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Complete type amalgamation for nonstandard finite groups

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We extend previous work on Hrushovski's stabilizer theorem and prove a measure-theoretic version of a well-known result of Pillay–Scanlon–Wagner on products of three types. This generalizes results of Gowers on products of three sets and yields model-theoretic proofs of existing asymptotic results for quasirandom groups. We also obtain a model-theoretic proof of Roth's theorem on the existence of arithmetic progressions of length 3 for subsets of positive density in suitable definably amenable groups, such as countable amenable abelian groups without involutions and ultraproducts of finite abelian groups of odd order.

Introduction

Szemerédi [1975] answered positively a question of Erdős and Turán by showing that every subset A of $\mathbb{N}$ of upper density

$$\limsup_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n} > 0$$

must contain an arithmetic progression of length k for every natural number k. For $k = 3$, the existence of arithmetic progressions of length 3 (in short 3-AP) was already proven by Roth [1953] in what is now called Roth's theorem on arithmetic progressions (not to be confused with Roth's theorem on diophantine approximation of algebraic integers). There has been (and still is) impressive work done on understanding Roth's and Szemerédi's theorems, explicitly computing lower bounds for the density as well as extending these results to more general settings. In the second direction, it is worth mentioning Green and Tao's result [2008] on the existence of arbitrarily long finite arithmetic progressions among the subset of prime numbers, which however has upper density 0.

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In the noncommutative setting, proving single instances of Szemerédi's theorem, particularly Roth's theorem, becomes highly nontrivial. Note that the sequence (a, ab, ab^2) can be seen as a 3-AP, even for noncommutative groups. Gowers [2008, Question 6.5] asked whether the proportion of pairs (a, b) in $\mathrm{PSL}_2(q)$, for q a prime power, such that a, ab and ab^2 all lie in a fixed subset A of density δ, approximately equals δ^3. Gowers's question was positively answered by Tao [2013] and later extended to arbitrary nonabelian finite simple groups by Peluse [2018]. For arithmetic progressions (a, ab, ab^2, ab^3) of length 4 in $\mathrm{PSL}_2(q)$, a partial result was obtained in [Tao 2013], whenever the element b is diagonalizable over the finite field $\mathbb{F}_q$ (which happens half of the time).

A different generalization of Roth's theorem, present in [Sanders 2009; Henriot 2016], concerns the existence of a 3-AP in finite sets of small doubling in abelian groups. Recall that a finite set A of a group has doubling at most K if the productset $A \cdot A = \{ab\}_{a,b \in A}$ has cardinality $|A \cdot A| \leq K|A|$. More generally, a finite set has tripling at most K if $|A \cdot A \cdot A| \leq K|A|$. If A has tripling at most K, the comparable set $A \cup A^{-1} \cup \{\mathrm{id}_G\}$ (of size at most $2|A| + 1$) has tripling at most $(CK^C)^2$ with respect to some explicit absolute constant $C > 0$, so we may assume that A is symmetric and contains the neutral element. Archetypal sets of small doubling are approximate subgroups, that is, symmetric sets A such that $A \cdot A$ is covered by finitely many translates of A.

The model-theoretic study of approximate subgroups originated in Hrushovski's striking paper [2012], which contained the so-called stabilizer theorem, adapting techniques of stability theory to an abstract measure-theoretic setting. Hrushovski's work has led to several remarkable applications of model theory to additive combinatorics.

In classical geometric model theory, and more generally, in a group G definable in a simple theory, Hrushovski's stabilizer of a generic type over an elementary substructure M is the connected component G^{00}_M, that is, the smallest type-definable subgroup over M of bounded index (bounded with respect to the saturation of the ambient universal model). Generic types in G^{00}_M are called *principal types*. If the theory is stable, there is a unique principal type, but this need not be the case for simple theories. However, Pillay, Scanlon and Wagner [1998, Proposition 2.2] noticed that for every three principal types p, q and r in a simple theory over an elementary substructure M, there are independent realizations a of p and b of q over M such that $a \cdot b$ realizes r. The main ingredient in their proof is a clever application of 3-complete amalgamation (also known as *the independence theorem*) over the elementary substructure M.

For the purpose of the present work, we shall not define what a general complete amalgamation problem is, but a variation of it, restricting the problem to conditions given by products with respect to the underlying group law:

Question. Fix a natural number $n \geq 2$. For each nonempty subset F of $\{1, \ldots, n\}$, let p_F be a principal generic (that is, weakly random) type over the elementary substructure M. Can we find (under suitable conditions) an independent (weakly random) tuple $(a_1, \ldots, a_n)$ of G^n such that for all $\varnothing \neq F \subseteq \{1, \ldots, n\}$, the element a_F realizes p_F, where a_F stands for the product of all a_i, with i in F, written with the indices in increasing order?

The above formulation resonates with [Green and Tao 2008, Theorem 5.3] for quasirandom groups and agrees for $n = 2$ with the aforementioned result of Pillay, Scanlon and Wagner.

In this work, we give a (partial) positive solution for $n = 2$ (Theorem 3.10) to the above question for definable groups equipped with a definable Keisler measure satisfying Fubini (e.g., ultraproducts of groups equipped with the associated counting measure localized with respect to a distinguished finite set, as in Example 1.5). As a by-product, we obtain a measure-theoretic version of the result of Pillay, Scanlon and Wagner (Theorem 3.10):

Main Theorem. Given a pseudofinite subset X of small tripling in a sufficiently saturated group G and a countable elementary substructure M, for every weakly random type q and almost all pairs (p, r) of weakly random types over M concentrated in the subgroup $\langle X \rangle$ generated by X, there is a weakly random pair (a, b) over M in $p \times q$ with $a \cdot b$ realizing r, whenever $\mathrm{Cos}(p) \cdot \mathrm{Cos}(q) = \mathrm{Cos}(r)$, where $\mathrm{Cos}(p)$ is the coset of $\langle X \rangle^{00}_M$ determined by the type p.

The result of Pillay, Scanlon and Wagner holds for all such pairs (p, r) of generic types. Unfortunately, our techniques can only prove the analogous result outside a set of measure 0. Whilst we do not know how to obtain the result for all pairs (p, r) of weakly random types over M, our results however suffice to reprove model-theoretically some known results. Using a model-theoretic analog of Croot–Sisask's almost periodicity [Croot and Sisask 2010, Corollary 1.2] (Corollary 3.2), we easily deduce a nonquantitative version of Roth's theorem (Theorem 3.14) on 3-AP for finite subsets of small doubling in abelian groups with trivial 2-torsion, which resembles previous work of Sanders [2009, Theorem 7.1] and generalizes a result of Frankl, Graham and Rödl [1987, Theorem 1].

In Section 4, we reprove model-theoretically results valid for ultra-quasirandom groups, that is, asymptotic limits of quasirandom groups, already studied in [Bergelson and Tao 2014], and later in [Palacín 2020]. In particular, in Corollary 4.8 we give nonquantitative model-theoretic proofs of [Gowers 2008, Theorems 3.3 and 5.3]. In Section 5, we explore further this analogy to extend some of the results of Gowers to a local setting, without imposing that the group is an ultraproduct of quasirandom groups (see Corollaries 5.12 and 5.13).

We assume throughout the text a certain familiarity with basic notions in model theory. Sections 1, 2 and 3 contain the model-theoretic core of the paper, whilst Sections 4 and 5 contain applications to additive combinatorics.

1. Randomness and Fubini

Most of the material in this section can be found in [Halmos 1974; Hrushovski 2012; Massicot and Wagner 2015; Simon 2015].

We work inside a sufficiently saturated model $\mathbb{U}$ of a complete first-order theory (with infinite models) in a countable language $\mathcal{L}$, that is, the model $\mathbb{U}$ is saturated and strongly homogeneous with respect to some sufficiently large cardinal κ. All sets and tuples are taken inside $\mathbb{U}$.

A subset X of $\mathbb{U}^n$ is definable over the parameter set A if there exists a formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and a tuple $a = (a_1, \ldots, a_m)$ in A such that an n-tuple b belongs to X if and only if $\varphi(b, a)$ holds in $\mathbb{U}$. As usual, we identify a definable subset of $\mathbb{U}$ with a formula defining it. Unless explicitly stated, when we use the word definable, we mean definably possibly with parameters. It follows that a subset X is definable over the parameter set A if and only if X is definable (over some set of parameters) and invariant under the action of the group of automorphisms $\mathrm{Aut}(\mathbb{U}/A)$ of $\mathbb{U}$ fixing A pointwise. The subset X of $\mathbb{U}$ is type-definable if it is the intersection of a bounded number of definable sets, where bounded means that its size is strictly smaller than the degree of saturation of $\mathbb{U}$.

For our applications we mainly consider the case where the language $\mathcal{L}$ contains the language of groups and the universe of our ambient model is a group. Nonetheless, our model-theoretic setting works as well for an arbitrary definable group, that is, a group whose underlying set and its group law are both definable.

Definition 1.1. A *definably amenable pair* (G, X) consists of an underlying definable group G together with the following data:

- a definable subset X of G;
- the (boolean) ring $\mathcal{R}$ of definable sets contained in the subgroup $\langle X \rangle$ generated by X, that is, the subcollection $\mathcal{R}$ is closed under finite unions and relative set-theoretic differences;
- a finitely additive measure μ on $\mathcal{R}$ invariant under both left and right translation with $\mu(X) = 1$. (Note that we require translation invariance under both actions).

Note that the subgroup $\langle X \rangle$ generated by the subset X need not be definable, but it is *locally definable*, for the subgroup $\langle X \rangle$ is a countable union of definable sets of the form

$$X^{\odot n} = \underbrace{X_1 \cdots X_1}_{n},$$

where X_1 is the definable set $X \cup X^{-1} \cup \{\mathrm{id}_G\}$. Furthermore, every definable subset Y of $\langle X \rangle$ is contained in some finite product $X^{\odot n}$, by compactness and saturation of the ambient model.

Remark 1.2. Model-theoretic compactness implies that the finitely additive measure μ satisfies Carathéodory's criterion, so there exists a unique σ-additive measure on the σ-algebra generated by $\mathcal{R}$. On the other hand, for every definable set Y of $\mathcal{R}$ over any set of parameters C, the measure μ extends to a regular Borel finite measure on the Stone space $S_Y(C)$ of complete types over C containing the C-definable set Y; see [Simon 2015, p. 99].

We denote the above extension of μ again by μ, though there will be (most likely) Borel sets of infinite measure, as noticed by Massicot and Wagner:

Fact 1.3 [Massicot and Wagner 2015, Remark 4]. *The subgroup $\langle X \rangle$ is definable if and only if $\mu(\langle X \rangle)$ is finite.*

Throughout the paper, we always assume that the language $\mathcal{L}$ is rich enough (see [Starchenko 2017, Definition 3.19]) to render the measure μ definable without parameters.

Definition 1.4. The measure μ of a definably amenable pair (G, X) is *definable without parameters* if for every $\mathcal{L}$-formula $\varphi(x, y)$, every natural number $n \geq 1$ and every $\epsilon > 0$, there is a partition of the $\mathcal{L}$-definable set

$$\{y \in \mathbb{U}^{|y|} \mid \varphi(\mathbb{U}, y) \subseteq X^{\odot n}\}$$

into $\mathcal{L}$-formulae $\rho_1(y), \ldots, \rho_m(y)$ such that whenever a pair (b, b') in $\mathbb{U}^{|y|} \times \mathbb{U}^{|y|}$ realizes $\rho_i(y) \wedge \rho_i(y')$, then

$$|\mu(\varphi(x, b)) - \mu(\varphi(x, b'))| < \epsilon.$$

The above definition is a mere adaptation of [Starchenko 2017, Definition 3.19] to the locally definable context, by imposing that the restriction of μ to every definable subset $X^{\odot n}$ is definable in the sense of [Starchenko 2017, Definition 3.19]. In particular, a definable measure of a definably amenable pair (G, X) is *invariant*, that is, its value is invariant under the action of $\mathrm{Aut}(\mathbb{U})$. Notice that whenever the measure μ is definable, given a definable subset $\varphi(x, b)$ of measure r and a value $\epsilon > 0$, the tuple b lies in some definable subset which is contained in

$$\left\{y \in \mathbb{U}^{|y|} \mid r - \epsilon \leq \mu(\varphi(\mathbb{U}, y)) \leq r + \epsilon\right\}.$$

Assuming that μ is definable, its extension to the σ-algebra generated by the definable subsets of $\langle X \rangle$ is again invariant under left and right translations, as well as under automorphisms. Indeed, every automorphism τ of $\mathrm{Aut}(\mathbb{U})$ (likewise for left and right translations) gives rise to a measure μ^τ, such that $\mu^\tau(Y) = \mu(\tau(Y))$

for every measurable subset Y of $\langle X\rangle$. Since μ^τ agrees with μ on $\mathcal{R}$, we conclude that the σ-additive measure μ^τ is μ by the uniqueness of the extension. Thus, the measure of a Borel subset Y in the space of types containing a fixed definable set Z in $\mathcal{R}$ depends solely on the type of the parameters defining Y.

Example 1.5. Let $(G_n)_{n\in\mathbb{N}}$ be an infinite family of groups, each with a distinguished finite subset X_n. Expand the language of groups to a language $\mathcal{L}$ including a unary predicate and set M_n to be an $\mathcal{L}$-structure with universe G_n, equipped with its group operation, and interpret the predicate as X_n. Following [Hrushovski 2012, Section 2.6] we can further assume that $\mathcal{L}$ has predicates $Q_{r,\varphi}(y)$ for each r in $\mathbb{Q}^{\geq 0}$ and every formula $\varphi(x, y)$ in $\mathcal{L}$ such that $Q_{r,\varphi}(b)$ holds if and only if the set $\varphi(M_n, b)$ is finite with $|\varphi(M_n, b)| \leq r|X_n|$. Note that if the original language was countable, so is the extension $\mathcal{L}$.

Consider now the ultraproduct M of the $\mathcal{L}$-structures $(M_n)_{n\in\mathbb{N}}$ with respect to some nonprincipal ultrafilter $\mathcal{U}$. Denote by G and X the corresponding interpretations in a sufficiently saturated elementary extension $\mathbb{U}$ of M. For each $\mathcal{L}$-formula $\varphi(x, y)$ and every tuple b in $\mathbb{U}^{|y|}$ such that $\varphi(\mathbb{U}, b)$ is a subset of $\langle X\rangle$, define

$$\mu(\varphi(x, b)) = \inf\{r \in \mathbb{Q}^{\geq 0} \mid Q_{r,\varphi}(b) \text{ holds}\},$$

where we assign ∞ if $Q_{r,\varphi}(b)$ holds for no value r. This is easily seen to be a finitely additive definable measure on the ring $\mathcal{R}$ of definable subsets of $\langle X\rangle$ which is invariant under left and right translation. In particular, the pair (G, X) is definably amenable.

Throughout this paper we consider two main examples:

(a) The set X equals G itself, which happens whenever the subset $X_n = G_n$ for $\mathcal{U}$-almost all n in $\mathbb{N}$. The normalized counting measure μ defined above is a definable Keisler measure [Keisler 1987] on the pseudofinite group G. Note that in this case the ring of sets $\mathcal{R}$ coincides with the Boolean algebra of all definable subsets of G.

(b) For $\mathcal{U}$-almost all n, the set X_n has *small tripling*: there is a constant $K > 0$ such that $|X_n X_n X_n| \leq K|X_n|$. The noncommutative Plünnecke–Ruzsa inequality [Tao 2008, Lemma 3.4] yields that $|X_n^{\odot m}| \leq K^{O_m(1)}|X_n|$, so the measure $\mu(Y)$ is finite for every definable subset Y of $\langle X\rangle$, since Y is then contained in $X^{\odot m}$ for some m in $\mathbb{N}$. In particular, the corresponding σ-additive measure μ is again σ-finite.

Whilst each subset X_n in the example (b) must be finite, we do not impose that the groups G_n are finite. If the set X_n has tripling at most K, the set $X^{\odot 1} = X_n \cup X_n^{-1} \cup \{\mathrm{id}_G\}$ has size at most $2|X_n| + 1$ and tripling at most $(CK^C)^2$ with respect to some explicit absolute constant $C > 0$. Thus, taking ultraproducts, both structures (G, X) and $(G, X^{\odot 1})$ will have the same sets of positive measure (or

density), though the values may differ. Hence, we may assume that in a definably amenable pair (G, X) the corresponding definable set X is symmetric and contains the neutral element of G.

The above example can be adapted to consider countable amenable groups.

Example 1.6. Recall that a countable group is *amenable* if it is equipped with a sequence $(F_n)_{n\in\mathbb{N}}$ of finite sets of increasing cardinalities (so $\lim_{n\to\infty} |F_n| = \infty$) such that for all g in G,

$$\lim_{n\to\infty} \frac{|F_n \cap g \cdot F_n|}{|F_n|} = 1.$$

Such a sequence of finite sets is called a left Følner sequence. The archetypal example of an amenable group is $\mathbb{Z}$ with left Følner sequence $F_n = \{-n, \ldots, n\}$.

By [Namioka 1964, Corollary 5.3], if a group is amenable, then there is a distinguished left Følner sequence where each F_n is symmetric. In particular, the sequence $(F_n)_{n\in\mathbb{N}}$ is also a right Følner sequence:

$$\lim_{n\to\infty} \frac{|F_n \cap F_n \cdot g|}{|F_n|} = 1 \quad \text{for all } g \text{ in } G.$$

Notice also that a subsequence of a Følner sequence is again Følner and so is the sequence $(F_n \times F_n)_{n\in\mathbb{N}}$ in the group $G \times G$. Given an amenable group G with a distinguished Følner sequence $(F_n)_{n\in\mathbb{N}}$ consisting of symmetric sets as well as a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the ultralimit

$$\mu(Y) = \lim_{n\to\mathcal{U}} \frac{|Y \cap F_n|}{|F_n|}$$

induces a finitely additive measure on the Boolean algebra of subsets of G which is invariant under left and right translation.

Starting from a fixed countable language $\mathcal{L}$ expanding the language of groups, we can render the above measure definable, similarly as in Example 1.5. Hence, we can consider every countable amenable group G as a definably amenable pair, setting $X = G$.

Example 1.7. Every stable group G has finitely satisfiable generics (is fsg) and thus is equipped with a unique left and right translation invariant Keisler measure which is generically stable (see [Hrushovski et al. 2013; Simon 2015, Example 8.34]).

Similarly, a compact semialgebraic Lie group $G(\mathbb{R})$, or more generally a definably compact group G definable in an o-minimal expansion of a real closed field, is again fsg. If the group is the $\mathbb{R}$-rational points of a compact semialgebraic Lie group, this measure coincides with the normalized Haar measure.

Hence, we can consider in these two previous cases (stable and o-minimal compact) the group G as a definably amenable pair, setting $X = G$.

If a group G is definable, so is every finite cartesian product. Moreover, the construction in Examples 1.5 and 1.6 can also be carried out for a finite cartesian product to produce for every $n \geq 1$ in $\mathbb{N}$ a definably amenable pair (G^n, X^n), where $\langle X^n \rangle = \langle X \rangle^n$, equipped with a definable σ-finite measure μ_n. Thus, the following assumption is satisfied by our Examples 1.5, 1.6 and 1.7.

Assumption 1. For every $n \geq 1$, the pair (G^n, X^n) is definably amenable for a definable σ-finite measure μ_n in a compatible fashion: the measure μ_{n+m} extends the corresponding product measure $\mu_n \times \mu_m$.

The definability condition in Definition 1.4 implies that the function

$$F^{\varphi}_{\mu_n, C} : S_m(C) \to \mathbb{R}, \quad \operatorname{tp}(b/C) \mapsto \mu_n(\varphi(x, b)),$$

is well-defined and continuous for every $\mathcal{L}_C$-formula $\varphi(x, y)$ with $|x| = n$ and $|y| = m$ such that $\varphi(x, y)$ defines a subset of $\langle X \rangle^{n+m}$. Therefore, for such $\mathcal{L}_C$-formulae $\varphi(x, y)$, consider the $\mathcal{L}_C$-definable subset $Y = \{y \in \langle X \rangle^m \mid \exists x\, \varphi(x, y)\}$ and the corresponding clopen subset $[Y]$ of $S_m(C)$. Thus, we can consider the following measure ν on $\langle X \rangle^{n+m}$:

$$\nu(\varphi(x, y)) = \int_{q \in [Y]} F^{\varphi}_{\mu_n, C}(q)\, d\mu_m = \int_{y \in Y} \mu_n(\varphi(x, y))\, d\mu_m.$$

By an abuse of notation, we write $\int_{\langle X \rangle^m} \mu_n(\varphi(x, y))\, d\mu_m$ for $\int_Y \mu_n(\varphi(x, y))\, d\mu_m$.

For the pseudofinite measures described in Example 1.5, the above integral equals the ultralimit

$$\lim_{k \to \mathcal{U}} \frac{1}{|X_k|^m} \sum_{y \in \langle X_k \rangle^m} \frac{|\varphi(x, y)|}{|X_k|^n},$$

so ν equals μ_{n+m} and consequently Fubini–Tonelli holds; see (the proof of) [Bergelson and Tao 2014, Theorem 19]. The same holds whenever the measure is given by densities with respect to a Følner sequence in an amenable group, as in Example 1.6. For arbitrary definably amenable pairs, whilst the measure ν extends the product measure $\mu_n \times \mu_m$, it need not be a priori μ_{n+m} [Starchenko 2017, Remark 3.28]. Keisler [1987, Theorem 6.15] exhibited a Fubini–Tonelli type theorem for general Keisler measures under certain conditions. These conditions hold for the unique generically stable translation invariant measure of an fsg group (see Example 1.7). We will impose a further restriction on the definably amenable pairs we consider, taking Examples 1.5, 1.6 and 1.7 as a guideline.

Assumption 2. For every definably amenable pair (G, X) and its corresponding compatible system of definable measures $(\mu_n)_{n \in \mathbb{N}}$ on the Cartesian powers of $\langle X \rangle$, the Fubini condition holds: whenever a definable subset of $\langle X \rangle^{n+m}$ is given by an

$\mathcal{L}_C$-formula $\varphi(x, y)$ with $|x| = n$ and $|y| = m$, we have

$$\mu_{n+m}(\varphi(x, y)) = \int_{\langle X\rangle^m} \mu_n(\varphi(x, y))\, d\mu_m = \int_{\langle X\rangle^n} \mu_m(\varphi(x, y))\, d\mu_n.$$

(Note that the above integrals do not run over the locally definable sets $\langle X\rangle^m$ and $\langle X\rangle^n$, but rather over definable subsets, for $\varphi(x, y)$ is itself definable).

Whilst this assumption is stated for definable sets, it extends to certain Borel sets, whenever the language $\mathcal{L}_C$ is countable.

Remark 1.8. Assume that $\mathcal{L}_C$ is countable and fix a natural number $k \geq 1$. Following [Conant et al. 2023, Definition 2.6], for every Borel subset Z of $S_{n+m}(C)$ of types $q(x, y)$ with $|x| = n$ and $|y| = m$, set

$$Z(x, b) = \{p \in S_n(\mathbb{U}) \mid \mathrm{tp}(a, b/C) \text{ belongs to } Z \text{ for some } a \text{ realizing } p_{\restriction C,b}\}.$$

Note that $Z(x, b)$ only depends on $\mathrm{tp}(b/C)$ by [Conant et al. 2023, Lemma 2.7]. If Z is contained in the clopen set determined by the $\mathcal{L}_C$-definable set $(X^{\odot k})^{n+m}$, we define analogously as before a function

$$F^Z_{\mu_n,C} : S_m(C) \to \mathbb{R}, \quad \mathrm{tp}(b/C) \mapsto \mu_n(Z(x, b)).$$

This function is Borel, and thus measurable, by the definability of the measure as well as the monotone convergence theorem, for it agrees with $F^{\varphi}_{\mu_n,C}$ whenever Z is the clopen $[\varphi]$. Furthermore, the identity

$$\mu_{n+m}(Z(x, y)) = \int_{\langle X\rangle^m} \mu_n(Z(x, y))\, d\mu_m = \int_{\langle X\rangle^n} \mu_m(Z(x, y))\, d\mu_n,$$

holds by a straightforward application as in [Bergelson and Tao 2014, Theorem 20] of the monotone class theorem, using the fact that $\mu(X^{\odot k})$ is finite. In particular, the above identity of integrals holds for every Borel set of finite measure by regularity.

Remark 1.9. The examples listed in Examples 1.5, 1.6 and 1.7 satisfy both Assumptions 1 and 2.

Henceforth, the language is countable and all definably amenable pairs satisfy Assumptions 1 and 2.

Adopting some terminology from additive combinatorics, we use the word *density* for the value of the measure of a subset in $\mathcal{R}$ of a definably amenable pair (G, X). A (partial) type is said to be *weakly random* if it contains a definable subset in $\mathcal{R}$ of positive density but no definable subset in $\mathcal{R}$ of density 0. Note that every weakly random partial type $\Sigma(x)$ over a parameter set A implies a definable set $X^{\odot k}$ in $\mathcal{R}$ for some k in $\mathbb{N}$ and thus it can be completed to a weakly random complete type over any arbitrary set B containing A, since the collection of formulae

$$\Sigma(x) \cup \{X^{\odot k} \setminus Z \mid Z \text{ in } \mathcal{R} \text{ is } B\text{-definable of density } 0\}$$

is finitely consistent. Thus, weakly random types exist (yet the partial type $x = x$ is not weakly random whenever $G \neq \langle X \rangle$). As usual, we say that an element b of G is weakly random over A if $\mathrm{tp}(b/A)$ is.

Weakly random elements satisfy a weak notion of transitivity.

Lemma 1.10. *Let b be weakly random over a set of parameters C and a be weakly random over C, b. The pair (a, b) is weakly random over C.*

Proof. We need to show that every C-definable subset Z of $\langle X \rangle^{n+m}$ containing the pair (a, b) has positive density with respect to the product measure μ_{n+m}, where $n = |a|$ and $m = |b|$. Since a is weakly random over C, b, the fiber Z_b of Z over b has measure $\mu_n(Z_b) = 2r$ for some real number $0 < r$. Hence b belongs to a C-definable subset Y of

$$\{y \in \mathbb{U}^m \mid r \leq \mu_n(Z_y) \leq 3r\},$$

by the definability of the measure. In particular, the measure $\mu_m(Y)$ is strictly positive. Thus,

$$\mu_{n+m}(Z) = \int_{\langle X \rangle^m} \mu_n(Z_y)\, d\mu_m \geq \int_Y \mu_n(Z_y)\, d\mu_m \geq \mu_m(Y) r > 0,$$

as desired. □

Note that the tuple b above may not be weakly random over C, a. To remedy the failure of symmetry in the notion of randomness, we introduce *random* types, which will play a fundamental role in Section 3. Though random types already appear in [Hrushovski 2013, Subsection 2.23] (see also [Hrushovski 2012, Subsection 2.20]), we take the opportunity here to recall Hrushovski's definition of *ω-randomness*. All the ideas here until the end of this section are due to Hrushovski and we are merely writing down some of the details for the sake of the presentation.

Fix some countable elementary substructure M and some Y in $\mathcal{R}$ definable over M (so $Y \subseteq (X^{\odot k})$ for some k in $\mathbb{N}$). As in Remark 1.2, we denote by $S_{Y^m}(M)$ the compact subset of the space of types over M containing the M-definable subset Y^m.

Definition 1.11. Denote by $\mathcal{B}_M^Y$ the smallest Boolean algebra of subsets of $S_{Y^m}(M)$, as m varies, containing all clopen subsets of $S_{Y^m}(M)$ and closed under the following operations:

- The preimage of a set $W \subseteq S_{Y^m}(M)$ in $\mathcal{B}_M^Y$ under the natural continuous map $S_{Y^n}(M) \to S_{Y^m}(M)$ given by the restriction to a choice of m coordinates belongs again to $\mathcal{B}_M^Y$.
- If $Z \subseteq S_{Y^{n+m}}(M)$ belongs to $\mathcal{B}_M^Y$, then so does

  $$(F^Z_{\mu_n, M})^{-1}(\{0\}) = \{\mathrm{tp}(b/M) \in S_{Y^m}(M) \mid \mu_n(Z(x, b)) = 0\},$$

  with $Z(x, b)$ as in Remark 1.8.

Note that each element of $\mathcal{B}_M^Y$ is a Borel subset of the appropriate space of types by Remark 1.8. Furthermore, it is countable since it can be inductively built from the Boolean algebras of clopen subsets of the $S_{Y^m}(M)$'s by adding in the next step all Borel sets of the form $(F^Z_{\mu_n,M})^{-1}(\{0\})$ and closing under Boolean operations. The collection $\mathcal{B}_M^Y$ contains new sets which are neither open nor closed.

Definition 1.12. Let Y in $\mathcal{R}$ be definable over the countable elementary substructure M. An n-tuple a of elements in Y is *random* over $M \cup B$, where B is some countable subset of parameters, if $\mu_n(Z(x, b)) > 0$ for every finite subtuple b in B and every Borel subset Z in $\mathcal{B}_M^Y$ with $\mathrm{tp}(a, b/M)$ in Z.

For $B = \varnothing$, we simply say that the tuple is *random* over M.

Remark 1.13. Since $\mathcal{B}_M^Y$ contains all clopen sets given by M-definable subsets, it is easy to see that a tuple random over $M \cup B$ is weakly random over $M \cup B$, which justifies our choice of terminology (instead of using the term *wide type* from [Hrushovski 2012]).

Randomness is preserved under the group law: if a is an element of $\langle X\rangle$ random over $M \cup B$, then so are a^{-1} and $b \cdot a$ for every element b in $B \cap \langle X\rangle$.

Furthermore, note that randomness is a property of the type: if a and a' have the same type over $M \cup B$, then a is random over $M \cup B$ if and only if a' is.

Remark 1.14. Since $\mathcal{B}_M^Y$ is countable, the σ-additivity of the measure yields that every measurable subset of $S_{Y^m}(M \cup B)$, with B countable, of positive density contains a random element over $M \cup B$. In particular, every weakly random definable subset of Y^m contains random elements over M, B.

Randomness is a symmetric notion.

Lemma 1.15 [Hrushovski 2013, Exercise 2.25]. *Let Y in $\mathcal{R}$ be definable over the countable elementary substructure M. A finite tuple (a, b) of elements in Y is random over M if and only if b is random over M and a is random over M, b.*

Proof. Assume first that (a, b) is random over M. Clearly so is b by Fubini and Remark 1.8. Thus we need only prove that a is random over M, b. Suppose for a contradiction that $\mu_{|a|}(Z(x, b)) = 0$ for some $Z \subseteq S_{Y^{|a|+|b|}}(M)$ of $\mathcal{B}_M^Y$ containing $\mathrm{tp}(a, b)$. The type of the pair (a, b) belongs to

$$\begin{aligned}\widetilde{Z} &= Z \cap \pi^{-1}\big((F^Z_{\mu_{|a|},M})^{-1}(\{0\})\big)\\ &= Z \cap \big\{\mathrm{tp}(c, d/M) \in S_{Y^{|a|+|b|}}(M) \mid \mu_{|a|}(Z(x, d)) = 0\big\},\end{aligned}$$

where $\pi = \pi_{|a|+|b|,|b|}$ is the corresponding restriction map. Now, the set $\widetilde{Z}$ belongs to $\mathcal{B}_M^Y$ and contains (a, b), so it cannot have density 0. However, Remark 1.8 yields

$$0 < \mu_{|a|+|b|}(\widetilde{Z}) = \int_{Y^{|b|}} \mu_{|a|}(\widetilde{Z}(x, d))\, d\mu_{|b|} \leq \int_{Y^{|b|}} \mu_{|a|}(Z(x, d))\, d\mu_{|b|} = 0,$$

which gives the desired contradiction.

Assume now that b is random over M and a is random over M, b. Suppose for a contradiction that $\mathrm{tp}(a, b/M)$ lies in some Borel $Z(x, y)$ of $\mathcal{B}^Y_M$ with $\mu_{|a|+|b|}(Z) = 0$. By Remark 1.8,

$$0 = \mu_{|a|+|b|}(Z) = \int_{Y^{|b|}} \mu_{|a|}(Z(x, d))\, d\mu_{|b|},$$

so $\mu_{|a|}(Z(x, d)) = 0$ for $\mu_{|b|}$-almost all types $\mathrm{tp}(d/M)$ in $S_{Y^{|b|}}(M)$. Hence, the set $(F^Z_{\mu_{|a|}, M})^{-1}(\{0\})$ has measure $\mu_{|b|}(Y^{|b|})$. Since a is random over M, b, we have that $\mu_{|a|}(Z(x, b)) > 0$, so $\mathrm{tp}(b/M)$ belongs to the complement of $(F^Z_{\mu_{|a|}, M})^{-1}(\{0\})$, which belongs to $\mathcal{B}^Y_M$ and has $\mu_{|b|}$-measure 0. We conclude that the element b is not random over M, which gives the desired contradiction. □

Symmetry of randomness will allow us in Sections 3 and 4 to transfer ideas arisen from the study of definable groups in simple theories to the pseudofinite context as well as to definably compact groups definable in o-minimal expansions of real closed fields. Whilst weak randomness is not symmetric, a weak form of symmetry holds (as pointed out by the anonymous referee, to whom we would like to express our sincere gratitude again).

Lemma 1.16 (the referee's lemma). *Let Y in $\mathcal{R}$ be a subset of positive density definable over the countable elementary substructure M. Given two finite tuples a and b of elements in Y with a weakly random over M and b random over M, a, then a is weakly random over M, b.*

Proof. Assuming otherwise, there is an M-definable set Z containing (a, b) such that the fiber Z_b has $\mu_{|a|}$-measure 0. Definability of the measure Definition 1.4 yields that the set

$$W = (F^Z_{\mu_{|a|}, M})^{-1}(\{0\}) = \{\mathrm{tp}(d/M) \in S_{Y^{|b|}}(M) \mid \mu_{|a|}(Z_d) = 0\}$$

is closed and thus it can be written as a countable intersection $W = \bigcap_{m\in\mathbb{N}} W_m$ of M-definable sets with $W_{m+1} \subseteq W_m$. Now, the closed set $[Z(x, y)] \cap W(y)$ belongs to $\mathcal{B}^Y_M$ and contains $\mathrm{tp}(a, b/M)$, so $\mu_{|b|}([Z(a, y)] \cap W(y)) > 0$, since b is random over M, a.

Claim. *There exists some M-definable subset V containing a such that*

$$\mu_{|b|}([Z(a', y)] \cap W(y)) > 0$$

for all a' in V.

Note that V has positive density, for $\mathrm{tp}(a/M)$ is weakly random.

Proof of Claim. Assume for a contradiction that this is not the case. Since both the language and M are countable, we may list all M-definable subsets containing a as $\{V_n\}_{n\in\mathbb{N}}$ with $V_{n+1} \subseteq V_n$. Therefore, for every n in $\mathbb{N}$ there is some a_n in V_n with

$\mu_{|b|}([Z(a_n, y)] \cap W(y)) < 1/(n+1)$. As W is a countable intersection of the W_m's, there is some m_n in $\mathbb{N}$ such that

$$\mu_{|b|}(Z(a_n, y) \cap W_{m_n}(y)) < \frac{1}{n+1}.$$

Notice that we may construct the sequence such that $m_{n+1} > m_n$. Set

$$\theta_<(Z, W_{m_n}) = \left\{ x \in Y^{|a|} \mid \mu_{|b|}(Z(x, y) \cap W_{m_n}(y)) < \frac{1}{n+1} \right\}$$

and define $\theta_\leq(Z, W_{m_n})$ analogously. By definability of the measure, there is some M-definable subset $\theta(Z, W_{m_n})$ such that

$$\theta_<(Z, W_{m_n}) \subseteq \theta(Z, W_{m_n}) \subseteq \theta_\leq(Z, W_{m_n}).$$

In particular, we have that $\theta(Z, W_{m_{n+1}}) \subseteq \theta(Z, W_{m_n})$ for $m_{n+1} > m_n$. Now, the collection of $\mathcal{L}_M$-formulae $\{V_n(x) \wedge \theta(Z, W_{m_n})(x)\}_{n\in\mathbb{N}}$ cannot be consistent, for it would yield the existence of a tuple a' realizing $\mathrm{tp}(a/M)$ with

$$\mu_{|b|}([Z(a', y)] \cap W(y)) \leq \mu_{|b|}(Z(a', y) \cap W_{m_n}(y)) \leq \frac{1}{n+1}$$

for every n in $\mathbb{N}$, so $\mu_{|b|}([Z(a', y)] \cap W(y)) = 0 < \mu_{|b|}([Z(a, y)] \cap W(y))$, which is a contradiction. By compactness, there exists some ℓ in $\mathbb{N}$ such that no realization of V_ℓ satisfies some $\theta(Z, W_{m_j})$ with $j \leq \ell$. However, the element a_ℓ belongs to $V_\ell \cap \theta_<(Z, W_{m_\ell})$, so a_ℓ lies in every $\theta(Z, W_{m_j})$ with $j \leq \ell$, which gives the desired contradiction. $\square_{\text{Claim}}$

Consider now the closed set $W' = [V(x)] \cap [Z(x, y)] \cap W$. The Fubini condition (Remark 1.8) yields that

$$\begin{aligned} 0 &\overset{\text{Claim}}{<} \int_{\mathrm{tp}(c/M)\in[V]} \mu_{|b|}([Z(c, y)] \cap W)\, d\mu_{|b|} \\ &= \mu(W') = \int_{\mathrm{tp}(d/M)\in W} \mu_{|a|}(V(x) \cap Z(x, d))\, d\mu_{|a|} \\ &\leq \int_{\mathrm{tp}(d/M)\in W} \mu_{|a|}(Z(x, d))\, d\mu_{|a|} = 0. \end{aligned}$$

We deduce from the above contradiction that a lies in no definable set Z_b over M, b of density 0, so a is weakly random over M, b, as desired. $\square$

2. Forking and measures

As in Section 1, we work inside a sufficiently saturated structure and a definably amenable pair (G, X) in a fixed countable language $\mathcal{L}$ satisfying Assumptions 1 and 2, though the classical notions of forking and stability do not require the presence of a group nor of a measure.

Recall that a definable set $\varphi(x, a)$ *divides* over a subset C of parameters if there exists an indiscernible sequence $(a_i)_{i\in\mathbb{N}}$ over C with $a_0 = a$ such that the intersection $\bigcap_i \varphi(x, a_i)$ is empty. Archetypal examples of dividing formulae are of the form $x = a$ for some element a not algebraic over C. Since dividing formulae need not be closed under finite disjunctions, witnessed for example by a circular order, we say that a formula $\psi(x)$ *forks* over C if it belongs to the ideal generated by the formulae dividing over C, that is, if ψ implies a finite disjunction of formulae, each dividing over C. A type *divides* (resp. *forks*) over C if it contains an instance which does.

Remark 2.1. Since the measure is invariant under automorphisms and σ-finite, no definable subset of $\langle X\rangle$ of positive density divides, and thus no weakly random type forks over the empty set; see [Hrushovski 2012, Lemma 2.9 and Example 2.12].

Nonforking need not define a tame notion of independence. For example it need not be symmetric, yet it behaves extremely well with respect to certain invariant relations, called stable.

Definition 2.2. An A-invariant relation $R(x, y)$ is *stable* if there exists no A-indiscernible sequence $(a_i, b_i)_{i\in\mathbb{N}}$ such that $R(a_i, b_j)$ holds if and only if $i < j$.

A straightforward Ramsey argument yields that the collection of invariant stable relations is closed under Boolean combinations. Furthermore, an A-invariant relation is stable if there is no A-indiscernible sequence as in the definition of length some fixed infinite ordinal.

The following remark will be very useful in the following sections.

Remark 2.3 [Hrushovski 2012, Lemma 2.3]. Suppose that the type $\mathrm{tp}(a/M, b)$ does not fork over the elementary substructure M and that the M-invariant relation $R(x, y)$ is stable. Then the following are equivalent:

(a) The relation $R(a, b)$ holds.

(b) The relation $R(a', b)$ holds, whenever $a' \equiv_M a$ and $\mathrm{tp}(a'/Mb)$ does not fork.

(c) The relation $R(a', b)$ holds, whenever $a' \equiv_M a$ and $\mathrm{tp}(b/Ma')$ does not fork.

(d) The relation $R(a', b')$ holds, whenever $a' \equiv_M a$ and $b' \equiv_M a$ such that $\mathrm{tp}(a'/M, b')$ or $\mathrm{tp}(b'/M, a')$ does not fork.

A clever use of the Krein–Milman theorem on the locally compact Hausdorff topological real vector space of all σ-additive probability measures allowed Hrushovski to prove the following striking result (the case $\alpha = 0$ is an easy consequence of the inclusion-exclusion principle).

Fact 2.4 [Hrushovski 2012, Lemma 2.10 and Proposition 2.25]. *Given a real number α and $\mathcal{L}_M$-formulae $\varphi(x, z)$ and $\psi(y, z)$ with parameters over an elementary*

substructure M, the M-invariant relation on the definably amenable pair (G, X)

$$R^{\alpha}_{\varphi,\psi}(a,b) \;\Leftrightarrow\; \mu_{|z|}(\varphi(a,z)\wedge\psi(b,z)) = \alpha$$

is stable. In particular, for any partial types $\Phi(x,z)$ and $\Psi(y,z)$ over M, the relation

$$Q_{\Phi,\Psi}(a,b) \;\Leftrightarrow\; \Phi(a,z)\wedge\Psi(b,z) \text{ is weakly random}$$

is stable.

Strictly speaking, Hrushovski's result in its original version is stated for arbitrary Keisler measures (in any theory). To deduce the statement above it suffices to normalize the measure $\mu_{|z|}$ by $\mu_{|z|}((X^{|z|})^{\odot k})$ for some natural number k such that $(X^{|z|})^{\odot k}$ contains the corresponding instances of $\varphi(x,z)$ and $\psi(y,z)$.

We finish this section with a summarized version of Hrushovski's stabilizer theorem tailored to the context of definably amenable pairs. Before stating it, we first need to introduce some notation.

Definition 2.5. Let X be a definable subset of a definable group G and let M be an elementary substructure. We denote by $\langle X\rangle^{00}_M$ the intersection of all subgroups of $\langle X\rangle$ type-definable over M and of bounded index.

If a subgroup of bounded index type-definable over M exists, the subgroup $\langle X\rangle^{00}_M$ is again type-definable over M and has bounded index; see [Hrushovski 2012, Lemmata 3.2 and 3.3]. Furthermore, it is also normal in $\langle X\rangle$ [Hrushovski 2012, Lemma 3.4].

Fact 2.6 [Hrushovski 2012, Theorem 3.5; Montenegro et al. 2020, Theorem 2.12]. *Let (G, X) be a definably amenable pair and let M be an elementary substructure. The subgroup $\langle X\rangle^{00}_M$ exists and equals*

$$\langle X\rangle^{00}_M = (p\cdot p^{-1})^2$$

for any weakly random type p over M, where we identify a type with its realizations in the ambient structure $\mathbb{U}$. Furthermore, the set $pp^{-1}p$ is a coset of $\langle X\rangle^{00}_M$. For every element a in $\langle X\rangle^{00}_M$ weakly random over M, the partial type $p\cap a\cdot p$ is weakly random. In particular, every weakly random element in $\langle X\rangle^{00}_M$ over M lies in $p\cdot p^{-1}$.

If the definably amenable pair we consider happens to be as in the first case of Example 1.5 or a stable group as in Example 1.7, our notation coincides with the classical notation G^{00}_M.

3. On 3-amalgamation and solutions of $xy = z$

As in Section 1, we fix a definably amenable pair (G, X) satisfying Assumptions 1 and 2. Throughout this section, we work over some fixed elementary substructure M.

We denote by $\mathrm{supp}_M(\mu)$ the *support* of μ, that is, the collection of all weakly random types over M contained in $\langle X\rangle$.

Note that each coset of the subgroup $\langle X\rangle^{00}_M$ of Definition 2.5 is type-definable over M and hence M-invariant, though it need not have a representative in M. Thus, every type p over M contained in $\langle X\rangle$ must determine a coset of $\langle X\rangle^{00}_M$. We denote by $\mathrm{Cos}(p)$ the coset of $\langle X\rangle^{00}_M$ of $\langle X\rangle$ containing some (and hence every) realization of p. The following result resonates with [Terry and Wolf 2019, Corollary 1] and [Conant et al. 2020, Theorem 1.3] beyond the definable context.

Proposition 3.1. *Consider an M-invariant subset S of $\langle X\rangle$ such that the relation $u \cdot v \in S$ is stable, as in Definition 2.2. The set S must be, up to M-definable sets of measure* 0, *a union of cosets of $\langle X\rangle^{00}_M$, that is, if an element g in $\langle X\rangle$ belongs to S with $q = \mathrm{tp}(g/M)$ in $\mathrm{supp}_M(\mu)$, then every element h in $\mathrm{Cos}(q)$ weakly random over M belongs to S as well.*

Our proof is mostly an adaptation of [Pillay et al. 1998, Proposition 2.2]. Whilst the authors used the independence theorem from simple theories, we use the stability of the M-invariant relation S instead.

Proof. Assume that the element g as above belongs to the stable M-invariant relation S. Let h be in $\mathrm{Cos}(\mathrm{tp}(g/M))$ weakly random over M and choose a realization b of $\mathrm{tp}(h/M)$ weakly random over M, g. Now, the elements g and b both lie in the same coset of $\langle X\rangle^{00}_M$, so the difference $b \cdot g^{-1}$ lies in $\langle X\rangle^{00}_M$ and is weakly random over M, g. Since weakly random types do not fork, the type $\mathrm{tp}(b \cdot g^{-1}/M, g)$ does not fork over M.

Fact 2.6 yields that the partial type $\mathrm{tp}(g/M) \cap (b \cdot g^{-1}) \cdot \mathrm{tp}(g/M)$ is weakly random. Choose therefore some element g_1 realizing $\mathrm{tp}(g/M)$ weakly random over M, g, b such that $b \cdot g^{-1} \cdot g_1 \equiv_M g$. By invariance of S, we have that $b \cdot g^{-1} \cdot g_1$ belongs to S as well.

Summarizing, the M-invariant relation $\bar{S} = \{(u, v) \in \langle X\rangle \times \langle X\rangle \mid u \cdot v \in S\}$ holds for the pair $(b \cdot g^{-1}, g_1)$ with $\mathrm{tp}(g_1/M, b \cdot g^{-1})$ weakly random and hence nonforking over M. Since the above relation is stable, for any pair (w, z) such that

$$w \equiv_M b \cdot g^{-1}, \quad z \equiv_M g_1 \quad \text{and} \quad \mathrm{tp}(w/M, z) \text{ nonforking over } M,$$

the relation $\bar{S}$ must also hold. Setting now $w = b \cdot g^{-1}$ and $z = g$, we deduce that $b = b \cdot g^{-1} \cdot g$ belongs to S. As the element h realizes $\mathrm{tp}(b/M)$, we conclude by M-invariance that h belongs to S, as desired. □

Given now two M-definable subsets A and B, the relation

$$R^{\alpha}_{A,B}(u, v) \;\Leftrightarrow\; \text{“}\mu(uA \cap vB) = \alpha\text{”}$$

is stable by Fact 2.4. So, setting $S = \{g \in \langle X\rangle \mid \mu(A \cap gB) = \alpha\}$, Proposition 3.1 yields immediately the following result, which we personally think resonates with Croot–Sisask's almost-periodicity [Croot and Sisask 2010, Corollary 1.2].

Corollary 3.2. *Given two M-definable subsets A and B, the values $\mu(A \cap gB)$ and $\mu(A \cap hB)$ agree for any two weakly random elements g and h over M within the same coset of $\langle X\rangle^{00}_M$.* □

Given now two types p_1 and p_2 over M and an element g of $\langle X\rangle$ such that the partial type $p_1 \cdot g \cap p_2$ is consistent, it follows that the type $\mathrm{tp}(g/M)$ determines the coset $\mathrm{Cos}(p_1)^{-1} \cdot \mathrm{Cos}(p_2)$, so $\mathrm{Cos}(p_1) \cdot \mathrm{Cos}(\mathrm{tp}(g/M)) = \mathrm{Cos}(p_2)$. The following result can be seen as a sort of converse. Notice that

$$S = \{g \in \langle X\rangle \mid p_1 \cdot g \cap p_2 \text{ is weakly random over } M\}$$

is M-invariant and $u \cdot v \in S$ is stable, by Fact 2.4.

Corollary 3.3. *Let p, q and r be three coset-compatible types in* $\mathrm{supp}_M(\mu)$, *that is*,

$$\mathrm{Cos}(p) \cdot \mathrm{Cos}(q) = \mathrm{Cos}(r).$$

If $p \cdot g \cap r$ is weakly random for some element g in $\langle X\rangle$ with $\mathrm{tp}(g/M)$ *in* $\mathrm{supp}_M(\mu)$, *then so is $p \cdot h \cap r$ for every weakly random element h whose type over M is concentrated in* $\mathrm{Cos}(q)$. □

The above result was first observed for principal generic types in a simple theory in [Pillay et al. 1998, Proposition 2.2] and later generalized to nonprincipal types in [Martin-Pizarro and Pillay 2004, Lemma 2.3]. For weakly random types with respect to a pseudofinite Keisler measure, a preliminary version was obtained in [Palacín 2020, Proposition 3.2] for ultra-quasirandom groups.

For the rest of this section, we assume that M is countable. Fix some k in $\mathbb{N}$ and consider $Y = (X^{\odot k})$. The value k should be chosen large enough to ensure that all the products and inverses of elements in the subsequent statements still belong to Y. By an abuse of language, we use the word *random* to mean a random type with respect to the corresponding class $\mathcal{B}^Y_M$ as in Definitions 1.11 and 1.12.

Remark 3.4. It follows immediately from Remark 1.14 that the Borel set of random types over M is dense in the compact Hausdorff space of weakly randoms concentrated on Y, that is, the space $[Y] \cap \mathrm{supp}_M(\mu)$, where $[Y]$ is the clopen set given by the M-definable set Y. We denote by $\mathrm{R}(\mathcal{B}^Y_M)$ the collection of random types over M concentrated on Y.

Lemma 3.5. *Given M-definable subsets A and B of Y of positive density, there exists some random element g over M with $\mu(Ag \cap B) > 0$.*

Proof. By Remark 1.14, let c be random in B over M and choose now g^{-1} in $c^{-1}A$ random over M, c. The element g is also random over M, c. By symmetry of randomness, the pair (c, g) is random over M, so c is random over M, g. Clearly the element c lies in $Ag \cap B$, so the set $Ag \cap B$ has positive density, as desired. □

Remark 3.6. Notice that the above results yields the existence of an element h random over M such that $hA \cap B$, and thus $A \cap h^{-1}B$, has positive density. Indeed, apply the statement to the definable subsets B^{-1} and A^{-1}.

For any two fixed types p and r in $\mathrm{supp}_M(\mu)$, the statement

$$\text{“}p \cdot y \cap r \text{ is weakly random and } y \text{ is weakly random”}$$

as a property of y is finitely consistent. Indeed, given finitely many M-definable subsets $A_1, \ldots, A_n$ in p and $B_1, \ldots, B_n$ in r, the M-definable subsets $A = \bigcap_{1 \leq i \leq n} A_i$ and $B = \bigcap_{1 \leq i \leq n} B_i$ lie in p and r, respectively, so they both have positive density. By Lemma 3.5, there exists a random element g in $\langle X \rangle$ over M with $A_i g \cap B_j$ of positive density for all $1 \leq i, j \leq n$.

However, the condition “$p \cdot y \cap r$ is weakly random” is a G_δ-condition on y, namely

$$\bigcap_{\substack{A \in p \\ B \in r}} \{y \in A^{-1}B \mid \mu(A \cdot y \cap B) > 0\}.$$

Thus, we cannot use compactness to deduce from the above argument that we fulfill the conditions of Corollary 3.3 for all weakly random types p, q and r. We are grateful to Angus Matthews for pointing out a mistake in a previous version of this paper.

To circumvent the aforementioned issue, we use the so-called *disintegration theorem*, which allows us to fulfill the conditions of Corollary 3.3 for almost all pairs of types p and r. Whilst there are plenty of excellent references on this subject worth being named, we just refer to [Bogachev 2007; Simmons 2012].

Remark 3.7. Given n in $\mathbb{N}$ consider a set Ω and a surjective map $F : S_{Y^n}(M) \to \Omega$ such that the set $\{(p, q) \in S_{Y^n}(M) \times S_{Y^n}(M) \mid F(p) = F(q)\}$ is closed. For example, consider a type-definable equivalence relation $E(x, y)$ on $Y^n \times Y^n$ with parameters over M and set $p \sim q$ if and only if

$$p(x) \cup q(y) \cup E(x, y) \text{ is consistent.}$$

The relation $\sim$ is a closed equivalence relation on $S_{Y^n}(M)$, so set Ω to be the collection of $\sim$-equivalence classes and F the natural projection map.

We can now equip Ω with the final topology with respect to F, so a subset C of Ω is closed if and only if $F^{-1}(C)$ is closed in the topological space $S_{Y^n}(M)$. It is immediate to see that Ω with this topology becomes a compact Hausdorff separable

space. Furthermore, we can define a measure on Ω, the *push-forward measure* $F_*\mu$, given by $F_*\mu(B) = \mu(F^{-1}(B))$ for every Borel subset B of Ω.

Fact 3.8 (disintegration theorem). *Consider the normalized measure μ_{Y^n} on the space of types $S_{Y^n}(M)$, so it becomes a probability space. Given $F : S_{Y^n}(M) \to \Omega$ as in Remark 3.7, there exists a* ***disintegration*** *of μ_{Y^n} by a (uniquely determined) family of Radon conditional probability measures on $S_{Y^n}(M)$ with respect to the continuous function $F : S_{Y^n}(M) \to \Omega$, i.e., there exists a mapping*

$$(Z, t) \mapsto \nu(Z, t) = \mu_t(Z),$$

where Z is a Borel set of $S_{Y^n}(M)$ and t is an element of Ω, with the following properties:

(a) *for all t in Ω, the measure μ_t is a Borel inner regular probability measure on $S_{Y^n}(M)$;*

(b) *for every measurable subset Z of $S_{Y^n}(M)$, the function $t \mapsto \mu_t(Z)$ is measurable with respect to the measure $F_*\mu_{Y^n}$;*

(c) *each measure μ_t is concentrated on the fiber $F^{-1}(t)$, that is, the measure $\mu_t(S_{Y^n}(M) \setminus F^{-1}(t)) = 0$, so $\mu_t(Z) = \mu_t(Z \cap F^{-1}(t))$ for every Borel subset Z of $S_{Y^n}(M)$;*

(d) *for every measurable function $f : S_{Y^n}(M) \to \mathbb{R}$, we have that*

$$\int_{S_{Y^n}(M)} f \, d\mu_{Y^n} = \int_{t\in\Omega} \int_{F^{-1}(t)} f \, d\mu_t \, dF_*\mu_{Y^n}.$$

In particular, setting f to be the characteristic function $\mathbb{1}_Z$ of the measurable subset Z of $S_{Y\times Y}(M)$, we have that

$$\mu_{Y^n}(Z) = \int_{t\in\Omega} \mu_t(Z) \, dF_*\mu_{Y^n}.$$

Lemma 3.9. *Consider the natural restriction map*

$$\pi : S_{Y^2}(M) \to S_Y(M) \times S_Y(M), \quad q(y_1, y_2) \mapsto (q_{\restriction y_1}(y_1), q_{\restriction y_2}(y_2)).$$

Every pair of types (p, r) of $S_Y(M) \times S_Y(M)$ outside of a $\pi_*\mu_{Y^2}$-measure 0 set can be completed to a random type of $S_{Y^2}(M)$.

Proof. Let $\mathrm{R}(\mathcal{B}_M^{Y^2})$ be the Borel set of random types on $S_{Y^2}(M)$. It follows from Remark 1.14 that $\mu_{Y^2}(\mathrm{R}(\mathcal{B}_M^{Y^2})) = 1$. Apply now the disintegration theorem (Fact 3.8) with $\Omega = S_Y(M) \times S_Y(M)$ and $F = \pi$, and deduce from

$$1 = \mu_{Y^2}(\mathrm{R}(\mathcal{B}_M^{Y^2})) = \int_{(p,r)\in S_Y(M)\times S_Y(M)} \mu_{(p,r)}(\mathrm{R}(\mathcal{B}_M^{Y^2})) \, d\pi_*\mu_{Y^2}$$

that $\mu_{(p,r)}(\mathrm{R}(\mathcal{B}_M^{Y^2})) = 1$ for $\pi_*\mu_{Y^2}$-almost all pairs (p, r), since each function $\mu_{(p,r)}$ takes values in the interval $[0, 1]$. In particular, the set $\pi^{-1}(p, r) \cap \mathrm{R}(\mathcal{B}_M^{Y^2})$ is nonempty by Fact 3.8(c). Every such completion yields a random pair (a, b) over M, with a realizing p and b realizing r, as desired. □

Theorem 3.10. *For every pair of types (p, r) of $S_Y(M) \times S_Y(M)$ outside of a $\pi_*\mu_{Y^2}$-measure 0 set and every weakly random type $q = \mathrm{tp}(b/M)$ concentrated on Y with $\mathrm{Cos}(p) \cdot \mathrm{Cos}(q) = \mathrm{Cos}(r)$, there is a realization a of p weakly random over M, b such that $a \cdot b$ realizes r.*

Proof. By Lemma 3.9, for every pair (p, r) of $S_Y(M) \times S_Y(M)$ outside of a $\pi_*\mu_{Y^2}$-measure 0 set there exists a random pair (c, d) over M, with c realizing p and d realizing r. By Remark 1.13 and Lemma 1.15, the pair $(c^{-1} \cdot d, d)$ is random over M, so the partial type $p \cdot (c^{-1} \cdot d) \cap r$ admits a random realization, and thus it is weakly random. The element $c^{-1} \cdot d$ is (weakly) random over M and belongs to $\mathrm{Cos}(q)$, since p, q and r are coset-compatible. We can thus apply Corollary 3.3 to deduce that $p \cdot b \cap r$ is weakly random. Choose some realization f of this partial type weakly random over M, b and notice that the element $a = f \cdot b^{-1}$ is weakly random over M, b and realizes p. By construction, the product $a \cdot b = f$ realizes r, as desired. □

Whilst Theorem 3.10 holds for almost all types (p, r), the corresponding $\pi_*\mu_{Y^2}$-measure 0 set could possibly contain all diagonal pairs (p, p), with p in $\mathrm{supp}_M(\mu)$. We conclude this section with an elementary observation, the consequences of which will be explored in detail in Section 4.

Remark 3.11. Fix a countable elementary substructure M. If there exists a random pair (a, b) over M with $a \equiv_M b$, then there exists a random type concentrated in $\langle X \rangle_M^{00}$. Indeed, the element $b^{-1} \cdot a$ is random over M by Remark 1.13 and Lemma 1.15. Clearly, the element $g = b^{-1} \cdot a$ lies in $\langle X \rangle_M^{00}$, as desired.

Question. Is there a random pair (a, b) over M with $a \equiv_M b$? More generally, is there a random type concentrated in $\langle X \rangle_M^{00}$?

A digression: Roth's theorem on arithmetic progressions

We now show how Corollary 3.2 yields solutions to the equation $x \cdot z = y^2$ in subsets of positive density for every definably amenable pair such that the squaring function $x \mapsto x^2$ preserves randomness.

Definition 3.12. The function $f : X \to G$ in the definably amenable pair (G, X) *preserves randomness* if for every element a in X and every subset C of parameters, we have that a is (weakly) random over C if and only if $f(a)$ is (weakly) random over C (so $f(a)$ must lie in $\langle X \rangle$).

Remark 3.13. Examples 1.5, 1.6 and 1.7 always have the property that the square function preserves randomness if the map $f : X \to G$ defined by $f(x) = x^2$ has finite fibers. This is always the case whenever X has distinct squares as in [Sanders 2017, Theorem 1.5] or if G is abelian and there are only finitely many involutions in $\langle X \rangle$.

Theorem 3.14. *Consider a definably amenable pair (G, X) such that the square function preserves randomness. If the definable subset A of X has positive density, then the set*

$$\{(x_1, x_2) \in A \times A \mid x_1 \cdot x_2 \in A^2\}$$

has positive μ_2-density, where $A^2 = \{x^2\}_{x \in A}$.

Assume A is definable over the countable elementary substructure M. Every pair (a, c) in the above set random over M gives raise to a *generalized* 3-AP in A. Indeed, the product $a \cdot c$ belongs to A^2 so $a \cdot c = b^2$ for some b in A. Since the square function preserves randomness, we have that b is random over M, a by Lemma 1.15. Set now $g = b^{-1} \cdot a = b \cdot c^{-1}$ and observe that the elements c, $g \cdot c$ and $g \cdot c \cdot g$ all belong to A. If the group is abelian, this is an actual 3-AP as in the introduction.

Proof. We may assume that A is definable over a countable elementary substructure M, so it contains a weakly random type p over M. Choose some weakly random element g in $\langle X \rangle^{00}_M$. By Fact 2.6, the partial type $p \cdot g \cap p$ is weakly random, so the set $A \cdot g \cap A$ has positive density. By Remark 1.14, choose an element a in $A \cdot g \cap A$ random over M, g and notice that $b = a \cdot g^{-1}$ lies in A as well.

Since squaring preserves randomness, the element a^2 is also random over M, g and hence so is $a \cdot b = a^2 \cdot g^{-1}$ by Remark 1.13. By Lemma 1.16, the element g is weakly random over $M, a \cdot b$, and hence $a^2 = (a \cdot b) \cdot g$ is weakly random over $M, a \cdot b$. We deduce that a is weakly random over $M, a \cdot b$, for squaring preserves randomness. Furthermore, multiplying on the left by $(a \cdot b)^{-1}$ we conclude that b^{-1}, and hence b, is weakly random over $M, a \cdot b$.

Note that b belongs to $A^{-1} \cdot (a \cdot b) \cap A$, so this intersection must have positive density. Corollary 3.2 yields that the set $A^{-1} \cdot a^2 \cap A$ has positive measure, for a^2 and $a \cdot b$ lie in the same coset modulo $\langle X \rangle^{00}_M$. Choose now some random element a_1 in A over M, a with $a_1^{-1} \cdot a^2 = a_2$ in A. Remark 1.13 and Lemma 1.15 yield that the pair (a_1, a_2) is random over M. Thus, the M-definable set

$$\{(x_1, x_2) \in A \times A \mid x_1 \cdot x_2 \in A^2\}$$

has positive μ_2-measure, as desired. □

Question. Consider a definably amenable pair (G, X) such that the square function preserves randomness and let M be a countable elementary substructure M. Given

an M-definable subset A of $\langle X\rangle$ of positive density, does the M-definable set

$$\{(x_1, x_2) \in A \times A \mid x_2 \cdot x_1^{-1} \cdot x_2 \in A\}$$

have positive μ_2-density? Equivalently, is there a random pair (a, b) in $A \times A$ over M with $b \cdot a^{-1} \cdot b$ in A?

Such a pair (a, b) as above yields a 3-AP in A of the form $(a, a \cdot g, a \cdot g^2)$ with $g = a^{-1}b$. We do not currently know whether the above question has a positive answer, though it is the case for ultra-quasirandom groups, by [Tao 2013].

Remark 3.15. The proof of Theorem 3.14 in the abelian context yields immediately the existence of solutions to translation-invariant equations of the form

$$n_1x_1 + \cdots + n_mx_m = ky,$$

whenever $k = \sum_{j=1}^m n_j$ and each of the maps $x \mapsto n_1x$, $x \mapsto kx$ and $x \mapsto n'x$ preserves randomness, with $n' = \sum_{j=2}^m n_j$. That is, for every M-definable subset A of X of positive density, the set

$$\mathcal{E}(A) = \left\{(x_1, \ldots, x_m) \in A \times \overset{m}{\cdots} \times A \mid n_1x_1 + \cdots + n_mx_m = kc \text{ for some } c \text{ in } A\right\}$$

has positive μ_m-measure. Indeed, choose g, a and b as in the proof of Theorem 3.14, so $g = a - b$. If we denote by $\ell A = \{\ell d\}_{d\in A}$, we will first show that the set $n_1A + n'a \cap kA$ has positive density: By Corollary 3.2, we need only show that $n_1A + ka - n_1b \cap kA$ has positive density. Now, the element

$$ka - n_1b = n'a + n_1g$$

is random over M, g, since a is random over M, g. So g, and thus kg, is weakly random over $M, ka - n_1b$ by Lemma 1.16. Since

$$kg = ka - kb = (ka - n_1b) - n'b,$$

we deduce that $-n'b$, and hence n_1b, is weakly random over $M, ka - n_1b$. Hence, the element $ka = n_1b + (ka - n_1b)$ is weakly random over $M, ka - n_1b$ and belongs to $n_1A + ka - n_1b \cap kA$, as desired.

Choose now $a_2, \ldots, a_m$ realizations of $\mathrm{tp}(a/M)$ with each a_j weakly random over $M, a, g, a_2, \ldots, a_{j-1}$. Hence, the differences $a_j - a$ all belong to $\langle X\rangle_M^{00}$, by Fact 2.6. Corollary 3.2 and the above paragraph yield that $n_1A + \sum_{j=2}^m n_ja_j \cap kA$ has positive density, so choose an element a_1 in A weakly random over $M, a_2, \ldots, a_m$ exemplifying that the above intersection has positive density. The weakly random type $\mathrm{tp}(a_1, \ldots, a_m/M)$ contains the M-definable set $\mathcal{E}(A)$, as desired.

4. Ultra-quasirandomness revisited

Given a definably amenable pair (G, X) with $\langle X \rangle = G$, a straightforward application of compactness yields that $X^{\odot n} = G$ for some natural number n, so X generates G in finitely many steps. Up to scaling the σ-finite measure, we may assume that $G = X$, so $\mu(G) = 1$. This observation, together with Examples 1.5(a) and 1.7, motivates the following notion.

Definition 4.1. Let (G, X) be a definably amenable pair with $X = G$. We say that the pair is *generically principal* if $G = G^{00}_M$ for some elementary substructure M.

In an abuse of notation, we simply say that the group G is generically principal.

Remark 4.2. By [Martin-Pizarro and Palacín 2019, Corollary 2.6], a group G is generically principal if and only if $G = G^{00}_M$ for every elementary substructure M, so we may assume that M is countable.

In particular, a generically principal group contains trivially random elements concentrated in $\langle X \rangle^{00}_M = G$ for every countable elementary substructure M.

Example 4.3. Three known classes of groups are generically principal:

- Connected stable groups, such as every connected algebraic group over an algebraically closed field.
- Simple definably compact groups definable in some o-minimal expansion of a real closed field, such as $\mathrm{PSL}_n(\mathcal{R})$.
- Ultra-quasirandom groups, introduced in [Bergelson and Tao 2014]. Let us briefly recall this notion. A finite group is *d-quasirandom*, with $d \geq 1$, if all its nontrivial representations have degree at least d. An ultraproduct of finite groups $(G_n)_{n \in \mathbb{N}}$ with respect to a nonprincipal ultrafilter $\mathcal{U}$ is said to be *ultra-quasirandom* if for every integer $d \geq 1$, the set $\{n \in \mathbb{N} \mid G_n \text{ is } d\text{-quasirandom}\}$ belongs to $\mathcal{U}$.

The work of Gowers [2008, Theorem 3.3] yields that every definable subset A of positive density of an ultra-quasirandom group $G(M)$ is not *product-free*, i.e., it contains a solution to the equation $xy = z$, and thus the same holds in every elementary extension. Therefore, no weakly random type over an elementary substructure is product-free and thus $G = G^{00}_N$ over any elementary substructure N by [Martin-Pizarro and Palacín 2019, Corollary 2.6], so ultra-quasirandom groups are generically principal.

Proposition 3.1 and its corollaries yield now a short proof of the result mentioned in the above paragraph.

Lemma 4.4. *The following conditions are equivalent for a definably amenable pair (G, G):*

(a) *The group G is generically principal.*

(b) *Given two definable subsets A and B of positive density, we have that $A \cdot B$ has measure* 1. *In particular, whenever the definable subset C has positive measure, so is $G = A \cdot B \cdot C$.*

(c) *There is no definable product-free set of positive density.*

Proof. For (a) $\Rightarrow$ (b): Given two subsets A and B of positive density definable over some countable elementary substructure M, we need only show that every weakly random element g lies in $A \cdot B$. Now, Lemma 3.5 yields that there exists some random element h over M with $\mu(A \cap hB^{-1}) > 0$. Corollary 3.2 gives that every element g of G weakly random over M satisfies that $\mu(A \cap gB^{-1}) > 0$ as well. So the definable set $A \cdot B$ has measure 1, as desired.

For the second assertion, given a definable set C of positive density, let g in G be arbitrary. Now,

$$\mu(A \cdot B \cap gC^{-1}) = \mu(gC^{-1}) = \mu(C) > 0,$$

so g belongs to $A \cdot B \cdot C$, as desired.

The implication (b) $\Rightarrow$ (c) is clear, taking A and B to be the same set. Thus, we are left to consider the implication (c) $\Rightarrow$ (a). Suppose that $G \neq G^{00}_M$ for some countable elementary substructure M and take a weakly random type p in a nontrivial coset $\mathrm{Cos}(p)$ of G^{00}_M. Note that $p^{-1} \cdot p \cdot p \subseteq \mathrm{Cos}(p)$. A standard compactness argument yields the existence of some M-definable set A in p such that id_G does not lie in $A^{-1} \cdot A \cdot A$, so A is product-free. Since p is weakly random, the definable subset A has positive density. □

The following result on weak mixing, already present as is in [Bergelson and Tao 2014], was implicit in [Gowers 2008]. It will play a crucial role in studying some instances of complete amalgamation of equations in a group.

Corollary 4.5 (cf. [Bergelson and Tao 2014, Lemma 33]). *Let G be a generically principal group. Given two definable subsets A and B of positive density,*

$$\mu(A \cap gB) = \mu(A)\mu(B)$$

for μ-almost all elements g.

Proof. As before, fix some countable elementary substructure M such that both A and B are M-definable. We may assume that the measure μ is also definable over M. By Corollary 3.2, set $\alpha = \mu(A \cap gB)$ for some (or equivalently, every) weakly random element g over M. Notice that $\alpha > 0$ by Remark 3.6.

The subset

$$Z = \{x \in A \cdot B^{-1} \mid \mu(A \cap xB) = \alpha\}$$

is type-definable over M and contains all weakly random elements over M. Clearly, the measure $\mu(Z) \leq \mu(AB^{-1})$ and the latter equals 1, by Lemma 4.4. If we have

$\mu(Z) < \mu(A \cdot B^{-1})$, there is an M-definable set $\widetilde{Z}$ with $Z \subseteq \widetilde{Z} \subseteq A \cdot B^{-1}$ such that $\mu(A \cdot B^{-1} \setminus \widetilde{Z}) > 0$. Thus, the set $A \cdot B^{-1} \setminus \widetilde{Z}$ has positive density and it must contain a weakly random element over M, which gives the desired contradiction, so $\mu(Z) = \mu(A \cdot B^{-1}) = 1$.

Consider now the set $W = \{(a, z) \in A \times A \cdot B^{-1} \mid z = a \cdot b^{-1} \text{ for some } b \text{ in } B\}$. Note a belongs to $A \cap z \cdot B$ and z lies in aB^{-1} if (a, z) belongs to W. If we denote by μ_2 the normalized measure in $G \times G$, an easy computation yields that

$$\mu_2(W) = \int_{z \in A \cdot B^{-1}} \mu(A \cap zB) = \alpha \mu(A \cdot B^{-1}) \overset{4.4}{=} \alpha.$$

By Fubini, we also have that

$$\alpha = \mu_2(W) = \int_{a \in A} \mu(aB^{-1}) = \int_{a \in A} \mu(B) = \mu(A)\mu(B),$$

which gives the desired conclusion. □

A standard translation using Łoś's theorem yields the following finitary version:

Corollary 4.6 (cf. [Gowers 2008, Lemma 5.1; Bergelson and Tao 2014, Proposition 3]). *For every positive δ, ϵ and η there is some integer $d = d(\delta, \epsilon, \eta)$ such that for every finite d-quasirandom group G and subsets A and B of G of density at least δ, we have that*

$$\left|\{x \in G \mid |A \cap xB||G| < (1 - \eta)|A||B|\}\right| < \epsilon|G|.$$

Proof. Assume for a contradiction that the statement does not hold, so there are some fixed positive numbers δ, ϵ and η such that for each natural number d we find two subsets A_d and B_d of a finite d-quasirandom group G_d, each of density at least δ, such that the cardinality of the subset

$$\mathcal{X}(G_d) = \{x \in G_d \mid |A_d \cap xB_d||G_d| < (1 - \eta)|A_d||B_d|\}$$

is at least $\epsilon|G_d|$.

Following the approach of Example 1.5(a), we consider a suitable expansion $\mathcal{L}$ of the language of groups and regard each group G_d as an $\mathcal{L}$-structure N_d. Choose a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and consider the ultraproduct $N = \prod_{\mathcal{U}} N_d$. The language $\mathcal{L}$ is chosen in such a way that the sets $A = \prod_{\mathcal{U}} A_d$ and $B = \prod_{\mathcal{U}} B_d$ are $\mathcal{L}$-definable in the ultra-quasirandom group $G = \prod_{\mathcal{U}} G_d$. Furthermore, the normalized counting measure on G_d induces a definable Keisler measure μ on G, taking the standard part of the ultralimit. By Corollary 4.5, for μ-almost all g in G, we have $\mu(A \cap gB) = \mu(A)\mu(B)$. Hence, the type-definable set

$$\Sigma = \{x \in G \mid \mu(A \cap xB) \leq (1 - \eta)\mu(A)\mu(B)\}$$

does not contain any weakly random type. By compactness, it is contained in a definable set W whose density is 0, and in particular its density is strictly less than the fixed value ϵ. Since every element in the ultraproduct of the sets $\mathcal{X}(G_d)$ clearly lies in Σ, we conclude by Łoś's theorem that $|\mathcal{X}(G_d)| \leq |W(G_d)| < \epsilon|G_d|$ for infinitely many d, which yields the desired contradiction. □

The following result is a verbatim adaption of [Gowers 2008, Theorem 5.3] and may be seen as a first attempt to solve complete amalgamation problems whilst restricting the conditions to those given by products.

Theorem 4.7. *Fix a natural number $n \geq 2$. For each nonempty subset F of $\{1, \ldots, n\}$, let A_F be a subset of positive density of the generically principal group G. The set*

$$\mathcal{X}_n = \big\{(a_1, \ldots, a_n) \in G^n \mid a_F \in A_F \text{ for all } \varnothing \neq F \subseteq \{1, \ldots, n\}\big\}$$

has measure $\prod_F \mu(A_F)$ with respect to the measure μ_n on G^n, where a_F stands for the product of all a_i with i in F written with the indices in increasing order.

Proof. We reproduce Gowers's proof of [Gowers 2008, Theorem 5.3] and proceed by induction on n. For $n = 2$, set $B = A_{\{2\}}$ and $C = A_{\{1,2\}}$. A pair (a, b) lies in $\mathcal{X}_2$ if and only if a belongs to $A_{\{1\}}$ and b to $B \cap a^{-1}C$. Thus

$$\mu_2(\mathcal{X}_2) = \int_{A_{\{1\}}} \mu(B \cap a^{-1}C)\, d\mu \overset{4.5}{=} \mu(B)\mu(C)\mu(A_{\{1\}}),$$

as desired. For the general case, for any a in $A_{\{1\}}$, set $B_{F_1}(a) = A_{F_1} \cap a^{-1}A_{1,F_1}$, for $\varnothing \neq F_1 \subseteq \{2, \ldots, n\}$. Corollary 4.5 yields that $\mu(B_{F_1}(a)) = \mu(A_{F_1})\mu(A_{1,F_1})$ for μ-almost all a in $A_{\{1\}}$. A tuple $(a_1, \ldots, a_n)$ in G^n belongs to $\mathcal{X}_n$ if and only if the first coordinate a_1 lies in $A_{\{1\}}$ and the tuple $(a_2, \ldots, a_n)$ belongs to

$$\mathcal{X}_{n-1}(a_1) = \big\{(x_2, \ldots, x_n) \in G^{n-1} \mid x_{F_1} \in B_{F_1}(a_1) \text{ for all } \varnothing \neq F_1 \subseteq \{2, \ldots, n\}\big\}.$$

By induction, the set $\mathcal{X}_{n-1}(a)$ has constant μ_{n-1}-measure $\prod_{F_1} \mu(A_{F_1})\mu(A_{1,F_1})$, where F_1 now runs through all nonempty subsets of $\{2, \ldots, n\}$. Thus

$$\mu_n(\mathcal{X}_n) = \int_{A_1} \mu_{n-1}(\mathcal{X}_{n-1}(a_1))\, d\mu = \mu(A_1) \prod_{F_1} \mu(A_{F_1})\mu(A_{1,F_1}) = \prod_F \mu(A_F),$$

which yields the desired result. □

A standard translation using Łoś's theorem (we refer to the proof of Corollary 4.6 to avoid repetitions) yields the following finitary version, which was already present in a quantitative form in [Gowers 2008].

Corollary 4.8 (cf. [Gowers 2008, Theorem 5.3]). *Fix a natural number $n \geq 2$. For every $\varnothing \neq F \subseteq \{1, \ldots, n\}$ let $\delta_F > 0$ be given. For every $\eta > 0$ there is some integer*

$d = d(n, \delta_F, \eta)$ such that for every finite d-quasirandom group G and subsets A_F of G of density at least δ_F, we have that

$$|\mathcal{X}_n| \geq \frac{1-\eta}{|G|^{2^n-1-n}} \prod_F |A_F|,$$

where $\mathcal{X}_n$ is defined as in Theorem 4.7 with respect to the group G.

The above corollary yields in particular that

$$\left|\{(a,b,c) \in A \times B \times C \mid ab = c\}\right| > \frac{1-\eta}{|G|}|A||B||C|$$

as first proved by Gowers [2008, Theorem 3.3], which implies that the number of such triples is a proportion (uniformly on the densities and η) of $|G|^2$.

To conclude this section we answer affirmatively the question in the introduction for generically principal groups, whenever all the types are based over a common countable elementary substructure.

Theorem 4.9. *Fix a natural number $n \geq 2$ and a countable elementary substructure M of the generically principal definably amenable pair (G, X). For each nonempty subset F of $\{1, \ldots, n\}$, let p_F be a weakly random type over M. There exists a weakly random n-tuple $(a_1, \ldots, a_n)$ in G^n such that a_F realizes p_F for all nonempty $F \subseteq \{1, \ldots, n\}$, where a_F stands for the product of all a_i with i in F written with the indices in increasing order.*

Proof. Since M is countable, enumerate all the formulae occurring in each type p_F in a decreasing way, that is, write $p_F = \{A_{F,k}\}_{k\in\mathbb{N}}$ with $A_{F,k+1} \subseteq A_{F,k}$ for every natural number k. We want to show that the set

$$\mathcal{X}_n = \{(x_1, \ldots, x_n) \in G^n \mid p_F(x_F) \text{ for all } \varnothing \neq F \subseteq \{1, \ldots, n\}\}$$

is weakly random over M, that is, we need to prove that the partial type

$$\{\neg\psi(x_1, \ldots, x_n)\}_{\psi\in\Sigma} \cup \{x_F \in A_{F,k}\}_{F\in\mathcal{P},\, k\in\mathbb{N}}$$

is consistent, where $\mathcal{P} = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\varnothing\}$ and Σ is the set of $\mathcal{L}_M$-formulae of μ_n-measure 0. By compactness, since the subsets $A_{F,k}$ are enumerated decreasingly, we need only consider a finite subset of the above partial type where the level k_0 is the same for each of the subsets A_{F,k_0} of positive density. By Theorem 4.7 the set

$$\mathcal{X}_{n,k_0} = \{(a_1, \ldots, a_n) \in G^n \mid a_F \in A_{F,k_0} \text{ for all nonempty } F \subseteq \{1, \ldots, n\}\}$$

has μ_n-measure $\prod_F \mu(A_{F,k_0}) > 0$, so we conclude the desired result. □

5. Local ultra-quasirandomness

In this final section, we adapt some of the ideas present in Section 4 to arbitrary finite groups.

Theorem 3.10 holds in any definably amenable pair for *almost all* three weakly random types, whenever their cosets modulo G_M^{00} are product-compatible. Thus, it yields asymptotic information for subsets of positive density in arbitrary finite groups satisfying certain regularity conditions, which force some completions in the ultraproduct to be in a suitable position to apply our main Theorem 3.10. We present two examples of such regularity notions. Our intuition behind these notions is purely model-theoretic and we ignore whether it is meaningful from a combinatorial perspective. We would like to express our gratitude to Julia Wolf (and indirectly to Tom Sanders) for pointing out that our previous definition of principal subsets did not extend to the abelian case.

Definition 5.1. Let A be a definable subset of $\langle X\rangle$ of positive density in a definably amenable pair (G, X). We say that A is *principal* over the parameter set B if

$$\mu(A \cap (Y \cdot Y)) > 0$$

whenever Y is a B-definable neighborhood of the identity (that is, the set Y is symmetric and contains the identity) such that finitely many left translates of Y cover $A \cdot A^{-1} \cdot A \cdot A^{-1}$.

Analogously, we say that A is *hereditarily principal over the parameter set* B if all of its B-definable subsets of positive density are principal.

Remark 5.2. Let A be a definable subset of $\langle X\rangle$ of positive density of a definably amenable pair (G, X) such that $\mu(A \cap (Y \cdot Y)) = \mu(A)$, whenever Y is a definable neighborhood of the identity which covers $A \cdot A^{-1} \cdot A \cdot A^{-1}$ with finitely many left translates. Then the set A is hereditarily principal over any subset of parameters.

Proof. Let A_0 be a definable subset of A of positive measure. Notice that there is a maximal finite subset F of $(AA^{-1})^2$ with the property that $\mu(xA_0 \cap yA_0) = 0$ for any two distinct x and y in F. In particular, the set $(AA^{-1})^2$ is contained in $F \cdot A_0 \cdot A_0^{-1}$. Thus, any definable neighborhood Y of the identity such that finitely many left translates of cover $A_0A_0^{-1}A_0A_0^{-1}$ also cover $AA^{-1}AA^{-1}$, so $\mu(A \cap (Y \cdot Y)) = \mu(A)$ by assumption on A. Hence $\mu(A_0 \cap (YY)) = \mu(A_0) > 0$, as desired. □

Example 5.3. If G is generically principal, every definable subset A of positive density is hereditarily principal over any parameter set. Indeed, Lemma 4.4 yields that $G = A \cdot A^{-1} \cdot A \cdot A^{-1}$. Therefore, finitely many translates of the neighborhood Y must cover G, so Y has positive measure and hence $\mu(Y \cdot Y) = 1$ by Lemma 4.4.

By the previous remark, the definable subset A satisfies that $\mu(A \cap (Y \cdot Y)) = \mu(A)$, so A is hereditarily principal over any subset of parameters.

Example 5.4. Fix some enumeration $(q_n)_{n\in\mathbb{N}}$ of all the primes and consider the family of groups $(G_n = \mathrm{PSL}_2(q_n) \times \mathbb{Z}_2)_{n\in\mathbb{N}}$, each equipped with the distinguished subset $X_n = \mathrm{PSL}_2(q_n) \times \{\bar{0}\}$. This family produces a definably amenable pair (G, X), as in Example 1.5. Note that

$$G = \mathrm{PSL}_2(\mathbb{F}) \times \mathbb{Z}_2 \quad \text{and} \quad X = \mathrm{PSL}_2(\mathbb{F}) \times \{\bar{0}\}$$

for some infinite (pseudofinite) field $\mathbb{F}$. Over any elementary substructure M we have that G_M^{00} equals the simple group $X = \mathrm{PSL}_2(\mathbb{F}) \times \{\bar{0}\}$, which is clearly definable. The definable subset G is clearly principal yet not hereditarily principal, for the dense subset $X \cdot (0_{\mathrm{PSL}_2(\mathbb{F})}, \bar{1})$ does not intersect $X = G_M^{00}$.

Lemma 5.5. *Let M be a countable elementary substructure of a definably amenable pair (G, X).*

(a) *Principal definable sets over M contain weakly random principal types in $S_\mu(M)$, that is, types concentrated in $\langle X\rangle_M^{00}$.*

(b) *Every weakly random type over M containing a hereditarily principal definable set is principal.*

Proof. For (a), assume that the M-definable set A is principal over the model M. Note that we can write the type-definable subgroup $\langle X\rangle_M^{00}$ as a countable intersection

$$\langle X\rangle_M^{00} = \bigcap_{i\in\mathbb{N}} V_i,$$

where the decreasing chain $(V_i)_{i\in\mathbb{N}}$ consists of M-definable neighborhoods of the identity such that $V_{i+1} \cdot V_{i+1} \subseteq V_i$ for all i in $\mathbb{N}$. Since $\langle X\rangle_M^{00}$ has bounded index in the subgroup $\langle X\rangle$, compactness yields that finitely many translates of each V_i cover the subset $A \cdot A^{-1} \cdot A \cdot A^{-1}$ (yet the number of translates possibly depends on i). Hence, the type-definable subset $A \cap \langle X\rangle_M^{00}$ is weakly random, since A is principal, so A contains a weakly random type concentrated in $\langle X\rangle_M^{00}$, as desired.

For (b), suppose that the M-definable set A is hereditarily principal yet it contains a weakly random type q which does not concentrate on $\langle X\rangle_M^{00} = \bigcap_{i\in\mathbb{N}} V_i$, with the same notation as above. By compactness, this implies the existence of some i in $\mathbb{N}$ and some M-definable subset A_0 of A of positive density with $A_0 \cap V_i = \varnothing$. The subset $A_0 \cap (V_{i+1} \cdot V_{i+1})$ has in particular measure 0, so A_0 is not principal, contradicting our assumption on A. □

Proposition 5.6. *Consider a subset A of positive density definable over a countable elementary substructure M of a sufficiently saturated definably amenable pair (G, X). If A contains a weakly random type p concentrated in $\langle X\rangle_M^{00}$, then the subset*

$$\{(a, b) \in A \times A \mid a \cdot b \in A\}$$

has positive μ_2-measure. In particular, if A is principal, then the above set of pairs has positive μ_2-measure.

Notice that the definable set A above cannot be product-free, for the equation $x \cdot y = z$ has a solution in A.

Proof. The proof is an immediate application of Fact 2.6. Indeed, for every realization a of p, the partial type $p \cap a^{-1} \cdot p$ is weakly random (for the weakly random element a over M belongs to $\langle X \rangle^{00}_M$), so choose a weakly random element b over M, a realizing p such that $a \cdot b$ does as well. By Lemma 1.10, we obtain a weakly random type $\mathrm{tp}(a, b/M)$ with all three elements a, b and $a \cdot b$ in A, which yields immediately the desired result. □

Proposition 5.6 resonates with [Schur 1917, Hilfssatz] on the number of monochromatic triples $(x, y, x \cdot y)$ in any finite coloring (or cover) of the natural numbers $1, \ldots, N$, for N sufficiently large. In fact, by a standard application of Łoś's theorem, the above argument yields a nonquantitative version of the following result of Sanders [2019, Theorem 1.1]:

> *For every natural number $k \geq 1$ there is some $\eta = \eta(k) > 0$ with the following property: given any coloring on a finite group G with k many colors $A_1, \ldots, A_k$, there exists some color A_j, with $1 \leq j \leq k$, such that*
>
> $$\big|\{(a, b, c) \in A_j \times A_j \times A_j \mid a \cdot b = c\}\big| \geq \eta |G|^2.$$

Motivated by [Gowers 2008, Theorem 5.3] for (ultra-)quasirandom groups, we now provide a weaker version of it, taking all A_F's to be the same subset A, for $\varnothing \neq F \subseteq \{1, \ldots, n\}$ as in Corollary 4.8.

Corollary 5.7. *In a sufficiently saturated definably amenable pair (G, X) with associated measure μ, consider a definable subset A of X of positive density which is hereditarily principal over the parameter set G itself. For every countable elementary substructure M of (G, X) such that both the measure and the sets A are M-definable, there is a tuple $(a_1, \ldots, a_n)$ in G^n weakly random over M such that the product a_F (as in Theorem 4.9) lies in A for every subset F as above.*

An inspection of the proof shows that it suffices if the definable set A is hereditarily principal over N, where N is an $\aleph_1$-saturated elementary substructure of (G, X) containing M. This is not surprising, since an easy compactness argument shows that a set A which is hereditarily principal over an $\aleph_1$-saturated elementary substructure N of (G, X) must be hereditarily principal over the parameter set G itself.

Proof. We proceed by induction on the natural number n. Since both the base case $n = 3$ and the induction step have similar proofs, we assume that the statement of the corollary has already been shown for $n - 1$.

The set A is principal, so it contains a weakly random type concentrated in $\langle X\rangle^{00}_M$, by Lemma 5.5(a). As in the proof of Proposition 5.6, there is a weakly random element a_1 in A over M such that $A' = A \cap a_1^{-1} \cdot A$ has positive density. Notice that A' is no longer definable over M, yet it is again hereditarily principal over the parameter set G. By downwards Löwenheim–Skolem, choose some countable elementary substructure M_1 of (G, X) containing $M \cup \{a_1\}$. By induction, there is a tuple $(a_2, \ldots, a_n)$, weakly random over M_1, such that each product a_{F_1} lies in A' for every subset $\varnothing \neq F_1 \subseteq \{2, \ldots, n\}$. For $n = 3$, we obtain such a tuple by applying Proposition 5.6 to the principal M_1-definable set A'.

Lemma 1.10 yields now that the tuple $(a_1, \ldots, a_n)$ is weakly random over M. By construction, the product a_F lies in A for every subset $\varnothing \neq F \subseteq \{1, \ldots, n\}$, as desired. □

Motivated by the above result, we isolate a particular instance of a complete amalgamation problem (cf. the question in the introduction).

Question. Let M be a countable elementary substructure of a sufficiently saturated definably amenable pair (G, X) and p be a weakly random type in $\langle X\rangle^{00}_M$. Given a natural number n, is there a tuple $(a_1, \ldots, a_n)$ in G^n weakly random over M such that a_F realizes p for all $\varnothing \neq F \subseteq \{1, \ldots, n\}$, where a_F stands for the product, enumerated in an increasing order, of all a_i with i in F?

At the moment of writing, we do not have a solid guess what the answer to the above question will be. Following the lines of the proof of Corollary 5.7, the above question would have a positive answer if the following statement is true:

> *Let $p = \mathrm{tp}(a/M_0)$ be a weakly random type in $\langle X\rangle^{00}_{M_0}$, where M_0 is a countable elementary substructure of a saturated definably amenable pair (G, X). Then there are an elementary substructure M_1 containing $M_0 \cup \{a\}$ and a weakly random type q in $\langle X\rangle^{00}_{M_1}$ extending $p \cap a^{-1} \cdot p$*

Nonetheless, if the question could be positively answered, it would imply by a standard compactness argument a finitary version of Hindman's theorem [Hindman 1974].

Remark 5.8. If the above question has a positive answer, then for every natural numbers k and n there is some constant $\eta = \eta(k, n) > 0$ such that in any coloring on a finite group G with k many colors $A_1, \ldots, A_k$, there exists some color A_j, with $1 \leq j \leq k$ such that

$$\left|\{(a_1, \ldots, a_n) \in G^n \mid a_F \in A_j \text{ for all } \varnothing \neq F \subseteq \{1, \ldots, n\}\}\right| \geq \eta |G|^n,$$

where a_F stands for the product, enumerated in an increasing order, of all a_i with i in F.

We can now state the finitary versions of principal sets to provide finitary analogs of Proposition 5.6 and Corollary 5.7.

Definition 5.9. Fix $\epsilon > 0$ and k in $\mathbb{N}$. A finite subset A of a group G is (k, ϵ)-*principal* if

$$|A \cap (Y \cdot Y)| \geq \epsilon |A|$$

whenever Y is a neighborhood of the identity (that is, the set Y is symmetric and contains the identity) such that k many left translates (or equivalently, right translates) of Y cover $A \cdot A^{-1} \cdot A \cdot A^{-1}$.

We say that the finite subset A is *hereditarily* (k, ϵ)*-principal up to* ρ if all its subset of relative density at least ρ (in A) are (k, ϵ)-principal.

Example 5.10. Consider the finite group $G = \mathbb{Z}_n \times \mathbb{Z}_2$. The set G is clearly $(k, 1/k)$-principal for every natural number $k \neq 0$, yet it is not hereditarily $(2, 1/k)$-principal up to $1/2$ for any $k \neq 0$, for the subset $A = \mathbb{Z}_n \times \{\bar{1}\}$ does not intersect $Y = \mathbb{Z}_n \times \{\bar{0}\}$, which covers G in 2 steps.

Example 5.11. Given a subset A of a finite group G of density at least ϵ, the symmetric set AA^{-1} is $(k, \epsilon/k)$-principal. Indeed, if Y is a given neighborhood of the identity such that k many right translates of Y cover $(AA^{-1})^4$, then there exists some c in G such that $|Ac \cap Y| \geq |A|/k$ and so $|AA^{-1} \cap YY| \geq \epsilon |AA^{-1}|/k$, since $(Ac \cap Y)(Ac \cap Y)^{-1} \subseteq AA^{-1} \cap YY$.

Corollary 5.12. *Let $K > 0$ and $\delta > 0$ be given real numbers. There are real values $\epsilon = \epsilon(K, \delta) > 0$ and $\eta = \eta(K, \delta) > 0$ as well as a natural number $k = k(K, \delta)$ such that for every group G and a finite subset X of G of tripling at most K together with a (k, ϵ)-principal subset A of X of relative density at least δ with respect to X, the collection of triples*

$$\{(a, b) \in A \times A \mid a \cdot b \in A\}$$

has size at least $\eta |X|^2$.

Proof. Assume for a contradiction that the statement does not hold. Negating quantifiers, there are positive constants K and δ such that for each triple $\bar{\ell} = (k, n, m)$ of natural numbers there exists a group $G_{\bar{\ell}}$ and a finite subset $X_{\bar{\ell}}$ of $G_{\bar{\ell}}$ of tripling at most K as well as a $(k, 1/n)$-principal subset $A_{\bar{\ell}}$ of $X_{\bar{\ell}}$ of relative density at least δ such that the cardinality of the subset

$$\mathcal{Y}(G_{\bar{\ell}}) = \{(x, y) \in A_{\bar{\ell}} \times A_{\bar{\ell}} \mid x \cdot y \in A_{\bar{\ell}}\}$$

is bounded above by $|X_{\bar{\ell}}|^2/m$.

Following the approach of Example 1.5(b), we consider a suitable countable expansion $\mathcal{L}$ of the language of groups and regard each such group $G_{\bar{\ell}}$, with $\bar{\ell}$ of the form (k, k, k), as an $\mathcal{L}$-structure $N_{\bar{\ell}}$ in such a way that $\mathcal{L}$ contains predicates for $X_{\bar{\ell}}$ and $A_{\bar{\ell}}$. Identify now the set of such triples (k, k, k) with the natural numbers in a

natural way and choose a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Consider the ultraproduct $N = \prod_{\mathcal{U}} N_{\bar{\ell}}$. As outlined in Example 1.5, this construction gives rise to a definable amenable pair (G, X) with respect to a measure μ equipped with an $\varnothing$-definable subset A of X of positive density (at least δ) such that $\mu_2(\mathcal{Y}(G)) = 0$. Notice that A is now principal over the parameter set N, by Łoś's theorem.

Fix a countable elementary substructure M of N. By Proposition 5.6, the set

$$\mathcal{Y}(G) = \{(x, y) \in A \times A \mid x \cdot y \in A\}$$

has positive density with respect to μ_2, which contradicts the ultraproduct construction. □

The proof of the next result follows from Corollary 5.7 along the same lines as Corollary 5.12 by a standard ultraproduct argument using Łoś's theorem (and implicitly that a nonprincipal ultraproduct of finite sets is $\aleph_1$-saturated).

Corollary 5.13. *For a natural number $n \geq 3$, let real numbers $K > 0$ and $\delta_F > 0$, with $\varnothing \neq F \subseteq \{1, \ldots, n\}$, be given. There are $\epsilon = \epsilon(n, K, \delta_F) > 0$, $\rho = \rho(n, K, \delta_F)$ and $\eta = \eta(n, K, \delta_F) > 0$ as well as a natural number $k = k(n, K, \delta_F)$ such that for every group G and a finite subset X of G of tripling at most K together with a subset A of X of relative density at least δ, whenever*

$$\left|\{(a_1, \ldots, a_n) \in G^n \mid a_F \in A \text{ for all } \varnothing \neq F \subseteq \{1, \ldots, n\}\}\right| < \eta |X|^n,$$

where a_F stands for the product, enumerated in an increasing order, of all the a_i with i in F, then A cannot be hereditarily (k, ϵ)-principal up to ρ.

In order to extend Proposition 5.6 to pairs (a, b) in the cartesian product $A \times B$ with $a \cdot b$ in C, we introduce a new notion, which we refer to as compatibility for certain subsets in a definably amenable pair.

Definition 5.14. Let A, B and C be subsets of $\langle X \rangle$ of positive density in a definably amenable pair (G, X), all three definable over the countable elementary substructure M. We say that A and B are *compatible* with respect to C over M if there exists a random pair (a, b) in $A \times B$ over M such that $a \cdot b$ lies in the same coset modulo $\langle X \rangle^{00}_M$ as some element c of C which is weakly random over M.

It is clear that every two definable subsets A and B of positive density in a generically principal group G are compatible with respect to any subset C of positive density over any countable elementary substructure M containing the parameters of definition of all three sets. More generally, we have the following observation.

Remark 5.15. Given three definable subsets A, B and C of positive density at least $\delta > 0$ in a definably amenable pair (G, X), all three defined over a countable

elementary substructure M, every weakly random type of $\langle X\rangle^{00}_M$ is contained in

$$A \cdot A^{-1} \cap B \cdot B^{-1} \cap C \cdot C^{-1},$$

by Fact 2.6. Hence, the M-definable set

$$\{(x, y) \in (A \cdot A^{-1}) \times (B \cdot B^{-1}) \mid x \cdot y \in C \cdot C^{-1}\}$$

contains a pair (a_1, b_1) with a_1 and b_1 both in $\langle X\rangle^{00}_M$ weakly random over M such that b_1 is weakly random over M, a_1. Hence, the above set has positive density, so there exists a random pair (a, b) in $AA^{-1} \times BB^{-1}$ over M such that $a \cdot b$ belongs to CC^{-1}. Since $a \cdot b$ is (weakly) random over M, we deduce that $A \cdot A^{-1}$ and $B \cdot B^{-1}$ are compatible with respect to $C \cdot C^{-1}$.

Lemma 5.16. *Let A, B and C be subsets of $\langle X\rangle$ of positive density in a definably amenable pair (G, X), all three definable over the countable elementary substructure M.*

(a) *If for some element g in $\langle X\rangle^{00}_M$ weakly random over M, the definable subset*

$$Z_g = \{(a, b) \in A \times B \mid a \cdot b \in C \cdot g\}$$

has positive μ_2-measure, then A and B are compatible with respect to C over M.

(b) *If A and B are compatible with respect to C over M, then the M-definable set*

$$\{(a, b) \in A \times B \mid a \cdot b \in C\}$$

has positive μ_2-measure.

Proof. For (a), given a weakly random element g in $\langle X\rangle^{00}_M$, suppose that the definable set Z_g has positive density. By Remark 1.14, choose some (a, b) in Z_g random over M, g, so the element $c = a \cdot b \cdot g$ is again random over M by Remark 1.13 and Lemma 1.15. This immediately yields that A and B are compatible with respect to C over M.

For (b), suppose that A and B are compatible with respect to C over M, so by definition, there is a random pair (a, b) in $A \times B$ over M such that $a \cdot b$ lies in the same coset of $\langle X\rangle^{00}_M$ as some element c in C whose type over M is weakly random. By Lemma 1.15, the pair $(a^{-1}, a \cdot b)$ is a random pair over M, so the definable set $A^{-1} \cdot (a \cdot b) \cap B$ has positive measure, for it belongs to the weakly random type $\mathrm{tp}(b/M, a \cdot b)$. By Corollary 3.2, we deduce that $A^{-1} \cdot c \cap B$ has positive measure, so choose b_1 in B weakly random over M, c such that $c = a_1 \cdot b_1$. In particular, the M-definable set

$$\{(y, z) \in B \times C \mid z \cdot y^{-1} \in A\}$$

has positive μ_2-measure and so it contains a random pair (b_2, c_2) over M. The pair (a_2, b_2) of $A \times B$, with $a_2 = c_2 \cdot b_2^{-1}$ is again random over M by Lemma 1.15 and satisfies that $a_2 \cdot b_2$ belongs to C, as desired. □

Remark 5.17. If the definable set A has positive density and the pair (A, A) is compatible with respect to A over a countable elementary substructure M, then A is not product-free (cf. the corresponding comment after Proposition 5.6). On the other hand, is it the case that every principal definable set yields a compatible pair? Or are the two notions unrelated, even if they provide the same positive answer?

Lemma 5.16 yields a sufficient condition to ensure that the corresponding ultraproducts of finite subsets will be compatible. We have several candidates of finitary versions of compatibility, which will allow us to obtain a local version of [Gowers 2008, Theorem 5.3] to count the number of pairs in $A \times B$ such that the product $a \cdot b$ lies in the subset C of positive density, all within a finite subset of small tripling. However, it is unclear to us how combinatorially relevant our tentative definitions are, so we would rather leave the ultraproduct formulation as an open question: Is there a meaningful combinatorial definition (akin to Definition 5.9) of when two finite sets A and B are compatible with respect to the finite set C?

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