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of his theorem on finite subgroups of  $GL_n(\mathbb{C})$**

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# An exposition of Jordan's original proof of his theorem on finite subgroups of $GL_n(\mathbb{C})$

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*Dedicated to Udi Hrushovski on the occasion of his 60th birthday.*

We discuss Jordan's theorem on finite subgroups of invertible matrices and give an account of his original proof.

## 1. Introduction

In 1878 Camille Jordan [29] proved the following theorem:

**Theorem 1.1** (Jordan's theorem). *Let  $G$  be a finite subgroup of  $GL_n(\mathbb{C})$ . Then there is a normal abelian subgroup  $A$  in  $G$  of index bounded by a constant  $J(n)$  depending on  $n$  only.*

It is the purpose of this note to provide an account of Jordan's original proof of his result. Jordan's proof is purely algebraic, and quite different from the proofs found in most textbooks (such as [19] or [21]) that are based on a geometric argument due to Bieberbach [3]. Jordan's proof does not appear to have been discussed much elsewhere (with the exception of Dieudonné's notes in Jordan's collected works [20]) even as this year marks the hundredth anniversary of Jordan's death.

Jordan's motivation for proving this result came from the study of linear differential equations of order  $n$  with rational functions as coefficients and with algebraic solutions; in this context finite subgroups of  $GL_n$  arise naturally as monodromy groups and information such as Theorem 1.1 on the monodromy group translates immediately into structural properties for the solutions of the equation.<sup>1</sup> Prior to Jordan, Fuchs and Klein had studied the two dimensional case and Klein had given a complete list of finite subgroups of  $GL_2(\mathbb{C})$ . Jordan announced his result in [28], published it in [29] and later wrote a second article [30] to clarify his proof.

Jordan argued by induction on the dimension, but he gave no explicit bound on  $J(n)$  in his article, not even an inductive one. It is therefore understandable

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<sup>1</sup>For the full story of the motivations and context in which Jordan's theorem was proven, we refer the reader to the wonderful book by Jeremy Gray [24].

that mathematicians sought to find explicit bounds closer to the truth and this topic has been quite active in the last 146 years. Indeed, after Jordan's memoir, several authors gave new proofs of his theorem. The first of these appears to be Blichfeldt, who gave an entirely different proof of Jordan's result via the study of the  $p$ -Sylow subgroups, for which he established explicit bounds on their size in terms of  $p$  and  $n$  (see [4; 5; 6; 7; 35]). Subsequently Bieberbach [3] came up with yet another very different and purely geometric argument, which was later refined by Frobenius [23]. This third proof is much slicker and it is the one that can be found, with some variants, in most textbooks that treat the question, such as [19, Chapter V; 41, Chapter 8]. Blichfeldt himself later combined it with his previous approach to improve his bounds on  $J(n)$ ; see [35] and [46; 26; 22].

Bieberbach's argument starts with what people refer to nowadays as Weyl's unitary trick (i.e., the observation that a finite, or compact, subgroup  $G$  of  $\mathrm{GL}_n(\mathbb{C})$  can be conjugated inside the compact unitary group  $\mathbb{U}_n(\mathbb{C})$  by averaging a hermitian product over  $G$ ). Then one makes use of a volume packing argument in combination with the *commutator shrinking property* of Lie groups, i.e., the fact that commutators of elements close to the identity in  $\mathbb{U}_n(\mathbb{C})$  are themselves close, and in fact much closer, to the identity. This commutator shrinking property has inspired several other authors [50; 10; 31; 1] and is nowadays a crucial tool in the study of discrete subgroups of Lie groups and in Riemannian geometry. We refer the reader to [42, Theorem A] or [14, §2] for a proof of Jordan's theorem via this argument.

Jordan's original proof, on the other hand, was based on a purely algebraic idea, which should be traced back to Klein's method for the classification of the finite subgroups of rotations of the 2-sphere (and isometries of Plato's solids), as described in Klein's famous book on the icosahedron [32]. Basically, one enumerates the elements of  $G$  according to the shape and size of their centralizers and one can thus write a class equation involving the order of  $G$  and of the centralizers of its elements. Inducting on dimension, this yields a diophantine equation of the form

$$\frac{1}{g} = \frac{1}{q_1} + \cdots + \frac{1}{q_k} - \frac{b}{a}, \quad (1-1)$$

where  $g = [G : \Phi]$  is the index of the center  $\Phi$  of  $G$  in  $G$ ,  $a$ ,  $b$  and  $k$  are integers that are bounded in terms of  $n$  only and each  $q_i$  is the cardinality of a certain subgroup of  $G/\Phi$ . It is easy to check that any equation of this form forces  $g$  to be bounded (in terms of  $n$ ) and Jordan then discusses the boundedly many cases that may arise. Although more cumbersome, this method gives potentially much more information on the finite subgroup  $G$ . For example, Jordan used it to list all finite subgroups of  $\mathrm{GL}_3(\mathbb{C})$ , giving an explicit set of generators for each class of groups, after examining some 47 different cases.<sup>2</sup>

<sup>2</sup>In fact Jordan missed some groups; see [4; 5; 6; 7; 20].

With the advent of the classification of finite simple groups, B. Weisfeiler [49] and more recently M. Collins [16] have found tight bounds for  $J(n)$ . For example, Collins proved that if  $n \geq 71$ , then  $J(n)$  can be taken to be  $(n + 1)!$ . This is tight, i.e.,  $(n + 1)!$  is always a lower bound for  $J(n)$ , because the symmetric group on  $n + 1$  letters acts irreducibly on the hyperplane  $\sum_{i=1}^{n+1} x_i = 0$  by permuting the  $n + 1$  coordinates.

Schur [44] extended Jordan's theorem, proving that it holds assuming only that the group  $G$  is torsion (i.e., every element has finite order). In particular, every finitely generated torsion subgroup of  $GL_n(\mathbb{C})$  is finite. This is sometimes called the Jordan–Schur theorem; see [47; 19].

In another direction initiated by Brauer and Feit [13], Larsen and Pink [34] gave a vast generalization of Jordan's theorem to finite linear groups in characteristic  $p$ , which avoids the classification of finite simple groups. Interestingly enough, part of their proof is very much akin to Jordan's original argument. See also [11] for a recent use of theorems of Jordan and Larsen–Pink type in the study of finite group actions on elementary abelian  $p$ -groups with finite Morley rank.

Finally, we mention that there are nonlinear analogues of Jordan's theorem for finite subgroups of homeomorphisms of manifolds (conjectured by E. Ghys) and for finite subgroups of birational automorphisms of algebraic varieties. In these cases, it has recently been shown that there is a nilpotent subgroup of index bounded only in terms of the dimension of the manifold or variety. But “nilpotent” cannot be replaced by “abelian”. We refer the reader to the preprints [18; 25] and references therein for these exciting recent developments.

This paper is organized as follows: in Section 2 we define the notion of  $M$ -fan and state the version of Theorem 1.1 that will be used as induction hypothesis. In Sections 3 and 4 we write the corresponding class equation and complete Jordan's proof of Theorem 1.1. In Section 5 we give an illustration of the method by specializing to the case of  $n = 2$  and we derive the classical results of Klein for finite subgroups of  $SO_3(\mathbb{R})$ . In Section 6 we discuss a nonstandard treatment of Jordan's proof, which is very close to Jordan's original formulation of his proof, and in the last section we briefly survey bounds for  $J(n)$  from a historical perspective.

## 2. A reformulation of Theorem 1.1

As we will see below, Jordan's argument uses nothing about the field  $\mathbb{C}$  and in fact his proof carries over to an arbitrary field provided we assume that every element of  $G$  is semisimple, i.e., diagonalizable in some field extension. So we let  $K$  be an arbitrary field, which we assume algebraically closed without loss of generality.

Let us first reformulate Theorem 1.1 in the form originally proved by Jordan. For this we need to introduce a couple of definitions.

**Definition 2.1.** By a *root torus*, we mean a subgroup of  $\mathrm{GL}_n(K)$  which is conjugate to a subgroup of the diagonal matrices defined by a set of equalities between the diagonal entries.

For example, the subgroup of diagonal matrices

$$\{g = \mathrm{diag}(a_1, \dots, a_6) \mid a_i \in K^*, a_1 = a_2, a_5 = a_6\}$$

is a root torus of  $\mathrm{GL}_6(K)$ .

**Definition 2.2.** Let  $G$  be a finite subgroup of  $\mathrm{GL}_n(K)$ . Given  $M \geq 2$ , we say that a subgroup  $F$  of  $G$  is an  $M$ -fan if it is conjugate to a subgroup of the diagonal matrices  $\{g = \mathrm{diag}(a_1, \dots, a_n) \mid a_i \in K^*\}$  such that for every pair of indices  $i, j$  the set of ratios  $a_i(g)/a_j(g)$  is either reduced to  $\{1\}$  or achieves at least  $M$  distinct values as  $g$  varies in  $F$ .

The terminology *fan* is a liberal translation of Jordan's *faisceau*.<sup>3</sup> Note that the subgroup  $\Phi$  of all scalar matrices in  $G$  is clearly an  $M$ -fan, for any  $M \geq 2$ .

Note that every  $M$ -fan  $F$  is contained in a unique minimal root torus  $S_F$  defined by the same equalities between diagonal elements, such as  $a_i = a_j$ , as those that hold in  $F$ . In particular  $G \cap S_F$  is itself an  $M$ -fan and every maximal  $M$ -fan in  $G$  has this form.

We can now state an alternative, slightly more precise, version of Theorem 1.1:

**Theorem 2.3** (Jordan's theorem, second form). *Given  $n \in \mathbb{N}$ , there are constants  $M = M(n)$ ,  $N = N(n) \geq 1$  such that the following holds. Let  $K$  be an algebraically closed field, and let  $G$  be a finite subgroup of  $\mathrm{GL}_n(K)$  such that every element of  $G$  is diagonalizable. Then  $G$  contains a unique maximal  $M$ -fan. Call it  $\mathcal{F}$ . We have  $[G : \mathcal{F}] \leq N$ .*

The proof of Theorem 2.3 spans the next two sections. Before we start, a number of simple remarks are in order:

- (1) Since  $\mathcal{F}$  is unique, it must be normal in  $G$ .
- (2) To see that Theorem 2.3 implies Theorem 1.1, it only remains to check that if  $K = \mathbb{C}$ , then every element of  $G$  is diagonalizable. This is indeed the case, because every element of  $G$  has finite order and is thus diagonalizable over  $\mathbb{C}$ .
- (3) Although we prove the result in any characteristic, it is worth mentioning that the case of positive characteristic follows from the case when  $K = \mathbb{C}$ , because if  $G$  is as in Theorem 2.3, then  $|G|$  is prime to  $\mathrm{char}(K)$  and thus  $G$  admits an embedding in  $\mathrm{GL}_d(\mathbb{C})$ . See for instance [39, Proof of Theorem C] or [21, Theorem 3.8].

<sup>3</sup>We are grateful to the referee for suggesting this translation. In fact the word *faisceau* is used throughout Jordan's other works to mean sometimes "subgroup", sometimes "abelian subgroup".

- (4) The proof of Theorem 2.3 proceeds by induction on the dimension. The letter  $F$  denotes a fan and we reserve the letter  $\mathcal{F}$  for maximal fans.
- (5) Since  $S_{\mathcal{F}}$  is normalized by  $G$ ,  $G$  must permute the eigenspaces of  $S_{\mathcal{F}}$ . So if  $G$  acts primitively on  $K^n$  (i.e., does not permute the components of any nontrivial direct sum decomposition of  $K^n$ ), then  $S_{\mathcal{F}}$  must be reduced to scalar matrices and those have bounded index in  $G$ .
- (6) If  $g \in GL_n(K)$  normalizes  $\mathcal{F}$ , then it must normalize the root torus  $S_{\mathcal{F}}$  too. In particular  $G$  lies in the normalizer of a root torus  $S_{\mathcal{F}}$  and  $[G : G \cap S_{\mathcal{F}}] \leq N$ .
- (7) The abelian normal subgroup  $A$  in Theorem 1.1 can be taken to be *characteristic* in  $G$ . A theorem of Chermak–Delgado [27, Theorem 1.41] asserts that in any finite group  $G$  with an abelian subgroup of index  $i$ , there is an abelian characteristic subgroup of index at most  $i^2$ . So we could make  $A$  characteristic at the expense of changing  $J(n)$  into  $J(n)^2$ . Another route is to observe that, if  $M > n!$ , the maximal  $M$ -fan  $\mathcal{F}$  in Theorem 2.3 commutes with every normal abelian subgroup of  $G$ . Hence the subgroup generated by all  $\alpha(\mathcal{F})$ ,  $\alpha \in \text{Aut}(G)$ , is abelian, characteristic, and of smaller index.

As seen from items (1) and (2) above, Theorem 2.3 implies Theorem 1.1. It turns out that one can also derive Theorem 2.3 from Theorem 1.1 directly and we explain this in the paragraph below. To be more precise, since we have only stated Theorem 1.1 over  $\mathbb{C}$  while Theorem 2.3 is also valid in positive characteristic, we are going to prove that Theorem 2.3 follows from the assertion that any finite subgroup of  $GL_n(K)$  made of diagonalizable elements admits a normal abelian subgroup of index at most  $J(n)$ . Jordan's original proof goes by proving Theorem 2.3 first, because its formulation is more adequate for the induction scheme.

*Proof of the equivalence of Theorems 1.1 and 2.3.* Assume the conclusion of Theorem 1.1. Since every element of  $G$  is diagonalizable and  $A$  is abelian,  $A$  is simultaneously diagonalizable and  $K^n$  decomposes as a direct sum of weight spaces (i.e., joint eigenspaces) for  $A$ . Since  $A$  is normal in  $G$ , these eigenspaces are permuted by  $G$  and thus  $G$  lies in the normalizer  $N(S)$  of the root torus  $S$  that acts on  $K^n$  by a scalar multiple on each one of the weight spaces of  $A$ . Note that  $A \leq S$ . Moreover, if  $F$  is an  $M$ -fan with  $M > J(n)$  and  $m := [F : F \cap S] \leq [G : G \cap S] \leq J(n)$ , we have  $f^m \in S$  for all  $f \in F$ . Thus  $F \cap S$  is an  $M/m$ -fan lying in  $S$ . Since  $M/m > 1$ , this implies that  $F$  itself lies in  $S$ . Hence every  $M$ -fan is contained in  $S$ . Finally, viewing  $S$  as a diagonal subgroup it is straightforward to check that the subgroup generated by all  $M$ -fans in  $S$  is itself an  $M$ -fan. Hence it is the unique maximal  $M$ -fan in  $G$ . But  $G \cap S$  contains some  $M$ -fan with index at most  $(M - 1)^{n-1}$  as follows by intersecting the kernels of the homomorphisms  $g \mapsto a_i(g)/a_j(g)$  at most  $n - 1$  times. Hence  $[G : \mathcal{F}] \leq J(n)(M - 1)^{n-1}$ . This completes the claims of Theorem 2.3 with  $N = J(n)^n$ ,  $M = J(n) + 1$ . □

### 3. Jordan's fundamental equation

In this section we begin the proof of Theorem 2.3 and obtain Jordan's fundamental equation (3-3) below, which expresses an enumeration of the elements of  $G$  into various classes, which we are about to describe. The proof of Theorem 2.3 will be completed in the next section after a discussion of the fundamental equation.

We proceed by induction on the dimension  $n$ .

If  $n = 1$ , then  $\text{GL}_1(K) = K^*$  is abelian and there is nothing to prove. We now assume the theorem proven for all dimensions  $< n$ .

Observe that, by the argument at the end of the last section, it is enough to establish the conclusion of Theorem 1.1, namely the existence of an abelian normal subgroup of index bounded by some function  $J(n)$ , as this automatically implies the conclusion of Theorem 2.3 with  $N(n) = J(n)^n$  and  $M(n) = J(n) + 1$ .

If  $G$  preserves a direct sum decomposition  $K^n = K^r \oplus K^{n-r}$ , with  $1 < r < n$ , then we may use the induction hypothesis in the obvious way applying it to the projections  $\pi_r(G)$  and  $\pi_{n-r}(G)$  to  $\text{GL}_r(K)$  and  $\text{GL}_{n-r}(K)$ , respectively. The conclusion of Theorem 1.1 then easily follows as soon as  $J(n) \geq \max_{0 < r < n} N(r)N(n-r)$  and that of Theorem 2.3 too as we have just said.

We repeatedly use the last observation for subgroups of  $G$  that preserve such a decomposition. If  $g \in G$  is not a scalar matrix, then the centralizer  $C_G(g)$  preserves the eigenspace decomposition of  $g$  on  $K^n$ . We can therefore apply this observation to  $C_G(g)$  and conclude from the induction hypothesis that  $C_G(g)$  contains a unique maximal  $M$ -fan (for all  $M$  larger than a number depending on  $n$  only). That is:

**Lemma 3.1.** *If  $g \in G$  is not a scalar matrix, then the centralizer  $C_G(g)$  contains a unique maximal  $M$ -fan.*

We can thus set the following definition:

**Definition 3.2.** An element  $g$  is said to be *associated with* an  $M$ -fan  $F$  if  $F$  lies in the centralizer  $C_G(g)$  and is the unique maximal  $M$ -fan of  $C_G(g)$ .

We denote by  $F_g$  the  $M$ -fan associated with  $g$ . This definition makes sense (so far, thanks to the induction hypothesis) as soon as  $g$  is not a scalar matrix in  $\text{GL}_n(K)$  by the remarks above the definition. Note that, by maximality,  $F_g$  must contain the subgroup  $\Phi$  of  $G$  of all scalar matrices in  $G$ . Moreover, setting  $N := N(n-1)^2$ , it follows from the induction hypothesis that

$$[C_G(g) : F_g] \leq N. \quad (3-1)$$

These remarks also have the following three consequences:

**Lemma 3.3.** *If  $F$  is an  $M$ -fan of  $G$  not entirely made of scalar matrices, then  $F$  is contained in a unique maximal  $M$ -fan  $\mathcal{F}$  of  $G$ .*



*Proof.* Let  $f \in F$  be a nonscalar element. If  $F_1$  is an  $M$ -fan containing  $F$ , then  $F_1$  must commute with all elements of  $F$  and thus lie in  $C_G(f)$ , the centralizer of  $f$ . Therefore  $F_1$  must lie in the unique maximal  $M$ -fan of  $C_G(f)$ .  $\square$

Let  $F$  be an  $M$ -fan of  $G$  not contained in the scalar matrices  $\Phi$  and let  $\mathcal{F}$  be the maximal  $M$ -fan of  $G$  containing  $F$ . Since  $\mathcal{F}$  is contained in the centralizer  $C_G(F)$ , it must be the maximal  $M$ -fan there too and, by the induction hypothesis, we must have  $[C_G(F) : \mathcal{F}] \leq N$ .

**Lemma 3.4.** *Suppose  $\Phi \not\leq F \not\leq \mathcal{F}$ . Then the number  $n_F$  of elements of  $G$  associated with  $F$  is divisible by  $|\mathcal{F}|$  and  $n_F/|\mathcal{F}| \leq N$ .*

*Proof.* If  $n_F = 0$  there is nothing to prove, so we assume  $n_F \geq 1$ . Every element associated with  $F$  lies in the centralizer  $C_G(F)$ . Moreover, if  $g \in C_G(F)$  is associated with  $F$  and  $f \in \mathcal{F}$ , then  $gf$  is also associated with  $F$ , i.e.,  $F_{gf} = F_g = F$ . Indeed, since  $F \not\leq \mathcal{F}$  we must have  $gf \notin \Phi$  (as otherwise  $C_G(g) = C_G(f)$  contains  $\mathcal{F}$ ) and by Lemma 3.1 there is a unique maximal  $M$ -fan  $F_{gf}$  in  $C_G(gf)$ . Since  $F \subset C_G(gf)$  we have  $F \subset F_{gf} \subset C_G(gf)$ . Moreover,  $F_{gf}$  is contained in  $\mathcal{F}$  and must therefore commute with  $f$ , and hence also with  $g$ . It follows that  $F \subset F_{gf} \subset C_G(g)$  and  $F = F_{gf}$  by maximality of  $F$ .

Consequently, the set of elements of  $G$  associated with  $F$  is a union of cosets of  $\mathcal{F}$  all lying in  $C_G(F)$ . Since  $C_G(F)$  contains  $\mathcal{F}$  as a subgroup of index at most  $N$  the result follows.  $\square$

And for maximal fans we have:

**Lemma 3.5.** *Let  $\mathcal{F} \neq \Phi$  be a maximal  $M$ -fan in  $G$ . Then the number  $n_{\mathcal{F}}$  of nonscalar elements  $g$  in  $G$  which are associated with  $\mathcal{F}$  is  $n_{\mathcal{F}} = |C_G(\mathcal{F})| - |\Phi|$ , and  $[C_G(\mathcal{F}) : \mathcal{F}] \leq N$ .*

*Proof.* A nonscalar element  $g$  is associated with  $\mathcal{F}$  if and only if  $\mathcal{F} \leq C_G(g)$ , i.e.,  $g \in C_G(\mathcal{F})$ . The bound follows from (3-1).  $\square$

The strategy of Jordan's proof consists in enumerating the elements of  $G$  according to their associated  $M$ -fan. Let  $\Phi$  be the scalar matrices in  $G$ . We may decompose  $G$  as the *disjoint* union

$$G = \Phi \cup_F \{g \mid g \text{ associated with } F\},$$

where the union is taken over fans arising as maximal  $M$ -fans of centralizers of nonscalar elements of  $G$ . We split this union into four disjoint parts,

$$G = \Phi \cup G_1 \cup G_2 \cup G_3,$$

where  $G_1$  is the subset of those  $g$ 's not in  $\Phi$  such that  $F_g = \Phi$ , and  $G_2$  is the subset of those  $g$ 's not in  $\Phi$  such that  $F_g$  contains  $\Phi$  strictly but is not the maximal  $M$ -fan  $\mathcal{F}_g$  which contains it by Lemma 3.3, and finally  $G_3$  is the remaining subset of those

$g$ 's not in  $\Phi$  for which  $F_g$  is not  $\Phi$  and is maximal in  $G$ . We now consider each subset  $G_i$  one after the other.

(1) We first enumerate the elements of  $G_1$ , that is, the  $g$ 's outside  $\Phi$  which are associated with  $\Phi$ . This subset is invariant under conjugation by  $G$ . Also it is clearly a union of cosets of  $\Phi$ , for if  $\phi \in \Phi$ , then  $C_G(g\phi) = C_G(g)$  and thus  $g\phi$  is also associated with  $\Phi$ . It follows that conjugation by  $G$  permutes those  $\Phi$ -cosets.

The stabilizer  $N_G(g\Phi)$  of a  $\Phi$ -coset  $g\Phi$  under the  $G$ -action by conjugation must contain  $C_G(g)$  as a subgroup of index at most  $n$ . Indeed, if  $h \in G$  has  $hg\Phi h^{-1} = g\Phi$ , then  $hgh^{-1} = g\phi$  for some  $\phi \in \Phi$ . It follows that  $\det(\phi) = 1$  and thus  $\phi$  is an  $n$ -th root of unity. We conclude that  $[N_G(g\Phi) : C_G(g)] \leq n$ .

Thus the number of elements in the  $G$ -conjugacy class of the coset  $g\Phi$  equals

$$\frac{|G|}{|N_G(g\Phi)|} |\Phi| = |G| \frac{1}{[N_G(g\Phi) : C_G(g)]} \frac{1}{[C_G(g) : \Phi]} = |G| \frac{1}{\lambda}.$$

Enumerating all such conjugacy classes, we find

$$|G_1| = |G| \left( \frac{1}{\lambda_1} + \dots + \frac{1}{\lambda_{k_1}} \right),$$

where each  $\lambda_i$  is a positive integer of size at most  $nN$  by (3-1) and the remark above.

(2) We now pass to the subset  $G_2$ . Clearly  $G_2$  is stable under conjugation by  $G$ . Let  $F$  be an  $M$ -fan of  $G$  with maximal  $M$ -fan  $\mathcal{F}$  such that  $\Phi \not\subseteq F \not\subseteq \mathcal{F}$ . Let  $n_F$  be the number of  $g$ 's which are associated with  $F$ . By Lemma 3.4,  $n_F/|\mathcal{F}|$  is an integer of size at most  $N$ .

Grouping together the fans that are conjugate to  $F$ , we obtain  $|G|/|N_G(F)|$  different fans, where  $N_G(F)$  is the normalizer of  $F$  in  $G$ . Note that

$$[N_G(F) : C_G(F)] \leq n! \tag{3-2}$$

since  $N_G(F)$  permutes the weight spaces of  $F$  and hence a subgroup of index at most  $n!$  will preserve them and thus commute with  $F$ .

It follows that the number of elements that are associated with a fan lying in the  $G$ -conjugacy class of  $F$  equals

$$n_F \frac{|G|}{|N_G(F)|} = |G| \frac{1}{[N_G(F) : C_G(F)]} \frac{n_F/|\mathcal{F}|}{[C_G(F) : \mathcal{F}]} = |G| \frac{\nu}{\mu},$$

and thus enumerating the different conjugacy classes

$$|G_2| = |G| \left( \frac{\nu_1}{\mu_1} + \dots + \frac{\nu_{k_2}}{\mu_{k_2}} \right),$$

where the  $\nu_i \leq N$  and  $\mu_i \leq n!N$  are positive integers.

(3) Finally we consider the subset  $G_3$  of those nonscalar  $g$ 's such that  $F_g$  is maximal in  $G$  and different from  $\Phi$ . Clearly this set is invariant under conjugation by  $G$ . Given a maximal  $M$ -fan  $\mathcal{F}$ , the number  $n_{\mathcal{F}}$  of elements of  $G$  which are associated with  $\mathcal{F}$  equals  $|C_G(\mathcal{F})| - |\Phi|$  according to Lemma 3.5.

Setting  $\omega = [N_G(\mathcal{F}) : C_G(\mathcal{F})]$  and  $q = [N_G(\mathcal{F}) : \Phi]$ , the number of elements that are associated with a maximal fan conjugate to  $\mathcal{F}$  is

$$n_{\mathcal{F}} \frac{|G|}{|N_G(\mathcal{F})|} = |G| \left( \frac{1}{\omega} - \frac{1}{q} \right),$$

where  $\omega$  and  $q$  are positive integers with  $\omega \leq n!$  and  $q = [C_G(\mathcal{F}) : \Phi] \omega \geq 2\omega$ .

Summing over the conjugacy classes, we get

$$|G_3| = |G| \left( \left( \frac{1}{\omega_1} - \frac{1}{q_1} \right) + \dots + \left( \frac{1}{\omega_{k_3}} - \frac{1}{q_{k_3}} \right) \right).$$

Combining all three cases, we have thus completed our enumeration of  $G$  and we obtain:

**Proposition 3.6** (Jordan's fundamental equation). *Let  $G$  be a finite subgroup of  $GL_n(K)$  all of whose elements are diagonalizable, and  $\Phi$  the subgroup of scalar matrices in  $G$ . Then there are positive integers  $q_i$  dividing  $g := |G|/|\Phi|$  such that*

$$|G| = |\Phi| + |G| \sum_{i=1}^{k_1} \frac{1}{\lambda_i} + |G| \sum_{i=1}^{k_2} \frac{v_i}{\mu_i} + |G| \sum_{i=1}^{k_3} \left( \frac{1}{\omega_i} - \frac{1}{q_i} \right), \tag{3-3}$$

where  $k_i, \lambda_i, v_i, \mu_i$  and  $\omega_i$  are nonnegative integers of size at most  $2n!N$  (recall that  $N = N(n-1)^2$  is the bound from Theorem 2.3 under the induction hypothesis).

In particular,

$$\frac{1}{g} = \frac{1}{q_1} + \dots + \frac{1}{q_{k_3}} - \frac{b}{a}, \tag{3-4}$$

where  $\frac{b}{a}$  is an irreducible fraction whose numerator and denominator are bounded in terms of  $n$  only.

To prove Proposition 3.6 it remains only to show the bound on the number  $k_i$  of elements in each sum and then derive (3-4). But this follows from the equation (3-3) and from the bounds previously obtained, because

$$\frac{1}{\omega_i} - \frac{1}{q_i} \geq \frac{1}{2\omega_i}$$

for each  $i = 1, \dots, k_3$  and thus each term in the above sums contributes at least  $|G|/2n!N$ , forcing  $k_1 + k_2 + k_3 \leq 2n!N$ .

Showing (3-4) is a simple matter of rearranging (3-3):

$$\frac{|G|}{|\Phi|} \left( \sum_{i=1}^{k_1} \frac{1}{\lambda_i} + \sum_{i=1}^{k_2} \frac{v_i}{\mu_i} + \sum_{i=1}^{k_3} \frac{1}{\omega_i} - 1 \right) = \sum_{i=1}^{k_3} \frac{|G|}{|\Phi|} \frac{1}{q_i} - 1. \tag{3-5}$$

Then we let

$$g := \frac{|G|}{|\Phi|} \quad \text{and} \quad \frac{b}{a} := \sum_{i=1}^{k_1} \frac{1}{\lambda_i} + \sum_{i=1}^{k_2} \frac{v_i}{\mu_i} + \sum_{i=1}^{k_3} \frac{1}{\omega_i} - 1,$$

where  $\frac{b}{a}$  is an irreducible fraction. We thus get (3-4).

Note further that  $a$  is bounded in terms of  $n$  only: indeed it cannot exceed the least common multiple of at most  $2n!N$  integers of size at most  $n!N$ . A similar bound holds for  $b$ . This completes the proof of Proposition 3.6.

#### 4. Proof of Theorem 2.3

It remains to discuss the fundamental equation (3-3) according to the possible values of the integers  $\lambda_i, v_i, \mu_i, \omega_i$  and  $q_i$ .

The proof rests on the following elementary lemma about fractions:

**Lemma 4.1.** *Consider the following equation, where all variables are positive integers:*

$$\frac{1}{g} = \frac{1}{q_1} + \dots + \frac{1}{q_k} - \frac{b}{a}.$$

Suppose that  $q_i < g$  for all  $i$ . Then  $g \leq f(k, a)$ , where  $f(k, a)$  is a function of  $k$  and  $a$  only. One may take  $f(k, a) = (k!a)^{2^k}$ .

*Proof.* The proof proceeds by induction on  $k$ . If  $k = 1$ , then  $\frac{1}{q_1} \geq \frac{b}{a}$  implies  $q_1 \leq a$  and  $\frac{1}{g} \geq \frac{1}{q_1 a} \geq \frac{1}{a^2}$ , so  $g \leq a^2 =: f(1, a)$ .

Suppose the lemma proven for all indices  $\leq k - 1$ . Without loss of generality, we may assume that  $\frac{1}{q_1} \leq \dots \leq \frac{1}{q_k}$ . Then  $\frac{1}{q_2} + \dots + \frac{1}{q_k} < \frac{b}{a}$ , for  $q_1 < g$ .

It follows that

$$\frac{b}{a} > \frac{1}{q_2} + \dots + \frac{1}{q_k} \geq \frac{1}{q_k}.$$

We may thus write  $\frac{c}{d} = \frac{b}{a} - \frac{1}{q_k}$ , where  $c, d$  are positive integers and  $\frac{c}{d}$  is an irreducible fraction.

Since  $\frac{1}{g} \leq \frac{k}{q_k} - \frac{b}{a}$ , we get  $q_k \leq ka$  and thus  $d = \text{lcm}(a, q_k) \leq ka^2$ . We obtain

$$\frac{1}{g} = \frac{1}{q_1} + \dots + \frac{1}{q_{k-1}} - \frac{c}{d}.$$

Applying the induction hypothesis we conclude  $g \leq f(k - 1, ka^2) =: f(k, a)$ .  $\square$

We now complete the proof of Theorem 2.3. If  $k_3 = 0$ , then we see from (3-4) that  $\frac{-b}{a} = \frac{1}{g}$  so  $b = -1$  and  $g = a$  is bounded in terms of  $n$  only by Proposition 3.6. Hence  $\Phi$  has bounded index in  $G$  and we are done.

Assume  $k_3 \geq 1$ . If  $q_i = g$  for some  $i$ , then

$$G = N_G(\mathcal{F}_i) \quad \text{and} \quad [G : \mathcal{F}_i] = \omega_i [C_G(\mathcal{F}_i), \mathcal{F}_i] \leq n!N$$

by (3-1). So  $\mathcal{F}_i$  is the desired abelian normal subgroup of bounded index and we are done.

The right-hand side of (3-5) is a nonnegative positive integer. If it is zero, then  $k_3 = 1$ ,  $q_1 = g$  and we fall back to the previous case. Otherwise it is positive and thus  $b > 0$ , so that we are in the situation of Lemma 4.1. We conclude that  $g$  is bounded in terms of  $n$  only and again we are done.

Theorem 2.3 is now proven in full.

**Remark.** We mention in passing that the proof of Landau's theorem [33] that there are only finitely many finite groups  $G$  with exactly  $k$  conjugacy classes  $\text{cl}_1, \dots, \text{cl}_k$  is based on a similar, and easier, diophantine equation, namely

$$1 = \frac{1}{q_1} + \dots + \frac{1}{q_k},$$

where  $q_i = |G|/|\text{cl}_i|$ . This is an instance of an *Egyptian fraction* [9], and a simple argument [36] implies that

$$k \geq \frac{1}{\log 4} \log \log |G|.$$

### 5. Platonic solids and the finite subgroups of $SO_3(\mathbb{R})$

As an illustration and for the sake of comparison, we recall in this section a proof of the classification of finite subgroups of  $SO_3(\mathbb{R})$  following Klein's method, as given in many textbooks, e.g., [45; 51].

Let  $G$  be a finite subgroup of  $SO_3(\mathbb{R})$ . Every nontrivial element of  $G$  is a rotation around some axis. Let  $X$  be the set of all axes that arise as axes of rotations in  $G$ . Clearly  $G$  permutes  $X$  because if  $x_h$  is the axis of  $h \in G$ , then  $gx_h = x_{ghg^{-1}}$ . The determination of all possible groups  $G$  proceeds via a double counting argument, or class equation, which enumerates the elements of  $G$  according to their fixed axis. Given an axis  $x \in X$ , let  $G_x$  be the subset of elements of  $G$  whose axis is  $x$ , to which we adjoin the identity. Then  $G_x$  is a subgroup. It usually coincides with the centralizer of  $x$ , except when  $x$  is a flip (i.e., has angle  $\pi$ ). Enumerating the elements of  $G$  starting with the identity element, we can write

$$|G| = 1 + \sum_{x \in X} (|G_x| - 1).$$

To go further, we group together the terms corresponding to two axes that are  $G$ -congruent (i.e.,  $x \sim y$  if there is  $g \in G$  with  $y = gx$ ). We obtain

$$|G| = 1 + \sum_{\text{classes of } x \in X} \frac{|G|}{|\text{Stab}_x|} (|G_x| - 1), \tag{5-1}$$

where  $\text{Stab}_x$  is the stabilizer of  $x$  in  $G$ . It is a subgroup of  $G$ . Now observe that an element  $g$  of  $G$  which preserves  $x$  may be only of two possible forms: either it fixes both poles of  $x$ , in which case  $g$  belongs to  $G_x$ , or it permutes the two poles of  $x$ . It follows that  $G_x$  is a subgroup of  $\text{Stab}_x$  of index either 1 or 2. Let  $x_1, \dots, x_r, x_{r+1}, \dots, x_{r+s}$  be a set of representatives of the  $G$ -orbits in  $X$  such that  $[\text{Stab}_{x_i} : G_{x_i}] = 1$  if  $1 \leq i \leq r$  and  $[\text{Stab}_{x_i} : G_{x_i}] = 2$  if  $r + 1 \leq i \leq r + s$ . Setting  $g_i = |G_{x_i}|$ , dividing by  $|G|$  in (5-1) we obtain

$$1 = \frac{1}{|G|} + \sum_{i=1}^r \left(1 - \frac{1}{g_i}\right) + \frac{1}{2} \sum_{i=r+1}^{r+s} \left(1 - \frac{1}{g_i}\right), \tag{5-2}$$

or equivalently, in the form of Jordan's fundamental equation (3-4),

$$\frac{1}{|G|} = \frac{1}{n_1} + \dots + \frac{1}{n_{r+s}} - \frac{b}{a}, \tag{5-3}$$

where  $\frac{b}{a} = r + \frac{s}{2} - 1$ , and  $n_i = g_i$  for  $i \leq r$ ,  $n_i = 2g_i$  for  $i > r$ .

It remains to discuss equation (5-3) according to the possible values of the  $g_i$ 's. Since  $g_i \geq 2$ , we get from (5-2) that  $1 > \frac{r}{2} + \frac{s}{4}$ , from which it follows immediately that  $r \leq 1$  and  $2r + s \leq 3$ , so  $\frac{b}{a} \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ . Since  $n_i$  divides  $|G|$ , (5-3) forces  $\frac{b}{a} > 0$  (hence  $\frac{b}{a} = \frac{1}{2}$ ), unless  $\frac{b}{a} = 0$  and  $r + s = 1$ . This last case can only occur if  $r + \frac{s}{2} = 1$ , forcing  $r = 1, s = 0$ . We now examine the various possibilities.

- $\frac{b}{a} = 0$  and  $r = 1, s = 0$ . Then  $g_1 = |G|$  and  $G$  is a *cyclic group* of rotations around a single axis.
- $\frac{b}{a} = \frac{1}{2}$ , and  $r = 1, s = 1$ ,

$$\frac{1}{2} + \frac{1}{|G|} = \frac{1}{n_1} + \frac{1}{n_2}.$$

Since  $g_2 \geq 2$ , we have  $n_2 \geq 4$ . This forces  $n_1 < 4$  and hence  $n_1 = 2, 3$ . There are thus two cases:

- (1) If  $n_1 = 2$ , then  $n_2 = 2g_2 = |G|$  and  $G$  is a *dihedral group* of order  $2n$ , with  $n = g_2$  an *odd integer*.  $G$  is the group of orientation preserving isometries of a regular  $n$ -gon. Moreover in our case  $n$  is *odd* because there are only two  $G$ -orbits of axes.
- (2) If  $n_1 = 3$ , then one checks that  $n_2 = 4$  and  $|G| = 12$ . Here  $G$  is the group of orientation preserving isometries of a *regular tetrahedron*.

- $\frac{b}{a} = \frac{1}{2}, r = 0, s = 3,$

$$\frac{1}{2} + \frac{1}{|G|} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}.$$

We may assume  $n_1 \leq n_2 \leq n_3$ . This forces  $n_1 < 6$ . But  $n_1 = 2g_1 \geq 4$ . So  $n_1 = 4$  and thus

$$\frac{1}{4} + \frac{1}{|G|} = \frac{1}{2g_2} + \frac{1}{2g_3}.$$

We have the following cases:

- (1)  $g_2 = 2$ , then  $|G| = n_3 = 2g_3$  and  $G$  is a *dihedral group* of order  $2n$ , with  $n = g_3$  an *even integer*.  $G$  is the group of orientation preserving isometries of a regular  $n$ -gon. Moreover in our case  $n$  is *even* because there are exactly three  $G$ -orbits of axes. For example if  $n = 2$ ,  $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$  and every nontrivial element is a flip around one of three mutually orthogonal axes.
- (2)  $g_2 = 3$ , then

$$\frac{1}{6} + \frac{2}{|G|} = \frac{1}{g_3}.$$

This forces  $g_3 < 6$ , and hence three cases:

- (a)  $g_3 = 3$ , then  $|G| = 12$  and  $G$  preserves a regular tetrahedron. This however is in contradiction with the assumption  $s = 3$ , since there are only two  $G$ -orbits of axes in this case. So this case cannot occur.
- (b)  $g_3 = 4$ , then  $|G| = 24$  and  $G \simeq S_4$  is the group of orientation preserving isometries of a *cube* or regular *octahedron*.
- (c)  $g_3 = 5$ ,  $G \simeq A_5$  is the group of orientation preserving isometries of a regular *icosahedron* or *dodecahedron*.

### 6. Nonstandard analysis and Jordan's unlimited numbers

As we have seen, Jordan gave no explicit bound on  $J(n)$  in his article. Of course, this is not due to any fundamental ineffectiveness in the proof. Indeed, if one very carefully follows Jordan's argument, then it is possible to obtain in this way a bound in the form of a tower of exponentials, i.e., a tower

$$10^{10 \dots 10}$$

of length  $n$ ; see [2]. In fact Jordan himself seems to have been dissatisfied with his original exposition and devoted a second article [30], where he rewrote his proof and explained why it is effective (even though he still did not supply a concrete bound). From a purely epistemological viewpoint, it is however interesting to consider how Jordan gets away with not writing down any bound whatsoever in his original memoir. In fact, in order to convince the reader of the soundness of

his argument, he introduces a distinction between two kinds of numbers, which he calls limited and unlimited. Let us quote him [29, p. 114]:

It is important for the study thereafter to make precise the meaning we attach to the words *limited* and *unlimited*. They are not synonymous to *finite* and *infinite*. We will say that a number is limited if it is smaller than a certain bound that has been determined. It follows from this definition that a finite number, about which we have no data, is unlimited; but it becomes limited as soon as we manage to assign a bound to it.

This way, instead of saying that a certain quantity is bounded in terms of  $n$  only, he says that the quantity is limited, while if it is not, it is unlimited and this is somehow leading to a contradiction when considering the class equation (3-4). A century and a half after Jordan, it is hard not to see there the premise of a way of thinking that prefigures nonstandard analysis, where a new kind of number, the unlimited ones, is given an existence of its own.

In fact, it is possible to give a nonstandard treatment of Jordan's proof, which we now sketch. Starting with a sequence of possible counterexamples to the theorem, one may take their ultraproduct, which becomes a certain infinite, pseudofinite subgroup  $\widehat{G}$  of  $\text{GL}_n(\widehat{K})$ . Here  $\widehat{K}$  is the ultraproduct of algebraically closed fields  $K_i$ . A fan in  $\widehat{G}$  is defined to be an internal diagonalizable subgroup  $F$  such that each root  $\alpha_{ij} : F \rightarrow \widehat{K}$ ,  $f \mapsto \lambda_i(f)/\lambda_j(f)$ , is either infinite or trivial. The induction hypothesis on the dimension allows one to assume that the centralizer  $C(g)$  of every nonscalar element  $g \in \widehat{G}$  admits a unique normal maximal fan  $F_g$  whose index is finite. Arguing as in Jordan's proof, we can partition the elements of  $\widehat{G}$  into four parts: scalars  $\Phi$ , elements  $g \notin \Phi$  with  $F_g = \Phi$ , elements  $g \notin \Phi$  with  $\Phi \not\leq F_g$  and  $F_g$  not maximal in  $\widehat{G}$ , elements  $g \notin \Phi$  with  $F_g$  maximal. Exploiting the fact that  $\widehat{G}$  is pseudofinite, the second and third parts form a proportion of  $r|\widehat{G}|$  of  $\widehat{G}$ , while the last part forms a proportion  $(r' - \sum q_i^{-1})|\widehat{G}|$ , where  $r, r'$  are finite (standard) rationals and  $q_i = |N_{\widehat{G}}(F_{g_i})/\Phi|$  are a finite number of (nonstandard) divisors of  $|\widehat{G}/\Phi|$ . Denoting by  $g$  the nonstandard integer  $|\widehat{G}/\Phi|$ , we thus have the equation

$$\frac{1}{g} = r'' + \sum_i \frac{1}{q_i},$$

where  $r'' = 1 - (r + r')$ . Taking the difference with standard parts we see that this implies

$$\frac{1}{g} = \sum_i' \frac{1}{q_i}$$

for a subsum of the original sum. However, the  $q_i$ 's are divisors of  $g$  and the only way this can happen is if  $q_i = g$  for some  $i$ . By (3-1) and (3-2), this means that  $F_{g_i}$  is normal in  $\widehat{G}$  and of finite index. This contradiction ends the proof.



The reader curious to take a look at Jordan's original article will see that the above nonstandard treatment is in fact much closer to Jordan's own formulation of his proof than the exposition we have given of it in Section 3. Indeed Jordan does not talk about  $M$ -fans, but only defines fans. And he does so exactly as we did in the nonstandard treatment above only using the word "illimited" in place of the word "infinite". Of course this definition can only make sense rigorously if we place ourselves in a nonstandard universe to begin with. So his proof is resolutely nonstandard since its very inception. His original formulation [29, §40 p. 114] then reads as follows:

**Theorem 6.1** (Jordan's theorem, original formulation). *A finite subgroup  $G$  of linear substitutions admits a unique maximal fan. It is normal and its index is a limited number.*

To finish, we stress the key role of the finiteness of  $G$  in Jordan's theorem. In Jordan's proof it is exploited arithmetically via the class equation. This is to be contrasted with Bieberbach's geometric argument via the commutator shrinking property, where finiteness is exploited via the element closest to the identity.

We end this section by mentioning in passing some related recent developments around a question of Zilber [52, Problem 6.3] regarding pseudofinite groups. Recently, Nikolov, Schneider and Thom [38] proved that every homomorphism from a pseudofinite group to a compact Lie group has abelian-by-finite image, thus answering a conjecture of Pillay [40, Conjecture 1.7] and Zilber's question by the same token.

Of course such a strong statement is more than enough to establish Jordan's theorem itself following the nonstandard approach sketched above, at least when the characteristic of  $K$  is zero. Indeed, in this case  $\widehat{K}$  can be taken to be isomorphic to  $\mathbb{C}$  as any ultraproduct of countable algebraically closed fields of characteristic zero. Furthermore, we may assume that  $\widehat{G}$  lies in the internal set of unitary matrices  $\cup_n(\widehat{K})$ , in other words that  $\widehat{G}$  is a subgroup of a compact Lie group. By Nikolov–Schneider–Thom, this implies that  $\widehat{G}$  is abelian-by-finite, which is the desired contradiction.

Of course the theorem of Nikolov, Schneider and Thom lies much deeper than Jordan's theorem, because it applies to any pseudofinite group and not only those lying in some  $GL_n$  for some fixed  $n$ . Their proof relies on the deep results of Nikolov and Segal [37] about commutator width in finite groups.

### 7. Bounds on $J(n)$

To conclude, we briefly survey the history around Jordan's theorem and how bounds on  $J(n)$  have sharpened over time:

- Jordan (1878): no bound (in fact: tower of exponentials [2]).
- Blichfeldt (1905):  $\exp(O(n^3))$ .
- Bieberbach (1911):  $(1 + 32^4 n^{10})^{2n^2}$ .
- Frobenius (1911):  $(\sqrt{8n} + 1)^{2n^2}$ .
- Blichfeldt (1917):  $n!6^{(n-1)(\pi(n+1)+1)}$  ( $\pi(x)$  denoting the number of prime numbers  $\leq x$ ).
- Weisfeiler (1984):  $(n + 1)!e^{O(\log n)^2}$  (using CFSG).
- Collins (2007):  $(n + 1)!$  for  $n \geq 71$  (using CFSG).

In [4, p. 396] Blichfeldt, who had just completed his dissertation under Sophus Lie, shows that no prime  $p \geq n(2n - 1)$  divides the order  $g$  (modulo the center) of a finite primitive subgroup of  $\mathrm{GL}_n(\mathbb{C})$  (see also [8, §64]). In [5, p. 321] he obtains bounds for the  $p$ -exponent of  $g$ . He shows [7, Theorem 16, p. 42] that  $g$  is a divisor of  $n!(2 \cdot 3 \cdots p \cdots)^{n-1}$ , where the product extends over all primes  $p < n(2n - 1)$ . This is of order at most  $\exp(O(n^3))$ , and this implies a bound of the same magnitude for  $J(n)$  for arbitrary finite subgroups (implicit in [5, §12, p. 320] and [6, p. 232]).

In his 1917 monograph [8], Blichfeldt furthers his earlier results, incorporating a geometric argument inspired by Bieberbach's argument [3] and Frobenius' improvement [23]. He shows in [8, §73] that an abelian subgroup of a primitive group must have order at most  $6^{n-1}$  times the size of the group of scalar matrices. This is based on a lemma [8, §70] according to which in a finite primitive subgroup, any transformation whose eigenvalues are concentrated in an arc of length at most  $\frac{2\pi}{3}$  centered at one of them on the unit circle, must be scalar. This lemma is closely related to the Bieberbach–Frobenius proofs. And finally he derives the bound  $n!6^{(n-1)(\pi(n+1)+1)}$  on  $J(n)$ , where  $\pi(n)$  is the number of primes  $\leq n$ . See [22, Chapter 30] for a thorough treatment of Blichfeldt's bound and [42; 43] for recent improvements on Blichfeldt's lemma. In fact Blichfeldt claimed that 6 could be replaced by 5, but no proof of this has appeared. The three-author book [35], which is dedicated to Camille Jordan, also contains a summary of these results.

Other excellent expositions of Blichfeldt's bound are contained in [46] and [26]. See also [21, Chapter 5] for a treatment of Blichfeldt's earlier results and a proof of Jordan's theorem using the Bieberbach–Frobenius argument.

Brauer [12] conjectured that Blichfeldt's bound could be improved to one of the form  $e^{O(n \log n)}$ , and indeed he was able to achieve it under certain hypotheses. In fact, for finite solvable subgroups Dornhoff proved an exponential bound  $2^{4n/3} 3^{10n/9-1/3}$  that is even sharp for infinitely many  $n$ 's [22, Theorem 36.4].

Nevertheless, Blichfeldt's second bound of the form  $e^{O(n^2/\log n)}$  seems to be the best one available without the classification of finite simple groups (CFSG). This

small looking gain of a factor  $(\log n)^2$  in the exponent compared to the Bieberbach–Frobenius bound can sometimes prove important, as we have found out in [15].

Shortly before disappearing while hiking in Chile,<sup>4</sup> B. Weisfeiler announced a bound on  $J(n)$  of  $e^{O(n \log n)}$  quality [49]. His unpublished manuscript has now been typed up and is available online [48]. Finally, more recently, M. Collins [16; 17] has improved the bound for  $n$  large to one that is sharp, namely  $(n + 1)!$ , thus closing a long chapter in the history of finite linear groups.

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