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Additive reducts of real closed fields and strongly bounded structures

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Given a real closed field R, we identify exactly four proper reducts of R which expand the underlying (unordered) R-vector space structure. Towards this theorem we introduce the new notion of strongly bounded reducts of linearly ordered structures: a reduct $\mathcal{M}$ of a linearly ordered structure $\langle R; <, \dots \rangle$ is called *strongly bounded* if every $\mathcal{M}$-definable subset of R is either bounded or cobounded in R. We investigate strongly bounded additive reducts of o-minimal structures and prove the above theorem on additive reducts of real closed fields.

1. Introduction

The study of *ordered* additive reducts of real closed fields starts with the work of Pillay, Scowcroft and Steinhorn [Pillay et al. 1989], followed by Marker, Peterzil and Pillay [Marker et al. 1992]. The motivation behind the work here is a conjecture about *unordered* such reducts from [Peterzil 1993]. Before stating the conjecture, let us clarify our usage of the notion of "reduct" here.

Definition 1.1. Given two structures $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{M}$ is a *reduct of* $\mathcal{N}$ (or, $\mathcal{N}$ is an *expansion of* $\mathcal{M}$), denoted by $\mathcal{M} \dot{\subseteq} \mathcal{N}$, if $\mathcal{M}$ and $\mathcal{N}$ have the same universe and every set that is definable in $\mathcal{M}$ is also definable in $\mathcal{N}$ (where definability allows parameters). We say that $\mathcal{M}$ and $\mathcal{N}$ are *interdefinable*, denoted by $\mathcal{M} \doteq \mathcal{N}$, if $\mathcal{M}$ is a reduct of $\mathcal{N}$ and $\mathcal{N}$ is reduct of $\mathcal{M}$.

We say $\mathcal{M}$ is *a proper reduct* of $\mathcal{N}$ (or $\mathcal{N}$ a *proper expansion* of $\mathcal{M}$) if $\mathcal{M} \dot{\subseteq} \mathcal{N}$ and not $\mathcal{M} \doteq \mathcal{N}$.

Below, we let Λ_R be the family of all R-linear maps $\lambda_a(x) = \alpha x$ for all $\alpha \in R$. Our ultimate goal here is to prove the following:

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Theorem 1.2. *Let R be a real closed field. Then the only reducts between the vector space $\langle R; +, \Lambda_R\rangle$ and the field $\langle R; <, +, \cdot\rangle$ are as follows*:

$$\mathcal{R}_{\mathrm{alg}} = \langle R; +, \cdot, <\rangle,$$
$$\mathcal{R}_{\mathrm{sb}} = \langle R; +, <, \Lambda_R, \mathfrak{B}\rangle,$$
$$\mathcal{R}_{\mathrm{semi}} = \langle R; +, <, \Lambda_R\rangle, \qquad \mathcal{R}_{\mathrm{bd}} = \langle R; +, <^*, \Lambda_R, \mathfrak{B}\rangle,$$
$$\mathcal{R}^*_{\mathrm{lin}} = \langle R; +, <^*, \Lambda_R\rangle,$$
$$\mathcal{R}_{\mathrm{lin}} = \langle R; +, \Lambda_R\rangle,$$

where $<^*$ is the linear order on the interval $(0, 1)$ *and $\mathfrak{B}_{\mathrm{sa}}$ the collection of all bounded semialgebraic sets over R.*

Remark 1.3. (1) The definable sets in $\mathcal{R}_{\mathrm{alg}}$ are called semialgebraic, while those definable in $\mathcal{R}_{\mathrm{semi}}$ are semilinear. The structure $\mathcal{R}_{\mathrm{sb}}$ above is called *semibounded*, as it expands the ordered vector space by a collection of bounded sets. Semibounded structures were studied in several articles, for example, [Edmundo 2000; Belegradek 2004; Peterzil 2009].

(2) Notice that because all the above structures expand the full underlying R-vector space, then once $<^*$ is definable then the restriction of $<$ to every bounded interval is definable.

(3) A similar project, in the setting of Presburger arithmetic, was carried out in [Conant 2018], where it was proven that there are no proper reducts between $\langle \mathbb{Z}; +\rangle$ and $\langle \mathbb{Z}; <, +\rangle$. We expect that in arbitrary models of Presburger arithmetic, an analogous result to Theorem 1.2 holds, with the intermediate reducts corresponding to possible restrictions of $<$ to infinite subintervals.

Some of the work towards the proof of Theorem 1.2 can be read off earlier results. In particular, the fact that the semibounded reduct $\mathcal{R}_{\mathrm{sb}}$ is the only proper reduct between $\mathcal{R}_{\mathrm{semi}}$ and $\mathcal{R}_{\mathrm{alg}}$ was proven over $\mathbb{R}$ in [Peterzil 1993] and can be deduced for arbitrary real closed fields from [Edmundo 2000] (see Fact 5.1 below). However, the bulk of the work here is to show that if a reduct $\mathcal{M}$ of $\mathcal{R}_{\mathrm{alg}}$ does not define the full order then it is necessarily a reduct of $\mathcal{R}_{\mathrm{bd}}$. Towards that, we introduce a new notion of "a strongly bounded structure" in a more general setting, and most of our results here are about such structures.

Definition 1.4. Let $\mathcal{R} = \langle R; <, \dots\rangle$ be a linearly ordered structure. A reduct $\mathcal{M} = \langle R; \dots\rangle$ of $\mathcal{R}$ is called *strongly bounded* if every $\mathcal{M}$-definable $X \subseteq R$ is either bounded or cobounded (namely, $R \setminus X$ is bounded).

Remark 1.5. (1) The term "strongly bounded" was chosen to reflect a combination of a semibounded structure with a strongly minimal one. Almost all of our work here concerns strongly bounded additive reducts of o-minimal structures, where the underlying linear order is dense. Analogous definitions could be given for, say, models of Presburger arithmetic if one wishes to study all reducts which expand the underlying ordered group.

(2) The definition of a strongly bounded structure requires an ambient linear order. Thus it might not seem amenable to working in elementarily equivalent structures. However, in practice we only work in sufficiently saturated elementary extensions of a strongly bounded $\mathcal{M}$ as above, and thus we may assume that this elementary extension is also a reduct of a linearly ordered elementary extension of $\mathcal{R}$.

By definition, if $\mathcal{M}$ is a strongly bounded reduct of a linearly ordered structure then the ordering $<$ is not definable in $\mathcal{M}$. We prove several results about strongly bounded reducts of o-minimal structures (see, for example, Theorems 4.5 and 4.27):

Theorem. *Let $\langle R; <, +, \dots \rangle$ be an o-minimal expansion of an ordered group and let $\mathcal{M} = \langle R, +, \dots \rangle$ be a strongly bounded reduct.*

(1) *Every $\mathcal{M}$-definable subset of R^n is already definable in $\langle R; +, \Lambda_{\mathcal{M}}, \mathfrak{B}^* \rangle$, where Λ_M is the collection of $\mathcal{M}$-definable endomorphisms of $\langle R, + \rangle$ and $\mathfrak{B}^*$ is the collection of all $\mathcal{M}$-definable bounded sets.*

(2) *For every $\mathcal{N} \equiv \mathcal{M}$, the model theoretic algebraic closure equals the definable closure.*

2. Proper expansions of $\mathcal{R}_{\text{lin}}$

In this section we assume that $\mathcal{R}_{\text{omin}}$ is an o-minimal expansion of a real closed field R and $\mathcal{M} = \langle R; +, \dots \rangle$ is an additive reduct of $\mathcal{R}_{\text{omin}}$.

Theorem 2.1. *If $\mathcal{M}$ is not a reduct of $\mathcal{R}_{\text{lin}} = \langle R; +, \Lambda_R \rangle$, then $<^*$ is definable in $\mathcal{M}$.*

Proof. It is sufficient to prove that some interval $[0, b]$ is $\mathcal{M}$-definable, for $b > 0$.

Claim 2.2. $\text{Th}(\mathcal{M})$ *is unstable.*

Proof. This is based on work of Hasson, Onshuus and Peterzil [Hasson et al. 2010].

Assume towards contradiction that $\text{Th}(\mathcal{M})$ is stable. By [Hasson et al. 2010, Theorem 1], every 1-dimensional stable structure interpretable in an o-minimal structure is necessarily 1-based. So $\mathcal{M}$ is 1-based. By [Hrushovski and Pillay 1987, Theorem 4.1], it follows that every $\mathcal{M}$-definable set is a boolean combination of cosets of definable subgroups of R^n. Every definable subgroup of $\langle R^n; + \rangle$ in an o-minimal structure is an R-vector subspace of R^n and therefore every $\mathcal{M}$-definable set is definable in $\mathcal{R}_{\text{lin}}$, a contradiction. Hence $\mathcal{M}$ is unstable. $\square$

Because $\mathcal{M}$ is unstable, it is in particular not strongly minimal. This generally implies that in some elementary extension of $\mathcal{M}$, we have an $\mathcal{M}$-definable subset in one variable which is infinite and coinfinite. However, o-minimal structures eliminate $\exists^\infty$, and therefore so does $\mathcal{M}$. It follows that there is some $\mathcal{M}$-definable subset of R itself which is infinite and coinfinite. Call this set Y.

In the special case where both Y and $R \setminus Y$ are unbounded in R we can prove a stronger result which will be used several times here, and thus we state it separately.

Lemma 2.3. *Assume that $Y \subseteq R$ is definable in an o-minimal expansion of an ordered group. If both Y and $R \setminus Y$ are unbounded then the full linear order is definable in $\langle R; +, Y\rangle$.*

Proof. By o-minimality, Y has the form

$$Y := I_1 \cup I_2 \cup \cdots \cup I_n \cup L, \tag{1}$$

such that for every $i \in \{1, \dots, n\}$, $I_i := (a_i, b_i)$, L is a finite set and in addition $-\infty \le a_1 < b_1 < a_2 < \cdots < a_n < b_n \le +\infty$. Without loss of generality $L = \varnothing$.

Since both Y and $R \setminus Y$ are unbounded, Y has the form (1) above and without loss of generality, we may assume that $I_1 = (-\infty, b_1)$, and $I_i = (a_i, b_i)$ for $i \in \{2, \dots, n\}$.

By replacing Y by $Y - b_1$ we may assume that $b_1 = 0$, and then

$$-Y \cap Y = (-b_n, -a_n) \cup \cdots \cup (-b_2, -a_2) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n).$$

So $(-Y \cap Y) \cap [(-Y \cap Y) + (a_n + b_n)]$ equals the interval $I_n = (a_n, b_n)$ in Y. Replace Y by $Y_1 := Y \setminus I_n$; now Y_1 contains an unbounded ray together with $n-2$ bounded intervals. Continuing in this way we obtain a ray $(-\infty, 0)$ that is definable, so we can define $<$. $\square$

In the remaining case, either Y or $R \setminus Y$ are bounded, so we assume that Y is bounded, and as above

$$Y := (a_1, b_1) \cup \cdots \cup (a_, b_n),$$

with $a_i, b_i \in M$.

Let $\alpha := b_n - b_1$. The set $(Y + \alpha) \cap Y$ defines a single interval whose right endpoint is b_n. So, we are done. If Y is unbounded then replace Y by $R \setminus Y$ and finish as before. Hence, we have showed that $<^*$ is definable in $\mathcal{M}$. $\square$

3. Reducts of $\mathcal{R}_{\text{alg}}$ which are not semilinear

Here R is a real closed field and $\mathcal{R}_{\text{alg}} = \langle R; <, +, \cdot\rangle$. Before the next theorem we recall previous work from [Loveys and Peterzil 1993] (see a corrected and more general proof in [Belegradek 2004]), which will be used in its proof.

Given $a > 0$ in R, let $I = (-a, a)$. Denote by $+^*$ the partial function obtained by intersecting the graph of $+$ with I^3, and for each $\alpha \in R$, let λ^*_α be the partial function obtained by intersecting the graph of λ_α with I^2. Finally, let $<^*$ be the restriction of $<$ to I^2. Notice that for each $X \subseteq R^n$ such that $\langle R; <, +, \cdot, X\rangle$ is o-minimal, the structure

$$\mathcal{I} = \langle \mathrm{I}; <^*, +^*, \{\lambda^*_\alpha\}_{\alpha \in R}, X \cap I^n\rangle$$

is o-minimal as well.

In [Loveys and Peterzil 1993] the structure $\langle I; <^*, +^*\rangle$ was called a group-interval and its o-minimal expansions were studied there.

A partial endomorphism (p.e. for short) of this group-interval was a function $f : I \to I$ which respects addition when defined: namely, if $x, y, x +^* y \in I$ then $f(x +^* y) = f(x) +^* f(y)$.

Notice that in our setting every $\mathcal{I}$-definable p.e. is necessarily the restriction of λ_α for some $\alpha \in R$. Indeed, if $f : I \to I$ is an $\mathcal{I}$-definable p.e. then it is not hard to verify that

$$H = \{r \in R : \exists \varepsilon > 0\, \forall x \in (-\varepsilon, \varepsilon)\; f(rx) = rf(x)\}$$

is a semialgebraic subgroup of $\langle R, +\rangle$ which contains all integers.

O-minimality of the real field implies that $H = R$ and therefore f is the restriction of an R-linear map, namely the restriction of λ_α for some R.

Now, without going through their precise definition of "a linear theory", it was shown in [Loveys and Peterzil 1993, Proposition 4.2] that if $\mathrm{Th}(\mathcal{I})$ is linear then every $\mathcal{I}$-definable set is already defined in the structure $\langle I; +^*, <^*, \{\lambda^*_\alpha\}_{\alpha \in R}\rangle$ (possibly together with additional parameters). Thus if $\mathrm{Th}(\mathcal{I})$ is linear then $X \cap I^n$ is a semilinear set.

The following proposition seems to be obvious but for the sake of completion we include a proof in the Appendix.

Fact 3.1. *Let R be a real closed field and $X \subseteq R^n$ a definable set in an o-minimal expansion of $\langle R; <, +, \cdot\rangle$. If X is not semilinear then, in $\mathcal{M} = \langle R; <^*, +, \Lambda_R, X\rangle$, there exists a definable bounded set which is not semilinear.*

Theorem 3.2. *If $X \subseteq R^n$ is semialgebraic and not definable in $\mathcal{R}_{\mathrm{semi}}$, then every bounded R-semialgebraic set is definable in $\langle R; +, \Lambda_R, X\rangle$.*

Proof. Let $\mathcal{M} := \langle R; +, \Lambda_R, X\rangle$. By Theorem 2.1, the relation $<^*$ is definable in $\mathcal{M}$. Let us first see that $\mathcal{M}$ defines a real closed field on some interval.

By Fact 3.1, we may assume that $X \cap I^n$ is not semilinear, for some bounded interval $I = (-a, a)$. Consider the o-minimal structure

$$\mathcal{I} := \langle I; <^*, +^*, \{\lambda^*_\alpha\}_{\alpha \in R}, X \cap I^n\rangle,$$

as we described before stating the theorem. We noted that if $\mathrm{Th}(\mathcal{I})$ is linear then the set $X \cap I^n$ must be semilinear. Because $X \cap I^n$ is not semilinear then $\mathrm{Th}(\mathcal{I})$ is not linear in the sense of [Loveys and Peterzil 1993]. Therefore, by [Peterzil and

Starchenko 1998, Theorem 1.2], a real closed field is $\mathcal{I}$-definable, and hence also $\mathcal{M}$-definable, on some interval $J \subseteq I$.

Without loss of generality, assume that $J = (-a_0, a_0)$, $a_0 > 0$. Denote the field by

$$\mathcal{J} = \langle J, \oplus, \odot \rangle.$$

The structure $\mathcal{J}$ is $\mathcal{M}$-definable. By [Peterzil 1993, Corollary 2.4], every R-semialgebraic subset of J^k, $k \in \mathbb{N}$, is definable in $\mathcal{J}$, and therefore in $\mathcal{M}$.

Let $B \subseteq (-b, b)^n$ for some $b > 0$ in R. Using scalar multiplication from Λ_R, we can contract $(-b, b)$ into $(-a_0, a_0)$, so it is definable in $\mathcal{J}$. It follows that B is definable in $\mathcal{M}$. □

4. Strongly bounded structures

The ultimate goal of this section is to prove:

Theorem 4.1. *Let R be a real closed field. If $X \subseteq R^n$ is semialgebraic and not definable in $\mathcal{R}_{\text{bd}} = \langle R; <^*, +, \Lambda_R, \mathfrak{B}_{\text{sa}} \rangle$, then $<$ is definable in $\langle R; +, \Lambda_R, X \rangle$.*

We are going to work in a more general setting than that of a real closed field. Recall that a strongly bounded reduct of a linearly ordered $\langle R; <, \dots \rangle$ is one in which every definable subset of R is bounded or cobounded. Below, we will mostly be interested in strongly bounded reducts of o-minimal structures. By Lemma 2.3 and the definition of a strongly bounded structure, we have:

Lemma 4.2. *Let $\mathcal{R}_{\text{omin}} = \langle R; <, +, \dots \rangle$ be an o-minimal expansion of an ordered group. If $\mathcal{M} = \langle R; +, \dots \rangle$ is a reduct of $\mathcal{R}_{\text{omin}}$ then $\mathcal{M}$ is strongly bounded if and only if $<$ is not definable in $\mathcal{M}$.*

So in order to prove Theorem 4.1 it is sufficient to prove that if $X \subseteq R^n$ is definable in a strongly bounded $\mathcal{M} = \langle R; +, \dots \rangle$ then X is definable in $\langle R; +, \Lambda_{\mathcal{M}}, \mathfrak{B}_{\mathcal{M}} \rangle$, where $\mathfrak{B}_{\mathcal{M}}$ is the collection of all $\mathcal{M}$-definable bounded sets. A more precise and slightly stronger theorem — Theorem 4.5 — will be proved soon. We first make a general observation which we shall exploit repeatedly.

Definability of "boundedness". For $X \subseteq T \times R^n$, $T \subseteq R^m$ and $t \in T$, we let

$$X_t = \{a \in R^n : \langle t, a \rangle \in X\}.$$

The following general result will be very useful here.

Proposition 4.3. *Let $\mathcal{M} = \langle R; +, \dots \rangle$ be any reduct of an o-minimal expansion of an ordered group. If $\{X_t : t \in T\}$ is an $\mathcal{M}$-definable family of subsets of R^n, then the set*

$$\{t \in T : X_t \text{ is bounded in } R^n\}$$

is definable in $\mathcal{M}$.

Proof. Note that a set $Y \subseteq R^n$ is bounded if and only if for each i, the image of Y under the projection map $\pi_i : \langle y_1, \ldots, y_n \rangle \mapsto y_i$ is bounded in R. Thus, it is sufficient to prove the result under the assumption that all X_t are subsets of R.

By o-minimality, each $X_t \subseteq R$ is unbounded if and only if it contains an unbounded ray. Thus, it is easy to see that

$$\{t \in T : X_t \text{ is bounded}\} = \{t \in T : \exists a\, (a + X_t \cap X_t = \varnothing)\},$$

and hence the set is definable in $\mathcal{M}$. $\square$

The strongly bounded setting. We first clarify and somewhat generalize our setting.

Let $\mathcal{R}_{\mathrm{omin}} = \langle R, <, +, \ldots \rangle$ denote an o-minimal expansion of an ordered group in the language $\mathcal{L}_{\mathrm{omin}}$, and let $\mathcal{M} = \langle R; +, \ldots \rangle$ denote a strongly bounded reduct of $\mathcal{R}_{\mathrm{omin}}$, in the language $\mathcal{L}$, such that $\mathrm{acl}_{\mathcal{M}}(\varnothing)$ contains at least one nonzero element (it follows that $\mathrm{acl}_{\mathcal{M}}(\varnothing)$ is infinite).

Definition 4.4. An interval $(a, b) \subseteq R$ is called a *$\varnothing$-interval in $\mathcal{M}$* if $a, b \in \mathrm{acl}_{\mathcal{M}}(\varnothing)$. A subset $X \subseteq R^n$ is called *$\varnothing$-bounded in $\mathcal{M}$* if X is contained in some I^n, for I a $\varnothing$-interval in $\mathcal{M}$.

Our standing assumption is that for every $\varnothing$-interval $I \subseteq R$, the restricted order $<\restriction I$ is $\varnothing$-definable in $\mathcal{M}$. Notice that, using Theorem 2.1, this is true when $\mathcal{M}$ is elementarily equivalent to a reduct of a real closed field which properly expands $\mathcal{R}_{\mathrm{lin}}$.

We let $\Lambda_{\mathcal{M}}$ be the collection of all $\mathcal{M}$-definable endomorphisms of $\langle R, + \rangle$, defined over $\varnothing$. We let $\mathcal{L}_{\mathrm{bd}}(\mathcal{M})$ be the language consisting of $\{+, \{\lambda\}_{\lambda \in \Lambda_{\mathcal{M}}}\}$, augmented by a predicate for every $\varnothing$-definable, $\varnothing$-bounded set in $\mathcal{M}$.

By expanding $\mathcal{L}$ and $\mathcal{L}_{\mathrm{omin}}$ by function symbols and predicates for $\varnothing$-definable sets, we may assume that

$$\mathcal{L}_{\mathrm{bd}} \subseteq \mathcal{L} \subseteq \mathcal{L}_{\mathrm{omin}}.$$

We let $\mathcal{M}_{\mathrm{bd}}$ be the reduct of $\mathcal{M}$ to $\mathcal{L}_{\mathrm{bd}}$.

Our ultimate goal in this section is to prove:

Theorem 4.5. *For $\mathcal{M}$ strongly bounded as above, every definable subset of R^n is definable in $\mathcal{M}_{\mathrm{bd}}$.*

One of our main difficulties in working with strongly bounded structures is the failure of global cell decomposition. For instance, the set $R \setminus \{0\}$ cannot be decomposed definably into definable cells in a strongly bounded structure, because no ray is definable there.

Another difficulty is the fact that a priori we do not know whether the model theoretic algebraic closure equals the definable closure in strongly bounded structures. However, we shall eventually show in Theorem 4.27 that $\mathrm{acl} = \mathrm{dcl}$ in this setting.

We assume for the rest of this section that $\mathcal{M}$ is strongly bounded as above.

Definable subsets of R in strongly bounded structures. Notice that although the full order is not definable in $\mathcal{M}$, a basis for the $<$-topology on R and the product topology on R^n is definable in $\mathcal{M}$, using the restricted order. Thus we have:

Lemma 4.6. *If $\{X_t : t \in T\}$ is an $\mathcal{M}$-definable family of subsets of R^n, then the families*

$$\{\mathrm{Cl}(X_t) : t \in T\}, \qquad \{\mathrm{Int}(X_t) : t \in T\}, \qquad \{\mathrm{Fr}(X_t) : t \in T\}$$

are definable in $\mathcal{M}$.

Every $\mathcal{M}$-definable $X \subseteq R$ is a union of finitely many pairwise disjoint *maximal* open subintervals of X (which are possibly not $\mathcal{M}$-definable) and a finite set. Below, when we say that I *is an interval in* X we mean that I is one of these open components of X.

Definition 4.7. Let $Y \subseteq R$ be an $\mathcal{M}$-definable set. We define

$$\partial^-(Y) := \{y \in R : y \text{ is a left endpoint of an interval in } Y\},$$
$$\partial^+(Y) := \{y \in R : y \text{ is a right endpoint of an interval in } Y\}.$$

Lemma 4.8. *If $\{Y_t : t \in T\}$ is an $\mathcal{M}$-definable family of bounded subsets of R, then the families $\{\partial^-(Y_t) : t \in T\}$, $\{\partial^+(Y_t) : t \in T\}$ are $\mathcal{M}$-definable over the same parameter set.*

Proof. We fix an $\mathcal{M}$-definable $<\restriction(0, a_0)$ for some $a_0 > 0$. We define $\partial^-(Y_t)$ by the formula

$$(x \notin Y_t \wedge \exists \varepsilon {<} a_0\, (x, x+\varepsilon) \subseteq Y_t)$$
$$\vee \big(x \in Y_t \wedge \exists \varepsilon {\leq} a_0\, (x-\varepsilon, x) \cap Y_t = \varnothing \wedge (x, x+\varepsilon) \subseteq Y_t\big).$$

Because of the definability of $<^*$ in $\mathcal{M}$, $\{\partial^-(Y_t) : t \in T\}$ is $\mathcal{M}$-definable. We similarly handle $\partial^+(Y_t)$. $\square$

The next theorem is an important component of our analysis of strongly bounded structures.

Theorem 4.9. *If $\{X_t : t \in T\}$ is an $\mathcal{M}$-definable family of bounded subsets of R, then there is a uniform bound on the length of each interval in X_t. Moreover, there exists such a bound in* $\mathrm{dcl}_{\mathcal{M}}(\varnothing)$.

Proof. By Proposition 4.3, every $\mathcal{M}$-definable family $\{X_t : t \in T\}$ of bounded subsets of R is a subfamily of a $\varnothing$-definable family of such sets. Namely, if $\varphi(x, t, a)$ is the formula defining the X_t's over a, as t varies, then we can consider the formula

$$\psi(x, t, y) : \varphi(x, t, y) \wedge \ \psi(R, t, y) \text{ is a bounded set.}$$

Thus, it is sufficient to prove the result for $\varnothing$-definable families.

By Lemma 4.6, we may assume that each X_t is an open set. We use induction on the maximum number n of intervals in X_t, for $t \in T$.

For $n = 1$, write $X_t = (a_t, b_t)$. Consider the family $\{X_t - a_t : t \in T\}$. By Lemma 4.8, the family is $\varnothing$-definable. Thus, the set $Y = \bigcup_{t \in T} X_t - a_t$ is an $\mathcal{M}$-definable interval, over $\varnothing$, whose left endpoint is 0. Because $\mathcal{M}$ is strongly bounded, this interval must be bounded, and hence its right endpoint is some $K \in M$. By Lemma 4.8, the point K is definable over $\varnothing$.

Consider now the case $n = k+1$, i.e., each X_t consists of at most $k+1$ pairwise disjoint open intervals. For each $t \in T$, let $D_t = \{c_1 - c_2 : c_1, c_2 \in \partial^-(X_t)\}$, an $\mathcal{M}$-definable set by Lemma 4.8.

Claim 4.10. *For each $t \in T$, there exists $d \in D_t$ such that $(X_t + d) \cap X_t$ is one of the intervals in X_t.*

Proof. Let $X_t = I_{1,t} \cup I_{2,t} \cup \cdots \cup I_{k+1,t}$, where each $I_{m,t} := (a_{m,t}, b_{m,t})$, such that

$$a_{1,t} < b_{1,t} < a_{2,t} < b_{2,t} < \cdots < a_{k+1,t} < b_{k+1,t}.$$

For an interval $I = (a, b)$, let $|I| = b - a$.

Let $d = a_{k+1,t} - a_{1,t}$. In the set $X_t + d$, for each m, the interval $I_{m,t}$ is shifted to $I_{m,t} + d$. So $(X_t + d) \cap X_t$ consists of either $I_{k+1,t}$ (when $|I_{k+1,t}| < |I_{1,t}|$) or $I_{1,t} + d$ (when $|I_{k+1,t}| > |I_{1,t}|$).

If it consists of I_{k+1} we are done. Otherwise we take

$$d' = a_{1,t} - a_{k+1,t} \in D_t$$

and then $(X_t + d') \cap X_t = I_{1,t}$.

So in both cases there exists $d \in D_t$ such that $X_t + d \cap X_t$ is one of the intervals in X_t. $\square$

We define the set

$$D'_t := \{d \in D_t : (X_t + d) \cap X_t \text{ is one of the intervals in } X_t\}.$$

Claim 4.11. *The family $\{D'_t : t \in T\}$ is an $\mathcal{M}$-definable family of nonempty sets.*

Proof. For $t \in T$, $d \in D'_t$ if and only if the following two statements hold:

(1) $\partial^-((X_t + d) \cap X_t) \subseteq \partial^-(X_t)$ and $|\partial^-((X_t + d) \cap X_t)| = 1$, and

(2) $\partial^+((X_t + d) \cap X_t) \subseteq \partial^+(X_t)$ and $|\partial^+((X_t + d) \cap X_t)| = 1$.

By Lemma 4.8, (1) and (2) are definable properties in $\mathcal{M}$. By Claim 4.10, each D'_t is nonempty. $\square$

We proceed with the proof of Theorem 4.9. Consider the $\mathcal{M}$-definable family

$$\{Y_{t,d} := X_t + d \cap X_t : d \in D'_t,\ t \in T\},$$

still defined in $\mathcal{M}$ over $\varnothing$. For every t and $d \in D'_t$, the set $Y_{t,d}$ consists of a single interval which is one of the intervals in X_t. By case $n = 1$ we know that there is a uniform bound w_1 on the length of each $Y_{t,d}$, which can be chosen in $\mathrm{dcl}_{\mathcal{M}}(\varnothing)$. We now define, still over $\varnothing$, the family

$$\{Z_{t,d} := X_t \setminus Y_{t,d} : d \in D'_t,\ t \in T\}.$$

Each subset $Z_{t,d}$ consists of at most k intervals among the $k+1$ intervals of X_t. By the induction hypothesis, we know that there is a uniform bound w_2 on the length of each interval, which we may choose in $\mathrm{dcl}_{\mathcal{M}}(\varnothing)$.

Thus the maximum of w_1, w_2, which is in $\mathrm{dcl}_{\mathcal{M}}(\varnothing)$, is the bound on the length of each interval of X_t, as t varies. This ends the proof of Theorem 4.9. $\square$

As a corollary we can now match, definably in $\mathcal{M}$, each left endpoint of an interval in X_t with the corresponding right endpoint:

Proposition 4.12. *Let $\{X_t : t \in T\}$ be an $\mathcal{M}$-definable family of bounded subsets of R, and let*

$$L_t = \{\langle a, b\rangle \in \partial^-(X_t) \times \partial^+(X_t) : \text{the interval } (a, b) \text{ is one the intervals of } X_t\}.$$

Then the family $\{L_t : t \in T\}$ is $\mathcal{M}$-definable.

Proof. By Theorem 4.9, there is a bound $K \in \mathrm{dcl}_{\mathcal{M}}(\varnothing)$ for the length of each interval in X_t, for all $t \in T$. For each $t \in T$, we have

$$\langle a, b\rangle \in L_t \iff a \in \partial^-(X_t) \quad \text{and} \quad b = \min(\partial^+(X_t) \cap [a, a+K]). \qquad (*)$$

By Lemma 4.8, $\partial^-(X_t)$ and $\partial^+(X_t)$ are definable families and since in $(*)$ we only use the order on $[0, K]$, the family $\{L_t : t \in T\}$ is definable in $\mathcal{M}$. $\square$

Remark 4.13. (1) Notice that Theorem 4.9 fails without the assumption that the X_t's are bounded sets. Namely, it is not true in general that the lengths of the bounded components of X_t are bounded in t. For example, the set $X_t = R \setminus \{-t, t\}$ has $(-t, t)$ as an open component, with unbounded length as $t \to \infty$.

Also, even if each X_t is bounded it is not true that the diameter of the X_t's is uniformly bounded. For example, take the family $\{(-t, t-1) \cup (t, t+1) : t \in R\}$ that is definable using $< \restriction (0, 1)$.

(2) We do not know whether Proposition 4.12 holds if we drop the assumption that the X_t's are bounded. Can we still match definably the left and right endpoints of the bounded components of X_t, when the X_t's are unbounded?

Affine sets and functions. Recall that $\mathcal{R}_{\mathrm{omin}}$ is an o-minimal expansion of an ordered divisible abelian group R, and we assume that $\mathcal{M} = \langle R; +, \dots\rangle$ is a strongly bounded reduct of $\mathcal{R}_{\mathrm{omin}}$ in which $<$ is $\varnothing$-definable on every $\varnothing$-interval. We let $<^*$ denote the ordering on some fixed interval we call $(0, 1)$.

Definition 4.14. Let $\langle R; <, +\rangle$ be an abelian ordered divisible group.

(1) A map $f : R^n \to R^k$ is *affine* if it is of the form $\ell(x) + d$ for $\ell : R^n \to R^k$ a homomorphism between $\langle R^n, +\rangle$ and $\langle R^k, +\rangle$, and $d \in R^k$.

(2) A (partial) function $f : R \to R$ is *eventually affine* if there exists $a > 0$ such that $(a, \infty) \subseteq \operatorname{dom}(f)$ and the restriction of f to $(a, +\infty)$ is affine.

(3) $X \subseteq R^n$ is *locally affine at $a \in X$* if there is an open neighborhood $U \ni a$ such that for all $x, y, z \in U \cap X$, $x - y + z \in X$. The *affine part of X* is the set

$$\mathscr{A}(X) = \{x \in X : X \text{ is locally affine at } x\}.$$

Notice that if X is the graph of an affine map then $\mathscr{A}(X) = X$. Conversely, if X is the graph of a definable function from an open subset of R^k into R^ℓ and $a = (a', f(a')) \in \mathcal{A}(X)$ then f is an affine map in a neighborhood of a_1.

Because a basis for the R^n-topology is definable in $\mathcal{M}$, we immediately have:

Lemma 4.15. *Let $\{X_t : t \in T\}$ be an $\mathcal{M}$-definable family of subsets of R^n, defined over $\varnothing$. Then the family $\{\mathscr{A}(X_t) : t \in T\}$ is $\mathcal{M}$-definable over $\varnothing$.*

Proposition 4.16. *Every $\mathcal{M}$-definable endomorphism $f : R \to R$ is $\varnothing$-definable.*

Proof. Assume that f is defined by an $\mathcal{M}$-formula $\varphi(x, y, a)$ over the parameter a. We show that f can be defined without parameters.

Since being an R-endomorphism is $\mathcal{M}$-definable, we may assume that there is some $\mathcal{M}$-definable $T \subseteq R^k$ such that for all $t \in T$, if $\varphi(R^2, t)$ is nonempty then it defines a nonzero endomorphism f_t of $\langle R; +\rangle$.

Assume first that the set of endomorphisms f_t defined by φ is finite. Define $t_1 E t_2$ if $f_{t_1} = f_{t_2}$, an $\mathcal{M}$-definable equivalence relation. Consider the functions near 0, and define $[t_1]_E < [t_2]_E$ if for all $x > 0$ sufficiently small, we have $f_{t_1}(x) < f_{t_2}(x)$. By o-minimality, we obtain a linear ordering of the finitely many E-classes, and since $<$ is $\mathcal{M}$-definable in a neighborhood of 0, this ordering is $\mathcal{M}$-definable. Thus, each f_t in this finite family of endomorphisms is $\varnothing$-definable.

Assume now that the family $\{f_t : t \in T\}$ is infinite, and we shall reach a contradiction. Consider the set $\{f_t(1) : t \in T\}$. By o-minimality it contains an open interval (a, b), and by replacing each f_t with $f_t - f_{t_0}$, for some $t_0 \in T$ for which $f_{t_0} \in (a, b)$, we may assume that the interval (a, b) contains 0 and the ordering on (a, b) is $\mathcal{M}$-definable (we think of $f_t(a)$ as "the slope" of f_t). Let $T_0 = \{t \in T : f_t(1) \in (0, b)\}$.

We write $t_1 \sim t_2$ if $f_{t_1} = f_{t_2}$, and let $[t]$ be the equivalence class of t. In abuse of notation we let $f_{[t]}$ denote the corresponding endomorphism of R.

By o-minimality, if $f_{t_1}(1) = f_{t_1}(1)$ then $f_{t_1} = f_{t_2}$. Thus we obtain an $\mathcal{M}$-definable function $t : (0, b) \to T_0/\sim$, defined by $f_{[t(x)]}(1) = x$. Namely, $f_{[t(x)]}$ is the endomorphism whose "slope" is x. Fix an element $d > 0$, and define $\sigma : (0, b) \to R$

by $\sigma(x) = f^{-1}_{[t(x)]}(d)$. Namely, $\sigma(x) = y$ if there exists $t \in T_0$ such that $f_t(1) = x$ and $f_t(y) = d$ (we may think of $\sigma(x)$ as "d/x"). The function σ is also $\mathcal{M}$-definable. For every $t \in T_0$, we have $f_t(1) > 0$, and hence $f_t(x) > 0$ if and only if $x > 0$. Therefore, σ is positive on $(0, b)$.

Claim. $\mathrm{Im}(\sigma)$ *is unbounded in* R.

Indeed, assume towards contradiction that $K = \sup(\mathrm{Im}(\sigma)) < \infty$. By our observation, $K > 0$. Choose $y_0 \in \mathrm{Im}(\sigma)$, $y_0 < K$ and sufficiently close to K such that $K < 2y_0$. By assumption, there exists $t_0 \in T_0$ and $x_0 > 0$ such that $f_{t_0}(1) = x_0$ and $f_{t_0}(y_0) = d$.

Let $t_1 \in T_0$ be such that $[t_1] = t(x_0/2)$. Then $f_{t_1}(1) = x_0/2 = f_{t_0}(1)/2$. It follows that $f_{t_1} = f_{t_0}/2$ and hence

$$f_{t_1}(2y_0) = f_{t_0}(2y_0)/2 = f_{t_0}(y_0) = d.$$

But then $f_{t_1}(1) = x_0/2$ and $f_{t_1}(2y_0) = d$, so by definition, $\sigma(x_0/2) = 2y_0 > K$, contradicting the assumption that K bounds $\mathrm{Im}(\sigma)$.

Thus, $\mathrm{Im}(\sigma)$ is an $\mathcal{M}$-definable set which is unbounded and positive, contradicting the assumption that $\mathcal{M}$ is strongly bounded. □

Definition 4.17. We denote by Λ_{omin} the set of all $\mathcal{R}_{\mathrm{omin}}$-definable endomorphisms $f : \langle R, +\rangle \to \langle R, +\rangle$, and we still let $\Lambda_{\mathcal{M}}$ denote the set of all $\mathcal{M}$-definable endomorphisms of R, which by Proposition 4.16, is necessarily $\varnothing$-definable. Let $\Lambda^*_{\mathrm{omin}}$ and $\Lambda^*_{\mathcal{M}}$ denote those nonzero endomorphisms.

Definable functions of one variable. Our goal is to describe definable functions in one variable, and prove that $\mathcal{M}$ has no definable "poles".

Proposition 4.18. *If* $g : R \to R$ *is an* $\mathcal{M}$*-definable partial function whose domain is cobounded and* $\mathrm{Im}(g)$ *is bounded, then* g *is constant on a cobounded set.*

Proof. By o-minimality, there exists $L \in R$ such that $\lim_{x\to+\infty} g(x) = L$. We shall see that $g \equiv L$ on a cobounded set.

The function g is definable in an o-minimal structure, so there exists $a_1 \in R$ such that $g \restriction (a_1, +\infty)$ is either constant or strictly monotone, and there exists a_2 such that g is constant or strictly monotone on $(-\infty, a_2)$.

If g is constant L on $(a_1, +\infty)$ then $\{x \in R : g(x) = L\}$ is unbounded and since $\mathcal{M}$ is strongly bounded the set must be cobounded and we are done. Assume towards contradiction that $g \restriction (a_1, \infty)$ is strictly monotone.

Assume first that g is strictly increasing on (a_1, ∞). Notice that the property of being locally increasing in a neighborhood of $x \in R$ is definable using $<^*$. Thus the set

$$\{x \in R : g \text{ is locally increasing at } x\}$$

is $\mathcal{M}$-definable, contains (a_1, ∞) and hence must be cobounded. It follows that g is strictly increasing on $(-\infty, a_2)$.

Because $\lim_{x\to\infty} g(x) = L$ and g is increasing, there exists $b \in R$ such that for all $x > b$, $L-1 < g(x) < L$. Because $<^*$ is $\mathcal{M}$-definable the set of all $x \in R$ such that $L-1 < g(x) < L$ is $\mathcal{M}$-definable, so must be cobounded. In particular, we may assume that $L-1 < g(x) < L$ for all $x < a_2$ and thus $g(x)$ has a limit $L_1 \in R$ as $x \to -\infty$.

But since g is increasing on $x < a_2$, it follows that $L_1 < L$ and in addition there exists $a_2' \leq a_2$ and $\varepsilon > 0$, such that for all $x < a_2'$,

$$L_1 < g(x) < L_1 + \varepsilon < L.$$

Using $<^*$ again, this is an $\mathcal{M}$-definable property of x so must hold also for all $x > a_1'$, contradicting the fact that $\lim_{x\to+\infty} g(x) = L$.

A similar argument works when g is eventually decreasing. □

Remark 4.19. By [Edmundo 2000], if $\mathcal{N} = \langle R; <, +, \dots \rangle$ is an o-minimal expansion of an ordered group in which every definable bounded function is eventually constant then $\mathcal{N}$ is *semibounded*, namely every definable set is definable using the underlying ordered vector space, together with all the definable bounded sets. This might suggest a fast deduction of Theorem 4.5 from Proposition 4.18. The problem of this approach is that we do not know that the definable functions in the strongly bounded $\mathcal{M} = \langle R; +, <^*, \dots \rangle$ are the same as in its expansion by the full $<$. Thus, we do not see how to apply Edmundo's theorem here.

Next, using almost identical arguments to [Edmundo 2000] we show that every $\mathcal{M}$-definable function $f : R \to R$ is affine on a cobounded set. For that, we recall some notation and facts, based on [Miller and Starchenko 1998].

Notation. For $\mathcal{R}_{\text{omin}}$-definable positive (partial) functions $f, g : R \to R$ such that $(a, \infty) \subseteq \operatorname{dom}(f), \operatorname{dom}(g)$, we write $f \leq g$ (or $f < g$) if $f(x) \leq g(x)$ (or $f(x) < g(x)$) for all large enough x.

We write $v(f) < v(g)$ if $|f| > |\lambda \circ g|$ for all $\lambda \in \Lambda^*_{\text{omin}}$ such that $\lambda > 0$. We also write $v(f) = v(g)$ if there are $\lambda_1, \lambda_2 \in \Lambda^*_{\text{omin}}$, both positive, such that

$$|\lambda_1 \circ g| \leq |f| \leq |\lambda_2 \circ g|.$$

This is easily seen to be an equivalence relation, which roughly says that the rate of growth of f and g at $+\infty$ is of the same scale. In the case where R expands a real closed field then $v(f) = v(g)$ if and only if f and g belong to the same Archimedean class with respect to R, namely there exists $r \in R$ such that $(1/r)|g| \leq f \leq r|g|$.

Finally, we write $\Delta(f) = f(x+1) - f(x)$.

Fact 4.20 [Edmundo 2000]. *For every $\mathcal{R}_{\text{omin}}$-definable function on an unbounded ray,*

(1) *if* $v(f) > v(x)$ *then* $\lim_{x\to\infty} \Delta(f) = 0$;

(2) *if* $v(f) < v(x)$ *then* $v(f^{-1}) > v(x)$;

(3) *if* $v(f) = v(x)$ *then* $\Delta(f)(x)$ *has a limit in* R *as* $x \to \infty$.

The following is just a warm-up towards Theorem 4.25. The proof follows closely the proof of [Edmundo 2000, Proposition 2.8], which uses results of [Miller and Starchenko 1998].

Lemma 4.21. *If* $f : R \to R$ *is* $\mathcal{M}$*-definable on a cobounded set, then* f *is eventually affine. Moreover, there exists a* $\varnothing$*-definable endomorphism* $\lambda \in \Lambda_{\mathcal{M}}$ *and* $A > 0$ *such that for all* x *with* $|x| > A$*, we have* $f(x) = \lambda(x) + d$ *for some* $d \in R$.

Proof. Assume towards contradiction that $f : R \to R$ is not eventually affine. Without loss of generality, f is eventually increasing, and by Proposition 4.18, it must approach $+\infty$. If $v(f) > v(x)$ then by Fact 4.20, $\lim_{x\to\infty} \Delta(f) = 0$. Since $\Delta(f) := f(x+1) - f(x)$ is definable in $\mathcal{M}$, it follows from Proposition 4.18 that it must be eventually 0 and therefore f is eventually affine.

If $v(f) < v(x)$ then by Fact 4.20, $v(f^{-1}) > v(x)$, where f^{-1} is taken to be the eventual compositional inverse of f, which is also definable in $\mathcal{M}$. Thus, as above, f^{-1} is eventually affine so also f is.

We are left with the case $v(f) = v(x)$. By Fact 4.20(3), the $\mathcal{M}$-definable function $\Delta(f)$ approaches a limit c in R. By Proposition 4.18, we have $\Delta(f)$ eventually constant, and thus, by o-minimality, f is eventually affine.

Thus, we showed so far that there exists a definable endomorphism $\lambda \in \Lambda_{\mathcal{M}}$ such that $f(x) = \lambda(x) + d$ for all $x > 0$ large enough. By Proposition 4.16, λ is $\varnothing$-definable. The set

$$\{x \in R : f(x) = \lambda(x) + d\}$$

is $\mathcal{M}$-definable and contains an unbounded ray so must be cobounded. □

Before the next proposition, we introduce a new notion.

Definition 4.22. Given $X \subseteq R^n$, let

$$\mathrm{Stab}_{\mathrm{bd}}(X) := \{a \in R^n : (a + X) \triangle X \text{ is bounded}\},$$

where $A \triangle B = A \cup B \setminus A \cap B$.

For a function f*, we let* $\Gamma(f)$ *denote its graph.*

By Proposition 4.3, if X is definable in $\mathcal{M}$ over A then so is $\mathrm{Stab}_{\mathrm{bd}}(X)$. The following facts are easy to verify:

Fact 4.23. (1) *For every* $X \subseteq R^n$, $\mathrm{Stab}_{\mathrm{bd}}(X)$ *is a subgroup of* $\langle R^n, +\rangle$.

(2) *If* $X \subseteq R^2$ *is the graph of an affine function* $f(x) = \lambda(x) + b$*, on a cobounded subset of* R*, then*

$$\mathrm{Stab}_{\mathrm{bd}}(X) = \Gamma(\lambda).$$

(3) *If a definable set $X \subseteq R^2$ is a finite union of graphs of affine functions, all of the form $\lambda + d$ for a fixed λ, and at least one of the functions is defined on an unbounded set, then* $\mathrm{Stab}_{\mathrm{bd}}(X) = \Gamma(\lambda)$.

The following statement would have been immediately true if definable sets in $\mathcal{M}$ admitted definable cell decomposition (with respect to the ambient ordering).

Proposition 4.24. *Assume that $X \subseteq R^2$ is $\mathcal{M}$-definable over A, and* $\dim(X) \leq 1$. *Assume that there exists an $\mathcal{R}_{\mathrm{omin}}$-definable endomorphism $\lambda : R \to R$, and some $a, d \in R$ such that graph of $\lambda(x) + d \restriction (a, \infty)$ is contained in X. Then λ is $\mathcal{M}$-definable (necessarily over $\varnothing$).*

Proof. Recall that $\mathscr{A}(X)$, the affine part of X is $\mathcal{M}$-definable over A. For large enough a, it contains $\Gamma(\lambda + d \restriction (a, \infty))$. So, without loss of generality, $X = \mathscr{A}(X)$.

We define for each $x, y \in X$, the relation $x \sim y$ if there exist open sets $U, V \ni 0$ in R^2 such that

$$(y - x) + (x + U \cap X) = y + V \cap X.$$

Said differently, up to translation, X has the same germ at x and at y. Because a basis for the R^2 topology is definable in $\mathcal{M}$, the relation $\sim$ is definable in $\mathcal{M}$.

Notice that for x large enough, all elements on $\Gamma(\lambda + d) \cap X$ are in the same $\sim$-class, so we may replace X by this $\sim$-class, which is $\mathcal{M}$-definable.

Thus, we may assume that all elements of X are $\sim$-equivalent, and X contains $\Gamma(\lambda + d \restriction (a, \infty))$. It follows that X is contained in finitely many translates of the graph of λ. Applying Fact 4.23(3), we conclude that $\mathrm{Stab}_{\mathrm{bd}}(X)$ is exactly the graph of λ, and thus the function $\lambda(x)$ is $\mathcal{M}$-definable. By Proposition 4.16, λ is $\varnothing$-definable. $\square$

Definable subsets of R^2. The next result is the main structure theorem of the paper.

Theorem 4.25. *Under our standing assumptions on $\mathcal{M}$, assume that $X \subseteq R^2$ is definable in $\mathcal{M}$ over a parameter set $A \subseteq R$, with* $\dim(X) \leq 1$. *Then there are $\lambda_1, \dots, \lambda_r \in \Lambda_{\mathcal{M}}$ and $\mathcal{M}$-definable finite sets $D_i \subseteq R$, $i = 1, \dots, r$, and $D \subseteq R$ all defined over A, such that*

(i) *For every $i = 1, \dots, r$, and $d \in D_i$, $\Gamma(\lambda_i + d) \setminus X$ is bounded (i.e., X contains the restriction of $\lambda_i + d$ to a cobounded set).*

(ii) *For every $d \in D$, $(\{d\} \times R) \setminus X$ is bounded.*

(iii) *The set*

$$X \setminus \left(\bigcup_{i=1}^{r} \bigcup_{d \in D_i} \Gamma(\lambda_i + d) \cup \bigcup_{d \in D} \{d\} \times R \right)$$

is bounded in R^2.

Proof. If X is bounded then there is nothing to prove, so we assume $\dim(X) = 1$ and X is unbounded. By the cell decomposition theorem in o-minimal structures, X can be decomposed into a finite union of cells of dimension 0 and 1. However, these cells are not in general definable in $\mathcal{M}$.

Assume first that X contains the graph of a function $f : (a, +\infty) \to R$, and let $\Psi(x, y)$ be the $\mathcal{M}$-formula that defines X.

Case (i): f is bounded at ∞.

In this case we prove a general statement:

Claim 4.26. *If* $\dim X \leq 1$ *and* X *contains the graph of a bounded function* $f : (a, \infty) \to R$ *then* f *is eventually constant.*

Proof. By o-minimality, $\lim_{x\to+\infty} f(x) = L$ for some $L \in R$.

By our standing assumption, $< \restriction (0, a_0)$ is $\mathcal{M}$-definable, for some $a_0 > 0$, and thus $<$ is definable on every interval of length $\leq a_0$. Let $X_L := R \times [L - a_0, L + a_0]) \cap X$. By o-minimality, there exists $m \in \mathbb{N}$ such that for all large enough $a \in R$, we have $|X_a| \leq m$. The set $Z = \{a \in R : |X_a| \leq m\}$ is definable in $\mathcal{M}$ and unbounded, so we may replace X_L by $X_L \cap Z \times R$, containing the graph of f. We call it X_L again.

Using the restricted order, we can partition X_L, definably in $\mathcal{M}$, into finitely many graphs of functions $g_1, g_2, \ldots, g_k$, $k \leq m$. For instance, we let

$$g_1(x) = \min\{y \in [L - a_0, L + a_0] : \langle x, y\rangle \in X_L\}$$

and continue similarly to obtain the other g_i's. For x large enough, the function f is one of those g_i's, and therefore it is $\mathcal{M}$-definable. Using Proposition 4.18 we get that f is eventually constant. □

Case (ii): $\lim_{x\to+\infty} f(x) = +\infty$.

We recall the proof of Lemma 4.21, and consider three cases: $v(f) > v(x)$, $v(f) < v(x)$ and $v(f) = v(x)$ (remembering though that we do not know yet that f is an $\mathcal{M}$-definable function).

Assume first that $v(f) > v(x)$. By Fact 4.20, $f(x + 1) - f(x) \to 0$, as $x \to \infty$. We want to capture $\Delta(f) = f(x + 1) - f(x)$ within an $\mathcal{M}$-definable set.

The formula

$$\varphi(x, y) := \exists z_1 \exists z_2\, (\Psi(x + 1, z_1) \wedge \Psi(x, z_2) \wedge (y = z_1 - z_2))$$

defines in $\mathcal{M}$ a new subset of R^2—call it $\Delta(X)$—which contains the graph of $\Delta(f)$ (but possibly more functions).

We first note that $\dim(\Delta(X)) = 1$. Indeed, for $a \in R$, $\Delta(X)_a$ is infinite if either X_a or X_{a+1} is infinite. Since only finitely many X_a's are infinite the same is true for $\Delta(X)$. Thus, the graph of $\Delta(f)$ is contained in the one-dimensional $\mathcal{M}$-definable set $\Delta(X)$, so by Claim 4.26, $\Delta(f)$ must be eventually constant, implying that f is eventually affine.

Assume now that $v(f) < v(x)$. The formula $\Upsilon(x, y) := \Psi(y, x)$ defines in $\mathcal{M}$ a new set X^{-1} containing the graph of f^{-1} (a partial function). The graph of f^{-1} is still contained in X^{-1} and we have $v(f^{-1}) > v(x)$. Thus, applying the case we already handled, we see that f^{-1}, and hence also f is eventually affine.

We are left with the case $v(f) = v(x)$. Using Fact 4.20(3), the function $\Delta(f)$ tends to a constant. Thus, as above, we may use the $\mathcal{M}$-definable set $\Delta(X)$ to deduce that $\Delta(f)$ is eventually constant and thus f is eventually affine.

So far we handled all cases where the unbounded cell in X is the graph of some function on a ray (a, ∞). The same reasoning applies to rays $(-\infty, a)$. Applying this reasoning to X^{-1}, we obtain in addition those functions which are eventually constant in X^{-1}, namely sets of the form $\{d\} \times R$ whose intersection with X is co-unbounded in $\{d\} \times R$. The set of all such d is clearly definable over A.

To summarize, we showed that every unbounded cell in X is either contained in the graph of an eventually affine function f definable in $\mathcal{M}$, or in $\{d\} \times R$ for some d. By Proposition 4.24, the function f has the form $\lambda(x) + d$ for $\lambda \in \Lambda_{\mathcal{M}}$. Thus, we have $\lambda_1, \ldots, \lambda_k \in \Lambda_{\mathcal{M}}$, and for each such $i = 1, \ldots, k$, the set D_i of $d \in R$ such that $\Gamma(\lambda_i + d) \cap X$ is unbounded, is $\mathcal{M}$-definable over A, and must be finite. For every such d, $\Gamma(\lambda_i + d) \setminus X$ is bounded.

The above proof handles all unbounded cells, so the set

$$X \setminus \left(\bigcup_{i=1}^{r} \bigcup_{d \in D_i} \Gamma(\lambda_i + d) \cup \bigcup_{d \in D} \{d\} \times R \right)$$

is bounded. $\square$

The algebraic closure and definable closure in strongly bounded structures. Even though the full ordering on R is not definable, we can still prove:

Theorem 4.27. *The algebraic closure in $\mathcal{M}$ equals the definable closure. Moreover, if $a \in \mathrm{acl}_{\mathcal{M}}(\bar{b})$ then a is in the $\mathcal{L}_{\mathrm{bd}}$-definable closure of $\bar{b}$.*

Proof. We use acl, dcl and $\mathrm{acl}_{\mathrm{bd}}$, $\mathrm{dcl}_{\mathrm{bd}}$ to denote the corresponding operations in $\mathcal{M}$ and $\mathcal{M}_{\mathrm{bd}}$, respectively. We prove by induction on n that if $a \in \mathrm{acl}(b_1, \ldots, b_n)$, for some $a, b_i \in R$, then $a \in \mathrm{dcl}_{\mathrm{bd}}(b_1, \ldots b_n)$.

We first handle the case $n = 0$, namely $a \in \mathrm{acl}(\varnothing)$. In this case, there is a finite $\varnothing$-definable set $A \subseteq M$ such that $a \in A$. Viewing the set A in $\mathcal{R}_{\mathrm{omin}}$, we can order the elements $a_1 < \cdots < a_n$. The interval (a_1, a_n) is a $\varnothing$-interval, and $< \restriction (a_1, a_n)$ is $\mathcal{M}_{\mathrm{bd}}$-definable over $\varnothing$, so each a_i is in $\mathrm{dcl}_{\mathrm{bd}}(\varnothing)$.

We proceed by induction, and assume that we proved the result for $n-1$. Assume now that $a \in \mathrm{acl}(b_1, \ldots, b_{n-1}, b_n)$. Let $X \subseteq R^{n+1}$ be a $\varnothing$-definable set such that

$\langle b_1, \dots, b_n, a\rangle$ and $X_{b_1,\dots,b_n}$ has size m. Without loss of generality, for every b'_n, the set $X_{b_1,\dots,b_{n-1},b'_n}$ has size m.

Let $b' = (b_1, \dots, b_{n-1})$ and consider the set $X_{b'} = \{\langle x, y\rangle \in R^2 : \langle b', x, y\rangle \in X\}$. By our assumption, $\dim(X_{b'}) \leq 1$ and $\langle b_n, a\rangle \in X_{b'}$.

We now apply Theorem 4.25. We obtain finitely many $\varnothing$-definable endomorphisms $\lambda_1, \dots, \lambda_k \in \Lambda_{\mathcal{M}}$ and for each $i = 1, \dots, k$, we have a b'-definable finite set A_i such that

$$X_{b'}^{\mathrm{bd}} = X_{b'} \setminus \left(\bigcup_{i=1}^{k} \bigcup_{d\in A_i} \Gamma(\lambda_i + d)\right)$$

is bounded in R^2.

Since $|b'| = n-1$, it follows by induction that every $d \in A_i$ is in $\mathrm{dcl}_{\mathrm{bd}}(b')$. Assume first that $\langle b_n, a\rangle$ is in the graph of one of the $\lambda_i + d$, $d \in A_i$, namely $a = \lambda_i(b_n) + d$. Because λ_i is $\varnothing$-definable and $d \in \mathrm{dcl}_{\mathrm{bd}}(b')$ it follows that $a \in \mathrm{dcl}_{\mathrm{bd}}(b_1, \dots, b_n)$.

We are left with the case $\langle b_n, a\rangle \in X_{b'}^{\mathrm{bd}}$. The set $X_{b'}^{\mathrm{bd}}$ is b'-definable so we may assume that $X_{b'} = X_{b'}^{\mathrm{bd}}$ is bounded (but possibly not $\varnothing$-bounded). Let π_1, π_2 be the projection of $X_{b'}$ onto the first and second coordinates. Each of these is a finite union of points and pairwise disjoint bounded open intervals. Let

$$\pi_1(X_{b'}) = F_1 \cup \bigcup_{i=1}^{k} (a_i, b_i) \quad \text{for } F_1 \text{ finite and } a_1 < b_1 < \cdots < a_k < b_k,$$

and

$$\pi_2(X_{b'}) = F_2 \cup \bigcup_{j=1}^{r} (c_j, d_j) \quad \text{for } F_2 \text{ finite and } c_1 < d_1 < \cdots < c_r < d_r.$$

By Theorem 4.9, there is a fixed $K \in \mathrm{dcl}(\varnothing)$ such that for all $i, = 1, \dots, k$ and $j = 1, \dots, r$, we have $b_i - a_i, d_j - c_j \leq K$.

By Lemma 4.8, the sets $\{a_i\}, \{b_i\}, \{c_j\}, \{d_j\}$ are all finite and $\mathcal{M}$-definable over b', and thus, by induction each of these endpoints is in $\mathrm{dcl}_{\mathrm{bd}}(b')$. Assume that $\langle b_n, a\rangle \in X \cap (a_i, b_i) \times (c_j, d_j)$ for some $i = 1, \dots, k$ and $j = 1, \dots, r$. We replace X by the b'-definable set $X_1 = X - \langle a_i, c_j\rangle \cap (0, b_i - a_i) \times (0, d_j - c_j) \subseteq (0, K)^2$. Notice that $\langle b_n - a_i, a - c_j\rangle \in X'$, and the fiber in X' over $b_n - a_i$ is finite. Because the ordering on $(0, K)$ is $\mathcal{M}_{\mathrm{bd}}$-definable over $\varnothing$, we have $a - c_j \in \mathrm{dcl}_{\mathrm{bd}}(b', b_n - a_i)$, but since $a_i, c_j \in \mathrm{dcl}_{\mathrm{bd}}(b')$ we have $a \in \mathrm{dcl}_{\mathrm{bd}}(b', b_n)$. This ends the proof that $\mathrm{acl} = \mathrm{dcl}_{\mathrm{bd}}$ in $\mathcal{M}$. □

Definable subsets of R^n. We are now ready to prove the main theorem, under the assumptions outlined on p. 387.

Theorem 4.28. *If $X \subseteq R^n$ is $\mathcal{M}$-definable over $A \subseteq R$ then X is definable in $\mathcal{M}_{\mathrm{bd}}$ over A.*

Proof. It is sufficient to prove the result in $\mathcal{N} \succ \mathcal{M}$, so by replacing $\mathcal{R}_{\text{omin}}$ (thus also its reducts) by a sufficiently saturated extension, we may assume that $\mathcal{M}$ is ω-saturated.

We prove the result by induction on n. For $X \subseteq R$, the set X is either bounded or cobounded, so we may assume that it is bounded. Thus, it can be written as a disjoint union

$$(a_1, b_1) \cup \cdots \cup (a_n, b_n) \cup F,$$

with $a_1 < b_1 < \cdots < a_n < b_n$ and F finite. By Lemma 4.8, each a_i and b_i is in $\operatorname{acl}_{\mathcal{M}}(A)$, so by Theorem 4.27, it belong to $\operatorname{dcl}_{\text{bd}}(A)$. Similarly, $F \subseteq \operatorname{dcl}_{\text{bd}}(A)$. By Theorem 4.9, there is $K \in \operatorname{dcl}_{\text{bd}}(\varnothing)$ such that all intervals (a_i, b_i) are of length at most K. But then each interval $(0, b_i - a_i)$ is contained in a $\varnothing$-interval, hence definable in $\mathcal{M}_{\text{bd}}$ over A, so also (a_i, b_i) is $\mathcal{M}_{\text{bd}}$-definable over A. It follows that X is definable in $\mathcal{M}_{\text{bd}}$.

We now use induction on n. Given $X \subseteq R^{n+1}$ that is $\mathcal{M}$-definable over A, we consider, for each $t \in R^n$, the set

$$X_t = \{b \in R : \langle t, b \rangle \in X\} \subseteq R.$$

By the case $n = 1$, each X_t is $\mathcal{M}_{\text{bd}}$-definable over At. Thus, by compactness and saturation, we can find $\mathcal{L}_{\text{bd}}$-formulas over A, $\varphi_1(t, x), \ldots, \varphi_k(t, x)$ such that for every $t \in R^n$, one of the $\varphi_i(t, x)$ defines X_t. Let

$$T_i = \Big\{t \in R^n : \exists x \,\big(\langle t, x \rangle \in X \wedge \forall x\, (x \in X_t \leftrightarrow \varphi_i(t, x))\big)\Big\}.$$

The set T_i is $\mathcal{M}$-definable, over A, and thus, by induction, it is $\mathcal{M}_{\text{bd}}$-definable over A by some $\psi_i(t)$. The formula $\varphi_i(t, x) \wedge \psi_i(t)$ defines $X \cap T_i \times R$, so X is definable in $\mathcal{M}_{\text{bd}}$ over A. □

A comment on failure of definable choice in strongly bounded $\mathcal{M}$. Recall that a structure $\mathcal{M}$ has definable choice if for every definable family $\{X_t : t \in T\}$ of sets, there is a definable function $f : T \to \bigcup X_t$ such that $f(t) \in X_t$ and if $t_1 = t_2$ then $f(t_1) = f(t_2)$. Equivalently, every definable equivalence relation has a definable set of representatives. This fails in strongly bounded $\mathcal{M}$, because the relation $xEy \Leftrightarrow y = -x$ on R cannot have a definable set of representatives. If it did then it would contain either a positive or a negative ray (without its inverse).

We believe that elimination of imaginaries similarly fails.

5. Conclusion: The proof of Theorem 1.2

We are now ready to collect the results proved thus far in order to prove Theorem 1.2.

Recall that we want to prove that the only reducts between $\mathcal{R}_{\text{lin}}$ and $\mathcal{R}_{\text{alg}}$ are as follows:

$$\mathcal{R}_{\mathrm{alg}} = \langle R; +, \cdot, < \rangle,$$
$$\mathcal{R}_{\mathrm{sb}} = \langle R; +, <, \Lambda_R, \mathfrak{B} \rangle,$$
$$\mathcal{R}_{\mathrm{semi}} = \langle R; +, <, \Lambda_R \rangle, \qquad \mathcal{R}_{\mathrm{bd}} = \langle R; +, <^*, \Lambda_R, \mathfrak{B} \rangle,$$
$$\mathcal{R}^*_{\mathrm{lin}} = \langle R; +, <^*, \Lambda_R \rangle,$$
$$\mathcal{R}_{\mathrm{lin}} = \langle R; +, \Lambda_R \rangle.$$

First, we note that using [Edmundo 2000] we can generalize [Peterzil 1993, Theorem 1.1] from $\mathbb{R}$ to arbitrary real closed fields, and show:

Fact 5.1. *Let R be a real closed field. The only reduct between $\mathcal{R}_{\mathrm{semi}}$ and $\mathcal{R}_{\mathrm{alg}}$ is $\mathcal{R}_{\mathrm{sb}}$.*

Proof. Assume that $\mathcal{M}$ is a reduct of R_{alg} which properly expands $\mathcal{R}_{\mathrm{semi}}$. By [Edmundo 2000, Fact 1.6], either $\mathcal{M}$ is a reduct of $\mathcal{R}_{\mathrm{sb}}$ or a real closed field $F = \langle R; \oplus, \odot \rangle$ whose universe R is definable in $\mathcal{M}$. Assume the latter, and then since the field is semialgebraic then, again by [Peterzil 1993, Corollary 2.4], every semialgebraic subset of R is definable in F and hence in $\mathcal{M}$. Thus, $\mathcal{M} \doteq \mathcal{R}_{\mathrm{alg}}$.

If $\mathcal{M}$ is a reduct of $\mathcal{R}_{\mathrm{sb}}$ which is not semilinear then by Theorem 3.2, every bounded R-semialgebraic set is definable in $\mathcal{M}$, and thus $\mathcal{M} \doteq \mathcal{R}_{\mathrm{sb}}$. $\square$

We now consider an arbitrary reduct $\mathcal{M}$ of $\mathcal{R}_{\mathrm{alg}}$. Our goal is to show that $\mathcal{M}$ is one of the reducts in the above list.

First, if $\mathcal{M}$ is stable then by Claim 2.2, $\mathcal{R}_{\mathrm{lin}} \doteq \mathcal{M}$. If $\mathcal{M}$ is unstable then by Theorem 2.1, $<^*$ is definable in $\mathcal{M}$. So $\mathcal{R}^*_{\mathrm{lin}} \dot{\subseteq} \mathcal{M}$. So, we may assume that $<^*$ is definable in $\mathcal{M}$, i.e., $\mathcal{R}^*_{\mathrm{lin}} \dot{\subseteq} \mathcal{M}$.

Case 1: $\mathcal{M}$ is strongly bounded and $\mathcal{M} \dot{\subseteq} \mathcal{R}_{\mathrm{semi}}$.

We claim that $\mathcal{M} \doteq \mathcal{R}^*_{\mathrm{lin}}$. Indeed, because $\mathcal{M}$ is strongly bounded then, by Theorem 4.5, $\mathcal{M} \doteq \mathcal{M}_{\mathrm{bd}}$. Because $\mathcal{M} \dot{\subseteq} \mathcal{R}_{\mathrm{semi}}$, every $\mathcal{M}$-definable set is semilinear, and in particular this is true for each of the $\varnothing$-bounded sets in $\mathcal{M}_{\mathrm{bd}}$. However, it is easy to verify that every bounded semilinear set is definable in $\mathcal{R}^*_{\mathrm{lin}}$, so the whole structure $\mathcal{M}_{\mathrm{bd}}$ is a reduct of $\mathcal{R}^*_{\mathrm{lin}}$, and thus so is $\mathcal{M}$ as well. The converse $\mathcal{R}^*_{\mathrm{lin}} \dot{\subseteq} \mathcal{M}$ is already assumed.

Case 2: $\mathcal{M}$ is strongly bounded and $\mathcal{M} \dot{\nsubseteq} \mathcal{R}_{\mathrm{semi}}$.

We claim that $\mathcal{M} \doteq \mathcal{R}_{\mathrm{bd}}$. As in Case 1, every $\mathcal{M}$-definable set is definable in $\mathcal{M}_{\mathrm{bd}}$. Because $\mathcal{M}$ is a reduct of $\mathcal{R}_{\mathrm{alg}}$ then $\mathcal{M}_{\mathrm{bd}}$ is a reduct of $\mathcal{R}_{\mathrm{bd}}$ and so $\mathcal{M} \dot{\subseteq} \mathcal{R}_{\mathrm{bd}}$. By the assumption that $\mathcal{M} \dot{\nsubseteq} \mathcal{R}_{\mathrm{semi}}$, we know that there is an $\mathcal{M}$-definable semialgebraic set which is not semilinear, so by Theorem 3.2, we get that every bounded semialgebraic set is definable in $\mathcal{M}$, hence $\mathcal{R}_{\mathrm{bd}} \dot{\subseteq} \mathcal{M}$.

Next we assume that $\mathcal{M}$ is *not* strongly bounded.

Case 3: $\mathcal{M}$ is not strongly bounded and $\mathcal{M} \dot{\subsetneq} \mathcal{R}_{\text{semi}}$.

By Lemma 2.3, the linear order $<$ is definable in $\mathcal{M}$, so, since $\mathcal{R}^*_{\text{semi}} \dot{\subseteq} \mathcal{M}$, we have $\mathcal{R}_{\text{semi}} \doteq \mathcal{M}$.

Case 4: $\mathcal{M}$ is not strongly bounded and $\mathcal{M} \dot{\nsubseteq} \mathcal{R}_{\text{semi}}$.

As in Case 3, the linear order $<$ is definable in $\mathcal{M}$, so $\mathcal{R}_{\text{semi}} \dot{\subseteq} \mathcal{M}$. So we know that $\mathcal{M}$ is a reduct of $\mathcal{R}_{\text{alg}}$ which properly expands $\mathcal{R}_{\text{semi}}$. By Fact 5.1, either $\mathcal{M} \doteq \mathcal{R}_{\text{alg}}$ or $\mathcal{M} \doteq \mathcal{R}_{\text{bd}}$.

This completes the proof that if $\mathcal{M}$ is a reduct of $\mathcal{R}_{\text{alg}}$ expanding $\mathcal{R}_{\text{lin}}$, then it is one of the reducts in the above diagram.

It is left to see that all reducts in the above diagram are distinct. Because $\mathcal{R}_{\text{lin}}$ is stable and $\mathcal{R}^*_{\text{lin}}$ is unstable, these two are distinct. Also, the fact that $\mathcal{R}^*_{\text{lin}}$ and $\mathcal{R}_{\text{bd}}$ are distinct is easy to verify (e.g., the unit circle is definable in $\mathcal{R}_{\text{bd}}$ but not in $\mathcal{R}^*_{\text{lin}}$). The fact that $\mathcal{R}_{\text{bd}}$ is different than $\mathcal{R}_{\text{sb}}$ and $\mathcal{R}_{\text{semi}}$ follows from the next lemma.

Lemma 5.2. *Let R be a real closed field. If $\mathcal{B}^*$ is any collection of bounded subsets of R^n, $n \in \mathbb{N}$, then $<$ is not definable in $\mathcal{M} = \langle R; +, \Lambda_R, \mathcal{B}^* \rangle$.*

Proof. We use a similar idea to [Peterzil 1992] Assume towards a contradiction that $<$ is definable in $\mathcal{M}$, and let $\mathcal{N} = \langle R : +, <, \Lambda_R, \mathfrak{B}^* \rangle$.

Let $\psi(x, y, \bar{a})$, $\bar{a} \in R$ be the $\mathcal{M}$-formula that defines $<$. Namely,

$$\mathcal{N} \models \forall x \forall y \, (\psi(x, y, \bar{a}) \leftrightarrow x < y).$$

Let $\widetilde{\mathcal{N}} = \langle \widetilde{R}; +, <, \Lambda_R, \mathfrak{B}^* \rangle \succ \mathcal{N}$ be an $|N|^+$-saturated elementary extension whose reduct to the $\mathcal{M}$-language is $\widetilde{\mathcal{M}}$. It follows that $\psi(x, y, \bar{a})$ defines $<$ in $\widetilde{\mathcal{N}}$ as well.

We show that there is an automorphism of $\widetilde{\mathcal{M}}$ which fixes $\bar{a}$, thus leaving $\psi(\widetilde{R} \times \widetilde{R}, \bar{a})$ invariant, and yet not respecting $<$, leading to a contradiction.

The group $\langle \widetilde{R}, + \rangle$ is a vector space over R. We define an R-vector subspace of $\widetilde{R}$ by

$$\mathcal{A} = \{x \in \widetilde{R} : \exists \alpha \in R \, (|x| < \lambda_\alpha(1))\}.$$

So, by Zorn's lemma, there exists an R-vector space $V \subseteq \widetilde{R}$ such that $\widetilde{R} = \mathcal{A} \oplus V$, and by the saturation assumption, V is nontrivial. Now we define the following automorphism of the R-vector space $\widetilde{R}$: on $\mathcal{A}$ we define $\tau_1(v) = v$, on $\langle V, + \rangle$ we define $\tau_2(v) = -v$, and we let $\tau : \widetilde{R} \to \widetilde{R}$ be

$$\tau(v_1 + v_2) = \tau_1(v_1) + \tau_2(v_2) = v_1 - v_2.$$

This automorphism fixes all elements in $\mathcal{A}$ and in particular fixes all sets in $\mathfrak{B}^*$ pointwise, but does not respect $<$ (as positive elements in V are sent to negative ones). In model theoretic language τ is an automorphism of the structure $\widetilde{\mathcal{M}}$ which fixes $\bar{a}$ (since $\bar{a} \in \mathcal{A}$). However, τ does not preserve $<$, contradiction. □

This completes the proof of Theorem 1.2. □

Appendix: The proof of Fact 3.1

Fact 3.1. *Let R be a real closed field and $X \subseteq R^n$ a definable set in an o-minimal expansion of $\langle R; <, +, \cdot \rangle$. If X is not definable in $\mathcal{R}_{\text{semi}}$ then, in the structure $\mathcal{M} = \langle R; <^*, +, \Lambda_R, X \rangle$ there exists a definable bounded set which is not definable in $\mathcal{R}_{\text{semi}}$.*

Proof. We believe that this is known so we shall be brief. We prove the result by induction on $\dim(X)$, where the case $\dim X = 0$ is trivially true. Consider the affine part of X, $\mathscr{A}(X)$, which is definable in $\mathcal{M}$.

Assume first $\mathscr{A}(X)$ is not dense in X. Then there is an open box $U \subseteq R^n$ such that $U \cap X \neq \varnothing$ and $U \cap \mathscr{A}(X) = \varnothing$. We claim that $U \cap X$ is not semilinear. Indeed, if it were then $\mathscr{A}(U \cap X)$ must be nonempty, but because $U \cap X$ is relatively open in X then

$$\mathscr{A}(U \cap X) = U \cap \mathscr{A}(X) = \varnothing,$$

a contradiction.

Thus, $U \cap X$ above is not semilinear. and this gives the desired box when $\mathscr{A}(X)$ is not dense in X.

We assume then that $\mathscr{A}(X)$ is dense in X, and consider two cases: $\mathscr{A}(X)$ is either semilinear or not. If it were semilinear then necessarily $X \setminus \mathscr{A}(X)$ is not semilinear, and because of the density assumption, $\dim(X \setminus \mathscr{A}(X)) < \dim(X)$ and we can finish by induction.

Thus, we are left with the case that $\mathscr{A}(X)$ is not semilinear. For simplicity, we may assume now that $X = \mathscr{A}(X)$. We recall the $\mathcal{M}$-definable relation $a \sim b$ from the proof of Proposition 4.24, defined by letting $a \sim b$ if X has the same germ at a and b, up to translation.

Because $X = \mathscr{A}(X)$, each $\sim$-class is open in X, and thus there are finitely many classes, at least one of which is not semilinear. Thus, we may assume that $X = \mathscr{A}(X)$ consists of a single $\sim$-class. It follows that there is some R-subspace $L \subseteq R^n$, $\dim L = \dim X$, such that X is contained in a finite union of cosets of L. Thus each definably connected component of X is contained in a single such coset of L.

Each L is definable in $\mathcal{M}$ using Λ_R, so the intersection of X with each of these cosets is definable in $\mathcal{M}$. One of these intersections is not semilinear, so we may assume that $X \subseteq c + L$ for some c. Because $\dim X = \dim L$, and $\mathscr{A}(X) = X$, then X is open in $c + L$. We claim that $\operatorname{Fr}(X) \subseteq c + L$ is not semilinear: Indeed, $\operatorname{Fr}(X)$ is a closed subset of $c + L$, and X consists of finitely many components of $c + L \setminus \operatorname{Fr}(X)$. If $\operatorname{Fr}(X)$ were semilinear then each of its components would also be, so X would be semilinear.

Thus, $\operatorname{Fr}(X)$ is not semilinear, and definable in $\mathcal{M}$. By o-minimality,

$$\dim(\operatorname{Fr}(X)) < \dim(X).$$

Therefore, by induction we may find an $\mathcal{M}$-definable bounded set which is not semilinear. □

In fact, a stronger result is true: If $X \subseteq R^n$ is definable in an o-minimal expansion of the field R and not semilinear, then there is some bounded open box $U \subseteq R^n$ such that $U \cap X$ is not semilinear (we omit the proof here as we do not need it). Notice that this last statement fails if we replace "not semilinear" by "not semialgebraic", as Rolin's example from [Le Gal and Rolin 2009] shows: There exists a definable function $f : \mathbb{R} \to \mathbb{R}$ in an o-minimal expansion of the real field such that the restriction of f to every bounded interval is semialgebraic but f itself is not semialgebraic.

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