

# Model Theory

no. 1

vol. 3

2024

*MAT*

**Bounded ultrimaginary independence  
and its total Morley sequences**

James E. Hanson





## Bounded ultramaginary independence and its total Morley sequences

James E. Hanson

We investigate the following model-theoretic independence relation:  $b \downarrow_A^{\text{bu}} c$  if and only if  $\text{bdd}^u(Ab) \cap \text{bdd}^u(Ac) = \text{bdd}^u(A)$ , where  $\text{bdd}^u(X)$  is the class of all ultramaginarities bounded over  $X$ . In particular, we sharpen a result of Wagner to show that  $b \downarrow_A^{\text{bu}} c$  if and only if  $\langle \text{Autf}(\mathbb{M}/Ab) \cup \text{Autf}(\mathbb{M}/Ac) \rangle = \text{Autf}(\mathbb{M}/A)$ , and we establish full existence over hyperimaginary parameters (i.e., for any set of hyperimaginaries  $A$  and ultramaginarities  $b$  and  $c$ , there is a  $b' \equiv_A b$  such that  $b' \downarrow_A^{\text{bu}} c$ ). Extension then follows as an immediate corollary.

We also study *total*  $\downarrow^{\text{bu}}$ -Morley sequences (i.e.,  $A$ -indiscernible sequences  $I$  satisfying  $J \downarrow_A^{\text{bu}} K$  for any  $J$  and  $K$  with  $J + K \equiv_A^{\text{EM}} I$ ), and we prove that an  $A$ -indiscernible sequence  $I$  is a total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  if and only if whenever  $I$  and  $I'$  have the same Lascar strong type over  $A$ ,  $I$  and  $I'$  are related by the transitive, symmetric closure of the relation “ $J + K$  is  $A$ -indiscernible”. This is also equivalent to  $I$  being “based on”  $A$  in a sense defined by Shelah (1980) in his study of simple unstable theories.

Finally, we show that for any  $A$  and  $b$  in any theory  $T$ , if there is an Erdős cardinal  $\kappa(\alpha)$  with  $|Ab| + |T| < \kappa(\alpha)$ , then there is a total  $\downarrow^{\text{bu}}$ -Morley sequence  $(b_i)_{i < \omega}$  over  $A$  with  $b_0 = b$ .

### Introduction

A central theme in neostability theory is the importance of various kinds of “generic” indiscernible sequences — usually with Michael Morley’s name attached to them — such as Morley sequences in stable and simple theories, strict Morley sequences in NIP and  $\text{NTP}_2$  theories, tree Morley sequences in  $\text{NSOP}_1$  theories, and  $\downarrow^{\text{b}}$ -Morley sequences in rosy theories. A very broad question one might ask is this: How generically can we build indiscernible sequences in *arbitrary* theories?

Over a model  $M$ , we can always extend a given type  $p(x) \in S_x(M)$  to a global  $M$ -invariant type  $q(x) \supset p(x)$  and then use this to generate a sequence  $(b_i)_{i < \omega}$  satisfying  $b_i \models q \upharpoonright Mb_{<i}$  for each  $i < \omega$ . In some cases the particular choice of  $q(x)$  matters, but typically these sequences are robustly generic. Sequences produced in

MSC2020: 03C45.

Keywords: ultramaginarities, bounded closure, total Morley sequences.

this way have a certain property, which is that they are *based on  $M$  in the sense of Simon*; i.e., for any  $I$  and  $J$  with  $I \equiv_M J \equiv_M b_{<\omega}$ , there is a  $K$  such that  $I + K$  and  $J + K$  are both  $M$ -indiscernible. In NIP theories, the sequences with this property are precisely the sequences generated by an invariant type [Simon 2015, Proposition 2.38]. Over an arbitrary set of parameters  $A$ , however, there may fail to be any indiscernible sequences based on  $A$ . In the dense circular order, for instance, there are no indiscernible sequences based on  $\emptyset$ . Other technical issues also arise when working over arbitrary sets, such as the necessity of considering Lascar strong types over and above ordinary types.

A notion of independence  $\downarrow^*$  is said to satisfy *full existence* if for any  $A$ ,  $b$ , and  $c$ , there is a  $b' \equiv_A b$  such that  $b' \downarrow_A^* c$ . Together with a common model-theoretic application of the Erdős-Rado theorem (Fact 1.2), this implies that for any  $A$  and  $b$ , one can build an  $\downarrow^*$ -Morley sequence, an  $A$ -indiscernible sequence  $(b_i)_{i < \omega}$  with  $b_0 = b$  satisfying  $b_i \downarrow_A^* b_{<i}$  for each  $i < \omega$  (assuming  $\downarrow^*$  also satisfies right monotonicity). Model-theoretically tame theories often have full existence for powerful independence notions, such as nonforking, but this does fail in some notable tame contexts.

One independence notion that is known to satisfy full existence in arbitrary theories is that of *algebraic independence* [Adler 2009, Proposition 1.5]:  $b \downarrow_A^a c$  if  $\text{acl}^{\text{eq}}(Ab) \cap \text{acl}^{\text{eq}}(Ac) = \text{acl}^{\text{eq}}(A)$ . A natural modification of this concept is *bounded hyperimaginary independence*:  $b \downarrow_A^b c$  if  $\text{bdd}^{\text{heq}}(Ab) \cap \text{bdd}^{\text{heq}}(Ac) = \text{bdd}^{\text{heq}}(A)$ . Despite perhaps sounding like an intro-to-model-theory exercise, the combinatorics necessary to prove full existence for  $\downarrow^a$  are somewhat subtle. It was established in [Conant and Hanson 2022] that  $\downarrow^a$  satisfies full existence in continuous logic and, relatedly, that  $\downarrow^b$  satisfies full existence in discrete (and continuous) logic, answering a question of Adler [2005, Question A.8]. While the relations of  $\downarrow^a$  and  $\downarrow^b$  are algebraically nice,<sup>1</sup> they seem to lack semantic consequences outside of certain special theories (such as those with a canonical independence relation in the sense of Adler [2005, Lemma 3.2]).

While being able to build  $\downarrow^*$ -Morley sequences is certainly good, in many applications the important property is really that of being a *total  $\downarrow^*$ -Morley sequence*,<sup>2</sup> which is an  $A$ -indiscernible sequence satisfying  $b_{\geq i} \downarrow_A^* b_{<i}$  for every  $i < \omega$ . When  $\downarrow^*$  lacks the algebraic properties necessary to imply that all  $\downarrow^*$ -Morley sequences are total  $\downarrow^*$ -Morley sequences, it can in general be difficult to ensure their existence. Total  $\downarrow^a$ -Morley sequences arise in Adler's characterization of canonical independence relations. And building total  $\downarrow^K$ -Morley sequences, where  $\downarrow^K$  is

<sup>1</sup>In the sense of the algebra of an independence relation, not the sense of the algebra in “algebraic closure”.

<sup>2</sup>This use of the term “total” in the context of Morley sequences was originally introduced in [Kaplan and Ramsey 2020].

the relation of non-Kim-forking, is a crucial technical step in Kaplan and Ramsey’s proofs [2020] of the symmetry of Kim-forking and the independence theorem in NSOP<sub>1</sub> theories.

In simple theories, Morley sequences over  $A$  are not generally based on  $A$  in the sense of Simon. They do however nearly satisfy this property. If  $I$  and  $J$  are Morley sequences over  $A$  with  $I \equiv_A^L J$ ,<sup>3</sup> then there are  $I'$  and  $K$  such that  $I + I'$ ,  $I' + K$ , and  $J + K$  are  $A$ -indiscernible. In an NSOP<sub>1</sub> theory  $T$ , if  $I$  is a tree Morley sequence over  $M \models T$  and  $J \equiv_M I$ , then we can find  $K_0$ ,  $K_1$ , and  $K_2$  such that  $I + K_0$ ,  $K_1 + K_0$ ,  $K_1 + K_2$ , and  $J + K_2$  are all  $M$ -indiscernible (see Proposition 4.28). These facts suggest the consideration of the following equivalence relation, originally introduced in [Shelah 1980, Definition 5.1]: Let  $\approx_A$  be the transitive, symmetric closure of the relation “ $I + J$  is  $A$ -indiscernible”. The intuition is that what it means for an  $A$ -indiscernible sequence  $I$  to be “based on  $A$ ” is that there are few  $\approx_A$ -classes among the realizations of  $\text{tp}(I/A)$ . We say that  $I$  is *based on  $A$  in the sense of Shelah* if there does not exist a sequence  $(I_i)_{i < \kappa}$  (with  $\kappa$  large) such that  $I_i \equiv_A I$  for each  $i < \kappa$  and  $I_i \not\approx_A I_j$  for each  $i < j < \kappa$ . A simple compactness argument shows that  $I$  is based on  $A$  in the sense of Shelah if and only if the set of realizations of  $\text{tp}(I/A)$  decomposes into a bounded number of  $\approx_A$ -classes. Buechler [1997, Definition 2.4]<sup>4</sup> used this relation to define a notion of canonical base. He focused on  $\emptyset$ -indiscernible sequences and gave the following definition:  $A$  is a *canonical base* of the  $\emptyset$ -indiscernible sequence  $I$  if any automorphism  $\sigma \in \text{Aut}(\mathbb{M})$  fixes  $A$  pointwise if and only if it fixes the  $\approx_{\emptyset}$ -class of  $I$ . One difficulty with this concept, of course, is that not all indiscernible sequences have canonical bases in this sense (even in  $T^{\text{eq}}$ , e.g., [Adler 2005, Example 3.13]).

Two of the problems we have mentioned—the lack of canonical bases for indiscernible sequences and the lack of semantic consequences of  $\downarrow^a$  and  $\downarrow^b$ —can both be solved by an extremely blunt move: the introduction of ultrimaginary parameters. An *ultrimaginary* is an equivalence class of an arbitrary invariant equivalence relation (as opposed to a type-definable equivalence relation, as in the definition of hyperimaginaries). Every indiscernible sequence  $I$  trivially has an ultrimaginary canonical base in the sense of Buechler, i.e., the  $\approx_{\emptyset}$ -class of  $I$  itself.

Another appealing aspect of ultrimaginaries is that they characterize Lascar strong type in the same way that hyperimaginaries characterize Kim–Pillay strong type. An ultrimaginary  $[b]_E$  is *bounded over  $A$*  if it has boundedly many conjugates under  $\text{Aut}(\mathbb{M}/A)$ . We will write  $\text{bdd}^u(A)$  for the class of ultrimaginaries bounded over  $A$ . In general, it turns out that  $b$  and  $c$  have the same Lascar strong type

<sup>3</sup>The equivalence relation  $\equiv_A^L$  is the transitive closure of the relation “there is a model  $M \supseteq A$  such that  $b \equiv_M c$ ”. If  $b \equiv_A^L c$ , we say that  $b$  and  $c$  have the same *Lascar strong type* over  $A$ .

<sup>4</sup>This preprint is difficult to track down. The relevant ideas are developed further in [Adler 2005, Section 3.2], which is easily available.

over  $A$  if and only if they “have the same type over  $\text{bdd}^u(A)$ ”, once this concept is defined precisely.

Pure analogical thinking might lead one to consider the following independence notion:  $b \downarrow_A^{\text{bu}} c$  if  $\text{bdd}^u(Ab) \cap \text{bdd}^u(Ac) = \text{bdd}^u(A)$ . This notion is implicit in a result of Wagner [2015, Proposition 2.12], which we restate and expand slightly (Proposition 2.4):  $b \downarrow_A^{\text{bu}} c$  if and only if  $\langle \text{Autf}(\mathbb{M}/Ab) \cup \text{Autf}(\mathbb{M}/Ac) \rangle = \text{Autf}(\mathbb{M}/A)$  (where  $\langle X \rangle$  is the group generated by  $X$ ). This characterization is clearly semantically meaningful, and moreover it allows one to discuss  $\downarrow^{\text{bu}}$  without actually mentioning ultrimaginaries at all. One way to see why this equivalence works is the fact that ultrimaginaries are “dual” to co-small sets of automorphisms; a group  $G \leq \text{Aut}(\mathbb{M})$  is *co-small* if there is a small model  $M$  such that  $\text{Aut}(\mathbb{M}/M) \leq G$ . For every co-small group  $G$ , there is an ultrimaginary  $a_E$  such that  $\text{Aut}(\mathbb{M}/a_E) = G$  (Proposition 1.7).

As  $\downarrow^{\text{bu}}$  lacks finite character, *total  $\downarrow^{\text{bu}}$ -Morley sequences over  $A$*  seem to be correctly defined as  $A$ -indiscernible sequences  $(b_i)_{i < \omega}$  with the property that for any  $I + J \equiv_A^{\text{EM}} b_{<\omega}$ ,<sup>5</sup> we have that  $I \downarrow_A^{\text{bu}} J$ . The automorphism group characterization of  $\downarrow^{\text{bu}}$ , together with its the nice algebraic properties and the malleability of indiscernible sequences, leads to a pleasing characterization of total  $\downarrow^{\text{bu}}$ -Morley sequences over sets of hyperimaginary parameters (Theorem 4.8), the equivalence of the following.

- $(b_i)_{i < \omega}$  is a total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ .
- For some infinite  $I$  and  $J$ , we have that  $I + J \equiv_A^{\text{EM}} b_{<\omega}$  and  $I \downarrow_A^{\text{bu}} J$ .
- For any  $I$ ,  $I \approx_A b_{<\omega}$  if and only if there is  $I' \equiv_A^{\text{L}} I$  such that  $b_{<\omega} + I'$  is  $A$ -indiscernible.
- $b_{<\omega}$  is based on  $A$  in the sense of Shelah; i.e.,  $[b_{<\omega}]_{\approx_A} \in \text{bdd}^u(A)$ .

The condition in the third bullet point is a natural mutual generalization of Lascar strong type and Ehrenfeucht–Mostowski type (Definition 4.5). Theorem 4.8 also tells us that when total  $\downarrow^{\text{bu}}$ -Morley sequences exist, they act as particularly uniform witnesses of Lascar strong type (Proposition 4.3).

Of course this all leaves two critical questions: Does  $\downarrow^{\text{bu}}$  always satisfy full existence? And, even if it does, can we actually build total  $\downarrow^{\text{bu}}$ -Morley sequences in any type over any set under any theory? The bluntness of ultrimaginaries leaves us without one of the most important tools in model theory, compactness. Furthermore,  $\downarrow^{\text{bu}}$ 's lack of finite character gives us less leeway in applying the Erdős-Rado theorem to construct indiscernible sequences with certain properties;

<sup>5</sup> $I \equiv_A^{\text{EM}} J$  means that  $I$  and  $J$  have the same *Ehrenfeucht–Mostowski type* over  $A$  (i.e., for any increasing tuples  $\bar{b} \in I$  and  $\bar{c} \in J$  of the same length,  $\bar{b} \equiv_A \bar{c}$ ). Note that  $I$  and  $J$  do not need to have the same order type.

we now need to be more concerned with the particular order types of the sequences involved.

Using some of the indiscernible tree technology from [Kaplan and Ramsey 2020], we are able to prove that  $\downarrow^{\text{bu}}$  does satisfy full existence over arbitrary sets of (hyperimaginary) parameters in arbitrary (discrete or continuous) theories (Theorem 3.6).<sup>6</sup> With regards to building total  $\downarrow^{\text{bu}}$ -Morley sequences, Theorem 4.8 tells us that we don't need to worry *too* much about order types. All we need to get a total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  is an  $A$ -indiscernible sequence  $(b_i)_{i < \omega + \omega}$  with  $b_{\geq \omega} \downarrow_A^{\text{bu}} b_{< \omega}$ . This is fortunate because constructing ill-ordered  $\downarrow^{\text{bu}}$ -Morley sequences directly seems daunting. Unfortunately,  $\omega + \omega$  appears to be about one  $\omega$  further than we can go without a large cardinal. What we do get is this (Theorem 4.22): For any  $A$  and  $b$  in any theory  $T$ , if there is an Erdős cardinal  $\kappa(\alpha)$  with  $|Ab| + |T| < \kappa(\alpha)$  (for any  $\alpha \geq \omega$ ), then there is a total  $\downarrow^{\text{bu}}$ -Morley sequence  $(b_i)_{i < \omega}$  over  $A$  with  $b_0 = b$ . Without a large cardinal, the best we seem to be able to do (Proposition 4.17) is a half-infinite, half-arbitrary-finite approximation of a total  $\downarrow^{\text{bu}}$ -Morley sequence, which we call a *weakly total  $\downarrow^{\text{bu}}$ -Morley sequence*. These sequences also serve as uniform witnesses of Lascar strong type without any set-theoretic hypotheses (Corollary 4.18).

## 1. Ultrimaginaries

Here we will set definitions and conventions, and we also take the opportunity to collect some basic facts about ultrimaginaries which are likely folklore, although we could not find explicit references.

Fix a theory  $T$  and a set-sized monster model  $\mathbb{M} \models T$ .

**Definition 1.1.** An *invariant equivalence relation of arity  $\kappa$*  is an equivalence relation  $E$  on  $\mathbb{M}^x$  (with  $|x| = \kappa$ ) such that for any  $a, b, c, d \in \mathbb{M}^x$  with  $ab \equiv cd$ ,  $aEb$  if and only if  $cEd$ .

An *ultrimaginary of arity  $\kappa$*  is a pair  $(E, a_E)$  consisting of an invariant equivalence relation  $E$  (of arity  $\kappa$ ) and an  $E$ -equivalence class  $a_E$  of some tuple  $a \in \mathbb{M}^x$ . By an abuse of notation, we will write  $a_E$  for the pair  $(E, a_E)$ , and we may also write  $[a]_E$  if necessary for notational clarity.

Given an ultrimaginary  $a_E$ ,  $\text{Aut}(\mathbb{M}/a_E)$  is the set of automorphisms  $\sigma \in \text{Aut}(\mathbb{M})$  with the property that  $a_E(\sigma \cdot a)$ . We write  $\text{Autf}(\mathbb{M}/a_E)$  for the group generated by  $\{\sigma \in \text{Aut}(\mathbb{M}/M) : M \preceq \mathbb{M}, \text{Aut}(\mathbb{M}/M) \leq \text{Aut}(\mathbb{M}/a_E)\}$ .

We say that  $b_F$  is *definable over  $a_E$*  if  $b_F$  is fixed by every automorphism in  $\text{Aut}(\mathbb{M}/a_E)$ . We write  $\text{dcl}^{\text{u}}(a_E)$  for the class of all ultrimaginaries definable

<sup>6</sup>Although this result partially supersedes a result in [Conant and Hanson 2022] (full existence for  $\downarrow^{\text{a}}$  in continuous logic and  $\downarrow^{\text{b}}$  in discrete or continuous logic), the proof there gives more detailed numerical information which may be especially useful in the metric context.

over  $a_E$ . For any  $\kappa$ , we write  $\text{dcl}_\kappa^u(a_E)$  for the set of elements of  $\text{dcl}^u(a_E)$  of arity at most  $\kappa$ . We say that  $b_F$  and  $c_G$  are *interdefinable over  $a_E$*  if  $b_F \in \text{dcl}^u(a_E c_G)$  and  $c_G \in \text{dcl}^u(a_E b_F)$ .

We say that  $b_F$  is *bounded over  $a_E$*  if the  $\text{Aut}(\mathbb{M}/a_E)$ -orbit of  $b_F$  is bounded.<sup>7</sup> We write  $\text{bdd}^u(a_E)$  for the class of all ultraimaginaris bounded over  $a_E$ . We write  $\text{bdd}_\kappa^u(a_E)$  for the set of elements of  $\text{bdd}^u(a_E)$  of arity at most  $\kappa$ . We say that  $b_F$  and  $c_G$  are *interbounded over  $a_E$*  if  $b_F \in \text{bdd}^u(a_E c_G)$  and  $c_G \in \text{bdd}^u(a_E b_F)$ .

We write  $a_E \equiv b_E$  to mean that there is an automorphism  $\sigma \in \text{Aut}(\mathbb{M})$  with  $\sigma \cdot a_E = b_E$ . We write  $b_F \equiv_{a_E} c_F$  to mean that  $a_E b_F \equiv a_E c_F$  (i.e., there is  $\sigma \in \text{Aut}(\mathbb{M}/a_E)$  such that  $\sigma \cdot b_F = c_F$ ).

Note that real elements, imaginaries, and hyperimaginaries can all be regarded as ultraimaginaris.

An easy counting argument shows that  $\text{bdd}^u$  is a closure operator (i.e., for any  $a_E, b_F$ , and  $c_G$ , if  $b_F \in \text{bdd}^u(a_E)$  and  $c_G \in \text{bdd}^u(b_F)$ , then  $c_G \in \text{bdd}^u(a_E)$ ).

We will also sometimes define an invariant equivalence relation  $E$  on the realizations of a single type  $p(x)$  over  $\emptyset$ . Equivalence classes of such can be thought of as ultraimaginaris by using the same trick that is commonly used with hyperimaginaries: Consider the invariant equivalence relation  $E'(x, y)$  defined by  $x = y \vee (E(x, y) \wedge x \models p \wedge y \models p)$ .

For the sake of clarity, we will reserve the notation  $a_E$  for ultraimaginaris and write hyperimaginaries in the same way we write real elements. For the sake of cardinality issues, we will also take all hyperimaginaries to be quotients of countable tuples by countably type-definable equivalence relations. It is a standard fact that every hyperimaginary is interdefinable with some set of hyperimaginaries of this form.

**Fact 1.2** [Shelah 1980].<sup>8</sup> *Let  $(b_i)_{i < \lambda}$  be a sequence of tuples with  $|b_i| < \kappa$  and let  $A$  be some set of parameters. If  $\lambda \geq \beth_{(2^\kappa + |A| + |T|)^+}$ , then there is an  $A$ -indiscernible sequence  $(b'_i)_{i < \omega}$  such that for every  $n < \omega$ , there are  $i_0 < \dots < i_n < \kappa$  such that  $b'_0 \dots b'_n \equiv_A b_{i_0} \dots b_{i_n}$ .*

**Lemma 1.3.** *Let  $M$  be a model. If  $a_E \in \text{bdd}^u(M)$ , then  $a_E \in \text{dcl}^u(M)$ .*

*Proof.* Assume that  $a_E \notin \text{dcl}^u(M)$ . Let  $p(x)$  be a global  $M$ -invariant type extending  $\text{tp}(a/M)$ . Assume that there are  $a_0$  and  $a_1$  realizing  $\text{tp}(a/M)$  such that  $a_0 \not\equiv a_1$ . For any  $i > 1$ , given  $a_{<i}$ , let  $a_i \models p \upharpoonright M a_{<i}$ . Since  $a_i a_j \equiv_M a_i a_k$  for any  $j, k < i$ , we must have that  $a_i \not\equiv a_j$  for any  $j < i$ . Since we can do this indefinitely, we have that  $a_E$  is not bounded over  $M$ .  $\square$

<sup>7</sup>Specifically, by Proposition 1.4, this is equivalent to  $b_F$  having at most  $2^{|ab|+|T|}$  conjugates over  $a_E$ .

<sup>8</sup>See [Tent and Ziegler 2012, Lemma 7.2.12] for a modern presentation of the result.

**Proposition 1.4.** *For any ultraimaginarys  $a_E$  and  $b_F$ , the following are equivalent.*

- (1)  $b_F \notin \text{bdd}^u(a_E)$ .
- (2) *There is an  $a$ -indiscernible sequence  $(c_i)_{i < \omega}$  such that  $c_0 \equiv_{a_E} b$  and  $c_i \not\equiv c_j$  for each  $i < j < \omega$ .*
- (3)  $|\text{Aut}(\mathbb{M}/a_E) \cdot b_F| > 2^{|ab|+|T|}$ .

*Proof.* (3) $\Rightarrow$ (2). Let  $(b_F^i)_{i < (2^{|ab|+|T|})^+}$  be an enumeration of  $\text{Aut}(\mathbb{M}/a_E) \cdot b_F$ . Let  $M \supseteq a$  be a model with  $|M| \leq |a| + |T|$ . Let  $x$  be a tuple of variables of the same length as  $b$ . There are at most  $2^{|ab|+|T|}$  types in  $S_x(M)$ . Therefore, there must be  $i < j < (2^{|ab|+|T|})^+$  such that  $b^i \equiv_M b^j$ . Let  $p(x)$  be a global  $M$ -invariant type extending  $\text{tp}(b^i/M)$ , and let  $(c_i)_{i < \omega}$  be a Morley sequence generated by  $p(x)$  over  $Mb^i b^j$ . Since  $b^i \not\equiv b^j$ , we must have that  $c_0 \not\equiv b^i$ . Therefore  $c_i \not\equiv c_j$  for any  $i < j < \omega$ , and so  $(c_i)_{i < \omega}$  is the required  $a$ -indiscernible sequence.

(2) $\Rightarrow$ (1). Given an  $a$ -indiscernible sequence  $(c_i)_{i < \omega}$  as in the statement of the proposition, we can extend it to an  $a$ -indiscernible sequence  $(c_i)_{i < \lambda}$  for any  $\lambda$ . These sequences will still satisfy that  $c_i \not\equiv c_j$  for any  $i < j < \lambda$ , so  $b_F$  has an unbounded number of  $\text{Aut}(\mathbb{M}/a_E)$ -conjugates and  $b_F \notin \text{bdd}^u(a_E)$ .

(1) $\Rightarrow$ (3). This is immediate from the definition of  $\text{bdd}^u(a_E)$ .  $\square$

**Corollary 1.5.** *For any  $\lambda$ ,  $\text{bdd}_\lambda^u(a_E)$  has cardinality at most  $2^{|a|+2^{\lambda+|T|}}$ .*

*Proof.* For each  $\alpha \leq \lambda$ ,  $|S_{\alpha+\alpha}(T)| \leq 2^{\lambda+|T|}$ . Since an invariant equivalence relation on  $\alpha$ -tuples is specified by a subset of  $S_{\alpha+\alpha}(T)$ , this implies that for each  $\alpha \leq \lambda$ , there are at most  $2^{2^{\lambda+|T|}}$  invariant equivalence relations on  $\alpha$ -tuples. Therefore the total number of invariant equivalence relations on tuples of length at most  $\lambda$  is  $\lambda \cdot 2^{2^{\lambda+|T|}} = 2^{2^{\lambda+|T|}}$ . For each such  $F$ , the set  $\{b_F : b_F \in \text{bdd}_\lambda^u(a_E)\}$  has cardinality at most  $2^{|a|+\lambda+|T|}$  by Proposition 1.4. Finally,  $2^{2^{\lambda+|T|}} \cdot 2^{|a|+\lambda+|T|} = 2^{|a|+2^{\lambda+|T|}}$ .  $\square$

**Co-small groups of automorphisms.** Here we will see that ultraimaginarys are essentially the same thing as reasonable subgroups of  $\text{Aut}(\mathbb{M})$ .

**Definition 1.6.** A group  $G \leq \text{Aut}(\mathbb{M})$  is *co-small* if there is a small model  $M$  such that  $\text{Aut}(\mathbb{M}/M) \leq G$ .

Clearly for any ultraimaginary  $a_E$ ,  $\text{Aut}(\mathbb{M}/a_E)$  is co-small. The converse is true as well.

**Proposition 1.7.** *For any co-small  $G$ , if  $\text{Aut}(\mathbb{M}/M) \leq G$ , then there is an ultraimaginary  $a_E$  such that  $G = \text{Aut}(\mathbb{M}/a_E)$  where  $a$  is some enumeration of  $M$ .*

*Proof.* Let  $M$  be a small model witnessing that  $G$  is co-small. Consider the binary relation defined on realizations of  $\text{tp}(M)$  (in some fixed enumeration) defined by  $E(M_0, M_1)$  if and only if there is  $\sigma \in \text{Aut}(\mathbb{M})$  and  $\tau \in G$  such that  $\sigma \cdot M = M_0$

and  $\sigma\tau \cdot M = M_1$ . We need to verify that  $E$  is an invariant equivalence relation. Reflexivity is obvious.

*Invariance.* Suppose that  $E(M_0, M_1)$ , as witnessed by  $\sigma \in \text{Aut}(\mathbb{M})$  and  $\tau \in G$ . Fix  $\sigma' \in \text{Aut}(\mathbb{M})$ . We then have that  $\sigma'\sigma \cdot M = \sigma' \cdot M_0$  and  $\sigma'\sigma\tau \cdot M = \sigma' \cdot M_1$ , whence  $E(\sigma' \cdot M_0, \sigma' \cdot M_1)$ .

*Symmetry.* If  $\sigma \cdot M = M_0$  and  $\sigma\tau \cdot M = M_1$  with  $\sigma \in \text{Aut}(\mathbb{M})$  and  $\tau \in G$ , then  $\sigma\tau\tau^{-1} \cdot M = M_0$  and  $\sigma\tau \cdot M = M_1$ . We have  $\sigma\tau \in \text{Aut}(\mathbb{M})$  and  $\tau^{-1} \in G$ , so  $E(M_1, M_0)$ .

*Transitivity.* Suppose that for  $\sigma, \sigma' \in \text{Aut}(\mathbb{M})$  and  $\tau, \tau' \in G$ , we have that  $\sigma \cdot M = M_0$ ,  $\sigma\tau \cdot M = \sigma' \cdot M = M_1$ , and  $\sigma'\tau' \cdot M = M_2$ . This implies that  $(\sigma\tau)^{-1}\sigma' = \tau^{-1}\sigma^{-1}\sigma' \in \text{Aut}(\mathbb{M}/M) \leq G$ . Since  $\tau \in G$  as well, we have that  $\sigma^{-1}\sigma' \in G$ . Therefore  $\sigma^{-1}\sigma'\tau' \in G$ . Finally,  $\sigma\sigma^{-1}\sigma'\tau' \cdot M = M_2$ , so  $E(M_0, M_2)$ .

Consider the ultraimaginary  $M_E$ . For any  $\tau \in G$ , we clearly have  $E(M, \tau \cdot M)$ , so  $G \leq \text{Aut}(\mathbb{M}/M_E)$ . Conversely, suppose that  $\alpha \in \text{Aut}(\mathbb{M}/M_E)$ . By definition, this implies that  $E(M, \alpha \cdot M)$ , so there are  $\sigma \in \text{Aut}(\mathbb{M})$  and  $\tau \in G$  such that  $\sigma \cdot M = M$  and  $\sigma\tau \cdot M = \alpha \cdot M$ . Therefore  $\sigma, \tau^{-1}\sigma^{-1}\alpha \in \text{Aut}(\mathbb{M}/M) \leq G$ . Since  $\tau^{-1} \in G$ , we therefore have that  $\alpha \in G$ .  $\square$

**Corollary 1.8.** *If  $b_F \in \text{bdd}^u(a_E)$ , then there is  $c_G \in \text{bdd}^u(a_E)$  of arity at most  $|a| + |T|$  such that  $b_F$  and  $c_G$  are interdefinable over  $\emptyset$ . Furthermore,  $c$  can be taken to be an enumeration of any model of size at most  $|a| + |T|$  containing  $a$ .*

*Proof.* There is a model  $M \supseteq a$  with  $|M| \leq |a| + |T|$ . By Lemma 1.3, we have that  $\text{Aut}(\mathbb{M}/M) \leq \text{Aut}(\mathbb{M}/b_F)$ , so by Proposition 1.7, we have that there is  $c_G$  with arity at most  $|a| + |T|$  which satisfies that  $\text{Aut}(\mathbb{M}/c_G) = \text{Aut}(\mathbb{M}/b_F)$  (i.e.,  $c_G$  and  $b_F$  are interdefinable over  $\emptyset$ ). Furthermore, we can take  $c$  to be an enumeration of  $M$ .  $\square$

**Definition 1.9.** For any co-small group  $G$ , we write  $\llbracket G \rrbracket$  for some arbitrary ultraimaginary  $a_E$  of minimal arity satisfying  $G = \text{Aut}(\mathbb{M}/a_E)$ . We will write  $\text{dcl}^u \llbracket G \rrbracket$  and  $\text{dcl}_\lambda^u \llbracket G \rrbracket$  for  $\text{dcl}^u(\llbracket G \rrbracket)$  and  $\text{dcl}_\lambda^u(\llbracket G \rrbracket)$  and likewise with  $\text{bdd}^u$ . (Note that  $\text{dcl}^u \llbracket G \rrbracket$  and  $\text{bdd}^u \llbracket G \rrbracket$  only depend on  $G$ , not on the particular choice of  $\llbracket G \rrbracket$ .)

It is immediate from Proposition 1.7 that for any co-small  $G$  and  $H$ ,  $\llbracket G \rrbracket \in \text{dcl}^u \llbracket H \rrbracket$  if and only if  $G \geq H$ . A similar statement for  $\text{bdd}^u$  is given in Proposition 1.12.

Now we can see that intersections of  $\text{dcl}^u$ -closed sets (and therefore also  $\text{bdd}^u$ -closed sets) have semantic significance in arbitrary theories, in that intersections correspond to joins in the lattice of co-small groups of automorphisms.

**Proposition 1.10.** *For any  $a_E, b_F, c_G$ , and  $c'_G$ , the following are equivalent.*

- (1)  $c_G \equiv_{\text{dcl}_\lambda^u(a_E) \cap \text{dcl}_\lambda^u(b_F)} c'_G$  for all  $\lambda$ .
- (2) There is  $\sigma \in \langle \text{Aut}(\mathbb{M}/a_E) \cup \text{Aut}(\mathbb{M}/b_F) \rangle$  such that  $\sigma \cdot c_G = c'_G$ .

(3) *There is a sequence  $(a^i b^i c^i)_{i \leq n}$  such that  $a^0 = a, b^0 = b, c^0 = c, c_G^n = c'_G$ , and for each  $i < n$ ,*

- *if  $i$  is even, then  $a^i = a^{i+1}$  and  $b_F^i c_G^i \equiv_{a_E^i} b_F^{i+1} c_G^{i+1}$  and*
- *if  $i$  is odd, then  $b^i = b^{i+1}$  and  $a_E^i c_G^i \equiv_{b_F^i} a_E^{i+1} c_G^{i+1}$ .*

*Proof.* Let  $H = \langle \text{Aut}(\mathbb{M}/a_E) \cup \text{Aut}(\mathbb{M}/b_F) \rangle$ .

**Claim.**  $\text{dcl}_\lambda^u(a_E) \cap \text{dcl}_\lambda^u(b_F)$  and  $\llbracket H \rrbracket$  are interdefinable (i.e.,  $\text{dcl}_\lambda^u(a_E) \cap \text{dcl}_\lambda^u(b_F) \subseteq \text{dcl}^u(\llbracket H \rrbracket)$  and  $\llbracket H \rrbracket \in \text{dcl}^u(\text{dcl}_\lambda^u(a_E) \cap \text{dcl}_\lambda^u(b_F))$ ) for all sufficiently large  $\lambda$ .

*Proof of claim.* Clearly  $\llbracket H \rrbracket \in \text{dcl}^u(a_E) \cap \text{dcl}^u(b_F)$ , so  $\llbracket H \rrbracket \in \text{dcl}_\lambda^u(a_E) \cap \text{dcl}_\lambda^u(b_F)$  for all sufficiently large  $\lambda$ .

Conversely, suppose that  $d_I \in \text{dcl}^u(a_E) \cap \text{dcl}^u(b_F)$ . Any  $\sigma \in H$  is a product of elements of  $\text{Aut}(\mathbb{M}/a_E)$  and  $\text{Aut}(\mathbb{M}/b_F)$ , so it must fix  $d_I$ . Thus  $\text{Aut}(\mathbb{M}/d_I) \geq H$  and hence  $d_I \in \text{dcl}^u(\llbracket H \rrbracket)$ .  $\square_{\text{claim}}$

So now we have that  $c_G \equiv_{\text{dcl}_\lambda^u(a_E) \cap \text{dcl}_\lambda^u(b_F)} c'_G$  holds for sufficiently large  $\lambda$  if and only if  $c_G \equiv_{\llbracket H \rrbracket} c'_G$ . Also note that  $c_G \equiv_{\text{dcl}_\lambda^u(a_E) \cap \text{dcl}_\lambda^u(b_F)} c'_G$  for sufficiently large  $\lambda$  and only if the same holds for any  $\lambda$ . Therefore (1) and (2) are equivalent.

There is a  $\sigma \in H$  with  $\sigma \cdot c_G = c'_G$  if and only if there are  $\alpha_0, \dots, \alpha_{n-1} \in \text{Aut}(\mathbb{M}/a_E)$  and  $\beta_0, \dots, \beta_{n-1} \in \text{Aut}(\mathbb{M}/b_F)$  such that  $\sigma = \alpha_{n-1} \beta_{n-1} \dots \beta_1 \alpha_0 \beta_0$ .

For (2) $\Rightarrow$ (3), assume that there are such  $\bar{\alpha}$  and  $\bar{\beta}$  for which

$$\alpha_{n-1} \beta_{n-1} \alpha_{n-2} \dots \beta_1 \alpha_0 \beta_0 \cdot c_G = c'_G.$$

Let  $a^0 b^0 c^0 = abc$ ,  $a^1 b^1 c^1 = \alpha_{n-1} \cdot (a^0 b^0 c^0)$ ,  $a^2 b^2 c^2 = \alpha_{n-1} \beta_{n-1} \cdot (a^0 b^0 c^0)$ , and so on up to  $a^{2n} b^{2n} c^{2n} = \alpha_{n-1} \beta_{n-1} \alpha_{n-2} \dots \beta_1 \alpha_0 \beta_0 \cdot (a^0 b^0 c^0)$ . Clearly we have that  $c_G^{2n} = c'_G$ , so we just need to verify that  $(a^i b^i c^i)_{i \leq 2n}$  is the required sequence. If  $i < 2n$  is even, then  $\alpha_i \in \text{Aut}(\mathbb{M}/a_E)$ , so  $a_E^i = a_E^{i+1}$ . Furthermore,  $b_F^0 c_G^0 \equiv_{a_E^0} \alpha_i \cdot (b_F^0 c_G^0)$ , so by invariance,

$$\alpha_{n-1} \beta_{n-1} \dots \beta_{i+1} \cdot (b_F^0 c_G^0) \equiv_{\alpha_{n-1} \beta_{n-1} \dots \beta_{i+1} \cdot a_E^0} \alpha_{n-1} \beta_{n-1} \dots \beta_{i+1} \alpha_i \cdot (b_F^0 c_G^0),$$

which is the same as  $b_F^i c_G^i \equiv_{a_E^i} b_F^{i+1} c_G^{i+1}$ . If  $i < 2n$  is odd, then the same argument tells us that  $b_F^i = b_F^{i+1}$  and  $a_E^i c_G^i \equiv_{b_F^i} a_E^{i+1} c_G^{i+1}$ .

For (3) $\Rightarrow$ (2), the above argument is reversible. Fix  $(a_E^i b_F^i c_G^i)_{i \leq 2n}$  satisfying the conditions of (3). First of all we can find  $\alpha_{n-1} \in \text{Aut}(\mathbb{M}/a_E)$  such that  $\alpha_{n-1}^{-1} \cdot (a_E^1 b_F^1 c_G^1) = a_E^0 b_F^0 c_G^0$ . Then we can find  $\beta_{n-1} \in \text{Aut}(\mathbb{M}/b_F)$  such that  $\beta_{n-1}^{-1} \alpha_{n-1}^{-1} \cdot (a_E^2 b_F^2 c_G^2) = a_E^0 b_F^0 c_G^0$ . Then we can find  $\alpha_{n-2} \in \text{Aut}(\mathbb{M}/a_E)$  such that  $\alpha_{n-2}^{-1} \beta_{n-1}^{-1} \alpha_{n-1}^{-1} \cdot (a_E^3 b_F^3 c_G^3) = a_E^0 b_F^0 c_G^0$ . Continuing inductively in this way, we find  $\alpha_0, \dots, \alpha_{n-1} \in \text{Aut}(\mathbb{M}/a_E)$  and  $\beta_0, \dots, \beta_{n-1} \in \text{Aut}(\mathbb{M}/b_F)$  such that the same equalities as in the (2) $\Rightarrow$ (3) proof hold. Therefore there is a  $\sigma \in H$  (namely  $\alpha_{n-1} \beta_{n-1} \alpha_{n-2} \dots \beta_1 \alpha_0 \beta_0$ ) such that  $\sigma \cdot c_G = c'_G$ .  $\square$

A similar statement is true for arbitrary families of ultrimaginaries: If  $(a_{E_i}^i)_{i \in I}$  is a (possibly large) family of ultrimaginaries, then  $c_G \equiv_{\bigcap_{i \in I} \text{dcl}_\lambda^u(a_{E_i}^i)} c'_G$  if and only if there is a  $\sigma \in \left( \bigcup_{i \in I} \text{Aut}(\mathbb{M}/a_{E_i}^i) \right)$  such that  $\sigma \cdot c_G = c'_G$ . There is also an analog of (3), but it is more awkward to state.

**Lascar strong type.**

**Definition 1.11.** For any co-small group  $G \leq \text{Aut}(\mathbb{M})$ , let  $G_f$  be the group generated by all groups of the form  $\text{Aut}(\mathbb{M}/M) \leq G$  with  $M$  a small model. For any ultrimaginary  $a_E$ , let  $\text{Aut}_f(\mathbb{M}/a_E) = \text{Aut}(\mathbb{M}/a_E)_f$ .

We say that  $b_F$  and  $c_F$  have the same *Lascar strong type over  $a_E$* , written  $b_F \equiv_{a_E}^L c_F$ , if there is  $\sigma \in \text{Aut}_f(\mathbb{M}/a_E)$  such that  $\sigma \cdot b_F = c_F$ .

**Proposition 1.12.** *For any co-small groups  $G$  and  $H$ ,  $\llbracket G \rrbracket \in \text{bdd}^u \llbracket H \rrbracket$  if and only if  $G \geq H_f$ .*

*Proof.* Assume that  $\llbracket G \rrbracket \in \text{bdd}^u \llbracket H \rrbracket$ . Note that for a model  $M$ , by Lemma 1.3, we have that  $\llbracket G \rrbracket \in \text{bdd}^u(M)$  if and only if  $G \geq \text{Aut}(\mathbb{M}/M)$ . Therefore, for any model  $M$  with  $\llbracket H \rrbracket \in \text{bdd}^u(M)$ , we must have that  $\llbracket G \rrbracket \in \text{bdd}^u \llbracket H \rrbracket \subseteq \text{bdd}^u(M)$  and so  $G \geq \text{Aut}(\mathbb{M}/M)$ . Since  $\llbracket H \rrbracket \in \text{bdd}^u(M)$  if and only if  $H \geq \text{Aut}(\mathbb{M}/M)$ , we have that  $G \geq H_f$ .

Conversely, assume that  $G \geq H_f$ . This implies that for any small model  $M$  with  $\llbracket H \rrbracket \in \text{bdd}^u(M)$ , we have  $H_f \geq \text{Aut}(\mathbb{M}/M)$ , so  $G \geq \text{Aut}(\mathbb{M}/M)$  and  $\llbracket G \rrbracket \in \text{dcl}^u(M)$ . Fix some such model  $N$ . Assume for the sake of contradiction that  $\llbracket G \rrbracket \notin \text{bdd}^u \llbracket H \rrbracket$ . For any  $\lambda$ , we can find  $(\sigma_i)_{i < \lambda}$  in  $H = \text{Aut}(\mathbb{M}/\llbracket H \rrbracket)$  such that  $\sigma_i \cdot \llbracket G \rrbracket \neq \sigma_j \cdot \llbracket G \rrbracket$  for each  $i < j < \lambda$ . Since  $\llbracket G \rrbracket = a_E$  for some  $a$  with  $|a| \leq |N|$  by Proposition 1.7, we have that if  $\lambda$  is larger than  $2^{|N|+|T|}$ , there must be  $i < j < \lambda$  such that  $\sigma_i \cdot \llbracket G \rrbracket \equiv_N \sigma_j \cdot \llbracket G \rrbracket$ . Let  $N' = \sigma_i^{-1} \cdot N$ .  $N'$  is now a model satisfying  $\text{Aut}(\mathbb{M}/N') \leq G$ . So  $\llbracket G \rrbracket \in \text{dcl}^u(N')$ , but  $\llbracket G \rrbracket \equiv_{N'} \sigma_i^{-1} \sigma_j \cdot \llbracket G \rrbracket$  and  $\llbracket G \rrbracket \neq \sigma_i^{-1} \sigma_j \cdot \llbracket G \rrbracket$ , which is a contradiction.  $\square$

An important fact about ultrimaginaries is that  $\text{bdd}^u$  has the same relationship with Lascar strong types that  $\text{bdd}^{\text{heq}}$  has with Kim–Pillay strong types.

For any  $a_E$  and  $b_F$ , by an abuse of notation, we'll write  $[b_F]_{\equiv_{a_E}^L}$  for  $[ab]_G$ , where  $G(ab, a'b')$  holds if and only if  $aEa'$  and  $b_F \equiv_{a_E}^L b'_F$ . Note in particular that  $[b_F]_{\equiv_{a_E}^L} = [b'_F]_{\equiv_{a_E}^L}$  if and only if  $b_F \equiv_{a_E}^L b'_F$ .

**Proposition 1.13.** *For any ultrimaginaries  $a_E$ ,  $b_F$ , and  $c_F$ , the following are equivalent.*

- (1)  $b_F \equiv_{a_E}^L c_F$ .
- (2)  $b_F \equiv_{\text{bdd}_\lambda^u(a_E)} c_F$  for all  $\lambda$ .
- (3)  $b_F \equiv_{\text{bdd}_{|a|+|T|}^u(a_E)} c_F$ .

*Proof.* To see that (1) implies (3), fix a model  $M$  with  $a_E \in \text{bdd}^u(M)$  and some automorphism  $\sigma \in \text{Aut}(\mathbb{M}/M)$ . By Lemma 1.3, we have that  $\text{Aut}(\mathbb{M}/M) \leq \text{Aut}(\mathbb{M}/\text{bdd}_{|a|+|T|}^u(a_E))$ . Therefore  $b_F \equiv_{\text{bdd}_{|a|+|T|}^u(a_E)} \sigma \cdot b_F$ . By induction, we therefore have that  $b_F \equiv_{a_E}^L c_F$  implies  $b_F \equiv_{\text{bdd}_{|a|+|T|}^u(a_E)} c_F$ .

Corollary 1.8 implies that  $\text{Aut}(\mathbb{M}/\text{bdd}_\lambda^u(a_E)) \geq \text{Aut}(\mathbb{M}/\text{bdd}_{|a|+|T|}^u(a_E))$  for all  $\lambda$ , so (3) implies (2).

To see that (2) implies (1), note that  $[b_F]_{\equiv_{a_E}^L} \in \text{bdd}_\lambda^u(a_E)$  for some sufficiently large  $\lambda$  (because there are a bounded number of Lascar strong types over  $a_E$ ). Thus if  $b_F \equiv_{\text{bdd}_\lambda^u(a_E)} c_F$ , we must have  $[b_F]_{\equiv_{a_E}^L} = [c_F]_{\equiv_{a_E}^L}$  or, in other words,  $b_F \equiv_{a_E}^L c_F$ .  $\square$

## 2. Bounded ultrimaginary independence

**Definition 2.1.** Given sets of ultrimaginaries  $A$ ,  $B$ , and  $C$ , we write  $B \downarrow_A^{\text{bu}} C$  to mean that  $\text{bdd}^u(AB) \cap \text{bdd}^u(AC) = \text{bdd}^u(A)$ .

Recall that  $\text{bdd}^u$  is a closure operator (i.e., if  $c_G \in \text{bdd}^u(b_F)$  and  $b_F \in \text{bdd}^u(a_E)$ , then  $c_G \in \text{bdd}^u(a_E)$ ). We will ultimately show (in Proposition 2.3) that the following are equivalent:  $b_F \downarrow_{a_E}^{\text{bu}} c_G$ ,  $\text{bdd}_\kappa^u(a_E b_F) \cap \text{bdd}_\kappa^u(a_E c_G) = \text{bdd}_\kappa^u(a_E)$  for all  $\kappa$ , and  $\text{bdd}_\kappa^u(a_E b_F) \cap \text{bdd}_\kappa^u(a_E c_G) = \text{bdd}_\kappa^u(a_E)$  for  $\kappa = |T| + |abc|$ .  $\downarrow^{\text{bu}}$  satisfies some of the familiar properties of  $\downarrow^a$ .

**Proposition 2.2.** Fix ultrimaginaries  $a_E, b_F, c_G$ , and  $e_I$ .

- (Invariance) If  $a_E b_F c_G \equiv a'_E b'_F c'_G$ , then  $b_F \downarrow_{a_E}^{\text{bu}} c_G$  if and only if  $b'_F \downarrow_{a'_E}^{\text{bu}} c'_G$ .
- (Symmetry)  $b_F \downarrow_{a_E}^{\text{bu}} c_G$  if and only if  $c_G \downarrow_{a_E}^{\text{bu}} b_F$ .
- (Monotonicity) If  $b_F c_G \downarrow_{a_E}^{\text{bu}} d_H e_I$ , then  $b_F \downarrow_{a_E}^{\text{bu}} d_H$ .
- (Transitivity) If  $b_F \downarrow_{a_E}^{\text{bu}} c_G$  and  $d_H \downarrow_{a_E b_F}^{\text{bu}} c_G$ , then  $b_F d_H \downarrow_{a_E}^{\text{bu}} c_G$ .
- (Normality) If  $b_F \downarrow_{a_E}^{\text{bu}} c_G$ , then  $a_E b_F \downarrow_{a_E}^{\text{bu}} a_E c_G$ .
- (Anti-reflexivity) If  $b_F \downarrow_{a_E}^{\text{bu}} b_F$ , then  $b_F \in \text{bdd}^u(a_E)$ .

*Proof.* Everything except transitivity is immediate. The argument for transitivity is the same as the argument for transitivity of  $\downarrow^a$ : Assume that  $b_F \downarrow_{a_E}^{\text{bu}} c_G$  and  $d_H \downarrow_{a_E b_F}^{\text{bu}} c_G$ . Let  $e_I$  be an element of  $\text{bdd}^u(a_E b_F d_H) \cap \text{bdd}^u(a_E c_G)$ . This implies that it is an element of  $\text{bdd}^u(a_E b_F d_H) \cap \text{bdd}^u(a_E b_F c_G)$ , so by assumption it is an element of  $\text{bdd}^u(a_E b_F)$ . But this means that it's in both  $\text{bdd}^u(a_E b_F)$  and  $\text{bdd}^u(a_E c_G)$ , so, by assumption again, it is an element of  $\text{bdd}^u(a_E)$ .  $\square$

Part of the goal of this paper is to prove full existence and therefore also extension for  $\downarrow^{\text{bu}}$  (although only over hyperimaginary bases).

- (Full existence over hyperimaginaries) For any set of hyperimaginaries  $A$  and ultrimaginaries  $b_E$  and  $c_F$ , there is  $c'_F \equiv_A c_F$  such that  $b_E \downarrow_A^{\text{bu}} c'_F$ .

- (Extension over hyperimaginaries) For any set of hyperimaginaries  $A$  and ultrimaginaries  $b_E, c_F$ , and  $d_G$ , if  $b_E \downarrow_A^{\text{bu}} c_F$ , then there is  $b'_E \equiv_{Ac_F} b_E$  such that  $b'_E \downarrow_A^{\text{bu}} c_F d_G$ .

A fairly general argument will allow us to upgrade  $\equiv_A$  to  $\equiv_A^L$  in the above two conditions, which we establish in Theorem 3.6 and Corollary 3.8.

Finite character fails very badly, of course: As considered in [Wagner 2015, Example 2.8], if  $E$  is the equivalence relation on  $\omega$ -tuples of equality on cofinitely many indices, then for some sequences  $(a_i)_{i < \omega}$ , we will have  $a_{< n} \downarrow^{\text{bu}} [a_{< \omega}]_E$  for all  $n$ , yet  $a_{< \omega} \not\downarrow^{\text{bu}} [a_{< \omega}]_E$ . Given the existence of higher and higher cardinality generalizations of the previous example (e.g., equality on co-countably many indices on  $\omega_1$ -tuples), local character seems unlikely except possibly in the presence of large cardinals. We do have some control over the relevant cardinalities, however.

**Proposition 2.3.** *For any  $a_E, b_F$ , and  $c_G$ ,  $b_F \downarrow_{a_E}^{\text{bu}} c_G$  if and only if*

$$\text{bdd}_\lambda^{\text{u}}(a_E b_F) \cap \text{bdd}_\lambda^{\text{u}}(a_E c_G) = \text{bdd}_\lambda^{\text{u}}(a_E),$$

where  $\lambda = |ab| + |T|$ .

*Proof.* Let  $\lambda = |ab| + |T|$ . If  $b_F \downarrow_{a_E}^{\text{bu}} c_G$ , then  $\text{bdd}_\lambda^{\text{u}}(a_E b_F) \cap \text{bdd}_\lambda^{\text{u}}(a_E c_G) = \text{bdd}_\lambda^{\text{u}}(a_E)$ .

Conversely, assume that  $b_F \not\downarrow_{a_E}^{\text{bu}} c_G$ . There is some

$$d_H \in (\text{bdd}^{\text{u}}(a_E b_F) \cap \text{bdd}^{\text{u}}(a_E c_G)) \setminus \text{bdd}^{\text{u}}(a_E).$$

By Corollary 1.8, there is  $e_I$  of arity at most  $\lambda$  such that  $d_H$  and  $e_I$  are interdefinable. This means that  $e_I \in (\text{bdd}_\lambda^{\text{u}}(a_E b_F) \cap \text{bdd}_\lambda^{\text{u}}(a_E c_G)) \setminus \text{bdd}_\lambda^{\text{u}}(a_E)$ . Therefore  $\text{bdd}_\lambda^{\text{u}}(a_E b_F) \cap \text{bdd}_\lambda^{\text{u}}(a_E c_G) \neq \text{bdd}_\lambda^{\text{u}}(a_E)$ .  $\square$

The following characterization of  $\downarrow^{\text{bu}}$  (and the manner of proof) is essentially due to Wagner [2015].

**Proposition 2.4.** *For any ultrimaginaries  $a_E, b_F$ , and  $c_G$ , the following are equivalent.*

- (1)  $b_F \downarrow_{a_E}^{\text{bu}} c_G$ .
- (2) For any  $b'_F \equiv_{a_E}^L b_F$ , there are  $b^0, c^0, b^1, c^1, \dots, c^{n-1}, b^n$  such that  $b^0 = b$ ,  $c^0 = c$ ,  $b^n = b'$ , and for each  $i < n$ ,  $b^i \equiv_{a_E c_G}^L b^{i+1}$  and  $c^i \equiv_{a_E b_F}^L c^{i+1}$  if  $i < n - 1$ .
- (3)  $\langle \text{Autf}(\mathbb{M}/a_E b_F) \cup \text{Autf}(\mathbb{M}/a_E c_G) \rangle = \text{Autf}(\mathbb{M}/a_E)$ .

*Proof.* Let  $H = \langle \text{Autf}(\mathbb{M}/a_E b_F) \cup \text{Autf}(\mathbb{M}/a_E c_G) \rangle$ .

$\neg(3) \Rightarrow \neg(1)$ . Assume  $H \neq \text{Autf}(\mathbb{M}/a_E)$ , which implies that  $H < \text{Autf}(\mathbb{M}/a_E) = \text{Autf}(\mathbb{M}/a_E)_f$ . By Proposition 1.12, we have that  $\llbracket H \rrbracket \notin \text{bdd}^{\text{u}}[\llbracket \text{Autf}(\mathbb{M}/a_E) \rrbracket] = \text{bdd}^{\text{u}}(a_E)$ . But since  $\text{Autf}(\mathbb{M}/a_E b_F) = \text{Autf}(\mathbb{M}/a_E b_F)_f \leq H$  and  $\text{Autf}(\mathbb{M}/a_E c_G) = \text{Autf}(\mathbb{M}/a_E c_G)_f \leq H$ , we have that  $\llbracket H \rrbracket \in \text{bdd}^{\text{u}}(a_E b_F) \cap \text{bdd}^{\text{u}}(a_E c_G)$  again by Proposition 1.12.

(3) $\Rightarrow$ (1). Suppose  $H = \text{Autf}(\mathbb{M}/a_E)$ . Fix an ultrimaginary  $d_I \in \text{bdd}^u(a_E b_F) \cap \text{bdd}^u(a_E c_G)$ . By Proposition 1.12, we have that  $H \leq \text{Autf}(\mathbb{M}/a_E d_I) \leq \text{Autf}(\mathbb{M}/a_E)$ , which implies that  $\text{Autf}(\mathbb{M}/a_E d_I) = H$ . Hence by Proposition 1.12,  $d_I \in \text{bdd}^u(a_E)$ . Since we can do this for any such ultrimaginary, we have that  $b_F \downarrow_{a_E}^{\text{bu}} c_G$ .

(1) $\Rightarrow$ (2). Let  $b_{F^*}^* = \llbracket \text{Autf}(\mathbb{M}/a_E b_F) \rrbracket$  and  $c_{G^*}^* = \llbracket \text{Autf}(\mathbb{M}/a_E c_G) \rrbracket$ . Note that  $\text{bdd}^u(a_E b_F) = \text{dcl}^u(b_{F^*}^*)$  and  $\text{bdd}^u(a_E c_G) = \text{dcl}^u(c_{G^*}^*)$  (by Definition 1.9 and Proposition 1.12). In particular, we have that  $\text{dcl}^u(b_{F^*}^*) \cap \text{dcl}^u(c_{G^*}^*) = \text{bdd}^u(a_E)$ . Fix  $b'_F \equiv_{a_E}^L b_F$ . By passing to a different representative of the  $F$ -equivalence class  $b'_F$ , we may assume that  $b' \equiv_{a_E}^L b$ . Fix  $c'$  such that  $bc \equiv_{a_E}^L b'c'$ . By Proposition 1.13, we have that  $b'c' \equiv_{\text{bdd}^u(a_E)} bc$  for all  $\lambda$ , so  $b'c' \equiv_{\text{dcl}^u_\lambda(b_{F^*}^*) \cap \text{dcl}^u_\lambda(c_{G^*}^*)} bc$  for all  $\lambda$ . Therefore, by Proposition 1.10, we can find a sequence  $(b^{*i} c^{*i} b^i c^i)_{i \leq n}$  such that  $b^{*0} = b^*$ ,  $c^{*0} = c^*$ ,  $b^0 c^0 = bc$ ,  $b^n c^n = b'c'$ , and for each  $i < n$ ,

- if  $i$  is even,  $b^{*i} = b^{*i+1}$  and  $c^{*i} b^i c^i \equiv_{b^{*i}} c^{*i+1} b^{i+1} c^{i+1}$  and
- if  $i$  is odd,  $c^{*i} = c^{*i+1}$  and  $b^{*i} b^i c^i \equiv_{c^{*i}} b^{*i+1} b^{i+1} c^{i+1}$ .

This implies, by induction, that  $b^i c^i \equiv_{a_E b_F}^L b^{i+1} c^{i+1}$  and  $b_F^i = b_F^{i+1}$  for each even  $i$  and  $b^i c^i \equiv_{a_E c_G}^L b^{i+1} c^{i+1}$  and  $c_G^i = c_G^{i+1}$  for each odd  $i$ , so  $b^0 c^1 b^2 c^3 \dots c^{n-1} b^n$  is the sequence required by the proposition (after reindexing).

(2) $\Rightarrow$ (1). Assume (2), but also assume for the sake of contradiction that (1) fails. Let  $d_H$  be an element of  $(\text{bdd}^u(a_E b_F) \cap \text{bdd}^u(a_E c_G)) \setminus \text{bdd}^u(a_E)$ . Since  $d_H$  is not bounded over  $a_E$ , there must be some  $d'_H \equiv_{a_E}^L d_H$  such that  $d'_H \notin \text{bdd}^u(a_E b_E) \cap \text{bdd}^u(a_E c_G)$ . Find  $b'_F$  such that  $b_F d'_H \equiv_{a_E}^L b'_F d'_H$ . Let  $b^0, c^0, b^1, c^1, \dots, c^{n-1}, b^n$  be as in (2), with  $b^n = b'$ . Find  $d^{1/2}, d^1, d^{3/2}, d^2, \dots, d^{n-(1/2)}, d^n$  such that  $d^{1/2} = d$  and for each  $i < n$ ,

- $b_F^i d_H^{i+(1/2)} \equiv_{a_E c_G}^L b_F^{i+1} d^{i+1}$  and
- $c_G^i d_H^{i+1} \equiv_{a_E b_F^{i+1}}^L c_G^{i+1} d_H^{i+(3/2)}$  if  $i < n-1$ .

We now have that  $b'_F d'_H \equiv_{a_E}^L b_F d_H \equiv_{a_E}^L b'_F d_H^n$ , so in particular,  $d'_H \equiv_{a_E b'_F}^L d_H^n$ . For some  $i < n$ , consider  $e_I \in \text{bdd}^u(a_E b_F^i) \cap \text{bdd}^u(a_E c_G^i)$ . Since  $e_I \in \text{bdd}^u(a_E c_G^i)$  and since  $b_F^i d_H^{i+(1/2)} \equiv_{\text{bdd}^u_\lambda(a_E c_G^i)} b_F^{i+1} d^{i+1}$  for all  $\lambda$  (by Proposition 1.13), we must have that  $b_F^i d_H^{i+(1/2)} \equiv_{a_E e_I} b_F^{i+1} d^{i+1}$  and so  $e_I \in \text{bdd}^u(a_E b_F^{i+1})$  as well. By the reverse argument and since we can do this for any such ultrimaginary, we get that

$$\text{bdd}^u(a_E b_F^i) \cap \text{bdd}^u(a_E c_G^i) = \text{bdd}^u(a_E b_F^{i+1}) \cap \text{bdd}^u(a_E c_G^i).$$

Likewise, for any  $i < n-1$ , we get

$$\text{bdd}^u(a_E b_F^{i+1}) \cap \text{bdd}^u(a_E c_G^i) = \text{bdd}^u(a_E b_F^{i+1}) \cap \text{bdd}^u(a_E c_G^{i+1}).$$

Therefore  $d_H^n \in \text{bdd}^u(a_E b_F^n) \cap \text{bdd}^u(a_E c_G^{n-1})$ , so since  $d_H^n \equiv_{a_E b_F^n}^L d'_H$  and so  $d_H^n \equiv_{\text{bdd}_\lambda^u(a_E b_F^n)} d'_H$  for every  $\lambda$  (by Proposition 1.13), we must also have

$$d'_H \in \text{bdd}^u(a_E b_F^n) \cap \text{bdd}^u(a_E c_G^{n-1}) = \text{bdd}^u(a_E b_F) \cap \text{bdd}^u(a_E c_G),$$

which is a contradiction.  $\square$

### 3. Full existence

We will use the tree bookkeeping machinery from [Kaplan and Ramsey 2020], with some minor extensions (the notation  $\mathcal{T}_\alpha^*$  and  $\mathcal{F}_\alpha$ ).

**Definition 3.1.** For any ordinal  $\alpha$ ,  $\mathcal{L}_{s,\alpha}$  is the language

$$\{\trianglelefteq, \wedge, <_{\text{lex}}, P_0, P_1, \dots, P_\beta (\beta < \alpha), \dots\},$$

with  $\trianglelefteq$  and  $<_{\text{lex}}$  binary relations,  $\wedge$  a binary function, and each  $P_\beta$  a unary relation.

For any ordinal  $\alpha$ , we write  $\mathcal{T}_\alpha^*$  for the set of functions  $f$  with codomain  $\omega$  and finite support such that  $\text{dom}(f)$  is an end segment of  $\alpha$ . (For the sake of some minor edge cases, we will regard the empty functions in various  $\mathcal{T}_\alpha^*$ 's as distinct objects.) We write  $\mathcal{T}_\alpha$  for the set of functions  $f \in \mathcal{T}_\alpha^*$  with  $\text{dom}(f) = [\beta, \alpha)$  for a nonlimit ordinal  $\beta$ . We write  $\mathcal{F}_{\alpha+1}$  (for forest) for  $\mathcal{T}_{\alpha+1} \setminus \{\emptyset\}$ .

We interpret  $\mathcal{T}_\alpha^*$  and  $\mathcal{T}_\alpha$  as  $\mathcal{L}_{s,\alpha}$ -structures by

- $f \trianglelefteq g$  if and only if  $f \subseteq g$ ;
- $f \wedge g = f \upharpoonright [\beta, \alpha) = g \upharpoonright [\beta, \alpha)$ , where  $\beta = \min\{\gamma : f \upharpoonright [\gamma, \alpha) = g \upharpoonright [\gamma, \alpha)\}$  (with the understanding that  $\min \emptyset = \alpha$ );
- $f <_{\text{lex}} g$  if and only if either  $f \triangleleft g$  or  $f$  and  $g$  are  $\trianglelefteq$ -incomparable,  $\text{dom}(f \wedge g) = [\gamma, \alpha)$ , and  $f(\gamma) < g(\gamma)$ ; and
- $P_\beta(f)$  holds if and only if  $\text{dom}(f) = [\beta, \alpha)$ .

We write  $\langle i \rangle_\alpha$  for the function  $\{(\alpha, i)\}$  (which is an element of  $\mathcal{T}_{\alpha+1}^*$ ). Given  $i < \omega$  and  $f \in \mathcal{T}_\alpha^*$  with  $\text{dom}(f) = [\beta + 1, \alpha)$ , we write  $f \frown i$  to mean the function  $f \cup \{(\beta, i)\}$  (which is an element of  $\mathcal{T}_\alpha^*$ ). Given  $i < \omega$  and  $f \in \mathcal{T}_\alpha^*$ , we write  $i \frown f$  to mean the function  $\{(\alpha, i)\} \cup f$  (which is an element of  $\mathcal{T}_{\alpha+1}^*$ ).<sup>9</sup>

For  $\alpha < \beta$ , we define the canonical inclusion map  $\iota_{\alpha\beta} : \mathcal{T}_\alpha \rightarrow \mathcal{T}_\beta$  by  $\iota_{\alpha\beta}(f) = f \cup \{(\gamma, 0) : \gamma \in \beta \setminus \alpha\}$ . (Note that  $\iota_{\alpha, \alpha+1}(f) = 0 \frown f$ .)

For  $\beta \leq \alpha$ , we write  $\zeta_\beta^\alpha$  for the function whose domain is  $[\beta, \alpha)$  with the property that  $\zeta_\beta^\alpha(\gamma) = 0$  for all  $\gamma \in [\beta, \alpha)$ . (Note that  $\zeta_\alpha^\alpha$  is  $\mathcal{T}_\alpha$ 's copy of the empty function.)

Given a family  $(b_f)_{f \in X}$ , we may refer to it briefly as  $b \in X$ .

<sup>9</sup>Note that this notation is not ambiguous when  $f$  is an empty function, as we are regarding the empty functions in different  $\mathcal{T}_\alpha^*$ 's as distinct objects.

**Definition 3.2.** Given  $X \subseteq \mathcal{T}_\alpha^*$ , we say that a family  $(b_f)_{f \in X}$  is *s-indiscernible over A* if for any tuples  $f_0 \dots f_{n-1}$  and  $g_0 \dots g_{n-1}$  in  $X$  with  $f_0 \dots f_{n-1} \equiv^{\text{qf}} g_0 \dots g_{n-1}$ ,  $b_{f_0} \dots b_{f_{n-1}} \equiv_A b_{g_0} \dots b_{g_{n-1}}$ , where quantifier-free type is in the language  $\mathcal{L}_{s,\alpha}$ . (Note that this does not entail that  $b_f$ 's on different levels are tuples of the same sort.)

Given  $f \in \mathcal{T}_\alpha$ , we write  $b_{\triangleright f}$  to refer to some fixed enumeration of the set  $\{b_g : g \in \mathcal{T}_\alpha, f \leq g\}$ . In particular, we choose this enumeration in a uniform way so that if  $(b_f)_{f \in \mathcal{T}_\alpha}$  is *s-indiscernible over A*, then for any  $f$  with domain  $[\beta + 1, \alpha)$ , the sequence  $(b_{\triangleright f \smallfrown i})_{i < \omega}$  is *A-indiscernible*. When  $f$  is an element of  $\mathcal{T}_\alpha^*$ , we will also write  $b_{\triangleright f}$  for some fixed enumeration of the set  $\{b_g : g \in \mathcal{T}_\alpha, f \subseteq g\}$ . One particular example of this will be sequences of the form  $(b_{\triangleright \zeta_{\beta+1}^\alpha \smallfrown i})_{i < \omega}$ , where  $\beta$  is a limit ordinal. This is essentially the only situation in which we need to consider  $\mathcal{T}_\alpha^*$ .

Note that for a limit ordinal  $\alpha$ ,  $(b_f)_{f \in \mathcal{T}_\alpha}$  is *s-indiscernible over A* if and only if  $(b_f)_{f \in \iota_{\beta,\alpha}(\mathcal{T}_\beta)}$  is *s-indiscernible over A* for every  $\beta < \alpha$ .

We will also need the following fact.

**Fact 3.3** (modeling property for *s-indiscernibles* [Kim et al. 2014, Theorem 4.3]). *Let  $X$  be  $\mathcal{T}_\alpha$  or  $\mathcal{F}_{\alpha+1}$ . For any  $(b_f)_{f \in X}$  and any set  $A$  of **hyperimaginaries**, there is a family of tuples  $(c_f)_{f \in X}$  that is *s-indiscernible over A* and **locally based on**  $b_{\in X}$  (i.e., for any finite tuple  $f_0 \dots f_{n-1}$  from  $X$  and any neighborhood  $U$  of  $\text{tp}(c_{f_0} \dots c_{f_{n-1}}/A)$  (in the appropriate type space), there is a tuple  $g_0 \dots g_{n-1}$  from  $X$  such that  $f_0 \dots f_{n-1} \equiv^{\text{qf}} g_0 \dots g_{n-1}$  and  $\text{tp}(b_{g_0} \dots b_{g_{n-1}}/A) \in U$ ).*

Note that while Fact 3.3 is normally formulated for discrete logic, the corresponding statement in continuous logic follows easily from a very soft general argument: Given a metric structure  $M$  and a tree  $(b_f)_{f \in X}$  of elements of  $M$ , find  $\alpha$  large enough that  $M$ ,  $\text{Th}(M)$ , and  $b_{\in X}$  are elements of  $V_\alpha$  and apply [Kim et al. 2014, Theorem 4.3] to  $V_\alpha$  as a discrete structure and get some *A-s-indiscernible* family  $(c_f^*)_{f \in X}$  of elements of an elementary extension  $V_\alpha^* \geq V_\alpha$ . These elements live inside a structure  $M^* \in V_\alpha^*$  that is internally a model of  $\text{Th}(M)$ . By taking the standard parts of each real-valued predicate in  $M^*$  and then completing with regards to the metric, we get a metric structure  $N$  that is an elementary extension of  $M$ . For each  $f \in X$ , let  $c_f$  be the image in  $N$  of  $c_f^*$  under the canonical map from  $M^*$  to  $N$ . It is straightforward to check that  $(c_f)_{f \in X}$  is the required *A-s-indiscernible* family.

Before proving full existence for  $\downarrow^{\text{bu}}$ , we will need a lemma.

**Lemma 3.4.** *Fix  $\alpha$  and  $\gamma > \alpha$ . Let  $(e_f)_{f \in \mathcal{F}_{\gamma+1}}$  be an *s-indiscernible* family of real tuples over a set  $A$  of **hyperimaginary** parameters. Let  $\lambda = |Ae_{\triangleright \zeta_\alpha^\gamma}| + |T|$ . Suppose that there is an ultrimaginary  $c_F$  such that  $c_F \in \text{bdd}_\lambda^u(Ae_{\triangleright \zeta_\alpha^\gamma}) \cap \text{bdd}_\lambda^u(Ae_{\triangleright 1 \smallfrown \zeta_\alpha^\gamma})$ . Then there is a model  $M$  with  $Ac_F \subseteq \text{dcl}^u(M)$  and  $|M| \leq \lambda$  such that  $(e_f)_{f \in \mathcal{F}_{\gamma+1}}$  is *s-indiscernible over M*.*

*Proof.* By Fact 3.3, we can find a set of real parameters  $B$  such that  $|B| \leq |A| + |T|$ ,  $A \subseteq \text{bdd}^{\text{hd}}(B)$ , and  $(e_f)_{f \in \mathcal{F}_{\gamma+1}}$  is  $s$ -indiscernible over  $B$ .

Let  $T'$  be a Skolemization of  $T$  with  $|T'| = |T|$ . Let  $\mathbb{M}'$  be the monster model of  $T'$ , which we may think of as an expansion of  $\mathbb{M}$ . By Fact 3.3, we can find  $(b'_f)_{f \in \mathcal{F}_{\gamma+1}}$  locally based on  $(e_f)_{f \in \mathcal{F}_{\gamma+1}}$  which is  $s$ -indiscernible over  $B$  (in  $T'$ ). By considering an automorphism of  $\mathbb{M}$  (in  $T$ ), we may assume that  $(b'_f)_{f \in \mathcal{F}_{\gamma+1}}$  actually is  $(e_f)_{f \in \mathcal{F}_{\gamma+1}}$ , so that  $(e_f)_{f \in \mathcal{F}_{\gamma+1}}$  is  $s$ -indiscernible over  $B$  (in  $T'$ ).

Find an automorphism  $\sigma \in \text{Aut}(\mathbb{M}'/B)$  satisfying  $\sigma \cdot e_{\geq(i+1)\gamma+1} = e_{\geq(i)\gamma+1}$  for every  $i < \omega$ . Let  $M$  be the Skolem hull of  $B \cup \sigma \cdot e_{\geq \zeta_\alpha^{\gamma+1}}$ . Note that  $(e_f)_{f \in \mathcal{F}_{\gamma+1}}$  is  $s$ -indiscernible over  $M$  (and therefore the same is true in  $T$ ). Furthermore, note that  $|M| \leq \lambda$ .

Let  $M_i$  be the Skolem hull of  $B e_{\geq i \sim \zeta_\alpha^\gamma}$  for both  $i \in \{0, 1\}$ . Note that  $c_F \in \text{bdd}_\lambda^{\text{u}}(M_1)$  and  $|M_1| \leq \lambda$ . Pass back to the theory  $T$ . Note that  $M$ ,  $M_0$ , and  $M_1$  are still models of  $T$ . By Corollary 1.8, there is an invariant equivalence relation  $G$  (in  $T$ ) such that  $c_F$  and  $[M_1]_G$  are interdefinable. Therefore we have that  $[M_1]_G \in \text{bdd}_\lambda^{\text{u}}(A e_{\geq \zeta_\alpha^{\gamma+1}}) \subseteq \text{bdd}^{\text{u}}(M_0) = \text{dcl}^{\text{u}}(M_0)$ . Find an automorphism  $\tau \in \text{Aut}(\mathbb{M}/M_1)$  such that  $\tau(M_0) = M$  (which exists by indiscernibility).  $\tau$  witnesses that  $[M_1]_G \in \text{dcl}^{\text{u}}(M)$  and therefore  $c_F \in \text{dcl}^{\text{u}}(M)$ , so  $M$  is the required model.  $\square$

Now we are ready to prove full existence for  $\downarrow^{\text{bu}}$ , but we will take the opportunity to prove a certain technical strengthening which we will need later in the construction of  $\downarrow^{\text{bu}}$ -Morley trees.

**Lemma 3.5.** *If  $(b_f)_{f \in \mathcal{T}_\alpha}$  is a tree of real elements that is  $s$ -indiscernible over a set of hyperimaginaries  $A$ , then there is a  $\gamma > \alpha$  and a tree  $(e_f)_{f \in \mathcal{T}_{\gamma+1}}$  such that*

- $e_{\in \mathcal{T}_{\gamma+1}}$  is  $s$ -indiscernible over  $A$ ,
- for each  $f \in \mathcal{T}_\alpha$ ,  $b_f = e_{\iota_{\alpha, \gamma+1}(f)}$ , and
- $e_{\geq \zeta_\alpha^{\gamma+1}} \downarrow_A^{\text{bu}} e_{\geq 1 \sim \zeta_\alpha^\gamma}$ .

(Note that  $e_{\geq \zeta_\alpha^{\gamma+1}}$  is the original tree.)

*Proof.* If  $b_{\in \mathcal{T}_\alpha} \in \text{acl}(A)$ , then the statement is trivial, so assume that  $b_{\in \mathcal{T}_\alpha} \notin \text{acl}(A)$ .

Fix  $\lambda = |Ab_{\in \mathcal{T}_\alpha}| + |T|$ . By Proposition 2.3, we have that  $b_{\in \mathcal{T}_\alpha} \downarrow_A^{\text{bu}} c$  if and only if  $\text{bdd}_\lambda^{\text{u}}(Ab_{\in \mathcal{T}_\alpha}) \cap \text{bdd}_\lambda^{\text{u}}(Ac) = \text{bdd}_\lambda^{\text{u}}(A)$  for any  $c$ . Let  $\mu = |\text{bdd}_\lambda^{\text{u}}(Ab_{\in \mathcal{T}_\alpha}) \setminus \text{bdd}_\lambda^{\text{u}}(A)|^+$ .

We will build a family  $(e_f : f \in \iota_{\gamma+1, \mu}(\mathcal{T}_{\gamma+1}))$  inductively, where  $\gamma$  is some successor ordinal less than  $\mu$ . By an abuse of notation, we will systematically conflate the sets  $\iota_{\alpha, \mu}(\mathcal{T}_\alpha)$  and  $\mathcal{T}_\alpha$  (and likewise for  $\iota_{\alpha, \mu}(\mathcal{F}_{\alpha+1})$  and  $\mathcal{F}_{\alpha+1}$ ) for all  $\alpha < \mu$ . Note that in general this will mean that  $e_{\geq \zeta_\beta^\mu}$  is the same thing as  $e_{\in \mathcal{T}_\beta}$ .

Let  $e_f = b_f$  for all  $f \in \mathcal{T}_\alpha$ . Since  $b_{\in \mathcal{T}_\alpha} \notin \text{acl}(A)$ , we can find a family  $(d_f)_{f \in \mathcal{F}_{\alpha+1}}$  extending  $e_{\in \mathcal{T}_\alpha}$  such that  $(d_{\geq \zeta_{\alpha+1}^\mu \sim i})_{i < \omega}$  is a nonconstant  $A$ -indiscernible sequence. By Fact 3.3, we can define  $e_f$  for all  $f \in \mathcal{F}_{\alpha+1}$  in such a way that the family

$e_{\in \mathcal{F}_{\alpha+1}}$  is locally based on  $d_{\in \mathcal{F}_{\alpha+1}}$ . In particular,  $(e_{\geq \zeta_{\alpha+1}^\mu \frown i})_{i < \omega}$  will be a nonconstant  $A$ -indiscernible sequence.

At successor stage  $\beta + 1 \geq \alpha$ , assume that we have defined  $e_f$  for all  $f \in \mathcal{F}_{\beta+1}$  and that the family  $(e_f)_{f \in \mathcal{F}_{\beta+1}}$  is  $s$ -indiscernible over  $A$ . If there is no  $d_E \in \text{bdd}_\lambda^u(Ab_{\in \mathcal{T}_\alpha}) \setminus \text{bdd}_\lambda^u(A)$  such that the family  $(e_f)_{f \in \mathcal{F}_{\beta+1}}$  is  $s$ -indiscernible over  $Ad$ , let  $e_{\zeta_{\beta+1}^\mu} = \emptyset$  and  $\gamma = \beta$  and halt the construction. Otherwise, let  $e_{\zeta_{\beta+1}^\mu} = d$ . For later reference, let  $E_{\beta+1}$  be  $E$ . Note that the family  $e_{\in \mathcal{T}_{\beta+1}}$  is  $s$ -indiscernible over  $A$ . Since  $d_E \notin \text{bdd}_\lambda^u(A)$ , we can find, by Proposition 1.4, a sequence  $(\sigma_i)_{i < \omega}$  of elements of  $\text{Aut}(\mathbb{M}/A)$  such that  $(\sigma_i \cdot d)_{i < \omega}$  is an  $A$ -indiscernible sequence satisfying  $(\sigma_i \cdot d) \dot{\equiv}_{\beta+1} (\sigma_j \cdot d)$  for any  $i < j < \omega$ . Now choose  $(e_f)_{f \in \mathcal{F}_{\beta+2}}$  in such a way that  $e_{\in \mathcal{F}_{\beta+2}}$  extends what was already defined, is  $s$ -indiscernible over  $A$ , and is locally based on the family  $(c_f)_{f \in \mathcal{F}_{\beta+2}}$  defined by  $c_i \frown f = \sigma_i \cdot e_f$  for all  $f \in \mathcal{T}_{\beta+1}$  (which is possible by Fact 3.3). In particular, note that for any  $i < j < \omega$ , we still have that  $(e_{\zeta_{\beta+2}^\mu \frown i}, e_{\zeta_{\beta+2}^\mu \frown j}) \equiv_A (\sigma_0 \cdot d, \sigma_1 \cdot d)$  and so, in particular,  $e_{\zeta_{\beta+2}^\mu \frown i} \dot{\equiv}_{\beta+1} e_{\zeta_{\beta+2}^\mu \frown j}$ .

At limit stage  $\beta$ , we have constructed the family  $(e_f)_{f \in \mathcal{T}_\beta}$ . Note that this family is automatically  $s$ -indiscernible over  $A$ . Extend it to a family  $e_{\in \mathcal{F}_{\beta+1}}$  that is  $s$ -indiscernible over  $A$ . (This is always possible by Fact 3.3.)

**Claim.** For any  $\beta < \delta < \mu$ , if  $E_{\beta+1} = E_{\delta+1}$ , then  $e_{\zeta_{\beta+1}^\mu} \dot{\equiv}_{\beta+1} e_{\zeta_{\delta+1}^\mu}$ .

*Proof of claim.* The sequence  $(e_{\zeta_{\beta+2}^\mu \frown i})_{i < \omega}$  is  $e_{\zeta_{\delta+1}^\mu}$ -indiscernible. Since

$$e_{\zeta_{\beta+2}^\mu \frown 0} \dot{\equiv}_{\beta+1} e_{\zeta_{\beta+2}^\mu \frown 1},$$

it must be the case that  $e_{\zeta_{\delta+1}^\mu} \dot{\equiv}_{\beta+1} e_{\zeta_{\beta+2}^\mu \frown i}$  for all  $i < \omega$ . □<sub>claim</sub>

Let  $g$  be the partial function taking  $\beta$  to  $[e_{\zeta_{\beta+1}^\mu}]_{E_{\beta+1}}$ . By the claim, this is an injection into  $\text{bdd}_\lambda^u(Ab_{\in \mathcal{T}_\alpha}) \setminus \text{bdd}_\lambda^u(A)$ . By the choice of  $\mu$ ,  $g$ 's domain cannot be cofinal in  $\mu$ , so the construction must have halted at some  $\gamma < \mu$ .

Extend  $e_{\in \mathcal{T}_\gamma}$  to  $e_{\in \mathcal{F}_{\gamma+1}}$  in such a way that the resulting family is  $s$ -indiscernible over  $A$ . Set  $e_{\zeta_{\gamma+1}^\mu} = \emptyset$ .

**Claim.** For any  $c_F \in \text{bdd}_\lambda^u(Ae_{\geq \zeta_\alpha^\mu}) \setminus \text{bdd}_\lambda^u(A)$ ,  $c_F \notin \text{bdd}_\lambda^u(Ae_{\geq 1 \frown \zeta_\alpha^\gamma})$ .

*Proof of claim.* Assume there is some  $c_F \in (\text{bdd}_\lambda^u(Ae_{\geq \zeta_\alpha^\mu}) \cap \text{bdd}_\lambda^u(Ae_{\geq 1 \frown \zeta_\alpha^\gamma})) \setminus \text{bdd}_\lambda^u(A)$ . By Lemma 3.4, we can find a model  $M$  with  $Ac_F \subseteq \text{dcl}^u(M)$  and  $|M| \leq \lambda$  such that  $e_{\in \mathcal{F}_{\gamma+1}}$  is  $s$ -indiscernible over  $M$ . By Corollary 1.8, there is an invariant equivalence relation  $G$  such that  $c_F$  and  $[M]_G$  are interdefinable. But this means that we could have chosen  $[M]_G$  to be  $d_E$  at stage  $\gamma$ , contradicting the fact that the construction halted. Therefore no such  $c_F$  can exist. □<sub>claim</sub>

So, by the claim, we have that  $\text{bdd}_\lambda^u(Ae_{\geq \zeta_\alpha^\mu}) \cap \text{bdd}_\lambda^u(Ae_{\geq 1 \frown \zeta_\alpha^\gamma}) = \text{bdd}_\lambda^u(A)$ . Therefore, by the choice of  $\lambda$ ,  $e_{\geq \zeta_\alpha^\mu} \downarrow_A^{\text{bu}} e_{\geq 1 \frown \zeta_\alpha^\gamma}$ , as required. □

**Theorem 3.6** (full existence). *For any set of hyperimaginaries  $A$  and real tuples  $b$  and  $c$ , there is  $b' \equiv_A^L b$  such that  $b' \downarrow_A^{\text{bu}} c$ .<sup>10</sup>*

*Proof.* It is sufficient to show this in the special case that  $b = c$ . Specifically, given  $d$  and  $e$ , if we can find  $d'e' \equiv_A de$  such that  $d'e' \downarrow_A^{\text{bu}} de$ , then we have  $d' \downarrow_A^{\text{bu}} e$  by monotonicity. So fix a set of hyperimaginaries  $A$  and a real tuple  $b$ . Let  $B$  be a set containing realizations of all Lascar strong types extending  $\text{tp}(b/A)$ . We can now apply Lemma 3.5 to the family  $(B_f)_{f \in \mathcal{T}_0}$  with  $B_\emptyset = B$  to get a family  $(E_f)_{f \in \mathcal{T}_{\gamma+1}}$  such that  $E_{\sigma_0^{\gamma+1}} = B$  for some  $f \in \mathcal{T}_{\gamma+1}$ ,  $B \equiv_A B_f$ , and  $B \downarrow_A^{\text{bu}} B_f$ . Let  $\sigma$  be an automorphism fixing  $A$  taking  $B_f$  to  $B$ . Let  $B' = \sigma \cdot B$ .  $B'$  still contains realizations of all Lascar strong types extending  $\text{tp}(b/A)$ , so we can find  $b' \in B'$  with  $b' \equiv_A^L b$ , which is the required element.  $\square$

**Corollary 3.7.** *For any set of hyperimaginaries  $A$  and any ultraimaginaries  $b_E$  and  $c_F$ , there is  $b'_E \equiv_A^L b_E$  such that  $b'_E \downarrow_A^{\text{bu}} c_F$ .*

*Proof.* Apply Theorem 3.6 to  $b$  and  $c$  to get  $b' \equiv_A^L b$  such that  $b' \downarrow_A^{\text{bu}} c$ . We then have that  $\text{bdd}^u(b') \downarrow_A^{\text{bu}} \text{bdd}^u(c)$ , so by monotonicity,  $b'_E \downarrow_A^{\text{bu}} c_F$ .  $\square$

**Corollary 3.8** (extension). *For any set of hyperimaginaries  $A$  and any ultraimaginaries  $b_E$ ,  $c_F$ , and  $d_G$ , if  $b_E \downarrow_A^{\text{bu}} c_F$ , then there is  $b'_E \equiv_{Ac_F}^L b_E$  such that  $b'_E \downarrow_A^{\text{bu}} c_F d_G$ .*

*Proof.* By Corollary 3.7, we can find  $b'_E \equiv_{Ac_F}^L b$  such that  $b'_E \downarrow_{Ac_F}^{\text{bu}} d_G$ . By symmetry and transitivity, we have that  $b'_E \downarrow_A^{\text{bu}} c_F d_G$ .  $\square$

Compactness is essential in the proof of Fact 3.3 and therefore also Theorem 3.6, which raises the following question.

**Question 3.9.** *Does Theorem 3.6 hold when  $A$  is a set of ultraimaginaries?*

#### 4. Total $\downarrow^{\text{bu}}$ -Morley sequences

**Definition 4.1.** A  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  is an  $A$ -indiscernible sequence  $(b_i)_{i < \omega}$  such that  $b_i \downarrow_A^{\text{bu}} b_{<i}$  for each  $i < \omega$ .

A weakly total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  is an  $A$ -indiscernible sequence  $(b_i)_{i < \omega}$  such that for any finite  $I$  and any  $J$  (of any order type), if  $I + J \equiv_A^{\text{EM}} b_{<\omega}$ , then  $I \downarrow_A^{\text{bu}} J$ .<sup>11</sup>

<sup>10</sup>Anand Pillay has pointed out to us that Theorem 3.6 also follows from Theorem 615 of [Lascar 1982] (together with Wagner's characterization  $\downarrow^{\text{bu}}$  from [Wagner 2015] given in our Proposition 2.4). Theorem 615 is stated for countable sets of parameters in a countable theory, but it is clear that the proof generalizes to the uncountable case as well.

<sup>11</sup>Note that if we modified this definition to allow  $I$  to be any order type and require that  $J$  be finite, the resulting sequences would be precisely the order-reversals of the weakly total  $\downarrow^{\text{bu}}$ -Morley sequences as we have defined the term here (by symmetry of  $\downarrow^{\text{bu}}$ ).

A total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  is an  $A$ -indiscernible sequence  $(b_i)_{i < \omega}$  such that for any  $I$  and  $J$  (of any order type), if  $I + J \equiv_A^{\text{EM}} b_{< \omega}$ , then  $I \downarrow_A^{\text{bu}} J$ .

We could write down stronger and weaker forms of the  $\downarrow^{\text{bu}}$ -Morley condition, but we are really only interested in total  $\downarrow^{\text{bu}}$ -Morley sequences, as they seem to be a fairly robust class (see Theorem 4.8). Weakly total  $\downarrow^{\text{bu}}$ -Morley sequences seem to be the best we can get without large cardinals, however, which does raise the following question.

**Question 4.2.** *Is every weakly total  $\downarrow^{\text{bu}}$ -Morley sequence a total  $\downarrow^{\text{bu}}$ -Morley sequence?*

One immediate property of total  $\downarrow^{\text{bu}}$ -Morley sequences is that they act as universal witnesses of the relation  $\equiv_A^{\text{L}}$  in a strong way.

**Proposition 4.3.** *For any  $A$  and  $b$ , if there is a total  $\downarrow^{\text{bu}}$ -Morley sequence  $(b_i)_{i < \omega}$  over  $A$  with  $b_0 = b$ , then for any  $b'$ ,  $b' \equiv_A^{\text{L}} b$  if and only if there are  $I_0, J_0, I_1, \dots, J_{n-1}, I_n$  such that  $b \in I_0$ ,  $b' \in I_n$ , and, for each  $i < n$ ,  $I_i + J_i$  and  $I_{i+1} + J_i$  are both  $A$ -indiscernible and have the same EM-type as  $b_{< \omega}$ .*

*Proof.* Let  $I = (b_i)_{i < \omega}$ . We only need to prove that if  $b' \equiv_A^{\text{L}} b$ , then the required configuration exists (as the required configuration is clearly sufficient to witness that  $b' \equiv_A^{\text{L}} b$ ). Choose  $I'$  so that  $bI \equiv_A^{\text{L}} b'I'$ . Extend  $I$  to an  $A$ -indiscernible sequence  $I + J$  with  $I \equiv_A J$ . By assumption  $I \downarrow_A^{\text{bu}} J$ , so by Proposition 2.4, there are  $I_0, J_0, I_1, J_1, \dots, J_{n-1}, I_n$  such that  $I_0 = I$ ,  $J_0 = J$ ,  $I_n = I'$ , and for each  $i < n$ ,  $I_i \equiv_{A J_i}^{\text{L}} I_{i+1}$  and  $J_i \equiv_{A I_{i+1}}^{\text{L}} J_{i+1}$  if  $i < n$ . Since  $I_0 + J_0$  is  $A$ -indiscernible, we can show by induction that  $I_i + J_i$  and  $I_{i+1} + J_i$  are both  $A$ -indiscernible and have the same EM-type as  $I_0 = b_{< \omega}$ .  $\square$

A similar statement is true for weakly total  $\downarrow^{\text{bu}}$ -Morley sequences, which we will state in Corollary 4.18 after we have shown that weakly total  $\downarrow^{\text{bu}}$ -Morley sequences always exist without set-theoretic hypotheses.

#### **Characterization of total $\downarrow^{\text{bu}}$ -Morley sequences.**

**Definition 4.4.** For any set of parameters  $A$ , we write  $\approx_A$  for the transitive closure of the relation  $I \sim_A J$  that holds if and only if  $I$  and  $J$  are both infinite  $A$ -indiscernible sequences (of real or hyperimaginary elements) and either  $I + J$  or  $J + I$  is an  $A$ -indiscernible sequence.

By an abuse of notation, we write  $[I]_{\approx_A}$  for the ultrimaginary  $[AI]_E$ , where  $E$  is the equivalence relation on tuples of the same length as  $AI$  such that  $E(AI, BJ)$  holds if and only if  $A = B$  in our fixed enumeration and  $I \approx_A J$ .

Note that we do not in general require that  $I$  and  $J$  have the same order type. Also note that  $\approx_A$  is reflexive: For any infinite  $A$ -indiscernible sequence  $I$ , we can find an infinite sequence  $J$  such that  $I + J$  is also  $A$ -indiscernible. Then  $I \sim_A J \sim_A I$ , so  $I \approx_A I$ .

We will additionally need an appropriate Lascar strong type generalization of Ehrenfeucht–Mostowski type.

**Definition 4.5.** Given two infinite  $A$ -indiscernible sequences  $I$  and  $J$ , we say that  $I$  and  $J$  have the same *Lascar–Ehrenfeucht–Mostowski type* (or *LEM type*) over  $A$ , written  $I \equiv_A^{\text{LEM}} J$ , if there is some  $J' \equiv_A^L J$  such that  $I + J'$  is  $A$ -indiscernible.

To see that the name is justified, note that two infinite  $A$ -indiscernible sequences  $I$  and  $J$  have the same Ehrenfeucht–Mostowski type over  $A$  if and only if there is a  $J' \equiv_A J$  such that  $I + J'$  is  $A$ -indiscernible.

**Lemma 4.6.** *For any infinite order types  $O$  and  $O'$ ,  $I \approx_A J$  if and only if there are  $K_0, L_0, K_1, \dots, L_{n-1}, K_n$  such that*

- $K_0 = I$  and  $K_n = J$ ,
- for  $0 < i < n$ ,  $K_i$  is a sequence of order type  $O$ ,
- for  $i < n$ ,  $L_i$  is a sequence of order type  $O'$ , and
- for  $i < n$ ,  $K_i + L_i$  and  $K_{i+1} + L_i$  are  $A$ -indiscernible.

*Proof.* The  $\Leftarrow$  direction is obvious.

For the  $\Rightarrow$  direction, we will proceed by induction. First assume that  $I \sim_A J$ . If  $I + J$  is  $A$ -indiscernible, then find  $L$  of order type  $O'$  such that  $I + J + L$  is  $A$ -indiscernible. We then have that  $I + L$  and  $J + L$  are  $A$ -indiscernible. If  $J + I$  is  $A$ -indiscernible, then find  $L$  of order type  $O'$  such that  $J + I + L$  is  $A$ -indiscernible. We then have that  $I + L$  and  $J + L$  are  $A$ -indiscernible.

Now assume that we know the statement holds for any  $I$  and  $J$  such that there is a sequence  $I'_0, \dots, I'_n$  with  $I'_0 = I$ ,  $I'_n = J$ , and  $I'_i \sim_A I'_{i+1}$  for each  $i < n$ . Now assume that there is a sequence  $I'_0, \dots, I'_{n+1}$  with  $I'_0 = I$ ,  $I'_{n+1} = J$ , and  $I'_i \sim_A I'_{i+1}$  for each  $i \leq n$ . Apply the induction hypothesis to get  $K_0, L_0, K_1, \dots, L_{m-1}, K_m$  satisfying the properties in the statement of the lemma with  $K_0 = I$  and  $K_m = I'_n$ . Now since  $I'_n \sim_A I'_{n+1} = J$ , we can apply the  $n = 1$  case to get  $L_m$  such that  $I'_n + L_m$  and  $I'_{n+1} + L_m$  are both  $A$ -indiscernible. By compactness, we can find  $K_m^*$  of order type  $O$  such that  $K_m^* + L_m$  and  $K_m^* + L_{m-1}$  are both  $A$ -indiscernible. We then have that  $K_0, L_0, K_1, L_1, \dots, K_{m-1}, L_{m-1}, K_m^*, L_m, K_{m+1}$  is the required sequence, where  $K_{m+1} = J$ .  $\square$

**Proposition 4.7.** *Fix a set of hyperimaginary parameters  $A$ .*

- (1)  $\equiv_A^{\text{LEM}}$  is an equivalence relation on the class of infinite  $A$ -indiscernible sequences.
- (2) If  $I$  and  $J$  have the same order type, then  $I \equiv_A^L J$  if and only if  $I \equiv_A^{\text{LEM}} J$ .
- (3) If  $I \equiv_A^{\text{LEM}} J$ , then  $I \equiv_A^{\text{EM}} J$ .
- (4) If  $I \approx_A J$ , then  $I \equiv_A^{\text{LEM}} J$ .

*Proof.* Recall the following fact: If  $I$  and  $J$  have the same order type and  $I + J$  is  $A$ -indiscernible, then  $I \equiv_A^L J$ .<sup>12</sup>

(1). First, to see that  $\equiv_A^{\text{LEM}}$  is reflexive, note that if  $I$  is an infinite  $A$ -indiscernible sequence, then any infinite  $A$ -indiscernible extension  $I + I'$  will witness that  $I \equiv_A^{\text{LEM}} I$ . To see that  $\equiv_A^{\text{LEM}}$  is symmetric, assume that  $I \equiv_A^{\text{LEM}} J$ , and let  $J'$  be as in the definition of  $\equiv_A^{\text{LEM}}$ . Find  $I'$  such that  $IJ' \equiv_A^L I'J$ . Then extend  $I' + J$  to  $I' + J + I''$ , where  $I''$  has the same order type as  $I$ . We then have that  $I'' \equiv_A^L I' \equiv_A^L I$ , so  $J \equiv_A^{\text{LEM}} I$ . To see that  $\equiv_A^{\text{LEM}}$  is transitive, assume that  $I \equiv_A^{\text{LEM}} J$  and  $J \equiv_A^{\text{LEM}} K$ . Let this be witnessed by  $J'$  and  $K'$  such that  $I + J'$  and  $J + K'$  are  $A$ -indiscernible. Find  $K''$  with the same order type as  $K$  such that  $I + J' + K''$  is  $A$ -indiscernible. Then find  $K^*$  such that  $J'K'' \equiv_A^L JK^*$ . Note that both  $J + K^*$  and  $J + K'$  are  $A$ -indiscernible. By compactness, we can find  $K^{**}$  of the same order type as  $K$  such that  $K^{**} + J + K^*$  and  $K^{**} + J + K'$  are both  $A$ -indiscernible. By the above fact, we then have that  $K^* \equiv_A^L K^{**} \equiv_A^L K'$ . Finally,  $K' \equiv_A^L K$  by assumption, so we have that  $K'' \equiv_A^L K$  and therefore that  $I \equiv_A^{\text{LEM}} K$ .

(2) is immediate from the fact. (3) is obvious.

For (4), it is sufficient to show that  $I \sim_A J \Rightarrow I \equiv_A^{\text{LEM}} J$ . This follows immediately from the fact that  $I \equiv_A^L I$  and  $J \equiv_A^L J$ .  $\square$

Now we will see that total  $\downarrow^{\text{bu}}$ -Morley sequences over  $A$  are precisely those which are “as generic as possible” in terms of  $\approx_A$  (i.e., their  $\equiv_A^{\text{LEM}}$ -equivalence class decomposes into a single  $\approx_A$ -equivalence class).

**Theorem 4.8.** *For any  $A$ -indiscernible sequence  $(b_i)_{i < \omega}$  (with  $A$  a set of hyperimaginary parameters), the following are equivalent.*

- (1)  $b_{<\omega}$  is a total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ .
- (2) There exists a pair of infinite sequences  $I$  and  $J$  (of any, possibly distinct order types) such that  $I + J \equiv_A^{\text{EM}} b_{<\omega}$  and  $I \downarrow_A^{\text{bu}} J$ .
- (3) For any  $K$ ,  $K \approx_A b_{<\omega}$  if and only if  $K \equiv_A^{\text{LEM}} b_{<\omega}$ .
- (4)  $[b_{<\omega}]_{\approx_A} \in \text{bdd}^u(A)$ .

*Proof.* (1) $\Rightarrow$ (2). This is immediate from the definition.

(2) $\Rightarrow$ (3). First note that if  $K \approx_A b_{<\omega}$ , then  $K \equiv_A^{\text{LEM}} b_{<\omega}$  by Proposition 4.7. Let  $I$  and  $J$  be as in the statement of (2). By compactness, we may find  $I' \equiv_A b_{<\omega}$  such that  $I' + I + J$  is  $A$ -indiscernible. By applying an automorphism fixing  $A$ , we

<sup>12</sup>To see this, assume that  $I$  and  $J$  have the same order type and  $I + J$  is  $A$ -indiscernible for some set of hyperimaginary parameters. Let  $M$  be a model with  $A \subseteq \text{bdd}^{\text{heq}}(M)$ . We can find an  $M$ -indiscernible sequence  $I' + J'$  finitely based on  $I + J$ . In particular, this will have  $I' + J' \equiv_A I + J$ . Therefore we can find a model  $M' \equiv_A M$  such that  $I + J$  is  $M'$ -indiscernible. We then have that  $I \equiv_{M'} J$ , whereby  $I \equiv_A^L J$ .

may assume that  $b_{<\omega} + I + J$  is  $A$ -indiscernible. Fix  $K$  such that  $K \equiv_A^{\text{LEM}} b_{<\omega}$ . By compactness, we can find a  $K' \equiv_A K$  such that  $b_{<\omega} + I + K' + J$  is  $A$ -indiscernible. We have that  $K \equiv_A^{\text{LEM}} b_{<\omega} \sim_A K'$  and therefore  $K \equiv_A^L K'$  by Proposition 4.7. Let  $a_E \in \text{bdd}^u(AI)$  be an ultrimaginary satisfying  $\text{dcl}^u(a_E) = \text{bdd}^u(AI)$ . Likewise, let  $b_F \in \text{bdd}^u(AJ)$  be an ultrimaginary satisfying  $\text{dcl}^u(b_F) = \text{bdd}^u(AJ)$ .<sup>13</sup> Since  $\text{dcl}^u(a_F) \cap \text{dcl}^u(b_F) = \text{bdd}(A)$ , we have that  $K \equiv_{\text{dcl}^u_\lambda(Ia_F) \cap \text{dcl}^u_\lambda(Jb_F)} K'$  for all  $\lambda$ . Therefore, by Proposition 1.10, we can find a sequence  $(I^i a^i J^i b^i K^i)_{i \leq n}$  satisfying that  $I^0 a^0 = Ia$ ,  $J^0 b^0 = Jb$ ,  $K^0 = K'$ ,  $K^n = K$ , and for each  $i < n$ ,

- if  $i$  is even, then  $I^i a^i = I^{i+1} a^{i+1}$  and  $J^i b^i K^i \equiv_A J^{i+1} b^{i+1} K^{i+1}$  and
- if  $i$  is odd, then  $J^i b^i = J^{i+1} b^{i+1}$  and  $I^i a^i K^i \equiv_A I^{i+1} a^{i+1} K^{i+1}$ .

By induction, we have that  $I^i + K^i + J^i$  is  $A$ -indiscernible for each  $i \leq n$ . We therefore have that

$$K' = K^0 \sim_A I^0 \sim_A J^1 \sim_A I^2 \sim_A J^3 \sim_A \dots \sim_A L \sim_A K^n = K,$$

where  $L$  is either  $I^n$  or  $J^n$ . Therefore  $K' \approx_A K$ .

(3) $\Rightarrow$ (1). Assume that for any  $K \equiv_A^{\text{LEM}} b_{<\omega}$ ,  $K \approx_A b_{<\omega}$ . Let  $I$  and  $J$  be infinite sequences satisfying  $I+J \equiv_A^{\text{EM}} b_{<\omega}$ . By applying an automorphism fixing  $A$  to  $I+J$ , we may assume that  $b_{<\omega} + I + J$  is  $A$ -indiscernible. Fix some  $I' \equiv_A^L I$ . We have that  $I' \equiv_A^{\text{LEM}} b_{<\omega}$ , which by assumption implies that  $I' \approx_A b_{<\omega}$ . Since  $b_{<\omega} \sim_A I$ , we have that  $I \approx_A I'$ . By Lemma 4.6, we can find  $K_0, L_0, K_1, L_1, \dots, L_{n-1}, K_n$  such that  $K_0 = I$ ,  $K_n = I'$ ,  $L_0$  has the same order type as  $J$ ,  $K_i$  has the same order type as  $I$  for each  $i \leq n$ ,  $L_i$  has the same order type as  $J$  for each  $i < n$ , and  $K_i + L_i$  and  $K_{i+1} + L_i$  are  $A$ -indiscernible for each  $i < n$ . Let  $K_{-1} = I$  and  $L_{-1} = J$ . We now have that for each nonnegative  $i < n$ ,  $K_{i-1} \equiv_{AL_{i-1}}^L K_i$  and  $L_{i-1} \equiv_{AK_i}^L L_i$ .<sup>14</sup> Therefore  $K_{-1}, L_{-1}, K_0, L_0, \dots, L_{n-1}, K_n$  is precisely the kind of sequence needed to apply Proposition 2.4 (with the indices shifted down by 1). Since we can do this for any  $I' \equiv_A^L I$ , we have that  $I \downarrow_A^{\text{bu}} J$ .

(3) $\Rightarrow$ (4). Let  $x$  be a tuple of variables in the same sorts as  $b_{<\omega}$ . There are at most  $2^{|Ab_{<\omega}|+|T|}$  many Lascar strong types in  $x$  over  $A$ . (3) implies therefore that there are at most  $2^{|Ab_{<\omega}|+|T|}$  many  $\approx_A$  classes with representatives that realize  $\text{tp}(b_{<\omega}/A)$ . Therefore  $[c_{<\omega}]_{\approx_A} \in \text{bdd}^u(A)$  for any  $c_{<\omega} \equiv_A b_{<\omega}$  and so a fortiori  $[b_{<\omega}]_{\approx_A} \in \text{bdd}^u(A)$ .

(4) $\Rightarrow$ (3). Let  $I \equiv_A^{\text{LEM}} b_{<\omega}$ . Find  $I'$  such that  $I \equiv_A^L I'$  and  $b_{<\omega} + I'$  is  $A$ -indiscernible. Since  $[b_{<\omega}]_{\approx_A} \in \text{bdd}^u(A)$ , we must have, by Proposition 1.4, that there are at most  $2^{|Ab_{<\omega}|+|T|}$  conjugates of  $[b_{<\omega}]_{\approx_A}$  under  $\text{Aut}(\mathbb{M}/A)$ . For any  $I'' \equiv_A I'$ , we can find

<sup>13</sup>We can take  $a_E$  to be  $\llbracket \text{Autf}(\mathbb{M}/AI) \rrbracket$  and  $b_F$  to be  $\llbracket \text{Autf}(\mathbb{M}/AJ) \rrbracket$  by Definition 1.9 and Proposition 1.12.

<sup>14</sup>For  $i = 0$ , we have that  $K_{-1} \equiv_{AL_{-1}}^L K_0$  trivially, since  $K_{-1} = I = K_0$ .

$c_{<\omega} \equiv_A b_{<\omega}$  such that  $I'' \sim_A c_{<\omega}$ . Therefore there are at most  $2^{|Ab_{<\omega}|+|T|}$  conjugates of  $[I']_{\approx_A}$  under  $\text{Aut}(\mathbb{M}/A)$  as well, and so  $[I']_{\approx_A} \in \text{bdd}^u(A)$  by Proposition 1.4 again. By Proposition 1.13, there must be an automorphism  $\sigma \in \text{Aut}(\mathbb{M}/A, [I']_{\approx_A})$  such that  $\sigma \cdot I' = I$ . Therefore  $[I']_{\approx_A} = [I]_{\approx_A}$  and hence  $I \approx_A b_{<\omega}$ .  $\square$

**Building ((weakly) total)  $\downarrow^{\text{bu}}$ -Morley sequences.** Given that  $\downarrow^{\text{bu}}$  satisfies full existence, an immediate, familiar Erdős-Rado argument gives that  $\downarrow^{\text{bu}}$ -Morley sequences exist, but in the end we will need a technical strengthening of this result.

**Proposition 4.9.** *If  $(b_f)_{f \in \mathcal{T}_\alpha}$  is a family of real elements that is  $s$ -indiscernible over a set of hyperimaginaries  $A$ , then there is a family  $(c_f)_{f \in \mathcal{F}_{\alpha+1}}$  such that*

- $c_{\in \mathcal{F}_{\alpha+1}}$  is  $s$ -indiscernible over  $A$ ,
- $c_{\iota_{\alpha, \alpha+1}(f)} = b_f$  for each  $f \in \mathcal{T}_\alpha$ , and
- the sequence  $(c_{\geq(i)})_{i < \omega}$  is an  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ .

*Proof.* Let  $\kappa$  be sufficiently large to apply Erdős-Rado to a sequence of tuples of the same length as  $b_{\in \mathcal{T}_\alpha}$  over the set  $A$ .

Let  $\gamma(0) = \alpha$ . Let  $c_f^0 = b_f$  for all  $f \in \mathcal{T}_{\gamma(0)} = \mathcal{T}_\alpha$ . Let  $g_0 = \emptyset$  (as an element of  $\mathcal{T}_\alpha$ ).

At successor stage  $\beta + 1$ , assume we have  $(c_f^\beta)_{f \in \mathcal{T}_{\gamma(\beta)}}$  which is  $s$ -indiscernible over  $A$  and which satisfies  $c_{\iota_{\gamma(\delta), \gamma(\beta)}(f)}^\beta = c_f^\delta$  for all  $\delta < \beta$ . By Lemma 3.5, we can build a family  $(c_f^{\beta+1})_{f \in \mathcal{T}_{\gamma(\beta+1)}}$  (for some successor ordinal  $\gamma(\beta+1) > \gamma(\beta)$ ) such that

- $(c_f^{\beta+1})_{f \in \mathcal{T}_{\gamma(\beta+1)}}$  is  $s$ -indiscernible over  $A$ ,
- for each  $f \in \mathcal{T}_{\gamma(\beta)}$ ,  $c_f^{\beta+1} = c_{\iota_{\gamma(\beta), \gamma(\beta+1)}(f)}^{\beta+1}$ , and
- $c_{\geq \zeta_{\gamma(\beta)}^{\gamma(\beta+1)}}^{\beta+1} \downarrow_A^{\text{bu}} c_{\geq 1 \sim \zeta_{\gamma(\beta)}^{\gamma(\beta+1)-1}}^{\beta+1}$ .

Let  $g_{\beta+1} \in \mathcal{T}_{\gamma(\beta+1)}^*$  be  $1 \frown \zeta_\alpha^{\gamma(\beta+1)-1}$ . Note that  $g_{\beta+1} \geq h$ . Also note that by construction we have that

$$c_{\geq g_{\beta+1}}^{\beta+1} \downarrow_A^{\text{bu}} \{c_{\geq \iota_{\gamma(\delta), \gamma(\beta+1)}(g_\delta)}^{\beta+1} : \delta \in (\beta+1) \setminus \text{lim}(\beta+1)\},$$

since  $\iota_{\gamma(\delta), \gamma(\beta+1)}(g_\delta) \geq \zeta_{\gamma(\beta)}^{\gamma(\beta+1)}$  for all nonlimit  $\delta < \beta+1$ .

At limit stage  $\beta$ , let  $\gamma(\beta) = \sup_{\delta < \beta} \gamma(\delta)$  and let  $(c_f^\beta)_{f \in \mathcal{T}_{\gamma(\beta)}}$  be the direct limit of  $(c_f^\delta)_{f \in \mathcal{T}_{\gamma(\delta)}}$  for  $\delta < \beta$ . Leave  $g_\beta$  undefined.

Stop once we have  $(c_f^\kappa)_{f \in \mathcal{T}_{\gamma(\kappa)}}$ . Consider the sequence  $(c_{\geq \iota_{\gamma(\beta), \gamma(\kappa)}(g_\beta)}^\kappa)_{\beta \in \kappa \setminus \text{lim } \kappa}$ .<sup>15</sup> By our choice of  $\kappa$  and a standard application of the Erdős-Rado theorem, we can find a family  $(c_f)_{f \in \mathcal{F}_{\alpha+1}}$  such that the sequence  $(c_{\geq(i)})_{i < \omega}$  is  $A$ -indiscernible and

<sup>15</sup>We write  $\text{lim } \alpha$  for the set of limit ordinals in  $\alpha$ .

for every increasing tuple  $\bar{i} < \omega$ , there is  $\bar{\beta} \in \kappa \setminus \lim \kappa$  such that  $c_{\geq \langle i_0 \rangle_\alpha} \dots c_{\geq \langle i_k \rangle_\alpha} \equiv_A c_{\geq \langle \gamma(\beta_0), \gamma(\kappa) \rangle_\alpha} \dots c_{\geq \langle \gamma(\beta_k), \gamma(\kappa) \rangle_\alpha}$ .

In particular, note that this implies that

$$c_{\geq \langle i \rangle_\alpha} \downarrow_A^{\text{bu}} \{c_{\geq \langle j \rangle_\alpha} : j < i\}$$

for every  $i < \omega$ . Clearly by applying an automorphism, we may assume that  $c_{\geq \langle \alpha+1 \rangle_\alpha}(f) = b_f$  for each  $f \in \mathcal{T}_\alpha$ , so all we need to do is show that the family  $c_{\in \mathcal{F}_{\alpha+1}}$  is  $s$ -indiscernible over  $A$ .

Since the sequence  $(c_{\geq \langle i \rangle_\alpha})_{i < \omega}$  is  $A$ -indiscernible, it is sufficient, by induction, to show the following statement: For any sequence  $\bar{f}_0, \bar{f}_1, \dots, \bar{f}_k, \dots, \bar{f}_\ell$  of tuples of elements of  $\mathcal{F}_{\alpha+1}$  satisfying  $\bar{f}_i \geq \langle i \rangle_\alpha$  for all  $i \leq \ell$  and any  $\bar{h} \geq \langle k \rangle_\alpha$  such that  $\bar{f}_k$  and  $\bar{h}$  realize the same quantifier-free type, we have that  $c_{\bar{f}_k}$  and  $c_{\bar{h}}$  realize the same type over  $Ac_{\bar{f}_0} \dots c_{\bar{f}_{k-1}} c_{\bar{f}_{k+1}} \dots c_{\bar{f}_\ell}$ .

So let  $\bar{f}_0, \dots, \bar{f}_\ell$  and  $\bar{h}$  be as in the statement. By construction, there are  $\beta_0, \dots, \beta_\ell$  such that  $c_{\geq \langle i \rangle_\alpha} \equiv_A c_{\geq \langle \gamma(\beta_i), \gamma(\kappa) \rangle_\alpha}^{K}$  for each  $i \leq \ell$ . Let  $\bar{f}'_0, \dots, \bar{f}'_\ell, \bar{h}'$  be the corresponding elements of  $\mathcal{T}_{\gamma(\kappa)}$ . (So, in particular,  $\bar{f}'_i \geq \beta_{\beta_i}$  for each  $i \leq \ell$  and  $\bar{h}' \geq \beta_{\beta_k}$ .) We now have that  $\bar{f}'_k$  and  $\bar{h}'$  realize the same quantifier-free type. Therefore, by the  $s$ -indiscernible of  $c_{\in \mathcal{T}_{\gamma(\kappa)}}^{K}$ , we have that  $c_{\bar{f}'_k}^{K}$  and  $c_{\bar{h}'}^{K}$  realize the same type over  $Ac_{\bar{f}'_0}^{K} \dots c_{\bar{f}'_{k-1}}^{K} c_{\bar{f}'_{k+1}}^{K} \dots c_{\bar{f}'_\ell}^{K}$ . From this the required statement follows, and we have that  $c_{\in \mathcal{F}_{\alpha+1}}$  is  $s$ -indiscernible over  $A$ .  $\square$

**Corollary 4.10.** *For any set of hyperimaginaries  $A$  and any real tuple  $b$ , there is an  $\downarrow^{\text{bu}}$ -Morley sequence  $(b_i)_{i < \omega}$  over  $A$  with  $b_0 = b$ .*

*Proof.* Apply Proposition 4.9 to the tree  $(b_f)_{f \in \mathcal{T}_0}$  defined by  $b_\emptyset = b$ .<sup>16</sup>  $\square$

The order type  $\omega$  is essential, however; Erdős-Rado only guarantees the existence of sequences that satisfy the relevant condition on finite tuples. Fortunately, this is more than sufficient for the following weak “chain condition”.

**Lemma 4.11.** *If  $(b_i)_{i < \omega}$  is an  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  that is moreover  $Ac$ -indiscernible, then  $b_0 \downarrow_A^{\text{bu}} c$ .*

*Proof.* Fix  $\lambda$ . Let  $\mu = |\text{bdd}_\lambda^u(Ac) \setminus \text{bdd}_\lambda^u(A)|$ . Extend  $b_{< \omega}$  to  $(b_i)_{i < \mu^+}$ . We still have that for any  $i < j < \mu^+$ ,  $b_i \downarrow_A^{\text{bu}} b_j$  (since this is only a property of  $\text{tp}(b_i b_j / A)$ ). Therefore the sets  $\text{bdd}_\lambda^u(Ab_i) \setminus \text{bdd}_\lambda^u(A)$  are pairwise disjoint. Since there are  $\mu^+$  many of them, one of them must be disjoint from  $\text{bdd}_\lambda^u(Ac) \setminus \text{bdd}_\lambda^u(A)$ . Therefore by indiscernibility, we must have  $b_0 \downarrow_A^{\text{bu}} c$ .  $\square$

We will not use the following corollary of Lemma 4.11, but it is worth pointing out.

<sup>16</sup>This can also be proven directly by the standard argument for the existence of Morley sequences.

**Corollary 4.12.** *If  $I$  is a total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  that is  $Ac$ -indiscernible, then  $I \downarrow_A^{\text{bu}} c$ .*

*Proof.* Extend  $I$  to an  $Ac$ -indiscernible sequence  $I_0 + I_1 + I_2 + \dots$  with  $I_0 = I$ . Since  $I$  is totally  $\downarrow^{\text{bu}}$ -Morley, we have that  $(I_i)_{i < \omega}$  is an  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ . So by Lemma 4.11, we have  $I = I_0 \downarrow_A^{\text{bu}} c$ .  $\square$

Parts (2) and (3) of following definition are equivalent to [Kim et al. 2014, Definitions 2.1, 3.4] in our context; this formulation is used implicitly in [Kaplan and Ramsey 2020] and its equivalence to the standard definition is discussed in [Kaplan and Ramsey 2020, Remark 5.8]. The rest of it is based on [Kaplan and Ramsey 2020, Definition 5.7], although we have had to modify the definition of restriction slightly in order to deal with limit ordinals more smoothly.

**Definition 4.13.** Fix a family  $(b_f)_{f \in \mathcal{T}_\alpha}$ .

(1) For  $w \subseteq \alpha$ , the restriction of  $\mathcal{T}_\alpha$  to the set of levels  $w$  is given by

$$\mathcal{T}_\alpha \upharpoonright w = \{f \in \mathcal{T}_\alpha : \min \text{dom}(f) \in w, \beta \in \text{dom}(f) \setminus w \Rightarrow f(\beta) = 0\}.$$

(2) A family  $(b_f)_{f \in \mathcal{T}_\alpha}$  is *str-indiscernible over  $A$*  if it is  $s$ -indiscernible over  $A$  and satisfies that for any finite  $w, v \subseteq \alpha \setminus \lim \alpha$  with  $|w| = |v|$ ,  $b_{\in \mathcal{T}_\alpha \upharpoonright w}$  and  $b_{\in \mathcal{T}_\alpha \upharpoonright v}$  realize the same type over  $A$  (where we take  $b_{\in \mathcal{T}_\alpha \upharpoonright w}$  to be enumerated according to  $<_{\text{lex}}$ , which is a well-ordering on  $\mathcal{T}_\alpha \upharpoonright w$  for finite  $w$ ).

(3) We say that  $b_{\in \mathcal{T}_\alpha}$  is  $\downarrow^{\text{bu}}$ -spread-out over  $A$  if for any  $f \in \mathcal{T}_\alpha^*$  (with  $\text{dom}(f) = [\beta + 1, \alpha)$  for some  $\beta < \alpha$ ), the sequence  $(b_{\geq f \frown i})_{i < \omega}$  is an  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ .

(4) We say that  $b_{\in \mathcal{T}_\alpha \upharpoonright w}$  is  $\downarrow^{\text{bu}}$ -spread-out over  $A$  if for any  $f \in \mathcal{T}_\alpha^*$  (with  $\text{dom}(f) = [\beta + 1, \alpha)$  for some  $\beta < \alpha$  and satisfying that  $(f \frown i)_{i < \omega}$  is a sequence of elements of  $\mathcal{T}_\alpha \upharpoonright w$ ), the sequence  $(b_{\geq f \frown i})_{i < \omega}$  is an  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  (where we interpret  $b_f$  as  $\emptyset$  if  $f \notin \mathcal{T}_\alpha \upharpoonright w$ ).

(5)  $b_{\in \mathcal{T}_\alpha}$  is an  $\downarrow^{\text{bu}}$ -Morley tree over  $A$  if it is  $\downarrow^{\text{bu}}$ -spread-out and *str-indiscernible* over  $A$ .

Note that if  $b_{\in \mathcal{T}_\alpha}$  is  $\downarrow^{\text{bu}}$ -spread-out over  $A$ , then any restriction  $b_{\in \mathcal{T}_\alpha \upharpoonright w}$  is also  $\downarrow^{\text{bu}}$ -spread-out over  $A$  (even for infinite  $w$ ). Also note that, by a basic compactness argument, if  $\alpha$  is infinite and  $(b_f)_{f \in \mathcal{T}_\alpha}$  is *str-indiscernible* over  $A$ , then for any  $\beta$ , we can find a tree  $(c_f)_{f \in \mathcal{T}_\beta}$  which is *str-indiscernible* over  $A$  such that for any  $w \in [\alpha]^{< \omega}$  and  $v \in [\beta]^{< \omega}$  with  $|w| = |v|$ ,  $b_{\in \mathcal{T}_\alpha \upharpoonright w} \equiv_A c_{\in \mathcal{T}_\beta \upharpoonright v}$ .

**Proposition 4.14.** *For any set of hyperimaginaries  $A$ , real tuple  $b$ , and  $\kappa$ , there is a tree  $(b_f)_{f \in \mathcal{T}_\kappa}$  that is  $\downarrow^{\text{bu}}$ -spread-out and  $s$ -indiscernible over  $A$  such that for each  $f \in \mathcal{T}_\kappa$ ,  $b_f \equiv_A b$ .*

*Proof.* Let  $(b_f^0)_{f \in \mathcal{T}_0}$  be defined by  $b_\emptyset^0 = b$ . This is vacuously  $\downarrow^{\text{bu}}$ -spread-out and  $s$ -indiscernible over  $A$ .

At successor stage  $\alpha + 1$ , given  $(b_f^\alpha)_{f \in \mathcal{T}_\alpha}$  which is  $\downarrow^{\text{bu}}$ -spread-out and  $s$ -indiscernible by Proposition 4.9, we can find an extension  $(b_f^{\alpha+1})_{f \in \mathcal{F}_{\alpha+1}}$  satisfying  $b_{i_{\alpha,\alpha+1}(f)}^{\alpha+1} = b_f^\alpha$  for all  $f \in \mathcal{T}_\alpha$  such that  $b_{\in \mathcal{F}_{\alpha+1}}^{\alpha+1}$  is  $s$ -indiscernible over  $A$  and  $(b_{\geq(i)_\alpha}^{\alpha+1})_{i < \omega}$  is an  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ . By Fact 3.3, we can find  $b_\emptyset^{\alpha+1} \equiv_A b$  such that the tree  $(b_f^{\alpha+1})_{f \in \mathcal{T}_{\alpha+1}}$  is  $s$ -indiscernible over  $A$ . By construction, we now have that  $(b_f^{\alpha+1})_{f \in \mathcal{T}_{\alpha+1}}$  is  $\downarrow^{\text{bu}}$ -spread-out over  $A$ .

At limit stage  $\alpha$ , let  $(b_f^\alpha)_{f \in \mathcal{T}_\alpha}$  be the direct limit of  $(b_f^\beta)_{f \in \mathcal{T}_\beta}$  for  $\beta < \alpha$ . It is immediate from the definitions that  $b_{\in \mathcal{T}_\alpha}^\alpha$  is  $\downarrow^{\text{bu}}$ -spread-out and  $s$ -indiscernible over  $A$ .

Once we have constructed  $(b_f^\kappa)_{f \in \mathcal{T}_\kappa}$ , let  $b_f = b_f^\kappa$  for each  $f \in \mathcal{T}_\kappa$ . We have that  $b_{\in \mathcal{T}_\kappa}$  is the required tree by induction.  $\square$

By the same argument as in [Kaplan and Ramsey 2020, Lemma 5.10], we get the following.

**Fact 4.15.** *Fix a set of real parameters  $A$ , and let  $(b_f)_{f \in \mathcal{T}_\kappa}$  be a family of tuples of real parameters of the same length that is  $s$ -indiscernible over  $A$ . If  $\kappa \geq \beth_{\lambda+}(\lambda)$  (where  $\lambda = 2^{|Ab_f|+|T|}$ ), then there is an  $str$ -indiscernible tree  $(c_f)_{f \in \mathcal{T}_\omega}$  such that*

*for any  $w \in [\omega]^{<\omega}$ , there is  $v \in [\kappa]^{<\omega}$  such that  $(b_f)_{f \in \mathcal{T}_\kappa \upharpoonright v} \equiv_A (c_f)_{f \in \mathcal{T}_\omega \upharpoonright w}$ .  $(*)_A$*

Note that Fact 4.15 generalizes to continuous logic by the same soft argument as in the discussion after Fact 3.3.

**Lemma 4.16.** *Suppose that a family of tuples of real elements  $(b_f)_{f \in \mathcal{T}_\kappa}$  is  $\downarrow^{\text{bu}}$ -spread-out and  $s$ -indiscernible over a set of hyperimaginaries  $A$  with all  $b_f$  tuples of the same length. If  $\kappa \geq \beth_{\lambda+}(\lambda)$  (where  $\lambda = 2^{|Ab_f|+|T|}$ ), then there is an  $\downarrow^{\text{bu}}$ -Morley tree  $(c_f)_{f \in \mathcal{T}_\omega}$  over  $A$  such that condition  $(*)_A$  from Fact 4.15 holds.*

*Proof.* Find a model  $M$  with  $|M| \leq |A| + \aleph_0$  such that  $A \subseteq \text{bdd}^{\text{heq}}(M)$ . Apply Fact 4.15 with  $M$  as the base to the family  $(b_f)_{f \in \mathcal{T}_\kappa}$  to get a tree  $(c_f)_{f \in \mathcal{T}_\omega}$  that is  $str$ -indiscernible over  $M$  and satisfies  $(*)_M$ . This is enough to imply that  $c_{\in \mathcal{T}_\omega}$  is  $str$ -indiscernible over  $A$  and satisfies  $(*)_A$ . Furthermore, since the tree  $c_{\in \mathcal{T}_\omega}$  has height  $\omega$  and since  $b_{\in \mathcal{T}_\kappa}$  is  $\downarrow^{\text{bu}}$ -spread-out over  $A$ ,  $(*)_A$  implies that  $c_{\in \mathcal{T}_\omega}$  is  $\downarrow^{\text{bu}}$ -spread-out over  $A$ . Therefore  $c_{\in \mathcal{T}_\omega}$  is an  $\downarrow^{\text{bu}}$ -Morley tree over  $A$ .  $\square$

**Proposition 4.17.** *If  $(b_f)_{f \in \mathcal{T}_\omega}$  is a family of tuples of real elements that is an  $\downarrow^{\text{bu}}$ -Morley tree over a set of hyperimaginaries  $A$ , then  $(b_{\zeta_\beta^\omega})_{\beta < \omega}$  is a weakly total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ .*

*Proof.* Fix a linear order  $O$ . Let  $c_\alpha = b_{\zeta_\alpha^\omega}$  for each  $\alpha < \omega$ .

For each positive  $n < \omega$  and each  $i < j < \omega$ , we have that  $b_{\geq \zeta_n^\omega \frown i} \downarrow_A^{\text{bu}} b_{\geq \zeta_n^\omega \frown j}$  and that the sequence  $(b_{\geq \zeta_n^\omega \frown i})_{i < \omega}$  is  $Ac_{\geq n}$ -indiscernible. By compactness, we can

find  $(c_i)_{i \in O}$  such that  $(c_i)_{i \in \omega+O}$  is  $A$ -indiscernible and such that  $(b_{\geq \zeta_n^\omega \frown i})_{i < \omega}$  is  $A_{c \in [n, \omega)+O}$ -indiscernible for each  $n < \omega$ .

Therefore, by Lemma 4.11, we have that  $c_{< n} \downarrow_A^{\text{bu}} c_{\in [n, \omega)+O}$ . Hence,  $(b_{\zeta_\beta^\omega})_{\beta < \omega}$  is a weakly total  $\downarrow^{\text{bu}}$ -Morley sequence.  $\square$

**Corollary 4.18.** *For any set of hyperimaginaries  $A$  and tuple of real elements  $b$ , there is an  $A$ -indiscernible sequence  $(b_i)_{i < \omega}$  with  $b_0 = b$  such that for any  $b' \equiv_A^I b$  and  $n < \omega$ , there are  $I_0, J_0, I_1, J_1, \dots, J_{k-1}, I_k$  with*

- $b$  the first element of  $I_0$ ,
- $b'$  the first element of  $I_k$ ,
- $|I_i| = n$  for all  $i \leq k$ ,
- $J_i$  infinite for all  $i < k$ , and
- $I_i + J_i$  and  $I_{i+1} + J_i$  realizing the same EM-type over  $A$  as  $b_{< \omega}$  for all  $i < k$ .

We can also arrange it so that  $I_i$  is infinite for all  $i \leq k$ ,  $|J_i| = n$  for all  $i < k$ , and  $I_i + J_i$  and  $I_{i+1} + J_i$  realize the same EM-type over  $A$  as  $b_{< \omega}$  in the reverse order for all  $i < k$  (with the same choice of  $b_{< \omega}$  but possibly a different  $k$ ).

*Proof.* By Lemma 4.16 and Proposition 4.17, we can find a sequence  $(b_i)_{i < \omega}$  with  $b_0 = b$  that is a weakly total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ . Fix  $n < \omega$ , and write  $b_{< \omega}$  as  $I + J$  with  $|I| = n$ . By repeating the proof of Proposition 4.3, we get the required configuration of  $I_i$ 's and  $J_i$ 's.

For the final statement, by compactness, we can find an indiscernible sequence  $K$  of order type  $\omega$  which has  $b$  as its first element and realizes the reverse of the EM-type of  $b_{< \omega}$  over  $A$ . Fix an  $n < \omega$ . If we partition  $K$  as  $I + J$  where  $|J| = n$  and again repeat the proof of Proposition 4.3, we get the second required configuration of  $I_i$ 's and  $J_i$ 's.  $\square$

To go further, we will need the following fact from [Silver 1971]. Recall that the statement  $\kappa \rightarrow (\alpha)_\gamma^{< \omega}$  means that whenever  $f : [\kappa]^{< \omega} \rightarrow \gamma$  is a function, there is a set  $X \subseteq \kappa$  of order type  $\alpha$  such that for each  $n < \omega$ ,  $f$  is constant on  $[X]^n$ .

**Fact 4.19** [Silver 1971, Chapter 4]. *For any limit ordinal  $\alpha$ , if  $\kappa$  is the smallest cardinal satisfying  $\kappa \rightarrow (\alpha)_2^{< \omega}$ , then for any  $\gamma < \kappa$ ,  $\kappa \rightarrow (\alpha)_\gamma^{< \omega}$ . Furthermore,  $\kappa$  is strongly inaccessible.*

The smallest cardinal  $\lambda$  satisfying  $\lambda \rightarrow (\alpha)_2^{< \omega}$  is called the Erdős cardinal  $\kappa(\alpha)$ . In the specific case of  $\alpha = \omega$ , we will also need the following lemma.

**Lemma 4.20.** *If  $\kappa \rightarrow (\omega)_\gamma^{< \omega}$ , then  $(\gamma^\kappa)^+ \rightarrow (\omega+1)_\gamma^{< \omega}$ . In particular, if  $\kappa(\omega)$  exists, then  $(2^{\kappa(\omega)})^+ \rightarrow (\omega+1)_\gamma^{< \omega}$  for any  $\gamma < \kappa(\omega)$ .*

*Proof.* Fix a set  $X$  of cardinality  $(\gamma^\kappa)^+$  and a coloring  $f : [X]^{< \omega} \rightarrow \gamma$ . Fix an ordering  $(x_\alpha)_{\alpha < (\gamma^\kappa)^+}$  of  $X$ . Recall that a subset  $Y \subseteq X$  is *end-homogeneous* if for any  $\delta_0 < \dots < \delta_{n-1} < \alpha < \beta < (\gamma^\kappa)^+$ ,  $f(\{x_{\delta_0}, \dots, x_{\delta_{n-1}}, x_\alpha\}) = f(\{x_{\delta_0}, \dots, x_{\delta_{n-1}}, x_\beta\})$ .

By [Erdős et al. 1984, Lemma 15.2], there is an end-homogeneous set  $Y \subseteq X$  of order type  $\kappa + 1$ . Let  $(y_\alpha)_{\alpha < \kappa+1}$  be an enumeration of  $Y$  in order. Let  $g(A) = f(A \cup \{y_\kappa\})$ . By assumption, there is a  $g$ -homogeneous subset  $Z \subseteq Y$  of order type  $\omega$ . Therefore, by construction,  $Z \cup \{y_\kappa\}$  is the required  $f$ -homogeneous subset of order type  $\omega + 1$ .

The last statement follows from the fact that  $\kappa(\omega)$  is strongly inaccessible and cardinal arithmetic (i.e.,  $2^{\kappa(\omega)} = \gamma^{\kappa(\omega)}$  for  $\gamma > 1$  with  $\gamma < \kappa(\omega)$ ).  $\square$

**Lemma 4.21.** *Suppose  $(b_f)_{f \in \mathcal{T}_\lambda}$  is  $\downarrow^{\text{bu}}$ -spread-out and  $s$ -indiscernible over  $A$  with all  $b_f$  tuples of the same length. If  $\lambda \rightarrow (\omega + 1)_{2^{|Ab|+|T|}}^{<\omega}$ , then there is a set  $X \subseteq \lambda \setminus \lim \lambda$  with order type  $\omega + 1$  such that  $b_{\in \mathcal{T}_\lambda \upharpoonright X}$  is an  $\downarrow^{\text{bu}}$ -Morley tree over  $A$ .*

*Proof.* Let  $t$  be the function on  $[\lambda \setminus \lim \lambda]^{<\omega}$  that takes  $w \in [\lambda \setminus \lim \lambda]^{<\omega}$  to  $\text{tp}(b_{\in \mathcal{T}_\lambda \upharpoonright w}/A)$ . By assumption, we can find  $X \subset \lambda \setminus \lim \lambda$  of order type  $\omega + 1$  such that  $t$  is homogeneous on  $X$ .  $b_{\in \mathcal{T}_\lambda \upharpoonright X}$  is  $s$ -indiscernible over  $A$  and  $\downarrow^{\text{bu}}$ -spread-out over  $A$ , since these properties are both preserved by passing to restrictions.  $\square$

**Theorem 4.22.** *For any  $A$  and  $b$  in any theory  $T$ , if there is a cardinal  $\lambda$  satisfying  $\lambda \rightarrow (\omega + 1)_{2^{|Ab|+|T|}}^{<\omega}$ , then there is a total  $\downarrow^{\text{bu}}$ -Morley sequence  $(b_i)_{i < \omega}$  over  $A$  with  $b_0 = b$ .*

*In particular, it is enough if there is an Erdős cardinal  $\kappa(\alpha)$  such that  $|Ab|+|T| < \kappa(\alpha)$  (for any limit  $\alpha \geq \omega$ ).*

*Proof.* If the Erdős cardinal  $\kappa(\alpha)$  exists and  $|Ab|+|T| < \kappa(\alpha)$ , then by Fact 4.19, we have  $2^{|Ab|+|T|} < \kappa(\alpha)$  as well. Then if  $\alpha = \omega$ , we have that  $(2^{\kappa(\omega)})^+ \rightarrow (\omega + 1)_{2^{|Ab|+|T|}}^{<\omega}$  by Lemma 4.20. If  $\alpha > \omega$ , we clearly have  $\kappa(\alpha) \rightarrow (\omega + 1)_{2^{|Ab|+|T|}}^{<\omega}$  by Fact 4.19. So in any such case we have the required  $\lambda$ .

Let  $\lambda$  be a cardinal such that  $\lambda \rightarrow (\omega + 1)_{2^{|Ab|+|T|}}^{<\omega}$  holds. By Proposition 4.14, we can build a tree  $(b_f)_{f \in \mathcal{T}_\lambda}$  that is  $s$ -indiscernible and  $\downarrow^{\text{bu}}$ -spread-out over  $A$ . By Lemma 4.21 and the choice of  $\lambda$ , we can extract an  $\downarrow^{\text{bu}}$ -Morley tree  $(c_f)_{f \in \mathcal{T}_{\omega+1}}$  from this.

By compactness, we can extend this to a tree  $(c_f)_{f \in \mathcal{T}_{\omega+\omega}}$  that is  $str$ -indiscernible over  $A$ . We still have that for any  $i < j < \omega$ ,

$$c_{\succeq_{\zeta_{\omega+1}}^{\omega+\omega} \frown i} \downarrow_A^{\text{bu}} c_{\succeq_{\zeta_{\omega+1}}^{\omega+\omega} \frown j}$$

but now we also have that the  $(c_{\succeq_{\zeta_{\omega+1}}^{\omega+\omega} \frown i})_{i < \omega}$  is  $A \cup \{c_{\zeta_{\omega+i}^{\omega+\omega}} : i < \omega\}$ -indiscernible, by  $str$ -indiscernibility of the full tree  $c_{\in \mathcal{T}_{\omega+\omega}}$ . Therefore, by Lemma 4.11,

$$c_{\succeq_{\zeta_{\omega+1}}^{\omega+\omega} \frown 0} \downarrow_A^{\text{bu}} \{c_{\zeta_{\omega+i}^{\omega+\omega}} : i < \omega\},$$

so in particular,

$$\{c_{\zeta_i^{\omega+\omega}} : i < \omega\} \downarrow_A^{\text{bu}} \{c_{\zeta_{\omega+i}^{\omega+\omega}} : i < \omega\}.$$

Let  $d_i = c_{\xi^{\omega+\omega}}$  for each  $i < \omega + \omega$ . We have that  $(d_i)_{i < \omega+\omega}$  is  $A$ -indiscernible. Furthermore, by Theorem 4.8, we have that  $d_{<\omega}$  is a total  $\downarrow^{\text{bu}}$ -Morley sequence. By applying an automorphism, we get the required  $b_{<\omega}$ .  $\square$

So if we assume that for every  $\lambda$ , there is a  $\kappa$  such that  $\kappa \rightarrow (\omega + 1)_\lambda^{<\omega}$ , we get that Lascar strong type is always witnessed by total  $\downarrow^{\text{bu}}$ -Morley sequences in the manner of Proposition 4.3.

The use of large cardinals in Theorem 4.22 leaves an obvious question.

**Question 4.23.** *Does the statement “for every  $A$  and  $b$ , there is a total  $\downarrow^{\text{bu}}$ -Morley sequence  $(b_i)_{i < \omega}$  over  $A$  with  $b_0 = b$ ” have any set-theoretic strength? What if we add cardinality restrictions, such as  $|A| + |T| \leq \aleph_0$  and  $|b| < \aleph_0$ ?*

**Total  $\downarrow^{\text{bu}}$ -Morley sequences in tame theories.** Lemma 4.11 can be used to show that  $\downarrow^{\text{d}}$  implies  $\downarrow^{\text{bu}}$  (where  $b \downarrow_A^{\text{d}} c$  means that  $\text{tp}(b/Ac)$  does not divide over  $A$ ), something which was previously established for bounded hyperimaginary independence,  $\downarrow^{\text{b}}$ , in [Conant and Hanson 2022, Corollary 4.13] and which was originally folklore for algebraic independence,  $\downarrow^{\text{a}}$ .<sup>17</sup>

**Proposition 4.24.** *For any real elements  $A$ ,  $b$ , and  $c$ , if  $b \downarrow_A^{\text{d}} c$ , then  $b \downarrow_A^{\text{bu}} c$ .*

*Proof.* Let  $(c_i)_{i < \omega}$  be an  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$  with  $c_0 = c$ . Since  $b \downarrow_A^{\text{d}} c$ , we may assume that  $c_{<\omega}$  is  $Ab$ -indiscernible. Hence, by Lemma 4.11,  $b \downarrow_A^{\text{bu}} c$ .  $\square$

**Corollary 4.25.** *If  $(b_i)_{i < \omega}$  is a (nondividing) Morley sequence over  $A$ , then it is a total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ .*  $\square$

In simple theories, we get the converse (Proposition 4.27). Recall that  $B \downarrow_A^{\text{b}} C$  means  $\text{bdd}^{\text{heq}}(AB) \cap \text{bdd}^{\text{heq}}(AC) = \text{bdd}^{\text{heq}}(A)$ .

**Lemma 4.26.** *Let  $T$  be a simple theory. For any  $A$ ,  $b$ , and  $c$ ,  $b \downarrow_A^{\text{f}} C$  if and only if there is an  $AC$ -indiscernible sequence  $(b_i)_{i < \omega}$  with  $b_0 = b$  such that for any  $J$  and  $K$  with  $J + K \equiv_A^{\text{EM}} b_{<\omega}$ ,  $J \downarrow_A^{\text{b}} K$ .*

*Proof.* (The argument here is similar to the proof of [Adler 2005, Lemma 3.2], but we will give a proof for the sake of completeness.) If  $b \downarrow_A^{\text{f}} C$ , then we can build an  $AC$ -indiscernible  $\downarrow^{\text{f}}$ -Morley sequence  $(b_i)_{i < \omega}$  over  $A$  with  $b_0 = b$  (since  $T$  is simple). By some forking calculus, we have that  $J \downarrow_A^{\text{f}} K$  for any  $J$  and  $K$  with  $J + K \equiv_A^{\text{EM}} b_{<\omega}$ . Therefore, by [Conant and Hanson 2022, Corollary 4.13],  $J \downarrow_A^{\text{b}} K$  for any such  $J$  and  $K$  as well.

Conversely, assume that there is an  $AC$ -indiscernible sequence  $(b_i)_{i < \omega}$  with  $b_0 = b$  such that for any  $J$  and  $K$  with  $J + K \equiv_A^{\text{EM}} b_{<\omega}$ ,  $J \downarrow_A^{\text{b}} K$ . Let  $\kappa$  be a regular cardinal such that every type (in the same sort as  $C$ ) does not fork over some set of cardinality less than  $\kappa$ . Let  $(b_i)_{i < \kappa + \kappa^*}$  be an  $AC$ -indiscernible sequence

<sup>17</sup>There is an incorrect proof of this in the literature. To the author’s knowledge, the first correct published proof of this is in [Conant and Hanson 2022, Theorem 4.11].

extending  $b_{<\omega}$ , where  $\kappa^*$  is an order-reversed copy of  $\kappa$ . Now we clearly have that  $b_{<\kappa} \downarrow_A^b b_{\in\kappa^*}$ . By local character, there is a set  $D \subseteq Ab_{<\kappa}$  with  $|D| < \kappa$  such that  $C \downarrow_D^f Ab_{<\kappa}$ . Since  $\kappa$  is regular, there is a  $\lambda < \kappa$  such that  $D \subseteq Ab_{<\lambda}$ . Therefore, by base monotonicity,  $C \downarrow_{Ab_{<\lambda}}^f Ab_{<\kappa}$ . Since  $b_{\geq\lambda}$  is  $Ab_{<\lambda}C$ -indiscernible, we have that  $C \downarrow_{Ab_{<\lambda}}^f Ab_{\in\kappa+\kappa^*}$ . Therefore, by base monotonicity again,  $C \downarrow_{Ab_{<\kappa}}^f Ab_{\in\kappa+\kappa^*}$ . By the symmetric argument,  $C \downarrow_{Ab_{\in\kappa^*}}^f Ab_{\in\kappa+\kappa^*}$  as well.

In simple theories, forking is characterized by canonical bases in the following way:  $E \downarrow_D^f F$  (with  $D \subseteq F$ ) holds if and only if  $\text{cb}(\text{tp}(E/\text{bdd}^{\text{heq}}(F))) \in \text{bdd}^{\text{heq}}(D)$  [Kim 2014, Lemma 4.3.4]. Therefore, we have that  $\text{cb}(\text{tp}(C/\text{bdd}^{\text{heq}}(Ab_{\in\kappa+\kappa^*}))) \in \text{bdd}^{\text{heq}}(Ab_{<\kappa}) \cap \text{bdd}^{\text{heq}}(Ab_{\in\kappa^*})$ , but  $\text{bdd}^{\text{heq}}(Ab_{<\kappa}) \cap \text{bdd}^{\text{heq}}(Ab_{\in\kappa^*}) = \text{bdd}^{\text{heq}}(A)$  by assumption. So  $C \downarrow_A^f b_{\in\kappa+\kappa^*}$ , whence  $C \downarrow_A^f b_0$  and hence  $b_0 \downarrow_A^f C$ , as required.  $\square$

**Proposition 4.27.** *Let  $T$  be a simple theory. For any  $A$  and  $A$ -indiscernible sequence  $I$ , the following are equivalent.*

- (1)  $I$  is an  $\downarrow^f$ -Morley sequence over  $A$ .
- (2) For any  $J$  and  $K$  with  $J + K \equiv_A^{\text{EM}} I$ ,  $J \downarrow_A^b K$ .
- (3)  $I$  is a total  $\downarrow^{\text{bu}}$ -Morley sequence over  $A$ .

*Proof.* (1) $\Rightarrow$ (3) is Corollary 4.25. (3) $\Rightarrow$ (2) is obvious. For (2) $\Rightarrow$ (1), assume that (2) holds. Fix  $(b_i)_{i < \omega + \omega} \equiv_A^{\text{EM}} I$ .  $(b_i)_{\omega \leq i < \omega + \omega}$  is  $Ab_{<\omega}$ -indiscernible. Therefore by Lemma 4.26,  $b_\omega \downarrow_A^f b_{<\omega}$ , and we have that  $b_{<\omega + \omega}$ , and therefore  $I$ , is an  $\downarrow^f$ -Morley sequence over  $A$ .  $\square$

On the other hand, there are easy examples in NIP theories (such as DLO) of total  $\downarrow^{\text{bu}}$ -Morley sequences that are not strict Morley sequences (i.e., sequences  $b_{<\omega}$  satisfying that  $b_i \downarrow_A^f b_{<i}$  and  $b_{<i} \downarrow_A^f b_i$  for all  $i < \omega$ ). Fix a model  $M$  of DLO and let  $(a_i b_i)_{i < \omega}$  be a sequence of elements above  $M$  satisfying  $a_i < a_{i+1} < b_{i+1} < b_i$  for all  $i < \omega$ . This is a total  $\downarrow^{\text{bu}}$ -Morley sequence since it is generated by an  $M$ -invariant type, but it is clearly not a strict Morley sequence. DLO can also be used to show that not every  $\downarrow^b$ -Morley sequence in a rosy theory is a total  $\downarrow^{\text{bu}}$ -Morley sequence (e.g., [Adler 2005, Example 3.13] is an  $\downarrow^b$ -Morley sequence since  $\text{p}$ -forking in DLO is trivial but fails to even be an  $\downarrow^b$ -Morley sequence).

In  $\text{NSOP}_1$  theories, we do get that tree Morley sequences are total  $\downarrow^{\text{bu}}$ -Morley sequences.

**Proposition 4.28.** *Let  $T$  be an  $\text{NSOP}_1$  theory, and let  $M \models T$ . If  $I$  is a tree Morley sequence over  $M$ , then it is a total  $\downarrow^{\text{bu}}$ -Morley sequence over  $M$ .*

*Proof.* Let  $J$  be a sequence realizing the same EM-type as  $I$  over  $M$ . Find  $K \equiv_M I$  such that  $K \downarrow_M^K IJ$ . Let  $I', J'$ , and  $K'$  have the same order type such that  $I + I'$ ,  $J + J'$ , and  $K + K'$  are all  $M$ -indiscernible. Since these are tree Morley sequences,

we have that  $I \downarrow_M^K I'$ ,  $J \downarrow_M^K J'$ , and  $K \downarrow_M^K K'$ . Therefore, by the independence theorem for NSOP<sub>1</sub> theories, we can find  $I''$  and  $J''$  such that  $I + I''$ ,  $K + I''$ ,  $K + J''$ , and  $J + J''$  are all  $M$ -indiscernible, so  $I \approx_M J$ .

Since we can do this for any such  $J$ , we have that  $I$  is a total  $\downarrow^{\text{bu}}$ -Morley sequence by Theorem 4.8 and the fact that Lascar strong types are types over models.  $\square$

The converse is unclear. The argument in the context of simple theories relies on the existence of canonical bases for types.

**Question 4.29.** *If  $T$  is NSOP<sub>1</sub>, is every total  $\downarrow^{\text{bu}}$ -Morley sequence over  $M \models T$  a tree Morley sequence over  $M$ ?*

## References

- [Adler 2005] H. Adler, “Explanation of independence”, 2005. arXiv math/0511616
- [Adler 2009] H. Adler, “A geometric introduction to forking and thorn-forking”, *J. Math. Log.* **9**:1 (2009), 1–20. MR Zbl
- [Buechler 1997] S. Buechler, “Canonical bases in some supersimple theories”, preprint, 1997.
- [Conant and Hanson 2022] G. Conant and J. Hanson, “Separation for isometric group actions and hyperimaginary independence”, *Fund. Math.* **259**:1 (2022), 97–109. MR Zbl
- [Erdős et al. 1984] P. Erdős, A. Hajnal, A. Máté, and R. Rado, “Trees and positive ordinary partition relations”, Chapter IV, pp. 80–104 in *Combinatorial set theory: partition relations for cardinals*, Studies in Logic and the Foundations of Mathematics **106**, North-Holland, Amsterdam, 1984. MR Zbl
- [Kaplan and Ramsey 2020] I. Kaplan and N. Ramsey, “On Kim-independence”, *J. Eur. Math. Soc.* **22**:5 (2020), 1423–1474. MR Zbl
- [Kim 2014] B. Kim, *Simplicity theory*, Oxford Logic Guides **53**, Oxford University Press, 2014. MR Zbl
- [Kim et al. 2014] B. Kim, H.-J. Kim, and L. Scow, “Tree indiscernibilities, revisited”, *Arch. Math. Logic* **53**:1-2 (2014), 211–232. MR Zbl
- [Lascar 1982] D. Lascar, “On the category of models of a complete theory”, *J. Symbolic Logic* **47**:2 (1982), 249–266. Zbl
- [Shelah 1980] S. Shelah, “Simple unstable theories”, *Ann. Math. Logic* **19**:3 (1980), 177–203. MR Zbl
- [Silver 1971] J. H. Silver, “Some applications of model theory in set theory”, *Ann. Math. Logic* **3**:1 (1971), 45–110. MR Zbl
- [Simon 2015] P. Simon, *A guide to NIP theories*, Lecture Notes in Logic **44**, Cambridge University Press, 2015. MR Zbl
- [Tent and Ziegler 2012] K. Tent and M. Ziegler, *A course in model theory*, Lecture Notes in Logic **40**, Cambridge University Press, 2012. MR Zbl
- [Wagner 2015] F. O. Wagner, “Plus ultra”, *J. Math. Log.* **15**:2 (2015), art. id. 1550008. MR Zbl

Received 26 Mar 2022. Revised 16 Oct 2023.

JAMES E. HANSON:

jhanson9@umd.edu

Department of Mathematics, University of Maryland, College Park, MD, United States





# Model Theory

msp.org/mt

## EDITORS-IN-CHIEF

Martin Hils Westfälische Wilhelms-Universität Münster (Germany)  
hils@uni-muenster.de  
Rahim Moosa University of Waterloo (Canada)  
rmoosa@uwaterloo.ca

## EDITORIAL BOARD

Sylvy Anscombe Université Paris Cité (France)  
sylvy.anscombe@imj-prg.fr  
Alessandro Berarducci Università di Pisa (Italy)  
berardu@dm.unipi.it  
Emmanuel Breuillard University of Oxford (UK)  
emmanuel.breuillard@gmail.com  
Artem Chernikov University of California, Los Angeles (USA)  
chernikov@math.ucla.edu  
Charlotte Hardouin Université Paul Sabatier (France)  
hardouin@math.univ-toulouse.fr  
François Loeser Sorbonne Université (France)  
francois.loeser@imj-prg.fr  
Dugald Macpherson University of Leeds (UK)  
h.d.macpherson@leeds.ac.uk  
Alf Onshuus Universidad de los Andes (Colombia)  
aonshuus@uniandes.edu.co  
Chloé Perin The Hebrew University of Jerusalem (Israel)  
perin@math.huji.ac.il

## PRODUCTION

Silvio Levy (Scientific Editor)  
production@msp.org

---


See inside back cover or [msp.org/mt](https://msp.org/mt) for submission instructions.

---

Model Theory (ISSN 2832-904X electronic, 2832-9058 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

---

MT peer review and production are managed by EditFlow<sup>®</sup> from MSP.

PUBLISHED BY  
 **mathematical sciences publishers**  
nonprofit scientific publishing  
<https://msp.org/>

© 2024 Mathematical Sciences Publishers

# Model Theory

no. 1    vol. 3    2024

Complete type amalgamation for nonstandard finite groups AMADOR MARTIN-PIZARRO and DANIEL PALACÍN	1
Bounded ultraimaginary independence and its total Morley sequences JAMES E. HANSON	39
Quelques modestes compléments aux travaux de Messieurs Mark DeBonis, Franz Delahan, David Epstein et Ali Nesin sur les groupes de Frobenius de rang de Morley fini BRUNO POIZAT	71
Nonelementary categoricity and projective locally o-minimal classes BORIS ZILBER	101
A Pila–Wilkie theorem for Hensel minimal curves VICTORIA CANTORAL FARFÁN, KIEN HUU NGUYEN, MATHIAS STOUT and FLORIS VERMEULEN	119
Model theory in compactly generated (tensor-)triangulated categories MIKE PREST and ROSE WAGSTAFFE	147