and $N \trianglelefteq G$ is a normal subgroup. Hence $A^G \leq N$. Now Proposition 5.4 implies that $A^G \approx N$. Note that

$$\mathrm{MR}(N) = \mathrm{MR}(J) + \mathrm{MR}(A).$$

Moreover, $J \subseteq N$, therefore $J^{\cdot 2} \subseteq N$ and thus $\mathrm{MR}(N) \geq 2n - k$. Note that A acts without fixed points on any line $\lambda \in \Lambda_i$, and therefore $\mathrm{MR}(A) \leq k$. In conclusion

$$n - k \leq \mathrm{MR}(N) - \mathrm{MR}(J) = \mathrm{MR}(A) \leq k,$$

and therefore $n \leq 2k$. Now Proposition 5.19 implies that $H < G$ splits. □

Corollary 6.19. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type. If H is a minimal group, i.e., if H does not contain an infinite proper definable subgroup, then $H < G$ splits.*

Proof. The assumptions and Lemma 6.16 imply that $N_{\mathrm{Cen}(i)}(\lambda) = \mathrm{Cen}(i)$ holds for all i in J and $\lambda \in \Lambda_i$. If $H = \mathrm{Cen}(i)$, then $H = \bigcap_{\lambda \in \Lambda_i} N_H(\lambda)$ and H is abelian. Therefore Proposition 6.18 implies that $H < G$ splits. □

We can use Zilber's field theorem to find interpretable fields in Frobenius groups of odd type.

Proposition 6.20 [Borovik and Nesin 1994, Theorem 9.1]. *Let $G = A \rtimes H$ be a group of finite Morley rank, where A and H are infinite definable abelian subgroups and A is H-minimal, i.e., there are no definable infinite H-invariant subgroups. Assume that H acts faithfully on A. Then there is an interpretable field K such that $A \cong K_+$, $H \leq K^*$, and H acts by multiplication.*

Let $\lambda = \ell_{ij}$ be a line. Then $N_{\mathrm{Cen}(i)}(\lambda)$ is infinite and acts on $\lambda^{\cdot 2} = \mathrm{Cen}(ij)$ by conjugation. Take a minimal subgroup $A \leq N_{\mathrm{Cen}(i)}(\lambda)$. Since the action of A on $\mathrm{Cen}(ij)$ has no fixed points, we can find an infinite A-minimal subgroup $B \leq \mathrm{Cen}(ij)$ on which A acts faithfully. Moreover, B must be abelian, because $\mathrm{Cen}(ij)$ is an abelian group. Hence, by Proposition 6.20, there is an interpretable field K such that $B \cong K_+$, $A \leq K^*$, and A acts by multiplication.

In particular, if the line λ is strongly minimal, then K is strongly minimal and $A \cong K^*$.

If A is a proper subgroup of K^*, then K is a bad field, i.e., an infinite field of finite Morley rank such that K^* has a proper infinite definable subgroup. By [Baudisch et al. 2009], bad fields of characteristic 0 exist. However, it follows from work of Wagner [2001] that if $\mathrm{char}(K) \neq 0$, then K^* is a good torus, i.e., every definable subgroup of K^* is the definable hull of its torsion subgroup. We refer to [Cherlin 2005] for properties of these good tori.

Theorem 6.21. *Let $H < G$ be a connected Frobenius group of finite Morley rank and odd type. Fix Λ such that (J, Λ) is a generic mock hyperbolic reflection space. Moreover, assume that H has a definable nilpotent normal subgroup N such that $N \cap N_H(\lambda)$ is infinite for all $\lambda \in \Lambda_i$.*

If all lines in Λ are strongly minimal or if G does not interpret a bad field of characteristic 0, *then $H < G$ splits.*

Proof. We may assume that N is connected. Let T be a maximal good torus in N. As a consequence of the structure of nilpotent groups of finite Morley rank [Borovik and Nesin 1994, Theorems 6.8 and 6.9], T must be central in N. By [Cherlin 2005, Theorem 1], any two maximal good tori are conjugate. Therefore T is the unique maximal good torus in N. Since a connected subgroup of a good torus is a good torus, the assumptions (and the previous discussion) imply that $N_H(\lambda) \cap T$ is infinite for all lines $\lambda \in \Lambda_i$. By [Cherlin 2005, Lemma 2], the family $\{N_H(\lambda) \cap T : \lambda \in \Lambda_i\}$ is finite. Hence, after replacing Λ by a generic subset $\Lambda' \subseteq \Lambda$, we may assume that $\{N_H(\lambda) \cap T : \lambda \in \Lambda_i\}$ consists of a unique infinite abelian normal subgroup of H. Now Proposition 6.18 implies that $H < G$ splits. □

Frobenius groups of degenerate type. We now use mock hyperbolic spaces to study Frobenius groups of finite Morley rank and degenerate type. A geometry with similar properties, but defined on the whole group, was used by Frécon in his result on the nonexistence of bad groups of Morley rank 3.

Lemma 6.22. *Let $H < G$ be a connected Frobenius group of Morley rank n and of degenerate type. Suppose the Frobenius complement H is abelian and of Morley rank k. Then $n \geq 2k+1$, and if $n = 2k+1$, then G contains a definable normal subgroup N of Morley rank $k+1$.*

Proof. Note that G is uniquely 2-divisible, and hence $a \otimes b = a^{1/2}ba^{1/2}$ defines a K-loop structure on G. Let $L = (G, \otimes)$ denote the corresponding K-loop, and set $\mathcal{A} = G \times \langle \epsilon \rangle < \mathrm{Aut}(L)$, where ϵ is given by inversion. Now let $\mathcal{G}$ be the quasidirect product $\mathcal{G} = L \rtimes_Q \mathcal{A}$.

By Theorem 4.7, the involutions J in $\mathcal{G}$ form a partial mock hyperbolic reflection space, and since $\bigcup_{g \in G} H^g \subseteq G$ is a generic subset of G, the involutions must form a generic mock hyperbolic reflection space. Moreover, $\mathrm{MR}(J) = n$ and each line has Morley rank k. Now the lemma follows from Theorem 5.5. □

Theorem 6.23. *Let $H < G$ be a connected Frobenius group of Morley rank n and of degenerate type. Suppose the Frobenius complement H is abelian and of Morley rank k. Then $n \geq 2k+1$.*

If $n = 2k+1$, then G splits as $G = N \rtimes H$ for some definable connected normal subgroup N of Morley rank $k+1$. Moreover, if N is solvable, then there is an interpretable field K of characteristic $\neq 2$ such that $G = K_+ \rtimes H$, $H \leq K^*$, and H acts on K_+ by multiplication.

Proof. By the previous lemma, we may assume $n = 2k+1$. Then G contains a definable normal subgroup N of rank $k+1$, and we may assume that N is connected.

Note that $\mathrm{MR}\big(\bigcup_{g \in G}(N \cap H)^g\big) = k+1+\mathrm{MR}(N \cap H)$ and $\mathrm{MR}(N) = k+1$. Therefore $N \cap H$ must be finite. If $N \cap H$ is nontrivial, then $(N \cap H) < N$ is a connected Frobenius group, and hence $N \cap H$ must be connected. Therefore $N \cap H = \{1\}$.

The semidirect product $N \rtimes H$ has rank $2k+1$, and hence is generic in G. Therefore $G = N \rtimes H$ splits.

Now assume that N is solvable. Then N is nilpotent since, by [Borovik and Nesin 1994, Theorem 11.29], a solvable complement of a split Frobenius group of finite Morley rank is nilpotent. Moreover, $\mathrm{Cen}(u) \leq N$ for all $u \in N \setminus \{1\}$ by [Borovik and Nesin 1994, Theorem 11.32] (since G is solvable). Note that u^G cannot be generic in N, because G does not contain involutions. Therefore $\mathrm{MR}(u^G) \leq k$, and hence $\mathrm{MR}(\mathrm{Cen}(u)) \geq k+1$. Thus $\mathrm{Cen}(u) = N$, so N is abelian.

We now show that N is H-minimal: Let $A \leq N$ be a H-invariant subgroup. We may assume that A is connected. Given $a \in A \setminus \{1\}$, we have $\mathrm{Cen}(a) \cap H = \{1\}$, and therefore $a^H \subseteq A$ has rank k. If A has rank k, then a^H is generic in A, and therefore A must contain an involution. This is a contradiction. Therefore $A = \{1\}$ or $A = N$, and hence N is H-minimal.

By Proposition 6.20, there must be an interpretable field K such that $N = K_+$, $H \leq K^*$, and H acts on N by multiplication. □

7. Sharply 2-transitive groups of finite Morley rank

Let G be a sharply 2-transitive group of finite Morley rank with $\mathrm{char}(G) \neq 2$, and let J denote the set of involutions in G. By Corollary 3.4 (or by Theorem 6.11), the set J forms a mock hyperbolic reflection space.

We set $n = \mathrm{MR}(J)$ and $k = \mathrm{MR}(\mathrm{Cen}(ij))$ for involutions $i \neq j \in J$. Note that k does not depend on the choice of i, and j and $k = n$ if and only if G is split.

Now we assume that G is not split. By [Borovik and Nesin 1994, Proposition 11.71], we have $0 < 2k < n$, and we will improve this inequality below.

Since G acts sharply 2-transitively on J, it is easy to see that $\mathrm{MR}(G) = 2n$ and $\mathrm{MR}(J^{\cdot 2}) = 2n - k$. Moreover, G and $\mathrm{Cen}(ij)$ have Morley degree 1 by [Borovik and Nesin 1994, Lemma 11.60].

Proposition 7.1. (a) *The set iJ is indecomposable for all $i \in J$.*

(b) *$\langle J^{\cdot 2}\rangle$ is a definable connected subgroup. In particular, there is a bound m such that any $g \in \langle J^{\cdot 2}\rangle$ is a product of at most m translations.*

Proof. (a) Since $\mathrm{MD}(G) = 1$, the set J is indecomposable by [Borovik and Nesin 1994, Corollary 5.25], and hence iJ is indecomposable too.

(b) Since $\langle J^{\cdot 2}\rangle = \langle iJ\rangle$, (b) follows from Zilber's indecomposability theorem using (a). □

Remark 7.2. By Proposition 7.1(b), it is easy to see that the nonsplit examples of sharply 2-transitive groups of characteristic 0 constructed in [Rips and Tent 2019] do not have finite Morley rank.

Lemma 7.3. *For any $g \in G \setminus J$, the set $\{i \in J : gi$ has a fixed point$\}$ is generic in J.*

Proof. Let $g \in G$. For any $j \in J$ there is a unique $i_j \in J$ swapping j and j^g. Then gi_j centralizes j, so has a fixed point. If $i_j = i_p$ for some $j \neq p \in J$, then by sharp 2-transitivity it follows that $g = i_j = i_p \in J$. Hence for $g \notin J$, the i_j for $j \in J$ are pairwise distinct, and hence $\{i_j : j \in J\}$ has Morley rank n. □

Let $\mu : G^3 \to G$ be the multiplication map, i.e., $\mu(g_1, g_2, g_3) = g_1 g_2 g_3$.

Lemma 7.4. $$\mathrm{MR}(J^{\cdot 3}) > \mathrm{MR}(J^{\cdot 2}).$$

Proof. Note that $\mathrm{MR}(J^{\cdot 3}) = \mathrm{MR}(iJ^{\cdot 3}) \geq \mathrm{MR}(J^{\cdot 2}) > \mathrm{MR}(J) = n$, and hence J is not a generic subset of $J^{\cdot 3}$.

For $\alpha \in J^{\cdot 3}$, we let $X_\alpha = \{i \in J : i\alpha \in J^{\cdot 2}\}$ be the set of all involutions i such that $i\alpha$ is a translation. By Lemma 7.3 and Remark 3.1, $\mathrm{MR}(X_\alpha) < n$ for all $\alpha \in J^{\cdot 3} \setminus J$.

Let $\mathrm{MR}(J^{\cdot 3}) = 2n - k + l$ for some $l \geq 0$. There is a generic set of $\alpha \in J^{\cdot 3} \setminus J$ such that $\mathrm{MR}(\mu^{-1}(\alpha) \cap (J \times J \times J)) = n + k - l$. Set $X = X_\alpha$ for such an $\alpha \in J^{\cdot 3} \setminus J$. If $irs = \alpha$, then $\mathrm{MR}(\{j \in J : rs \in jJ\}) = k$, and hence $\mathrm{MR}(\mu^{-1}(\alpha)) = \mathrm{MR}(X) + k$. Therefore we have $\mathrm{MR}(X) = n - l$, and hence $l \geq 1$ by Lemma 7.3. □

Corollary 7.5. *Let G be a nonsplit sharply 2-transitive group of finite Morley rank. If the lines are strongly minimal, then G is simple and a counterexample to the Cherlin–Zilber conjecture.*

Proof. Let $N \trianglelefteq G$ be a normal subgroup. If N contains an involution, then $J \subseteq N$, and hence $J^{\cdot 2} \subseteq N$. Now assume N does not contain an involution. Fix $u \in N$ and $i \in J$. Then $1 \neq u^{-1}iui \in N \cap J^{\cdot 2}$, and hence $J^{\cdot 2} \subseteq N$, since all translations are conjugate. Therefore $J^{\cdot 2} \subseteq N$ holds true in both cases. Since $iJ^{\cdot 3} \subseteq \langle iJ \rangle \subseteq N$ and $\mathrm{MR}(J^{\cdot 2}) = 2n - 1 < \mathrm{MR}(J^{\cdot 3}) = \mathrm{MR}(iJ^{\cdot 3}) \leq \mathrm{MR}(G) = 2n$ (Lemma 7.4), this implies $N = G$. This shows that G must be simple.

Assume towards a contradiction that G is an algebraic group over an algebraically closed field K. If the K-rank of G is at least 2, then the torus contains commuting involutions, contradicting Remark 3.1(c). If the K-rank of G is 1, then G is isomorphic to $\mathrm{PSL}_2(K)$ and also contains commuting involutions, e.g., $x \mapsto -1/x$ and $x \mapsto -x$ are commuting involutions in $\mathrm{PSL}_2(K)$. □

Note that a sharply 2-transitive group of finite Morley rank in characteristic different from 2 is not a bad group in the sense of Cherlin, since for any translation $\sigma \in J^{\cdot 2}$ the group $N_G(\mathrm{Cen}(\sigma)) = \mathrm{Cen}(\sigma) \rtimes N_{\mathrm{Cen}(\sigma)}(\mathrm{Cen}(\sigma))$ is solvable, but not nilpotent.

If G is a sharply 2-transitive group of finite Morley rank and $\mathrm{char}(G) \neq 2$ with n, k and J be as before, then by Theorem 6.11, G splits if $n \leq 2k + 1$. Thus, we obtain:

Corollary 7.6. *If G is a sharply 2-transitive group and $\mathrm{MR}(G) = 6$, then G is of the form $\mathrm{AGL}_1(K)$ for some algebraically closed field K of Morley rank 3.*

Proof. If $\mathrm{char}(G) \neq 2$, then, by Theorem 6.11, G splits and the result follows from [Altınel et al. 2019]. If $\mathrm{char}(G) = 2$, then G is split by [Altınel et al. 2019] and any point stabilizer has Morley rank 3. Since the point stabilizers do not contain involutions, they are solvable by [Frécon 2018]. By [Borovik and Nesin 1994, Corollary 11.66], an infinite split sharply 2-transitive group of finite Morley rank whose point stabilizer contains an infinite normal solvable subgroup must be standard. □

8. Further remarks

A finite uniquely 2-divisible K-loop is the same as a finite B-loop in the sense of Glauberman [1964]. As a consequence of Glauberman's Z^*-theorem [1966] finite B-loops are solvable. Following Glauberman, we say that a K-loop L is *half-embedded* in some group G if it is isomorphic to a K-loop arising from a uniquely 2-divisible twisted subgroup of G as in Proposition 4.4. B-loops and uniquely 2-divisible K-loops can always be half-embedded in some group and that group can be chosen to be finite if the loop is finite [Glauberman 1964, Theorem 1 and Corollary 1]. This allows us to restate Glauberman's result for twisted subgroups:

Proposition 8.1 [Glauberman 1966]. *Let G be a group, and let $L \subseteq G$ be a finite uniquely 2-divisible twisted subgroup. Then $\langle L \rangle$ is solvable.*

As a consequence finite mock hyperbolic spaces must consist of a single line:

Proposition 8.2. *Suppose J forms a finite mock hyperbolic reflection space in a group G. Then J consists of a single line.*

Proof. We may assume that G acts faithfully on J. Let $i \in J$ be an involution. Since J acts regularly on itself, the square map on iJ must be injective and hence bijective as a consequence of finiteness. Now it is easy to check that iJ is a finite uniquely 2-divisible twisted subgroup in G. Therefore $\langle iJ \rangle$ is solvable by Proposition 8.1. Moreover, $\mathrm{Cen}(i) \leq N_G(\langle iJ \rangle)$ and G can be decomposed as $G = iJ\,\mathrm{Cen}(i)$. Therefore $\langle iJ \rangle$ is a solvable normal subgroup of G. It follows that G contains a nontrivial abelian normal subgroup. Now Theorem 2.14 implies that J consists of a single line. □

In the context of groups of finite Morley rank, we do not know if every uniquely 2-divisible K-loop of finite Morley rank can be definably half-embedded into a group of finite Morley rank. The following would be a finite Morley rank version of Glauberman's theorem:

Conjecture 8.3. Let G be a connected group of finite Morley rank with a definable uniquely 2-divisible twisted subgroup L of Morley degree 1 such that $G = \langle L \rangle$. Then G is solvable.

Note that this conjecture would imply the Feit–Thompson theorem for connected groups of finite Morley rank: if G is a connected group of finite Morley rank of degenerate type, then G is uniquely 2-divisible, and hence Conjecture 8.3 (applied to $L = G$) would imply that G is solvable.

Moreover, it would imply that Frobenius groups of finite Morley rank split: for Frobenius groups of degenerate type this would follow from solvability. If G is a connected Frobenius group of finite Morley rank and odd type with involutions J and lines Λ, then it suffices to show that G has a nontrivial definable solvable normal subgroup (in that case G has a nontrivial abelian normal subgroup and hence splits by Theorem 2.14). Note that iJ is a uniquely 2-divisible twisted subgroup. If G is sharply 2-transitive, then Proposition 7.1 shows that $\langle iJ\rangle$ is definable and connected and hence should be solvable by Conjecture 8.3.

For the general case consider the family $\mathcal{F}_i = \{\operatorname{Cen}(ij)^0 : j \in J \setminus \{i\}\}$. By Zilber's indecomposability theorem the subgroup $N = \langle H : H \in \mathcal{F}_i\rangle$ is definable and connected. Moreover, it is easy to see that $N \cap iJ$ must be generic in iJ and N must be normalized by $\operatorname{Cen}(i)$. Therefore N must be a normal subgroup of $G = iJ\operatorname{Cen}(i)$, and clearly $N = \langle N \cap iJ\rangle$. Therefore Conjecture 8.3 would imply that N is solvable.

If Frobenius groups of odd and degenerate type split, then Remark 6.3 shows that Frobenius groups of even type also split.

If the twisted subgroup in the statement of Conjecture 8.3 is strongly minimal, then we show that G must be 2-nilpotent:

Proposition 8.4. *Let G be a connected group of finite Morley rank with a definable strongly minimal uniquely 2-divisible twisted subgroup L such that $G = \langle L\rangle$. Then G is 2-nilpotent.*

Proof. Let $x \otimes y = x^{1/2}yx^{1/2}$ be the corresponding K-loop structure on L. If $(L, \otimes)$ is an abelian group, then [Kiechle 2002, Theorem 6.14, part (3)] implies $[[a, b], c] = 1$ for all $a, b, c \in L$, and therefore $G = \langle L\rangle$ must be 2-nilpotent. Therefore it suffices to show that $(L, \otimes)$ is an abelian group.

Put $T = N_G(L)/\operatorname{Cen}(L)$. Then $T \leq \operatorname{Aut}((L, \otimes))$, and we may consider the quasidirect product $\mathcal{G} = L \rtimes_Q T$. As stated in Proposition 4.5, the group $\mathcal{G} = L \rtimes_Q T$ acts transitively and faithfully on L by

$$(a, \alpha)(x) = a \otimes \alpha(x),$$

and T is the stabilizer of $1 \in L$. Note that $L' = L \times \{1\}$ is a uniquely 2-divisible twisted subgroup of $\mathcal{G}$. Hence $a \otimes' b = a^{1/2}ba^{1/2}$ defines a K-loop structure on L'. By [Kiechle 2002, Theorem 6.15], the K-loops $(L, \otimes)$ and $(L', \otimes')$ are isomorphic. Therefore it suffices to show that $(L', \otimes')$ is an abelian group.

Hrushovski's analysis of groups acting on strongly minimal sets [Borovik and Nesin 1994, Theorem 11.98] shows that $\operatorname{MR}(\mathcal{G}) \leq 3$. Moreover, if $\operatorname{MR}(\mathcal{G}) = 3$, then T acts sharply 2-transitively on $L \setminus \{1\}$, which is impossible, since T is a group of automorphisms of $(L, \otimes)$.

If $\operatorname{MR}(\mathcal{G}) = 2$, then $L \rtimes_Q T$ is a standard sharply 2-transitive group $K_+ \rtimes K^*$ (and the corresponding permutation groups coincide). Since L' acts without fixed points and the fixed-point-free elements of $K_+ \rtimes K^*$ are precisely the elements of K_+, L' is contained in K_+. Therefore $\otimes'$ agrees with the group structure on K_+, and hence $(L', \otimes')$ is an abelian group.

Now assume $\mathrm{MR}(\mathcal{G}) = 1$. We argue similarly to the proof of [Glauberman 1964, Lemma 5, part (v)].

Consider the finite twisted subgroup $L'' = \{a\mathcal{G}^0 : a \in L'\}$ of $\mathcal{G}/\mathcal{G}^0$. Since L' is uniquely 2-divisible, the map $L'' \to L''$, $a \mapsto a^2$ is surjective and hence a bijection, since L'' is finite. Hence we may define a K-loop structure $x\mathcal{G}^0 \otimes'' y\mathcal{G}^0 = x^{1/2}yx^{1/2}\mathcal{G}^0$ on L''. The natural map $L' \to L''$ is a surjective homomorphism from $(L', \otimes')$ to $(L'', \otimes'')$ with kernel $L' \cap \mathcal{G}^0$.

In particular, $L' \cap \mathcal{G}^0$ is a normal subloop of L'. Since $L'/(L' \cap \mathcal{G}^0)$ is finite and $\mathrm{MD}(L') = 1$, this implies $L' = L' \cap \mathcal{G}^0$, and hence $L' \subseteq \mathcal{G}^0$. The group $\mathcal{G}^0$ is strongly minimal and thus abelian. Therefore $\otimes'$ agrees with the group structure on $\mathcal{G}^0$, and therefore $(L', \otimes')$ is an abelian group. □

The proof of Proposition 8.4 in fact shows the following:

Corollary 8.5. *Let G be a group of finite Morley rank, and let $L \subseteq G$ be a definable uniquely 2-divisible twisted subgroup of G.*

(a) *If* $\mathrm{MD}(L) = 1$, *then* $L \subseteq G^0$.

(b) *If L is strongly minimal, then the associated K-loop $(L, \otimes)$ is an abelian group, and hence $\langle L\rangle$ is 2-nilpotent (without assuming that $\langle L\rangle$ is definable).*

In particular, if $(L, \otimes)$ is a strongly minimal uniquely 2-divisible K-loop such that L can be definably half-embedded into a group of finite Morley rank, then $(L, \otimes)$ is an abelian group.

Question 8.6. This suggests the following two questions:

(a) Suppose G and L satisfy the assumptions of Proposition 8.4. Must G be abelian?

(b) Is every strongly minimal (uniquely 2-divisible) K-loop an abelian group?

References

[Altınel et al. 2008] T. Altınel, A. V. Borovik, and G. Cherlin, *Simple groups of finite Morley rank*, Mathematical Surveys and Monographs **145**, American Mathematical Society, Providence, RI, 2008. MR Zbl

[Altınel et al. 2019] T. Altınel, A. Berkman, and F. O. Wagner, "Sharply 2-transitive groups of finite Morley rank", preprint, 2019, available at https://hal.archives-ouvertes.fr/hal-01935537.

[Bachmann 1959] F. Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer, 1959. MR Zbl

[Baudisch et al. 2009] A. Baudisch, M. Hils, A. Martin-Pizarro, and F. O. Wagner, "Die böse Farbe", *J. Inst. Math. Jussieu* **8**:3 (2009), 415–443. MR Zbl

[Borovik and Nesin 1994] A. Borovik and A. Nesin, *Groups of finite Morley rank*, Oxford Logic Guides **26**, Clarendon, 1994. MR Zbl

[Borovik et al. 2007] A. Borovik, J. Burdges, and G. Cherlin, "Involutions in groups of finite Morley rank of degenerate type", *Selecta Math.* (*N.S.*) **13**:1 (2007), 1–22. MR Zbl

[Cherlin 2005] G. Cherlin, “Good tori in groups of finite Morley rank”, *J. Group Theory* **8**:5 (2005), 613–621. MR Zbl

[Frécon 2018] O. Frécon, “Simple groups of Morley rank 3 are algebraic”, *J. Amer. Math. Soc.* **31**:3 (2018), 643–659. MR Zbl

[Glauberman 1964] G. Glauberman, “On loops of odd order”, *J. Algebra* **1** (1964), 374–396. MR Zbl

[Glauberman 1966] G. Glauberman, “Central elements in core-free groups”, *J. Algebra* **4** (1966), 403–420. MR Zbl

[Jaligot 2001] E. Jaligot, “Full Frobenius groups of finite Morley rank and the Feit–Thompson theorem”, *Bull. Symbolic Logic* **7**:3 (2001), 315–328. MR Zbl

[Kerby 1974] W. Kerby, *On infinite sharply multiply transitive groups*, Hamburger Mathematische Einzelschriften **6**, Vandenhoeck & Ruprecht, Göttingen, 1974. MR Zbl

[Kiechle 2002] H. Kiechle, *Theory of K-loops*, Lecture Notes in Mathematics **1778**, Springer, 2002. MR Zbl

[Neumann 1940] B. H. Neumann, “On the commutativity of addition”, *J. London Math Soc.* **15** (1940), 203–208. MR Zbl

[Poizat 1987] B. Poizat, *Groupes stables*: *une tentative de conciliation entre la géométrie algébrique et la logique mathématique*, Nur al-Mantiq wal-Ma’rifah **2**, Bruno Poizat, Lyon, 1987. MR Zbl

[Rips and Tent 2019] E. Rips and K. Tent, “Sharply 2-transitive groups of characteristic 0”, *J. Reine Angew. Math.* **750** (2019), 227–238. MR Zbl

[Rips et al. 2017] E. Rips, Y. Segev, and K. Tent, “A sharply 2-transitive group without a non-trivial abelian normal subgroup”, *J. Eur. Math. Soc.* **19**:10 (2017), 2895–2910. MR Zbl

[Sozutov et al. 2014] A. I. Sozutov, E. B. Durakov, and E. V. Bugaeva, “On some near-domains and sharply doubly transitive groups”, *Tr. Inst. Mat. Mekh.* **20**:2 (2014), 277–283. In Russian. MR

[Tent and Ziegler 2016] K. Tent and M. Ziegler, “Sharply 2-transitive groups”, *Adv. Geom.* **16**:1 (2016), 131–134. MR Zbl

[Wagner 2001] F. Wagner, “Fields of finite Morley rank”, *J. Symbolic Logic* **66**:2 (2001), 703–706. MR Zbl

[Wagner 2017] F. Wagner, “Bad groups”, preprint, 2017. arXiv 1703.01764

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