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Using an infinitary version of the hypergraph removal lemma due to Towsner, we prove a model-theoretic higher amalgamation result. In particular, we obtain an independent amalgamation property which holds in structures that are measurable in the sense of Macpherson and Steinhorn, but which is not generally true in structures that are supersimple of finite SU-rank. We use this to show that some of Hrushovski's non-locally-modular, supersimple ω-categorical structures are not MS-measurable.

1. Introduction

Towsner [2018] gives an infinitary version of the hypergraph removal lemma (quoted as Theorem 2.3 here), stated as a rather general measure-theoretic result. We use this to prove a model-theoretic higher amalgamation result (Theorem 2.4), again in the presence of a definable measure. In particular, we obtain an independent amalgamation property (Corollary 3.2; quoted below as Corollary 1.1) which holds in structures that are measurable in the sense of Macpherson and Steinhorn.

The statement of this independent amalgamation property makes no mention of measure and it makes sense in any supersimple structure of finite SU-rank. However, it is not generally true in structures which are supersimple of finite SU-rank. In Theorem 4.7, we use a Hrushovski construction to produce a structure which is ω-categorical, supersimple of SU-rank 1 and which does not satisfy the conclusion of Corollary 3.2. It follows that this structure is not MS-measurable.

The question of whether any (nontrivial) ω-categorical Hrushovski construction can be MS-measurable is open, and this is an important special case of the more general question of whether ω-categorical MS-measurable structures are necessarily one-based. Paolo Marimon [2022a; 2022b] has used a different and more generally applicable approach to show that a much wider class of ω-categorical, supersimple Hrushovski constructions are not MS-measurable. It is also unknown whether any of the ω-categorical Hrushovski constructions can be pseudofinite. In Remarks 4.6 we note that, as a by-product of our approach to non-MS-measurability, we obtain

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information about what coarse pseudofinite dimension would have to be in such a structure, if it were pseudofinite.

We begin with a rough outline of what we mean by a "higher amalgamation property". This is adapted to the form of the Towsner's paper, so is slightly different from other presentations (for example in [Hrushovski 2012]).

Suppose L is a first-order language and $\mathcal{M}$ is an L-structure with domain M and $C \subseteq M$. Let T denote the theory of $\mathcal{M}$. We will assume that $\mathcal{M}$ is "large" (for example $\aleph_1$-saturated, if L is countable) and C has smaller cardinality than that of M. Suppose $n \geq 2$ is a natural number. In an *n-amalgamation problem* over C we are looking for an n-tuple $\bar{b} = (b_1, \ldots, b_n)$ which satisfies certain constraints on subtuples $\bar{b}_I = (b_i : i \in I)$ with $I \subseteq [n] = \{1, \ldots, n\}$ of size $n-1$. The constraints should be in terms of the parameters C, say in the form of satisfying a type, or partial type, $\Phi_I(\bar{x}_I)$, over C. Here, $\bar{x} = (x_1, \ldots, x_n)$ is an n-tuple of variables and $x_I = (x_i : i \in I)$. So, subject to reasonable compatibility requirements, such as $\Phi_I(\bar{x}_I)$ and $\Phi_J(\bar{x}_J)$ having the same restriction to $\bar{x}_{I\cap J}$, we are looking for a solution $\bar{b} \models \bigwedge_I \Phi_I(\bar{x}_I)$, or, in terms of the sets $A_I = \{\bar{a} \in M^n : \mathcal{M} \models \Phi_I(\bar{a})\}$, an element of $\bigcap_I A_I$. If the Φ_I are complete types over C, we might refer to this as a *type-amalgamation* problem.

There are well-known variations on this. If $\mathcal{M}$ carries a notion of independence (or dimension on definable sets) then in an *independent* n-amalgamation problem over C, we are also looking for the b_i to be independent over C. Of course, in this case, the individual constraints $\Phi_I(\bar{x}_I)$ should have solutions which are independent over C. For example, if T is stable, then for all n, any independent type-amalgamation problem over a model (with n complete types over the model) has a solution. If T is simple, then this is true for $n = 2, 3$ (the case $n = 3$ is of course the independence theorem of Kim and Pillay). However, there are examples of supersimple theories of finite SU-rank which do not have independent 4-amalgamation over a model.

Our main result, Theorem 2.4, is an n-amalgamation property which holds in a general context where the set of n-tuples from which we are looking for a solution carries a well-behaved probability measure (see Section 2A for a precise statement). The general form of the statement is that we assume there our n-amalgamation problem has "degenerate" solutions $\bar{b} = (b_1, \ldots, b_n)$, where the b_i are interalgebraic over C. The conclusion is that the set of all solutions is of positive measure (and in particular, there are solutions where the b_i are not interalgebraic). Of course, for this to work, we need to ensure that there are enough solutions to the Φ_I: in the above notation, we require that the measure of A_I is positive, for each $(n-1)$-set I.

If $\mathcal{M}$ is an MS-measurable structure (see Section 3B for definitions and background) there is a strong interaction between dimension and measure. The structure $\mathcal{M}$ is supersimple of finite SU-rank and each definable subset has an associated dimension (which can be taken as SU-rank for the purposes of this introduction). Each definable set also carries a (definable) probability measure on its definable subsets with the property that a subset has positive measure if and only if it has the same dimension as the ambient definable set.

From Theorem 2.4 we obtain the following independent amalgamation result (Corollary 3.2), which holds in any MS-measurable M.

Corollary 1.1. *Suppose* $\mathcal{M}$ *is an MS-measurable structure and* $S_1, \ldots, S_n$ *are infinite* C*-definable sets, for some finite* $C \subset M$. *Let* $S = S_1 \times \cdots \times S_n$ *and for* $I \subset [n] = \{1, \ldots, n\}$, *let* $\pi_I : S \to \prod_{i \in I} S_i$ *be the projection map. Suppose* $E \subseteq S$ *is a* C*-definable subset such that*

(a) *if* $I \subset [n]$ *and* $|I| = n - 1$, *then* $\dim(\pi_I(E)) = \sum_{i \in I} \dim(S_i)$, *and*

(b) *if* $(b_1, \ldots, b_n) \in E$, *then* $b_i \in \operatorname{acl}(C \cup \{b_j : j \neq i\})$.

Then

$$\dim\big\{\bar{b} \in S : \pi_I(\bar{b}) \in \pi_I(E) \textit{ for all } I \textit{ with } |I| = n - 1\big\} = \dim(S).$$

Note that this does not tell us anything if $\mathcal{M}$ has trivial algebraic closure. Note also that it does not refer to the measure, so it makes sense in any supersimple theory (more properly, any S_1-theory) of finite SU-rank. In Section 4 we give an example of a supersimple structure of SU-rank 1 which does not satisfy the above result: so we have an independent amalgamation result which holds in MS-measurable structures, but which is not generally true in finite rank supersimple structures.

This paper is a revised version of some unpublished notes written in 2011–2012. The original version made use of Towsner's unpublished article [2010] and proved Theorem 2.4 under stronger assumptions on the definability of the measure and the behaviour of the measure under projection maps with finite fibres. In 2019, I sent a copy of the notes to Ehud Hrushovski, who observed that these assumptions could be weakened. He also gave examples of additional contexts in which the weaker assumptions would hold: see Section 3C here.

Towsner's published paper [2018] contains a reworking of [Towsner 2010] which involves a weaker assumption on the definability of the measure. In revising the original notes, I have therefore rewritten the proof of Theorem 2.4 to follow the approach and notation of [Towsner 2018].

The structure of the paper is as follows. In Section 2A we give the necessary notation and background to state Towsner's version of the hypergraph removal lemma from [Towsner 2018]. In Section 2B we deduce the main result, Theorem 2.4, from this. Our result is related to a standard deduction of Szemerédi's theorem from the hypergraph removal lemma: we make this explicit in Section 3A. In Section 3B, we discuss MS-measurability and prove Corollary 3.2, stated above. Additional examples in NIP theories are mentioned briefly in Section 3C. In Section 4, we discuss the ω-categorical Hrushovski constructions and their relationship to various open questions around MS-measurable ω-categorical structures. The main result of the section is Theorem 4.7, where we construct an ω-categorical structure which is of SU-rank 1 and which does not satisfy the amalgamation property in Corollary 3.2.

2. An amalgamation theorem for measured structures

2A. *Measured structures.* The following setup is taken from Towsner's paper [2018]. Chapter 1 of [Kallenberg 1997] is a convenient reference for the basic measure theory we need.

We work with a structure $\mathcal{M}$ with domain M. The following notation is introduced in [Towsner 2018, Section 2]. If V is a finite set of indices, then a V-tuple from M is a function $\bar{a}_V : V \to M$ and we denote the set of these by M^V. A V-tuple of variables will generally be denoted by $\bar{x}_V$. If $I \subseteq V$ then $\bar{a}_I \in M^I$ is the restriction of this to I. If U and W are disjoint sets, we write $\bar{a}_U \cup \bar{a}_W$ for the $(U \cup W)$-tuple extending $\bar{a}_U$ and $\bar{a}_W$. If $B \subseteq M^{U \cup W}$ and $\bar{a}_W \in M^W$, then $B(\bar{a}_W)$ denotes the fibre (or "slice") $\{\bar{a}_U \in M^U : \bar{a}_U \cup \bar{a}_W \in B\}$.

In what follows, V is a fixed finite set of indices $V = \{1, \dots, n\} = [n]$ for some $n \in \mathbb{N}$. We often denote $\bar{a}_V$ or $\bar{x}_V$ simply by $\bar{a}$ or $\bar{x}$, dropping the reference to V.

Definition 2.1 [Towsner 2018, Definition 4.1]. Suppose that for each $U \subseteq V$ we have a Boolean algebra $\mathcal{B}^0_U$ of subsets of M^U such that

- $\varnothing \in \mathcal{B}^0_U$;
- $\mathcal{B}^0_U \times \mathcal{B}^0_W \subseteq \mathcal{B}^0_{U \cup W}$ for disjoint $U, W \subseteq V$;
- if $U, W \subseteq V$ are disjoint, $\bar{a}_W \in M^W$ and $B \in \mathcal{B}^0_{U \cup W}$, then $B(\bar{a}_W) \in \mathcal{B}^0_U$.

For $I \subseteq V$ we define $\mathcal{B}^0_{V,I}$ to be the Boolean algebra generated by subsets

$$\{\bar{a}_V \in M^V : \bar{a}_I \in B\},$$

where $B \in \mathcal{B}^0_I$.

In all cases we will drop the superscript 0 to indicate the σ-algebra generated by the Boolean algebra.

The main result we need from [Towsner 2018] is Theorem 2.3 below. When we use this, $\mathcal{B}^0_V$ will consist of the parameter-definable subsets of M^V, so the reader may assume this from now on. We then refer to the elements of $\mathcal{B}_V$ as Borel sets. If $X \subseteq M$, then $\mathcal{B}^0_V(X)$ will denote the X-definable subsets of M^V, and we use a corresponding variation in the notation for the algebras introduced above. We will assume sufficient saturation, so that it makes sense to identify a formula defining a Borel set with its solution set in M. In particular, if the language is countable, we assume that $\mathcal{M}$ is $\aleph_1$-saturated. If the model is multisorted, then we can restrict each variable to having values in a particular sort.

Suppose, with the above notation, that $\nu = \nu^V$ is a probability measure on $(M^V; \mathcal{B}_V)$. If $I \subseteq V$, let ν^I denote the pushforward measure on $(M^I; \mathcal{B}_I)$. So for $A \in \mathcal{B}_I$, we have $\nu^I(A) = \nu(\pi_I^{-1}(A))$, where $\pi_I : M^V \to M^I$ is the projection map.

Recall that if ν is a probability measure on a σ-algebra $\mathcal{B}$ of subsets of a set N, then $L^\infty(\mathcal{B})$ denotes the space of $\mathcal{B}$-measurable functions $N \to \mathbb{R}$ which are essentially bounded, that is, are bounded outside a set of measure 0.

Henceforth, we shall assume that the following conditions on ν hold.

- Definability: For all $J \subseteq V$ and $B \in \mathcal{B}_V$, the function $x_J \mapsto \nu^{V\setminus J}(B(x_J))$ is $\mathcal{B}_J$-measurable.
- Fubini: Suppose $J \subseteq V$ and $f \in L^\infty(\nu^V)$. Then $\int f\, d\nu^V = \iint f\, d\nu^J\, d\nu^{V\setminus J}$.

Remarks 2.2. (1) It would be more correct to refer to the definability condition as "Borel definability", but we will not do this.

(2) It suffices to check that the definability property holds for all $B \in \mathcal{B}_V^0$, as the set of elements of $\mathcal{B}_V$ for which it holds is a σ-subalgebra.

(3) The definability property is a weaker requirement than asking that ν be invariant (over the empty set, or a small submodel).

(4) The definability property implies that, in the statement of the Fubini condition, the map

$$\bar{x}_{V\setminus J} \mapsto \int f(\bar{x}_J \bar{x}_{V\setminus J})\, d\nu^J(\bar{x}_J)$$

is $\mathcal{B}_{V\setminus J}$-measurable for almost all $\bar{x}_{V\setminus J} \in M^{V\setminus J}$. This is a standard argument using approximation by indicator functions of sets in $\mathcal{B}_V$. The same sort of argument shows that it suffices to check the Fubini condition in the case where f is the indicator function 1_B of a set $B \in \mathcal{B}_V^0$.

The following is Towsner's infinitary analogue of the hypergraph removal lemma. We refer to Towsner [2010; 2018] for a discussion of the origins of the proof and the finitary versions of this. The statement follows by combining Theorem 5.3 and Lemma 5.4 of [Towsner 2018]. Theorem 5.3 of [Towsner 2018] holds under weaker conditions than the Fubini property (involving the notion of J-*regularity* of ν^V), but we will not make use of this. Lemma 5.4 of [Towsner 2018] states that the definability and Fubini conditions imply J-regularity of $\nu^V$for all $J \subseteq V$.

Theorem 2.3 [Towsner 2018, Theorem 5.3]. *Suppose $\mathcal{M}$ is sufficiently saturated and $\mathcal{B}_V^0$ consists of the definable subsets of M^V. Suppose ν^V is a probability measure on $\mathcal{B}^V$ which satisfies the definability and Fubini conditions. Let $k < n = |V|$ and $\mathcal{J} = [V]^k$, the set of k-subsets from V.*

Let $A_I \in \mathcal{B}_{V,I}$ for $I \in \mathcal{J}$. Suppose there is $\delta > 0$ such that whenever $B_I \in \mathcal{B}_{V,I}^0$ are such that $\nu^V(A_I \setminus B_I) < \delta$, then $\bigcap_{I\in\mathcal{J}} B_I \neq \varnothing$.

Then $\nu^V\big(\bigcap_{I\in\mathcal{J}} A_I\big) > 0$. □

2B. ***A model-theoretic corollary.*** In the following, we give model-theoretic conditions which allows us to verify the hypotheses in Theorem 2.3. The setup is:

- $\mathcal{M}$ is an $\aleph_1$-saturated structure in a countable language L.
- $V = \{1, \ldots, n\}$ is a set of indices (each associated to a particular sort); we let $J = \{1, \ldots, n-1\} \subseteq V$, and $\mathcal{J}$ is the set of $(n-1)$-subsets of V.
- For each $I \subseteq V$, $\mathcal{B}_I^0$ is the Boolean algebra of M-definable subsets of M^I.
- $\nu = \nu^V$ is a probability measure on $\mathcal{B}_V$ which satisfies the definability and Fubini conditions.

For $I \subseteq V$ let π_I denote the projection map $M^V \to M^I$ and denote by ν^I the pushforward measure induced on $\mathcal{B}_I$ by ν. Each ν^I also satisfies the corresponding definability and Fubini properties.

Theorem 2.4. *With the above notation and assumptions, suppose $E \in \mathcal{B}_V$ is such that*

(a) $\nu^J(\pi_J(E)) > 0$;

(b) *there is $l \in \mathbb{N}$ such that for all $I \in \mathcal{J}$ and $\bar{a} \in M^I$, we have that $\pi_I^{-1}(\bar{a}) \cap E$ has at most l elements*;

(c) *there is $k > 0$ such that if $F \in \mathcal{B}_V^0$, then $\nu^J(\pi_J(F \cap E)) \leq k\nu^I(\pi_I(F \cap E))$ for all $I \in \mathcal{J}$.*

Then $\nu^V(\{\bar{b} \in M^V : \pi_I(\bar{b}) \in \pi_I(E)$ for all $I \in \mathcal{J}\}) > 0$.

Remarks 2.5. We make some comments about the conditions on E. By the second condition, we should not expect that $\nu(E) > 0$. However, suppose that we also have a measure λ on the definable subsets of E with $\lambda(E) > 0$ and $r, s > 0$ such that for all $F \in \mathcal{B}_0^V$ and $I \neq J$ we have

$$r\nu^J(\pi_J(F \cap E)) \leq \lambda(F \cap E) \leq s\nu^I(\pi_I(F \cap E)).$$

Then $\nu^J(\pi(F \cap E)) \leq (s/r)\nu^I(\pi_I(F \cap E))$, so the third condition holds.

In general, without assuming the existence of such a λ, we can define a measure ν_I^J on $\pi_J(E)$ by setting $\nu_I^J(X) = \nu^I\big(\pi_I(\pi_J^{-1}(X) \cap E)\big)$. Condition (c) implies that ν^J is absolutely continuous with respect to ν_I^J and k is a bound on the Radon–Nikodým derivative.

Before proving Theorem 2.4 we note the following lemmas.

Lemma 2.6. *With the notation as in Theorem 2.4, suppose $F \subseteq E$ is a countable intersection of sets in $\mathcal{B}_V^0$ with E. Then*:

(1) $\nu^J(\pi_J(F)) \leq k\nu^I(\pi_I(F))$.

(2) *If $J \neq I \in \mathcal{J}$ and $C \in \mathcal{B}_I^0$ then*

$$\nu^J\big(\pi_J(F \setminus \pi_I^{-1}(C))\big) \geq \nu^J(\pi_J(F)) - k\nu^I(C \cap \pi_I(F)).$$

(3) *If* $J \neq I \in \mathcal{J}$ *and* $B \in \mathcal{B}_I^0$, *then*

$$\nu^J\big(\pi_J(F \cap \pi_I^{-1}(B))\big) \geq \nu^J(\pi_J(F)) - k\nu^I(\pi_I(F) \setminus B).$$

Proof. (1) Write $F = E \cap \bigcap_{i<\omega} F_i$, where each F_i is in $\mathcal{B}_V^0$. We can assume that $F_i \supseteq F_{i+1}$. Then $\aleph_1$-saturation implies $\pi_J(F) = \bigcap_{i<\omega} \pi_J(E \cap F_i)$ and $\nu^J(\pi_J(F)) = \inf\{\nu^J(\pi_J(E \cap F_i)) : i < \omega\}$. By assumption on E, we have $\nu^J(\pi_J(E \cap F_i)) \leq k\nu^I(\pi_I(E \cap F_i))$ for each i; taking the limit gives what we require.

(2) By (1) we have

$$\nu^J\big(\pi_J(\pi_I^{-1}(C) \cap F)\big) \leq k\nu^I(\pi_I(F) \cap C).$$

Of course, $\pi_J(F) = \pi_J(\pi_I^{-1}(C) \cap F) \cup \pi_J(F \setminus \pi_I^{-1}(C))$, so

$$\nu^J(\pi_J(F)) \leq \nu^J\big(\pi_J(\pi_I^{-1}(C) \cap F)\big) + \nu^J\big(\pi_J(F \setminus \pi_I^{-1}(C))\big).$$

Putting these together gives the required result.

(3) Apply (2), taking C to be the complement of B. □

Lemma 2.7. *Suppose* $E \in \mathcal{B}_V$ *with* $\nu^J(\pi_J(E)) > 0$ *and, for all* $\bar{a} \in E$, *we have that* $\pi_J^{-1}(\pi_J(\bar{a})) \cap E$ *has at most* l *elements. Let* $X \subseteq M$ *be a countable set over which* E *is definable.*

(1) *There is some* $F \in \mathcal{B}_V(X)$ *with* $F \subseteq E$, *a natural number* r *and an* $L(X)$-*formula* $\psi(\bar{x})$ *such that* $\nu^J(\pi_J(F)) > 0$, *and if* $\bar{a} \in F$, *then* $\psi(\bar{a}_J, x_n)$ *isolates* $\mathrm{tp}^{\mathcal{M}}(a_n/\bar{a}_J X)$, *and this type has precisely* r *solutions in* $\mathcal{M}$. *The set* F *can be taken to be a countable intersection of sets in* $\mathcal{B}_V^0(X)$ *with* E.

(2) *If* X *is chosen so that* r *in* (1) *is minimal, then for countable* $Y \supseteq X$ *and for almost all* $\bar{a}_J \in \pi_J(F)$, *if* $(\bar{a}_J, a_n) \in F$, *then* $\psi(\bar{a}_J, x_n)$ *isolates* $\mathrm{tp}^{\mathcal{M}}(a_n/\bar{a}_J Y)$ (*and therefore this type has the same solutions as* $\mathrm{tp}^{\mathcal{M}}(a_n/\bar{a}_J X)$).

Proof. (1) For each V-variable formula $\psi(\bar{x}) \in L(X)$ and $r \leq l$, consider the set $E_{\psi,r}$ consisting of those $(a_1, \ldots, a_n) \in E$ such that the formula $\psi(a_1, \ldots, a_{n-1}, x_n)$ isolates $\mathrm{tp}(a_n/a_1, \ldots, a_{n-1}, X)$ and this type has r solutions in $\mathcal{M}$. As E is defined over X, all of these solutions lie in E. Note that $E_{\psi,r}$ is defined by the conjunction of E, and

$$\bigwedge_{\chi \in L(X)} \Big(\psi(x_1, \ldots, x_n) \wedge (\exists^{=r} x_n)\psi(x_1, \ldots, x_n) \wedge (\forall y)\big(\psi(x_1, \ldots, x_{n-1}, y) \to \big(\chi(x_1, \ldots, x_n) \leftrightarrow \chi(x_1, \ldots, x_{n-1}, y)\big)\big)\Big),$$

so is in $\mathcal{B}_V(X)$. Moreover, $\bigcup_{\psi;\, r \leq l} E_{\psi,r} = E$ (by the algebraicity). So as this is a countable union, there are ψ and $r \leq l$ with $\nu^J(\pi_J(E_{\psi,r})) > 0$. Then $F = E_{\psi,r}$ has the required properties.

(2) Let $Y \supseteq X$ be a countable subset of M and consider

$$E' = \{\bar{a} \in F : \psi(\bar{a}_J, x_n) \text{ does not isolate } \operatorname{tp}(a_n/\bar{a}_J Y)\}.$$

As in (1), we have $E' \in \mathcal{B}_V(Y)$. Suppose for a contradiction that $\nu^J(\pi_J(E')) > 0$. Applying (1) we obtain $F' \in \mathcal{B}_V(Y)$ with $F' \subseteq E'$ and $\nu^J(\pi_J(F')) > 0$, some $r' \in \mathbb{N}$ and an $L(Y)$-formula ψ' such that for all $\bar{a} \in F'$, $\psi'(\bar{a}_J, x_n)$ isolates $\operatorname{tp}(a_n/\bar{a}_J Y)$ and the latter has r' solutions. By definition of E' we have $r' < r$ and this contradicts the choice of r. Thus $\nu^J(\pi_J(E')) = 0$ and the result follows. $\square$

We now prove Theorem 2.4.

Proof of Theorem 2.4. From Lemma 2.7(2), there is a countable subset X of M containing the parameters for E and a countable intersection F of X-definable sets with E such that

- $\nu^J(\pi_J(F)) > 0$;
- if $(a_1, \dots, a_{n-1}, a_n), (a_1, \dots, a_{n-1}, a_n') \in F$, then $\operatorname{tp}^{\mathcal{M}}(a_n/a_1, \dots, a_{n-1}, X) = \operatorname{tp}^{\mathcal{M}}(a_n'/a_1, \dots, a_{n-1}, X)$;
- if $Y \supseteq X$ is countable, then for almost all $\bar{a} \in F$, the types $\operatorname{tp}^{\mathcal{M}}(a_n/\bar{a}_J X)$ and $\operatorname{tp}^{\mathcal{M}}(a_n/\bar{a}_J Y)$ have the same solutions.

To see the second point here, note that the two types are isolated by the same formula, so must be equal. The other points are directly from Lemma 2.7.

For $I \in \mathcal{J}$, let $A_I = \pi_I^{-1}(\pi_I(F))$. So of course, $A_I \in \mathcal{B}_{V,I}$ and $F \subseteq \bigcap_{I \in \mathcal{J}} A_I$. We verify that the hypotheses of Theorem 2.3 hold.

Let $\delta > 0$ (to be fixed later) and $B_I \in \mathcal{B}^0_{V,I}$ with $\nu^V(A_I \setminus B_I) < \delta$. Note that $\nu^V(A_I) = \nu^I(\pi_I(F))$ and similarly $\nu^I(\pi_I(F) \setminus \pi_I(B_I)) = \nu^V(A_I \setminus B_I)$. Therefore, with k as in condition (c) of Theorem 2.4 and $I \neq J$, Lemma 2.6(3) gives

$$\nu^J(\pi_J(F \cap B_I)) \geq \nu^J(\pi_J(F)) - k\nu^I(\pi_I(F) \setminus \pi_I(B_I)) > \nu^J(\pi_J(F)) - k\delta.$$

This also holds with $I = J$, as $k \geq 1$.

Now let $\eta = \nu^J(\pi_J(F))$ (so $\eta > 0$, by choice of F) and $\delta = \frac{1}{2}\eta kn$. We obtain, for all $I \in \mathcal{J}$,

$$\nu^J(\pi_J(F \cap B_I)) \geq \left(1 - \tfrac{1}{2}n\right)\eta.$$

The measure of the union of the complements of the sets $\pi_J(F \cap B_I)$ in $\pi_J(F)$ is therefore at most $\frac{1}{2}\eta$, and so

$$\nu^J\left(\bigcap_{I \in \mathcal{J}} \pi_J(F \cap B_I)\right) \geq \tfrac{1}{2}\eta.$$

Let Y be the union of X and the parameter sets of the B_I. Then we can find $\bar{b}_J = (b_1, \dots, b_{n-1}) \in \bigcap_I \pi_J(F \cap B_I)$ such that if $(\bar{b}_J, b_n) \in F$ and $(\bar{b}_J, b_n') \in F$, then they have the same type over Y. Indeed, almost all $\bar{b}_J \in \pi_J(F)$ have this property, by our conditions on F.

Take $b_n \in M$ with $\bar{b} = (b_1, \dots, b_{n-1}, b_n) \in F$. We show that $(b_1, \dots, b_n) \in \bigcap_I B_I$, and thus the hypotheses of Theorem 2.3 hold.

Clearly $\bar{b} \in B_J$. Take $I \neq J$. Because $(b_1, \dots, b_{n-1}) \in \pi_J(F \cap B_I)$, there is $(b'_1, b'_2, \dots, b'_n) \in F \cap B_I$ such that $(b'_1, \dots, b'_{n-1}) = (b_1, \dots, b_{n-1})$. Therefore $(b_1, \dots, b_{n-1}, b_n), (b_1, \dots, b_{n-1}, b'_n) \in F$, and thus b_n and b'_n have the same type over $Y \cup \{b_1, \dots, b_{n-1}\}$. As B_I is defined over Y and $(b_1, \dots, b_{n-1}, b'_n) \in B_I$, it follows that $(b_1, \dots, b_{n-1}, b_n) \in B_I$, as required.

We have shown $\bigcap_I B_I \neq \varnothing$, so Theorem 2.3 applies to give that $\nu\big(\bigcap_I A_I\big) > 0$. As $\bigcap_I A_I \subseteq \{\bar{b} \in \mathcal{M}^V : \pi_I(\bar{b}) \in \pi_I(E) \text{ for all } I \in \mathcal{J}\}$, we have the result. □

3. Examples and applications

3A. ***Pseudofinite structures and Szemerédi's theorem.*** In [Towsner 2010] (and [Towsner 2018, Section 5]), the structure $\mathcal{M}$ is an ultraproduct of finite structures $(F_i : i < \omega)$ and the measures arise by taking the standard part of ultraproducts of normalised counting measures on the F_i. The original language is enriched to ensure definability of the measure. The Fubini property then follows, as we are dealing with counting measures.

In [Towsner 2010, Section 2], Szemerédi's theorem is deduced from Theorem 2.3 in the following way (we do not give the details: the point is to explain where the statement of Theorem 2.4 comes from). The original language is that of abelian groups (written additively) and there is a predicate $A(\cdot)$ for a subset of the group. Each F_i is cyclic of prime order (increasing with i) and $A[F_i]$ is some subset of F_i. Denoting the ultraproduct (in the enriched language) by G, the main assumption is that the measure of $A[G]$ is strictly positive.

So $(G, +)$ is a torsion-free, divisible abelian group, and if $n \in \mathbb{N}$ and $n \geq 3$, we have a definable measure ν^n on the definable subsets of G^n which satisfies the hypotheses of Theorem 2.4. The measure is invariant under definable bijections (in particular, under translations and taking i-th roots). Let

$$E = \left\{ \left(x_1, \dots, x_{n-1}, \sum_{i<n} x_i \right) : \sum_{i<n} i x_i \in A \right\}.$$

This is definable, and in the notation of Theorem 2.4, $\pi_J(E) = \nu^1(A) > 0$ (using the divisibility of G and invariance of the measure under definable bijections). The projection maps π_I (with $|I| = n-1$) are injective on E and thus the remaining two conditions in Theorem 2.4 hold (with $k = l = 1$).

So, by Theorem 2.4, there is some $\bar{b} = (b_1, \dots, b_n) \in G^n$ such that $\pi_I(\bar{b}) \in \pi_I(E)$ for all I of size $n-1$, and by positivity of the measure, we can take $d = b_n - \sum_{i<n} b_j$

to be nonzero. The definition of E means that if we set $a = \sum_{i<n} ib_i$, then $a, a+d, \ldots, a+(n-1)d \in A$. Therefore, as $d \neq 0$, we have an n-term arithmetic progression in A.

3B. ***An amalgamation result in MS-measurable structures.*** The notion of a *measurable structure* was introduced by Macpherson and Steinhorn [2008], following on from observations of Chatzidakis, van den Dries and Macintyre in [Chatzidakis et al. 1992]. Elwes and Macpherson [2008] give a survey of results and open questions. Following [Kestner and Pillay 2011], we refer to this notion as *MS-measurability*.

We recall the definition of MS-measurability [Macpherson and Steinhorn 2008, Definition 5.1]. For a (first-order) L-structure $\mathcal{M}$ we denote by $\mathrm{Def}(\mathcal{M})$ the collection of all nonempty parameter definable subsets of M^n (for all $n \geq 1$).

Definition 3.1. A structure $\mathcal{M}$ is *MS-measurable* if there is a *dimension–measure* function $h : \mathrm{Def}(\mathcal{M}) \to \mathbb{N} \times \mathbb{R}^{>0}$ satisfying the following, where we write $h(X) = (\dim(X), \mu(X))$:

(i) If X is finite (and nonempty) then $h(X) = (0, |X|)$.

(ii) For every formula $\phi(\bar{x}, \bar{y})$ there is a finite set $D_\phi \subseteq \mathbb{N} \times \mathbb{R}^{>0}$ of possible values for $h(\phi(\bar{x}, \bar{a}))$ (with $\bar{a} \in M^n$), and for each such value, the set of $\bar{a}$ giving this value is 0-definable.

(iii) Fubini property: Suppose $X, Y \in \mathrm{Def}(\mathcal{M})$ and $f : X \to Y$ is a definable surjection. By (ii), Y can be partitioned into disjoint definable sets $Y_1, \ldots, Y_r$ such that $h(f^{-1}(y))$ is constant, equal to (d_i, m_i), for $y \in Y_i$. Let $h(Y_i) = (e_i, n_i)$. Let c be the maximum of $d_i + e_i$ and suppose this is attained for $i = 1, \ldots, s$. Then $h(X) = (c, m_1 n_1 + \cdots + m_s n_s)$.

In the above, $\dim(X)$ is the *dimension* and $\mu(X)$ the *measure* of X. Clearly we can normalise and assume that $\mu(M) = 1$. We also extend the definition so that $\mu(\varnothing) = 0$. Note that MS-measurability is a property of the theory of M, so any elementary extension or submodel of $\mathcal{M}$ is MS-measurable if $\mathcal{M}$ is. As observed in [Macpherson and Steinhorn 2008, Remark 5.2], measurability implies supersimplicity and dimension dominates D-rank, but is not necessarily equal to it. By [Macpherson and Steinhorn 2008, Proposition 5.10], the dimension–measure function extends to definable subsets of $\mathcal{M}^{\mathrm{eq}}$.

We suppose (for convenience) that L is countable and suppose that $\mathcal{M}$ is an $\aleph_1$-saturated MS-measurable structure with dimension–measure function $h = (\dim, \mu)$. Let $S \in \mathrm{Def}(\mathcal{M})$ be infinite and let $\mathcal{B}^0_S$ denote the set of definable subsets of S. For $D \in \mathcal{B}^0_S$ we define

$$\nu^S(D) = \begin{cases} \mu(D)/\mu(S) & \text{if } \dim(D) = \dim(S), \\ 0 & \text{otherwise.} \end{cases}$$

If $X_1, X_2 \in \mathcal{B}_S^0$ are disjoint, then (iii) of Definition 3.1 (with Y a two-point set) shows that $\nu^S(X_1 \cup X_2) = \nu^S(X_1) + \nu^S(X_2)$. So ν^S is a finitely additive probability measure on $\mathcal{B}_S^0$ and it therefore extends uniquely to a probability measure on $\mathcal{B}_S$, which we will also denote by ν^S.

Now suppose that $S_1, \ldots, S_n \in \mathrm{Def}(\mathcal{M})$ are infinite and $S = S_1 \times \cdots \times S_n$. If $I \subseteq V = \{1, \ldots, n\}$, let S_I be the product of the S_i for $i \in I$. As previously, $\pi_I : S \to S_I$ is the projection map. By considering this, (iii) in Definition 3.1 gives that $\dim(S) = \dim(S_I) + \dim(S_{V \setminus I})$ and $\mu(S) = \mu(S_I)\mu(S_{V \setminus I})$.

Let $\nu = \nu^V = \nu^S$. If $I \subseteq V$, then the pushforward measure ν^I on $\mathcal{B}_{S_I}$ obtained from ν and π_I is equal to ν^{S_I}, as defined above. Indeed, it suffices to check this for $D \in \mathcal{B}_{S_I}^0$. If $\dim(D) = \dim(S_I)$, then

$$\begin{aligned}\nu^I(D) &= \nu^V(D \times S_{V \setminus I}) \\ &= \mu(D \times S_{V \setminus I})/\mu(S) = \mu(D)\mu(S_{V \setminus I})/\mu(S) = \mu(D)/\mu(S_I),\end{aligned}$$

and this is equal to $\nu^{S_I}(D)$. If $\dim(D) < \dim(S_I)$, then $\dim(D \times S_{V \setminus I}) < \dim(S)$, so both $\nu^I(D)$ and $\nu^{S_I}(D)$ are zero.

The definability and Fubini properties given in Section 2A hold for the ν^I, using (ii) and (iii) of Definition 3.1 (see Remarks 2.2).

From Theorem 2.4 we obtain the following, which can be seen as a weak form of independent n-amalgamation:

Corollary 3.2. *Suppose $\mathcal{M}$ is an MS-measurable structure and $S_1, \ldots, S_n \in \mathrm{Def}(\mathcal{M})$ are infinite and defined over a finite set $C \subset M$. Let $S = S_1 \times \cdots \times S_n$ and suppose $E \subseteq S$ is a C-definable subset such that*

(a) $\dim(\pi_I(E)) = \sum_{i \in I} \dim(S_i)$ *for all* $I \in [n]^{n-1}$, *and*

(b) *if* $(b_1, \ldots, b_n) \in E$, *then* $b_i \in \mathrm{acl}(C \cup \{b_j : j \neq i\})$.

Then

$$\dim\{\bar{b} \in S : \pi_I(\bar{b}) \in \pi_I(E) \textit{ for all } I \in [n]^{n-1}\} = \dim(S).$$

Remarks 3.3. Assumptions (a) and (b) in Corollary 3.2 imply that the S_i have the same dimension. Indeed, $\sum_{j<n} \dim(S_j) = \dim(\pi_J(E)) = \dim(E) = \dim(\pi_I(E)) = \sum_{i \in I} \dim(S_i)$ for all $I \in [n]^{n-1}$. So $\dim(S_j) = \dim(S_n)$ for all $j < n$.

We now prove Corollary 3.2.

Proof. We may assume that $\mathcal{M}$ is $\aleph_1$-saturated. We check that the three conditions of Theorem 2.4 hold.

By (a), $\dim(\pi_J(E)) = \dim(S_J)$, so $\nu^J(\pi_J(E)) = \mu(\pi_J(E))/\mu(S_J) > 0$.

As E is definable, by compactness we have a uniform bound l on the algebraicity in assumption (b). This gives the second condition required by Theorem 2.4.

Suppose $I \in [n]^{n-1}$. The restriction of the projection map $E \to \pi_I(E)$ has finite fibres, of size at most l. Suppose $X \subseteq E$ is definable. If we decompose $\pi_I(X)$ according to the size of the fibres $X \to \pi_I(X)$ and apply (i) and (iii) of Definition 3.1, we obtain

$$\mu(\pi_I(X)) \leq \mu(X) \leq l\mu(\pi_I(X)).$$

Thus

$$\mu(\pi_J(X)) \leq \mu(X) \leq l\mu(\pi_I(X)).$$

If $\dim(X) = \dim(E)$, then $\dim(\pi_I(X)) = \dim(\pi_I(E)) = \dim(S_I)$ (by (a)) and we obtain

$$\nu^J(\pi_J(X)) \leq l\frac{\mu(S_I)}{\mu(S_J)}\nu^I(\pi_I(X)).$$

If $\dim(X) < \dim(E)$ then the inequality is also true, as both sides are zero. So we have the third condition required by Theorem 2.4.

So, by Theorem 2.4,

$$\nu^V(\{\bar{b} \in S : \pi_I(\bar{b}) \in \pi_I(E) \text{ for all } I \in [n]^{n-1}\}) > 0,$$

and the conclusion follows. □

3C. ***Further examples.*** If Th($\mathcal{M}$) is NIP, then *generically stable* measures (see [Hrushovski and Pillay 2011] or [Simon 2015]) provide examples of measures satisfying the definability and Fubini conditions. More precisely, suppose $\nu_{x_1}, \ldots, \nu_{x_n}$ are generically stable measures for $\mathcal{M}$ (in the indicated variables) and let $\nu^V = \nu_{x_1} \otimes \cdots \otimes \nu_{x_n}$. Then ν^V has the definability and Fubini properties, and therefore Theorems 2.3 and 2.4 hold. It would be interesting to know whether either of these results is saying something new, or at least nontrivial, in this context.

4. MS-measurability and the Hrushovski construction

In [Elwes and Macpherson 2008, Definition 3.13], a complete theory is defined to be *unimodular* if in any model $\mathcal{M}$, whenever $f_i : X \to Y$ are definable k_i-to-1 surjections in $\mathcal{M}^{\mathrm{eq}}$ (for $i = 1, 2$), then $k_1 = k_2$. (See [Kestner and Pillay 2011] for comments on this and, in particular, on why it should more properly be termed weak unimodularlity.) An MS-measurable structure is necessarily superstable of finite SU-rank and unimodular, and Question 7 of [Elwes and Macpherson 2008] asks whether the converse holds. Unimodularity is implied by ω-categoricity [Elwes and Macpherson 2008, Proposition 3.16], and in a similar vein, Question 2 of [Elwes and Macpherson 2008] asks whether a MS-measurable ω-categorical structure is necessarily one-based. For both of these questions the key examples to be considered are Hrushovski's non-locally-modular supersimple ω-categorical structures [1997; 1988]. In this section we apply Corollary 3.2 to show that *some* of Hrushovski's examples are *not* MS-measurable. In particular, this answers Question 7 of [Elwes and Macpherson 2008]: there is a supersimple, finite rank unimodular theory (even, ω-categorical, SU-rank 1) which is not MS-measurable.

4A. ***The Hrushovski construction for ω-categorical structures.*** We recall briefly some details of the construction method. The original version is in [Hrushovski 1988], where it is used to provide a counterexample to Lachlan's conjecture, and in [Hrushovski 1997], where it is used to construct a nonmodular, supersimple $\aleph_0$-categorical structure. The book [Wagner 2000] is a very convenient reference for this (see Section 6.2.1). Generalisations and reworkings of the method (particularly relating to simple theories) are also to be found in [Evans 2002]. We will restrict to the simplest form of the construction appropriate for producing ω-categorical structures of SU-rank 1.

We work with a finite relational language $L = \{R_i : i \leq m\}$. For later use, it will be convenient to assume that this contains some 3-ary relation R. Recall that if B and C are L-structures with a common substructure A then the *free amalgam* $B \amalg_A C$ of B and C over A is the L-structure whose domain is the disjoint union of B and C over A and whose atomic relations are precisely those of B together with those of C. Let $\overline{\mathcal{K}}$ be the class of L-structures and denote by $\mathcal{K}$ the finite structures in $\overline{\mathcal{K}}$.

For $A \in \mathcal{K}$ define the *predimension* $\delta(A)$ to be equal to $|A| - \sum_i |R_i[A]|$. If $A \subseteq B \in \mathcal{K}$ write $A \leq B$ to mean $\delta(A) < \delta(B')$ for all $A \subset B' \subseteq B$. (We sometimes say that A is *self-sufficient* in B.) For structures in $\mathcal{K}$, one has

- if $X \subseteq B$ and $A \leq B$, then $X \cap A \leq X$;
- if $A \leq B \leq C$, then $A \leq C$.

Consequently (see [Wagner 2000, Corollary 6.2.8]), for each $B \in \mathcal{K}$ there is a closure operation given by $\mathrm{cl}_B(X) = \bigcap\{A : X \subseteq A \leq B\} \leq B$ for $X \subseteq B$. Of course, if $B \leq C \in \mathcal{K}$ and $X \subseteq B$, then $\mathrm{cl}_B(X) = \mathrm{cl}_C(X)$.

The relation $\leq$ can be extended to infinite structures so that the above properties still hold: if $M \in \overline{\mathcal{K}}$ and $A \subseteq M$, write $A \leq M$ to mean that $A \cap X \leq X$ for all finite $X \subseteq M$.

If $A, B \in \overline{\mathcal{K}}$, an embedding $\alpha : A \to B$ with $\alpha(A) \leq B$ is called a $\leq$-embedding.

Now consider $\overline{\mathcal{K}}_0$, the class of $B \in \overline{\mathcal{K}}$ with $\varnothing \leq B$. Equivalently, if $A \subseteq B$ is finite and nonempty, then $\delta(A) > 0$. Let $\mathcal{K}_0$ be the finite structures in $\overline{\mathcal{K}}_0$. Any structure B in $\overline{\mathcal{K}}_0$ carries a notion of *dimension* d^B associated to the predimension δ and a notion of d^B*-independence*. If $X, Y \subseteq B$ are finite, write $d^B(X) = \delta(\mathrm{cl}_B(X)) = \min\{\delta(Y) : X \subseteq Y \subseteq B\}$ and $d^B(X/Y) = d^B(X \cup Y) - d^B(Y)$. If the ambient structure B is clear from the context, then we omit it from the notation. Say that finite X and Z are d*-independent* over Y (in B) if $d^B(X/YZ) = d^B(X/Y)$. In particular, this implies $\mathrm{cl}_B(XY) \cap \mathrm{cl}_B(YZ) = \mathrm{cl}_B(Y)$. (Here, we use the usual shorthand of YZ for $Y \cup Z$.) For the particular predimension which we have given, it can be shown that cl_B satisfies the exchange condition, and therefore gives a pregeometry; furthermore, d^B is the dimension in this pregeometry.

We look at a version of the construction (also from [Hrushovski 1997]) where closure is uniformly locally finite. For this, we have a continuous, increasing $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ with $f(x) \to \infty$ as $x \to \infty$ and we consider $\mathcal{K}_f = \{A \in \mathcal{K}_0 : \delta(X) \geq f(|X|)$ for all $X \subseteq A\}$. For suitable choice of f (call these *good* f), $(\mathcal{K}_f, \leq)$ has the free $\leq$-amalgamation property: if $A_0 \leq A_1, A_2 \in \mathcal{K}_f$, then $A_i \leq A_1 \amalg_{A_0} A_2 \in \mathcal{K}_f$. In this case we have an associated *generic structure* M_f (see [Wagner 2000, Theorem 6.2.13]). This is a countable structure characterised by the following properties:

(i) M_f is the union of a chain of finite self-sufficient substructures, all in $\mathcal{K}_f$.

(ii) $\leq$-extension property: If $A \leq M_f$ is finite and $A \leq B \in \mathcal{K}_f$, then there is a $\leq$-embedding $\beta : B \to M_f$ with $\beta(a) = a$ for all $a \in A$.

Equivalently, $\mathcal{K}_f$ is the class of finite substructures of M_f, and isomorphisms between finite self-sufficient substructures of M_f extend to automorphisms of M_f (we refer to the latter property as $\leq$-homogeneity). Because of the function f, closure in M_f is uniformly locally finite and (using free amalgamation and the $\leq$-extension property) it is equal to algebraic closure [Wagner 2000, Lemma 6.2.17]. It then follows from $\leq$-homogeneity that M_f is ω-categorical and the type of a tuple is determined by the isomorphism type of its closure.

Remarks 4.1. To construct good functions, we can take f which are piecewise smooth, and where the right derivative f' satisfies $f'(x) \leq 1/(x+1)$ and is nonincreasing. The latter condition implies that $f(x+y) \leq f(x) + yf'(x)$ (for $y \geq 0$). It can be shown that under these conditions, $\mathcal{K}_f$ has the free $\leq$-amalgamation property. (This is originally from [Hrushovski 1988]; see also [Wagner 2000, Example 6.2.27] or [Evans 2002, Lemma 3.3].)

Remarks 4.2 ([Hrushovski 1997]; see also [Wagner 2000, Example 6.2.27; Evans 2002, Corollary 2.24, Theorem 3.6]). If f also satisfies the slower growth condition

$$f(3x) \leq f(x) + 1,$$

then the structure M_f is supersimple of SU-rank 1. Moreover, for tuples $\bar{a}$ and $\bar{b}$ in M_f, we have $\mathrm{SU}(\mathrm{tp}(\bar{b}/\bar{a})) = d(\bar{b}/\bar{a})$. To see the latter, note that (by additivity of both sides) it suffices to prove this when $\bar{b}$ is a single element b. Now, $d(b/\bar{a})$ is a natural number and at most $\delta(b)$, so is 0 or 1. If it is 0, then $b \in \mathrm{acl}(\bar{a})$ so $\mathrm{SU}(b/\bar{a}) = 0$. Thus, it suffices to show that if $\mathrm{tp}(b/\bar{a})$ divides over $\varnothing$, then $d(b/\bar{a}) < d(b/\varnothing)$. This is done (in greater generality) in the above references.

4B. *The dimension function.* For the rest of the section suppose that f is a good function as in Remarks 4.1 and M_f is the corresponding generic structure. We suppose that $h = (\dim, \mu) : \mathrm{Def}(M_f) \to \mathbb{R}^{>0}$ is a dimension–measure function. In this subsection we relate dim to the dimension d coming from the predimension (which will be the same as SU-rank if M_f is simple), and the measure will not be used.

Notation 4.3. For tuples $\bar{a}$ and $\bar{b}$ in M_f, let $\mathrm{loc}(\bar{b}/\bar{a})$, the *locus of* $\bar{b}$ *over* $\bar{a}$, be the set of realisations in M_f of $\mathrm{tp}_L(\bar{b}/\bar{a})$, the L-type of $\bar{b}$ over $\bar{a}$. By ω-categoricity, this is definable by an L-formula with parameters from $\bar{a}$. Let $\dim(\bar{b}/\bar{a})$ denote the dimension of this set.

The Fubini property in MS-measurability implies that dim is additive: $\dim(\bar{b}/\bar{a}) = \dim(\bar{a}\bar{b}/\varnothing) - \dim(\bar{a}/\varnothing)$. We also have $\dim(M_f^n) = n \dim(M_f)$. Note the existence of dim-generic points: if $D \in \mathrm{Def}(M_f)$ is definable over a finite tuple $\bar{a}$, then $\dim(D) = \max\{\dim(\bar{b}/\bar{a}) : \bar{b} \in D\}$. From this we deduce that if $D' \subseteq D$ is definable, then $\dim(D') \leq \dim(D)$. A further property of dim which we require is the weak algebraicity property that if $\bar{b} \in \mathrm{acl}(\bar{a})$, then $\dim(\bar{b}/\bar{a}) = 0$. Of course, d also has these properties.

Under these assumptions on dim (and the given conditions on f) we will show that dim is just a scaled version of the dimension d.

Theorem 4.4. *Suppose* $f'(x) \leq \frac{1}{2}(1/(x+1))$. *If* $\bar{a}$ *and* $\bar{b}$ *are finite tuples in* M_f, *then we have*

$$\dim(\bar{b}/\bar{a}) = \dim(M_f) d(\bar{b}/\bar{a}).$$

The theorem follows from the following (always assuming the given condition on f).

Proposition 4.5. *Let* $\bar{a}, b \in M_f$ *with* $b \notin \mathrm{acl}(\bar{a})$ *and* $P = \mathrm{loc}(b/\bar{a})$. *Then, for every* $r \in \mathbb{N}$ *and* $\bar{y} \in M^r$, *there is some* $\bar{x} \in P^{r+2}$ *with* $\bar{y} \in \mathrm{acl}(\bar{x}\bar{a})$.

We note that Marimon (unpublished work) shows that Theorem 4.4 holds for a wider class of Hrushovski constructions than we give here.

First we show how Theorem 4.4 follows from the proposition.

Proof of Theorem 4.4. By the additivity property of both dim and d, it will suffice to prove the statement when $\bar{b} = b$ is a single element. If $b \in \mathrm{acl}(\bar{a})$, then the statement holds as both sides of the equation are zero, by the weak algebraicity property of dim and d. So now suppose that $b \notin \mathrm{acl}(\bar{a})$. Let $P = \mathrm{loc}(b/\bar{a})$, as in Proposition 4.5. Consider

$$Y = \{\bar{y} = (y_1, \ldots, y_r) \in M_f^r : y_1, \ldots, y_r \in \mathrm{acl}(\bar{x}\bar{a}) \text{ for some } \bar{x} \in P^{2+r}\}.$$

By ω-categoricity, this set is definable by an L-formula with parameters from $\bar{a}$ (for example, it is invariant under automorphisms of M_f fixing $\bar{a}$). Thus (by existence of generic points for dim) there is $\bar{c} \in Y$ with $\dim(Y) = \dim(\bar{c}/\bar{a})$. By definition of Y, there are $b_1, \dots, b_{r+2} \in P$ with $\bar{c} \in \mathrm{acl}(\bar{a}b_1 \cdots b_{r+2})$. It follows (using weak algebraicity) that

$$\dim(Y) = \dim(\bar{c}/\bar{a}) \le \dim(b_1 \cdots b_{r+2}/\bar{a}) \le \dim(M_f^{r+2}) = (r+2)\dim(M_f).$$

But, by Proposition 4.5, we have $Y = M_f^r$. So

$$r \dim(M_f) = \dim(M_f^r) = \dim(Y) \le (r+2)\dim(M_f).$$

Dividing by $(r+2)$ and letting $r \to \infty$, we obtain that $\dim(b/\bar{a}) = \dim(M_f)$. As $d(b/\bar{a}) = 1$, this gives $\dim(b/\bar{a}) = \dim(M_f)d(b/\bar{a})$, as required. □

The proof of Proposition 4.5 is a technical argument with Hrushovski constructions, so we relegate it to a separate section (Section 4D). Marimon's approach [2022a; 2022b] to proving non-MS-measurability of other examples of ω-categorical Hrushovski constructions avoids the need for a result such as Theorem 4.4.

Remarks 4.6. It is an open problem to determine whether any of the M_f are (or are not) pseudofinite. We note that Theorem 4.4 provides some information relevant to this question. Suppose that f is a good function with $f'(x) \le \frac{1}{2}(1/(x+1))$ and $\mathcal{K}_f$ is the corresponding amalgamation class with generic structure M_f. *Assume* that M_f is elementarily equivalent to an ultraproduct $\mathcal{M} = \prod_U F_i$ of finite structures. Following [Hrushovski 2013], if $\Phi(\bar{x})$ is a formula with parameters from M, then the *coarse pseudofinite dimension* $\Delta(\Phi(\bar{x}))$ is the standard part of the nonstandard real $\prod_U \log|\Phi(F_i)|/\log|F_i|$. We will show that for every L-formula $\Phi(\bar{x})$ (without parameters), we have $\Delta(\Phi(\bar{x})) = d(\Phi(\bar{x}))$.

In principle, we could deduce the result from Theorem 4.4 as Δ has the properties required in the proof of Theorem 4.4, as long as we expand the language by dimension quantifiers so that it becomes continuous (see [Hrushovski 2013, Section 2.7]). However, it seems clearer to give a fuller argument which is essentially a modification of that given for Theorem 4.4.

If $\bar{a}$ is a finite tuple in M_f, let $\Phi_{\bar{a}}(\bar{x})$ denote an L-formula isolating $\mathrm{tp}(\bar{a}/\varnothing)$ (the L-type of $\bar{a}$ in M_f). Such a formula exists, by ω-categoricity. If $\bar{b}$ is another tuple, then $\Phi_{\bar{a}\bar{b}}(\bar{a}, \bar{y})$ isolates $\mathrm{tp}(\bar{b}/\bar{a})$.

Claim. *Suppose $\bar{a}$ is a k-tuple in M_f and $b \in M_f$. Suppose $\bar{u}$ is a k-tuple in $\mathcal{M}$ and $\mathcal{M} \models \Phi_{\bar{a}}(\bar{u})$. Then $\Delta(\Phi_{\bar{a}b}(\bar{u}, y)) = d(b/\bar{a})$.*

Proof of claim. If $d(b/\bar{a}) = 0$ then b is algebraic over $\bar{a}$. The size of $\mathrm{acl}(\bar{a})$ is bounded uniformly (actually, in k), so $\Phi_{\bar{a}b}(\bar{u}, y)$ has finitely many solutions in $\mathcal{M}$. Thus its pseudofinite dimension is 0.

Now suppose that $b \notin \mathrm{acl}(\bar{a})$, so that $d(b/\bar{a}) = 1$. Let $r \in \mathbb{N}$. There is a formula $C_r(y, x_1 \cdots x_{r+2}\bar{z})$ such that if $\Phi_{\bar{a}b}(\bar{a}'b_i)$ (for $i \le r+2$), then $C_r(M_f, b_1 \cdots b_{r+2}, \bar{a}')$ is $\mathrm{acl}(b_1 \cdots b_{r+2}, \bar{a}')$. Let $K(r)$ bound the size of this algebraic closure.

The set Y in the proof of Theorem 4.4 is defined by $Y(\bar{y}, \bar{a})$, where $Y(\bar{y}, \bar{z})$ is the formula

$$\exists x_1, \ldots, x_{r+2} \bigwedge_{i \le r+2} \Phi_{\bar{a}b}(\bar{z}x_i) \wedge \bigwedge_{j \le r} C_r(y_j, x_1 \cdots x_{r+2}\bar{z}).$$

Moreover, by Proposition 4.5,

$$M_f \models (\forall \bar{z})\big(\Phi_{\bar{a}}(\bar{z}) \to \forall y_1, \ldots, y_r\, Y(y_1 \cdots y_r \bar{z})\big),$$

so this formula also holds in $\mathcal{M}$.

Suppose $\bar{u} \in \mathcal{M}$ and $\mathcal{M} \models \Phi_{\bar{a}}(\bar{u})$. Denote by $\bar{u}_i$ the k-tuple of i-th coordinates (in F_i) in $\bar{u}$. From the definition of Y, for almost all i, we have

$$|Y(F_i, \bar{u}_i)| \le K(r)^r |\Phi_{\bar{a}b}(\bar{u}_i, F_i)|^{r+2}.$$

Thus, as $Y(\mathcal{M}, \bar{u}) = M^r$, for almost all i,

$$K(r)^r |\Phi_{\bar{a}b}(\bar{u}_i, F_i)|^{r+2} \ge |F_i|^r.$$

As $|F_i| \to \infty$, we obtain $\Delta(\Phi_{\bar{a}b}(\bar{u}, y)) \ge r/(r+2)$. But r here is arbitrary, and thus $\Delta(\Phi_{\bar{a}b}(\bar{u}, y)) \ge 1$. The reverse inequality is trivial, so we have the claim. □

Now suppose that $\bar{b} = (b_1, \ldots, b_n)$ is an n-tuple in M_f. We show that if $\bar{u}$ is a tuple in $\mathcal{M}$ and $\mathcal{M} \models \Phi_{\bar{a}}(\bar{u})$, then $\Delta(\Phi_{\bar{a}\bar{b}}(\bar{u}, \bar{y})) = d(\bar{b}/\bar{a})$. The required formula for general L-definable sets follows as each such is a finite union of pairwise disjoint sets of this form. We may assume that $d(\bar{b}/\bar{a}) = n$ and we prove the result by induction on n. Let D be $\Phi_{\bar{a}\bar{b}}(\bar{u}, \mathcal{M}) \subseteq \mathcal{M}^n$ and $E = \Phi_{\bar{a}b_1 \cdots b_{n-1}}(\bar{u}, \mathcal{M})$. Let $f : D \to E$ be projection onto the first $n-1$ coordinates. By the claim, the fibres of f have coarse pseudofinite dimension $d(b_n/b_1 \cdots b_{n-1}\bar{a}) = 1$. By the induction hypothesis, $\Delta(E) = n-1$. Thus, by Lemma 2.8(4) of [Hrushovski 2013], $\Delta(D) = n-1+1 = n$, as required. (In order to apply the results from [Hrushovski 2013], we need to first enrich the language so that Δ becomes continuous, but this has no effect on the dimension of formulas in the original language.)

4C. *A structure which is not MS-measurable.*

Theorem 4.7. *There is an ω-categorical, supersimple structure M_f of* SU*-rank* 1 *which does not satisfy the amalgamation property in Corollary 3.2. In particular, M_f is not MS-measurable.*

Proof. We choose f so that $\mathcal{K}_f$ is a free amalgamation class; the generic M_f is supersimple of SU-rank 1; the independent amalgamation property, Corollary 3.2, does not hold. We are only interested in providing an example, so we choose economy of effort over elegance.

Take L to have a 3-ary relation R, a 10-ary relation S and a 11-ary relation U. Let $f(x) = \log_8(x+1)$. Then $f'(x) = \frac{1}{\ln 8}(1/(x+1)) < \frac{1}{2}(1/(x+1))$, and therefore, by Remarks 4.1, $\mathcal{K}_f$ is a free amalgamation class and the hypothesis on f in Theorem 4.4 holds. We also have $f(3x) \leq f(x) + 1$, so by Remarks 4.2, the generic M_f is supersimple, with d-independence being the same as nonforking, and M_f is of SU-rank 1.

Consider the L-structure A with points $a_1, \dots, a_{10}, u_1, \dots, u_r$, where $r = 8^9 - 11$, and relations $S(a_1, \dots, a_{10})$ and $U(a_1, \dots, a_{10}, u_i)$ (for $i \leq r$). Then $\delta(A) = 9$ and $|A| = 8^9 - 1$, so $\delta(A) \geq f(|A|)$. It is easy to check that for any $X \subset A$ we have $\delta(X) \geq f(|X|)$, so $A \in \mathcal{K}_f$. Moreover (in the notation of Corollary 3.2), for each $I \in [10]^9$, the tuple $\bar{a}_I$ is d-independent (in A) and has closure A. Note also that if $I \in [10]^8$, then $\bar{a}_I \leq A$.

Suppose, for a contradiction, that the conclusion of Corollary 3.2 holds, where dim is given by SU-rank (in this case, given by the dimension function d). We will apply this where $n = 10$, $S = M_f^{10}$ and

$$E = \{\alpha(a_1 \cdots a_{10}) \mid \alpha : A \to M_f \text{ is an } \leq\text{-embedding}\}.$$

Note that E is $\varnothing$-definable, the algebraic closure (equal to the $\leq$-closure) of every element of E is isomorphic to A, and (by the $\leq$-homogeneity of M_f) all elements of E have the same type over $\varnothing$.

Therefore, if the conclusion of Corollary 3.2 holds, there exists a d-independent set $B_0 = \{b_1, \dots, b_{10}\}$ of distinct elements of M_f with the property that for each $I \in [10]^9$ we have $\mathrm{acl}_{M_f}(\mathcal{B}_I) \cong A$ (via an isomorphism taking $\bar{b}_i \mapsto \bar{a}_i$), where $B_I = \{b_i : i \in I\}$. Let $B = \mathrm{acl}(B_0)$. By the d-independence, $\delta(B) = 10$ and we have $\mathrm{acl}(B_I) \cap \mathrm{acl}(B_{I'}) = B_I \cap B_{I'} = B_{I \cap I'}$ for $I \neq I' \in [10]^9$.

Thus

$$\begin{aligned} |B| &\geq |B_0| + \left| \bigcup_{I \in [10]^9} \mathrm{acl}(B_I) \setminus B_0 \right| \\ &= |B_0| + \sum_{I \in [10]^9} |\mathrm{acl}(B_I) \setminus B_0| \\ &\geq 10 + 10(8^9 - 1 - 9) = 10.8^9 - 90. \end{aligned}$$

So

$$f(|B|) \geq \log_8(10.8^9 - 89) > 10 = \delta(B),$$

and thus $B \notin \mathcal{K}_f$, a contradiction. So the amalgamation property in the conclusion of Corollary 3.2 does not hold, and in particular, M_f is not MS-measurable. □

4D. ***Proof of Proposition 4.5.*** Before the proof, we give a technical lemma.

Lemma 4.8. *Suppose R is a 3-ary relation in L and $f'(x) \leq \frac{1}{2}(1/(x+1))$. Let $A \leq C, T \in \mathcal{K}_f$ (with $A \neq C, T$), and let E be the free amalgam of C and T over A. Suppose $t_1, \dots, t_r \in T \setminus A$ are d-independent over A, and let $c \in C \setminus A$.*

Let $F = E \cup \{s_1, \ldots, s_r\}$ with additional relations $R(c, s_i, t_i)$ (for $1 \leq i \leq r$). Then $As_1 \cdots s_r, C, T \leq F$ and $F \in \mathcal{K}_f$.

Proof. Suppose $C \subset V \subseteq F$. If $V \cap T = A$, then (by construction) $\delta(V) = \delta(C) + |V \setminus C|$; if $V \cap T \supset A$ then $\delta(V) \geq \delta(C) + \delta(V \cap T) - \delta(A) > \delta(C)$. In either case, $\delta(V) > \delta(C)$, so $C \leq F$. A similar argument shows $T \leq F$.

By free amalgamation, it is enough to prove the rest of the lemma in the case where $T = \mathrm{cl}_T(At_1 \cdots t_r)$ and $C = \mathrm{cl}_C(Ac)$. So henceforth assume this. Suppose $As_1 \cdots s_r \subset V \subseteq F$ has $\delta(V) \leq \delta(As_1 \cdots s_r) = \delta(A) + r$. We can assume that $V \leq F$. Clearly $c \in V$ and therefore $t_1, \ldots, t_r \in V$. It follows that $V = F$. But $\delta(F) = \delta(A) + r + 1$, a contradiction.

Finally we show that $F \in \mathcal{K}_f$. Let $X \subseteq F$. We need to show $\delta(X) \geq f(|X|)$. As $X \cap (T \cup C)$ is the free amalgamation of $X \cap T$ and $X \cap C$ over $X \cap A$, the structure X is of the same form as F (possibly together with some points s_i not lying in any relation in X). So it will suffice to prove that $\delta(F) \geq f(|F|)$.

<u>Case 1</u>: Suppose $|T \setminus A| \leq r|C \setminus A|$.

Note that $|F| = |C| + |T \setminus A| + r$ and $\delta(F) = \delta(C) + r$. As $C \in \mathcal{K}_f$ we have $\delta(C) \geq f(|C|)$. Furthermore, as the graph of f lies below its tangent at any point, and $f'(x) \leq \frac{1}{2}(1/(x+1)) \leq 1/(x+1)$, we have

$$\begin{aligned} f(|F|) &\leq f(|C|) + (|T \setminus A| + r) f'(|C|) \\ &\leq f(|C|) + \frac{1}{(|C|+1)} r(|C \setminus A| + 1) \leq \delta(C) + r = \delta(F), \end{aligned}$$

as required.

<u>Case 2</u>: Suppose $|T \setminus A| \geq r|C \setminus A|$.

This is similar. We have $|F| = |T| + |C \setminus A| + r$ and $\delta(F) = \delta(T) + 1$. Then

$$\begin{aligned} f(|F|) &\leq f(|T|) + (|C \setminus A| + r) f'(|T|) \\ &\leq f(|T|) + \frac{1}{2|T|}(|C \setminus A| + r) \leq \delta(T) + 1 = \delta(F), \end{aligned}$$

using the fact that $|T \setminus A| \geq |C \setminus A|, r$. □

Proof of Proposition 4.5. Recall that we are assuming that the language L contains a 3-ary relation symbol R, so we can use the previous lemma. Let $A = \mathrm{acl}(\bar{a})$ and $B = \mathrm{acl}(Ab)$.

First, we note that it is enough to prove the proposition in the case where $\bar{y}$ is d-independent over $\bar{a}$ (that is, $d(\bar{y}/\bar{a}) = r$). To see this, take $\bar{y}_1 \subseteq \bar{y}$ which is d-independent over $\bar{a}$ and has $\bar{y} \in \mathrm{acl}(\bar{y}_1\bar{a})$; extend this to an r-tuple $\bar{y}'$ which is d-independent over $\bar{a}$. If $\bar{x} \in P^{r+2}$ has $\bar{y}_1 \in \mathrm{acl}(\bar{a}\bar{x})$, then $\bar{y} \in \mathrm{acl}(\bar{a}\bar{x})$.

<u>Step 1</u>: We first assume that $\bar{y} = (s_1, \ldots, s_r)$ is d-independent over $\bar{a}$ and $A\bar{y} \leq M_f$. We shall show that there is $(b_0, \ldots, b_r) \in P^{r+1}$ with $\bar{y} \in \mathrm{acl}(\bar{a}, b_0, \ldots, b_r)$.

We apply Lemma 4.8 with T the free amalgam of r copies B_j $(1 \le j \le r)$ of B over A and C another copy of B. Let $b_1, \dots, b_r, b_0$ be the corresponding copies of b (over A) inside $B_1, \dots, B_r, C$ respectively. Let F be the disjoint union over A of $A\bar{y}$, C and T, but with the extra relations $R(b_j, s_j, b_0)$, where $1 \le j \le r$, as in the lemma. Then, by the lemma,

(i) $A\bar{y} \le F$;

(ii) $B_j \le F$; and

(iii) $F \in \mathcal{K}_f$.

Then by (i), (iii) and the $\le$-extension property we can assume $F \le M_f$; by (ii), we then have $\bar{x} = (b_0, b_1, \dots, b_r) \in P^{r+1}$; then, because of the relations $R(b_j, s_j, b_0)$ we have $s_j \in \operatorname{acl}(b_0, b_j, A)$, so $\bar{y} \in \operatorname{acl}(\bar{a}\bar{x})$, as required.

Step 2: Now let $\bar{y} = (t_1, \dots, t_r)$ be d-independent over A and let $T = \operatorname{acl}(A\bar{y})$. Let C be a copy of B over A with c the copy of b over A inside C, and let F be constructed as in the lemma. As in Step 1, we can assume that $F \le M_f$. So $c \in P$ and $\bar{y} \in \operatorname{acl}(\bar{a}, c, s_1, \dots, s_r)$. But by Step 1 (and $As_1 \cdots s_r \le F$) the tuple $(s_1, \dots, s_r)$ is in $\operatorname{acl}(\bar{a}\bar{z})$ for some $\bar{z} \in P^{r+1}$. The result follows. □

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