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Appendix by Philip Dittmann

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Galois groups of large simple fields

Anand Pillay and Erik Walsberg
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For Ehud Hrushovski, on his 60th birthday.

Suppose that K is an infinite field which is large (in the sense of Pop) and whose first-order theory is simple. We show that K is *bounded*, namely has only finitely many separable extensions of any given finite degree. We also show that any genus 0 curve over K has a K-point and if K is additionally perfect then K has trivial Brauer group. These results give evidence towards the conjecture that large simple fields are bounded PAC. Combining our results with a theorem of Lubotzky and van den Dries we show that there is a bounded PAC field L with the same absolute Galois group as K. In the appendix we show that if K is large and NSOP_∞ and v is a nontrivial valuation on K then (K, v) has separably closed Henselization, so in particular the residue field of (K, v) is algebraically closed and the value group is divisible. The appendix also shows that formally real and formally p-adic fields are SOP_∞ (without assuming largeness).

1. Introduction

Throughout K is a field. Large fields were introduced by Pop [1996], one definition being that K is *large* if any algebraic curve defined over K with a smooth (nonsingular) K-point has infinitely many K-points. Finite fields, number fields, and function fields are not large. Local fields, Henselian fields, quotient fields of Henselian domains, real closed fields, separably closed fields, pseudofinite fields, infinite algebraic extensions of finite fields, and fields which satisfy a local-global principle (in particular pseudo-real closed and pseudo-p-adically closed fields) are all large. All infinite fields whose first-order theory is known to be "tame" or well-behaved in various senses are large. Let K^{sep} be a separable closure of K. We say that K is *bounded* if for any n there are only finitely many degree n extensions of K in K^{sep}, or equivalently if the absolute Galois group $\mathrm{Aut}(K^{\mathrm{sep}}/K)$ of K has only finitely many open subgroups of any given finite index. When K is also perfect, this is also called Serre's property (F). (Other authors use "bounded" to mean that

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K has only finitely many extensions of each degree.) Koenigsmann has conjectured that bounded fields are large [Junker and Koenigsmann 2010, p. 496].

Recall that K is *pseudoalgebraically closed* (PAC) if any geometrically integral K-variety V has a K-point (and hence the set $V(K)$ of K-points is Zariski dense in V). We mention in passing that a PAC field need not be perfect; if $p = \mathrm{Char}(K)$, and $a \in K$ is not a p-th power, then Spec $K[x]/(x^p - a)$ is not geometrically integral [Poonen 2017, Example 2.2.9]. PAC fields are large, by definition. PAC fields were introduced by Ax [1968], who showed that pseudofinite fields are bounded PAC. Infinite algebraic extensions of finite fields are also bounded PAC [Fried and Jarden 2005, Corollary 11.2.4]. In either case PAC follows from the Hasse–Weil estimates.

On the model-theoretic side, we have various "tame" classes of first-order theories T, the most "perfect" being stable theories, and some others being simple theories and NIP theories. It is a well-known theorem of Shelah that a theory is stable if and only if it is both simple and NIP. Good examples come from theories of fields. We say that a first-order structure, in particular a field, is stable (simple, NIP) if its theory is stable (simple, NIP). In general we consider fields as structures in a language expanding the language of rings, although in the following sentence they are considered in precisely the language of rings. Separably closed fields are stable and bounded PAC fields are simple. There is a considerable amount of work on NIP fields, which include real closed and p-adically closed fields, but this does not concern us in the present paper. We now recall two longstanding open conjectures.

Conjecture 1.1. (1) Infinite stable fields are separably closed.

(2) Infinite simple fields are bounded PAC.

Our general idea is that Conjecture 1.1 is both true and tractable after making the additional assumption of largeness. It is shown in [Johnson et al. 2020] that a large stable field is separably closed. We describe another proof of this result in Section 4A. Here we consider (2), and prove:

Theorem 1.2. *Suppose that K is large and simple. Then there is a bounded* PAC *field L of the same characteristic as K such that the absolute Galois group of L is isomorphic* (*as a topological group*) *to the absolute Galois group of K.*

Note that Theorem 1.2 implies that K is bounded. We prove this separately.

Theorem 1.3. *If K is large and simple then K is bounded.*

The assumption that K is simple can be replaced by the more general assumption that the field K is definable in some model M of a simple theory. If we also require M to be highly saturated we can take K to be type-definable (over a small set of parameters) in M. The latter will follow from our proofs and references and we will not talk about it again. Theorem 1.3 generalizes the theorem of Chatzidakis that a simple PAC field is bounded, which is proven via quite different methods in [Chatzidakis 1999]. Poizat [1983] proved that an infinite stable bounded field is separably closed. Combining Poizat's result with Theorem 1.3 we get the above mentioned result of [Johnson et al. 2020] that large stable fields are separably closed.

Theorem 1.3 is reasonably sharp. The restriction to separable extensions is necessary. If K is separably closed of infinite imperfection degree and $\mathrm{Char}(K) = p > 0$ then K is large, stable, and has infinitely many extensions of degree p. There is an emerging body of work on a generalization of simplicity known as NSOP_1. All known NSOP_1 fields are PAC. Theorem 1.3 fails over NSOP_1 fields as there are unbounded PAC NSOP_1 fields (equivalently, there are PAC fields that are NSOP_1 but not simple). For example if K is characteristic zero, PAC, and the absolute Galois group of K is a free profinite group on $\aleph_0$ generators then K is unbounded and NSOP_1 [Chernikov and Ramsey 2016, Corollary 6.2].

A profinite group G is *projective* if any continuous surjective homomorphism $H \to G$ with H profinite has a section. Ax [1968] showed that the absolute Galois group of a perfect PAC field is projective, and Jarden [1972, Lemma 2.1] proved this for nonperfect PAC fields.

Theorem 1.4. *If K is large and simple then the absolute Galois group of K is projective.*

Theorem 1.2 follows from Theorem 1.3, Theorem 1.4, and the theorem of Lubotzky and van den Dries that for any field K and projective profinite group G there is a PAC field extension L of K such that the absolute Galois group of L is isomorphic to G (see [Fried and Jarden 2005, Corollary 23.1.2]). An earlier version of this paper proved Theorem 1.4 under the additional assumption that K is perfect. Philip Dittmann showed us how to remove this assumption.

Theorem 1.5. *Suppose that K is perfect, large, and simple. Then the Brauer group of K is trivial. It follows that*

(1) *any finite-dimensional division algebra over K is a field, and*

(2) *any Severi–Brauer K-variety V has a K-point.*

We recall the definition of Severi–Brauer variety. Let K^{alg} be an algebraic closure of K. Given a K-variety V we let $V_{K^{\mathrm{alg}}}$ be the base change $V \times_K \operatorname{Spec} K^{\mathrm{alg}}$ of V to a K^{alg}-variety. A *Severi–Brauer variety* is a K-variety V such that $V_{K^{\mathrm{alg}}}$ is isomorphic (over K^{alg}) to $\dim V$-dimensional projective space. A Severi–Brauer variety is geometrically integral, so (2) is a modest step towards the conjecture that large simple fields are PAC. Theorem 1.5 was proven for supersimple fields in [Pillay et al. 1998]; our proof closely follows that in [Pillay et al. 1998], so we do not recall the definition of the Brauer group. (Supersimple fields are perfect, but large simple fields need not be perfect.) Items (1) and (2) of Theorem 1.5 are well-known consequences of triviality of the Brauer group. We refer to [Poonen 2017, Sections 1.5 and 4.5.1] for the definition of the Brauer group and these facts.

Suppose $\mathrm{Char}(K) \neq 2$, then we say that a *conic* over K is a smooth irreducible projective K-curve of genus 0. One-dimensional Severi–Brauer varieties are exactly conics [Poonen 2017, Example 4.5.8]. Thus Theorem 1.6 generalizes the one-dimensional case of Theorem 1.5.2 to imperfect fields.

Theorem 1.6. *Suppose that K is large and simple*, $\mathrm{Char}(K) \neq 2$, *and C is a conic over K. Then C has a K-point*, *hence* (*by largeness*) *$C(K)$ is infinite.*

Let us mention some other earlier work around the conjectures on stable and simple fields described above. One of the first results on deducing algebraic results from model-theoretic hypotheses was Macintyre's theorem [1971] that infinite fields with ω-stable theory are algebraically closed (generalized to superstable fields in [Cherlin and Shelah 1980]). Macintyre's Galois-theoretic method has been used in many later works including the result on large stable fields [Johnson et al. 2020] mentioned above. Supersimple theories are simple theories in which there are not infinite forking chains of types, whereby any complete type has an ordinal valued dimension called the SU-rank. This gives a so-called "surgical dimension" as in [Pillay and Poizat 1995] from which one deduces that an infinite field with *supersimple* theory is perfect and bounded. So insofar as Conjecture 1.1(2) is restricted to supersimple theories, it remained to prove that supersimple theories are PAC, and some partial results were obtained in [Pillay et al. 1998; Martin-Pizarro and Pillay 2004] for example. A theme of the current paper is that, other than perfection of K, any results on supersimple fields also hold over large simple fields.

The conclusions of Theorems 1.4, 1.5, and 1.6 are properties of PAC fields. Another well-known consequence of a field K being PAC is that the Henselization of any nontrivial valuation on K is separably closed; see [Fried and Jarden 2005, Corollary 11.5.9]. In an earlier draft of this paper we showed that any nontrivial valuation on a large simple field has separably closed Henselization. Dittmann generalized this to Theorem 1.7, which is proven in the Appendix.

Theorem 1.7. *Suppose that K is large and* NSOP_∞. *Then any nontrivial valuation on K has separably closed Henselization. In particular*, *any nontrivial valuation on K has algebraically closed residue field and divisible value group.*

NSOP_∞ is a weakening of simplicity; see the Appendix for a definition and some discussion. NSOP_1 implies NSOP_∞ and essentially every known theory without the strict order property is NSOP_∞. It is natural to ask if Theorem 1.7 holds without the assumption of largeness. In the Appendix we give the following partial generalization.

Theorem 1.8. *If K admits a p-valuation then K is* SOP_∞. *In particular if K is a subfield of a finite extension of $\mathbb{Q}_p$ then K is* SOP_∞.

See Section A2 for the definition of a p-valuation. The proof of Theorem 1.8 uses diophantine work of Anscombe, Dittmann, and Fehm [Anscombe et al. 2020] in place of largeness. The results of [Anscombe et al. 2020] are p-adic analogues of classical results on sums of squares. In Section A2 we give a similar argument using Lagrange's four-square theorem to show that a formally real field is SOP_∞. If K admits a valuation with formally real residue field then K is formally real [Bochnak et al. 1998, Corollary 10.1.9]. Thus if K admits a valuation with formally real residue field then K is SOP_∞.

2. Large fields and definability

2A. ***Algebraic conventions.*** We let K^* be the set of nonzero elements of K and $\mathrm{Char}(K)$ be the characteristic of K. A *K-variety* is a separated, reduced K-scheme of finite type. We let $\dim V$ be the usual algebraic dimension and $V(K)$ be the set of K-points of a K-variety V. We let $\mathbb{A}^n$ be n-dimensional affine space over K (recall that $\mathbb{A}^n(K) = K^n$). We often assume irreducibility of the relevant K-varieties. A K-curve is a one-dimensional K-variety. A *morphism* is a morphism of K-varieties.

2B. ***Largeness.*** Large fields were introduced by Florian Pop. A survey appears in [Pop 2014], which starts by saying that large fields are fields over which (or in which) one can do a lot of "interesting mathematics". So largeness looks like a field-arithmetic tameness notion. The field K is *large* if every irreducible K-curve with a smooth (also called nonsingular) K-point has infinitely many K-points.

Fact 2.1 [Pop 1996]. *The following are equivalent*:

(1) *K is large.*

(2) *K is existentially closed in $K((t))$.*

(3) *If an irreducible K-variety V has a smooth K-point then $V(K)$ is Zariski dense in V.*

Fact 2.2 [Pop 2014, Proposition 2.7]. *An algebraic extension of a large field is large.*

Fact 2.3 allows us to pass to elementary extensions.

Fact 2.3 [Pop 2014, Proposition 2.1]. *Large fields form an elementary class.*

2C. ***Existentially étale sets.*** Let W be a K-variety. The authors of [Johnson et al. 2020] introduced the *étale open topology* on $W(K)$. If K is not large then the étale open topology is always discrete and if K is large then the étale open topology on $W(K)$ is nondiscrete whenever $W(K)$ is infinite. Our original proofs were given in terms of this topology, but at present we mostly avoid the topology and give proofs from scratch. We use properties of certain special existentially definable subsets of $W(K)$. A subset X of $W(K)$ is an *EE* set if there is a K-variety V and an étale morphism $f : V \to W$ such that $X = f(V(K))$. It is shown in [Johnson et al. 2020] that the EE subsets of $W(K)$ form a basis for the étale open topology. (In [Johnson et al. 2020] EE sets are referred to as "étale images".)

If W is smooth and $V \to W$ is an étale morphism then V is also smooth. At present we are mainly concerned with subsets of $K^n = \mathbb{A}^n(K)$, so we may restrict attention to smooth K-varieties. We quickly recall what we need from this setting. Let V, W be smooth irreducible K-varieties. An *étale morphism* $f : V \to W$ is a morphism such that the differential df_a is an isomorphism $TV_a \to TW_{f(a)}$ for all $a \in V$. In particular if $f : \mathbb{A}^n \to \mathbb{A}^n$ is a morphism then f is étale at $a \in K^n$ if and only if the Jacobian of f at a is invertible. The general notion of an étale morphism between not necessarily smooth varieties is more complicated but is not needed here.

Fact 2.4 [EGA IV$_4$ 1967, Proposition 17.1.3]. *Suppose W_1, W_2, V are smooth K-varieties and $f_i : W_i \to V$ is an étale morphism for $i \in \{1, 2\}$. Let W be the fiber product $W_1 \times_V W_2$ and $f : W \to V$ be the canonical map. Then W is a smooth K-variety and f is étale.*

We have $(W_1 \times_V W_2)(K) = \{(a_1, a_2) \in W_1(K) \times W_2(K) : f_1(a_1) = f_2(a_2)\}$, from which it easily follows that the image of $(W_1 \times_V W_2)(K)$ under f agrees with $f_1(W_1(K)) \cap f_2(W_2(K))$. Corollary 2.5 follows.

Corollary 2.5. *Suppose that W is a smooth K-variety. Then the collection of EE subsets of $W(K)$ is closed under finite intersections.*

Corollary 2.5 holds for an arbitrary K-variety, but we do not need this.

Lemma 2.6. *Suppose that K is large, W is a smooth irreducible K-variety, and X is a nonempty EE subset of $W(K)$. Then X is Zariski-dense in W. In particular any nonempty EE subset of K^n is Zariski dense in K^n.*

The identity morphism $W \to W$ is étale, so Lemma 2.6 generalizes the fact that if K is large and W is a smooth irreducible K-variety with $W(K) \neq \varnothing$ then $W(K)$ is Zariski dense in W.

Proof. Let V be a K-variety and $f : V \to W$ be an étale morphism such that $X = f(V(K))$. Suppose that X is not Zariski dense in V. Then X is contained in a proper closed subvariety Y of W. As W is irreducible we have $\dim Y < \dim W$. Note that $f^{-1}(Y)$ is a closed subvariety of V containing $V(K)$. As f is étale it is finite-to-one, hence $\dim V = \dim W$ and $\dim f^{-1}(Y) = \dim Y < \dim W$. So $f^{-1}(Y)$ is a proper closed subvariety of V containing $V(K)$. This contradicts Fact 2.1. □

Corollary 2.7 follows from Corollary 2.5 and Lemma 2.6.

Corollary 2.7. *Suppose that K is large. Let W be a smooth irreducible K-variety and $X_1, \dots, X_n$ be EE subsets of $W(K)$ with $\bigcap_{i=1}^k X_i \neq \varnothing$. Then $\bigcap_{i=1}^k X_i$ is Zariski dense in W. In particular, if $X_1, \dots, X_k$ are EE subsets of K^n with $\bigcap_{i=1}^k X_i \neq \varnothing$ then $\bigcap_{i=1}^k X_i$ is Zariski dense in K^n.*

Fact 2.8 is proven in [Johnson et al. 2020] for arbitrary K-varieties.

Fact 2.8. *Let W be a smooth K-variety, $g : W \to W$ be a K-variety isomorphism, and X be an EE subset of $W(K)$. Then $g(X)$ is also an EE subset of $W(K)$.*

Proof. Let V be a smooth K-variety and $f : V \to W$ be an étale morphism such that $X = f(V(K))$. Note that g is étale as any K-variety isomorphism is étale. So $g \circ f : V \to W$ is étale as a composition of étale morphisms is étale. □

We will apply Corollary 2.9 below.

Corollary 2.9. *Suppose that X is an EE subset of K^n, $a = (a_1, \dots, a_n) \in K^n$, and $b = (b_1, \dots, b_n) \in (K^*)^n$. Then*

$$X + a = \{(c_1 + a_1, \dots, c_n + a_n) : (c_1, \dots, c_n) \in X\},$$
$$bX = \{(b_1c_1, \dots, b_nc_n) : (c_1, \dots, c_n) \in X\}$$

are EE subsets of K^n.

Proof. The morphisms $\mathbb{A}^n \to \mathbb{A}^n$ given by $(x_1, \dots, x_n) \mapsto (x_1 + a_1, \dots, x_n + a_n)$ and also by $(x_1, \dots, x_n) \mapsto (b_1x_1, \dots, b_nx_n)$ are K-variety isomorphisms. Apply Fact 2.8. □

3. Fields with simple theory

We recall some basic results about fields K whose first-order theory is simple, and then make an additional observation under the assumption of largeness. For simple theories see [Kim and Pillay 1997; Casanovas 2011], and for groups definable in (models of) simple theories, see in addition [Pillay 1998; Pillay et al. 1998]. We recall the relevant portions of this theory.

3A. ***Conventions and basic definitions.*** Our model-theoretic notation is standard. We let L be a first-order language, T be a complete consistent L-theory, and $\overline{M}$ be a highly saturated model of T. For now, $x, y, z, \dots$ range over finite tuples of variables, $a, b, c, \dots$ range over finite tuples of parameters from $\overline{M}$, and $A, B, C, \dots$ range over small subsets of $\overline{M}$. "Definable" means "definable in $\overline{M}$, possibly with parameters". We sometimes identify definable sets with the formulas defining them.

Given an L-formula $\varphi(x, y)$ and a suitable tuple b we say that $\varphi(x, b)$ *divides over* a set A of parameters if $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent for some infinite A-indiscernible sequence $(b_i : i < \omega)$ with $b_0 = b$. A partial type $\Sigma(x)$ divides over A if some formula in Σ divides over A. The theory T is *simple* if for any small set A of parameters and complete type $\Sigma(x)$ there is $A_0 \subseteq A$ such that $|A_0| \leq |T|$ and $\Sigma(x)$ *does not divide* over A_0. Simplicity may also be defined in terms of the combinatorial *tree property*, but we do not need this. It is worth mentioning that simplicity is incompatible with the existence of a definable partial ordering which contains an infinite chain. It follows that real closed fields and nonseparably closed Henselian fields are not simple. Nondividing yields a good notion of independence in simple theories: a is *independent* from B over A if $\mathrm{tp}(a/B, A)$ does not divide over A.

3B. *Generics in definable groups.* In this section we summarize [Pillay 1998, Section 3], although we introduce things in a different order and use somewhat different terminology. Suppose that T is simple and G is an infinite group definable over $\varnothing$ in $\overline{M}$. A definable subset X of G is (*left*) *f-generic* if every left translate gX of X does not divide over $\varnothing$ and a complete type $\Sigma(x)$ concentrated on G is (left) f-generic if every formula in $\Sigma(x)$ is left f-generic. Note that if a definable $X \subseteq G$ is f-generic then aX is f-generic for any $a \in G$. Note that in [Pillay 1998] "generic" is used for "f-generic". (The language was changed after some more recent work on groups definable in NIP theories.)

Fact 3.1. *Suppose that T is simple, G is an $\varnothing$-definable group in $\overline{M}$, A, B are small sets of parameters, and $a \in G$.*

(1) *Left f-genericity is equivalent to right f-genericity (so we just say f-generic).*

(2) *If $X \subseteq G$ is f-generic then X is f-generic in any expansion of $\overline{M}$ by constants.*

(3) $\mathrm{tp}(a/A)$ *is f-generic if whenever $b \in G$ is independent from a over A then the product ba is independent of $A \cup \{b\}$ over $\varnothing$.*

(4) *If $A \subseteq B$ and a is independent from B over A, then* $\mathrm{tp}(a/B)$ *is f-generic if and only if* $\mathrm{tp}(a/A)$ *is f-generic.*

(5) *If $b \in B$ then* $\mathrm{tp}(a/A, b)$ *is f-generic if and only if* $\mathrm{tp}(ba/A, b)$ *is f-generic.*

(6) *An A-definable subset X of G is f-generic if and only if it is contained in an f-generic complete type over A.*

Fact 3.2 is immediate from the definitions and Fact 3.1.

Fact 3.2. *If X is a definable subset of G which is not f-generic then we have $\bigcap_{i=1}^{k} g_i X = \varnothing$ for some $g_1, \dots, g_k \in G$.*

Lemma 3.3. *Suppose that T is simple, M is a model of T, G is an $\varnothing$-definable group in M, H is a subgroup of G with $|G/H| \geq \aleph_0$, and X is a definable subset of G such that $X \subseteq aH$ for some $a \in G$. Then X is not f-generic. In particular, an infinite index definable subgroup of G is not f-generic.*

Proof. Let $(g_i : i < \omega)$ be a sequence of elements of G which lie in distinct cosets of H. So $g_i X \cap g_j X = \varnothing$ when $i \neq j$. After passing to a highly saturated elementary extension and applying Ramsey and saturation we obtain a sequence $(h_i : i < \omega)$ of elements of G which is indiscernible over the defining parameters of X and satisfies $h_i X \cap h_j X = \varnothing$ when $i \neq j$. So X is not f-generic. □

Lemma 3.4. *Suppose that T is simple, X is a definable subset of G, $\approx$ is a definable equivalence relation on X, and each $\approx$-class is f-generic. Then there are only finitely many $\approx$-classes.*

Proof. Suppose towards a contradiction that there are infinitely many $\approx$-classes. Let c be a finite tuple of parameters over which X and $\approx$ are definable. Then there is an $\approx$-class D with canonical parameter d such that $d \notin \operatorname{acl}(c)$. Let $\varphi(x, d, c)$ be a formula defining D and $(d_i : i < \omega)$ be an infinite sequence of realizations of $\operatorname{tp}(d/c)$ which is indiscernible over c and satisfies $d_0 = d$. Then $((c, d_i) : i < \omega)$ is indiscernible, and the formulas $\varphi(x, d_i, c)$ are pairwise inconsistent, so $\varphi(x, d, c)$ divides over $\varnothing$. This contradicts that $\varphi(x, d, c)$ defines the set D which is an f-generic subset of K^n. □

We now prove Lemma 3.5, which we could not find in the literature.

Lemma 3.5. *Suppose T is simple and G, H are $\varnothing$-definable groups in $\overline{M}$. Fix a small set A of parameters and $(a, b) \in G \times H$. Then* $\operatorname{tp}((a, b)/A)$ *is f-generic in $G \times H$ if and only if the following conditions hold*:

(1) $\operatorname{tp}(a/A)$ *is an f-generic type of G*,

(2) $\operatorname{tp}(b/A)$ *is an f-generic type of H*,

(3) *and a is independent from b over A.*

Proof. The definitions and "forking calculus" easily show that (1), (2), and (3) together imply that $\operatorname{tp}((a, b)/A)$ is f-generic in $G \times H$. The difficulty lies in showing that all f-generic types of $G \times H$ are of this form. We suppose that $\operatorname{tp}((a, b)/A)$ is f-generic in $G \times H$. It follows directly that $\operatorname{tp}(a/A)$ and $\operatorname{tp}(b/A)$ are f-generic types of G and H, respectively. It remains to prove that a is independent from b over A. Suppose that $(c, d) \in G \times H$, $\operatorname{tp}(c/A)$, $\operatorname{tp}(d/A)$ are f-generic in G, H, respectively, and (c, d) is independent from (a, b) over A. By Fact 3.1 ca is independent from db over $\varnothing$. As $\operatorname{tp}((a, b)/A)$ is f-generic in $G \times H$, and (a, b) is independent from (c, d) over A, we see that (ca, db) is independent from A, c, d over $\varnothing$. It follows that a is independent from b over A, c, d, and then that a is independent from b over A. □

3C. ***Generics in definable fields.*** Now suppose K is an infinite field definable (say over $\varnothing$) in $\overline{M} \models T$. Everything we say remains true for K a type-definable field in $\overline{M}$. We have two attached groups, the additive group $(K, +)$ and the multiplicative group $(K^*, \times)$ (recall that $K^* = K \setminus \{0\}$). A definable $X \subseteq K$ is *additively f-generic* if it is f-generic in $(K, +)$ and is *multiplicatively f-generic* if $X \cap K^*$ is an f-generic in $(K^*, \times)$, and we make the analogous definitions for a type concentrated on K. The first two claims of Fact 3.6 are from [Pillay et al. 1998, Proposition 3.1]. Uniqueness of f-generic types in stable fields is [Poizat 2001, Theorem 5.10].

Fact 3.6. *Suppose that T is simple. Let X be a definable subset of K, A be a small set of parameters, and $p = \mathrm{tp}(a/A)$ for some $a \in K$. Then the following are equivalent*:

(1) *X is an additive f-generic.*

(2) *X is a multiplicative f-generic.*

Furthermore the following are equivalent:

(1) *p is an additive f-generic*

(2) *p is a multiplicative f-generic.*

If T is stable then there is a unique additive f-generic type over K.

We let D_n be the group $((K^*)^n, \times)$. Corollary 3.7 is a higher-dimensional version of Fact 3.6. The first claim of Corollary 3.7 follows from Fact 3.6, Lemma 3.5, and induction on n. The second claim follows from the first claim and Fact 3.1.5.

Corollary 3.7. *Suppose that T is simple*, *A is a small set of parameters*, *$a = (a_1, \dots, a_n) \in K^n$*, *and $p(x) = \mathrm{tp}(a/A)$. Then p is an f-generic type of $(K^n, +)$ if and only if p is an f-generic type of D_n. So if $X \subseteq K^n$ is definable*, *then X is f-generic in $(K^n, +)$ if and only if $X \cap D_n$ is f-generic in D_n.*

Proposition 3.8 is our main tool when dealing with large simple fields.

Proposition 3.8. *Suppose that T is simple and K is large. Let X be a definable subset of K^n which contains a nonempty EE subset. Then X is f-generic for $(K^n, +)$*, *and is hence f-generic for D_n.*

Thus if T is simple and large then any definable subset of K^n with nonempty interior in the étale open topology is f-generic. If K is perfect, bounded PAC then a definable subset of K^n is f-generic if and only if it has nonempty interior in the étale open topology [Walsberg and Ye 2023].

Proof. Suppose towards a contradiction that X is not f-generic for $(K^n, +)$. By Corollary 3.7, $X \cap D_n$ is not f-generic for D_n. We may suppose that X contains $\bar{0} = (0, \dots, 0)$ as both EE subsets and f-generic subsets of K^n are closed under additive translation (by Corollary 2.9 and definitions). Let $X' = X \cap D_n$. By Corollary 3.7, X' is not f-generic in D_n. By Fact 3.2 there are $g_1, \dots, g_k \in D_n$ such that $\bigcap_{i=1}^k g_i X' = \varnothing$. Then $\bigcap_{i=1}^k g_i X$ is nonempty, as it contains $\bar{0}$, but is contained in $K^n \setminus D^n$ and is hence not Zariski dense in K^n. This contradicts Corollary 2.7. □

Fact 3.9 will be crucial for Theorem 1.5. It is proven in [Pillay et al. 1998].

Fact 3.9. *Suppose that T is simple. Let H be a finite index definable subgroup of $(K^*, \times)$ and H_1, H_2 be cosets of H. Then $H_1 + H_2$ contains K^*, namely every nonzero element of K is of the form $a + b$, where $a \in H_1$ and $b \in H_2$.*

4. Proof of Theorem 1.3

This section is the proof of Theorem 1.3. Our proof follows the strategy of the "Remarque" at the end of [Pillay and Poizat 1995] which outlines another proof, suggested by Chatzidakis, of the main result of that paper (that fields equipped with a certain "surgical dimension" are bounded). Remember that when we say that K is bounded we mean that for every n, K has only finitely many extensions of any given degree inside K^{sep}. We first make a few reductions. Fact 4.1 is well-known, but we include a proof for the sake of completeness.

Fact 4.1. *The following are equivalent*:

(1) *K is bounded.*

(2) *For any n there are only finitely many degree n separable extensions of K up to K-algebra isomorphism.*

Proof. By the primitive element theorem a degree n separable extension L of K is of the form $L = K(\alpha)$, where α is a root of a separable irreducible monic degree n polynomial $p(x) \in K[x]$. So L has at most n distinct conjugates over K in K^{sep}; the fact easily follows. □

We set some notation. Given $a = (a_0, \dots, a_{n-1}) \in K^n$ we let $p_a(x)$ denote the polynomial $x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$. We let U be the set of $a \in K^n$ such that p_a is separable and irreducible in $K[x]$. Note that U is definable. Given $p \in K[x]$ we let (p) be the ideal in $K[x]$ generated by p. For each $a \in U$ the field extension $K(\alpha)$ generated over K by a root α of p_a is isomorphic to $K[x]/(p_a)$. For $a, b \in U$, we write $a \approx b$ if $K[x]/(p_a)$ is isomorphic over K to $K[x]/(p_b)$. So K has finitely many separable extensions of degree n if and only if there are only finitely many $\approx$-classes.

Remark 4.2. The equivalence relation $\approx$ on U is definable in K.

Proof. The field $K[x]/(p_a)$ is uniformly interpretable in K (as a varies), as an n-dimensional vector space over K (with basis $1, \alpha, \dots, \alpha^{n-1}$ for α a root of $p_a(x)$ and the appropriate multiplication). Now note that if $a, b \in U$ then $a \approx b$ if and only if p_b has a root in $K[x]/(p_a)$. □

Next we have the main result needed to obtain Theorem 1.3:

Theorem 4.3. *Suppose that $a \in U$ and let D be the $\approx$-class of a. Then there is an EE subset X of K^n such that $a \in X \subseteq D$.*

Equivalently: every $\approx$-class is étale open.

Proof of Theorem 4.3. Fix $a \in U$, and let $\alpha \in K^{\mathrm{sep}}$ be a root of $p_a(x)$. Let $\bar{x} = (x_0, \dots, x_{n-1})$ and $\beta(\bar{x}) = x_0 + \alpha x_1 + \cdots + x_{n-1}\alpha^{n-1}$. Let $\alpha = \alpha_1, \dots, \alpha_n$ be the K-conjugates of α, namely the roots of $p_a(x)$ (which are distinct). We write $\beta_i(\bar{x})$ for $x_0 + x_1\alpha_i + \cdots + x_{n-1}\alpha_i^{n-1}$. So, for $b \in K^n$, $\beta_1(b), \dots, \beta_n(b)$ are the K-conjugates of $\beta(b)$.

Let V be the set of $b = (b_0, b_1, \dots, b_{n-1}) \in K^n$ such that $K(\beta(b)) = K(\alpha)$. Note that $b \in V$ if and only if $\beta(b)$ is a root of $p_c(x)$ for some (in fact unique) $c \in U$ such that $c \approx a$. Note further that $b \in V$ if and only if $1, \beta(b), \dots, \beta(b)^{n-1}$ are linearly independent over K, hence V is a Zariski open subset of K^n.

Let $e_1, \dots, e_n \in \mathbb{Z}[\bar{x}]$ be the elementary symmetric polynomials in n variables, i.e.,

$$e_k(\bar{x}) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}.$$

Given $b = (b_0, \dots, b_{n-1}) \in K^n$ we let

$$G(b) = \bigl(-e_1(\beta_1(b), \dots, \beta_n(b)), e_2(\beta_1(b), \dots, \beta_n(b)), \dots, (-1)^n e_n(\beta_1(b), \dots, \beta_n(b))\bigr).$$

Claim 4.4. *There are $G_1, \dots, G_n \in K[\bar{x}]$ such that $G(b) = (G_1(b), \dots, G_n(b))$ for all $b \in K^n$, and if $b \in V$ then $G(b) \approx a$.*

The first claim of Claim 4.4 follows as G is symmetric in $\alpha_1, \dots, \alpha_n$. The second claim follows as $p_{G(b)}$ is the monic polynomial with roots $\beta_1(b), \dots, \beta_n(b)$. Claim 4.5 below is crucial.

Claim 4.5. *$G(0, 1, 0, \dots, 0) = a$ and the Jacobian of G at $(0, 1, 0, \dots, 0)$ is invertible.*

Given a polynomial function $f : K^n \to K^n$ we let $\mathrm{Jac}_f(a)$ be the Jacobian of f and $|\mathrm{Jac}_f(a)|$ be the Jacobian determinant of f at $a \in K^n$.

Proof. It is clear that $G(0, 1, 0, \dots, 0) = a$ and $(0, 1, 0, \dots, 0) \in V$. Let $L = K(\alpha)$. To show that the Jacobian of G at $(0, 1, 0, \dots, 0)$ is invertible we first produce maps $D, E, F : L^n \to L^n$ such that G agrees with the restriction of $D \circ E \circ F$ to V. We define $F : L^n \to L^n$ by

$$F(b_0, \dots, b_{n-1}) = (b_0 + b_1\alpha_1 + \cdots + b_{n-1}\alpha_1^{n-1}, \dots, b_0 + b_1\alpha_n + \cdots + b_{n-1}\alpha_n^{n-1}).$$

$E : L^n \to L^n$ is given by

$$E(b) = (e_1(b), \dots, e_n(b)),$$

and $D : L^n \to L^n$ is given by

$$D(b_0, \dots, b_{n-1}) = (-b_0, b_1, -b_2, \dots, (-1)^n b_{n-1}).$$

So if $b \in V$ then $G(b) = (D \circ E \circ F)(b)$. Note that F and D are linear, so Jac_F and Jac_D are constant. Applying the chain rule we have

$$\begin{aligned}\mathrm{Jac}_G(0, 1, 0, \ldots, 0) &= \mathrm{Jac}_D \,\mathrm{Jac}_E(F(0, 1, 0, \ldots, 0))\,\mathrm{Jac}_F\\ &= \mathrm{Jac}_D\,\mathrm{Jac}_E(\alpha_1, \ldots, \alpha_n)\,\mathrm{Jac}_F\,.\end{aligned}$$

It is clear that $|\mathrm{Jac}_D| \in \{-1, 1\}$. Furthermore, Jac_F is a Vandermonde matrix

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$

So Jac_F is invertible as $\alpha_1, \ldots, \alpha_n$ are distinct. Finally, by [Lascoux and Pragacz 2002],

$$|\mathrm{Jac}_E(\alpha_1, \ldots, \alpha_n)| = \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j).$$

This is nonzero as the α_i are distinct, so $\mathrm{Jac}_E(\alpha_1, \ldots, \alpha_n)$ is invertible. □

We now deduce Theorem 4.3. Let O be the open subvariety of $\mathbb{A}^n$ given by $|\mathrm{Jac}_G(\bar{x})| \neq 0$. So G gives an étale morphism $O \to \mathbb{A}^n$. Then $O(K) \cap V$ is a Zariski open subset of K^n, which is nonempty by Claim 4.4. Let W be an open subvariety of $\mathbb{A}^n$ such that $W(K) = O(K) \cap V$. The restriction of G to W is an étale morphism $W \to \mathbb{A}^n$. Let $X = G(W(K))$. So X is a nonempty EE subset of K^n contained in the $\approx$-class of a. As a was an arbitrary member of U, this concludes the proof of Theorem 4.3. □

Finally we can complete the proof of Theorem 1.3.

Proof. Let T be a simple theory, M be a model of T, and K be an infinite field definable in M. As remarked at the beginning of this section, it suffices to fix n and show that K has only finitely many separable extensions of degree n, and thus that the definable equivalence relation $\approx$ on the definable set $U \subset K^n$ has only finitely many classes. After possibly passing to an elementary extension we may suppose that M is highly saturated. By Theorem 4.3 and Proposition 3.8, every $\approx$-class is f-generic for $(K^n, +)$. By Lemma 3.4 there are only finitely many $\approx$-classes. □

4A. ***Another proof that large stable fields are separably closed.*** We give a proof that large stable fields are separably closed that avoids Macintyre's Galois-theoretic argument. We first prove Lemma 4.6. We continue to use the notation of the previous section.

Lemma 4.6. *Let Y be the set of $a \in K^n$ such that p_a has n distinct roots in K. Then Y is an EE subset of K^n.*

Proof. Let V be the open subvariety of $\mathbb{A}^n$ given by $x_i \neq x_j$ for all $1 \leq i < j \leq n$. Let $H : K^n \to K^n$ be given by $H(b) = (-e_1(b), e_2(b), \dots, (-1)^n e_n(b))$. So $p_{H(a)}$ is the polynomial with roots $a_1, \dots, a_n$ for any $a = (a_1, \dots, a_n) \in V(K)$. It follows from [Lascoux and Pragacz 2002] that $|\mathrm{Jac}_H(a)|$ agrees up to sign with $\prod_{1 \leq i < j \leq n}(a_i - a_j)$ for any $a = (a_1, \dots, a_n) \in K^n$. So $\mathrm{Jac}_H(a)$ is invertible for all $a \in V(K)$. Thus $H(V(K))$ is an EE subset of K^n. □

We now show that a large stable field is separably closed.

Proof. Suppose that K is large and not separably closed. Fact 3.6 and Lemma 3.5 together show that if K is stable then for each $n \geq 1$ there is a unique n-ary type over K which is generic for $(K^n, +)$. It follows by Proposition 3.8 that if K is stable then any two nonempty EE subsets of K^n have nonempty intersection. As K is not separably closed there is a separable, irreducible, and nonconstant $p \in K[x]$. Suppose that p is monic and fix $a \in K^n$ such that $p = p_a$. By Theorem 4.3 there is an EE subset X of K^n such that $a \in X$ and p_b is separable and irreducible for any $b \in X$. Let Y be the set of $b \in K^n$ such that p_b has n distinct roots in K; by Lemma 4.6, Y is an EE subset of K^n. So X, Y are disjoint nonempty EE subsets of K^n, hence K is unstable. □

The proof above easily adapts to show that an infinite superstable field is algebraically closed. We describe this proof, assuming some familiarity with superstability. We let $\dim_U Z$ be the U-rank of a definable set Z. Suppose that K is infinite and superstable. A superstable field is perfect, so it suffices to show that K is separably closed. Suppose otherwise and fix n such that there is a nonconstant separable irreducible $p \in K[x]$. Let X, Y be as in the proof above. Note that both X and Y contain a set of the form $f(W(K))$, where W is a dense open subvariety of $\mathbb{A}^n$ and $f : W \to \mathbb{A}^n$ is étale. So $\dim_U W(K) = \dim_U K^n$ and the induced map $W(K) \to K^n$ has finite fibers as f is étale. Hence $\dim_U X = \dim_U K^n = \dim_U Y$. So X, Y are both f-generic in $(K^n, +)$, which contradicts uniqueness of generic types.

4B. ***Topological corollaries.*** Suppose that v is a nontrivial Henselian valuation on K. It follows from the classical Krasner's lemma that each $\approx$-class is open in the v-adic topology on K^n. See for example [Poonen 2017, 3.5.13.2] for a treatment of the case when K is a local field, which easily generalizes to the Henselian case. It is shown in [Johnson et al. 2020] that if K is not separably closed then the v-adic topology on each K^n agrees with the étale open topology. So Corollary 4.7 generalizes this consequence of Krasner's lemma.

Corollary 4.7. *Fix $a \in K^n$ such that p_a is separable and irreducible. Then the set of $b \in K^n$ such that $K[x]/(p_b)$ is K-algebra isomorphic to $K[x]/(p_a)$ is an* *étale open neighborhood of a. So the set of $a \in K^n$ such that p_a is separable and irreducible is étale open.*

Fact 4.8 is proven in [Johnson et al. 2020] by an application of Macintyre's Galois-theoretical argument.

Fact 4.8. *If K is not separably closed then the étale open topology on K is Hausdorff.*

If K is separably closed then the étale open topology agrees with the Zariski topology on $V(K)$ for any K-variety V; equivalently, every EE subset of $V(K)$ is Zariski open. We give a proof of Fact 4.8 which avoids Galois theory. We apply the fact that if $V \to W$ is a morphism between K-varieties then the induced map $V(K) \to W(K)$ is étale open continuous.

Proof. Equip K with the étale open topology. Any affine transformation $x \mapsto ax+b$, $a \in K^*$, $b \in K$ gives a homeomorphism $K \to K$. Thus it is enough to produce two disjoint nonempty étale open subsets of K. The argument of Section 4A yields two disjoint nonempty étale open subsets X, Y of K^n. Fix $p \in X$ and $q \in Y$ and let $f : K \to K^n$ be given by $f(t) = (1-t)p + tq$. Then f is a continuous map between étale open topologies so $f^{-1}(X)$, $f^{-1}(Y)$ are disjoint nonempty étale open subsets of K. $\square$

Finally, we characterize bounded PAC fields amongst PAC fields.

Corollary 4.9. *Suppose that K is* PAC *and equip each K^n with the étale open topology. Then K is bounded if and only if any definable equivalence relation on K^n has only finitely many classes with interior.*

Note that Corollary 4.9 fails when "PAC" is replaced by "large". For example $\mathbb{Q}_p$ is bounded, the étale open topology on $\mathbb{Q}_p$ agrees with the p-adic topology, and the equivalence relation E where $E(a, b)$ if and only if $a, b \in \mathbb{Q}_p$ have the same p-adic valuation is definable and has infinitely many open classes.

Proof. Suppose that K is not bounded. Fix n such that K has infinitely many separable extensions of degree n. Let U and $\approx$ be as in the proof of Theorem 1.3. Then each $\approx$-class is open and there are infinitely many $\approx$-classes. Now suppose that K is bounded and E is a definable equivalence relation on K^n. Note that K is simple. By Proposition 3.8 any E-class with interior is f-generic. The proof of Lemma 3.4 shows that there are only finitely many f-generic E-classes. $\square$

5. Additional remarks and results

We discuss a few related topics and results, and prove Theorem 1.4. If $\mathrm{Char}(K) = p > 0$ then we let $\wp : K \to K$ be the *Artin–Schreier map* $\wp(x) = x^p - x$. This map is an additive homomorphism, so $\wp(K)$ is a subgroup of $(K, +)$. In this section we let $P_n = \{a^n : a \in K^*\}$ for each n. Some of our proofs below could be simplified by applying Scanlon's theorem [Kaplan et al. 2011] that an infinite stable field is Artin–Schreier closed, but we avoid this.

5A. ***Boundedness and large stable fields.*** It is a theorem of Poizat that an infinite bounded stable field is separably closed. Poizat's result and Theorem 1.3 together show that large stable fields are separably closed. Poizat's result is mentioned somewhat informally at the bottom of p. 347 in [Poizat 1983] and does not appear to be well-known, so we take the opportunity to clarify the matter. Fact 5.1 is [Poizat 1983, Lemma 4].

Fact 5.1. *Suppose that L is a finite Galois extension of K. Then the following hold.*

(1) *If $q \neq \mathrm{Char}(K)$ is a prime then there are only finitely many cosets H of P_q in $(K^*, \times)$ such that some (equivalently, any) $a \in H$ has a q-th root in L.*

(2) *Suppose $\mathrm{Char}(K) = p > 0$. Then there are only finitely many cosets H of $\wp(K)$ in $(K, +)$ such that some (equivalently, any) $a \in H$ is of the form $b^p - b$ for some $b \in L$.*

Fact 5.2 follows from Fact 5.1.

Fact 5.2. *Suppose that K is bounded. Then*

(1) *if $q \neq \mathrm{Char}(K)$ is prime then P_q has finite index in $(K^*, \times)$, and*

(2) *if $\mathrm{Char}(K) > 0$ then $\wp(K)$ has finite index in $(K, +)$.*

We sketch a proof. See [Fehm and Jahnke 2016, Lemma 2.2] for a proof of the characteristic zero case of Fact 5.2(1) via Galois cohomology.

Proof. We only prove (1) as the proof of (2) is similar. Suppose $a \in K^*$ and $\alpha \in K^{\mathrm{sep}}$ satisfies $\alpha^q = a$. Then α and its conjugates generate a degree $\leq q$ Galois extension of K. As K is bounded there are only finitely many such extensions. So by Fact 5.1 P_q has finite index in $(K^*, \times)$. □

Finally, Fact 5.3 is essentially proven in [Macintyre 1971] via a Galois-theoretic argument.

Fact 5.3. *Suppose that the following hold for any finite Galois extension L of K:*

(1) *the q-th power map $L^* \to L^*$ is surjective for any prime $q \neq \mathrm{Char}(K)$, and*

(2) *if $\mathrm{Char}(K) \neq 0$ then the Artin–Schreier map $L \to L$ is surjective.*

Then K is separably closed.

We now sketch a proof of Poizat's theorem.

Corollary 5.4. *Suppose that K is infinite, bounded, and stable. Then K is separably closed.*

Proof. We verify the conditions of Fact 5.3. Suppose that L is a finite Galois extension of K. Then L is bounded and stable (the latter holds as L is interpretable in K). As L is stable there is a unique additive (multiplicative) generic type over K (see Fact 3.6). It follows that there are no proper finite index definable subgroups of $(L^*, \times)$ or $(L, +)$. So by Fact 5.2 the q-th power map $L^* \to L^*$ is surjective for any prime $q \neq \mathrm{Char}(K)$ and if $\mathrm{Char}(K) > 0$ then the Artin–Schreier map $L \to L$ is surjective. □

We repeat that the below corollary follows from Fact 5.1 and Theorem 1.3.

Corollary 5.5. *Suppose that K is large and simple. Then*

(1) *if $q \neq \mathrm{Char}(K)$ is prime then P_q has finite index in $(K^*, \times)$, and*

(2) *if $\mathrm{Char}(K) > 0$ then $\wp(K)$ has finite index in $(K, +)$.*

Corollary 5.5(2) is proven more generally for infinite simple fields in [Kaplan et al. 2011]. We take the opportunity to sketch a direct proof of Corollary 5.5. We let $\mathbb{G}_m$ be the scheme-theoretic multiplicative group $\mathrm{Spec}\, K[x, x^{-1}]$, so $\mathbb{G}_m(K) = K^*$.

Proof. We first fix a prime $q \neq \mathrm{Char}(K)$. The morphism $\mathbb{G}_m \to \mathbb{A}^1$ given by $x \mapsto ax^q$ is étale for any $a \in K^*$. So any coset of P_q is an EE subset of K. By the special (and easier) case of Proposition 3.8 when $n = 1$, any coset of P_q is f-generic in $(K^*, \times)$. By Lemma 3.3, P_q has finite index in $(K^*, \times)$. Item (2) follows by a similar argument and the fact that the Artin–Schreier morphism $\mathbb{A}^1 \to \mathbb{A}^1$ is étale. □

Fehm and Jahnke construct an unbounded PAC field K such that the group of n-th powers has finite index in each finite extension of K [Fehm and Jahnke 2016, Proposition 4.4], so Theorem 1.3 does not follow from Corollary 5.5.

5B. ***Conics, Brauer group, and projectivity.*** Corollary 5.6 follows from Fact 3.9 and Corollary 5.5.

Corollary 5.6. *Suppose that K is large and simple, $a, b \in K^*$, and $p \neq \mathrm{Char}(K)$ is a prime. Then there are $c, d \in K$ such that $c^p + ad^p = b$.*

The proof in [Pillay et al. 1998] that conics over (infinite) supersimple fields have points now extends to proving Theorem 1.6.

Proof of Theorem 1.6. Let C be a conic, i.e., a smooth projective irreducible K-curve of genus 0. As $\mathrm{Char}(K) \neq 2$ we may assume that C is a closed subvariety of $\mathbb{P}^2$ given by the homogenous equation $ax^2 + by^2 = z$ for some $a, b \in K^*$. By Corollary 5.6 there are $c, d \in K$ such that $ac^2 + bd^2 = 1$. So $C(K)$ is nonempty. □

We let $\mathrm{Br}\, K$ be the Brauer group of K. Recall that the Brauer group of an arbitrary field is an abelian torsion group. Given a prime p we let $\mathrm{Br}_p\, K$ be the p-part of the Brauer group of K. Facts 5.7 and 5.8 both follow by the proof of [Pillay et al. 1998, Theorem 4.6].

Fact 5.7. *Let $p \neq \mathrm{Char}(K)$ be a prime. Suppose that whenever L is a finite separable extension of K and $a \in L^*$, then $\{b^p + ac^p : b, c \in L^*\}$ contains L^*. Then $\mathrm{Br}_p\, K$ is trivial.*

Fact 5.8. *Suppose that*

(1) *K is perfect, and*

(2) *if L is a finite extension of K, p is a prime, and $a \in L^*$, then $\{b^p + ac^p : b, c \in L^*\}$ contains L^*.*

Then the Brauer group of K is trivial.

We now prove Theorem 1.5.

Proof. It suffices to show that the second condition of Fact 5.8 is satisfied. Let L be a finite extension of K and p be a prime. Note that L is perfect as a finite extension of a perfect field is perfect; the case when $p = \mathrm{Char}(K)$ follows. Suppose that $p \neq \mathrm{Char}(K)$. Note that L is simple as L is interpretable in K and L is large by Fact 2.2. Apply Corollary 5.6. □

Theorem 1.4 follows from Proposition 5.9 as a field of cohomological dimension ≤ 1 has projective absolute Galois group [Gruenberg 1967].

Proposition 5.9. *If K is simple and large then K has cohomological dimension ≤ 1.*

See [Serre 1997, Chapter I, §3] for an overview of cohomological dimension. The proof of Proposition 5.9 is due to Philip Dittmann. The simpler case where K is assumed to be perfect was proved earlier by the authors. We do not know if every large simple field has trivial Brauer group.

Proof. Suppose K is simple and large. The same argument as in the proof of Theorem 1.5 shows that if $p \neq \mathrm{Char}(K)$ is a prime, L is a finite separable extension of K, and $a \in L^*$, then $\{b^p + ac^p : b, c \in L^*\}$ contains L^*. So by Fact 5.7, $\mathrm{Br}_p\, L$ is trivial for every finite extension L of K and prime $p \neq \mathrm{Char}(K)$. By [Serre 1997, II.2.3 Proposition 4], K has p-cohomological dimension ≤ 1 for every prime $p \neq \mathrm{Char}(K)$. By [Serre 1997, II.2.2 Proposition 3], any field L has $\mathrm{Char}(L)$-cohomological dimension ≤ 1. So K has cohomological dimension ≤ 1. □

Appendix: NSOP$_\infty$ fields

by Philip Dittmann

A theory T has the *fully finite strong order property* if there is a formula $\psi(x, y)$, with the two tuples of variables x and y having equal length, a model $M \models T$, and a sequence $(a_i)_{i \in \omega}$ of tuples in M satisfying $M \models \psi(a_i, a_j)$ for all $i < j$, and for any $n \geq 3$ the formula $\psi(x_1, x_2) \wedge \cdots \wedge \psi(x_{n-1}, x_n) \wedge \psi(x_n, x_1)$ is inconsistent with T. In short, the binary relation described by ψ admits infinite chains and does not admit cycles. In this situation we also say that T has or is SOP_∞. A structure M is SOP_∞ if its theory is. A theory or structure is NSOP_∞ if it is not SOP_∞.

In Shelah's terminology, T having the fully finite strong order property witnessed by ψ is equivalent to ψ having the n-strong order property SOP_n for T, for all $n \geq 3$ [Shelah 1996, Definition 2.5]. In particular, if T is complete and simple then T is NSOP_∞ [Shelah 1996, Claim 2.7]. The notion "fully finite strong order property" seems to have first appeared in an unpublished manuscript by Adler [2008], although it has by now also been used elsewhere [Conant and Terry 2016, Definition 2.1].

A1. *Valuations.* The Henselization of a PAC field with respect to any nontrivial valuation is separably closed [Fried and Jarden 2005, Corollary 11.5.9]. Thus the following can be seen as supporting evidence for the conjecture that large simple fields are PAC.

Theorem A.1. *Suppose that K is large and v is a nontrivial valuation on K. If (K, v) has nonseparably closed Henselization then K is* SOP_∞. *In particular, if either the residue field of v is not algebraically closed or the value group of v is not divisible then K is* SOP_∞.

The second claim of Theorem A.1 follows from the first as the Henselization of (K, v) has the same residue field and value group as (K, v) [Engler and Prestel 2005, Theorem 5.2.5], and a nontrivially valued separably closed field has algebraically closed residue field and divisible value group [Engler and Prestel 2005, Theorem 3.2.11]. We will make use of Fact A.2, proven in [Johnson et al. 2020, Theorem 6.15].

Fact A.2. *Let v be a nontrivial valuation on K. If the Henselization of (K, v) is not separably closed then the étale open topology refines the v-adic topology on K.*

The following argument using generics was used in a preliminary version of the main article to get the simple case of Theorem A.1. Suppose that K is simple and the Henselization of (K, v) is not separably closed. By Fact A.2, $\mathfrak{m}_v$ is an étale open neighborhood of 0, so there is an EE subset U of K satisfying $0 \in U \subset \mathfrak{m}_v$. By Proposition 3.8 the set U is f-generic for $(K, +)$. This contradicts Lemma 3.3 as $\mathfrak{m}_v$ is an infinite index subgroup of $(K, +)$. This argument does not generalize to large NSOP_1 fields as at present there is no theory of generics in NSOP_1 groups. (This is not straightforward: [Dobrowolski 2020] gives an example of a definable group in an NSOP_1 structure in which generics with respect to Kim forking do not exist.)

Lemma A.3. *Suppose that K is large and $U \subseteq K$ is an étale open neighborhood of zero. Then for any $n \geq 2$ there is $a \in K^*$ such that $a, a^2, \dots, a^n \in U$.*

Proof. For each $i \in \{2, \dots, n\}$ let $V_i = \{b \in K : b^i \in U\}$. Each map $K \to K$, $b \mapsto b^i$ is continuous with respect to the étale open topology, so each V_i is an étale open neighborhood of zero. Then $V = V_1 \cap \dots \cap V_n$ is an étale open neighborhood of zero. As K is large V contains a nonzero element of K. □

Proof of Theorem A.1. By Fact A.2 there is a nonempty EE subset U of K with $0 \in U \subseteq \mathfrak{m}_v$. Let $\psi(x, y)$ be the formula $(x \neq 0) \wedge (y \neq 0) \wedge (x^{-1}y \in U)$. Note that if $K \models \psi(a, b)$ then $b/a \in \mathfrak{m}_v$, hence $v(a) < v(b)$. We show that $\psi(x, y)$ witnesses SOP_∞. First suppose that $a_1, \dots, a_n \in K$ satisfy

$$\psi(a_1, a_2) \wedge \dots \wedge \psi(a_{n-1}, a_n) \wedge \psi(a_n, a_1).$$

Then we have $v(a_1) < v(a_2) < \dots < v(a_{n-1}) < v(a_n) < v(a_1)$, a contradiction. We now show that for each $n \geq 1$ there are $a_1, \dots, a_n \in K$ such that $K \models \psi(a_i, a_j)$ if and only if $i < j$. By Lemma A.3 there is $a \in K^*$ such that $a, a^2, \dots, a^n \in U$. Then $K \models \psi(a^i, a^j)$ for $i < j$. Thus the binary relation on K defined by ψ admits chains of arbitrary finite length, and in a saturated elementary extension of K we obtain an infinite chain. Thus K is SOP_∞. □

Recall that EE sets are existentially definable. Note that the formula ψ in the proof of Theorem A.1 is existential. This is optimal as a quantifier-free formula in an arbitrary field is stable. This is similar to the result, proven in [Johnson et al. 2020, Theorem 3.1], that an unstable large field admits an unstable existential formula. The witnesses for SOP_∞ produced in the next section are also existential.

If v is actually Henselian, the same technique as in the proof of Theorem A.1 gives a slightly stronger statement. This is presumably well-known to the experts, but appears not to be available in the literature.

Theorem A.4. *Suppose that v is a nontrivial Henselian valuation on K and K is not separably closed. Then K has the strict order property* [Shelah 1996, Definition 2.1].

Proof. Fact A.2 provides an EE subset U of K with $0 \in U \subseteq \mathfrak{m}_v$. By [Johnson et al. 2020, Theorem B] the v-topology on K agrees with the étale open topology, hence U is v-open, and in particular contains a ball around 0.[1] Therefore, for any element $c \in K^\times$ with $v(c)$ sufficiently large we have $cU \subsetneq U$. Thus the definable family $\{xU : x \in K\}$ contains the infinite chain $U \supsetneq cU \supsetneq c^2U \supsetneq \cdots$ under inclusion. Hence K has the strict order property. □

A2. ***Formally real and formally p-adic fields.*** Corollary 5.6 implies that if K is large, simple, and of characteristic zero then there are $a, b \in K$ such that $a^2 + b^2 = -1$.

[1]This argument does not seriously use the étale open topology — we only need that the topology given by v is definable in the field language. This latter fact is already implicit in [Prestel and Ziegler 1978, Remark 7.11].

So a large simple field cannot be formally real. Duret [1977] showed that formally real fields are unstable. Theorem A.5 generalizes these.

Theorem A.5. *Suppose that K is formally real. Then K is* SOP_∞.

Proof. Let $\varphi(x, y)$ be the formula

$$\exists z_1, z_2, z_3, z_4\,[x - y - 1 = z_1^2 + z_2^2 + z_3^2 + z_4^2].$$

We show that φ witnesses SOP_∞. An application of Lagrange's four-square theorem shows that $K \models \varphi(m, m')$ for all integers $m > m'$. Now suppose that $a_1, \dots, a_n \in K$ and we have $K \models \{\varphi(a_1, a_2), \dots, \varphi(a_{n-1}, a_n), \varphi(a_n, a_1)\}$. Then

$$-n = (a_1 - a_2 - 1) + (a_2 - a_3 - 1) + \cdots + (a_{n-1} - a_n - 1) + (a_n - a_1 - 1)$$

is a sum of squares, a contradiction. $\square$

Fix a prime p. A field K is p-adically closed if K is elementarily equivalent to a finite extension of $\mathbb{Q}_p$ and K is *formally p-adic* if K embeds into a p-adically closed field. An equivalent definition (which we shall not need) is that there exists a p-valuation v on K, i.e., v is of mixed characteristic, the residue field is a finite extension of $\mathbb{F}_p$, and the interval $[0, v(p)]$ in the value group is finite. Indeed, if v is a p-valuation on K then the so-called p-adic closure of (K, v) is an elementary extension of a finite extension of $\mathbb{Q}_p$. See [Prestel and Roquette 1984] for a comprehensive treatment of formally p-adic fields.

Theorem A.6. *Suppose that K is formally p-adic. Then K is* SOP_∞.

Proof. Let F be a finite extension of $\mathbb{Q}_p$ such that K embeds into an elementary extension of F. Let v be the unique extension of the p-adic valuation on $\mathbb{Q}_p$ to F and $\mathcal{O}_F$ be the valuation ring of v.

By [Anscombe et al. 2020, Propositions 4.7 and 4.8] (applied to the base field $K = \mathbb{Q}$, the prime $\mathfrak{p} = p$ of $\mathbb{Q}$, and the relative type τ of $F/\mathbb{Q}_p$ in the terminology there), there exists a parameter-free existential formula $\psi(x)$ such that $\psi(F) \subseteq \mathcal{O}_F$, and $\psi(\mathbb{Q}) = \mathbb{Z}_{(p)}$. (Note that the paper cited phrases the result in terms of a concrete "diophantine family" $D^{\tau}_{\mathfrak{p},t_{\mathfrak{p}},A,B}$, but this is effectively the same as an existential formula with parameters from the base field $\mathbb{Q}$ [Anscombe et al. 2020, Remark 3.2], and parameters from $\mathbb{Q}$ can be eliminated.)

Let $\varphi(x, y)$ be the formula

$$(y \neq 0) \wedge \exists z\,(\psi(z) \wedge y = p \cdot x \cdot z).$$

We show that $\varphi(x, y)$ witnesses SOP_∞ for K. Suppose $m < m'$ are integers. Then we have $\mathbb{Q} \models \varphi(p^m, p^{m'})$, since $p^{m'}/(p \cdot p^m) \in \mathbb{Z} \subseteq \psi(\mathbb{Q})$. Since φ is existential, we have $K \models \varphi(p^m, p^{m'})$. Thus the binary relation on K defined by φ admits an infinite chain.

Now suppose that K satisfies

$$\Theta = \exists x_1, \dots, x_n\, [\varphi(x_1, x_2) \wedge \cdots \wedge \varphi(x_{n-1}, x_n) \wedge \varphi(x_n, x_1)].$$

As Θ is existential and K embeds into an elementary extension of F, we have $F \models \Theta$. Hence there are $b_1, \dots, b_n \in F$ such that

$$F \models \{\varphi(b_1, b_2), \dots, \varphi(b_{n-1}, b_n), \varphi(b_n, b_1)\}.$$

As $\psi(F) \subseteq \mathcal{O}_F$, we see that $F \models \varphi(a, a')$ implies that $v(a) < v(a')$ for any $a, a' \in F$. Thus we have $v(b_1) < v(b_2) < \cdots < v(b_{n-1}) < v(b_n) < v(b_1)$, a contradiction. $\square$

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