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Higher internal covers

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We define and study a higher-dimensional version of model-theoretic internality, and relate it to higher-dimensional definable groupoids in the base theory.

1. Introduction

The model-theoretic notions of *internality* and the binding group came up originally in work of Zil'ber on categorical theories [13], and shortly after of Poizat [11] in the ω-stable context, where it was also noticed that differential Galois theory occurs as a special case. The stability hypothesis was completely removed in [3, Appendix B], where it was shown that the crucial hypothesis is *stable embeddedness* of the base theory.

Internality is a condition on a definable set $\boldsymbol{Q}$ in an expansion $\mathcal{T}^*$ of a theory $\mathcal{T}$ to "almost" be interpretable in $\mathcal{T}$: it is interpretable after adding a set of parameters to $\mathcal{T}^*$. In this situation, the theory provides a *definable* group $\boldsymbol{G}$ in $\mathcal{T}^*$, acting definably on $\boldsymbol{Q}$ as its group of automorphisms fixing all elements in the reduct $\mathcal{T}$. It is important here that the binding group $\boldsymbol{G}$ is defined in $\mathcal{T}^*$ rather than in $\mathcal{T}$: in applications, one often understands groups in $\mathcal{T}$ better than in $\mathcal{T}^*$. The group $\boldsymbol{G}$ itself is also internal to $\mathcal{T}$, and as a result can be identified with a definable group $\boldsymbol{H}$ in $\mathcal{T}$, but only noncanonically (and in general, only after adding parameters). In the context of differential Galois theory, this is related to the fact that the group of points of the (algebraic) Galois group of a differential equation does not act on the set of solutions, and its identification with the group of automorphisms is not canonical.

The noncanonicity was explained by Hrushovski in [4], where it is shown that the natural object that appears in this context is a definable *groupoid* in $\mathcal{T}$, with the different groups $\boldsymbol{H}$ occurring as the groups of automorphisms of each object. In fact, it is shown there that there is a correspondence between groupoids definable in the base theory $\mathcal{T}$ and internal sorts in expansions of $\mathcal{T}$. This correspondence is reviewed in Section 2. It is also suggested in [4] that sorts of $\mathcal{T}^*$ internal to $\mathcal{T}$ should be viewed as generalised sorts of $\mathcal{T}$, obtained as a quotient by the

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corresponding definable groupoid, just like an imaginary sort is obtained from a definable equivalence relation (which is a special case). In the current paper we try to follow this suggestion, by considering what should be the correct notion of internality, after viewing these new sorts as "legitimate" definable sorts.

Our approach is motivated by topology. There, a typical example of a groupoid arises as the fundamental groupoid $\pi_1(X)$ of a space X, i.e., the groupoid whose objects are the points of X, and whose morphisms are homotopy classes of paths. For sufficiently nice X, this groupoid can be described in terms of the category of *local systems* (locally constant sheaves) on X: each point of X determines a functor to the category of sets, satisfying suitable properties (for example, it commutes with products), and each path determines a map between such functors (which depends only on the homotopy class since the system is locally constant). We propose to view internality as analogous to this picture: definable sets in the theory corresponding to a definable groupoid $\boldsymbol{G}$ in $\mathcal{T}$ can be viewed as local systems (of definable sets) on $\boldsymbol{G}$, and conversely. This point of view is explained in Section 2.2.5 (the base theory $\mathcal{T}$ corresponds to a contractible space in this approach, so definable sets in it correspond to constant systems).

By definition, the local systems on X do not tell us anything about the homotopy type of X above homotopical dimension 1. To encode higher homotopical information, we may try looking at families of spaces rather than of sets. A space X is called *n-truncated* if $\pi_k(X, x)$ is trivial for all $k > n$ and base points $x \in X$. Such spaces are represented in homotopy theory by what we call in this paper n-groupoids (Definition 3.2.3 in the definable setting; these are equivalent to n-categories in the sense of [8, §2.3.4] which are groupoids). In the case $n = 1$, these are usual groupoids, and the previous paragraph suggests studying them by systems of 0-truncated spaces, i.e., sets. Going one dimension higher, one expects to recover 2-groupoids from systems of 1-truncated spaces. In the definable context, we decided to identify such spaces with internal sorts, we consider "local systems" of internal sorts, i.e., internal sorts of an expanded theory.

Our main result, Theorem 3.3.9, shows one direction of this expected correspondence: we associate to a 2-groupoid $\boldsymbol{G}$ in the theory $\mathcal{T}$ a theory $\mathcal{T}_{\boldsymbol{G}}$ expanding it, and a collection of internal sorts of $\mathcal{T}_{\boldsymbol{G}}$, which we view as "higher local systems". The statement is that the canonical 2-groupoid associated to this datum recovers (up to weak equivalence) the original one (part of the other direction is indicated briefly, but is mostly left for future work).

We mention that this result is one possible generalisation of the results of [4] to higher dimensions. Other such generalisations include the papers [1; 2; 12], but they all appear to go in different directions. We also mention that in the context of usual (rather than definable) homotopy theory, analogous results are well known (for example, the main part of Theorem 3.3.9 is really a version of the higher-dimensional Yoneda lemma), but the methods in the proof of these results do not translate easily to the definable setting. In fact, the situation here is more similar to the one described in [8, §6.5], though made simpler by the existence of models (i.e., we have "enough points").

1.1. ***Structure of the paper.*** It is very simple: in Section 2, we review the situation in the one-dimensional case. This serves both as a motivating analogy and to complete some background used later. Most of the material there appears in some form in [4] (sometimes implicitly), but we include a few easy remarks regarding morphisms and equivalence, interpretation in terms of "local systems", and a different description of the groupoid associated to an internal cover (which already appeared slightly differently in [7]).

In Section 3, we expose the higher-dimensional picture, concentrating on dimension 2. We first define our higher internal covers, then review the theory of (truncated) Kan complexes and n-categories, with a few remarks special to the definable setting, and then prove the main result mentioned above (Theorem 3.3.9).

1.2. ***Conventions and terminology.*** For simplicity, we assume our theories $\mathcal{T}$ to admit elimination of quantifiers. By a *$\mathcal{T}$-structure* we mean a substructure of some model of $\mathcal{T}$. If A is such a $\mathcal{T}$-structure, by $\mathcal{T}_A$ we mean the expansion of $\mathcal{T}$ by constants for the elements of A, along with the usual axioms describing A. If A was not mentioned, we mean "for some A".

We also assume $\mathcal{T}$ eliminates imaginaries (this could be included in the general treatment, but would complicate the exposition). Our usage of elimination of imaginaries is often in the (equivalent) form of the existence of internal Homs: for every two definable sets $\boldsymbol{X}$ and $\boldsymbol{Y}$, there are an ind-definable set $\underline{\mathbf{Hom}}(\boldsymbol{X}, \boldsymbol{Y})$ and map $\mathbf{ev} : \boldsymbol{X} \times \underline{\mathbf{Hom}}(\boldsymbol{X}, \boldsymbol{Y}) \to \boldsymbol{Y}$, identifying the A-points of $\underline{\mathbf{Hom}}(\boldsymbol{X}, \boldsymbol{Y})$, for each $\mathcal{T}$-structure A, with the set of A-definable maps from $\boldsymbol{X}$ to $\boldsymbol{Y}$. It follows that the subset $\underline{\mathbf{Iso}}(\boldsymbol{X}, \boldsymbol{Y})$ of definable isomorphisms is also ind-definable.

Finally, we assume that each theory is generated by one sort, and finitely many relations. Similar to the case in [4], it can be seen that this assumption is not restrictive, since all our constructions commute with adding structure.

We recall that an *interpretation* of a theory $\mathcal{T}_1$ in another theory $\mathcal{T}_2$ is a model of $\mathcal{T}_1$ in the definable sets of $\mathcal{T}_2$: it assigns definable sets to the elements of the signature of $\mathcal{T}_1$, so that the axioms in $\mathcal{T}_1$ hold (this is often called a *definition* in the literature, which is equivalent to an interpretation under our assumption of elimination of imaginaries in $\mathcal{T}_2$). If $i : \mathcal{T}_1 \to \mathcal{T}_2$ is such an interpretation, it thus assigns to each definable set $\boldsymbol{X}$ of $\mathcal{T}_1$ a definable set $i(\boldsymbol{X})$ of $\mathcal{T}_2$. Since i is a model of $\mathcal{T}_1$, it assigns definable functions of $\mathcal{T}_2$ to definable functions of $\mathcal{T}_1$, and composition to composition, and thus determines a functor from the category $\mathcal{D}ef(\mathcal{T}_1)$ of definable sets of $\mathcal{T}_1$ to $\mathcal{D}ef(\mathcal{T}_2)$. We normally identify i with this functor, writing for example

$i(\boldsymbol{X})$ for the interpretation of a definable set $\boldsymbol{X}$ of $\mathcal{T}_1$. We note that an expansion is a particular case of an interpretation. We remark that not every functor from $\mathcal{D}ef(\mathcal{T}_1)$ to $\mathcal{D}ef(\mathcal{T}_2)$ arises from an interpretation: For example, an interpretation preserves all finite (inverse) limits (which always exist in $\mathcal{D}ef(\mathcal{T}_1)$). This is the main property of such functors that we use in this paper. A detailed description of categories of the form $\mathcal{D}ef(\mathcal{T})$ and functors that arise from interpretation occurs in [10], but we do not require it.

Similarly, if $i, j : \mathcal{T}_1 \to \mathcal{T}_2$ are interpretations, a map from i to j is an elementary map of models (equivalently a homomorphism, by our assumption of quantifier elimination), given by definable maps in $\mathcal{T}_2$. Equivalently, this is a natural transformation of functors, when i and j are viewed in this way. Such a map is an isomorphism if it has an inverse. An interpretation is called a *bi-interpretation* if there is an interpretation in the other direction such that both compositions are isomorphic to the identity.

When $\mathcal{T}_1$ and $\mathcal{T}_2$ are given with fixed interpretations $i_k : \mathcal{T} \to \mathcal{T}_k$ of a theory $\mathcal{T}$, we have versions of these notions over $\mathcal{T}$: an interpretation $j : \mathcal{T}_1 \to \mathcal{T}_2$ is over $\mathcal{T}$ if $j \circ i_1 = i_2$, and given two such interpretations $j_1, j_2 : \mathcal{T}_1 \to \mathcal{T}_2$, a map $\alpha : j_1 \to j_2$ is over $\mathcal{T}$ if $\alpha_{i_1(X)} : j_1(i_1(\boldsymbol{X})) \to j_2(i_1(\boldsymbol{X}))$ is the identity for all definable sets $\boldsymbol{X}$ of $\mathcal{T}$ (more naturally, we could require a given isomorphism from $j \circ i_1$ to i_2, but in practice we can always assume it to be the identity, and do that to simplify notation).

2. A review of the classical theory

2.1. ***Stable embeddings and internal covers.*** We recall the following classical definition of internal covers:

Definition 2.1.1. An expansion $\mathcal{T}^*$ of a theory $\mathcal{T}$ is an *internal cover* if $\mathcal{T}$ is stably embedded in $\mathcal{T}^*$, and for some expansion $\mathcal{T}^*_A$ of $\mathcal{T}^*$ by a set of constants A, each definable set in $\mathcal{T}^*_A$ is definably isomorphic to a definable set in $\mathcal{T}_{A_0}$, for some set of parameters A_0.

We recall that *stably embedded* here means that for every definable set $\boldsymbol{X}$ in $\mathcal{T}$, every subset of $\boldsymbol{X}$ definable in $\mathcal{T}^*$ with parameters from $\mathcal{T}^*$ is definable in $\mathcal{T}$, with parameters from $\mathcal{T}$.

It was noted in [6] that this condition can be reformulated as follows: if $i : \mathcal{T} \to \mathcal{T}^*$ is an expansion and $\boldsymbol{X}, \boldsymbol{Y}$ are definable in $\mathcal{T}$, there is a natural ($\mathcal{T}^*$-definable) map $i(\underline{\mathbf{Hom}}_{\mathcal{T}}(\boldsymbol{X}, \boldsymbol{Y})) \to \underline{\mathbf{Hom}}_{\mathcal{T}^*}(i(\boldsymbol{X}), i(\boldsymbol{Y}))$, and i is stably embedded precisely if this map is a bijection (for all $\boldsymbol{X}, \boldsymbol{Y}$ definable in $\mathcal{T}$). We note that in this case, the restriction of this map to the subset $\underline{\mathbf{Iso}}(\boldsymbol{X}, \boldsymbol{Y})$ of isomorphisms is also a bijection, and that taking into account parameters, no new structure is induced on $\mathcal{T}^*$. In particular, for any $\mathcal{T}$-structure A, the expansion ${\mathcal{T}^*}_A$ is well defined.

The same definition can be applied to a more general interpretation, so we say that an interpretation $i : \mathcal{T}_1 \to \mathcal{T}_2$ is *stable* if for all definable sets $\boldsymbol{X}$ and $\boldsymbol{Y}$ of $\mathcal{T}_1$, the natural $\mathcal{T}_2$-definable map

$$i(\underline{\mathbf{Hom}}_{\mathcal{T}_1}(\boldsymbol{X}, \boldsymbol{Y})) \to \underline{\mathbf{Hom}}_{\mathcal{T}_2}(i(\boldsymbol{X}), i(\boldsymbol{Y}))$$

is a bijection. Explicitly, this means that each map definable with parameters from $i(\boldsymbol{X})$ to $i(\boldsymbol{Y})$ (in $\mathcal{T}_2$) "comes from" a unique map definable with parameters from $\boldsymbol{X}$ to $\boldsymbol{Y}$ (in $\mathcal{T}_1$). If i is viewed as a (left-exact) functor, as in Section 1.2, this is often stated as saying that i is Cartesian closed. In these terms, the definition of internal covers can be reformulated as follows:

Proposition 2.1.2. *An expansion $\mathcal{T}^*$ of a theory $\mathcal{T}$ is an internal cover if it is stable, and $\mathcal{T}^*$ admits a stable interpretation p in $\mathcal{T}_A$ over $\mathcal{T}$.*

As mentioned in Section 1.2, by "over $\mathcal{T}$" we mean that the restriction of p to $\mathcal{T}$ coincides with the expansion by constants.

Proof. Let $\boldsymbol{Q}$ be a definable set in $\mathcal{T}^*$, generating it over $\mathcal{T}$. Assume first that $\mathcal{T}^*$ is an internal cover of $\mathcal{T}$, so there is a sort $\boldsymbol{X}$ of $\mathcal{T}^*$, an expansion by a constant symbol $a \in \boldsymbol{X}$, and a definable bijection $g_a : \boldsymbol{Q} \to \boldsymbol{Q}_a$, with $\boldsymbol{Q}_a$ definable in $\mathcal{T}$. By stable-embeddedness, $\boldsymbol{Q}_a$ is definable by a parameter a_0 in $\mathcal{T}$. The assignment $\boldsymbol{Q} \mapsto \boldsymbol{Q}_a$ extends uniquely to an interpretation x_{a_0} of $\mathcal{T}^*$ in $\mathcal{T}_{a_0}$, over $\mathcal{T}$. Since g_a determines a definable isomorphism between $\boldsymbol{Q}$ and $\boldsymbol{Q}_a$ (and similarly for any definable set it generates), this interpretation is stable.

Conversely, assume we have a stable interpretation $p : \mathcal{T}^* \to \mathcal{T}_{A_0}$ over i. We still denote by i the extension of i to the expansion $\mathcal{T}_{A_0} \to \mathcal{T}^*_{A_0}$ which is the identity on A_0 (it is still stable). Setting $\boldsymbol{C} = p(\boldsymbol{Q})$, the set $\underline{\mathbf{Iso}}(p(\boldsymbol{Q}), p(i(\boldsymbol{C}))) = \underline{\mathbf{Iso}}(\boldsymbol{C}, \boldsymbol{C})$ is nonempty, since it contains the identity on $\boldsymbol{C}$. Since p is stable, the left-hand side admits a definable bijection with $\underline{\mathbf{Iso}}(\boldsymbol{Q}, i(\boldsymbol{C}))$, so is nonempty as well. Any point a of this set shows that $\mathcal{T}^*$ is an internal cover. □

2.2. *Definable groupoids.* A definable groupoid is denoted as $\boldsymbol{G} = \langle \boldsymbol{G}_0, \boldsymbol{G}_1 \rangle$, with a definable set $\boldsymbol{G}_0$ of objects and a definable set $\boldsymbol{G}_1$ of isomorphisms, where the domain and codomain maps are denoted $\boldsymbol{d}, \boldsymbol{c} : \boldsymbol{G}_1 \to \boldsymbol{G}_0$, respectively, and composition denoted by $\circ$. For objects $a, b \in \boldsymbol{G}_0$, we write $\boldsymbol{G}(a, b)$ for the a, b-definable set $\langle \boldsymbol{d}, \boldsymbol{c} \rangle^{-1}(\langle a, b \rangle)$ of morphisms from a to b. A map $f : \boldsymbol{G} \to \boldsymbol{H}$ of definable groupoids is a definable functor: a pair of maps $f_0 : \boldsymbol{G}_0 \to \boldsymbol{H}_0$ and $f_1 : \boldsymbol{G}_1 \to \boldsymbol{H}_1$ commuting with the domain, codomain and composition maps. We say that f is a *weak equivalence*, denoted $f : \boldsymbol{G} \xrightarrow{\sim} \boldsymbol{H}$, if it induces an equivalence of categories on all models (this terminology is generalised in Definition 3.2.7). Our groupoids are generally *not* assumed to be connected.

In [4, §2], a definable groupoid is attached to each internal cover. This groupoid also admits two descriptions. The first is as a definable groupoid $\boldsymbol{G}^*$ in $\mathcal{T}^*$; this construction depends on the choice of a definable set $\boldsymbol{X}$ in $\mathcal{T}^*$, as in the proof of Proposition 2.1.2. Given this choice, the groupoid can be described as follows:

Construction 2.2.1. The groupoid associated to the data above is described as follows:

Objects ($\boldsymbol{G}_0$): Complete types of elements $a \in \boldsymbol{X}$ over $\mathcal{T}$, along with an additional object $*$. Since $\mathcal{T}$ is stably embedded, this set of types is definable in $\mathcal{T}$ (it is definable, rather than pro-definable, by our finiteness assumption on the language in Section 1.2).

Morphisms ($\boldsymbol{G}_1$): The set of isomorphisms from $*$ to a type $p \in \boldsymbol{G}_0$ is given by the realisations of p. Given another type $q \in \boldsymbol{G}_0$, a morphism from p to q is given by a 2-type s extending p and q (over $\mathcal{T}$). Distinct realisations of s correspond to distinct ways of writing the morphism s as a composition of a morphism from p to $*$ and a morphism from $*$ to q.

Composition: Given a type $s(x, y) \in \boldsymbol{G}_1$ extending $p(x), q(y) \in \boldsymbol{G}_0$, and a type $t(y, z) \in \boldsymbol{G}_1$ extending $q(y)$ and $r(z) \in \boldsymbol{G}_0$, the internality assumption implies that there is a unique 3-type $u(x, y, z)$ extending all of them. The restriction u to x, z is the composition of s, t.

The composition of an isomorphism a from $*$ to $p \in \boldsymbol{G}_0$ with (the inverse of) another such isomorphism b to $q \in \boldsymbol{G}_0$ is the type of $\langle a, b\rangle$. The other compositions are determined by these conditions. □

We denote by $\boldsymbol{G}$ the full subgroupoid of $\boldsymbol{G}^*$ on the same objects excluding $*$. Then $\boldsymbol{G}$ is defined entirely in $\mathcal{T}$.

2.2.2. To give a second description, consider, for each $\mathcal{T}$-structure A, the groupoid $I(A) = I_{\mathcal{T}^*/\mathcal{T}}(A)$ whose objects are stable interpretations of $\mathcal{T}^*$ in $\mathcal{T}_A$, that are the identity on $\mathcal{T}_A$, and whose morphisms are isomorphisms of such interpretations, which are the identity when restricted to $\mathcal{T}$. Here again we may enlarge I to obtain I^*, by adding an additional object $*$, which is described explicitly as the identity interpretation of $\mathcal{T}^*$, and again morphisms are given by A^*-definable isomorphisms of interpretations over $\mathcal{T}$ (where A^* is now a $\mathcal{T}^*$-structure). The following statement appeared in a slightly different form in [7]:

Proposition 2.2.3. *With notation as in 2.2, for each $\mathcal{T}^*$-structure A, there is a fully faithful embedding $i_A : \boldsymbol{G}^*(A) \to I^*(A)$, preserving the vertex, and commuting with action by automorphisms on A. If A is a model, i_A is an equivalence of categories.*

Proof. This is essentially [4, Theorem 3.2]. The functor i_A was described in the proof of Proposition 2.1.2: to an object p of $\boldsymbol{G}(A)$, viewed as a type over $\mathcal{T}$ (definable over A_0), we attach the interpretation $x_b = x_p$ described there, with b any realisation of p (as explained there, x_b depends only on $p \in A_0$ and not on b).

Each such realisation determines an isomorphism g_b from x_b to $*$, again as above, which describes the functor on morphisms from p to $*$. If q is another object, with realisation c, i_A assigns ${g_c}^{-1} \circ g_b : x_p \to x_q$ to the type r of the pair (b, c). This depends only on r, since the code for this composition lies in $\mathcal{T}$, by stable embeddedness. This code also determines r completely, so the functor is fully faithful.

To prove the final statement, let $i : \mathcal{T}^* \to \mathcal{T}$ be any interpretation over a model M_0. The internality assumption implies that for some $p \in \boldsymbol{G}(M_0)$, the set Y of isomorphisms between $*$ and p is nonempty. Since M_0 is a model, there is a point b in $i(Y)(M_0)$. Then g_b is an isomorphism from x_b to i. □

To summarise, to each stable embedding of $\mathcal{T}$ in $\mathcal{T}^*$, we had attached a groupoid $I_{\mathcal{T}^*/\mathcal{T}}$ classifying stable interpretations of $\mathcal{T}^*$ back in $\mathcal{T}$. The embedding is an internal cover precisely if the groupoid is nonempty, and in this case, the groupoid I is equivalent to a definable one (and to the classical binding groupoid). Conversely, starting with a definable groupoid $\boldsymbol{G}$ in $\mathcal{T}$, there is an internal cover $\mathcal{T}^* = \mathcal{T}_{\boldsymbol{G}}$ and an equivalence $\boldsymbol{G} \to I_{\mathcal{T}^*/\mathcal{T}}$:

Construction 2.2.4. The theory $\mathcal{T}_{\boldsymbol{G}}$ expands $\mathcal{T}$ by an additional sort $\boldsymbol{X}$, a function symbol $c : \boldsymbol{X} \to \boldsymbol{G}_0$, and a function symbol $a : \boldsymbol{X} \times_{\boldsymbol{G}_0} \boldsymbol{G}_1 \to \boldsymbol{X}$, where $\boldsymbol{X} \times_{\boldsymbol{G}_0} \boldsymbol{G}_1 = \{\langle x, g\rangle \mid c(x) = d(g)\}$, and $\mathcal{T}_{\boldsymbol{G}}$ states that $\boldsymbol{X}$ is nonempty, and that the resulting structure is a groupoid $\boldsymbol{G}^*$ extending $\boldsymbol{G}$ by an additional object $*$, with elements $x \in \boldsymbol{X}$ viewed as morphisms from $*$ to $c(x)$, and a provides the composition of such elements with morphisms of $\boldsymbol{G}$.

Starting from this $\mathcal{T}_{\boldsymbol{G}}$, each element of $\boldsymbol{X}$ exhibits $\mathcal{T}_{\boldsymbol{G}}$ as an internal cover of $\mathcal{T}$. The type of such an element $x \in \boldsymbol{X}$ over $\mathcal{T}$ is given by $c(x)$, and the realisations of this type are indeed the morphisms from $*$ to $c(x)$, so Construction 2.2.1 indeed recovers $\boldsymbol{G}$. □

For example, when $\boldsymbol{G}$ is a definable group (groupoid with one object), the resulting $\mathcal{T}_{\boldsymbol{G}}$ is the theory of $\boldsymbol{G}$-torsors. We refer to [4, §3] for more details, but note again that our construction is slightly different when $\boldsymbol{G}$ is not connected: we always expand just by one additional object $*$, thus obtaining an internal cover (possibly incomplete), even in the nonconnected case.

Alternatively, Definition 3.3.1 is a generalisation that also applies to this case.

2.2.5. *Definable $\boldsymbol{G}$-sets.* If $\boldsymbol{G} = \langle \boldsymbol{G}_0, \boldsymbol{G}_1\rangle$ is a definable groupoid in $\mathcal{T}$, by a (left) *$\boldsymbol{G}$-set* we mean a definable set $\boldsymbol{X}$, a definable map $\pi : \boldsymbol{X} \to \boldsymbol{G}_0$ to the set of objects $\boldsymbol{G}_0$ of $\boldsymbol{G}$, and an "action" map $a : \boldsymbol{G}_1 \times_{\boldsymbol{G}_0} \boldsymbol{X} \to \boldsymbol{X}$, over $\boldsymbol{G}_0$, satisfying the usual action axioms (here, $\boldsymbol{G}_1 \times \boldsymbol{G}_0 \boldsymbol{X}$ is the definable subset of $\boldsymbol{G}_1 \times \boldsymbol{X}$ given by $d(g) = \pi(x)$, and "over $\boldsymbol{G}_0$" means that $c(g) = \pi(a(g, x))$ for all such pairs). Thus, a morphism $g : a \to b$ in $\boldsymbol{G}$ determines a bijection $a_g : \boldsymbol{X}_a \to \boldsymbol{X}_b$, where $\boldsymbol{X}_t = \pi^{-1}(t)$, and we sometimes write gx in places of $a_g(x)$ (a pair $\langle \boldsymbol{G}, \boldsymbol{X}\rangle$ as above is called a *concrete* *groupoid* in [4, §3]). We think of $\boldsymbol{G}$-sets as analogues of local systems over $\boldsymbol{G}$. A morphism from a $\boldsymbol{G}$-set $\boldsymbol{X}$ to another $\boldsymbol{G}$-set $\boldsymbol{Y}$ is a definable map from $\boldsymbol{X}$ to $\boldsymbol{Y}$ that commutes with π and a.

Let $\boldsymbol{X}$ be a $\boldsymbol{G}$-set. If $\boldsymbol{H} = \langle \boldsymbol{H}_0, \boldsymbol{H}_1 \rangle$ is another groupoid, and $i : \boldsymbol{G} \to \boldsymbol{H}$ is a definable map of groupoids, we set

$$i_!(\boldsymbol{X}) = \{\langle h, x\rangle \in \boldsymbol{H}_1 \times \boldsymbol{X} \mid i(\pi(x)) = \boldsymbol{d}(h)\}/\sim,$$

where $\langle h, gx\rangle \sim \langle h \circ i(g), x\rangle$ for $g \in \boldsymbol{G}_1$ satisfying $\boldsymbol{d}(g) = \pi(x)$ and $i(\boldsymbol{c}(g)) = \boldsymbol{d}(h)$. This is an $\boldsymbol{H}$-set, with structure map induced by $\langle h, x\rangle \mapsto \boldsymbol{c}(h)$ and action induced by $\langle h', \langle h, x\rangle\rangle \mapsto \langle h'h, x\rangle$. On the other hand, if $\boldsymbol{Y}$ is an $\boldsymbol{H}$-set, we set $i^*(\boldsymbol{Y}) = \boldsymbol{G}_0 \times_{\boldsymbol{H}_0} \boldsymbol{Y}$, with the projection to $\boldsymbol{G}_0$ as the structure map, and action given by $\langle g, y\rangle \mapsto i(g)y$ for $y \in \boldsymbol{Y}$ with $\pi(y) = i(\boldsymbol{d}(g))$. It is clear that both constructions are functorial, and as the notation suggests, $i_!$ is left adjoint to i^*.

With these notions, we have the following description of definable sets in $\mathcal{T}^*$ as local systems over $\boldsymbol{G}$:

Proposition 2.2.6. *If $\mathcal{T}^*$ is an internal cover of $\mathcal{T}$, corresponding to the definable groupoid $\boldsymbol{G}$ in $\mathcal{T}$, then the category of definable sets in $\mathcal{T}^*$ is equivalent to the category of $\boldsymbol{G}$-sets in $\mathcal{T}$. Definable sets from $\mathcal{T}$ correspond to themselves, with trivial action.*

Proof. To each definable set $\boldsymbol{X}^*$ in $\mathcal{T}^*$ we assign the definable set $\boldsymbol{X} = \coprod_{p \in \boldsymbol{G}_0} p(\boldsymbol{X}^*)$. It follows from the uniformity of p that $\boldsymbol{X}$ is definable in $\mathcal{T}$. By definition, $\boldsymbol{X}$ admits a definable map to $\boldsymbol{G}_0$. The action is given tautologically by the identification of the morphisms in $\boldsymbol{G}_0$ with maps of interpretations. Since each p is an interpretation, this is functorial in $\boldsymbol{X}^*$.

In the other direction, let $\boldsymbol{G}^*$ be the canonical extension of $\boldsymbol{G}$ in $\mathcal{T}^*$ (we identify $\boldsymbol{G}$ with its image in $\mathcal{T}^*$), let $i : \boldsymbol{G} \to \boldsymbol{G}^*$ be the inclusion, and let j be the inclusion of the canonical object $*$ of $\boldsymbol{G}^*$, along with its automorphism group $\boldsymbol{H}$, into $\boldsymbol{G}^*$. A definable $\boldsymbol{G}$-set $\boldsymbol{X}$ in $\mathcal{T}$, viewed again as embedded in $\mathcal{T}^*$, corresponds then to $\boldsymbol{X}^* = j^*(i_!(\boldsymbol{X}))$ (and the resulting action by $\boldsymbol{H}$ is the natural action by automorphisms). □

We note that each definable set in $\mathcal{T}^*$ comes equipped with an action of the binding group $\underline{\mathbf{Aut}}(*)$, and with it, the first direction could likewise be described as $\boldsymbol{X} = i^*(j_!(\boldsymbol{X}^*))$.

Corollary 2.2.7. *If $\boldsymbol{G}$ is a groupoid associated to an internal cover $\mathcal{T}^*$ of $\mathcal{T}$, then $\mathcal{T}^*$ is bi-interpretable with $\mathcal{T}_{\boldsymbol{G}}$ over $\mathcal{T}$.*

Proof. The definable groupoid $\boldsymbol{G}^*$ in $\mathcal{T}^*$ forms an interpretation of $\mathcal{T}_{\boldsymbol{G}}$ over $\mathcal{T}$. It is a bi-interpretation since both categories of definable sets are equivalent to the category of $\boldsymbol{G}$-sets in $\mathcal{T}$ (commuting with the above interpretation). □

2.2.8. *Pushouts.* Let $g : \boldsymbol{K} \to \boldsymbol{G}$ and $h : \boldsymbol{K} \to \boldsymbol{H}$ be maps of definable groupoids, and assume that g is fully faithful. We construct another definable groupoid $\boldsymbol{G} \otimes_{\boldsymbol{K}} \boldsymbol{H}$ that can be viewed as the pushout of $\boldsymbol{G}$ and $\boldsymbol{H}$ over $\boldsymbol{K}$ as follows: For objects, we let $(\boldsymbol{G} \otimes_{\boldsymbol{K}} \boldsymbol{H})_0 = \boldsymbol{G}_0 \amalg \boldsymbol{H}_0$. If a, b are two such objects, we define the morphisms as follows:

(1) If $a, b \in \boldsymbol{H}_0$, then $(\boldsymbol{G} \otimes_{\boldsymbol{K}} \boldsymbol{H})(a, b) = \boldsymbol{H}(a, b)$.

(2) If $a \in \boldsymbol{G}_0$ and $b \in \boldsymbol{H}_0$, morphisms from a to b are equivalence classes $v \otimes u$ of pairs $\langle v, u\rangle$, where $u \in \boldsymbol{G}(a, g(c))$, $v \in \boldsymbol{H}(h(c), b)$ for some $c \in \boldsymbol{K}_0$, and $\langle v, g(w) \circ u\rangle$ is equivalent to $\langle v \circ h(w), u\rangle$ for all $w \in \boldsymbol{K}_1$ for which the composition is defined. Morphisms from b to a are defined analogously.

(3) If $a, b \in \boldsymbol{G}_0$ are both in the essential image of g, a morphism from a to b is similarly defined as an equivalence class $u' \otimes v \otimes u$, with $u, u' \in \boldsymbol{G}_1$ and $v \in \boldsymbol{H}_1$.

(4) If either of $a, b \in \boldsymbol{G}_0$ is not in the essential image of g, then morphisms are the same as in $\boldsymbol{G}$.

The composition $(u' \otimes v \otimes u) \circ (u'_1 \otimes v_1 \otimes u_1)$ is defined as follows: There are $a, b \in \boldsymbol{K}_0$ such that $u \circ u'_1$ is a morphism from $g(a)$ to $g(b)$. Since g is fully faithful, it has the form $g(w)$ for a unique morphism w from a to b in $\boldsymbol{K}$. We define the composition to be $u' \otimes (v \circ h(w) \circ v_1) \otimes u_1$. It is clear that this is independent of the choices of representatives. The composition in the other cases is defined similarly.

There is an obvious map $h' : \boldsymbol{H} \to \boldsymbol{G} \otimes_{\boldsymbol{K}} \boldsymbol{H}$, and we define $g' : \boldsymbol{G} \to \boldsymbol{G} \otimes_{\boldsymbol{K}} \boldsymbol{H}$ by sending each object to itself, each morphism between objects not in the essential image of g to itself as well, and for $a, b \in \boldsymbol{G}_0$ in the essential image of g, and u a morphism from a to b, we set $g'(u) = (u \circ u'^{-1}) \otimes \mathbf{1}_{h(c)} \otimes u'$, where $u' : a \to g(c)$ is any morphism and $c \in \boldsymbol{K}_0$. We have an isomorphism α from $h' \circ h$ to $g' \circ g$, given on an object $c \in \boldsymbol{K}_0$ by $\mathbf{1}_{g(c)} \otimes \mathbf{1}_{h(c)}$. It is routine to check that everything is well defined, and also that the following statement holds.

Proposition 2.2.9. *Let $g : \boldsymbol{K} \to \boldsymbol{G}$, $h : \boldsymbol{K} \to \boldsymbol{H}$ and the rest of the notation be as above.*

(1) *Given definable maps of groupoids $g_1 : \boldsymbol{G} \to \boldsymbol{F}$ and $h_1 : \boldsymbol{H} \to \boldsymbol{F}$, and an isomorphism $\beta : h_1 \circ h \to g_1 \circ g$, there is a unique map of groupoids $f : \boldsymbol{G} \otimes_{\boldsymbol{K}} \boldsymbol{H} \to \boldsymbol{F}$ that coincides with g_1 and h_1 on the objects, $f \circ g' = g_1$, $f \circ h' = h_1$ and such that $f \cdot \alpha = \beta$.*

(2) *$h' : \boldsymbol{H} \to \boldsymbol{G} \otimes_{\boldsymbol{K}} \boldsymbol{H}$ is fully faithful. If g is a weak equivalence, then so is h'.*

Remark 2.2.10. We could make a similar construction where the set of objects is $\boldsymbol{G}_0 \amalg_{\boldsymbol{K}_0} \boldsymbol{H}_0$ in place of the disjoint union, and with α the identity. The last proposition provides a map from $\boldsymbol{G} \otimes_{\boldsymbol{K}} \boldsymbol{H}$ to this variant, which is easily seen to be a weak equivalence. We use the two constructions interchangeably. □

Remark 2.2.11. Without the assumption that one of the maps is fully faithful, the pushout need not be definable. For example, when all groupoids are groups, this is the usual free product with amalgamation. □

2.3. ***Maps of groupoids and of interpretations.*** With stable interpretations over $\mathcal{T}$, the assignment $\mathcal{T}^* \mapsto I_{\mathcal{T}^*/\mathcal{T}}$ is contravariantly functorial in $\mathcal{T}^*$, and fully faithful: a stable interpretation $i : \mathcal{T}_1 \to \mathcal{T}_2$ over $\mathcal{T}$ induces a functor $i^* : I_{\mathcal{T}_2/\mathcal{T}} \to I_{\mathcal{T}_1/\mathcal{T}}$ by composition.

In the other direction, if $f : \boldsymbol{G} \to \boldsymbol{H}$ is a map of definable groupoids, corresponding to internal covers $\mathcal{T}_{\boldsymbol{G}}$ and $\mathcal{T}_{\boldsymbol{H}}$, f determines a stable interpretation $i^f : \mathcal{T}_{\boldsymbol{H}} \to \mathcal{T}_{\boldsymbol{G}}$ over $\mathcal{T}$ that can be described in at least two ways:

(1) An interpretation of $\mathcal{T}_{\boldsymbol{H}}$ over $\mathcal{T}$ is determined by its value on the extended groupoid $\boldsymbol{H}^*$ defined in $\mathcal{T}_{\boldsymbol{H}}$. We set $i^f(\boldsymbol{H}^*) = \boldsymbol{G}^* \otimes_{\boldsymbol{G}} \boldsymbol{H}$ (with respect to the given map f). This makes sense since the inclusion of $\boldsymbol{G}$ in $\boldsymbol{G}^*$ is a weak equivalence, and is an interpretation since the embedding of $\boldsymbol{H}$ in $\boldsymbol{G}^* \otimes_{\boldsymbol{G}} \boldsymbol{H}$ is a weak equivalence that misses precisely one object $*$, and this completely determines its theory. To see that it is stable, we may first choose a parameter in $\boldsymbol{G}^*$. But then i^f is identified with one of the standard interpretations into $\mathcal{T}$.

(2) Alternatively, we may use Proposition 2.2.6 to identify definable sets in $\mathcal{T}_{\boldsymbol{G}}$ and in $\mathcal{T}_{\boldsymbol{H}}$ with $\boldsymbol{G}$- and $\boldsymbol{H}$-sets in $\mathcal{T}$. Then i^f is identified with f^* (in this approach, it is less direct to see that one gets a stable interpretation).

It is easy to verify that $(i^f)^* = f$ (after identifying $\boldsymbol{G}$ with its image in $I_{\mathcal{T}_G/\mathcal{T}}$ via Proposition 2.2.3, and similarly for $\boldsymbol{H}$). However, not every stable interpretation $i : \mathcal{T}_{\boldsymbol{H}} \to \mathcal{T}_{\boldsymbol{G}}$ (over $\mathcal{T}$) is of the form i^f for some $f : \boldsymbol{G} \to \boldsymbol{H}$. The other source of interpretations comes from the other operation described in 2.2.5: when f is a weak equivalence, the composition of f with the inclusion of $\boldsymbol{H}$ in $\boldsymbol{H}^*$ is a weak equivalence, so restricting to the image of f (on the objects), we obtain an interpretation i_f of $\boldsymbol{G}^*$ (hence of $\mathcal{T}_{\boldsymbol{G}}$).

Proposition 2.3.1. *Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be two definable groupoids, with associated covers $\mathcal{T}_{\boldsymbol{G}}$ and $\mathcal{T}_{\boldsymbol{H}}$. Then every stable interpretation $i : \mathcal{T}_{\boldsymbol{G}} \to \mathcal{T}_{\boldsymbol{H}}$ over $\mathcal{T}$ is obtained as a composition $i = i^f \circ i_g$, for some definable groupoid $\boldsymbol{K}$, definable map $f : \boldsymbol{H} \to \boldsymbol{K}$ and weak equivalence $g : \boldsymbol{G} \xrightarrow{\sim} \boldsymbol{K}$.*

In particular, if $i : \mathcal{T}_1 \to \mathcal{T}_2$ is a stable interpretation of internal covers over $\mathcal{T}$, we may choose definable groupoids $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ corresponding to the $\mathcal{T}_i$, so that i is induced by a map $f : \boldsymbol{G}_2 \to \boldsymbol{G}_1$ of groupoids (up to bi-interpretation).

A configuration of the form $\langle \boldsymbol{K}, f, g\rangle$ as above is called a *cospan* from $\boldsymbol{H}$ to $\boldsymbol{G}$.

Proof. $\boldsymbol{H}$ embeds in $I_{\mathcal{T}_{\boldsymbol{H}}/\mathcal{T}}$ via Proposition 2.2.3, which maps via i^* to $I_{\mathcal{T}_{\boldsymbol{G}}/\mathcal{T}}$. We set $f : \boldsymbol{H} \to \boldsymbol{K}$ to be the restriction of i^* to $\boldsymbol{H}$, where $\boldsymbol{K}$ denotes any definable weakly equivalent subgroupoid of $I_{\mathcal{T}_G/\mathcal{T}}$, which also contains $\boldsymbol{G}$. Then g is the inclusion of $\boldsymbol{G}$ in $\boldsymbol{K}$.

The last part follows (using Corollary 2.2.7) by choosing $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ arbitrarily, and then replacing $\boldsymbol{G}_1$ by $\boldsymbol{K}$ as above. □

As in the construction of the pushout, we may choose $\boldsymbol{K}$ so that its objects are the disjoint union of the objects of $\boldsymbol{G}$ and $\boldsymbol{H}$, and we always assume that this is the case. In the case when i is a bi-interpretation, we recover the notion of equivalence from [4, §3].

2.3.2. *Composition and isomorphisms.* Assume that for groupoids $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{H}$ in $\mathcal{T}$, we are given interpretations $i : \mathcal{T}_{\boldsymbol{F}} \to \mathcal{T}_{\boldsymbol{G}}$ and $j : \mathcal{T}_{\boldsymbol{G}} \to \mathcal{T}_{\boldsymbol{H}}$, represented by cospans $g_1 : \boldsymbol{F} \overset{\sim}{\dashrightarrow} \boldsymbol{K}_1$, $f_1 : \boldsymbol{G} \to \boldsymbol{K}_1$, $g_2 : \boldsymbol{G} \overset{\sim}{\dashrightarrow} \boldsymbol{K}_2$ and $f_2 : \boldsymbol{H} \to \boldsymbol{K}_2$ as in Proposition 2.3.1. Since g_2 is a weak equivalence, we may form the pushout $\boldsymbol{K} = \boldsymbol{K}_1 \otimes_{\boldsymbol{G}} \boldsymbol{K}_2$. By Proposition 2.2.9, the map from $\boldsymbol{K}_1$ to $\boldsymbol{K}$ is a weak equivalence, and therefore so is the composed map g. Hence, g along with the composed map $f : \boldsymbol{H} \to \boldsymbol{K}$ form a cospan that represents a stable interpretation of $\mathcal{T}_{\boldsymbol{F}}$ in $\mathcal{T}_{\boldsymbol{H}}$. To conform with the decision about the objects of the representing groupoid $\boldsymbol{K}$, we remove the intermediate two copies of $\boldsymbol{G}_0$, and denote the resulting groupoid by $\boldsymbol{K}_2 \circ \boldsymbol{K}_1 = \boldsymbol{K}_2 \circ_{\boldsymbol{G}} \boldsymbol{K}_1$ (though it does depend on the additional data). The following is a direct calculation.

Proposition 2.3.3. *In the above situation, the maps* $g : \boldsymbol{F} \overset{\sim}{\dashrightarrow} \boldsymbol{K}_2 \circ \boldsymbol{K}_1$ *and* $f : \boldsymbol{H} \to \boldsymbol{K}_2 \circ \boldsymbol{K}_1$ *represent the composed interpretation* $j \circ i$.

Finally, we consider isomorphisms of interpretations between (stable) interpretations of internal covers over $\mathcal{T}$.

Proposition 2.3.4. *Let* $i, j : \mathcal{T}_{\boldsymbol{G}_1} \to \mathcal{T}_{\boldsymbol{G}_2}$ *be two stable interpretations of internal covers over* $\mathcal{T}$. *Assume* i *is represented by a cospan* $i_1 : \boldsymbol{G}_1 \overset{\sim}{\dashrightarrow} \boldsymbol{H}_1$ *and* $i_2 : \boldsymbol{G}_2 \to \boldsymbol{H}_1$, *and* j *by* $j_1 : \boldsymbol{G}_1 \overset{\sim}{\dashrightarrow} \boldsymbol{H}_2$, $j_2 : \boldsymbol{G}_2 \to \boldsymbol{H}_2$, *with each set of objects of* $\boldsymbol{H}_n$ *the disjoint union of the objects of* $\boldsymbol{G}_1$ *and* $\boldsymbol{G}_2$ (*realised by the object parts of* i_k *and* j_k).

Then there is a natural bijection between isomorphisms $\alpha : i \to j$ (*over* $\mathcal{T}$) *and isomorphisms* $\widetilde{\alpha} : \boldsymbol{H}_1 \to \boldsymbol{H}_2$ *which are the identity on the images of* $\boldsymbol{G}_1, \boldsymbol{G}_2$.

As an example, if $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are groups, then each $\boldsymbol{H}_i$ corresponds to a $\boldsymbol{G}_1 - \boldsymbol{G}_2$ bitorsor, and an isomorphism of the corresponding interpretations corresponds to an isomorphism of such bitorsors.

Proof. Let $\boldsymbol{P}_l$ be the set of arrows in $\boldsymbol{H}_l$ with domain in $\boldsymbol{G}_1$ and codomain in $\boldsymbol{G}_2$. This is a $\boldsymbol{G}_1$-set, with structure given by the domain map and composition. The interpretation i takes $\boldsymbol{P}_1$ to the $\boldsymbol{G}_2$-set given by viewing the arrows in $\boldsymbol{P}_1$ in the other direction, and likewise with j and $\boldsymbol{P}_2$. So the map α maps $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$, compatibly with the composition. This is the same as giving an isomorphism $\widetilde{\alpha}$ as in the statement.

The rest of the structure is induced by the $\boldsymbol{P}_i$, so α is determined by $\widetilde{\alpha}$. Conversely, each $\widetilde{\alpha}$ as in the statement extends to an interpretation. □

We summarise most of the content of this section in the following theorem (mostly contained in [4, §3]):

Theorem 2.3.5. *Let $\mathcal{T}$ be a theory. Each internal cover $\mathcal{T}^*$ of $\mathcal{T}$ is bi-interpretable over $\mathcal{T}$ with an internal cover of the form $\mathcal{T}_{\boldsymbol{G}}$. An interpretation of $\mathcal{T}_{\boldsymbol{H}}$ in $\mathcal{T}_{\boldsymbol{G}}$ corresponds to a cospan from $\boldsymbol{G}$ to $\boldsymbol{H}$, and each such cospan determines an interpretation. Maps between interpretations correspond to maps between cospans.*

In particular, covers $\mathcal{T}_1$ and $\mathcal{T}_2$ are bi-interpretable over $\mathcal{T}$ if and only if the corresponding groupoids are equivalent.

More succinctly (and slightly more precisely), the bicategory of internal covers over $\mathcal{T}$ is equivalent to the bicategory of definable groupoids in $\mathcal{T}$ (with morphisms given by cospans and morphisms between them given by bitorsors). See also Remark 3.3.4.

Proof. This is a combination of Corollary 2.2.7 with Propositions 2.3.1 and 2.3.4. □

The description above exhibits the groupoid associated to an expansion as interpretations of $\mathcal{T}^*$ in $\mathcal{T}$. In [4], it was suggested that definable sets of an internal cover of $\mathcal{T}$ can be viewed as generalised imaginary sorts of $\mathcal{T}$. With this point of view, it is natural to ask for the structure classifying interpretations of such sorts as well. However, such generalised sorts have more structure: in addition to the sorts themselves and maps between them (interpretations), we also have maps between maps. The notion of equivalence should be modified as well: it is no longer reasonable to expect a bijection on the level of morphisms. In fact, as the 1-dimensional case already shows, it is not reasonable to expect even a map.

3. Generalised imaginaries

We now suggest how internal covers can play the role of definable sets in the above description, by going one dimension higher.

3.1. *Higher internal covers.*

Definition 3.1.1. Let $\mathcal{T}$ be a theory, $\mathcal{T}_1$ and $\mathcal{T}_2$ internal covers of $\mathcal{T}$. For every set of parameters A for $\mathcal{T}$, we denote by $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2)(A)$ the groupoid whose objects are stable interpretations of $\mathcal{T}_1$ in $\mathcal{T}_{2A}$, over $\mathcal{T}_A$, and whose morphisms are isomorphisms of interpretations over $\mathcal{T}$.

Thus, what we denoted by I above is $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^*, \mathcal{T})$. Similar to that case, $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2)$ is ind-definable in $\mathcal{T}$: if $\mathcal{T}_i = \mathcal{T}_{\boldsymbol{G}_i}$ for $\mathcal{T}$-definable groupoids $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, each interpretation above can be described like in Section 2.3 as given by certain definable maps $\boldsymbol{G}_i \to \boldsymbol{H}$, a definable condition. Similarly, isomorphisms between interpretations are given by the $\mathcal{T}$-definable families of maps as in Proposition 2.3.4, uniform in the $\boldsymbol{H}_i$ (see Section 3.3.6 for a more detailed description).

An interpretation between theories extends to internal sorts: if $i : \mathcal{T} \to \mathcal{S}$ is an interpretation, and $\widetilde{\mathcal{T}}$ is an internal cover of $\mathcal{T}$ associated to the groupoid $\boldsymbol{G}$ in $\mathcal{T}$, we denote by $i(\widetilde{\mathcal{T}})$ the internal cover of $\mathcal{S}$ associated to $i(\boldsymbol{G})$.

We now wish to define (slightly) higher analogues of stable embeddings and internal covers. One discrepancy with the 1-dimensional case occurs as follows: If $\mathcal{T}$ is an internal cover of $\mathcal{T}_0$, we might be interested in only part of the structure on $\mathcal{T}$ when considering, for example, the Galois group. As long as this partial structure includes the definable sets witnessing the internality, this can be done be replacing $\mathcal{T}$ with a reduct including only those definable sets. In the higher version, definable sets are replaced by definable groupoids in $\mathcal{T}$ (equivalently, internal covers), and again we may wish to restrict to a partial collection. However, there is no reason to expect that this partial collection is the full collection of definable groupoids in some reduct of $\mathcal{T}$. Furthermore, the internality condition for 0-definable sets automatically implies it for definable sets over parameters. Again, there is no reason to expect a similar statement for groupoids. For this reason, our definition depends on the auxiliary data Γ consisting of families of definable groupoids, which are the groupoids we wish to preserve. More precisely, we have the following.

Definition 3.1.2. Let $\mathcal{T}$ be a theory. The data of *distinguished covers* for $\mathcal{T}$ consists of the following:

(1) An ind-definable family Γ_0 of internal covers of $\mathcal{T}$ (equivalently, of definable groupoids in $\mathcal{T}$).

(2) An ind-definable family of interpretations over $\mathcal{T}$ between any two covers $\mathcal{T}_1, \mathcal{T}_2 \in \Gamma_0$, depending definably on $\mathcal{T}_1, \mathcal{T}_2$ and closed under composition (the full definable family is denoted by Γ_1).

(3) For every two interpretations $f, g : \mathcal{T}_1 \to \mathcal{T}_2$ in Γ_1, an ind-definable family of isomorphisms from f to g, closed under composition and restricting to the identity on $\mathcal{T}$. Again we assume that the family of all such isomorphisms is uniformly (ind-)definable in f, g, and denote it by Γ_2.

If $\mathcal{T}_0$ is a reduct of $\mathcal{T}$, we say that $\Gamma = \langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ is over $\mathcal{T}_0$ if the parameters for the ind-definable families above range over definable sets in $\mathcal{T}_0$.

We note that in terms of definable groupoids, the closure under composition translates to closure under the composition operation from Section 2.3.2.

If a theory $\mathcal{T}$ is given with a collection Γ of distinguished covers, we often omit further explicit reference to Γ, and call them *admissible covers*. We modify notions like bi-interpretation etc., to be with respect to Γ. In particular, the notation $\mathrm{Hom}_{\mathcal{T}}$ refers to admissible covers and admissible maps.

Definition 3.1.3. Let $i : \mathcal{T} \to \mathcal{S}$ be an interpretation, and let Γ be a collection of distinguished covers of $\mathcal{T}$. We say that the interpretation i is 2-*stable* (with respect to Γ) if for every two internal covers $\mathcal{T}_1, \mathcal{T}_2$ in Γ over each $\mathcal{T}$-structure A, the natural map $i(\mathrm{Hom}_{\mathcal{T}_A}(\mathcal{T}_1, \mathcal{T}_2)) \to \mathrm{Hom}_{\mathcal{S}_A}(i(\mathcal{T}_1), i(\mathcal{T}_2))$ is an equivalence.

If Γ is omitted, we take it to be all definable groupoids in $\mathcal{T}$, and all definable morphisms among them.

The expression $i(\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2))$ makes sense, since, as we had noted above, $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2)$ is definable in $\mathcal{T}$.

Proposition 3.1.4. *A stable interpretation $i : \mathcal{T}_1 \to \mathcal{T}_2$ is 2-stable.*

Proof. We may replace $\mathcal{T}$ by $\mathcal{T}_A$, and thus assume that $A = \varnothing$. Let $\mathcal{T}_1, \mathcal{T}_2$ be internal covers of $\mathcal{T}$. The statement is invariant when replacing each cover with a bi-interpretable one (over $\mathcal{T}$). Hence, we may assume that $\mathcal{T}_1 = \mathcal{T}_{\boldsymbol{G}_1}$ and $\mathcal{T}_2 = \mathcal{T}_{\boldsymbol{G}_2}$, the covers associated to definable connected groupoids $\boldsymbol{G}_1, \boldsymbol{G}_2$ in $\mathcal{T}$.

According to Proposition 2.3.1, we may choose $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ so that a stable interpretation of $i(\mathcal{T}_1)$ in $i(\mathcal{T}_2)$ corresponds to a definable map of groupoids from $i(\boldsymbol{G}_2)$ to $i(\boldsymbol{G}_1)$. Since i is stable, this map comes from a map in $\mathcal{T}$ (and similarly for morphisms). □

The definition of a 2-cover is analogous to that of an internal cover, as formulated in Proposition 2.1.2:

Definition 3.1.5. A 2-*internal cover* of a theory $\mathcal{T}$ consists of a theory $\mathcal{T}^*$, a collection Γ of families of internal covers of $\mathcal{T}^*$ over $\mathcal{T}$, and a stable embedding $\mathcal{T} \to \mathcal{T}^*$ such that $\langle \mathcal{T}^*, \Gamma \rangle$ admits a 2-stable interpretation p in $\mathcal{T}_A$, over $\mathcal{T}$ (for some set of parameters A).

More explicitly, we require that each internal cover in Γ is bi-interpretable, over parameters in $\mathcal{T}^*$, with one coming from $\mathcal{T}$, in a manner coherent with interpretations over $\mathcal{T}^*$. Or, via the equivalence with groupoids, that for each family of definable groupoids in Γ there is a set of parameters B in $\mathcal{T}^*$ such that each groupoid in the family is equivalent, over B, to one coming from $\mathcal{T}$ (again, in a coherent manner).

As in the 1-dimensional case, the typical examples come from higher-dimensional groupoids, which we review next.

3.2. ***Higher categories and higher groupoids.*** We recall a few definitions from homotopy theory and higher category theory, adapted to our language and setup. Though our main references are [8; 9], the ideas seem to originate in [5] (and in the main case of groupoids, much more classically). We are interested in the notions of n-category and n-groupoid discussed in [8, §2.3.4] (through most other parts of the paper we are interested in the case $n = 2$, but here it is convenient and harmless to work in general). There, they are defined as special cases of quasicategories and

Kan simplicial sets, respectively, but for us it is more convenient to use terminology that makes explicit the finite nature of these structures. The following definitions are a variant of the description in [8, §2.3.4.9], which gives an equivalent condition (in the case of simplicial sets).

For each $i \in \mathbb{N}$, we denote by $[i]$ the ordered set $\{0, \dots, i\}$. For $k \in [i]$, we identify k with the map $[0] \to [i]$ taking 0 to k (writing k_i if needed), and we let $\hat{k} = \hat{k}_i : [i-1] \to [i]$ be the unique increasing map with k not in the image. We fix a natural number n (one could also allow $n = \infty$ to obtain the usual definitions of quasicategories and spaces, but we do not use them).

Definition 3.2.1. The signature Σ_n of n-simplicial sets consists of

(1) a sort $\boldsymbol{G}_i$, for $0 \le i \le n+1$,

(2) for each weakly increasing map $t : [i] \to [j]$, where $i, j \le n+1$, a function symbol $d_t : \boldsymbol{G}_j \to \boldsymbol{G}_i$.

3.2.2. *Notation.* We define the following auxiliary notation. We let $\boldsymbol{G}_{-1}$ be the one element set. For each $0 \le m \le n+1$, and $i \le m$, the map $d_{\hat{i}} : \boldsymbol{G}_m \to \boldsymbol{G}_{m-1}$ is called the *i-th face map*. We denote by $\partial = \partial_m : \boldsymbol{G}_m \to \boldsymbol{G}_{m-1}^{m+1}$ the Cartesian product of these maps, and by $\partial^{\hat{i}} : \boldsymbol{G}_m \to \boldsymbol{G}_{m-1}^{m}$ the Cartesian product with i omitted. If $g \in \boldsymbol{G}_m$ and $t : [k] \to [m]$, we sometimes write g_t in place of $d_t(g)$ (in particular, for $t = \hat{i}$ or $t = i$).

For $m \ge -1$, the set $\boldsymbol{G}^\circ_{m+1}$ of $(m+1)$*-cycles* is the definable subset of ${\boldsymbol{G}_m}^{m+2}$ given by the conjunction of the equations $d_{\hat{i}}(x_j) = d_{\hat{k}}(x_l)$ for all $0 \le j, l \le m+1$, $0 \le i, k \le m$ satisfying $\hat{j} \circ \hat{i} = \hat{l} \circ \hat{k} : [m-1] \to [m+1]$. (So no conditions when $m = 0$. Note that $(m+1)$-cycles are potential boundaries of $(m+1)$-dimensional elements, but are themselves m-dimensional. This is compatible with the notation in [8].)

For each $0 \le i \le m+1$, the set $\Lambda_i^{m+1}(\boldsymbol{G})$ of *i-th* $(m+1)$*-horns* is the subset of ${\boldsymbol{G}_m}^{m+1}$ defined by the same conditions, with x_i omitted. Hence, the projection $\pi_{\hat{i}} : \boldsymbol{G}^\circ_m \to \boldsymbol{G}^m_{m-1}$ omitting the i-th coordinate takes values in $\Lambda_i^m(\boldsymbol{G})$.

We extend the notation by inductively setting $\boldsymbol{G}_m = \boldsymbol{G}^\circ_m$ for $m > n+1$, and $d_{\hat{i}} : \boldsymbol{G}_m \to \boldsymbol{G}_{m-1}$ the i-th projection ($0 \le i \le m$). Consequently, all the above notation makes sense for arbitrary natural number m.

Definition 3.2.3. Let $n \ge 1$. The theory $\mathcal{C}_n$ of n-categories in this signature says:

(1) $d_{t \circ s} = d_s \circ d_t$ for s, t composable, d_t is the identity whenever t is. It follows that ∂_m takes values in $\boldsymbol{G}^\circ_m$ and $\partial^{\hat{i}}_m$ in $\Lambda^m_i(\boldsymbol{G})$.

(2) For each $0 < m \le n+1$ and each $0 < i < m$, the map $\partial^{\hat{i}}_m : \boldsymbol{G}_m \to \Lambda^m_i(\boldsymbol{G})$ is surjective.

(3) For $m = n+1, n+2$, $\partial^{\hat{i}}_m$ is bijective for each $0 < i < m$.

The theory $\mathcal{G}_n$ of n-groupoids is the extension of $\mathcal{C}_n$ where the conditions above are required to also hold for $i = 0, m$.

Note that by the third condition, the set $\boldsymbol{G}_{n+1}$ is completely determined by the rest of the data. However, it is still convenient to have it for the statement of the axiom. It follows from the axioms that the unique map $d_f : \boldsymbol{G}_0 \to \boldsymbol{G}_m$ is injective, and we use it to identify $\boldsymbol{G}_0$ with its image in each $\boldsymbol{G}_m$, writing a or a^m for $d_f(a)$ (this map assigns to each object a the m-dimensional identity morphisms at a).

The first condition (when $n = \infty$) is the usual definition of a simplicial set, the second is the definition of quasicategory (or space, in the case of a groupoid, where it is called the Kan condition), and the third specifies that the object is an n-category, rather than a quasicategory. By a *definable n-category* or a *definable n-groupoid* in $\mathcal{T}$ we mean an interpretation in $\mathcal{T}$ of the respective theory.

The intuition is, roughly speaking, that the horns represent configurations of (higher) composable arrows, but the composition (represented by the element g) need not be uniquely determined, except on the highest dimension. We refer to the first chapters of [8] for further explanations, but explain how the case $n = 1$ of the formalism recovers usual categories and groupoids:

Example 3.2.4. A category can be viewed as a 1-category in the above sense by taking $\boldsymbol{G}_0$ the set of objects, $\boldsymbol{G}_1$ the set of morphisms, and $\boldsymbol{G}_2$ the set of pairs of composable morphisms (as we are forced by the axioms). The maps $d_{\hat{0}}, d_{\hat{1}} : \boldsymbol{G}_1 \to \boldsymbol{G}_0$ are the codomain and domain maps, the unique map $\boldsymbol{G}_0 \to \boldsymbol{G}_1$ assigns to each object its identity, and the maps $d_{\hat{0}}, d_{\hat{2}}, d_{\hat{1}} : \boldsymbol{G}_2 \to \boldsymbol{G}_1$ are the two projections and the composition. The only nontrivial instances of the third conditions are when $m = 2$ and $i = 1$, which asserts that any two composable arrows have a unique composition, and when $m = 3$, which corresponds to associativity of the composition.

Conversely, each 1-category determines a category by reversing this process (and likewise for groupoids). □

As in the 1-dimensional case, the axioms imply that for $0 < i < n+1$, the relation $\boldsymbol{G}^{\circ}_{n+1}$ is the graph of a "composition" function $c_i : \Lambda^n_i(\boldsymbol{G}) \to \boldsymbol{G}_n$, by projecting to the i-coordinate. For n-groupoids, we also have such maps for $i = 0, n+1$.

Remark 3.2.5. If $\boldsymbol{G}$ is an n-category, and $m > n$, our extension of the notation determines a canonical way of viewing $\boldsymbol{G}$ as a Σ_m structure, and as such it is an m-category. Consequently, we view $\boldsymbol{G}$ as an m-category for each $m > n$. If $\boldsymbol{G}$ were an n-groupoid, it would similarly be an m-groupoid for $m > n$. □

3.2.6. *Homotopy sets.* The definition of homotopy sets admits a definable version. Let $\boldsymbol{G}$ be an n-groupoid, and let $b \in \boldsymbol{G}^{\circ}_m$ ($m \geq 0$). We let $\boldsymbol{S}(\boldsymbol{G}, b) = \partial^{-1}_m(b)$ be the set of elements of $\boldsymbol{G}_m$ with boundary b. For $\alpha, \beta \in \boldsymbol{S}(\boldsymbol{G}, b)$, we write $\alpha \sim \beta$

(or $\alpha \sim_b \beta$) if some $h \in \boldsymbol{G}_{m+1}$ satisfies $h_{\hat{0}} = \alpha$, $h_{\hat{1}} = \beta$ and $h_{\hat{i}} = d_t(\alpha)_{\hat{i}}$, where $t : [m+1] \to [m]$ is the surjective map with $t(1) = 0$ (so h is a homotopy from α to β, relative to the boundary b). This is an equivalence relation by the Kan condition. Note that when $m \geq n$, this relation coincides with equality.

For $a \in \boldsymbol{G}_0$ and $k \geq 0$, we write $\boldsymbol{S}_k(\boldsymbol{G}, a)$ for $\boldsymbol{S}(\boldsymbol{G}, b)$, where b is the constant boundary with value a in $\boldsymbol{G}^\circ_k$. These are a-definable sets, whose elements correspond to the set of pointed maps from the k-sphere to $\boldsymbol{G}$ with base point a (note that $\boldsymbol{K}_0(\boldsymbol{G}, a) = \boldsymbol{G}_0$ does not actually depend on a). The k-th *homotopy set* of $\boldsymbol{G}$ at a is the quotient $\pi_k(\boldsymbol{G}, a) = \boldsymbol{S}_k(\boldsymbol{G}, a)/\sim$ (in the case of usual simplicial sets, this is one of the equivalent definitions by [9, 00W1]).

If $f : \boldsymbol{G} \to \boldsymbol{H}$ is a groupoid map (between definable n-groupoids in the theory $\mathcal{T}$), it commutes with all the structure above, and therefore induces definable maps of sets $\pi_k(f, a) : \pi_k(\boldsymbol{G}, a) \to \pi_k(\boldsymbol{H}, f(a))$.

Definition 3.2.7. A definable map $f : \boldsymbol{G} \to \boldsymbol{H}$ of n-groupoids is a *weak equivalence* if $\pi_k(f, a) : \pi_k(\boldsymbol{G}, a) \to \pi_k(\boldsymbol{H}, f(a))$ is a bijection for all $0 \leq k \leq n$ and $a \in \boldsymbol{G}_0$.

Remark 3.2.8. More explicitly, for nonempty $\boldsymbol{G}$, the map $f : \boldsymbol{G} \to \boldsymbol{H}$ is a weak equivalence if and only if the following conditions are satisfied for each $n \geq k \geq 0$ and each $a \in \boldsymbol{G}_0$:

(1) For every $g_0, g_1 \in \boldsymbol{S}_k(\boldsymbol{G}, a)$, if $f(g_1) \sim f(g_2)$ then $g_1 \sim g_2$.

(2) For every $h \in \boldsymbol{S}_k(\boldsymbol{H}, f(a))$, there is $g \in \boldsymbol{S}_k(\boldsymbol{G}, a)$ with $f(g) \sim h$.

Alternatively, f is a weak equivalence if and only if it induces a surjective map on $\boldsymbol{S}$-classes, i.e., for each $\boldsymbol{G}$-cycle b, and each $v \in \boldsymbol{S}(\boldsymbol{H}, f(b))$, there is $u \in \boldsymbol{S}(\boldsymbol{G}, b)$ with $f(u) \sim v$. (To prove these equivalences, it suffices to show that they hold in each model, where each of these conditions is equivalent to homotopy equivalence [9, 00WV].) □

Definition 3.2.9. The n-groupoids $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are *equivalent* if there are weak equivalences $f_1 : \boldsymbol{G}_1 \to \boldsymbol{H}$ and $f_2 : \boldsymbol{G}_2 \to \boldsymbol{H}$ for some $\boldsymbol{H}$.

Example 3.2.10. Let $\boldsymbol{G}, \boldsymbol{H}$ be definable groupoids, viewed as definable 1-groupoids as in Example 3.2.4. A map $f : \boldsymbol{G} \to \boldsymbol{H}$ is a functor. For $k = 0$, the first condition in Remark 3.2.8 says that if a, b are objects of $\boldsymbol{G}$, and there is a morphism between $f(a)$ and $f(b)$ in $\boldsymbol{H}$, then there is a morphism from a to b in $\boldsymbol{G}$. The second condition says that every object h of $\boldsymbol{H}$ has a morphism to an object in the image of f. Together, this part implies that f induces a bijection on isomorphism classes.

For $k = 1$, the first condition says that if g_1, g_2 are automorphisms of a such that $f(g_1) = f(g_2)$, then $g_1 = g_2$, i.e., that f is faithful. The second condition says that f is full. Hence, f is a weak equivalence if and only if it is a weak equivalence in the sense of Section 2.2. In particular, our notion of equivalence coincides with that in [4, §3]. □

As in 2.3.2, equivalence of n-groupoids is an equivalence relation: if $\boldsymbol{H}$ and $\boldsymbol{H}'$ witness that $\boldsymbol{G}_2$ is equivalent to $\boldsymbol{G}_1$ and $\boldsymbol{G}_3$, respectively, the pushout $\boldsymbol{H} \otimes_{\boldsymbol{G}_2} \boldsymbol{H}'$ witnesses the equivalence of $\boldsymbol{G}_1, \boldsymbol{G}_3$.

Remark 3.2.11. The group operation on $\pi_k(\boldsymbol{G}, a)$ (for $k > 0$) is also definable, but we will not use this. □

Remark 3.2.12. The equivalence of our definitions of homotopy groups and weak equivalence with other formulations that appear, for example, in [9, 00V2] does not hold in the definable setting, in general. For example, the analogue of Whitehead's theorem [9, 00WV] is usually false (as seen already in the one-dimensional setting). □

3.2.13. *Morphism groupoids.* Our next goal is to define the space of morphisms between two objects a, b of an n-category $\boldsymbol{G}$, and obtain a (weak) version of the Yoneda embedding that makes sense in the definable setting.

Let $\boldsymbol{G}$ be an n-category, and let $a, b \in \boldsymbol{G}_0$ be two objects. As in [8, §1.2.2], we define the Σ_{n-1}-structure $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, b)$ by

$$\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, b)_k = \{g \in \boldsymbol{G}_{k+1} \mid g_0 = a, g_{\hat{0}} = b^k\} \quad \text{for } k \leq n. \tag{1}$$

The structure maps are given by $t \mapsto d^{\boldsymbol{G}}{}_{t^+}$, where $t^+ : [u+1] \to [k+1]$ is given by $t^+(i+1) = t(i)+1$ for $i \in [u]$ and $t^+(0) = 0$. It is clear that $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, b)$ is uniformly definable over a, b when $\boldsymbol{G}$ is definable. It follows from [8, §§4.2.1.8, 2.3.4.18, 2.3.4.19] that this structure is equivalent to an $(n-1)$-groupoid, but since we are not working up to equivalence, we need to prove that it is already an $(n-1)$-groupoid by itself (which we do in Proposition 3.2.15 below).

If we fix a "generic" object $v \in \boldsymbol{G}_0$, the assignment $b \mapsto \underline{\mathbf{Hom}}^L(v, b)$ looks like the object part of the Yoneda embedding for usual categories. One could hope that this is part of a higher Yoneda embedding in our situation as well. However, since there is no composition function for morphisms in $\boldsymbol{G}$, such an embedding does not exist as a functor (it exists noncanonically for set-theoretic quasicategories, but not definably). Instead, we have the following situation (explained in [8, §2.1]): There is an n-category $\boldsymbol{G}_{v/}$ [8, §2.3.4.10], defined by $(\boldsymbol{G}_{v/})_k = \{g \in \boldsymbol{G}_{k+1} \mid g_0 = v\}$, and a map of n-categories $\pi : \boldsymbol{G}_{v/} \to \boldsymbol{G}$, given by $\pi(g) = g_{\hat{0}}$. By definition, the fibre of this map over $b \in \boldsymbol{G}_0$ is $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, b)$. Moreover, this map is a *left fibration* [8, §2.1.22]: given $g \in \Lambda^k_i(\boldsymbol{G}_{v/})$, for $i < k$, any "filling" $h \in \boldsymbol{G}_k$ of $\pi(g)$ (so that $\partial^{\hat{i}}(h) = \pi(g)$) can be lifted to a filling $\tilde{h} \in \boldsymbol{G}_{v/}$ with $\partial^{\hat{i}}(\tilde{h}) = g$ and $\pi(\tilde{h}) = h$. It follows from this that the association $b \mapsto \pi^{-1}(b)$ behaves like a functor of b, but this is only precisely true in the homotopy category.

We show that in the case that h above is invertible, the lifting property above holds for our definable version of equivalence. To do this, we show that the map π

behaves like a local system: the fibres can be continued along (suitable) contractible pieces. The pieces we have in mind are defined as follows:

Definition 3.2.14. The simplicial set D^l, for $l \geq 0$, is defined by $D^l_k = \{0, \dots, l\}^{\{0,\dots,k\}}$ (all maps, not necessarily increasing, from $[k]$ to $[l]$), with structure maps given by composition.

We often write elements of ${D^l}_k$ as words of length $k+1$ in the "digits" $0, \dots, l$. By the usual Yoneda lemma, maps $D^l \to D^m$ correspond (via composition) to functions $\{0, \dots, l\} \to \{0, \dots, m\}$. Note that homotopically, all these maps are weak equivalences, and in particular the map to the point D^0, so that all D^l are contractible.

We now extend the definition of morphisms. For $\boldsymbol{G}$ a definable n-category, let $a \in \boldsymbol{G}_0$ be an object, and let $f : D^l \to \boldsymbol{G}$ be a map of simplicial sets (perhaps over parameters). We define a Σ_{n-1}-structure $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f)$ as follows: for each $k \leq n$,

$$\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f)_k = \{\langle g, e\rangle \in \boldsymbol{G}_{k+1} \times {D^l}_k \mid g_0 = a, g_{\hat{0}} = f(e)\} \tag{2}$$

with structure maps given as before by $\langle g, e\rangle \mapsto \langle d_{t^+}(g), e \circ t\rangle$ for each weakly increasing function $t : [u] \to [k]$. In other words, $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f)$ is the pullback under f of the map $\pi : \boldsymbol{G}_{v/} \to \boldsymbol{G}$ described above. For $l = 0$ and f mapping the point D^0 to b, we recover the previous definition. In general, the projection determines a map $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f) \to D^l$ of simplicial sets, which can be viewed as the "restriction" of π to D^l. If $h : D^r \to D^l$ is a map of simplicial sets, there is an induced map $\hat{h} : \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f \circ h) \to \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f)$, given by $\hat{h}(\langle g, e\rangle) = \langle g, h(e)\rangle$.

Proposition 3.2.15. *Let $\boldsymbol{G}$, $a \in \boldsymbol{G}_0$ and $f : D^l \to \boldsymbol{G}$ be as above.*

(1) *The structure $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f)$ is an $(n-1)$-groupoid.*

(2) *For each map $h : \{0, \dots, r\} \to \{0, \dots, l\}$ (identified with the corresponding map $D^r \to D^l$), the induced map $\hat{h} : \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f \circ h) \to \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f)$ is a weak equivalence.*

Proof. (1) Let $\boldsymbol{H} = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f)$. It is clear that $\boldsymbol{H}$ is a simplicial definable set. To check the Kan condition, we prove a stronger claim, namely, that the projection $\pi : \boldsymbol{H} \to D^l$ is a Kan fibration: given a horn element $h \in \Lambda^m_i(\boldsymbol{H})$ and an element $d \in {D^l}_m$ with $\partial^{\hat{i}}_m(d) = \pi(h)$, there is $\tilde{d} \in \boldsymbol{H}_m$ with $\pi(\tilde{d}) = d$ and $\partial^{\hat{i}}_m(\tilde{d}) = h$.

Let $h \in \Lambda^m_i(\boldsymbol{H})$ be a horn element as above, with $0 \leq i \leq m \leq n+2$. Such an element is given by a matching sequence of elements $h^j = \langle g^j, e^j\rangle$, for $j \in [m]$, $j \neq i$, with $g^j \in \boldsymbol{G}_m$ and with $\pi(h) = \langle e^0, \dots, e^m\rangle$ an element of $\Lambda^m_i(D^l)$. In D^l, each such horn element comes from a *unique* element of ${D^l}_m$. Let $e \in {D^l}_m$ be this element, and let $g^{-1} = f(e) \in \boldsymbol{G}_m$.

We claim that $\tilde{g} = \langle g^{-1}, g^0, \dots, g^m \rangle \in \boldsymbol{G}_m^{m+1}$ is in $\Lambda_{i+1}^{m+1}(\boldsymbol{G})$. To show that, we need to show that if $\hat{b} \circ \hat{a} = \hat{d} \circ \hat{c} : [m-1] \to [m+1]$ for some $b, d \in [m+1]$, $b, d \neq i+1$ and $a, c \in [m]$, then $g^{b-1}{}_{\hat{a}} = g^{d-1}{}_{\hat{c}}$.

Assume first that $a, b, c, d \geq 1$. The assumption on h implies that

$$h^{b-1}{}_{\widehat{a-1}_{m-1}} = h^{d-1}{}_{\widehat{c-1}_{m-1}} \tag{3}$$

whenever a, b, c, d satisfy

$$\widehat{b-1}_m \circ \widehat{a-1}_{m-1} = \widehat{d-1}_m \circ \widehat{c-1}_{m-1}. \tag{4}$$

Equation (3) implies that

$$g^{b-1}{}_{\widehat{a-1}^{+}_{m-1}} = g^{d-1}{}_{\widehat{c-1}^{+}_{m-1}}$$

under this condition. But $\hat{j}^{+}_{m-1} = \widehat{j+1}_m$ for all $j \in [m-1]$, so we find that $g^{b-1}{}_{\hat{a}_m} = g^{d-1}{}_{\hat{c}_m}$ whenever equation (4) holds. But equation (4) is equivalent to

$$\hat{b}_{m+1} \circ \hat{a}_m = \hat{d}_{m+1} \circ \hat{c}_m \tag{5}$$

so we obtain the required condition when $a, b, c, d \geq 1$.

If $b = 0$ or $d = 0$, the corresponding element of $\boldsymbol{G}_m$ is g^{-1}. In this case, the condition follows from the definition of $\underline{\mathbf{Hom}}_{\boldsymbol{G}}^L(a, f)$. For example, if $b = 0$ we must have $c = 0$ and $d = a+1$, so we need to show that $g^a{}_{\hat{0}} = f(e)_{\hat{a}} = f(e_{\hat{a}}) = f(e^a)$, and we are done. If $a = 0$ or $c = 0$, the condition forces $b = 0$ or $d = 0$, so we are back to the same case.

This concludes the proof that $\tilde{g} \in \Lambda_{i+1}^{m+1}(\boldsymbol{G})$. If $i < m$, the Kan condition on $\boldsymbol{G}$ implies that we may find $g \in \boldsymbol{G}_{m+1}$ restricting to $\tilde{g}$. It follows that $g_0 = a$ and $g_{\hat{0}} = f(e)$, so that $\langle g, e \rangle$ solves the lifting problem. It follows from [8, §1.2.5.1] that the case $i < m$ is sufficient.

When $m = n$ or $m = n+1$, the injectivity follows similarly from injectivity for $\boldsymbol{G}$ (and for D^l).

(2) We use Remark 3.2.8. An element in $\underline{\mathbf{Hom}}_{\boldsymbol{G}}^L(a, f \circ h)_0$ is given by $g \in \boldsymbol{G}_1$ with $g_0 = a$ and $g_1 = f(h(e))$, where $e \in [u]$. Assume that $\langle s, c \rangle, \langle t, d \rangle \in \underline{\mathbf{Hom}}_{\boldsymbol{G}}^L(a, f \circ h)_k$ satisfy $s_{\hat{0}} = t_{\hat{0}} = f(h(c)) = f(h(d)) = h(e)$ and $s_{\hat{i}} = t_{\hat{i}} = g$ for $k \geq i > 0$, so that they are elements of $\boldsymbol{S}_k(\underline{\mathbf{Hom}}_{\boldsymbol{G}}^L(a, f \circ h))$. Assume also that we are given some $w \in \boldsymbol{G}_{k+2}$ satisfying $w_{\hat{1}} = s$, $w_{\hat{2}} = t$ and $w_{\hat{i}} = g$ for $i > 2$, and some $v \in D^u{}_{k+1}$ with $v_{\hat{0}} = c$, $v_{\hat{1}} = d$ and $v_{\hat{i}} = e$ for $i > 1$, and with $f(h(v)) = w_{\hat{0}}$ (this is a homotopy from $\langle s, c \rangle$ to $\langle t, d \rangle$). Then $\langle w, h(v) \rangle$ is a homotopy from $\langle s, h(c) \rangle$ to $\langle t, h(d) \rangle$. The argument for $k = 0$ is similar (using that D^u is connected).

For the second condition of Remark 3.2.8, let $g \in \boldsymbol{G}_1$ be such that $g_0 = a$, $g_1 = f(e)$ for some $e \in [l]$, so that $b = \langle g, e \rangle$ represents a basepoint of $\underline{\mathbf{Hom}}_{\boldsymbol{G}}^L(a, f)$, and let $\langle s, c \rangle \in \boldsymbol{S}_k(\underline{\mathbf{Hom}}_{\boldsymbol{G}}^L(a, f), b)$. Then $c \in \boldsymbol{S}_k(D^l, e)$ is the constant function e.

Let $e' \in [u]$, and let $\gamma \in D^l$ be some path from $h(e')$ to e. By the Kan condition above, there is an element s' of $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, f)$ above γ, restricting to s. This s' serves as a homotopy from s to an element over $h(e')$, which is thus in the image of $\hat{h}$. □

Corollary 3.2.16. *Let $\boldsymbol{G}$ be an n-category, $v, a, b \in \boldsymbol{G}_0$ objects. Each isomorphism $t \in \boldsymbol{G}_1$ from a to b determines an equivalence $e_t : \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, a) \to \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, b)$. If $s \in \boldsymbol{G}_1$ is another isomorphism from a to b, each isomorphism $m : t \to s$ determines an isomorphism $e_m : e_t \to e_s$.*

Proof. Apply Proposition 3.2.15 to maps from D^1 and from D^2 determined by t, s and e. □

We describe the equivalence explicitly in the 2-dimensional case, which is most relevant for us:

Example 3.2.17. Let $\boldsymbol{G}$ be a 2-groupoid, $v, a, b \in \boldsymbol{G}_0$ and $f \in \boldsymbol{G}_1$ with $f_0 = a$ and $f_1 = b$. Since $\boldsymbol{G}$ is a groupoid, there is $h \in \boldsymbol{G}_2$ (not necessarily unique) with $h_{\hat{0}} = f$ and $h_{\hat{1}} = b$. We denote $f^{-1} = h_{\hat{2}}$. By the 2-groupoid axioms, h has a uniquely determined inverse h^{-1}. Let $\gamma : D^1 \to \boldsymbol{G}$ be the unique map with $\gamma(101) = h$, so that $\gamma(01) = g$ and $\gamma(10) = g^{-1}$. Then $\boldsymbol{H} = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, \gamma)$ can be described as follows:

(1) $\boldsymbol{H}_0 = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, a)_0 \amalg \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, b)_0$ (this is just the union if $a \neq b$, but if $a = b$ we take disjoint copies).

(2) Let $\boldsymbol{X} = \{g \in \boldsymbol{G}_2 \mid g_0 = v, g_{\hat{0}} = f\}$, and let $\boldsymbol{X}^{-1} = \{g \in \boldsymbol{G}_2 \mid g_0 = v, g_{\hat{0}} = f^{-1}\}$ (again taking disjoint copies if $f = f^{-1}$). Then

$$\boldsymbol{H}_1 = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, a)_1 \amalg \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, b)_1 \cup \boldsymbol{X} \cup \boldsymbol{X}^{-1}$$

with $d^{\boldsymbol{X}}_{\hat{0}} = d^{\boldsymbol{G}}_{\hat{1}}$, $d^{\boldsymbol{X}}_{\hat{1}} = d^{\boldsymbol{G}}_{\hat{2}}$ and vice versa for $\boldsymbol{X}^{-1}$ (and the structure coming from $\underline{\mathbf{Hom}}^L$ on the other parts).

(3) Composition is defined again as in $\underline{\mathbf{Hom}}^L$ on the corresponding parts. The composition of $h \circ g$ for $g \in \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, a)$ and $h \in \boldsymbol{X}$ is the composition in $\boldsymbol{G}$ of the three elements $g, h, i \in \boldsymbol{G}_2$, where i is the identity morphism of the object f of $\underline{\mathbf{Hom}}^R_{\boldsymbol{G}}(a, b)$. Similarly for the compositions $g' \circ h$, $h' \circ g'$, $g \circ h'$, $h \circ h'$ and $h' \circ h$, for $g' \in \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(v, b)$ and $h' \in \boldsymbol{X}^{-1}$ (in each case, the two elements of $\boldsymbol{G}_2$ along with i form three faces of a 2-horn, with vertices a, a, b, v or a, b, b, v, and the result is the uniquely determined fourth face).

It is clear, by construction, that each of the inclusions of $\underline{\mathbf{Hom}}^L(v, a)$ and of $\underline{\mathbf{Hom}}^L(v, b)$ into $\boldsymbol{H}$ determine fully faithful functors. As in the general proof, they are also essentially surjective by the Kan property. □

Corollary 3.2.18. *Let $\boldsymbol{G}$ be a 2-groupoid, $\gamma : D^2 \to \boldsymbol{G}$ a fixed map, and $a \in \boldsymbol{G}_0$ a fixed vertex. Then*

$$\underline{\mathbf{Hom}}^L(a, \gamma) = \underline{\mathbf{Hom}}^L(a, \gamma \circ \hat{2}) \otimes_{\underline{\mathbf{Hom}}^L(a,\gamma\circ 1)} \underline{\mathbf{Hom}}^L(a, \gamma \circ \hat{0})$$

(*canonical isomorphism*), *and*

$$\underline{\mathbf{Hom}}^L(a, \gamma \circ \hat{1}) = \underline{\mathbf{Hom}}^L(a, \gamma \circ \hat{0}) \circ_{\underline{\mathbf{Hom}}^L(a,\gamma\circ 1)} \underline{\mathbf{Hom}}^L(a, \gamma \circ \hat{2}).$$

In other words, the composition of two morphism groupoids (in the sense of 2.3.2) is given by composition in the homotopy category.

Proof. By definition, both sides have the same sets of objects. Proposition 2.2.9 provides the required map, and since on both sides we also have a weak equivalence (by the second part of Proposition 2.2.9 and by Proposition 3.2.15), this map is an isomorphism. The second part again follows directly from the definition, as both sides are the restriction to the same set of objects. □

3.3. *The theory associated with a groupoid.* We continue to fix $n \in \mathbb{N}$. To each definable n-groupoid in the theory $\mathcal{T}$ we define an associated expansion $\mathcal{T}_{\boldsymbol{G}}$ of $\mathcal{T}$, directly generalising (a variant of) the one-dimensional case (Construction 2.2.4).

Definition 3.3.1. Let $\boldsymbol{G}$ be a definable n-groupoid in a theory $\mathcal{T}$. The expansion $\mathcal{T}_{\boldsymbol{G}}$ of $\mathcal{T}$ is obtained by adding additional sorts $\boldsymbol{G}_i^*$ for $0 \le i \le n+1$, function symbols $e_i : \boldsymbol{G}_i \to \boldsymbol{G}_i^*$, a constant symbol $* \in \boldsymbol{G}_0^*$, and the axioms expressing:

(1) $\boldsymbol{G}^*$ is an n-groupoid, and e_* is a map of simplicial sets (i.e., commutes with the structure maps). We identify $\boldsymbol{G}$ with its image.
(2) $\boldsymbol{G}_0^* = \boldsymbol{G}_0 \cup \{*\}$.
(3) The inclusion of $\boldsymbol{G}$ in $\boldsymbol{G}^*$ is a weak homotopy equivalence (Remark 3.2.8), and an isomorphism onto the full subgroupoid of $\boldsymbol{G}^*$ spanned by $\boldsymbol{G}_0$.

For each natural number r, there is a definable family $\Gamma_r = \underline{\mathbf{Hom}}^L(*, f)$ of groupoids, parametrised by the definable set of maps $f : D^r \to \boldsymbol{G}^*$. This is our collection Γ of admissible groupoids, in the sense of Definition 3.1.3.

We note that our choice of Γ does satisfy the assumption on composition, by Corollary 3.2.18.

As in the one-dimensional case, each object $a \in \boldsymbol{G}_0$ determines an interpretation ω_a over $\mathcal{T}_a$ determined by the requirement that $\omega_a(*) = a$, $\omega_a({\boldsymbol{G}^*}_i) = \boldsymbol{G}_i$ for $i = 1, 2$ and similarly for the face maps. We would like to show that the ω_a are objects in the 2-groupoid associated with $\mathcal{T}_{\boldsymbol{G}}$ over $\mathcal{T}$, namely:

Proposition 3.3.2. *For every object $a \in \boldsymbol{G}_0$, the interpretation $\omega_a : \mathcal{T}_{\boldsymbol{G}} \to \mathcal{T}_a$ is 2-stable. In particular, $\langle \mathcal{T}_{\boldsymbol{G}}, \Gamma \rangle$ is a 2-internal cover of $\mathcal{T}$.*

Proof. We need to show that over some parameter u, each Γ-admissible groupoid $\boldsymbol{H}$ is equivalent to $\omega_a(\boldsymbol{H})$, over some parameters from $\mathcal{T}$. Let u be any element of $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}^*}(*, a)_0$ (it is consistent that such a u exists: for any model M of $\mathcal{T}$ such that $a \in \boldsymbol{G}_0(\mathrm{M})$, $\mathrm{M} \circ \omega_a$ is a model of $\mathcal{T}_{\boldsymbol{G}}$ for which this set is nonempty).

Let $f : D^r \to \boldsymbol{G}^*$ be a map, and assume first that for some $i \in [r]$, $b = f(i) \in \boldsymbol{G}_0$. By the second item of Proposition 3.2.15, $\boldsymbol{H} = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}^*}(*, f)$ is equivalent (over no additional parameters) to $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}^*}(*, b)$, so we may assume that $f = b$. We may also assume that b is in the same connected component as a, because otherwise $\boldsymbol{H}$ is empty. According to Corollary 3.2.16, it follows that $\boldsymbol{H}$ is equivalent to $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}^*}(*, a)$. Again according to (a dual version of) Corollary 3.2.16, the fixed element u determines an equivalence from $\boldsymbol{H}$ to $\omega_a(\boldsymbol{H}) = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, a)$.

The remaining case is when f is the constant map $*$, so that $\boldsymbol{H} = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}^*}(*, *)$, and $\omega_a(\boldsymbol{H}) = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, a)$. The same argument as above shows that both are equivalent to $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}^*}(*, a)$ over u. □

We would like to prove that the association $a \mapsto \omega_a$ is the object part of an assignment that recovers (up to equivalence) $\boldsymbol{G}$. To do that, we need to define the 2-groupoid which is the target of this assignment. This will be the analogue of $I_{\mathcal{T}*/\mathcal{T}}$ from the one-dimensional case (2.2.2).

Definition 3.3.3. Let $\mathcal{T}^*$ be a stable expansion of a theory $\mathcal{T}$, with admissible family of distinguished internal covers Γ. The 2-groupoid associated to this datum is defined as follows:

(1) Objects are 2-stable interpretations of $\langle \mathcal{T}^*, \Gamma \rangle$ in $\mathcal{T}$, over $\mathcal{T}$.

(2) If x, y are two objects as above, a morphism $u : x \to y$ is given by a bi-interpretation $u_{\mathcal{T}'} : x(\mathcal{T}') \to y(\mathcal{T}')$ over $\mathcal{T}$, for each admissible internal cover $\mathcal{T}' \in \Gamma$. These bi-interpretations are given with isomorphisms $c_i : u_{\mathcal{T}_2} \circ x(i) \to y(i) \circ u_{\mathcal{T}_1}$ for every admissible interpretation $i : \mathcal{T}_1 \to \mathcal{T}_2$ between admissible covers $\mathcal{T}_1, \mathcal{T}_2$ in Γ (uniformly in families).

The isomorphisms are required to satisfy $c_{j \circ i} = y(j)(c_i) \circ c_j(x(i))$ for admissible interpretations $i : \mathcal{T}_1 \to \mathcal{T}_2$, $j : \mathcal{T}_2 \to \mathcal{T}_3$ as above (these make sense since, by definition, the c_i are definable maps in $y(\mathcal{T}_2)$, $y(j)$ is an interpretation of $y(\mathcal{T}_2)$ in $y(\mathcal{T}_3)$ and c_j is a map between interpretations of $x(\mathcal{T}_2)$, so can be applied to definable sets of the form $x(i)$).

(3) The 2-morphisms with edges $u : x \to y$, $v : y \to z$ and $w : x \to z$ are given by isomorphisms $v \circ u \to w$, all over $\mathcal{T}$.

(4) The "2-composition" of the 2-morphisms $\alpha : v \circ u \to w$, $\beta : s \circ v \to r$ and $\gamma : r \circ u \to t$ is given by

$$s \circ w \xrightarrow{s \cdot \alpha^{-1}} s \circ (v \circ u) = (s \circ v) \circ u \xrightarrow{\beta \cdot u} r \circ u \xrightarrow{\gamma} t,$$

where $\cdot$ stands for pointwise application (or *horizontal composition*) as above.

Applying the definition with $\mathcal{T}$ replaced by $\mathcal{T}_A$, for a $\mathcal{T}$-structure A, we obtain a 2-groupoid for each such structure A, which we denote $I^2(A) = I^2{}_{\mathcal{T}^*/\mathcal{T}}(A)$.

Using the equivalence between internal covers and definable groupoids, this can be described in terms of definable groupoids. We give an explicit description in 3.3.6 below.

Remark 3.3.4. Let $\mathcal{C}$ be the category of definable groupoids in $\mathcal{T}$, with weak equivalences as morphisms. We may form its bicategory of *cospans* for this category, as in [9, 0084]. By Proposition 2.3.4, it is equivalent (as a bicategory) to the category of internal covers and bi-interpretations. The 2-category in Definition 3.3.3 can be viewed as the *Duskin nerve* [9, 009T] of this bicategory (clear from the description in [9, 00A1]). In particular, it follows that this is indeed a 2-category. □

Proposition 3.3.5. *Let $\boldsymbol{G}$ be a definable 2-groupoid in a theory $\mathcal{T}$, and $\mathcal{T}_{\boldsymbol{G}} = \langle \mathcal{T}_{\boldsymbol{G}}, \Gamma \rangle$ the corresponding 2-internal cover. The association $a \mapsto \omega_a$ extends to a map $\omega : \boldsymbol{G}(A) \to I^2{}_{\mathcal{T}_G/\mathcal{T}}(A)$ of 2-groupoids, compatible with extensions of the structure A.*

Proof. Proposition 3.3.2 shows that for all $a \in \boldsymbol{G}_0(A)$, ω_a is indeed an object of $I^2(A)$. Given $t : a \to b$ in $\boldsymbol{G}_1(A)$, we define $\omega_t : \omega_a \to \omega_b$ as follows.

Let $\boldsymbol{H}$ be an admissible groupoid. As in the proof of Proposition 3.3.2, we assume that $\boldsymbol{H} = \boldsymbol{H}_c = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}^*}(*, c)$ for some $c \in \boldsymbol{G}_0(A)$, so that $\omega_a(\boldsymbol{H}) = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(a, c)$ and $\omega_b(\boldsymbol{H}) = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(b, c)$. By Corollary 3.2.16, t induces an (admissible) equivalence from $\omega_a(\boldsymbol{H})$ to $\omega_b(\boldsymbol{H})$, which we take to be $\omega_t(\boldsymbol{H})$. Our definition (and construction) ensures the compatibility under admissible maps $\boldsymbol{H} \to \boldsymbol{H}'$.

Similarly, let $\alpha \in \boldsymbol{G}_2(A)$, with edges $r : a \to b$, $s : b \to c$ and $t : a \to c$. We need to construct an isomorphism (over $\mathcal{T}$) from $\omega_s \circ \omega_r$ to ω_t. Consider the map $f : D^2 \to \boldsymbol{G}$ determined by α. We have $f(01) = r$, $f(12) = s$ and $f(02) = t$, so that for each object $d \in \boldsymbol{G}_0(A)$, the equivalence $\omega_r(\boldsymbol{H}_d) : \omega_a(\boldsymbol{H}_d) \to \omega_b(\boldsymbol{H}_d)$ is given by $\underline{\mathbf{Hom}}^L(f \circ h_{01}, d)$, where $h_{01} : [1] \to [2]$ is the inclusion (and similarly for s, t). Hence, $\underline{\mathbf{Hom}}^L(f, d)$ represents the composition $\omega_s \circ \omega_r$, and restriction to ω_t provides the required map.

This completes the construction of ω. The proof that this is a map of 2-groupoids (i.e., that it commutes with composition) is similar to the above, using D^4 in place of D^3, and the fact that it commutes with extension of scalars is obvious. □

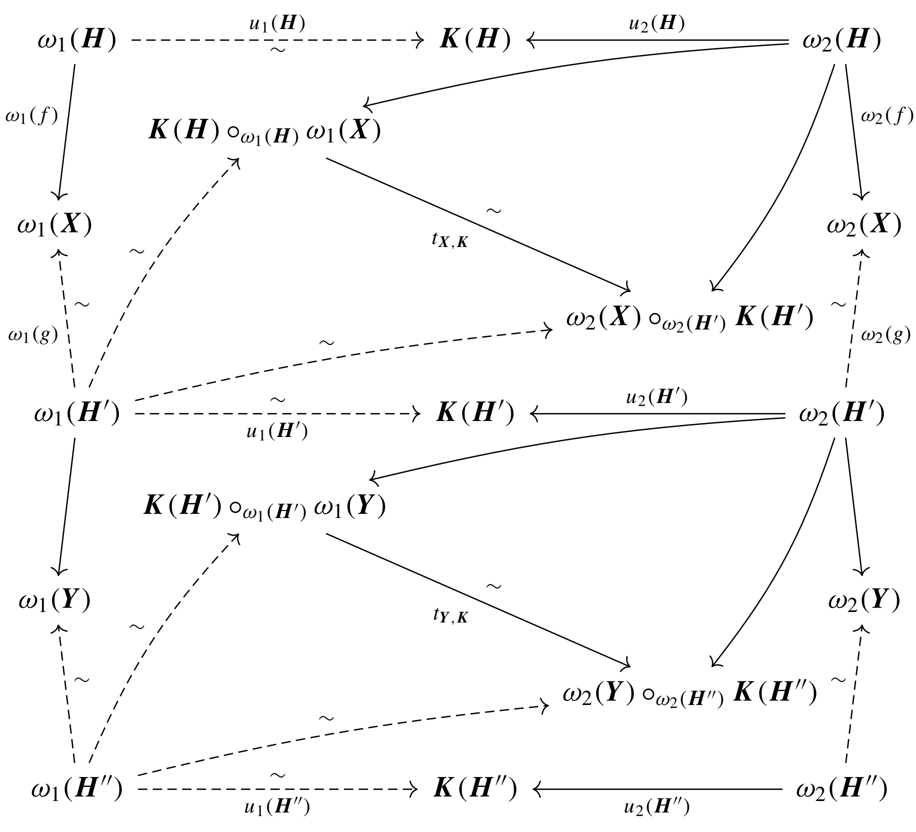
3.3.6. Our main goal is to prove that the map ω constructed in Proposition 3.3.5 is a weak equivalence. Similarly to the 1-dimensional case, it is generally only true in a model. As a preparation, we consider more explicitly the structure of I^2 from Definition 3.3.3, from a definable groupoid point of view.

Let ω_1 and ω_2 be two objects of $I^2_{\mathcal{T}^*/\mathcal{T}}$, i.e., 2-stable interpretations of $\mathcal{T}^*$ in $\mathcal{T}$. An isomorphism from ω_1 to ω_2 over $\mathcal{T}$ is given, according to Proposition 2.3.1, by a family $\boldsymbol{K}(\boldsymbol{H})$ of groupoids in $\mathcal{T}$, for each admissible groupoid $\boldsymbol{H}$ in $\mathcal{T}^*$, along with weak equivalences $u_i(\boldsymbol{H}) : \omega_i(\boldsymbol{H}) \overset{\sim}{\dashrightarrow} \boldsymbol{K}(\boldsymbol{H})$, all definable uniformly in $\boldsymbol{H}$. Given another admissible groupoid $\boldsymbol{H}'$, an admissible interpretation from $\mathcal{T}^*{}_{\boldsymbol{H}'}$ to $\mathcal{T}^*{}_{\boldsymbol{H}}$ is given, again by Proposition 2.3.1, by an admissible groupoid $\boldsymbol{X}$ and admissible maps $f : \boldsymbol{H} \to \boldsymbol{X}$ and $g : \boldsymbol{H}' \overset{\sim}{\dashrightarrow} \boldsymbol{X}$.

According to Definition 3.3.3, we are provided with definable isomorphisms (realising the isomorphisms c_i there, via Proposition 2.3.4)

$$t_{\boldsymbol{X},\boldsymbol{K}} : \boldsymbol{K}(\boldsymbol{H}) \circ_{\omega_1(\boldsymbol{H})} \omega_1(\boldsymbol{X}) \xrightarrow{\sim} \omega_2(\boldsymbol{X}) \circ_{\omega_2(\boldsymbol{H}')} \boldsymbol{K}(\boldsymbol{H}') \tag{6}$$

uniformly definable in $\boldsymbol{X}$, $\boldsymbol{K}$ (and the associated embeddings), and restricting to the identity on $\omega_1(\boldsymbol{H}')$ and on $\omega_2(\boldsymbol{H})$. The situation is depicted in the top part of the following diagram:



If $\boldsymbol{Y}$ determines a map to $\mathcal{T}^*{}_{\boldsymbol{H}'}$ from $\mathcal{T}^*{}_{\boldsymbol{H}''}$ for a further groupoid $\boldsymbol{H}''$, we have the maps

$$\begin{aligned} t_{\boldsymbol{X},\boldsymbol{K}} \otimes_{\omega_1(\boldsymbol{H}')} \mathbf{1}_{\omega_1(\boldsymbol{Y})} : \boldsymbol{K}(\boldsymbol{H}) \circ_{\omega_1(\boldsymbol{H})} \omega_1(\boldsymbol{X}) \circ_{\omega_1(\boldsymbol{H}')} \omega_1(\boldsymbol{Y}) \\ \xrightarrow{\sim} \omega_2(\boldsymbol{X}) \circ_{\omega_2(\boldsymbol{H}')} \boldsymbol{K}(\boldsymbol{H}') \circ_{\omega_1(\boldsymbol{H}')} \omega_1(\boldsymbol{Y}) \end{aligned} \tag{7}$$

and

$$\mathbf{1}_{\omega_2(\boldsymbol{X})} \otimes_{\omega_2(\boldsymbol{H}')} t_{\boldsymbol{Y},\boldsymbol{K}} : \omega_2(\boldsymbol{X}) \circ_{\omega_2(\boldsymbol{H}')} \boldsymbol{K}(\boldsymbol{H}') \circ_{\omega_1(\boldsymbol{H}')} \omega_1(\boldsymbol{Y}) \\ \xrightarrow{\sim} \omega_2(\boldsymbol{X}) \circ_{\omega_2(\boldsymbol{H}')} \omega_2(\boldsymbol{Y}) \circ_{\omega_2(\boldsymbol{H}'')} \boldsymbol{K}(\boldsymbol{H}''). \quad (8)$$

The groupoid $\boldsymbol{X} \circ_{\boldsymbol{H}'} \boldsymbol{Y}$ represents the composition of interpretations, and

$$\omega_i(\boldsymbol{X} \circ_{\boldsymbol{H}'} \boldsymbol{Y}) = \omega_i(\boldsymbol{X}) \circ_{\omega_i(\boldsymbol{H}')} \omega_i(\boldsymbol{Y})$$

(canonical identification), since ω_i is an interpretation. Under this identification, we require that

$$(\mathbf{1}_{\omega_2(\boldsymbol{X})} \otimes_{\omega_2(\boldsymbol{H}')} t_{\boldsymbol{Y},\boldsymbol{K}}) \circ (t_{\boldsymbol{X},\boldsymbol{K}} \otimes_{\omega_1(\boldsymbol{H}')} \mathbf{1}_{\omega_1(\boldsymbol{Y})}) = t_{(\boldsymbol{X} \circ_{\boldsymbol{H}'} \boldsymbol{Y}),\boldsymbol{K}}. \quad (9)$$

Finally, a 2-morphism is determined by a natural isomorphism between two maps as above (one a composition, which we already understand), so it is enough to describe those. Let ω_1, ω_2 and $\boldsymbol{K}$ be as above, and let $\boldsymbol{L}$ represent another morphism. A natural isomorphism is then given by a uniform family of isomorphisms $\alpha_{\boldsymbol{H}} : \boldsymbol{K}(\boldsymbol{H}) \to \boldsymbol{L}(\boldsymbol{H})$ over $\omega_i(\boldsymbol{H})$, which intertwine the maps $t_{\boldsymbol{X},\boldsymbol{K}}$ and $t_{\boldsymbol{X},\boldsymbol{L}}$ whenever $\boldsymbol{X}$ represents an interpretation. The 2-composition of three such suitable maps is described as in Definition 3.3.3, with composition replaced by pushouts as appropriate.

Remark 3.3.7. By definition, internality means that there is a nonempty definable *set* (i.e., a 0-groupoid) of isomorphisms between the internal sorts and sorts of the base theory. Similarly, the structure described above includes the description of a $\mathcal{T}^*$-definable 1-groupoid $\underline{\mathbf{Iso}}_{\mathcal{T}}(\widetilde{\mathcal{T}}^*, \widetilde{\mathcal{T}})$ of weak equivalences between admissible covers $\widetilde{\mathcal{T}}^*$ of $\mathcal{T}^*$ and covers $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ (nonempty for some $\widetilde{\mathcal{T}}$ if $\mathcal{T}^*$ is 2-internal). In terms of groupoids, the families $\boldsymbol{K}$ as in 3.3.6 are the objects, and the morphisms are the natural isomorphisms α. Furthermore, this groupoid itself is admissible. □

Example 3.3.8. Let $\mathcal{T}^* = \mathcal{T}_{\boldsymbol{G}}$ as in Proposition 3.3.5, and let $\omega_1 = \omega_a$ and $\omega_2 = \omega_b$ for some $a, b \in \boldsymbol{G}_0(A)$. Let $f : a \to b$ be a morphism in $\boldsymbol{G}(A)$ (identified with the corresponding map from D^1). Given an admissible groupoid $\boldsymbol{H} = \boldsymbol{H}_d = \underline{\mathbf{Hom}}^L(d, *)$, we let $\boldsymbol{K}_f(d) = \boldsymbol{K}_f(\boldsymbol{H}_d) = \underline{\mathbf{Hom}}^L_{\boldsymbol{G}}(d, f)$, with the canonical maps from $\omega_a(\boldsymbol{H}_d) = \underline{\mathbf{Hom}}^L(d, a)$ and $\omega_b(\boldsymbol{H}_d) = \underline{\mathbf{Hom}}^L(d, b)$ (these are weak equivalences by Proposition 3.2.15).

To give $\boldsymbol{K}$ the structure of an isomorphism from ω_a to ω_b, we need to supply the isomorphisms (6). If $\boldsymbol{H}' = \boldsymbol{H}_c = \underline{\mathbf{Hom}}^L(c, *)$ is another admissible groupoid in $\mathcal{T}_{\boldsymbol{G}}$, an admissible isomorphism from $\boldsymbol{H}_c$ to $\boldsymbol{H}_d$ is given by a groupoid $\boldsymbol{X}_g = \underline{\mathbf{Hom}}^L(g, *)$, with $g : c \to d$ in $\boldsymbol{G}$. Seeing as $\omega_x(\boldsymbol{X}_g) = \underline{\mathbf{Hom}}^L(g, x)$ for all x, such a structure consists of a definable family of maps

$$t_{g,f} : \underline{\mathbf{Hom}}^L(d, f) \circ_{\underline{\mathbf{Hom}}^L(d,a)} \underline{\mathbf{Hom}}^L(g, a) \xrightarrow{\sim} \underline{\mathbf{Hom}}^L(g, b) \circ_{\underline{\mathbf{Hom}}^L(c,b)} \underline{\mathbf{Hom}}^L(c, f).$$

Recall that t is the identity on objects, so we only need to define it on morphisms. Let $\langle u, v\rangle$ represent a morphism of $\underline{\mathbf{Hom}}^L(d, f) \circ_{\underline{\mathbf{Hom}}^L(d,a)} \underline{\mathbf{Hom}}^L(g, a)$. Let $h : c \to a$ be the domain of v. By the Kan property, there is a morphism $w \in \underline{\mathbf{Hom}}^L(c, f)$ whose domain is h. Then u, v, w form an element of $\Lambda^3_3(\boldsymbol{G})$, so composition provides a fourth face $y \in \underline{\mathbf{Hom}}^L(g, b)_1$. We let $t_{g,f}(u \otimes v) = y \otimes w$. If w' is a different choice in place of w, then $w' \circ w^{-1}$ is in $\underline{\mathbf{Hom}}^L(c, b)$, so the result represents the same morphism of $\underline{\mathbf{Hom}}^L(g, b) \circ_{\underline{\mathbf{Hom}}^L(c,b)} \underline{\mathbf{Hom}}^L(c, f)$. It is clear that t is well defined on the class $u \otimes v$, and uniformly definable in g, f.

To prove the identity (9), assume we are given another morphism $g' : c' \to c$, corresponding to an admissible interpretation represented by $\boldsymbol{Y} = \underline{\mathbf{Hom}}^L(g', *)$. Let $v' \in \underline{\mathbf{Hom}}^L(g', a) = \omega_1(\boldsymbol{Y})$. Proceeding with the notation above, we need to determine the image of $y \otimes w \otimes v'$ in

$$\underline{\mathbf{Hom}}^L(g, b) \circ_{\underline{\mathbf{Hom}}^L(c,b)} \underline{\mathbf{Hom}}^L(g', b) \circ_{\underline{\mathbf{Hom}}^L(c',b)} \underline{\mathbf{Hom}}^L(c', f).$$

As above, it is given by $y \otimes y' \otimes w'$, where $y' \in \underline{\mathbf{Hom}}^L(g', b) = \omega_2(\boldsymbol{Y})$ and $w' \in \underline{\mathbf{Hom}}^L(c', f) = \boldsymbol{K}_f(\boldsymbol{H}_{c'})$ represent the other two faces of a partial simplex with faces w and v' (this other simplex can be visualised as attached to the previous one at the face w). On the other hand, $\boldsymbol{X} \circ_{\boldsymbol{H}_c} \boldsymbol{Y}$ was identified (as in Corollary 3.2.18) with $\underline{\mathbf{Hom}}^L(g \circ g', *)$, for any composition $g \circ g'$. After choosing such a composition h, $v \otimes v'$ is identified with an element of $\underline{\mathbf{Hom}}^L(h, a)$ and $y \otimes y'$ with an element of $\underline{\mathbf{Hom}}^L(h, b)$, so that they become two faces of the simplex with vertices a, b, c, d, the other two being u and w', so that $u \otimes w' = t_{h,f}(v \otimes v', y \otimes y')$, as required.

Assume now that we are given a map $\gamma : D^2 \to \boldsymbol{G}$ corresponding to an element $w \in \boldsymbol{G}_2$, with edges $f = \gamma(01)$, $g = \gamma(12)$ and $h = \gamma(02)$. Given an element $u \in \boldsymbol{K}_f(\boldsymbol{H}_c) = \underline{\mathbf{Hom}}^L(c, f)$ and $v \in \boldsymbol{K}_g(\boldsymbol{H}_c) = \underline{\mathbf{Hom}}^L(c, g)$ (for an arbitrary $c \in \boldsymbol{G}_0$), the 2-composition applied to u, v and w provides an element of $\boldsymbol{K}_h(\boldsymbol{H}_c) = \underline{\mathbf{Hom}}^L(c, h)$. This process assembles into a family of isomorphisms

$$\alpha_\gamma : \boldsymbol{K}_g(\boldsymbol{H}_c) \circ \boldsymbol{K}_f(\boldsymbol{H}_c) \to \boldsymbol{K}_h(\boldsymbol{H}_c),$$

definable uniformly in γ and c. This completes the description (and a reformulation of the proof) of the map in Proposition 3.3.5 in terms of definable groupoids. □

We are now ready to prove our main result.

Theorem 3.3.9. *Let $\boldsymbol{G}$ be a 2-groupoid defined in a theory $\mathcal{T}$, and let $\mathcal{T}_{\boldsymbol{G}}$ be the associated theory (and admissible covers), as in Definition 3.3.1. Then $\mathcal{T}_{\boldsymbol{G}}$ is a 2-internal cover of $\mathcal{T}$, and for every model* M *of $\mathcal{T}$, the 2-groupoid of* M*-points $I^2_{\mathcal{T}_G/\mathcal{T}}(M)$ is weakly equivalent to $\boldsymbol{G}(M)$.*

Proof. The fact that $\mathcal{T}_{\boldsymbol{G}}$ is a 2-internal cover is Proposition 3.3.2. The map from $\boldsymbol{G}$ to $I^2_{\mathcal{T}_G/\mathcal{T}}$ was constructed in Proposition 3.3.5, and described in terms of groupoids in Example 3.3.8. We use this description to show that the map is a weak equivalence, using Remark 3.2.8. We assume that we are working over M, and proceed by considering the possible dimensions $0 \le k \le 2$.

$\underline{k=0}$: We need to show that any 2-stable interpretation ω of $\mathcal{T}_{\boldsymbol{G}}$ in $\mathcal{T}$ admits a coherent collection of bi-interpretations as in Definition 3.3.3(2) to some ω_a.

Since ω is an interpretation over $\mathcal{T}$, $\omega(\boldsymbol{G}^*)$ is a definable 2-groupoid in $\mathcal{T}$, containing $\boldsymbol{G}$, with the inclusion a weak equivalence. The proof now proceeds exactly as the proof of Proposition 3.3.2, with $\omega(\boldsymbol{G}^*)$ in place of $\boldsymbol{G}^*$.

$\underline{k=1}$: This is the main case, which can be viewed as a definable version of the Yoneda lemma. Let $a, b \in \boldsymbol{G}_0$, and assume we are given an equivalence from ω_a to ω_b. Hence, for every $c \in \boldsymbol{G}_0$ (some parameters), we are given a groupoid $\boldsymbol{K}(c)$ in $\mathcal{T}$ and weak equivalences $\underline{\mathbf{Hom}}^L(c,a) = \omega_a(\boldsymbol{H}_c) \to \boldsymbol{K}(c)$ and $\underline{\mathbf{Hom}}^L(c,b) = \omega_b(\boldsymbol{H}_c) \to \boldsymbol{K}_c$, uniformly in c, along with structure maps (6)

$$t_{g,\boldsymbol{K}} : \boldsymbol{K}(d) \circ_{\underline{\mathbf{Hom}}^L(d,a)} \underline{\mathbf{Hom}}^L(g,a) \to \underline{\mathbf{Hom}}^L(g,b) \circ_{\underline{\mathbf{Hom}}^L(c,b)} \boldsymbol{K}(c)$$

(all notation as in Example 3.3.8, except $\boldsymbol{K}$ is no longer known to be of the given form). We identify $\omega_a(\boldsymbol{H}_c), \omega_b(\boldsymbol{H}_c)$ with their images in $\boldsymbol{K}(c)$.

In particular, we have the identity morphism $\mathbf{1}_a$ of a as an object $\mathbf{1}_a \in \omega_a(\boldsymbol{H}_a)$, and by weak equivalence, an object $f : a \to b$ in $\omega_b(\boldsymbol{H}_a) \subseteq \boldsymbol{K}(a)$, along with a morphism $u : \mathbf{1}_a \to f$ in $\boldsymbol{K}(a)$. We show that $\boldsymbol{K}$ is isomorphic to $\boldsymbol{K}_f$, by a unique isomorphism.

To do that, let $c \in \boldsymbol{G}_0$ be an arbitrary object, and let v be a morphism of $\boldsymbol{K}_f(c) = \underline{\mathbf{Hom}}^L(c, f)$ (so a 2-morphism of $\boldsymbol{G}$). Denote by $g \in \underline{\mathbf{Hom}}^L(c,a)_0$ the domain of v. Then v can also be viewed as a morphism in $\underline{\mathbf{Hom}}^L(g,b)$, and on the other hand, we have the canonical morphism w from g to $\mathbf{1}_a$ in $\underline{\mathbf{Hom}}^L(g,a)$. Applying $t_{g,\boldsymbol{K}}$ to the morphism $u \otimes w \in \boldsymbol{K}(a) \circ_{\underline{\mathbf{Hom}}^L(a,a)} \underline{\mathbf{Hom}}^L(g,a)$, we may write $t_{g,\boldsymbol{K}}(u \otimes w)$ as $v \otimes x$ for a unique $x \in \boldsymbol{K}(c)$, which we take to be the image of v. By construction this map commutes with the structure maps t, and is unique with this property.

$\underline{k=2}$: We need to show that each isomorphism $\alpha : \boldsymbol{K}_g \circ \boldsymbol{K}_f \to \boldsymbol{K}_h$ with $f : a \to b$, $g : b \to c$ and $h : a \to c$ arises from a unique $\gamma : D^2 \to \boldsymbol{G}$, with boundary f, g, h (as in the end of Example 3.3.8). Uniqueness was already shown in the part $k = 1$. For existence, we apply $\alpha_b : \boldsymbol{K}_g(b) \circ \boldsymbol{K}_f(b) \to \boldsymbol{K}_h(b)$ to the element $\mathbf{1}_g \otimes \mathbf{1}_f$ (where $\mathbf{1}_g$ is the identity morphism of the object g of $\underline{\mathbf{Hom}}^L(b,c)$, viewed as an element of $\boldsymbol{G}_2$, and similarly for f), to obtain an element $\gamma_b \in \boldsymbol{K}_h(b)_1$, again viewed as a 2-morphism of $\boldsymbol{G}$. It is clear that the map α coincides with α_γ on the given maps, and then, again by the uniqueness statement, that $\alpha = \alpha_\gamma$ globally. □

3.3.10. *Recovering a definable* 2*-groupoid.* The main statement of classical internality starts with the assumption of internality, and produces a definable (nonempty) groupoid from it. The general outline of this construction was recalled in Section 2.2, and in Proposition 2.2.6 we indicated how this construction is useful in the description of definable sets in the cover.

In our approach, the construction of the (2-)groupoid is almost tautological: we defined a groupoid (or a 2-groupoid) associated to every stable expansion, and by definition, the expansion is an internal cover if the groupoid is nonempty. However, we still need to show that the groupoid is equivalent to a definable one, which we sketch below. The other part, describing the (admissible) 1-groupoids in the cover in terms of suitable definable fibrations in the base, is more involved, and we postpone most of the work here to future work.

Proposition 3.3.11. *Let $\mathcal{T}^*$ be a 2-internal cover of $\mathcal{T}$. Then the 2-groupoid $I^2_{\mathcal{T}^*/\mathcal{T}}$ associated to it is equivalent to a $\mathcal{T}$-definable one.*

Proof sketch. Assume $\mathcal{T}^*$ is a 2-internal cover of $\mathcal{T}$, and let $\omega : \mathcal{T}^* \to \mathcal{T}_A$ be a 2-stable interpretation. For simplicity we assume that Γ, the collection of admissible covers, consists of one definable family. As in 3.3.6, we have a fixed parameter u_0 and a uniform family $\boldsymbol{K} = \boldsymbol{K}_c$ of groupoids in $\mathcal{T}^*$ defined over u_0, along with (uniformly definable) weak equivalences $f_c : \boldsymbol{X}_c \xrightarrow{\sim} \boldsymbol{K}_c$ and $g_c : \omega(\boldsymbol{X}_c) \to \boldsymbol{K}_c$ for $\boldsymbol{X}_c$ members of Γ (note that c ranges over a definable set in $\mathcal{T}$ by assumption). Like in Remark 3.3.7, as u_0 varies, we obtain a family $\boldsymbol{K}_{u,c}$ of objects of a definable 1-groupoid $\boldsymbol{P}_c$, along with a map $\boldsymbol{P}_c \to \underline{\mathbf{Iso}}(\boldsymbol{X}, \omega(\boldsymbol{X}))$ for each member $\boldsymbol{X}$ of Γ. Furthermore, $\boldsymbol{P}_c$ itself is also in Γ. Applying the above map to $\boldsymbol{X} = \boldsymbol{P}_{c'}$, we obtain a family of definable maps of 1-groupoids $a : \boldsymbol{P}_c \to \underline{\mathbf{Iso}}(\boldsymbol{P}_{c'}, \omega_c(\boldsymbol{P}_{c'}))$.

The 2-groupoid $\boldsymbol{G}$ is constructed as follows: $\boldsymbol{G}_0$ is the definable set of parameters c as above. Each groupoid $\boldsymbol{P}_c$ will be isomorphic to $\underline{\mathbf{Hom}}^L(*, c)$ in the corresponding $\boldsymbol{G}^*$. Let c, d be two elements of $\boldsymbol{G}_0$. Given an object u of $\boldsymbol{P}_c$, the map a above produces a groupoid $\boldsymbol{K}_u$ as an object of $\underline{\mathbf{Iso}}(\boldsymbol{P}_d, \omega_c(\boldsymbol{P}_d))$, along with weak equivalences $f : \boldsymbol{P}_d \xrightarrow{\sim} \boldsymbol{K}_u$ and $g : \omega_c(\boldsymbol{P}_d) \to \boldsymbol{K}_u$. Let $\boldsymbol{Q}_{u,v}$ be the set of morphisms in $\boldsymbol{K}_u$ with domain $f(v)$. We set the morphisms from c to d to be the definable types space of $\boldsymbol{Q}_{u,v}$ over $\mathcal{T}$ (this is definable in $\mathcal{T}$ by stability of the embedding). Note that each such type includes, in particular, the information of the object of $\omega_c(\boldsymbol{P}_d)$ which is the codomain of any realisation (as an element of $\boldsymbol{Q}_{u,v}$).

Similarly, assume e is another element of $\boldsymbol{G}_0$, and w an object of $\boldsymbol{P}_e$. The elements of $\boldsymbol{G}_2$ with vertices c, d, e are defined as the types over $\mathcal{T}$ of triples $\boldsymbol{Q}_{u,v} \times \boldsymbol{Q}_{v,w} \times \boldsymbol{Q}_{u,w}$ (over all such u, v, w). The 2-composition is defined similarly, by considering 4-tuples. We skip the details of the construction, as well as the proof that the map determined by a is a weak equivalence. □

We mention also that this description can, in principle, be used to give an equivalent combinatorial definition of 2-internality (similar to the original definition of internality), but I could not find one sufficiently pleasant to write.

3.4. ***Questions.*** I mention a few natural questions that I hope to address in the future.

3.4.1. *Structure of admissible internal covers.* The definable version of the 2-groupoid $\boldsymbol{G}$ associated to a 2-internal cover $\mathcal{T}^*$ of $\mathcal{T}$ was only sketched above. Assuming it is properly described, it still needs to be seen that $\mathcal{T}_{\boldsymbol{G}}$ and $\mathcal{T}^*$ are, in some sense, equivalent. It would also be useful to describe the admissible covers (in either) as suitably defined "higher local systems" on $\boldsymbol{G}$. Both questions require that we name the precise closure properties on the collection of admissible covers: we already assumed that they are closed under finite inverse limit and definable mapping spaces, but it is not clear, for example, if some closure under quantifiers is required.

3.4.2. *Lax interpretations.* We had not run into the questions above because we required objects of the 2-groupoid I^2 to be actual interpretations. This works well in the example of $\mathcal{T}_{\boldsymbol{G}}$, but for general expansions it might make more sense to consider a larger class of "lax interpretations" that preserve only the admissible covers (possibly up to weak equivalence).

3.4.3. *Internal covers of $\mathcal{T}_{\boldsymbol{G}}$.* The requirement for introducing admissible covers was motivated above. However, in the case of $\mathcal{T}_{\boldsymbol{G}}$ it might still be true that essentially all internal covers are the ones described (up to covers that come from the base $\mathcal{T}$). Again, stating this precisely requires clarifying the structure of the collection of admissible covers.

3.4.4. *Relation to analysability.* The 2-groupoid $\boldsymbol{G}^*$ in the theory $\mathcal{T}$ provides an example of a 2-analysable set over $\mathcal{T}$. Can we describe (combinatorially) which 2-analysable covers occur in this way?

3.4.5. *Generalisation to higher dimensions.* This is rather clear: one continues by induction, defining an $(i+1)$-groupoid associated to a stable expansion by taking into account i-internal covers, and then defining the expansion to be an $(i+1)$-cover if this groupoid is nonempty. However, some of the proofs given above would be difficult to generalise, and it would be interesting to look for a smoother way. In any case, this only applies to each finite level, and it does not seem reasonable to expect a generalisation to arbitrary ∞-groupoids.

3.4.6. *Structure at $*$.* We did not consider the structure of the groupoid $\underline{\mathbf{Hom}}^L_{\boldsymbol{G}^*}(*,*)$ definable in $\mathcal{T}_{\boldsymbol{G}}$. On top of the groupoid structure, composition gives it a structure of a monoidal category up to homotopy (i.e., the homotopy category is monoidal). It also acts on all the admissible covers, so it is really a higher analogue of the binding group. However, we did not consider what could be a version of the Galois correspondence or of descent, as in the 1-dimensional case.

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