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Very ampleness in strongly minimal sets

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Dedicated to Boris Zilber, whose far-reaching vision is a never ending source of inspiration.

Inspired by very ampleness of Zariski geometries, we introduce and study the notion of a very ample family of plane curves in any strongly minimal set and the corresponding notion of a very ample strongly minimal set (characterized by the definability of such a family). We show various basic properties; for example, any strongly minimal set internal to an expansion of an algebraically closed field is very ample, and any very ample strongly minimal set nonorthogonal to a strongly minimal set Y is internal to Y. We then use very ampleness to characterize the *full relics* of an algebraically closed field K — those structures $\mathcal{M} = (M, \dots)$ interpreted in K which recover all constructible subsets of powers of M. Next we show that very ample strongly minimal sets admit very ample families of plane curves of all dimensions, and we use this to characterize very ampleness in terms of definable pseudoplanes. Finally, we show that nonlocally modular expansions of divisible strongly minimal groups are very ample, and we deduce — answering an old question of Martin (1988) — that in a pure algebraically closed field K there are no reducts between $(K, +, \cdot)$ and $(K, \cdot)$.

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1. Introduction

Many ideas in stability theory (such as Zilber's classification of totally categorical structures, see, e.g., [Zilber 1993]) build heavily on or are inspired by Zilber's weak trichotomy theorem, stating that any strongly minimal set is either locally modular or defines a rank 2 pseudoplane (recall that a *pseudoplane* is an abstract incidence relation $I \subset P \times L$ between "points" and "lines" such that each point lies on infinitely many lines, each line contains infinitely many points, any two lines have finite intersection, and any two points lie on only finitely many common lines). Indeed, Zilber's (strong) trichotomy conjecture, in one of its many variants, asserted that every uncountably categorical pseudoplane should be mutually interpretable with an algebraically closed field.

While Zilber's conjecture is false [Hrushovski 1993], it has been shown to hold in many instances. For example, Castle [2023] has recently shown that the conjecture holds assuming the pseudoplane is itself interpreted in an algebraically closed field of characteristic zero, and Hasson and Sustretov [2017] have shown the same for rank preserving interpretations in all characteristics.

Contexts where Zilber's trichotomy is known to hold are of a geometric nature, and the geometry plays a major part in many proofs of restricted versions of the trichotomy. Most such proofs, given a nonlocally modular strongly minimal set S, make use of almost faithful families of plane curves (families of one-dimensional subsets of S^2 with few infinite intersections), rather than pseudoplanes. The reason is that, although such objects are combinatorially harder to work with than definable pseudoplanes, the good geometric properties of S can be exploited to study and control them. And while S does always interpret a pseudoplane, the set of "points" of the pseudoplane might be imaginary, making it much harder (if at all possible) to use the geometric properties of S to construct a field.

However — in the above notation — the existence of a definable pseudoplane in S^2 could simplify the construction of the field in many instances. For example, it seems that Rabinovich's work [1993] would be much simpler if one assumes the set of points of the given pseudoplane to be the affine plane[1]. The lack of a pseudoplane on S^2 is also at the technical heart of [Castle 2023], and moreover explains a key reason that strongly minimal expansions of groups are historically easier to handle — since in the case of groups, one can always find such a pseudoplane (potentially after quotienting by a finite subgroup).

A crucial difference between a definable pseudoplane in S^2 and a 2-dimensional family of plane curves is that the latter allows the existence of *semi-indistinguishable points*. These are certain exceptional pairs of points incident to infinitely many

[1] See the discussion on p. 2–3 of [Rabinovich 1993]. Unfortunately, the manuscript [9] that is referred to in that text does not seem to be available.

common curves, which underlay many combinatorial intricacies of trichotomy proofs. Semi-indistinguishable points tend to prohibit the "full" recovery of the underlying geometry. Indeed, already in [Martin 1988], it was pointed out (in an example attributed to Hrushovski, see Example 3.6) that a reduct of an algebraically closed field need not be constructible (i.e., quantifier free definable in the language of rings) in any copy of the field it interprets (or even $\mathcal{M}$-definably isomorphic to such a constructible set).

In their seminal work, Hrushovski and Zilber [1996] identified a condition they call *very ampleness*, assuring that a Zariski Geometry, $\mathcal{Z}$, is algebraic in the above sense, i.e., that not only does it interpret a field, K, but it is definably isomorphic (as a Zariski geometry) to a smooth K-algebraic curve. Very ampleness (as defined by Hrushovksi and Zilber) does not go so far as to require a pseudoplane on the set Z^2, but it is a related notion — namely, it asserts the existence of a family of plane curves that separates points in Z^2 in an obvious sense.

In the present paper, we introduce a similar notion of very ampleness[2] for arbitrary strongly minimal sets, characterized by the presence of families of plane curves with no semi-indistinguishable pairs (Corollary 3.21). In fact, we show that the presence of one such family implies the existence of such families of arbitrary dimension (Corollary 6.5); using this, we conclude that a strongly minimal set D is very ample if and only if there is a definable pseudoplane whose points are (a generic subset of) D^2 (Proposition 6.7). Moreover, we show that in the context of Zariski geometries, our version of very ampleness is a necessary and sufficient condition for a Zariski geometry to be algebraic, and is thus equivalent to the original notion (Proposition 4.8).

We apply very ampleness to the study of structures interpretable in algebraically closed fields. We show that every algebraic curve over such a field is very ample (Corollary 3.29). We then show that a very ample strongly minimal set is internal to every strongly minimal set it interprets, a phenomenon allowing for the identification of very ample ACF-interpreted structures with algebraic curves (Proposition 4.1). In particular (noting the known instances of the Zilber trichotomy in the relevant structures, see [Castle 2023; Hasson and Sustretov 2017]), we conclude with an analogue of the main theorem on Zariski geometries (see Theorem 4.14):

Theorem 1. *Let K be an algebraically closed field, M be constructible over K, and $\mathcal{M} = (M, \dots)$ be a strongly minimal reduct of the full K-induced structure on M. Assume $\mathcal{M}$ satisfies the Zilber trichotomy*[3]*. Then the following are equivalent*:

(1) *$\mathcal{M}$ is very ample.*

[2] In [Hrushovski and Zilber 1996] the weaker notion of *ampleness*, equivalent to nonlocal modularity, is used. We do not use this term here.

[3] By "[A structure] $\mathcal{M}$ satisfies Zilber's trichotomy" we mean that every nonlocally modular strongly minimal set definable in $\mathcal{M}$ interprets an algebraically closed field. For ACF-relics, this assumption is redundant, since Zilber's trichotomy is true for those. But as a complete proof has not,

(2) *$\mathcal{M}$ is isomorphic, outside of a finite set, to the full K-induced structure on some irreducible algebraic curve over K.*

(3) *Every constructible subset of every power of M is definable in $\mathcal{M}$.*

For arbitrary ACF-interpreted structures, we get the next result (see Theorem 4.16).

Theorem 2. *Let K be an algebraically closed field, and assume the Zilber trichotomy holds for strongly minimal structures interpreted in K. Let M be constructible over K, and let $\mathcal{M} = (M, \dots)$ be an arbitrary reduct of the K-induced structure on M. Then the following are equivalent:*

(1) *$\mathcal{M}$ is almost strongly minimal, and every strongly minimal set in $\mathcal{M}$ is very ample.*

(2) *Every constructible subset of every power of M is definable in $\mathcal{M}$.*

We then apply Theorem 1 in the case of groups, recovering, in particular, the following (see Theorem 7.8):

Theorem 3. *Let K be an algebraically closed field, and let $(G, \cdot)$ be a one-dimensional divisible algebraic group over K. Let $\mathcal{G}^{\mathrm{Zar}}$ be the full K-induced structure on G, and let $\mathcal{G}^{\mathrm{lin}}$ be the structure endowing G with the group operation and all K-definable endomorphisms of G. Then there are no intermediate structures between $\mathcal{G}^{\mathrm{lin}}$ and $\mathcal{G}^{\mathrm{Zar}}$.*

This expands the main result in [Marker and Pillay 1990] and can be seen as an algebraically closed field analogue of a recent similar result of Abu Saleh and Peterzil [2023] for real closed fields.

For strongly minimal groups, divisibility is equivalent to unbounded exponent. In the case of finite exponent, the statement of Theorem 3 fails: in positive characteristic, Marker and Pillay [1990] give an expansion of $(K, +)^{\mathrm{lin}}$ that interprets a field but does not define multiplication (see the discussion after Corollary 1.8 *loc. cit.*).

Along the way, we prove several technical results that may be of interest on their own right. We single out the following (Theorem 7.3):

Theorem 4. *If $\mathcal{G}$ is a strongly minimal expansion of a group, then $\mathcal{G}$ is not locally modular if and only if there exists a definable $X \subseteq G^2$ that is not a finite boolean combination of cosets of definable subgroups of G.*

This result, while well known among experts, does not seem to exist in writing in full generality, and since we needed it in the present work we took the opportunity to give the details.

The paper is written with an eye toward the proofs of Theorems 1–4 above, but also as a possible reference for future work around Zilber's trichotomy and other

at the time of the writing of this paper, appeared in print, so we chose a formulation of the results whose correctness is independent of Zilber's restricted Trichotomy

questions related to the fine structure of strongly minimal sets and the structures they interpret. For that reason, some of the proofs are stated and proved in somewhat greater generality than is actually needed. Thus, for example, the existence of arbitrarily large very ample families of plane curves (Proposition 6.3) is not explicitly used in the text, but significant parts of its proof (e.g., very ampleness of algebraic curves, see Corollary 3.29) are essential for our arguments.

2. Notation and preliminaries

Throughout, we work in a saturated enough model of a stable theory. Except for the general definition of very ample types (invoked only in Corollary 6.12) the work could be carried out in any structure where all definable strongly minimal sets are stably embedded. Since stable embeddedness is equivalent to uniform stable embeddedness, given a strongly minimal set X there is no harm in assuming that all definable families of subsets of X^n we are considering are parameterized by X-definable sets.

We are using standard model theoretic terminology. For basic concepts such as Morley rank, strong minimality, local modularity, one-based theories, etc., we refer the reader to any textbook covering the first chapters of geometric stability theory such as [Marker 2002], [Pillay 1996], or [Tent and Ziegler 2012].

After fixing a structure $\mathcal{M}$, unless otherwise stated, the word *definable* refers to definability in $\mathcal{M}^{\mathrm{eq}}$ with parameters. All parameter sets are smaller than the level of saturation of $\mathcal{M}$, and are usually denoted, A, B. We use the standard model theoretic abuse of notation and write $a \in M$ instead of $a \in M^{|a|}$, allowing a to also be an element of an imaginary sort.

If X is a strongly minimal definable set, the notation X^{eq} refers to the union of all sorts formed by quotienting definable subsets of powers of X by definable equivalence relations. A definable set in X^{eq} is *stationary* if it has Morley degree one — equivalently, if it has a unique generic type over any set of parameters defining it. It follows by uniform stable embeddedness that every stationary set S in X^{eq} has a canonical base in X^{eq} (i.e., the canonical base of its generic type), which we denote $\mathrm{Cb}(S)$. We use dim and *dimension* to denote Morley rank of definable sets in X^{eq}. If $Y \subseteq Z$ are definable sets in X^{eq}, we say that S is large in Z if $\dim(Z \setminus Y) < \dim(Z)$ and Y is small in Z if $\dim(Y) < \dim(Z)$. A generic subset of Z is a definable subset of Z that is not small. Thus, if $Y \subset Z$ and Z is stationary, Y is generic in Z if and only if it is large in Z.

We will use implicitly the definability of Morley rank in X^{eq} for strongly minimal X; see [Baldwin 1973]. We will also use the resulting fact that, given a small parameter set A, a definable set C in X^{eq} is always a generic member of an A-definable family of $\mathcal{M}$-definable subsets of the same Morley rank.

By a *curve* (in X^{eq}) we mean a one-dimensional definable set, and by a *plane curve* in X, we mean a curve in X^2. Note that we do not require curves to be strongly

minimal. A plane curve C is *trivial* if one of its projections has an infinite fiber. If C and D are nontrivial plane curves, then their composition $D \circ C$ is — in analogy with the composition of functions — the curve $\{(x, z) : (\exists y)((x, y) \in C \wedge (y, z)) \in D\}$.

A definable family of plane curves $\mathcal{C} := \{C_t : t \in T\}$ is *faithful* if $t \neq t'$ implies $\dim(C_t \triangle C_t') = 0$. In greater generality, a definable family of definable sets in X^{eq} is faithful if the symmetric difference of any two members of the family is small in both. The total space (or the graph) of the family $\mathcal{C}$ is denoted $C := \{(x, t) \in X^2 \times T : x \in C_t\}$.

A definable set C is *almost contained* in a definable set C' if $\dim(C \setminus C) < \dim(C)$. The set C is *almost equal* to C' if C is almost contained in C' and C' is almost contained in C (equivalently, if $\dim(C \triangle C') < \dim(C) = \dim(C')$). This is a definable equivalence relation (on families of definable sets). It is now a standard and easy exercise to show that if C is any stationary set in X^{eq} with $c := \text{Cb}(X)$, then, up to a set of dimension smaller than $\dim(C)$, the set C is a member of the $\varnothing$-definable family $\varphi(x, y) \wedge \theta(y)$ (parameterized by $\theta(y)$), where $\theta(y) \in \text{tp}(c/\varnothing)$ isolates the type (in its rank and degree) and $\varphi(x, c)$ isolates the generic type of C over c in its rank and degree.

A strongly minimal set X is *locally modular*, if any definable faithful family of plane curves in X^2 is (at most) one-dimensional. By this we mean that if such a family is parameterized by a set T then $\dim(T)$, the dimension of the family, is at most one. We will systematically use (without further reference) the equivalence of local modularity and one-basedness, which — in turn — is equivalent to the fact that $\dim(\text{Cb}(S)) \leq 1$ for any strongly minimal $S \subseteq X^2$ [Pillay 1996, §II Proposition 2.6].

3. Very ampleness: first properties

3A. ***Definition and first examples.*** We start with a general definition, that we later specialize almost exclusively to the strongly minimal setting:

Definition 3.1. Let p be a stationary type, with canonical base c, and let A be a set of parameters. We say that p is *very ample over* A if for any two distinct realizations $x, y \models p$, x and c fork over Ay.

Definition 3.2. Let X be a strongly minimal set, definable over a set A. A strongly minimal $C \subset X^2$ is *very ample in* X *over* A if the generic type of C is very ample over A.

Definition 3.3. Let X be a strongly minimal set. We say that X is *very ample* if, for some set A such that X is A-definable, there is a very ample strongly minimal plane curve in X over A.

Definition 3.4. Let $\mathcal{M}$ be a strongly minimal structure. We say that $\mathcal{M}$ is very ample if its universe is very ample as a strongly minimal set.

Example 3.5. Let $(K, +, \cdot)$ be an algebraically closed field, and let $(a, b) \in K^2$ be generic. We claim that the line $L_{(a,b)}$ defined by $y = ax + b$ is very ample in K over $\varnothing$, and thus K is very ample. To see this, first note that the canonical base of the generic type of $L_{(a,b)}$ is just (a, b). Now let $z_1 = (x_1, y_2)$ and $z_2 = (x_2, y_2) \in L_{(a,b)}$ be distinct generics over (a, b). One then easily computes that $\dim(ab/z_2) = 1$, essentially because the family of lines through a point in the plane is one-dimensional. On the other hand, because any two distinct points determine exactly one line, we have $\dim(ab/z_1z_2) = 0$, which gives the desired dependence.

Example 3.6. Suppose $(K, +, \cdot)$ is an algebraically closed field, and consider the reduct $\mathcal{M}$ of K obtained by endowing K with the relations $x^2 + y^2 = z^2$ and $x^2y^2 = z^2$. Then $\mathcal{M}$ is nonlocally modular, but is not very ample. Indeed, suppose $C \subset K^2$ is a strongly minimal plane curve which is very ample over A, and let $c = \mathrm{Cb}(C)$. Let $(x, y) \in C$ be generic over Ac. Without loss of generality, assume x is generic in K over Ac (otherwise y is). Note that any function $\sigma : K \to K$ with the property that $\sigma(x) \in \{x, -x\}$ for all x is an automorphism of $\mathcal{M}$. In particular, there is an automorphism fixing Ac pointwise but sending x to $-x$, and y to some $z = \pm y$. It follows that $\mathrm{tp}(-x, z/Ac) = \mathrm{tp}(x, y/Ac)$, so $(-x, z)$ is generic in C over Ac. Then, by very ampleness, $(-x, z)$ should fork with c over Axy; but this is impossible because $(-x, z)$ is algebraic over (x, y).

Remark 3.7. We note that there is a significant difference between a strongly minimal set, X, being very ample (meaning that it admits a very ample plane curve) and the generic type of X being very ample. As we will see in the sequel, the existence of a (strongly minimal) very ample complete type is equivalent to the structure being non one-based. The former condition is stronger and is equivalent to the existence of a definable pseudoplane whose set of points is generic in X^2. Except for Corollary 6.12 the latter property is reserved for plane curves in X (and we will say that C is very ample in X), so hopefully no confusion will arise.

3B. ***Very ampleness and nonlocal modularity.*** It follows directly from the definition that a very ample strongly minimal set is not locally modular.

Lemma 3.8. *Let X be strongly minimal and definable over A, let C be a strongly minimal plane curve in X, and let $c = \mathrm{Cb}(C)$. If C is very ample in X over A, then* $\dim(c/A) \geq 2$. *In particular, if X is very ample, then X is not locally modular.*

Proof. Let x and y be independent generics in C over Ac. By very ampleness, c forks with x over Ay, which implies in particular that $\dim(c/Ay) \geq 1$.

Claim 3.8.1. $\dim(y/A) = 2$.

Proof. Otherwise, y would belong to a one-dimensional set over A, which must have C as a strongly minimal component. But this would force $c \in \mathrm{acl}(A)$, contradicting $\dim(c/Ay) \geq 1$. □

Now by the claim, and since $\dim(y/Ac) = 1$ by definition, it follows that y forks with c over A. By symmetry this implies

$$\dim(c/A) > \dim(c/Ay) \geq 1,$$

i.e., $\dim(c/A) \geq 2$. □

We will see later on, Corollary 6.12, that the weaker condition of X^{eq} containing a very ample type is equivalent to nonlocal modularity.

Remark 3.9. It also follows from this lemma that if a strongly minimal plane curve C is very ample in X over some set, then C is *nontrivial*, i.e., both projections $C \to X$ are finite-to-one. Indeed, otherwise C would agree up to finitely many points with $\{c\} \times X$ or $X \times \{c\}$ for some $c \in X$, implying that $c = \mathrm{Cb}(X)$ has dimension at most 1 over A.

Note that if X, Y are strongly minimal sets such that $Y \triangle X$ is finite, then for any nontrivial strongly minimal plane curve $C \subseteq X^2$ the set $C \setminus Y^2$ is finite, so $C \cap Y^2$ and C have the same generic type. In particular, if C is very ample in X then $C \cap Y^2$ is very ample in Y. Thus X is very ample if and only if Y is.

Before moving on we mention the following useful and related observation, which may help to further clarify the definition. Roughly, it shows that the non-very ampleness of a nonlocally modular strongly minimal set always arises from an interalgebraicity relation on the plane.

Lemma 3.10. *Let X be strongly minimal and A-definable. Let C be a plane curve in X, definable over $B \supset A$. Assume that for every strongly minimal component S of C we have* $\dim(\mathrm{Cb}(S)/A) \geq 2$. *Let $x, y \in C$ be any two distinct generics over a parameter set $B \supseteq A$. Then either x forks with B over Ay, or x and y are interalgebraic over A.*

Proof. Throughout we assume $A = \varnothing$. Let $S \subset C$ be a strongly minimal component containing x and let $s = \mathrm{Cb}(S)$. Note that $s \in \mathrm{acl}(B)$ and $\dim(x/s) = 1$ (since x is generic in C). Now, by assumption, $\dim(s) \geq 2$; in particular, $s \notin \mathrm{acl}(\varnothing)$. So $\mathrm{tp}(x/s)$ must fork over $\varnothing$, and thus $\dim(x) = 2$. We now argue in two cases:

Case 1: Suppose $x \in \mathrm{acl}(y)$. Since $\dim(x) = 2$, this implies $\dim(y) = 2$. Thus, x and y are interalgebraic, proving the lemma in this case.

Case 2: Suppose $x \notin \mathrm{acl}(y)$; we show that x forks with B over y. By assumption, we have $\dim(x/y) \geq 1$. Note also that $\dim(x/By) \leq 1$ (as $x \in C_t$). Now suppose toward a contradiction that $x \downarrow_y B$. Then clearly we have $\dim(x/y) = \dim(x/By) = 1$. Since $s \in \mathrm{acl}(B)$, also $\dim(x/Bsy) = 1$. In particular, $\mathrm{tp}(x/Bsy)$ (i.e., the generic type of S over Bsy) does not fork over y; so since $s = \mathrm{Cb}(S)$, we get that $s \in \mathrm{acl}(y)$. Since $y \in X^2$, this gives $\dim(s) \leq 2$. Combined with the assumption that $\dim(s) \leq 2$, we therefore have $\dim(s) = 2$, which implies that s and y are interalgebraic. But then $\dim(y/s) = \dim(y/Bs) = 0$, and since $s \in \mathrm{acl}(B)$ this implies $\dim(y/B) = 0$, contradicting that y is generic in C over B. □

3C. ***Extending the definition.*** In many applications, it is more convenient to work with families of plane curves that are not necessarily generically strongly minimal. We extend our definition of very ampleness to such (families of) curves. To do this, we observe that the choice of defining parameter for a plane curve, C, does not affect Definition 3.2 in any significant way.

Lemma 3.11. *Let X be a strongly minimal and A-definable set, and let C be an At-definable plane curve in X for some tuple t. The following are equivalent*:

(1) *For any two distinct $x, y \in C$ each generic over At, x forks with t over Ay.*

(2) *For any two distinct $x, y \in C$ each generic over At, $\dim(t/Axy) \leq \dim(t/A) - 2$.*

(3) *For every strongly minimal set $S \subset C$, $\dim(\mathrm{Cb}(S)/A) \geq 2$, and for any two distinct $x, y \in C$ each generic over At, x and y are not interalgebraic over A.*

Proof. Throughout, we assume $A = \varnothing$. By our standing assumption that X is stably embedded, we may assume that t is a tuple in X.

(1) $\Longrightarrow$ (2): Assume (1), and let x and y be distinct generics of C_t over t.

Claim 3.11.1. $\dim(y) = 2$.

Proof. Let y' be an independent realization of $\mathrm{tp}(y/t)$ over ty. Then by (1), y forks with t over y'. Since $\dim(y/t) = 1$ by assumption, we obtain $\dim(y/y') = 2$, which implies the claim. □

Now since $\dim(y) = 2$ and $\dim(y/t) = 1$, it follows that $\dim(t/y) = \dim(t) - 1$. By (1) again, t forks with x over y, which gives $\dim(t/xy) \leq \dim(t) - 2$, as desired.

(2) $\Longrightarrow$ (3): Assume (2). First let S be a strongly minimal component of C, with canonical base s. Let x and y be independent generics in S over st. Then $\dim(txy) = \dim(t) + 2$, and by (2) $\dim(t/xy) \leq \dim(t) - 2$, which combined gives $\dim(xy) = 4$. On the other hand $\dim(sxy) = \dim(s) + 2$, so since $\dim(xy) = 4$ we get $\dim(s/xy) = \dim(s) - 2$. In particular, $\dim(s) - 2 \geq 0$, so $\dim(s) \geq 2$.

Now to complete the proof of (3), suppose x and y are any two distinct generics of C over t. Then $\dim(txy) \geq \dim(tx) = \dim(t) + 1$, while by (2) $\dim(t/xy) \leq \dim(t) - 2$. It follows that $\dim(xy) \geq 3$, while if x and y were interalgebraic we would have $\dim(xy) \leq 2$. Thus, we have shown (3).

(3) $\Longrightarrow$ (1): Assume (3). Let x and y be a counterexample to (1). By (3), the hypotheses of Lemma 3.10 are satisfied, and applying the lemma gives that x and y are interalgebraic over A, contradicting (3). □

Corollary 3.12. *Let X be strongly minimal and A-definable. For $i = 1, 2$ let C_i be an At_i-definable plane curve in X for some tuple t_i. Assume that C_1 and C_2 have finite symmetric difference. Then the equivalent conditions of Lemma 3.11 hold for C_1 over At_1 if and only if they hold for C_2 over At_2.*

Proof. We use (3) of the lemma. Since the C_i have finite symmetric difference, the canonical bases of their strongly minimal components coincide. So, by symmetry, it suffices to show that the failure of the second clause of (3) for C_1 and t_1 implies the failure of the same clause for C_2 and t_2. To that end, let x and y be distinct generics in C_1 over At_1, and assume they are interalgebraic over A. Let (x', y') be an independent realization of $\mathrm{tp}(xy/At_1)$ over At_1t_2. Then x' and y' are independent generics over C_2 over At_2, and are also interalgebraic over A, as desired. □

The following is now well defined:

Definition 3.13. Let X be strongly minimal and A-definable. Let C be a plane curve in X. Then C is *very ample in X over A* if any of the equivalent conditions of Lemma 3.11 hold for C over At, for some (equivalently, any) tuple t such that C is At-definable.

It is immediate from the last corollary that Definitions 3.2 and 3.13 coincide for strongly minimal plane curves, since up to a finite set any such curve is definable over its canonical base. Let us also point out the following:

Corollary 3.14. *Let X be strongly minimal and definable over A. Let C be a plane curve in X. If C is very ample in X over A, then so is every strongly minimal component of C.*

Proof. Assume C, and all of its strongly minimal components, are definable over At for some tuple t, and let $S \subset C$ be any such component. Let $x \neq y$ be generics in S over At. Then x and y are also generics of C over At; so by the very ampleness of C over A, x forks with t over Ay, which shows that S is also very ample over A. □

The converse of the last corollary is not true: there can be non-very ample plane curves every component of which is very ample, as in the following example:

Example 3.15. Let $\mathcal{G} = (G, +, \dots)$ be a strongly minimal expansion of a group, and let $S \subset G^2$ be a $\varnothing$-definable strongly minimal plane curve whose generic type has trivial stabilizer (for example $\mathcal{G}$ could be the additive group of the complex field, and S could be the graph of $y = x^2$). Now let $a \in G^2$ be generic, let $S_a := S + a$, and let $C = S_a \cup -S_a$. It is easy to see that for $s \in S_a$ generic over a, the pair $s, -s \in C$ contradicts the very ampleness of C (by (3) of Lemma 3.11). However, each of S_a and $-S_a$ is very ample; see Example 3.18 below.

3D. ***Very ampleness in families.*** The reader may find our definition of very ampleness foreign in light of the analogous notion in [Hrushovski and Zilber 1996] (a precise statement on the relation of our notion of very ampleness and the original term in the context of Zariski geometries requires additional preparation, and can be found in Section 4B). We hope that the present subsection will help to explain for now why our definition captures the original idea of the term. To that end, we now define a notion of very ampleness for families of plane curves.

Notation. If S is definable and $X := \{X_t : t \in T\}$ is a definable family of subsets of S, then for $s \in S$ we denote the set $\{t \in T : s \in X_t\}$ by X^s.

Definition 3.16. Let X be strongly minimal and A-definable, and let $\mathcal{C} = \{C_t : t \in T\}$ be an A-definable family of plane curves in X.

(1) $\mathcal{C}$ is *very ample* if for any $x \neq y \in M^2$, we have $\dim(C^x \cap C^y) \leq \dim T - 2$.

(2) $\mathcal{C}$ is *generically very ample* if for any generic $t \in T$ over A and any distinct $x, y \in C_t$ generic over At, we have $\dim(t/Axy) \leq \dim T - 2$.

Observe that, in particular, a generically very ample family of plane curves is at least two-dimensional.

Example 3.17. Let $(K, +, \cdot)$ be an algebraically closed field. Then, similarly to Example 3.5, one shows easily that the family of lines $y = ax + b$ is very ample, because any two distinct points determine exactly one line.

The following example is well known and easy:

Example 3.18. Let $(G, +)$ be a strongly minimal expansion of a group, and let $C \subset G^2$ be a strongly minimal plane curve. Then the family of translates of C, $\{C + t : t \in G^2\}$, is very ample if and only if the generic type of C has trivial stabilizer. Indeed, $t_1 + C$ is almost equal to $t_2 + C$ if and only if $t_1 - t_2 \in \mathrm{Stab}(p)$ for p the generic type of C.

Proposition 3.19 below, in addition to the ensuing two useful corollaries, shows the relationship between the notions of very ampleness we have defined so far (of a plane curve, of a strongly minimal set, and of a family of plane curves).

Proposition 3.19. *Let X be strongly minimal and A-definable, let $\mathcal{C} = \{C_t : t \in T\}$ be an A-definable family of plane curves in X, and let $C \subset X^2 \times T$ be the graph of $\mathcal{C}$. Then the following are equivalent*:

(1) *$\mathcal{C}$ is generically very ample.*

(2) *There is a nongeneric A-definable set $Z \subset C$ such that $C - Z$ is the graph of a very ample family of plane curves in X.*

(3) *For any generic $t \in T$ over A, C_t is very ample in X over A.*

Proof. The equivalence of (1) and (3) is Lemma 3.11.

(1) $\Longrightarrow$ (2): Let $p(x, y, z)$ be a partial type over A asserting that $x \neq y \in X^2$ and that (x, z) and (y, z) are both generics in C over A. It is a restatement of (1) that for $(x, y, z) \models p$, we have $\dim(z/Axy) \leq \dim(T) - 2$. By compactness, this is witnessed by a finite part of p. We thus obtain a nongeneric $Z \subset C$ such that for $x \neq y$, we have $\dim((C - Z)^x \cap (C - Z)^y) \leq \dim T - 2$. This almost implies (2). Technically, though, one should add to Z all C_t having only finitely many points surviving in $C - Z$ (so that $C - Z$ is a family of plane curves); one should then

verify that Z is still nongeneric in C, and that set of "surviving" $t \in T$ in $C - Z$ is generic in T. Both of these are easy to do.

(2) $\Longrightarrow$ (1): Let Z be as in (2), and let $p(x, y, z)$ be the same type considered in the case above. We show that for $(x, y, z) \models p$ we have $\dim(z/Axy) \leq \dim T - 2$, which as above is equivalent to (1). So take such (x, y, z). We may assume that $(x, y, z) \perp_A B$ where Z is B-definable; thus (x, y, z) is generic in C over AB, so $(x, y, z) \in C - Z$. Then by very ampleness, $\dim(z/ABxy) \leq \dim(T) - 2$, and again since $(x, y, z) \perp_A B$ this implies $\dim(z/Axy) \leq 2$, as desired. □

To sum up, we show that a plane curve is very ample precisely when it coincides, up to a finite set, with a generic member of a very ample family.

Corollary 3.20. *Let X be strongly minimal and A-definable. Let C be a plane curve in X. Then the following are equivalent*:

(1) *C is very ample in X over A.*

(2) *C has finite symmetric difference with an A-generic member of an A-definable very ample family of plane curves in X.*

(3) *Whenever T is a stationary definable set, $\mathcal{C} = \{C_t : t \in T\}$ is an* $\operatorname{acl}(A)$*-definable family of plane curves in X, and $t \in T$ is a generic element over* $\operatorname{acl}(A)$ *such that C and C_t have finite symmetric difference, the family $\mathcal{C}$ is generically very ample.*

Proof. Throughout, we assume $A = \varnothing$. It is immediate from Proposition 3.19 that (2) implies (1) and (1) implies (3). To see that (3) implies (1), simply note that every plane curve in X is a generic member of some $\varnothing$-definable family of plane curves, and the parameter space for such a family can be made stationary after passing to $\operatorname{acl}(\varnothing)$. So it is enough to show that (1) implies (2).

Now suppose (1) holds. As above, we can realize C as a generic member of a $\varnothing$-definable family of plane curves, say $C_{t_0} \in \mathcal{C} = \{C_t : t \in T\}$. By compactness and the definability of dimension, we may assume that for each $t \in T$, for any distinct generics $x, y \in C_t$, we have $\dim(t/xy) \leq \dim T - 2$ – that is, that $\{C_t : t \in T\}$ is generically very ample. By Proposition 3.19, $\mathcal{C}$ becomes very ample after removing a nongeneric $\varnothing$-definable subset of its graph. More precisely, apply Proposition 3.19 to the restriction of the family T to an $\operatorname{acl}(\varnothing)$-definable stationary generic subset $T_0 \subseteq T$ containing t_0, then use the fact that the conjugates of T_0 over $\varnothing$ are also very ample, and therefore, so is their union, T. Since $t_0 \in T$ is generic, only finitely many points of C_{t_0} are removed, which is enough to prove (2). □

We conclude this section with a geometric characterization of very ampleness.

Corollary 3.21. *Let X be strongly minimal. Then X is very ample if and only if there is a very ample definable family of plane curves in X.*

Proof. If X is very ample, by definition there is a plane curve in X which is very ample over some set. Then, by the previous corollary, there is a very ample family of plane curves in X.

If on the other hand there is a very ample family, then by the proposition there is a very ample plane curve over some set. Then by Corollary 3.14 every strongly minimal component of that plane curve is very ample over the same set, so that by definition X is very ample. □

3E. ***Preservation properties.*** In this section, we develop three basic preservation properties of very ampleness; we then pose a fourth (stronger) property as a question, and answer this question in the context of expansions of fields. To begin we point out, generalizing the preservation under naming parameters (Lemma 3.11), that very ampleness is preserved under arbitrary expansions.

Lemma 3.22. *Any strongly minimal expansion of a very ample strongly minimal structure is very ample.*

Proof. Immediate by Corollary 3.21, since a very ample family is still definable in any expansion (technically we use here that the dimension of a definable set is unchanged in strongly minimal expansions, which is easy to check). □

Next we show that very ample plane curves are closed under "independent compositions," in an appropriate sense. This will be used later on to show that very ample strongly minimal structures admit very ample families of plane curves of arbitrarily large dimension.

Lemma 3.23. *Let X be strongly minimal and A-definable. Let C and D be plane curves in X, definable over As and At, respectively. Assume that s and t are independent over A. If C and D are each very ample over A, then so is $D \circ C$.*

Proof. Without loss of generality, assume $A = \varnothing$. Let (x_1, z_1), (x_2, z_2) be distinct generics in $D \circ C$ over st. It will suffice to show that

$$\dim(st/x_1z_1x_2z_2) \leq \dim(st) - 2.$$

Without loss of generality, assume $x_1 \neq x_2$. By definition of $D \circ C$ there are y_1, y_2 such that each $(x_i, y_i) \in C$ and each $(y_i, z_i) \in D$. Then each (x_i, y_i) is generic in C over st (so also over s), which by the very ampleness of C gives that

$$\dim(s/x_1y_1x_2y_2) \leq \dim(s) - 2.$$

Next, note that y_i and z_i are interalgebraic over t (for $i = 1, 2$), by the nontriviality of D; see Remark 3.9. Thus,

$$\begin{aligned}\dim(st/x_1z_1x_2z_2) &= \dim(sty_1y_2/x_1z_1x_2z_2)\\ &= \dim(t/x_1z_1x_2z_2) + \dim(y_1y_2/tx_1z_1x_2z_2) + \dim(s/tx_1y_1z_1x_2y_2z_2).\end{aligned}$$

In this last expression, note that the first term is at most $\dim(t)$, the second term is 0, and the third term is at most $\dim(s) - 2$ as we have seen above. Thus,

$$\dim(st/x_1z_1x_2z_2) \leq \dim(s) + \dim(t) - 2,$$

which, by the independence of s and t, is equivalent to the desired statement. □

Finally, we point out that very ampleness is preserved under finite-to-one functions.

Lemma 3.24. *Suppose X and Y are strongly minimal sets, and $f : X \to Y$ is a definable finite-to-one function. If X is very ample, then so is Y.*

Proof. Without loss of generality, assume X, Y, and f are $\varnothing$-definable. Let C be a strongly minimal plane curve in X which is very ample over some set A. Without loss of generality, we assume also that $A = \varnothing$. Let $c = \mathrm{Cb}(C)$. After editing finitely many points, we may assume C is definable over c.

Now let D be the image of C in Y^2 (applying f to each coordinate). So D is definable over c. Since f is finite-to-one, D is moreover strongly minimal. We claim that D is very ample in Y. To that end, suppose z, w are distinct generics in D over c. It will suffice to show that z forks with c over w. Now by assumption there are $x, y \in C$ with $f(x) = z$ and $f(y) = w$. Since f is finite-to-one, each of the pairs x, z and y, w are interalgebraic. So by dimension considerations, x and y are generics in C. Moreover, since $z \neq w$, it follows that $x \neq y$. Thus, by the very ampleness of C, x forks with c over y. Then, by interalgebraicity, z forks with c over w, as desired. □

Recall that two strongly minimal sets are *nonorthogonal* if there is a definable (possibly over additional parameters) finite-to-finite correspondence between them. Lemma 3.24 does not extend to the preservation of very ampleness under nonorthogonality, as can be seen from Example 3.6. Indeed, the (non-very ample) structure $\mathcal{M}$ from Example 3.6 is nonorthogonal to the strongly minimal set $K/\sim$, where $\sim$ is the relation $x^2 = y^2$ on K; but the induced structure on $K/\sim$ is a pure algebraically closed field, so very ample by Example 3.5.

As we will see later, the main reason very ampleness does not pass from $K/\sim$ to $\mathcal{M}$ in Example 3.6 is that K is not *internal* to $K/\sim$ (in the sense of the reduct, $\mathcal{M}$). We may thus ask the following:

Question 3.25. Is very ampleness preserved under internality? That is, if X is very ample, and Y is strongly minimal and internal to X, must Y be very ample?

To conclude this subsection, we now answer Question 3.25 whenever X has a definable field structure. We will later improve on this result in Corollary 4.2 and in the proof of Proposition 6.3. In order to smoothly recall the setting in this later proposition, we present the result here as a corollary of two general statements, Lemma 3.27 and Proposition 3.28.

Before proceeding, let us clarify:

Remark 3.26. By *expansion* below, we are assuming the field structure is part of the signature of $(K, +, \cdot, \dots)$. In general, for a strongly minimal structure, $\mathcal{M}$, admitting a definable field structure (with universe M), elimination of imaginaries will hold after naming any set of parameters defining a field structure on M.

Lemma 3.27. *Every strongly minimal expansion of an algebraically closed field eliminates imaginaries.*

Proof. Let $(K, +, \cdot, \dots)$ be an expansion of an algebraically closed field. Because of the field structure, it is automatic that $\operatorname{acl}(\varnothing)$ is infinite. So by weak elimination of imaginaries in strongly minimal structures, it suffices to code finite sets: that is, any such expansion $(K, +, \cdot, \dots)$ has elimination of imaginaries if and only if there are $\varnothing$-definable injections $K^{(n)} \to K^m$ for all n (and some m depending on n), where $K^{(n)}$ denotes the n-th symmetric power of K. But such functions already exist in the pure field $(K, +, \cdot)$, so we are done. □

Proposition 3.28. *Let $(K, +, \cdot, \dots)$ be a strongly minimal expansion of an algebraically closed field, and let $X \subset K^n$ be a $\varnothing$-definable set of dimension $r \geq 1$. Let $\mathcal{H} = \{H_t : t \in T\}$ be the family of hyperplanes in K^n, and let $t \in T$ be generic. Then $\dim(H_t \cap X) = r - 1$, and if $x, y \in H_t \cap X$ are distinct generic over t, then x and t fork over y.*

Proof. Throughout this proof, we freely use the fact that the strongly minimal expansion $(K, +, \cdot, \dots)$ preserves the dimensions of $(K, +, \cdot)$-definable sets, which is well known and easy to check. We also use that T can be identified by a large subset of $\mathbb{P}^n(K)$ (the set of projective hyperplanes intersecting K^n); in particular, it follows that T is stationary of dimension n. As is well known, for any $x \in K^n$ the set H^x (the hyperplanes through x) then has dimension $n - 1$, and for $x \neq y$ the set $H^x \cap H^y$ has dimension $n - 2$.

Now we prove the proposition in two claims.

Claim 3.28.1. $\dim(H_t \cap X) = r - 1$.

Proof. Let $x \in X$ be generic and $s \in H^x$ be generic over x. Then $\dim(xs) = r + n - 1$. Since $\dim(s) \leq n$, this forces $\dim(x/s) \geq r - 1$. In particular, $\dim(H_s \cap X) \geq r - 1$.

Let us verify that $\dim(H_s \cap X) = r - 1$. Suppose not. Then there is $y \in H_s \cap X$ with $\dim(y/xs) = r$. Thus $\dim(xys) = 2r + n - 1$. But $\dim(xy) \leq 2r$, so $\dim(s/xy) \geq n - 1$. As observed above, this is only possible if $x = y$; but clearly $x \neq y$, since $\dim(y/xs) = r \geq 1$.

So $\dim(H_s \cap X) = r - 1$, and thus $\dim(x/s) \leq r - 1$. Recalling that $\dim(xs) = r + n - 1$, we conclude that $\dim(s) = n$ and $\dim(x/s) = r - 1$. In particular, s is generic in T. Since T is stationary, $\operatorname{tp}(s) = \operatorname{tp}(t)$, so also $\dim(H_t \cap X) = r - 1$. □

Claim 3.28.2. *Let $x \neq y$ be generics in $H_t \cap X$ over t. Then x forks with t over y.*

Proof. By assumption $\dim(t) = n$, and by the previous claim, $\dim(y/t) = r - 1$. So $\dim(yt) = n + r - 1$. But $\dim(t) \leq r$ since $y \in X$; and since $t \in H^y$, we get $\dim(t/y) \leq n - 1$. So we must have $\dim(y) = r$ and $\dim(t/y) = n - 1$. But since $t \in H^x \cap H^x$, we also have $\dim(t/xy) \leq n - 2$, which proves the claim. □

Thus we have proved Proposition 3.28. □

Corollary 3.29. *Let K be a strongly minimal expansion of a field, and let X be a strongly minimal set internal to K. Then X is very ample.*

Proof. Adding parameters if necessary, we may assume X is $\varnothing$-definable. By Lemma 3.27, we may assume $X \subset K^n$ for some n. Then by Proposition 3.28 (applied to $X^2 \subset K^{2n}$), there is a very ample plane curve in X. So X is very ample. □

3F. ***When very ampleness can be guaranteed.*** As we have seen, our first example of a non-very ample structure (Example 3.6) still admits a very ample *sort* — that is, it becomes very ample after dividing by a definable equivalence relation. We thus think of K of that example as very ample "up to a finite cover," which is precisely the situation of Theorem B in [Hrushovski and Zilber 1996]. We do not know whether this is true of all strongly minimal nonlocally modular structures.

Question 3.30. Suppose X is strongly minimal and not locally modular.

(1) Can one always find a very ample strongly minimal set Y internal to X?

(2) More specifically, can one always find a definable equivalence relation $\sim$ on X so that $X/\sim$ is strongly minimal and very ample?

Later in this paper, we will see that for strongly minimal expansions of groups G, the answer to Question 3.30 (2) is positive. To show that, we pick a well-chosen two-dimensional family of plane curves $\{C_t : t \in T\}$ and show that the relation $I(x, y) \subset G^2$, given by $I(x, y)$ if and only if for some z and w we have $|C^{(x,z)} \cap C^{(y,w)}| = \infty$, is contained in the graph of an equivalence relation $\sim$ on G with finite classes, (in fact, the equivalence classes are cosets of a finite subgroup). Then we show that in the quotient $G/\sim$ the image of $\{C_t : t \in T\}$ is very ample. For a general strongly minimal structure, this strategy seems to work once we can find a definable two-dimensional family of plane curves where the analogous relation I is contained in an equivalence relation with finite classes. It can also be shown that we can always find families of plane curves where $I(x, y)$ is one-dimensional. However, we do not know how to extend I to an equivalence relation in general.

Proposition 3.32 shows that, as one might expect from the above discussion, nonlocally modular strongly minimal structures with elimination of imaginaries are very ample.

Definition 3.31. Let X be strongly minimal and A-definable. We say that X *eliminates imaginaries over* A if for any elements $a_1, \ldots, a_n \in X$ and any $c \in \mathrm{dcl}^{eq}(A\bar{a})$, there is a finite sequence $b_1, \ldots, b_m \in X$ with $\mathrm{dcl}^{eq}(A\bar{b}) = \mathrm{dcl}^{eq}(Ac)$.

Proposition 3.32. *Let X be strongly minimal, A-definable, and not locally modular. If X eliminates imaginaries over A, then X is very ample.*

Proof. By nonlocal modularity, there are a set $B \supset A$ and a strongly minimal plane curve $C \subset X^2$, such that $\dim(c/B) = 2$, where $c = \mathrm{Cb}(C)$. By stable embeddedness of X, there are $a_1, \ldots, a_n \in X$ with $c \in \mathrm{dcl}(A\bar{a})$. So by elimination of imaginaries, there are $b_1, \ldots, b_m \in X$ with $\mathrm{dcl}(A\bar{b}) = \mathrm{dcl}(Ac)$, and in particular, $\mathrm{dcl}(B\bar{b}) = \mathrm{dcl}(Bc)$. So $\dim(\bar{b}/B) = 2$, and we may extract a two element basis for $\bar{b}$ over B. Without loss of generality, let us assume $\dim(b_1b_2/B) = 2$. So $b_1b_2 \in \mathrm{dcl}(Bc)$ and $c \in \mathrm{acl}(Bb_1b_2)$.

Now let $x \in X^2$ be generic in C over Bc. So $\dim(cx/B) = 3$. Since $c \notin \mathrm{acl}(B)$, x forks with c over B, which implies that $\dim(x/B) = 2$, and thus $\dim(c/Bx) = 1$. By interalgebraicity, $\dim(b_1b_2/Bx) = 1$. Let $p = \mathrm{stp}(b_1b_2/Bx)$; so p is a minimal type in X^2. Let S be a strongly minimal plane curve whose generic type is p. So (b_1, b_2) is generic in S over $\mathrm{acl}(Bx)$.

Claim 3.32.1. *S is very ample in X over B.*

Proof. Let (b_1', b_2') be another generic of S over $\mathrm{acl}(Bx)$. So $(b_1', b_2') \models p$. Then there is some c' with $\mathrm{tp}(b_1b_2cx/B) = \mathrm{tp}(b_1'b_2'c'x/B)$. Since $b_1b_2 \in \mathrm{dcl}(Bc)$ and $(b_1', b_2') \neq (b_1, b_2)$, it follows that $c \neq c'$. Because $\mathrm{tp}(cx) = \mathrm{tp}(c'x)$ there is an automorphism mapping cx to $c'x$ and mapping $\mathrm{tp}(x/c)$ to $\mathrm{tp}(x/c')$. Since $c \neq c'$, $c = \mathrm{Cb}(x/c)$, and $c' = \mathrm{Cb}(x/c')$, the (stationary) types $\mathrm{tp}(x/c)$ and $\mathrm{tp}(x/c')$ have no common nonforking extension (essentially, by the definition of canonical bases), implying that x forks with c' over c. □

We have shown that X admits a very ample strongly minimal plane curve, so X is very ample. □

Remark 3.33. The formulation of the above proof using types is convenient, but may obscure the geometric idea. Let us now explain the proof in more geometric terms: we first use nonlocal modularity to find a faithful two-dimensional family of plane curves; we then use elimination of imaginaries to reparametrize this family by a definable set $T \subset X^n$, and by taking a projection $X^n \to X^2$, we further reparametrize (up to losing generic strong minimality of the curves) by X^2. We then "dualize" the resulting family (interchanging points and curves), and observe that the "dual" notion of faithfulness is precisely very ampleness.

Remark 3.34. The converse of Proposition 3.32 is false: Let $\mathcal{M} = (M, \ldots)$ be the full $\mathbb{C}$-induced structure on a smooth irreducible curve M over $\mathbb{C}$ of genus $g > 0$. Then $\mathcal{M}$ is very ample by Corollary 3.29, but does not eliminate the imaginary $\mathbb{C}$

over any set of parameters. Let us elaborate: First, by the Zilber trichotomy ([Hasson and Sustretov 2017] or [Castle 2023]), $\mathcal{M}$ interprets a set isomorphic to $\mathbb{C}$, and thus interprets a smooth curve X of genus 0. If X were embeddable into some M^n, then by composing with a projection one could obtain a nonconstant rational map $X \to M$, which (by completing, normalizing, and then comparing the genus of each curve) is easily seen to contradict the Riemann–Hurwitz formula.

4. Applications of very ampleness

We now turn toward applications of very ampleness, with an eye toward improved versions of the Zilber trichotomy for structures interpreted in algebraically closed fields. Unless explicitly stated otherwise, we work in a uncountable saturated stable structure $\mathcal{M}$. Throughout this section and Section 7 statements will often include the disclaimer "assuming Zilber's trichotomy" (for structures interpretable in algebraically closed fields) or the weaker "an algebraically closed field is interpretable". At the time of writing of this paper, the status of Zilber's trichotomy for such structures is as follows:

(1) Hasson and Sustretov [2017] have a proof of the trichotomy in the one-dimensional case.

(2) Castle [2023] has a proof of the trichotomy in characteristic 0 (for all dimensions). This result is independent of [Hasson and Sustretov 2017].

(3) These results are yet to be published.

(4) Castle, Hasson, and Ye [Castle et al. 2024] have a proof of the trichotomy in positive characteristic.

4A. ***Very ampleness and internality.*** The following proposition summarizes the effect of very ampleness in comparing strongly minimal sets. In fact, almost every application we make of very ampleness will, ultimately, follow from this result:

Proposition 4.1. *Let X and Y be strongly minimal sets, and assume X is very ample. If X is nonorthogonal to Y, then X is internal to Y.*

Proof. By nonorthogonality, there is a strongly minimal set $C \subset X \times Y$ such that both projections $C \to X$ and $C \to Y$ are finite-to-one. Without loss of generality (by adding finitely many points to C), assume that $C \to X$ is surjective. We then view C as a multivalued function $X \to Y$, mapping each $x \in X$ to the finite set $\{y \in Y : (x, y) \in C\}$. In particular, it follows that one can find a sort S in Y^{eq}, and a finite-to-one definable map $f : X \to S$. Absorbing parameters into the language, we may assume that f is $\varnothing$-definable.

Now by assumption there is a plane curve $C \subseteq X^2$ which is very ample over some set A. Applying f coordinatewise, we obtain a finite-to-one function $\tilde{f} : C \to S^2$.

Note that since f is $\varnothing$-definable, if $\tilde{f}(x_1, x_2) = \tilde{f}(y_1, y_2)$ then $(x_1, x_2), (y_1, y_2)$ are interalgebraic over $\varnothing$. So by condition (3) of Lemma 3.11, there are no distinct generics $(x_1, x_2), (y_1, y_2) \in C$ over A such that $\tilde{f}(x_1, x_2) = \tilde{f}(y_1, y_2)$. In other words, $\tilde{f}$ is generically injective. Letting $T = \mathrm{im}(\tilde{f})$ (potentially minus a finite set) and inverting $\tilde{f}$, we can then extract a definable function $g : T \to C$ with cofinite image. Then composing with a coordinate projection π, the resulting map $\pi \circ g : T \to X$ has cofinite image in X. This shows that X is internal to T, and since $T \subset S^2$, T is internal to Y. Thus, X is internal to Y. □

Before moving on we give two quick applications. First, the following is immediate from Proposition 4.1, Lemma 3.27, and Corollary 3.29:

Corollary 4.2. *Let K be a strongly minimal expansion of an algebraically closed field, and let X be a strongly minimal set nonorthogonal to K. Then the following are equivalent*:

(1) *X is very ample.*

(2) *X is internal to K.*

(3) *There is a definable embedding of X into some K^n.*

It follows from this last corollary, that if X is very ample and interprets an algebraically closed field, K, then any strongly minimal set Y interpretable in X is very ample. Indeed, by the corollary X is internal to K, and since Y is internal to X, Y too is internal to K, so by the corollary, again, Y is very ample. Thus, in particular, a counterexample to Question 3.25 must be a counterexample to Zilber's trichotomy.

Next, we give a partial answer to Question 3.30 for strongly minimal sets nonorthogonal to very ample sets.

Corollary 4.3. *Let X and Y be strongly minimal and nonorthogonal, and assume Y is very ample. Then*:

(1) *There is a very ample strongly minimal set internal to X.*

(2) *If, moreover, Y eliminates imaginaries (over some parameter set), then there is a definable equivalence relation $\sim$ on X with finite classes such that $X/\sim$ is very ample.*

Proof. (1) is immediate, because by Proposition 4.1, Y is internal to X. So we need only prove (2). For that, assume that Y eliminates imaginaries over some set. By nonorthogonality, one can find a finite-to-one map $f : X \to S$, where S is a sort in Y^{eq}. By elimination of imaginaries we may assume $S \subset Y^n$ for some n, so we have $f : X \to Y^n$. Then composing with an appropriate projection $Y^n \to Y$, we obtain a finite-to-one map $X \to Y$, which shows that (up to editing finitely many points) Y is a quotient of X. □

Remark 4.4. In particular, it follows from Corollary 4.3 that every nonlocally modular strongly minimal structure satisfying the Zilber trichotomy admits a positive answer to both parts of Question 3.30.

4B. ***Very ample Zariski geometries.*** In the present section, we show that our definition of very ampleness is equivalent, in the context of Zariski geometries, to the original definition of Hrushovski and Zilber. The results of this section are not needed in the sequel, and the reader only interested in the application may safely skip to the next section.

Recall that a strongly minimal Zariski geometry[4] is a structure $\mathcal{Z}$, in a language equipping it with a compatible system of Noetherian topologies on Z^n (all n), such that $\mathcal{Z}$ has quantifier elimination and satisfies the dimension theorem; see [Hrushovski and Zilber 1996] for the details. The Zariski geometry $\mathcal{Z}$ is ample if it is not locally modular; it is then shown [Hrushovski and Zilber 1996, Theorem B] that any ample Zariski geometry interprets an algebraically closed field K. To further identify $\mathcal{Z}$ with a smooth curve over K, the notion of very ampleness is introduced. We now give the definition (paraphrased) as it appears in [Hrushovski and Zilber 1996]. For the sake of clarity, we will refer to this version as Z-very ample.

Definition 4.5. Let $\mathcal{Z}$ be a Zariski geometry with universe Z. Then $\mathcal{Z}$ is *Z-very ample* if there are an irreducible closed set $E \subset Z^n$ for some n, and an irreducible closed set $C \subset E \times Z^2$, such that the following hold:

(1) For generic $e \in E$, the set $C(e)$ is irreducible and one-dimensional.

(2) For any distinct $a, b \in Z^2$ there is some $e \in E$ such that $C(e)$ contains exactly one of a, b.

The main theorem in [Hrushovski and Zilber 1996], equivalently stated, then asserts that every Z-very ample Zariski geometry is isomorphic (as a Zariski geometry) to a smooth curve over an algebraically closed field.

It is not directly mentioned in [Hrushovski and Zilber 1996] that the converse holds — namely, that every smooth curve is very ample. We thank Hrushovski for outlining the following proof:

Lemma 4.6. *Let Z be a smooth algebraic curve over an algebraically closed field. Then Z is Z-very ample.*

Proof. Since C is a curve it is quasiprojective; so we may assume $C \subset \mathbb{P}^n(K)$ for some n. Our goal is to find a family $C \subset E \times Z^2$ consisting of all hyperplane intersections in Z^2 (that is, sets $H_t \cap Z^2$ where H_t is a hyperplane in $\mathbb{P}^n(K)$). In this case, (1) follows by Bertini's theorem, and (2) follows since any two points in projective space can be separated by a hyperplane.

[4]We are referring to the term as appearing in [Hrushovski and Zilber 1996] and not to the more general one of Zilber [2010].

The only difficulty here is finding such a family which is irreducible, closed, and parametrized by a closed irreducible set in a power of Z (rather than the natural parametrization by $\mathbb{P}^n(K)$). For this, it will suffice to find a surjective morphism $Z^n \to \mathbb{P}^n(K)$, since then we can lift the natural parametrization from $\mathbb{P}^n(K)$ to Z^n.

To find such a morphism, we note that by composing with a surjection $(\mathbb{P}^1)^n \to \mathbb{P}^n$ (which exists because the former is a projective variety of dimension n), it will moreover suffice to find a surjection $C \to \mathbb{P}^1(K)$.

Finally, let us build a surjection $C \to \mathbb{P}^1(K)$. First choose any (dominant) morphism $f : C \to \mathbb{P}^1(K)$, and suppose $f(C)$ misses m points of $\mathbb{P}^1(K)$, say $x_1, \ldots, x_m$. Then the map $g \circ f : C \to \mathbb{P}^1(K)$ is surjective, where $g : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ is any ramified cover with generically $(m+1)$-sized fibers that does not ramify at any of $x_1, \ldots, x_m$. □

It follows by [Hrushovski and Zilber 1996] and Lemma 4.6 that a Zariski geometry $\mathcal{Z}$ is Z-very ample if and only if it is isomorphic to a smooth curve over an algebraically closed field. Our goal now is to show that the same statement holds using our notion of very ampleness in place of Z-very ampleness. To do this, we first need to inspect the proof in [Hrushovski and Zilber 1996], in order to extract the precise use of Z-very ampleness and replace it with a use of very ampleness. In fact, the proof in [Hrushovski and Zilber 1996] is achieved by separately proving the following three facts:

Fact 4.7. Let $\mathcal{Z}$ be a Zariski geometry with universe Z.

(1) If $\mathcal{Z}$ is not locally modular, then $\mathcal{Z}$ interprets an algebraically closed field (p. 47, first paragraph of the proof of Theorem A).

(2) If $\mathcal{Z}$ interprets the algebraically closed field K, and $\mathcal{Z}$ is Z-very ample, then Z is internal to K (from (1) to the end of the paragraph splitting p. 47–48).

(3) If $\mathcal{Z}$ interprets the algebraically closed field K, and $\mathcal{Z}$ is internal to K, then $\mathcal{Z}$ is isomorphic to a smooth curve over K (p. 48, the rest of the proof of Theorem A).

We now show:

Proposition 4.8. *Let $\mathcal{Z}$ be a Zariski geometry. Then the following are equivalent*:

(1) *$\mathcal{Z}$ is very ample.*

(2) *$\mathcal{Z}$ is Z-very ample.*

(3) *$\mathcal{Z}$ is isomorphic to a smooth curve over an algebraically closed field.*

Proof. As stated above, the equivalence of (2) and (3) is [Hrushovski and Zilber 1996, Theorem A] and Lemma 4.6. Moreover, the implication (3) $\Longrightarrow$ (1) follows from Corollary 3.29. So it will suffice to show (1) $\Longrightarrow$ (3).

Assume (1). Then by Lemma 3.8, $\mathcal{Z}$ is not locally modular. So by Fact 4.7 (1), $\mathcal{Z}$ interprets an algebraically closed field K. Then by Proposition 4.1, Z is internal to K. So by Fact 4.7 (3), $\mathcal{Z}$ is isomorphic to a smooth curve over K. Thus (3) holds, and we are done. □

4C. ***Very ample strongly minimal ACF-relics.*** We now turn toward applications of very ampleness to structures interpretable in algebraically closed fields. In [Loveys 2004, Theorem 7.1] Loveys gives a complete list (up to definable finite covers) of all locally modular nontrivial reducts of the algebraically closed field $\mathbb{C}$. If we are interested in the classification of nonlocally modular reducts of $\mathbb{C}$ only up to finite covers, then Rabinovich's theorem [1993] is the final word. But one could hope for a more precise classification of such reducts. One natural question is the identification of reducts that are not proper. To state this problem in somewhat greater generality, it is convenient to have:

Definition 4.9. Let $\mathcal{N}$ be any structure.

(1) An $\mathcal{N}$*-relic* is a reduct, $\mathcal{M}$, of the structure induced on some $\mathcal{N}$-definable set M.

(2) If $\mathcal{M}$ is an $\mathcal{N}$-relic, and X is $\mathcal{M}$-definable, then X is $\mathcal{N}/\mathcal{M}$*-full* if every $\mathcal{N}$-definable subset of every X^n is $\mathcal{M}$-definable.

(3) If $\mathcal{M} = (M, \dots)$ is an $\mathcal{N}$-relic, then $\mathcal{M}$ is *full in* $\mathcal{M}$ if the universe M is $\mathcal{N}/\mathcal{M}$-full.

In a recent unpublished work, Castle and M. Tran give a concrete characterization of fullness for reducts of $\mathbb{C}$ whose atomic sets are polynomial functions, though a similarly concrete identification of all full reducts of $\mathbb{C}$ is not available. In the next two subsections, we show, roughly, that the fullness of an ACF-relic is equivalent to very ampleness. In fact, in the main results of the current subsection, we show an equivalence between several properties of a strongly minimal ACF-relic, notably very ampleness, fullness, and bi-interpretability with the field.

We begin with a couple of straightforward and well-known facts about relics.

Lemma 4.10. *Let $\mathcal{M}$ be an $\mathcal{N}$-relic, and let X be $\mathcal{M}$-definable and $\mathcal{N}/\mathcal{M}$-full. If Y is $\mathcal{M}$-definable and internal to X (in the sense of $\mathcal{M}$), then Y is also $\mathcal{N}/\mathcal{M}$-full.*

Proof. Let $Z \subset Y^n$ be $\mathcal{N}$-definable. By internality, there are an $\mathcal{M}$-definable set $W \subset X^m$, for some m, and an $\mathcal{M}$-definable surjective function $f : W \to Y^n$. By the fullness of X, $f^{-1}(Z) \subset X^m$ is $\mathcal{M}$-definable. Then since f is surjective, $Z = f(f^{-1}(Z))$ is also $\mathcal{M}$-definable. Thus Y is $\mathcal{N}/\mathcal{M}$-full. □

Lemma 4.11. *Let $(K, +, \cdot)$ be an algebraically closed field, let $\mathcal{M} = (M, \dots)$ be a K-relic, and F an infinite $\mathcal{M}$-definable field. Then F is a pure field as seen from both K and $\mathcal{M}$. In particular, F is $K/\mathcal{M}$-full.*

Proof. Since $\mathcal{M}$ defines fewer sets than K, and $\mathcal{M}$ defines the field operations on F, it suffices to show that every K-definable subset of every F^n is definable from only the field operations on F. So let $X \subset F^n$ be K-definable. By [Poizat 1988], there is a K-definable field isomorphism $f : K \to F$. Then $f^{-1}(X) \subset K^n$ is definable from the field operations on K, and since f is an isomorphism, X is definable from the field operations on F. □

Using the previous two lemmas, we now give several equivalent characterizations of fullness for ACF-relics. These are likely well known.

Proposition 4.12. *Let K be an algebraically closed field, and let $\mathcal{M} = (M, \dots)$ be a K-relic. Then the following are equivalent*:

(1) *$\mathcal{M}$ is full in K.*

(2) *$\mathcal{M}$ is bi-interpretable with K over parameters.*

(3) *There is an infinite field F definable in $\mathcal{M}$, such that M is internal to F (in the sense of $\mathcal{M}$).*

(4) *$\mathcal{M}$ defines an infinite field, and for every infinite $\mathcal{M}$-definable field F, M is internal to F (in the sense of $\mathcal{M}$).*

Proof. (1) $\Longrightarrow$ (2): Assume $\mathcal{M}$ is full. First, we point out that $\mathcal{M}$ interprets an infinite field. Indeed, let X be any strongly minimal set in $\mathcal{M}$. By fullness, X is also strongly minimal in K. Thus, by quantifier elimination in K, after potentially deleting finitely many points we may assume X is a smooth irreducible curve, and we conclude by the main theorem on Zariski geometries [Hrushovski and Zilber 1996].

Now let F be an infinite $\mathcal{M}$-definable field. By [Poizat 1988], there is a K-definable isomorphism $f : K \to F$. Let $\mathcal{Y} = (Y, \dots)$ be the image of $\mathcal{M}$ under f (equivalently, the relic of F defined using the same formulas defining $\mathcal{M}$ in K).

We claim that the given interpretations of $\mathcal{M}$ in K, and of F in $\mathcal{M}$, provide a bi-interpretation of K and $\mathcal{M}$ over parameters. Indeed, we already know that K and F are definably isomorphic in K, as witnessed by f. So it remains to show that $\mathcal{Y}$ is $\mathcal{M}$-definably isomorphic to $\mathcal{M}$. But f gives a K-definable isomorphism $\mathcal{M} \to \mathcal{Y}$, and $\mathcal{Y}$ resides entirely in $\mathcal{M}^{\mathrm{eq}}$. So by the fullness of $\mathcal{M}$, this isomorphism is $\mathcal{M}$-definable.

(2) $\Longrightarrow$ (3): The statement of (3) is part of the definition of a bi-interpretation.

(3) $\Longrightarrow$ (4): Assuming (3), let F_1 be an infinite $\mathcal{M}$-definable field such that M is internal to F_1 in the sense of $\mathcal{M}$. To prove (4), it remains to show that if $\mathcal{M}$ defines another infinite field, say F_2, then M is internal to F_2.

By assumption, F_2 is internal to F_1, and thus nonorthogonal to F_1. But by Lemma 4.11, each F_i is very ample and strongly minimal. So by Proposition 4.1, F_1 is internal to F_2, and thus so is M.

(4) $\Longrightarrow$ (1): Assume (4). Let F be an infinite $\mathcal{M}$-definable field. By assumption, M is internal to F in the sense of $\mathcal{M}$. By Lemma 4.11, F is $K/\mathcal{M}$-full. So by Lemma 4.10, M is $K/\mathcal{M}$-full, i.e., $\mathcal{M}$ is full in K. □

Remark 4.13. Given a K-relic $\mathcal{M}$, fullness of $\mathcal{M}$ may, a priori. depend on the interpretation of $\mathcal{M}$ in K. However, as the other five conditions appearing in the statement do not depend on the interpretation, it follows that fullness is a property only of the abstract structure $\mathcal{M}$. In other words, if $\mathcal{M}$ is interpretable in an algebraically closed field, then either all possible interpretations are full or none are.

We now use Proposition 4.12 and the Zilber trichotomy to show that very ampleness characterizes fullness for strongly minimal relics. This can be seen as an analogue of the main theorem on Zariski geometries.

Theorem 4.14. *Let K be an algebraically closed field, and let $\mathcal{M} = (M, \dots)$ be a strongly minimal K-relic which satisfies Zilber's trichotomy. Then the following are equivalent*:

(1) *$\mathcal{M}$ is very ample.*

(2) *$\mathcal{M}$ is full in K.*

(3) *Up to deleting a finite set, $\mathcal{M}$ is isomorphic to an irreducible algebraic curve over K, with its induced structure from K.*

Proof. (1) $\Longrightarrow$ (2): Assume $\mathcal{M}$ is very ample. By Lemma 3.8, $\mathcal{M}$ is not locally modular, so by the Zilber trichotomy, $\mathcal{M}$ interprets an infinite field, say F. Then M and F are nonorthogonal, and by Lemma 4.11, F is strongly minimal. So by Proposition 4.1, M is internal to F. Then, by Proposition 4.12, $\mathcal{M}$ is full.

(2) $\Longrightarrow$ (3): Assuming $\mathcal{M}$ is full, the universe M is also strongly minimal in K. So by quantifier elimination in K, M agrees up to finitely many points with an irreducible curve. This makes (3) clear.

(3) $\Longrightarrow$ (1): Follows by Corollary 3.29. □

4D. ***More on full ACF-relics.*** It follows from Theorem 4.14 that strongly minimal relics of algebraically closed fields are full precisely when they are very ample. In this section, we give a similar characterization of fullness for a general (nonstrongly minimal) relic. In this case, the argument will be a little more delicate. In particular, we need to first develop general conditions under which fullness of a relic can be checked at the level of strongly minimal sets.

Proposition 4.15. *Let $\mathcal{M}$ be an $\mathcal{N}$-relic, where $\mathcal{M}$ and $\mathcal{N}$ are almost strongly minimal. Assume there is a strongly minimal set in $\mathcal{M}$ which is $\mathcal{N}/\mathcal{M}$-full. Then the following are equivalent*:

(1) *$\mathcal{M}$ is full in $\mathcal{N}$.*

(2) *Every strongly minimal set in $\mathcal{M}$ is $\mathcal{N}/\mathcal{M}$-full.*

(3) *Every strongly minimal set in $\mathcal{M}$ is also strongly minimal in $\mathcal{N}$.*

(4) *Every stationary definable set in $\mathcal{M}$ is also stationary in $\mathcal{N}$.*

Proof. Throughout, we use the following:

Claim 4.15.1. *Let X be any $\mathcal{M}$-definable set. Then* $\dim_{\mathcal{M}}(X) = \dim_{\mathcal{N}}(X)$.

Proof. Let F be an $\mathcal{N}/\mathcal{M}$-full strongly minimal set. By almost strong minimality, there is an $\mathcal{M}$-definable set Y internal to F (in the sense of $\mathcal{M}$) which is in $\mathcal{M}$-definable finite correspondence with X. Since finite correspondences preserve dimension, we may assume $X = Y$, i.e., that X is internal to F. But then by Lemma 4.10, X is $\mathcal{N}/\mathcal{M}$-full, which makes the lemma obvious. □

Notation. In light of Claim 4.15.1, for the rest of the proof we will drop all subscripts when denoting dimensions of definable sets.

We now prove the proposition.

(1) $\Longrightarrow$ (2): Clear.

(2) $\Longrightarrow$ (3): Clear.

(3) $\Longrightarrow$ (4): Assume (3), and let X be stationary in the sense of $\mathcal{M}$. If $\dim(X) = 0$, everything is clear, so assume $\dim(X) = d \geq 1$. Using coordinatization in almost strongly minimal structures, one can $\mathcal{M}$-definably write $X = \bigcup_{t \in T} X_t$, where T is strongly minimal in $\mathcal{M}$, for any $t \in T$ we have $\dim(X_t) = d - 1$, and X_t is stationary in $\mathcal{M}$ for generic $t \in T$. Specifically, after possibly adding a finite set of parameters, there is a strongly minimal set T_0 such that $M = \mathrm{acl}(T_0)$ so given a generic $x \in X$ there is $t_0 \in T_0$ such that x forks over t_0. Take $c := \mathrm{Cb}(\mathrm{stp}(x/t_0))$, T strongly minimal in $\mathrm{stp}(c)$, and $\varphi(x, t)$ a formula isolating $\mathrm{stp}(x/t_0)$ in its dimension and degree, X_t its set of realizations. Now, by (3), T is also strongly minimal in $\mathcal{N}$. Moreover, arguing by induction, we may assume that X_t is stationary in $\mathcal{N}$ for generic $t \in T$. Then, as the union of a stationary family of (generically) stationary sets, it follows easily that X is stationary in $\mathcal{N}$.

(4) $\Longrightarrow$ (1): Assume (4), and let $X \subset M^n$ be $\mathcal{M}$-definable. We argue by induction on $d = \dim(X)$. Let F be an $\mathcal{N}/\mathcal{M}$-full strongly minimal set. By almost strong minimality, there are a sort S in F^{eq} and a finite-to-one $\mathcal{M}$-definable function $f : M^n \to S$. Let $Y = f^{-1}(f(X))$. Note that $f(X)$ is $\mathcal{M}$-definable, by the fullness of F; thus Y is also $\mathcal{M}$-definable. Moreover, $X \subset Y$, and since f is finite-to-one, we have $\dim(X) = \dim(Y) = d$.

In the structure $\mathcal{M}$, write $Y = Y_1 \cup \cdots \cup Y_m$ as a union of stationary sets of dimension d. By (4), each Y_i is stationary in $\mathcal{N}$, so each Y_i is either almost contained in X or almost disjoint from X. Since $X \subset Y$ and $\dim(X) = \dim(Y)$, X is almost equal to the union (say Z) of those Y_i that are almost contained in X. But Z is

$\mathcal{M}$-definable, and X is a Boolean combination of Z, $X-Z$, and $Z-X$, the latter two having dimension less than d. It then follows by induction that X is $\mathcal{M}$-definable. □

We are now ready for the main theorem of this subsection.

Theorem 4.16. *Let K be an algebraically closed field, and assume the Zilber trichotomy holds for strongly minimal K-relics. Let $\mathcal{M} = (M, \dots)$ be a K-relic. Then the following are equivalent*:

(1) *$\mathcal{M}$ is full.*

(2) *There is a very ample strongly minimal set X in $\mathcal{M}$ such that M is internal to X (in the sense of $\mathcal{M}$).*

(3) *$\mathcal{M}$ is almost strongly minimal, and every strongly minimal set in $\mathcal{M}$ is very ample.*

Proof. (1) $\Longrightarrow$ (2): By Proposition 4.12, M is internal to an $\mathcal{M}$-definable infinite field, say F. By Lemma 4.11, F is strongly minimal and very ample.

(2) $\Longrightarrow$ (3): Let X be as in (2). Internality to X is a strengthening of almost strong minimality, so to show (3), it suffices to show that strongly minimal sets in $\mathcal{M}$ are very ample. So, let Y be strongly minimal in $\mathcal{M}$. By assumption, Y is internal to X in the sense of $\mathcal{M}$. So by Lemma 4.10, Y is $K/\mathcal{M}$-full. Then by Theorem 4.14 (applied to Y with its induced structure from $\mathcal{M}$), Y is very ample.

(3) $\Longrightarrow$ (1): Assume (3). We show fullness using Proposition 4.15. Since $\mathcal{M}$ and K are almost strongly minimal by assumption, it suffices to show that every strongly minimal set in $\mathcal{M}$ is $K/\mathcal{M}$-full. But this is automatic: if X is strongly minimal in $\mathcal{M}$, then by (3) X is very ample, so by Theorem 4.14, X is $K/\mathcal{M}$-full. □

Remark 4.17. The reader may be tempted to replace "almost strongly minimal" with "uncountably categorical" in the statement of Theorem 4.16. Let us point out, then, that almost strong minimality and unidimensionality are not equivalent for ACF-relics. Indeed, for any algebraically closed field, K, consider the two sorted relic $\mathcal{M}$ defined as follows: we define the two sorts to be $M := K^2$ and $U := K$. We then equip $\mathcal{M}$ with the binary relation V on M interpreted as

$$\{((x, y), (x, z)) : x, y, z \in K\},$$

the map

$$f((x, y), (x, z)) = y - z$$

from V to U, and the full field structure on U. (Essentially, we have defined a K-indexed family of copies of K, each of which has "forgotten where 0 is" but remembers an isomorphism with K after fixing any element to be 0.) Clearly, $\mathcal{M}$ is a K-relic. We leave it as an exercise to the interested reader to verify that $\mathcal{M}$ is uncountably categorical, but not almost strongly minimal.

5. Technical results on strongly minimal structures

Here we prove some useful facts about general strongly minimal structures, without assuming very ampleness or Zilber's trichotomy. Though these are technical results needed for the sequel, we hope they will be seen as interesting in their own right. Throughout this section $\mathcal{M}$ is assumed to be a strongly minimal structure.

We will use the following well-known notions.

Definition 5.1. Let p and q be stationary types over A and B, respectively.

(1) The *Morley product* of p and q, denoted $p \otimes q$, is $\mathrm{tp}(ab/AB)$, where $a \models p$, $b \models q$, $a \downarrow_A B$, and $b \downarrow_B Aa$.

(2) Given a positive integer k, the k-th *Morley power* of p, denoted p^k, is the k-fold Morley product of p with itself.

(3) The stationary types p and q are *interalgebraic* if $A = B$ and for $a \models p$, there is $b \models q$ with a and b interalgebraic over A.

5A. *Sweeping extensions.* Suppose that we are given a 3-dimensional stationary set X, and a high-dimensional family $\mathcal{C}$ of curves in X covering X generically. A subfamily $\mathcal{C}'$ of $\mathcal{C}$ (over additional parameters) could have high dimension, but concentrate on a forking subset of X. In the present section, we develop a notion of largeness of subfamilies aimed at avoiding precisely such situations.

The following notion appeared in an equivalent form in the first author's Ph.D. thesis, and later appeared implicitly in [Castle 2023]. Below, if p and q are complete types, by $p \subset q$ we mean that q is an extension of p.

Definition 5.2. Let $p \subset q$ be stationary types. We say that q is a *k-sweeping* extension of p if $p^k \subset q^k$.

Definition 5.3. Let $A \subset B$ and let p be a stationary type over B. We say that p *k-sweeps* over A if p is a k-sweeping extension of its restriction to $\mathrm{acl}(A)$.

Definition 5.4. If $A \subset B$ and a is a tuple, we say that a *k-sweeps* from B to A if $\mathrm{stp}(a/B)$ k-sweeps over A.

Note that if p is a stationary type over A, and q is a k-sweeping extension of p, then for any Morley sequence $\bar{a} = a_1, \ldots, a_k$ in p, there is an A-automorphism conjugate q' of q such that $\bar{a}$ is a Morley sequence in q'. We, thus, think of the conjugates of q as "filling out" p "to order k". In particular, k-sweeping extensions provide a workable measure of "largeness" for families of extensions of a type. Now, given a stationary set X and a family $\mathcal{C}$ of (generically stationary) subsets of X, we can ask whether the generic type of a generic member of the family is a k-sweeping extension of the generic type of X. This gives a stronger, and geometrically more accurate, notion of largeness for families of definable subsets of X, than just the dimension of the family.

The following facts are easily verified:

Fact 5.5. Let p, q, r, s be stationary types, A and B be parameter sets, a and b be tuples, and k be a positive integer.

(1) If $q \supset p$ is nonforking, then $q \supset p$ is k-sweeping.

(2) If $r \supset q$ and $q \supset p$ are k-sweeping, then $r \supset p$ is k-sweeping.

(3) If $q \supset p$ is k-sweeping, q and s are interalgebraic, p and r are interalgebraic, and $s \supset r$, then $s \supset r$ is k-sweeping.

(4) If ab k-sweeps from B to A, then a k-sweeps from B to A.

(5) If a k-sweeps from B to A, and $b \downarrow_{Aa} B$, then b k-sweeps from B to A.

We also give the following two lemmas, which essentially appeared in Castle's Ph.D. thesis:

Lemma 5.6. *Suppose that a k-sweeps from B to A. Then for any b such that $\dim(b/A) < k$, $a \downarrow_B b$ implies $a \downarrow_A b$.*

Proof. Assume $a \downarrow_B b$. Let $\bar{a} = a_1, \ldots, a_k$ be a Morley sequence in $\mathrm{stp}(a/Bb)$. Since $a \downarrow_B b$, $\bar{a}$ is also a Morley sequence in $\mathrm{stp}(a/B)$, and since a k-sweeps from B to A, $\bar{a}$ is also a Morley sequence in $\mathrm{stp}(a/A)$. Hence $\dim(\bar{a}b/A) \geq k \cdot \dim(a/A)$. Since $\dim(b/A) < k$, this implies $\dim(\bar{a}/Ab) > k \cdot (\dim(a/A) - 1)$. Hence, there is some i with $\dim(a_i/Ab) = \dim(a/A)$. But $\mathrm{tp}(a/Ab) = \mathrm{tp}(a_i/Ab)$, so $\dim(a/Ab) = \dim(a/A)$. □

Lemma 5.7. *Let $p \subset q$ be stationary types over A and B respectively. If $\dim(q) = \dim(p) - 1$, then q is a k-sweeping extension of p, where $k = \dim(\mathrm{Cb}(q)/A)$.*

Proof. Let $c = \mathrm{Cb}(q)$. Let $d = \dim(p)$. Let $\bar{a} = (a_1, \ldots, a_k)$ be a Morley sequence in q. So $\dim(c\bar{a}/A) = kd$, counting from left to right. Now if the lemma fails, then $\dim(\bar{a}/A) < kd$, so $\dim(c/A\bar{a}) > 0$. Let c' be an independent realization of $\mathrm{tp}(c/A\bar{a})$ over c. Then by assumption we get (1) $\dim(cc'\bar{a}/A) > kd$, and (2) $\dim(c'/Ac) > 0$. By (2) $c' \neq c$, so each a_i forks with Acc' over Ac, and thus $\dim(\bar{a}/Acc') \leq k(d-2)$. So by (1), $\dim(cc'/A) > 2k$, contradicting that $\dim(c/A) = \dim(c'/A) = k$. □

We conclude this subsection with an existence result for sweeping extensions. It is well known that nonlocal modularity is characterized by the presence of arbitrarily large families of one-dimensional subsets of M^2; see also Conditions (A), (B) and (C) on pp. 85–87 of [Hrushovski 1989]. By Lemma 5.7, this is equivalent to the generic type of M^2 having k-sweeping one-dimensional extensions for arbitrarily large k. Proposition 5.8 says that there is nothing special about one-dimensional subsets of M^2: namely, adopting sweeping as a notion of largeness, every definable set X admits arbitrarily large definable families of m-dimensional subsets, for any $0 < m \leq \dim X$.

Proposition 5.8. *Assume $\mathcal{M}$ is not locally modular. Let $0 < m \leq n$ and k be positive integers, and let p be a stationary type over A with $\dim(p) = n$. Then there is a k-sweeping extension $q \supset p$ with $\dim(q) = m$.*

Proof. We start with some reductions. If $m = n$, then p is a k-sweeping extension of itself. Now assume $m < n$. By Fact 5.5 (2), it suffices to assume $m = n - 1$. Now by weak elimination of imaginaries, there is a nonforking extension $p' \supset p$, say over $A' \supset A$, such that p is interalgebraic with the generic type p'' of M^n over A'.

By nonlocal modularity, there is a strongly minimal plane curve $C \subset M^2$, with canonical base c, such that $\dim(c/A') \geq k$. Let r be the generic type of C over $A'c$, and let $q' = r \otimes s$, where s is the generic type of M^{n-2} over $A'c$. So $\dim(q') = n-1$, and clearly $\mathrm{Cb}(q')$ is also c. Then by Lemma 5.7, and since $\dim(c/A') \geq k$, the extension $q' \supset p''$ is k-sweeping. Let $a \models q'$. So $a \models p''$. By interalgebraicity, there is $b \models p'$ with a and b interalgebraic over A'. Now let $q = \mathrm{stp}(a/A'c)$. By interalgebraicity, $\dim(q) = n - 1$, and by Fact 5.5 (3), $q \supset p'$ is k-sweeping. But by Fact 5.5 (1), $p' \supset p$ is also k-sweeping, thus by Fact 5.5 (2), so is $q \supset p$. □

Note that nonlocal modularity is crucial in the above proposition. Indeed, in any stable one-based theory, if $q \supset p$ are stationary types with $q^2 \supset p^2$, then q is a nonforking extension of p. To see this, suppose p is over A, let $c = \mathrm{Cb}(q)$, and let $(a, b) \models q^2$. By one-basedness, $c \in \mathrm{acl}(a) \cap \mathrm{acl}(b)$. But since $a \perp_A b$, this gives $c \in \mathrm{acl}(A)$, implying that q does not fork over A.

5B. ***Sets transverse to any family.*** The result of this subsection was inspired by the following question: given a definable family $\{X_t : t \in T\}$ of codimension r subsets of a definable set Z, can one always find an r-dimensional definable set $Y \subset Z$ having finite intersection with each X_t? This type of question arises, for example, when trying to extract a 2-dimensional very ample family of plane curves from a higher dimensional such family (see Proposition 6.1).

In order to answer the question above, it is convenient to study a more general question. Namely, for type-definable sets $X, Y \subset Z$, say that X and Y are *transverse in Z* if $\dim(X \cap Y) \leq \dim X + \dim Y - \dim Z$. We wish to investigate whether, given a type-definable family $\{X_t : t \in T\}$ of subsets of a definable set Z, one can find a definable $Y \subset Z$ of any prescribed dimension which is transverse to each X_t in Z.

As stated, it is not always possible to find such Y: indeed, if the prescribed dimension of Y is small enough (but still at least 0), transversality could require Y to be disjoint from each X_t, which is clearly impossible if the X_t cover Z. So at the least, we must allow Y to be nonempty. Conversely, Proposition 5.9 and Corollary 5.10 show that this is, essentially, the only obstacle.

Proposition 5.9. *Assume $\mathcal{M}$ is not locally modular. Fix nonnegative integers i, j, m, k with $i, j \leq m$. Let p be a stationary type over A of dimension m. Let*

$\{X_t : t \in T\}$ be a type-definable family of sets over A, with each $\dim(X_t) \leq i$. Then there are a set $B \supset A$ and a stationary type $q \supset p$ over B satisfying:

(1) $\dim(q) = j$.

(2) *If $j \geq 1$, then $q \supset p$ is a k-sweeping extension.*

(3) *If $a \models q$ and $a \in X_t$ for some t, then* $\dim(a/Bt) \leq \max\{0, i + j - m\}$.

Proof. For ease of notation, we assume throughout that $A = \varnothing$. Note that it is harmless to increase the value of k, so we may assume $k > \dim T$.

We proceed by induction on j. If $j = 0$ then conditions (2) and (3) are automatic, and one can simply take $q = \mathrm{tp}(a/a)$ where $a \models p$.

Now assume $j \geq 1$, and let $r \supset p$ satisfy the proposition with j replaced by $j-1$. Suppose r is over C. Let $c = \mathrm{Cb}(r)$. Since $j - 1 < j \leq m$, r is a forking extension of p, thus $\dim(c) \geq 1$. Then, by Proposition 5.8, there is a set B such that $\mathrm{tp}(c/B)$ is minimal and k-sweeps over $\varnothing$. Without loss of generality, B is algebraically closed. Finally, let r' be the nonforking extension of r over BC, and let q be the restriction of r to B. Since B is algebraically closed, q is stationary. We show in the ensuing four claims that q satisfies the requirements of the proposition.

Claim 5.9.1. $\dim(q) = j$.

Proof. Let $a \models r'$. We want to show that $\dim(a/B) = j$. Now since r' is a nonforking extension of r, we have

$$\dim(a/Bc) = \dim(r') = \dim(r) = j - 1.$$

Since $\dim(c/B) = 1$, this implies $j - 1 \leq \dim(a/B) \leq j$. So assume $\dim(a/B) = j - 1$. Then r' does not fork over B, and since $c = \mathrm{Cb}(r) = \mathrm{Cb}(r')$, this implies $c \in \mathrm{acl}(B)$, contradicting that $\dim(c/B) = 1$. □

Claim 5.9.2. *The stationary type q is a k-sweeping extension of p.*

Proof. Let $a \models r'$. It is enough to show that a k-sweeps from B to $\varnothing$. Now by assumption, c k-sweeps from B to $\varnothing$. Moreover, since $r' \supset r$ is nonforking, a and B are independent over c. So, by Fact 5.5 (5), a k-sweeps from B to $\varnothing$. □

Item (3) in the proposition will be the most difficult to establish. The key observation is:

Claim 5.9.3. *Let $a \models r'$ and $a \in X_t$ for some t. Then either $c \in \mathrm{acl}(Bt)$ or $i + (j-1) - m \geq 0$.*

Proof. By various instances of additivity, we have

$$\begin{aligned}\dim(c/t) &\leq \dim(ac/t) = \dim(a/t) + \dim(c/at) \\ &\leq \dim(a/t) + \dim(c/a) = \dim(a/t) + \dim(c) + \dim(a/c) - \dim(a) \\ &\leq i + \dim(c) + (j-1) - m.\end{aligned}$$

Rearranging,

$$\dim(c/t) - \dim(c) \leq i + (j-1) - m.$$

Now $\dim(c/B) = 1$ by assumption, so if $c \notin \operatorname{acl}(Bt)$, then $c \perp_B t$. But by construction, c k-sweeps from B to $\varnothing$, and $k > \dim(T) \geq \dim(t)$. So by Lemma 5.6, we get $c \perp_\varnothing t$, and the above simplifies to $0 \leq i + (j-1) - m$. □

Finally, we complete the proof of Proposition 5.9 by showing (3):

Claim 5.9.4. *Let $a \models q$ and $a \in X_t$ for some t. Then* $\dim(a/Bt) \leq \max\{0, i+j-m\}$.

Proof. After applying an automorphism fixing B, we may assume $a \models r'$. Let $a't' \models \operatorname{tp}(at/c)$ with $a't' \perp_c C$. So $a' \models r$ and $a' \in X_{t'}$. Combined with the inductive hypothesis, we get

$$\dim(a/ct) = \dim(a'/ct') = \dim(a'/Ct') \leq \max\{0, i + (j-1) - m\}.$$

We now proceed in two cases, according to the previous claim.

Case 1: Assume $c \in \operatorname{acl}(Bt)$. Then

$$\dim(a/Bt) \leq \dim(a/ct) \leq \max\{0, i + (j-1) - m\} \leq \max\{0, i + j - m\}.$$

Case 2: Assume $i + (j-1) - m \geq 0$. Then $\max\{0, i+(j-1)-m\} = i+(j-1)-m$, so $\dim(a/ct) \leq i + (j-1) - m$. But since $\dim(c/B) = 1$, we then have

$$\dim(a/Bt) \leq \dim(a/ct) + 1 = i + j - m,$$

as claimed. □

This finishes the proof of the proposition. □

Combining Proposition 5.9 with compactness, we obtain:

Corollary 5.10. *Let i, j, m be integers with $0 \leq i, j \leq m$. Let Z be a definable set of dimension m, and let $\{X_t : t \in T\}$ be a definable family of subsets of Z, with each* $\dim(X_t) \leq i$. *Then there is a definable set $Y \subset Z$ of dimension j such that for each $t \in T$, we have* $\dim(X_t \cap Y) \leq \max\{0, i + j - m\}$.

6. Very ample families of a prescribed dimension

As we have seen, very ampleness in strongly minimal structures is equivalent to the existence of very ample families of plane curves (Corollary 3.21). In applications, it may be convenient to have access to such families of a prescribed dimension (as is the case with nonlocal modularity, where for any $k > 1$ the existence of any k-dimensional family of plane curves implies the existence of families of all dimensions). In the present section, we show that this can always be achieved. Namely, we first use Proposition 5.9 to show that, given a k-dimensional very ample family of plane curves, we can extract an l-dimensional very ample subfamily for any $2 \leq l \leq k$. We then adapt the usual proof from the nonlocally modular setting (using

composition and a field configuration) to construct very ample families of arbitrarily large dimension. While the general proof structure is the same as in the nonlocally modular case, the need to preserve very ampleness will make each stage of the argument more delicate. In particular, once a field (say K) is interpreted in our set (say X), some care is needed to examine the precise relationship between X and K.

Once we have achieved the result described above, we end the section with a short application, in which (as mentioned in the introduction to the paper) we characterize very ampleness in terms of definable pseudoplanes.

6A. ***Going down.*** Here we prove that very ample families of plane curves admit very ample subfamilies of any prescribed dimension. The proof amounts to an application of Proposition 5.9.

Proposition 6.1. *Let X be strongly minimal and A-definable. Let C be a strongly minimal plane curve in X, let $c = \mathrm{Cb}(C)$, and let $m = \dim(c/A)$. Assume that C is very ample over A. Then for all $2 \leq j < m$, there is a set $B \supset A$ such that $\dim(c/B) = j$ and C is very ample over B.*

Proof. Without loss of generality, assume A is algebraically closed, as this does not affect any of the relevant computations. Let $p = \mathrm{tp}(c/A)$. So p is stationary of dimension m. Let a be generic in C over Ac, and let $T = X^2 \times X^2 \setminus \Delta$, where Δ is the diagonal $(v, v) \in X^2 \times X^2$. Let $\{X_t : t \in T\}$ be the A-type-definable family (in the variables v, v', z) defined as follows: for $t = (v, v')$, set $z \in X_t$ if $\mathrm{tp}(vz/A) = \mathrm{tp}(v'z/A) = \mathrm{tp}(ac/A)$. Then X_t is the set of A-automorphic images of C, passing generically through v and v'. It follows from the very ampleness of C over A that $\dim(X_t) \leq m - 2$ for all $t \in T$ (see Lemma 3.11).

Now let $i = m-2$ and $k = 1$, and let $B \supset A$ and q be as provided by Proposition 5.9 (for j as given in the statement of the current proposition). After applying an automorphism, we may assume that $c \models q$. So $\dim(c/B) = j$. We will show that C is very ample over B, proving the proposition.

To show very ampleness, let v, v' be distinct generics of C over c. By Lemma 3.11, it is enough to show that $\dim(c/Bvv') \leq j - 2$. But by definition we have $c \models q$ and $c \in X_{(v,v')}$, so $\dim(c/Bvv') \leq \max\{0, i + j - m\}$. But $i = m - 2$, so $i + j - m = j - 2$. Thus $\dim(c/Bvv') \leq \max\{0, j - 2\}$, and since $j \geq 2$, this implies $\dim(c/Bvv') \leq j - 2$. □

6B. ***Going up.*** Our next goal is to show that if X is a very ample strongly minimal set, then X has arbitrarily large very ample families of plane curves. Before proceeding, we need the following well-known lemma[5]:

[5]The argument goes back to the proof that nonlocal modularity implies the existence of faithful families of plane curves or arbitrarily large dimension. Similar arguments can be found in [Hasson and Sustretov 2017, Lemma 4.18] and [Eleftheriou et al. 2021, §3]

Lemma 6.2. *Let X be an A-definable strongly minimal set, and let C, D, and $E \subset D \circ C$ be strongly minimal plane curves in X, with canonical bases c, d, and e, respectively. Assume that $c \perp_A d$ and $\dim(e/A) \leq \dim(c/A) = \dim(d/A) = k$ for some $k \geq 2$. Then $k \in \{2, 3\}$, and there is a definable strongly minimal expansion of an algebraically closed field which is internal to X.*

Proof. Take $(x, z) \in E$ generic and y such that $(x, y) \in C$ and $(y, z) \in D$. The assumption implies that $\{c, d, e, x, y, z\}$ is a group configuration; see [Eleftheriou et al. 2021, Lemma 3.18, Lemma 3.20] for the details. By a result of Hrushovski (see [Poizat 2001, Theorem 3.27]) this implies that $k \leq 3$, and if $\dim(k) \geq 2$ (which holds by assumption), a strongly minimal field is definable in X^{eq}. □

We now give the main result of this subsection.

Proposition 6.3. *Suppose X is strongly minimal, A-definable, and very ample. Then for any $k \geq 2$ one can find a set $B \supset A$, and a strongly minimal plane curve C in X, such that $\dim(\mathrm{Cb}(C)/B) \geq k$ and C is very ample in X over B.*

Proof. For simplicity of notation, we assume that $A = \varnothing$. Assume the proposition fails. By very ampleness, there is a strongly minimal plane curve C in X (with canonical base c, say) which is very ample in X over some set B. Fix such (C, c, B) so that the value $k := \dim(c/B)$ is maximal (that this is possible follows by the assumed failure of the proposition). We may assume moving forward that $B = \varnothing$. Note also that $k \geq 2$ by Lemma 3.8.

Let $d \models \mathrm{tp}(c/B)$, with $c \perp d$. So $\dim(d) = k$, and $d = \mathrm{Cb}(D)$ for some strongly minimal plane curve D in X which is also very ample in X over $\varnothing$. Let $E \subset D \circ C$ be strongly minimal. Then, by Lemma 3.23 and Corollary 3.14, E is also very ample over $\varnothing$. So by the choice of k, letting $e = \mathrm{Cb}(E)$, we have $\dim(e) \leq k$. Thus, by Lemma 6.2, $k \in \{2, 3\}$ and there is a strongly minimal expansion $(K, +, \cdot, \dots)$ of an algebraically closed field internal to X.

Since K is internal to X, clearly X and K are nonorthogonal. So by Corollary 4.2, there is a definable embedding of X into some K^n. Let us choose, among all definable embeddings of cofinite subsets of X into any K^n, one which minimizes the value n. So, replacing X with a cofinite subset if necessary, we assume that $X \subset K^n$ and no cofinite subset of X embeds into any smaller power of K. Without loss of generality, we will assume that K and X are $\varnothing$-definable.

In what follows, it will be convenient to introduce the following notation: if $S \subseteq K^m$ is definable, we let S^{aff} be the intersection of all affine linear subspaces of K^n which contain a large subset of S. It is easy to see that S^{aff} is itself affine linear and contains a large subset of S, and moreover that S^{aff} is definable over any set of parameters defining S. Note also that by the choice of n, we have $X^{\mathrm{aff}} = K^n$ (since any proper affine linear subspace of K^n is in definable bijection with a lower power of K).

Let $\mathcal{H} = \{H_t : t \in T\}$ be the family of hyperplanes in K^{2n}. So $\dim(T) = 2n$. We note the following basic observation, for future reference:

Claim 6.3.1. *If $D \subseteq K^{2n}$ is a d-dimensional affine linear subspace, then we have* $\dim(\{t \in T : D \subseteq H_t\}) = 2n - d - 1$.

Now let $t \in T$ be generic. Then, by Proposition 3.28, $H_t \cap X^2$ is a very ample (over $\varnothing$) plane curve in X. Let $S \subset H_t \cap X^2$ be strongly minimal. Then, by Corollary 3.14, S is also very ample in X over $\varnothing$. So by the choice of k, letting $s = \mathrm{Cb}(S)$, we have $\dim(s) \leq k \leq 3$, and thus $\dim(s) \leq 3$.

Since S is a strongly minimal component of $H_t \cap X^2$, clearly $s \in \mathrm{acl}(t)$. So $\dim(ts) = \dim(t) = 2n$, and thus $\dim(t/s) \geq 2n - 3$. On the other hand, by definition H_t contains S, so since H_t is affine linear it must also contain S^{aff}. Let $d := \dim(S^{\mathrm{aff}})$; then by Claim 6.3.1, we have $\dim(t/s) \leq 2n - d - 1$. So, combining with $\dim(t/s) \geq 2n - 3$, we get $2n - 3 \leq 2n - d - 1$, and thus $d \leq 2$. Note also that if equality holds (i.e., $d = 2$), we have $2n - 3 \leq \dim(t/s) \leq 2n - 3$, so that $\dim(t/s) = 2n - 3$, and thus $\dim(s) = 3$.

Now, the main point of the proof is the following:

Lemma 6.4. *There is a definable strongly minimal algebraically closed field $(F, +_F, \cdot_F)$, whose underlying set F almost coincides with X.*

Proof. First note that if $\dim(X^{\mathrm{aff}}) = 1$, then X almost coincides with X^{aff}, and X^{aff} is in definable bijection with K; in particular, this is enough to prove the lemma (setting $F = X^{\mathrm{aff}}$ with the field structure inherited from K). So we may assume that $n = \dim(X^{\mathrm{aff}}) \geq 2$, and thus (as above) $\dim(s) = 3$. Our goal is to construct a definable transitive group action of a 3-dimensional group on a set almost equal to X, and apply Hrushovski's classification [Poizat 2001, Theorem 3.27] of such actions. We first proceed with the following two claims, which give us the group we will use:

Claim 6.4.1. *Let $\pi : K^{2n} \to K^n$ be either the leftmost or rightmost projection. Then the restriction of π to S^{aff} is surjective.*

Proof. By Remark 3.9, π is finite-to-one on S, and thus $\pi(S)$ contains a cofinite subset of X. By definition, S^{aff} contains a cofinite subset of S, so clearly $\pi(S^{\mathrm{aff}})$ contains a cofinite subset of $\pi(S)$. In particular, $\pi(S^{\mathrm{aff}})$ also contains a cofinite subset of X, and thus contains $X^{\mathrm{aff}} = K^n$ (because projections of affine sets are affine). □

We next use Claim 6.4.1 to show the following:

Claim 6.4.2. *The intersection S^{aff} is the graph of an invertible affine linear map $L : K^2 \to K^2$.*

Proof. By Claim 6.4.1 we have $d \geq n \geq 2$; so since we already had $d \leq 2$, we get $d = n = 2$. Then by Claim 6.4.1 again, each of the two projections $S^{\mathrm{aff}} \to K^2$ is a surjective linear map between two-dimensional affine linear spaces; thus each of these projections is a linear bijection, which implies the claim. □

Let $L : K^2 \to K^2$ be as in Claim 6.4.2. Since S projects almost onto X in both directions, and by the strong minimality of X, it follows that $L(X)$ almost coincides with X. By the strong minimality of S, it moreover follows that S almost coincides with the graph of the restriction of L to X. In particular, it follows easily that s is interdefinable with the canonical parameter of L.

Now let G be the generic stabilizer of X in $\mathrm{AGL}_2(K)$ — that is, the set of all invertible affine linear maps $g : K^2 \to K^2$ such that $g(X)$ almost coincides with X. Clearly, G is a $\varnothing$-definable subgroup of $\mathrm{AGL}_2(K)$. By the above remarks (since s is definable from L), we have $\dim(G) \geq \dim(s) = 3$. Let G^0 be the connected component of G; so $\dim(G^0) \geq 3$ and G^0 is $\varnothing$-definable.

We now have the group for our action; our next goal is to find an appropriate set which is acted on transitively. Now G^0 comes equipped with both an action on K^2, and a generic action on X: so if $g \in G^0$ and $x \in X$ then $g(x) \in K^2$ is always defined, while if g and x are independent generics then $g(x)$ is also a generic of X. Fix $x_0 \in X$ generic and independent of s, and let $G^0(x_0)$ be the orbit of x_0 under all of G^0. Note, then, that $G^0(x_0)$ is definable over x_0. Our aim is to show that $G^0(x_0)$ almost coincides with X, and subsequently build a field structure on $G^0(x_0)$. We will do this via a sequence of claims. To start, we show:

Claim 6.4.3. *The orbit $G^0(x_0)$ almost contains X.*

Proof. Since x_0 is generic over s, there is $y_0 \in X$ such that (x_0, y_0) is generic in S. So $y_0 = L(x_0)$. Now, by genericity, we have $\dim(x_0 y_0/s) = 1$; Note then that $\dim(x_0 y_0/\varnothing)$ cannot also be 1, because then we would have $s \in \mathrm{acl}(\varnothing)$, contradicting $\dim(s) = 3$. So $\dim(x_0 y_0) = 2$. In particular, y_0 is generic in X over x_0. Since $y_0 = L(x_0)$, we see that $y_0 \in G^0(x_0)$, and the claim follows. □

Using Claim 6.4.3, we now show:

Claim 6.4.4. *The orbit $G^0(x_0)$ is one-dimensional, and X is (up to a finite set) one of its strongly minimal components.*

Proof. Let D be the set of $g \in G^0$ with $g(x_0) \in X$. Then D is definable over x_0 and contains all generics of G^0 over x_0, which implies that D is generic in G^0. So there are $g_1, \ldots, g_m \in G^0$ such that the translates $g_i D$ cover G^0. Then $G^0(x_0)$ is contained in the union of the $g_i D(x_0)$, which by definition is contained in the union of the $g_i(X)$. Now each $g_i(X)$ is strongly minimal, which implies that $G^0(x_0)$ is contained in a one-dimensional definable set. The claim then follows by Claim 6.4.3. □

Finally, we show:

Claim 6.4.5. *The orbit $G^0(x_0)$ is strongly minimal, and thus almost coincides with X.*

Proof. As a transitive G^0-set, $G^0(x_0)$ is isomorphic to a coset space G/H, where H is the stabilizer of x_0. It is easy to see that this identification is definable (that is,

the usual identification from basic algebra is clearly definable). In this case, the strong minimality of G^0/H follows easily from the fact that G^0 is connected: indeed, given a partition of G^0/H into disjoint definable infinite sets Z_1 and Z_2, the preimages of the Z_i in G^0 would partition G^0 into two generic subsets, contradicting connectedness. □

Now by Claim 6.4.5, there is a definable (transitive by definition) action of G^0 on the strongly minimal set $G^0(x_0)$, and $\dim(G^0) \geq 3$. So by the classification of such actions [Poizat 2001, Theorem 3.27], $\dim(G^0) = 3$ and $G^0(x_0)$ is in definable bijection with $\mathbb{P}^1(F)$ for some definable, strongly minimal, algebraically closed field F. Deleting a point, we then recover $\mathbb{A}^1(F) = F$; then by Claim 6.4.5 again, X is (up to a definable bijection) almost equal to F, proving Lemma 6.4. □

Now by Lemma 6.4, we may assume after changing finitely many points that there is a definable field structure on the set X. It is then an easy exercise to find arbitrarily large very ample families of plane curves (for example, the family of degree d polynomial maps $X \to X$ is very ample for each d). In particular, there are very ample (over $\varnothing$) strongly minimal plane curves $C \subset X^2$ with $\dim(\mathrm{Cb}(C)) > k$, contradicting our initial assumption and thus proving Proposition 6.3. □

The main part of the proof is showing that the conclusion of Proposition 6.1 is true for strongly minimal sets definable in strongly minimal expansions of algebraically closed fields. It may be interesting to know whether a simpler proof of this fact exists in the case of pure algebraically closed fields.

Finally, we now deduce:

Corollary 6.5. *Let X be a very ample strongly minimal set. Then for every $k \geq 2$, there is a faithful, very ample family $\{C_t : t \in T\}$ of plane curves in X where $\dim(T) = k$.*

Proof. Assume X is A-definable. Applying Proposition 6.3 and then Proposition 6.1, we can find a set $B \supset A$ and a strongly minimal plane curve $C \subset X^2$ which is very ample in X over A and satisfies $\dim(\mathrm{Cb}(C)/B) = k$. After editing finitely many points of C and applying compactness, there is then a B-definable faithful family of plane curves in X whose generic members are the conjugates of C over B. The result then follows by applying Proposition 3.19 to this family. □

6C. ***Very ampleness and pseudoplanes.*** We now show, as described in the introduction, that a strongly minimal set X is very ample (i.e., admits a very ample plane curve) if and only if there is a definable pseudoplane on (a large subset of) X^2. Let us begin by formally recalling the definition.

Definition 6.6. A *pseudoplane* consists of sets P and L, and an incidence relation $I \subset P \times L$, satisfying the following:

(1) For each $l \in L$, there are infinitely many $p \in P$ with $(p, l) \in I$.

(2) For each $p \in P$, there are infinitely many $l \in L$ with $(p, l) \in I$.

(3) For each $l \neq l' \in L$, there are only finitely many $p \in P$ with $(p, l), (p, l') \in I$.

(4) For each $p \neq p' \in P$, there are only finitely many $l \in L$ with $(p, l), (p', l) \in I$.

One thinks of P as the set of "points" of a plane, and L as the set of "lines". As mentioned above, already Zilber in his thesis proved the "weak trichotomy theorem", asserting that a strongly minimal structure is not locally modular if and only if it defines a pseudoplane. However, if one wishes to make the natural identification $P := M^2$ (or a large subset thereof), one can only guarantee conditions (1)–(3) in the definition. We now point out that the missing data in order to identify P (generically) with M^2 is precisely very ampleness.

Proposition 6.7. *Let X be strongly minimal. Then the following are equivalent*:

(1) *X is very ample.*

(2) *There is a definable pseudoplane (P, L, I), where P is a generic subset of X^2.*

Proof. First, assume X is very ample. Then by Proposition 6.1, there is a definable faithful very ample family of plane curves $\mathcal{C} = \{C_t : t \in T\}$ in X, where $\dim(T) = 2$. Let $C \subset X^2 \times T$ be the graph of $\mathcal{C}$. Let P be the set of $x \in X^2$ such that the set $C^x \subset T$ has dimension 1. Let L be the set of $t \in T$ such that $C_t \cap P$ is infinite. Let $I = C \cap (P \times L)$. It is easy to check that P is generic in M^2 and L is cofinite in T. It then follows easily that axioms (1) and (2) of Definition 6.6 hold for (P, L, I) (these computations are essentially carried out in Section 2 of [Castle 2023]). To conclude, we note that (3) is a restatement of the faithfulness of $\mathcal{C}$, and (4) is a restatement of the very ampleness of $\mathcal{C}$.

Now assume (P, L, I) is a definable pseudoplane, where P is generic in X^2. We will show that $\{I_l : l \in L\}$ is a two-dimensional very ample family of plane curves. By (3) and (4) in the definition, it will suffice to show that $\dim(L) = 2$ and each $\dim(I_l) = 1$. These follows from the ensuing three claims. Throughout, we will assume (P, L, I) (and X) are $\varnothing$-definable.

Claim 6.7.1. *For all $l \in L$, we have* $\dim(I_l) = 1$.

Proof. By assumption $I_l \subset X^2$ and I_l is infinite, so we need only rule out the case that $\dim(I_l) = 2$. But in this case I_l is generic in X^2, so $\dim(X^2 - I_l) = 1$. By the finiteness of Morley degree, it follows easily (using (3)) that there can be only finitely many distinct $I_{l'}$ other than I_l. In other words, we get that L is finite, which clearly contradicts (2). □

Claim 6.7.2. $\dim(L) \leq 2$.

Proof. Let $l \in L$ be generic, and let p, p' be independent generics in I_l over l. By (4), $\dim(l/pp') = 0$, so $\dim(ppl') = \dim(pp') \leq 4$. But by the previous claim $\dim(pp'/l) = 2$, so it follows that $\dim(l) \leq 2$, and thus $\dim(L) \leq 2$. □

Claim 6.7.3. $\dim(L) \geq 2$.

Proof. Let $p \in P$ be generic, and let $l \in I^p$ be generic over p. By (2), we have $\dim(l/p) \geq 1$. Now if $\dim(L) < 2$, we are forced to conclude that $\dim(l/p) = \dim(l) = 1$. Thus, l and p are independent. By assumption $\dim(p) = 2$, so also $\dim(p/l) = 2$. But then $\dim(I_l) \geq 2$, contradicting Claim 6.7.1. □

This completes the proof of the proposition. □

Remark 6.8. The above proof would be fairly straightforward, and included much sooner in the paper, if we knew right away that very ampleness gave us a two-dimensional very ample family of plane curves. The reason we had to wait until now to present the result is that we needed Proposition 6.1 to get such a family.

Before moving on, we point out one more fact that is relevant to pseudoplanes. As we have seen, very ampleness implies nonlocal modularity of strongly minimal structures. In Corollary 6.12, we show, conversely, that nonlocal modularity (in fact, non-one-basedness) implies the existence of a very ample stationary type. This is true for stable theories, and the proof goes through *complete type-definable pseudoplanes*. Recall the following (see [Pillay 1996, §4, Definition 1.6]):

Definition 6.9. A *complete-type-definable pseudoplane* consists of a complete type of a pair of (potentially imaginary) tuples $p = \mathrm{tp}(b, c)$ such that

(1) $b \notin \mathrm{acl}(c)$.

(2) $c \notin \mathrm{acl}(b)$.

(3) If $c' \neq c$ and $\mathrm{tp}(bc) = \mathrm{tp}(bc')$, then $b \in \mathrm{acl}(cc')$.

(4) If $b' \neq b$ and $\mathrm{tp}(bc) = \mathrm{tp}(b'c)$, then $c \in \mathrm{acl}(bb')$.

The following is proved in [Pillay 1996, §4, Lemma 1.7]:

Fact 6.10. A stable theory is one-based if and only if it does not admit a complete-type-definable pseudoplane.

Remark 6.11. Fact 6.10 need not be true if we do not insist that a type-definable pseudoplane is concentrated on a complete type. Indeed, the existence of a (type) definable pseudoplane does not imply the existence of a complete type definable pseudoplane, and consequently the existence of a (type) definable pseudoplane in itself does not contradict one-basedness. Let X be an infinite set equipped with a definable surjection, $f : X \to X$ with all fibers infinite. Construct a pseudoplane (P, L, I) by setting $X = P = L$ and $I(p, l)$ if $f(p) = l$ or $f(l) = p$. Clearly, there is no complete type-definable pseudoplane contained in (P, L, I). We leave it as an exercise to the interested reader to find stable one-based structures where such definable pseudoplanes can be constructed.

We now show:

Corollary 6.12. *A stable theory admits a very ample nonalgebraic stationary type if and only if it is not one-based.*

Proof. Let T be a stable theory. First suppose T is one-based. We show that no nonalgebraic stationary type is very ample. To see this, let $p = \mathrm{tp}(a/c)$ be stationary, where $c = \mathrm{Cb}(p)$. Since p is not algebraic, there is $b \models p$ with $b \neq a$. Now by one-basedness, $c \in \mathrm{acl}(a)$, so automatically $c \mathop{\downarrow}_a b$, which shows that p is not very ample.

Now suppose T is not one-based. By Fact 6.10, there is a complete-type-definable pseudoplane $\mathrm{tp}(bc)$. Let $d = \mathrm{Cb}(\mathrm{stp}(b/c))$. Clearly, $d \in \mathrm{acl}(c)$. On the other hand, by Definition 6.9, any c-conjugate of $\mathrm{tp}(b/c)$ is not parallel to $\mathrm{tp}(b/c)$. Thus $c \in \mathrm{dcl}(d)$, so in particular, c is interalgebraic with d.

We now claim that $\mathrm{tp}(b/d)$ is very ample. To see this, let b' be any realization which is distinct from b'. We want to show that b' forks with d over b; by interalgebraicity, it is enough to show that b' forks with c over b. By symmetry, it is equivalent to show that $r = \mathrm{tp}(c/bb')$ is a forking extension of $q = \mathrm{tp}(c/b)$. But this is clear, since r is algebraic and q is not. □

7. Very ampleness in strongly minimal groups

In this final section, we give some results related to very ampleness in strongly minimal groups. We first prove that nonlocally modular strongly minimal groups admit nonaffine plane curves (a result that has been long assumed but does not exist in writing). We then use this result to show that Question 3.30 has a positive answer for groups, and that nonlocally modular expansions of divisible groups are already very ample; finally, we apply the very ampleness of divisible groups to characterize fullness of strongly minimal ACF-relics with definable divisible group structures, answering, in particular, an old question on expansions of the multiplicative group.

7A. ***Existence of nonaffine plane curves.*** Hrushovski and Pillay [1987] show that a stable group $\mathcal{G}$ is one-based if and only if every definable subset of G^n (any n) is *affine*, i.e., a boolean combination of cosets of definable subgroups of G^n. Restricted to strongly minimal (expansions of) groups, it has been well known among experts that non-one-basedness (equivalently, nonlocal modularity) even implies the existence of a nonaffine plane curve. While this result is often used in practice, there does not seem to exist a full treatment anywhere in writing[6]. It can, however, be deduced by a particular application of Proposition 5.8. Thus, we dedicate the present section to proving this result as a service to the community.

Throughout this subsection, we fix $\mathcal{G}$, a strongly minimal expansion of a group. We will say that a definable set $X \subseteq G^n$ is *affine* if it is a (finite) boolean combination

[6]The result is claimed in [Kowalski and Randriambololona 2016, Proposition 4.2], but there is a mistake in the proof that we do not see how to bridge.

of cosets of definable subgroups of G^n; and *almost affine* if it is almost equal to an affine set. Note that an affine stationary set is, up to a small correction, a coset of a definable group. This suggests the following definition: a stationary type in some G^n is *affine* if it is the generic type of a stationary affine definable set (equivalently, the generic type of a coset of a connected group). Note that affine sets are almost affine, and the converse holds for one-dimensional sets (since finite sets are affine); more generally, note that arbitrary finite Boolean combinations of affine sets are affine.

Notation 7.1. For any nonempty definable set $X \subset G^n$, we let $\mathrm{Stab}(X)$ denote the set of $g \in G^n$ such that $g + X$ almost coincides with X; so $\mathrm{Stab}(X)$ is a subgroup of G^n which is definable over any set of parameters defining X.

We will need the following, which is well known and is left as an easy exercise to the reader (see [Eleftheriou et al. 2021, Lemma 3.10] and Example 3.18 and, for (2), see also the proof of Lemma 7.4 and Lemma 7.6 below):

Lemma 7.2. *Let $\mathcal{G} = (G, +, \dots)$ be a strongly minimal expansion of a group, and let $C \subset G^2$ be a strongly minimal plane curve.*

(1) *The following are equivalent*:
   (a) *The family of translates $\{C + t : t \in T^2\}$ is faithful.*
   (b) *The family of translates $\{C + t : t \in T^2\}$ is very ample.*
   (c) *The group* Stab (C) *is trivial.*

(2) *The following are also equivalent*:
   (a) *C is affine.*
   (b) $\mathrm{Stab}(C)$ *is infinite.*

We will show:

Theorem 7.3. *Let $\mathcal{G} = (G, +, \dots)$ be a strongly minimal expansion of a group. Then the following are equivalent*:

(1) *$\mathcal{G}$ is locally modular.*

(2) *Every definable plane curve in $\mathcal{G}$ is affine.*

(3) *Every definable one-dimensional set $X \subset G^n$, for any n, is affine.*

(4) *Every stationary type in G^n, for every n, is affine.*

(5) *Every definable subset of each G^n is almost affine.*

(6) *Every definable subset of each G^n is affine.*

We will be brief, as many parts of the above proof are standard, and follow from either well known arguments or the main result in [Hrushovski and Pillay 1987]. Our contribution will be the quick proof of the implication (3) $\Longrightarrow$ (4) using sweeping extensions of stationary types. In fact there is also a rather easy direct

proof of the result of Hrushovski and Pillay [1987] restricted to the strongly minimal case (using only the equivalence of local modularity and one-basedness). So, for completeness, we now give this proof (so that our proof of Theorem 7.3 does not depend on [Hrushovski and Pillay 1987]).

Lemma 7.4. *The following are equivalent*:

(1) $\mathcal{G}$ *is locally modular.*

(2) *If* $\{X_t : t \in T\}$ *is a nonempty faithful family of definable subsets of a definable set* Y, *each of dimension* k, *then* $\dim(T) \leq \dim(Y) - k$.

(3) *Every definable set* $X \subset G^n$ (*for all* n) *is affine.*

Moreover, if (1)–(3) *hold, then every definable subgroup* $H \subset G^n$ (*for all* n) *is* $\mathrm{acl}(\varnothing)$*-definable.*

Proof. (1) $\Longrightarrow$ (2): Assume $\mathcal{G}$ is locally modular, and let $\{X_t : t \in T\}$, Y, and k be as in (2). Without loss of generality $\{X_t\}$ and Y are $\varnothing$-definable. Let $t \in T$ be generic and $x \in X_t$ generic over t. Then $\dim(xt) = \dim(T) + k$, and by one-basedness (equivalently local modularity) $t \in \mathrm{acl}(x)$, so $\dim(x) = \dim(T) + k$, thus $\dim(Y) \geq \dim(T) + k$, which is equivalent to (2).

(2) $\Longrightarrow$ (3): Let $X \subset G^n$ be stationary of dimension k, without loss of generality $\varnothing$-definable. Applying (2) shows that the family of translates of X is at most $(n-k)$-dimensional (i.e., a generic translate of X has canonical base of dimension at most $n-k$). It follows that $\dim(\mathrm{Stab}(X)) \geq k$, which easily implies that X is almost equal to a coset of $\mathrm{Stab}(X)$. By induction on Morley rank and degree, the conclusion follows for all X.

(3) $\Longrightarrow$ (1): Let S be any strongly minimal plane curve; we show that $\dim(\mathrm{Cb}(S)) \leq 1$, implying local modularity. Now we can write $S = C_{t_0}$ for some generic $t_0 \in T$, where $\mathcal{C} = \{C_t : t \in T\}$ is a $\varnothing$-definable (not necessarily faithful) family of plane curves, and $T \subset G^n$ for some n. Absorbing more constants to the language, if needed, we may assume T is stationary. It follows that the total space $C = \{(x, t) : x \in C_t\}$ of $\mathcal{C}$ is stationary, so (by (3)) almost coincides with a coset H of some connected definable subgroup of G^{n+2}. Note that (since H is a coset) there is some (necessarily $\varnothing$-definable) subgroup $K \leq G^2$ such that each nonempty fiber $H_t \subset G^2$ is a coset of K. In particular, since $t_0 \in T$ is generic, it follows that $S = C_{t_0}$ almost coincides with a coset of K. Thus $\mathrm{Cb}(S)$ is definable over a single element of the one-dimensional group G^2/K, which gives $\dim(\mathrm{Cb}(S)) \leq 1$, as desired.

So we have shown that (1)–(3) are equivalent. Now suppose $\mathcal{G}$ is locally modular. If there is a connected definable subgroup $H \subset G^n$ for some n (say of dimension k) which is not $\mathrm{acl}(\varnothing)$-definable, then there is a $\varnothing$-definable infinite faithful family $\{H_t : t \in T\}$ of k-dimensional subgroups of G^n, whose generic members are connected. Passing to a subfamily and adding parameters if necessary, we may

assume that $\dim(T) = 1$. Then the family $\{X_s : s \in S\}$ of all cosets of all H_t is a $\varnothing$-definable faithful $(n-k+1)$-dimensional family of k-dimensional subsets of G^n, which contradicts (2). □

Before proceeding, let us also isolate the following two well-known and easy facts:

Lemma 7.5. *Let $\mathcal{H}$ be a group of Morley rank 1 definable in an ω-stable structure. Then any definable subset of $\mathcal{H}$ is affine.*

Proof. Let $S \subseteq \mathcal{H}$ be any definable set, and let $H_1, \ldots, H_k$ be the cosets of H^0 (the definable connected component of H). Then each H_i is strongly minimal, so $S \cap H_i$ is either finite or cofinite for all i. Thus S has finite symmetric difference with $\cup\{H_i : \mathrm{RM}(S \cap H_i) = 1\}$. Since the right-hand side is an affine set and the symmetric difference is finite (therefore also affine), the lemma follows. □

Lemma 7.6. *Let p be a stationary type in G^n for some n. Then the following are equivalent*:

(1) *The stationary type p is affine.*

(2) *If $(x, y, z) \models p^3$, then $x + y - z \models p$.*

Proof. It is clear that (1) implies (2), by writing p as the generic type of a coset. Now assume (2) holds, and write p as the generic type of a stationary definable set S. Fix $z \in S$ generic over the parameters defining S. Then (2), equivalently stated, gives that $S - z$ is almost contained in $\mathrm{Stab}(S)$ (since given x, y, z as in (2) it is clear that $(x - z) \in \mathrm{Stab}(S)$). Thus, S is almost contained in $\mathrm{Stab}(S) + z$. On the other hand, since z is generic in S it follows by definition that $\mathrm{Stab}(S) + z$ is almost contained in S, so that in fact $S \sim \mathrm{Stab}(S) + z$. So p is equivalently the generic type of $\mathrm{Stab}(S) + s$, and thus (1) holds. □

Let us now proceed with the proof of Theorem 7.3:

Proof. The implications (1) $\Longrightarrow$ (2) and (6) $\Longrightarrow$ (1) are contained in Lemma 7.4. The implication (4) $\Longrightarrow$ (5) is clear, and (5) $\Longrightarrow$ (6) is immediate by induction on Morley rank. We show (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4).

(2) $\Longrightarrow$ (3): We have to prove that if every definable plane curve is affine, then so is every definable one-dimensional set. Let $C \subset G^n$ be definable of dimension 1. Since a finite union of affine sets is affine, we may assume C is strongly minimal. Since C is infinite, there is a projection $\pi : G^n \to G$ with cofinite image in G. Without loss of generality $\pi = \pi_1$ (the leftmost projection). Now for $i = 2, \ldots, n$, let $C_i = \pi_{1i}(C) \subset M^2$ (the image of C in the first and i-th coordinates). Then C_i is a strongly minimal plane curve, so by (2) almost coincides with a strongly minimal coset H_i. Deleting finitely many points from C if necessary, we may assume $C_i \subset H_i$ for all i.

Using that H_i is a coset and $\pi_1(C)$ is cofinite in G, it follows that $\pi_1(H_i) = G$. So for each i, there is an element $c_i \in G$ with $(0, c_i) \in H_i$. Then replacing C with $C - (0, c_2, \ldots, c_n)$ if necessary, we may assume each H_i is in fact a subgroup of G^2.

Now let $H = \{(x_1, \ldots, x_n) : (x_1, x_i) \in H_i \text{ for } i = 2, \ldots, n\}$. Then $C \subset H$, and it is easy to verify that H is a one-dimensional definable subgroup of G^n. So by Lemma 7.5, C is affine.

(3)$\Longrightarrow$(4): We have to show that if every definable one-dimensional set is affine, then every stationary type is affine. If $\mathcal{G}$ is locally modular this is immediate by Lemma 7.4. So assume $\mathcal{G}$ is not locally modular. The result now follows readily from the existence of 3-sweeping extensions: Let p be a stationary type in G^n for some n, say over A. If $\dim(p) = 0$, p is clearly affine, so assume $\dim(p) \geq 1$. Then by Proposition 5.8, p has a 3-sweeping stationary extension $q \supset p$ with $\dim(q) = 1$. It follows from (3) that q is affine. Now let $(x, y, z) \models q^3$, so by Lemma 7.6, $x + y - z \models q$. Since $q \supset p$ is 3-sweeping, we get $(x, y, z) \models p^3$ and $x + y - z \models p$. Then by Lemma 7.6 again, p is affine. □

7B. ***Very ampleness in groups.*** We now apply Theorem 7.3 to show that a nonlocally modular strongly minimal expansion of a group G admits a finite subgroup H such that the quotient G/H is very ample. We conclude that nonlocally modular expansions of divisible groups are already very ample.

Theorem 7.7. *Let $\mathcal{G} = (G, +, \ldots)$ be a nonlocally modular strongly minimal expansion of a group. Then*:

(1) *There is a finite subgroup $H \leq G$ such that G/H is very ample. In particular, $\mathcal{G}$ admits a very ample sort.*

(2) *If $\mathcal{G}$ is divisible, then $\mathcal{G}$ itself is very ample.*

Proof. First we prove (1). By Theorem 7.3, there is a strongly minimal plane curve $C \subset G^2$ which is not affine. Then by Lemma 7.2, the group $\mathrm{Stab}(C) \leq G^2$ is finite. Let $H \leq G$ be a finite subgroup such that $\mathrm{Stab}(C) \leq H^2$ (for example, H could be the group generated by all coordinates of elements of $\mathrm{Stab}(C)$). Let C/H be the image of C in $(G/H)^2$, by applying the projection $G \to G/H$ to both coordinates. It follows easily that, in the strongly minimal group (G/H), the set C/H is a strongly minimal plane curve with trivial stabilizer. So by Lemma 7.2 again, G/H is very ample.

Now assume further that G is divisible; we prove (2). Let H be as in (1), and let $h = |H|$. Then the map $xH \mapsto h \cdot x$ is a well-defined map from G/H to G, and is clearly definable. Moreover, by divisibility this map is surjective with finite fibers. Thus, we are in the situation of Lemma 3.24 with $X = G/H$ and $Y = G$. So since G/H is very ample, the lemma implies that so is G. □

7C. ***Expansions of algebraic groups.*** In the final subsection, we apply Theorem 7.7 to characterize ACF-definable expansions of one-dimensional divisible algebraic groups. Our motivation is as follows. Suppose $(K, +, \times)$ is an algebraically closed field, and let K^{lin} denote the structure $(K, +, \{\lambda_a : a \in K\})$, where λ_a is the map $x \mapsto ax$. Martin [1987] makes the following two conjectures:

(1) There are no intermediate structures between $(K, \times)$ and $(K, +, \times)$.

(2) If $\text{char}(K) = 0$ then there are no intermediate structures between K^{lin} and $(K, +, \times)$.

Conjecture (2) was proved by Marker and Pillay [1990]; however, it seems that (1) has not since been addressed. Our result in this subsection will in fact be a more general statement implying both (1) and (2). Namely, we show:

Theorem 7.8. *Let K be an algebraically closed field, and let $(G, \cdot)$ be the group of K-points of a one-dimensional divisible algebraic group over K. Let G^{lin} denote the structure endowing G with the group operation and all of its endomorphisms (as an algebraic group), and let G^{Zar} denote the full K-induced structure on G. Then there are no intermediate structures between G^{lin} and G^{Zar}.*

Before proceeding with the proof, let us note:

Remark 7.9. Since every endomorphism of the additive group is a scaling, and every endomorphism of the multiplicative group is a power map (thus definable from the group operation alone), we obtain (1) and (2) above. Moreover, note that Theorem 7.8 also applies to elliptic curves over K.

Proof. Let $\mathcal{G} = (G, \cdot, \dots)$ be a reduct of G^{Zar} which properly expands G^{lin}.

Claim 7.9.1. *There is a nonaffine $\mathcal{G}$-definable set $X \subset G^n$ for some n.*

Proof. It suffices to show that every connected definable subgroup of G^n is definable in G^{lin}. So, let $H \leq G^n$ be a connected definable subgroup, say of dimension d. So H is stationary, which implies there is an almost finite-to-one projection $H \to G^d$. Since H is a subgroup, this implies that $H \to G^d$ is everywhere finite-to-one, and the fibers of $H \to G^d$ are cosets of a finite group, say $K \leq H$. Let $k = |K|$, and let kH be the image of H under scaling by k. Then the projection $kH \to G^d$ is the graph of a definable function $f : G^d \to G^{n-d}$. Since kH is a subgroup, f is a homomorphism. In particular, each coordinate component of f (i.e., $G^d \to G^{n-d} \to G$) is a definable homomorphism from G^d to G, and each of those is built of d G^{lin}-definable G-endomorphisms, so is itself G^{lin}-definable. Thus f is G^{lin}-definable, and therefore so is kH.

Finally, let H' be the preimage of kH under scaling by K; so $H \leq H'$. Note that by divisibility, scaling by k is finite-to-one on G^n; thus $\dim(H) = \dim(kH) = \dim(H')$. Then, since H is connected, it must be the connected component $(H')^0$. In particular,

H is the image of H' under scaling by l for some l. Now to complete the proof, recall that kH is G^{lin}-definable; thus so is H' by definition, and thus (scaling by l) so is H. □

Now by the claim and the main result in [Hrushovski and Pillay 1987] (or Lemma 7.4 if desired), it follows that $\mathcal{G}$ is not locally modular. Then, by divisibility and Theorem 7.7, $\mathcal{G}$ is very ample. Finally, by [Hasson and Sustretov 2017] $\mathcal{G}$ satisfies the Zilber trichotomy; so by Theorem 4.14, $\mathcal{G}$ is full in K. In other words, $\mathcal{G}$ is interdefinable with G^{Zar}, as desired. □

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