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In difference algebra, basic definable sets correspond to prime ideals that are invariant under a structural endomorphism. The main idea of an article with Peterzil (*Proc. London Math. Soc.* **85**:2 (2002), 257–311) was that periodic prime ideals enjoy better geometric properties than invariant ideals, and to understand a definable set, it is helpful to enlarge it by relaxing invariance to periodicity, obtaining better geometric properties at the limit. The limit in question was an intriguing but somewhat ephemeral setting called virtual ideals. However, a serious technical error was discovered by Tom Scanlon's UCB seminar. In this text, we correct the problem via two different routes. We replace the faulty lemma by a weaker one that still allows recovering all results of the aforementioned paper for all virtual ideals. In addition, we introduce a family of difference equations ("cumulative" equations) that we expect to be useful more generally. Previous work implies that cumulative equations suffice to coordinatize all difference equations. For cumulative equations, we show that virtual ideals reduce to globally periodic ideals, thus providing a proof of Zilber's trichotomy for difference equations using periodic ideals alone.

Introduction

Boris Zilber developed a geometric description of $\aleph_1$-categorical theories, having a trichotomy at its heart. It is based on the dimension theory of Morley (shown to take finite values by Baldwin), but gives information of a radically new kind than an abstract dimension theory. Intuitively, a model of the theory is coordinatized by geometries that have either a graph-theoretic nature, or derive from linear algebra, or belong to algebraic geometry. Though it is only the minimal definable sets that are described in this way, Zilber (and later others) demonstrated an overwhelming effect on the structure globally.

Zilber conjectured that there is no fourth option. This turned out to be incorrect at the precise level of generality of $\aleph_1$-categorical structures. But it was established with additional hypotheses of a topological nature [11], and moreover

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proved to be meaningful and indeed to capture the nature of structures far beyond strong minimality. Appropriate versions hold for compact complex manifolds, for differentially closed and separably closed fields, for strongly minimal sets interpretable in algebraically closed fields of characteristic zero [1]; the latter closes in characteristic zero a line opened more than thirty years ago by Eugenia Rabinovich, in her Kemerovo PhD with Zilber. The trichotomy is also meaningful for unstable theories: see [12] for the o-minimal case. Many applications depend on the trichotomy, including Zilber's gem [15]. For difference equations, applications to diophantine geometry include [3; 4; 10; 13].

Thanks to Zilber's philosophy, when we made our first steps in the structure of difference equations in [2], we knew in advance what it is that we should aim to prove. The methods were informed by finite-rank stability and the nascent generalization to simplicity. But they also relied strongly on ramification divisors, and thus applied only in characteristic zero. Our approach in [7] to the positive characteristic case thus had to be different.

The trichotomy results of [11] are valid for stable structures with a finite dimension assigned to definable sets, satisfying a "dimension theorem" controlling dimensions of intersections. Now the model companion ACFA of the theory of difference fields is not stable, nor does the geometry of finite-dimensional sets satisfy the dimension theorem: the intersection of two such sets may have unexpectedly low dimension. For instance, the naive intersection of two surfaces in 3-space over the fixed field of the automorphism σ could be two lines interchanged by σ; within the fixed field their intersection point would be the only solution. Both of these pathologies are ameliorated as one relaxes σ to σ^m (going from the equation $\sigma(x) = F(x)$ to $\sigma^m(x) = F^{(m)}(x)$). At the limit, one has a *virtual structure*, defined and studied in [7]; under appropriate conditions, this structure is stable and the dimension theorem is valid. Proving this uses basic ideas from topological dynamics to obtain recurrent points that may not be periodic; see Lemma 2.8 for example. Using a generalization of the Zariski geometries of [11], one can then deduce the trichotomy theorem. The concrete form it takes here allows analyzing any difference equation via a tower of equations over fixed fields and equations of locally modular type.

In 2015, however, Tom Scanlon's Berkeley seminar recognized a problem with a key technical lemma, Lemma 3.7. We show below how to prove a somewhat weaker version of this lemma: where the wrong Lemma 3.7 asserted a unique component through a point, the corrected version, Lemma 2.16, implies that the number of such components is finite, indeed at most the degree of the normalization of the relevant variety in the base. All the main results of the paper remain valid with the same set of ideas, but considerable reorganization is required. One role of the present paper is to provide a lengthy erratum, explaining in detail how this may be done. Parts of this paper are thus technical and need to be read in conjunction with [7]. However, Section 2, which contains the main correction and in particular the key dimension theorem, is self-contained in the sense of quoting some results from [7] but not requiring entering into their proofs.

At the same time, we take the opportunity to present a setting ("cumulative equations") in which the limit structure is equivalent to an ordinary structure, in the sense that the associated algebraic object is an ordinary ring with its periodic ideals, rather than an abstract limit of such rings as in the virtual case. Results of [5] imply that this setting, while not fully general, suffices to coordinatize all difference equations. It may be of interest for other applications, in particular the study of limit structures for more equations that are not necessarily algebraic over equations of SU-rank one.

We expect that a trichotomy theorem can be proved for Zariski geometries based on Robinson structures. This has so far been worked out only in special cases; the most general treatment is contained in the unpublished PhD thesis of Elsner [9]. Consequently, the trichotomy follows from the basic cumulative case alone, though this is not the case for some of the other results: for finer statements such as a description of the fields definable in the limit structures, both in [7] and here, we use additional features of the specific structure.

Let S be a difference ring, generated by a finitely generated ring R. The main idea of [7] was that as n becomes more and more divisible, more σ^n-ideals appear, and their structures become progressively smoother. However there is also a countercurrent at work: the difference subring R_{σ^n} of (S, σ^n) generated by R may become smaller. This double movement leads to technical complexity. If, however, $\sigma(R)$ is contained in the ring generated by R and $\sigma^n(R)$ for any n, this problem does not arise. It is this behavior (slightly generalized to fraction fields) that we refer to as *cumulative*. It turns out that cumulative difference equations still represent all isogeny classes, and allow for considerable simplification.

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Plan of the paper. In Section 1 we mainly recall definitions and notation from [7]. Section 2 contains the proof of Proposition 2.6 of [7], as well as some useful auxiliary results and remarks. The cumulative case is done in the first half, the general case in the second half. Sections 3 and 4 are devoted to rereading [7] and making the necessary changes and adaptations: Section 3 deals with Sections 2 to 4 of [7], and Section 4 with the remainder of the paper.

1. Setting, notation, basic definitions

1.1. ***Setting and notation.*** In what follows, K is a sufficiently saturated existentially closed difference field, containing an algebraically closed difference subfield k_0, and

Ω a $|K|^+$-saturated existentially closed difference field containing K. We always work inside Ω.

If L is a field, then L^s and L^{alg} denote the separable and algebraic closure of the field L.

Conventions. Unless otherwise stated, all difference fields and rings will be *inversive*, i.e., the endomorphism σ is an automorphism; in other words, we take a difference ring to be a commutative ring with a $\mathbb{Z}$-action. Similarly, all difference ideals will be *reflexive*, i.e., if (R, σ) is a difference ring, a σ-ideal of R is an ideal I such that $\sigma(I) = \sigma^{-1}(I) = I$.

If k is a difference field, $X = (X_1, \dots, X_n)$, then $k[X]_\sigma$ denotes the inversive difference domain $k[\sigma^i(X_j) \mid i \in \mathbb{Z}, 1 \le j \le n]$ and $k(X)_\sigma$ its field of fractions. Similarly, if a is a tuple in Ω, $k[a]_\sigma$ and $k(a)_\sigma$ denote the inversive difference subring and subfield of Ω generated by a over k. Similar notation is used for difference rings. If a is an n-tuple, then $I_\sigma(a/k) = \{f \in k[X_1, \dots, X_n]_\sigma \mid f(a) = 0\}$. If $k(a)_\sigma$ has finite transcendence degree over k, the *limit degree* of a over k, denoted $\mathrm{ld}(a/k)$ or $\mathrm{ld}_\sigma(a/k)$, is $\lim_{n\to\infty}[k(a, \dots, \sigma^{n+1}(a)) : k(a, \dots, \sigma^n(a))]$.

If A is a subset of a difference ring S, then $(A)_{\sigma^m}$ denotes the (reflexive) σ^m-ideal of S generated by A. If $A \subset \Omega$, then $\mathrm{cl}_\sigma(A)$ denotes the perfect closure of the difference subfield of Ω generated by A, $\mathrm{acl}_\sigma(A)$ the (field-theoretic) algebraic closure of $\mathrm{cl}_\sigma(A)$, and $\mathrm{dcl}_\sigma(A)$ the model-theoretic definable closure of A. If A is a subring of a difference ring S, then A_σ denotes the (inversive) difference subring of S generated by A.

Recall that $\mathrm{acl}_\sigma(A)$ coincides with the model-theoretic algebraic closure $\mathrm{acl}(A)$, and that independence (in the sense of the difference field Ω) of A and B over a subset C coincides with the independence (in the sense of ACF) of $\mathrm{acl}(A)$ and $\mathrm{acl}(B)$ over $\mathrm{acl}(C)$.

If $m \ge 1$, then $\Omega[m]$ denotes the σ^m-difference field (Ω, σ^m). The languages $\mathcal{L}$ and $\mathcal{L}[m]$ are the languages $\{+, -, \cdot, 0, 1, \sigma\}$ and $\{+, -, \cdot, 0, 1, \sigma^m\}$. We view $\mathcal{L}[m]$ as a sublanguage of $\mathcal{L}$, and $\Omega[m]$ as a reduct of Ω. Recall that $\Omega[m]$ is also an existentially closed saturated difference field, by Corollary 1.12 of [2]. If a is a tuple of Ω and k a difference subfield of Ω, then $\mathrm{qftp}(a/k)$ denotes the quantifier-free type of a over k in the difference field Ω, and if $m \ge 1$, then $\mathrm{qftp}(a/k)[m]$ denotes the quantifier-free type of a over k in the difference field $\Omega[m]$. Similarly, if q is a quantifier-free type over k, then $q[m]$ denotes the set of $\mathcal{L}(k)[m]$ quantifier-free formulas implied by q.

Basic and semibasic types.

Definitions 1.2. We consider quantifier-free types $p, q, \dots$ over the algebraically closed difference field k_0, and integers $m, n \ge 1$.

(1) q satisfies (ALGm) if whenever a realizes q, then $\sigma^m(a) \in k_0(a)^{\mathrm{alg}}$.

(2) The *eventual SU-rank* of q, $\mathrm{evSU}(q)$, is $\lim_{m\to\infty} \mathrm{SU}(q[m!])$, where $\mathrm{SU}(q[m!])$ (the SU-rank of $q[m!]$) is computed in the $\sigma^{m!}$-difference field $\Omega[m!]$. For more details, see Section 1 in [7], starting with 1.10. Write $\mathrm{SU}(a/k_0)[n] := \mathrm{SU}(q[n])$, computed in the σ^n-difference field $\Omega[n]$ ($n \geq 1$, a realizing q). If D is a countable union of k-definable subsets of some cartesian power of Ω, then $\mathrm{evSU}(D) = \sup\{\mathrm{evSU}(a/k) \mid a \in D\}$.

(3) $p \sim q$ if and only if for some $m \geq 1$, $p[m] = q[m]$. The $\sim$-equivalence class of p is denoted by $[p]$ and is called a *virtual type*.

(4) $\mathcal{X}_p(K)$ denotes the set of tuples in K which realize $p[m]$ for some $m \geq 1$, and similarly for $\mathcal{X}_p(\Omega)$. We denote by X_p the underlying affine variety, i.e., the Zariski closure of $\mathcal{X}_p(\Omega)$ in affine space.

(5) A *basic* type is a quantifier-free type p over k_0, with evSU-rank 1, which satisfies (ALGm) for some m. Note that if p is basic, so is $p[n]$ for every n.

(6) A *semibasic type* is a quantifier-free type q such that if a realizes q, then there are tuples $a_1, \dots, a_n \in k_0(a)^{\mathrm{alg}}$ which realize basic types over k_0, are algebraically independent over k_0, and are such that $a \in k_0(a_1, \dots, a_n)^{\mathrm{alg}}$.

(7) The quantifier-free type q is *cumulative* if for some (any) realization a of q and every $m \geq 1$, $\sigma(a) \in k_0(a, \sigma^m(a))$. Note that this implies that $k_0(a)_\sigma = k_0(a)_{\sigma^m}$ for any $m \geq 1$, and that (ALGm) is equivalent to (ALG1).

Remarks 1.3. Let k be an inversive difference field.

(1) We will often use the following equivalences, for a tuple a:
 (i) $[k(a, \sigma(a)) : k(a)] = \mathrm{ld}(a/k)$.
 (ii) The fields $k(\sigma(a) \mid i \leq 0)$ and $k(\sigma^i(a) \mid i \geq 0)$ are linearly disjoint over $k(a)$.
 (iii) $I_\sigma(a/k)$ is the unique prime σ-ideal of $k[X]_\sigma$ extending the prime ideal $\{f(X, \sigma(X)) \in k[X, \sigma(X)] \mid f(a, \sigma(a)) = 0\}$ of $k[X, \sigma(X)]$ ($|X| = |a|$).

 Note that these equivalent conditions on the tuple a in the difference field Ω also imply the analogous conditions for the tuple a in the difference field $\Omega[m]$ for $m \geq 1$ (use (ii)).

(2) Let P be a prime ideal of $k[X, \sigma(X)]$ (X a tuple of variables) and assume that $\sigma(P \cap k[X]) = P \cap k[\sigma(X)]$. Then P extends to a prime σ-ideal of $k[X]_\sigma$. We will usually use it with the prime ideal $\sigma^{-1}(P)$ of $k[\sigma^{-1}(X), X)]$.

Proof. All these are straightforward remarks; see also Section 1.3 of [5] for the equivalence of (i) and (ii), and Sections 5.6 and 5.2 of [8] for the remaining items. □

1.4. ***Coordinate rings associated to quantifier-free types.*** (See also (3.5) and (3.6) in [7]). Let q be a quantifier-free type over k_0, in the tuple x of variables, fix a realization a of q. The *pair* $(R_q, R_{q,\sigma})$ *of coordinate rings associated to* q is defined as follows: Let $k_0(x)_\sigma$ be the fraction field of $k_0[X]_\sigma/I_\sigma(a/k_0)$, $k_0(x)$ its subfield generated by x over k_0. Then we define the ring $R_q := k_0(x) \otimes_{k_0} K$ and the σ^m-difference ring $R_{q,\sigma^m} := k_0(x)_{\sigma^m} \otimes_{k_0} K$ for $m \geq 1$. We often denote R_q and R_{q,σ^m} by $K\{x\}$ and $K\{x\}_{\sigma^m}$, and define in an analogous way the coordinate rings $k_1\{x\}$ and $k_1\{x\}_{\sigma^m}$ if k_1 is a difference field containing k_0.

Given semibasic types $q_1(x_1), \dots, q_n(x_n)$, we take the tensor product over K of their coordinate rings, and call them the coordinate rings associated to $(q_1, \dots, q_n)$. So, we have

$$R_{(q_1,\dots,q_n)} = K\{x_1\} \otimes_K \cdots \otimes_K K\{x_n\},$$
$$R_{(q_1,\dots,q_n),\sigma^m} = K\{x_1\}_{\sigma^m} \otimes_K \cdots \otimes_K K\{x_n\}_{\sigma^m}.$$

To a semibasic type q, we associate three new pairs of coordinate rings as follows. Say q is realized by a tuple a, and $a_1, \dots, a_n$ are as in the definition of semibasic given above. We let $p_i = \mathrm{qftp}(a_i/k_0)$, $r = \mathrm{qftp}(a_1, \dots, a_n/k_0)$, and $s = \mathrm{qftp}(a, a_1, \dots, a_n/k_0)$. Then we define

$$R_q^1 = R_{p_1} \otimes_K \cdots \otimes_K R_{p_n}, \qquad R_q^2 = R_r, \qquad R_q^3 = R_s,$$
$$R_{q,\sigma^m}^1 = R_{p_1,\sigma^m} \otimes_K \cdots \otimes_K R_{p_n,\sigma^m}, \qquad R_{q,\sigma^m}^2 = R_{r,\sigma^m}, \qquad R_{q,\sigma^m}^3 = R_{s,\sigma^m}.$$

These rings depend on the choice of the tuples $a_1, \dots, a_n$, but we may fix once and for all these tuples. Note that then $R_q^1 \subseteq R_q^2 \subseteq R_q^3 \supseteq R_q$, that R_q^2 is a localization of R_q^1, and that R_q^3 is integral algebraic over R_q^2 and over R_q. Similar statements hold for the associated difference rings. If q is basic, we define $R_q^i = R_q$ and $R_{q,\sigma^m}^i = R_{q,\sigma^m}$. We extend the notation to the more general coordinate rings $R_{(q_1,\dots,q_n)}$.

We say that a coordinate ring R_σ satisfies (ALGm) or is cumulative, if the semibasic types involved in the definition of R_σ all satisfy (ALGm) or are cumulative.

1.5. ***Convention.*** From now on, all quantifier-free types will satisfy (ALGm) for some $m \geq 1$, so that all coordinate rings will satisfy (ALGm).

Definitions 1.6. Let (R, R_σ) be a pair of coordinate rings, as defined above, and S a ring.

(1) Let P be a prime ideal of a ring S. The *dimension* of P, denoted by $\dim(P)$, is the Krull dimension of the ring S/P. If I is an ideal of S, the *dimension* of I, $\dim(I)$, is $\sup\{\dim(P) \mid P \supseteq I,\ P \in \mathrm{Spec}(S)\}$. If $S = R$, then $\dim(P)$ coincides with $\mathrm{tr.deg}_K \mathrm{Frac}(R/P)$.

(2) Let P be a prime ideal of a coordinate ring R_σ. The *virtual dimension of* P, denoted $\mathrm{vdim}(P)$, is $\dim(P \cap R)$. If R_σ satisfies (ALGm), it coincides with $\dim(P \cap R_{\sigma^m})$. Similarly, if I is an ideal of R_σ, then $\mathrm{vdim}(I) = \dim(I \cap R)$.

(3) A *virtual* [*perfect*], [*prime*] *ideal* of R_σ is a [perfect[1]], [prime] (reflexive) σ^m-ideal of R_{σ^m} for some $m \geq 1$.

(4) A [*perfect*], [*prime*] *periodic ideal* of R_σ is a [perfect], [prime] σ^m-ideal I of R_σ for some $m \geq 1$. A priori, not all virtual ideals extend to periodic ideals.

(5) Let I be an ideal of R. We say that I is *pure of dimension* d if all minimal primes over I have dimension d. Let I be an ideal of R_σ. We say that I is *virtually pure of dimension* d if $I \cap R$ is pure of dimension d.

(6) Let I be a virtual ideal of $R_\sigma = K\{x\}_\sigma$. Then $V(I)$ is the subset of $K^{|x|}$ defined by $a \in V(I)$ if and only if for some $m \geq 1$, for each $h \in I \cap R_{\sigma^m}$, viewed as a σ^m-polynomial, we have $h(a, \sigma^m(a), \ldots) = 0$. Thus $V(I)$ stands in bijection with $\bigcup_m \mathrm{Hom}_{\sigma^m}(R_{\sigma^m}/I, K)$, where Hom_{σ^m} refers to ring homomorphisms commuting with σ^m.

 Note that if $R_\sigma = R_q$ for some quantifier-free type q, then $V(0)$ is precisely $\mathcal{X}_q(K)$. We call vdim(0) (i.e., the Krull dimension of R) the (*virtual*) *dimension of* q.

2. Existence theorems for periodic ideals

The aim of this section is to give proofs of the results of [7] needed towards the proof of the trichotomy in positive characteristic, and in particular the very important Proposition 2.6 of [7]. We try to follow the plan of [7], and will occasionally refer to it. While the results of Chapter 2 are indeed correct, the problem is that our coordinate rings do not satisfy the required hypotheses. The mistake appears in Lemma 3.7.

Assumptions. The coordinate rings we consider are those associated to tensor products of coordinate rings of semibasic types whose corresponding basic types have virtual dimension e, *for some fixed integer* $e \geq 1$. A typical pair of coordinate rings is denoted (R, R_σ), without reference to the types involved in the construction.

As for types, we declare two virtual prime ideals P, Q equivalent, and write $P \sim Q$, if for some $m \geq 1$, $P \cap R_{\sigma^m} = Q \cap R_{\sigma^m}$. We retain, however, Definition 1.2(3) of virtual prime ideals; the equivalence classes are called virtual prime ideal classes.

Proposition 2.1 (addendum to Proposition 2.4 of [7]). *Let* (R, R_σ) *be a pair of coordinate rings.*

(1) *Let* P *and* Q *be virtual prime ideals. If* $V(P) = V(Q)$, *then* $P \sim Q$.

(2) *Let* P *be a prime* σ^m*-ideal of* R_{σ^m}. *Then for some* $\ell > 0$, P *extends to a prime* σ^ℓ*-ideal* Q *of* R_σ. *In particular, since* $V(Q) = V(P)$, *this shows that every set defined by a virtual prime ideal is also defined by a periodic prime ideal*

[1] A σ-ideal I of a difference ring R is perfect if whenever $a^n\sigma(a) \in I$, then $a \in I$.

of R_σ; i.e., every prime periodic ideal of R_{σ^m} is equivalent to a prime periodic ideal of R_σ.

Proof. (1) We may assume that P and Q are prime σ-ideals and that R satisfies (ALG1). Choose a (small) subfield k_1 of K such that for any $m \geq 1$, $P \cap R_{\sigma^m}$ and $Q \cap R_{\sigma^m}$ are generated by their intersection with $k_1\{x\}_{\sigma^m}$ (x the variables of R). By saturation of K, it contains a point a which is a generic point of $V(P)$ over k_1, i.e., with $\text{tr.deg}(k_1(a)/k_1) = \dim(P)$. Then a is in $V(Q)$, whence $\dim(Q) \geq \dim(P)$, and the symmetric argument tells us that these dimensions are equal, and that a is a generic of $V(Q)$ over k_1. Let ℓ be divisible by m and such that $P \cap R_{\sigma^\ell}$ and $Q \cap R_{\sigma^\ell}$ are prime σ^ℓ-ideals contained in $(x-a)_{\sigma^\ell}$. Then $I_{\sigma^\ell}(a/k_1) = P \cap k_1\{x\}_{\sigma^\ell} = Q \cap k_1\{x\}_{\sigma^\ell}$, which shows that $P \sim Q$.

(2) Let $\varphi : R_{\sigma^m} \to \Omega$ be a K-homomorphism of σ^m-difference rings with kernel P. If $p_1(x_1), \dots, p_n(x_n)$ are the semibasic types associated to R_σ, then

$$R_\sigma = k_0(x_1)_\sigma \otimes_{k_0} \cdots \otimes_{k_0} k_0(x_n)_\sigma \otimes_{k_0} K,$$

and R_{σ^m} corresponds to the subring $k_0(x_1)_{\sigma^m} \otimes_{k_0} \cdots \otimes_{k_0} k_0(x_n)_{\sigma^m} \otimes_{k_0} K$. Our map φ is entirely determined by its restrictions to each of the factors of the tensor product, and for $i = 1, \dots, n$, we let φ_i denote the restriction of φ to $k_0(x_i)_{\sigma^m}$. Since $k_0(x)_\sigma$ is finitely generated over $k_0(x)_{\sigma^m}$, Proposition 1.12(3) of [7] gives that for some $\ell > 0$ divisible by m, the σ^ℓ-embeddings $\varphi_i : k_0(x_i)_{\sigma^m} \to \Omega$ extend to σ^ℓ-embeddings $\psi_i : k_0(x)_\sigma \to \Omega$ for $i = 1, \dots, n$. Then define $\psi = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \otimes \text{id}_K$, and take $Q = \ker \psi$. □

Lemma 2.2. *Let R_σ be a coordinate ring, and $S_\sigma = R[c]_\sigma$ a difference ring, with $S = R[c]$ integral algebraic (and finitely generated) over R. If P is a prime σ-ideal of R_σ, then for some $\ell \geq 1$, $P \cap R_{\sigma^\ell}$ extends to a prime σ^ℓ-ideal of S_{σ^ℓ}.*

Proof. Replacing σ by σ^m for some m, we may assume that R_σ satisfies (ALG1).

Claim. *There is $m \geq 1$ such that for any $\ell \geq 1$, if $R' = R[\sigma(R), \dots, \sigma^m(R)]$, then $P \cap R'_{\sigma^\ell}$ is the unique prime σ^ℓ-ideal of R'_{σ^ℓ} which extends $P \cap R'[\sigma^\ell(R')]$.*

Proof of claim. Indeed, take $a \in \Omega$ such that $\text{Frac}(R_\sigma/P) \simeq K(a)_\sigma$ and m such that $[K(a, \dots, \sigma^{m+1}(a)) : K(a, \dots, \sigma^m(a))] = \text{ld}(a/K)$. Then if $b = (a, \dots, \sigma^m(a))$, we have $\text{ld}(b/K) = \text{ld}(a/K)$ and for $\ell \geq 1$, $\text{ld}_{\sigma^\ell}(b/k_0) = [K(b, \sigma^\ell(b)) : K(b)]$.

The claim now follows by the equivalences given in Remarks 1.3(1). □

For $n \geq 0$, let $R(n)$ and $S(n)$ denote the subrings of R_σ and S_σ generated respectively by $\sigma^i(R)$ and $\sigma^i(S)$, $-n \leq i \leq n$. Then each $S(n)$ is Noetherian, integral algebraic over $R(n)$, $S_\sigma = \bigcup_{n\in\mathbb{N}} S(n)$, and we have a natural map

$$\text{Spec}(S_\sigma) \to \prod_{n\in\mathbb{N}} \text{Spec}(S(n)).$$

For each $n \in \mathbb{N}$, the set X_n of prime ideals of $S(n)$ which extend $P \cap R(n)$ is finite and nonempty, and the natural map $\mathrm{Spec}(S(n+1)) \to \mathrm{Spec}(S(n))$ sends X_{n+1} to X_n. Hence $X := \varprojlim X_n$ is a closed, compact, nonempty subset of $\prod_{n\in\mathbb{N}} X_n$, and is the set of prime ideals of S_σ which extend P. As each X_n is finite, and the set X is stable under the (continuous) action of σ on $\mathrm{Spec}(S_\sigma)$, X contains a recurrent point Q. Let m be given by the claim, and consider $S(m)$. Then for some $\ell \geq 1$, we have $\sigma^\ell(Q) \cap S(m) = Q \cap S(m)$, and therefore, using Remarks 1.3(2), there is a prime σ^ℓ-ideal Q' of $S(m)_{\sigma^\ell}$ such that

$$Q' \cap S(m)[\sigma^{-\ell}(S(m))] = Q \cap S(m)[\sigma^{-\ell}(S(m))].$$

As Q contains $P \cap R'[\sigma^{-\ell}(R')]$ and has the same dimension, by the claim Q' must extend $P \cap R'_{\sigma^\ell}$, and therefore also $P \cap R_{\sigma^\ell}$. □

Remark 2.3. A consequence of our hypothesis on the dimension of the basic types is as follows: Let P be a virtual prime ideal of R_σ. Then $\dim(P \cap R)$ is divisible by e. Indeed, choose m such that $P \cap R_{\sigma^m}$ is a prime σ^m-ideal of R_{σ^m} and R_σ satisfies (ALGm). We may assume that $m = 1$. We use the notation and definition of Section 1.4, and recall that R^3 is finite integral algebraic over R. Thus, by Lemma 2.2, $P \cap R_\sigma$ extends to a periodic prime ideal of R^3_σ. This means that $\mathrm{Frac}(R_\sigma/P \cap R_\sigma)$ is equi-algebraic over K to a difference field which is generated over K by realizations of basic types of dimension e. Since basic types have evSU-rank 1, these realizations may be taken independent, and therefore $\mathrm{tr.deg}_K(\mathrm{Frac}(R_\sigma/P \cap R_\sigma))$ is a multiple of e, so that $\dim(P \cap R_\sigma)$ is a multiple of e. As R_σ is integral algebraic over R, $\dim(P \cap R)$ is a multiple of e.

The basic cumulative case. We now prove some results in the particular case when our coordinate rings are tensor products of coordinate rings of *basic cumulative* types; this assumption holds until Proposition 2.10. The proof in the general case follows the same lines, but is slightly more involved.

Note that the assumptions imply that all coordinate rings satisfy (ALG1), that all virtual ideals are periodic, and that $\sim$ coincides with equality.

Lemma 2.4. *Let I be an ideal of R of dimension d. Then there are only finitely many periodic prime ideals of R_σ which contain I and are of dimension d.*

Proof. A prime ideal of R_σ which contains I and is of dimension d must extend a prime ideal P of R of dimension d containing I. As R is Noetherian, there are only finitely many such prime ideals, and we may therefore assume that $I = P$ is prime, and extends to a periodic prime ideal of R_σ.

Then Proposition 3.10 of [7], together with Proposition 2.1, gives the result. □

Corollary 2.5. *Let I be an ideal of R_σ of dimension d. Then there are only finitely many periodic prime ideals of R_σ which contain I and are of dimension d.*

Proof. Such an ideal contains in particular $I \cap R$. The result then follows from Lemma 2.4. □

Corollary 2.6. *Let I be an ideal of R_σ of dimension d. Then there are periodic prime ideals $P_1, \ldots, P_s$ of R_σ of dimension d, and a finite subset F of I, such that if P is a periodic prime ideal of R_σ which contains F and is of dimension d, then $V(P) = V(P_i)$ for some i.*

Proof. By Lemma 2.4, if F is a finite subset of R_σ which generates an ideal of dimension d and $\mathrm{per}(F)$ denotes the set of prime periodic ideals of R_σ containing F and of dimension d, then $\mathrm{per}(F)$ is finite. Take a sufficiently large finite F such that $\mathrm{per}(F) = \mathrm{per}(I)$. □

Lemma 2.7. *Let I be a periodic ideal of R_σ of dimension d. Then I is contained in a periodic prime ideal of R_σ of dimension d.*

Proof. We may assume that $I = \sigma(I)$. Let $F \subset I$ and $P_1, \ldots, P_s$ be given by Corollary 2.6. Let X be the set of prime ideals of R_σ of dimension d containing I, and for $n \in \mathbb{N}$, let $R(n)$ be the subring of R_σ generated by $\sigma^i(R)$, $-n \le i \le n$, and X_n be the set of prime ideals of $R(n)$ containing $I \cap R(n)$ and of dimension d. Each X_n is finite and nonempty, and we have natural maps $X_{n+1} \to X_n$. Hence, $X = \varprojlim X_n$ is nonempty and compact. The automorphism σ acts continuously on X, and therefore has a recurrent point Q. Let n be such that $R(n)$ contains F. Then for some $m > 0$, we have $Q \cap R(n) = \sigma^m(Q) \cap R(n)$. By Remarks 1.3(2), there is a prime σ^m-ideal Q' of $R(n)_{\sigma^m}$ which extends $Q \cap R(n)[\sigma^{-m}(R(n))]$. But $R(n)_{\sigma^m} = R_\sigma$, and because Q' contains F and has dimension d, it must contain I. □

Lemma 2.8. *Let I be a periodic ideal of R_σ, with $I \cap R$ pure of dimension d. Then there are periodic prime ideals $P_1, \ldots, P_s$ of virtual dimension d such that $V(I) = V(P_1) \cup \cdots \cup V(P_s)$.*

Proof. We already know by Lemma 2.4 (and Proposition 2.1) that $V(I)$ has only finitely many irreducible components of dimension d, say $V(P_1), \ldots, V(P_s)$. It therefore suffices to show that every point of $V(I)$ is in one of these components.

Assume this is not the case. Let $a \in V(I)$, and $m \ge 1$ such that I is a σ^m-ideal and $Q = (x - a)_{\sigma^m} \supseteq I$. Without loss of generality, $m = 1$. For $n \in \mathbb{N}$, let $R(n)$ be the subring of R_σ generated by the rings $\sigma^i(R)$, $-n \le i \le n$. Then for each $n \in \mathbb{N}$, the ideal $I \cap R(n)$ is pure of dimension d, and therefore, the set X_n of prime ideals P of $R(n)$ of dimension d containing $I \cap R(n)$ and contained in Q is finite and nonempty. Moreover, if $P \in X_{n+1}$, then $P \cap R(n) \in X_n$. Hence, the compact subset $X = \varprojlim X_n$ of $\mathrm{Spec}(R_\sigma)$ is nonempty. It is the set of prime ideals of R_σ of dimension d, containing I and contained in Q. Let F be given by Corollary 2.6, and n such that $F \subset R(n)$ and Q does not contain any of the $P_i \cap R(n)$. As σ acts continuously on the compact set X, X has a recurrent point, say P. Then for some $m \geq 1$, $P \cap R(n) = \sigma^m(P) \cap R(n)$. As in the proof of Lemma 2.7, there is a prime σ^m-ideal P' of R_σ which extends $P \cap R(n)[\sigma^{-m}(R(n))]$, and therefore has dimension d, contains I and is not in the finite set $\{P_1, \ldots, P_s\}$. This gives us the desired contradiction. □

We define a topology on V, taking the closed sets to be the sets $V(I)$. (It is easy to see that the sets $V(I)$ are closed under intersections and under finite unions.) Then when s is taken minimal in Lemma 2.8, the $V(P_i)$ are the *irreducible components* of $V(I)$.

Lemma 2.9. *Write $R_\sigma = K\{x_1\} \otimes_K \cdots \otimes_K K\{x_m\}$, with $m \geq 2$. Let P be a prime σ-ideal of R_σ, and let Q be the ideal $Q = (x_1 - x_2)_\sigma$ corresponding to the diagonal on* Spec $K\{x_1\} \times$ Spec $K\{x_2\}$, *i.e., generated by the $x_{1,j} - x_{2,j}$. Then either $Q \subseteq P$, or every irreducible component of $V(P) \cap V(Q)$ has dimension* $\dim(P) - e$.

Proof. Assume $Q \nsubseteq P$, and consider the σ-ideal $I = P + Q$. Note that since Q is generated by elements of R, at least one of them is not in P. Thus $I \cap R$ is strictly bigger than $P \cap R$, so each component of $I \cap R$ has dimension $< \dim(P)$.

Let $R(n)$ be the subring of R_σ generated by $\sigma^i(R)$, $-n \leq i \leq n$, for $n \in \mathbb{N}$. Then each $R(n)$ is a localization of the affine coordinate ring of a smooth variety. (In our construction, *all* proper subvarieties defined over k_0, including the singular locus, were localized away. See the discussion in (5.18) of [7].)

Hence the dimension theorem holds: since $Q \cap R(n)$ has codimension e, all minimal prime ideals over $P \cap R(n) + Q \cap R(n)$ have dimension $\geq \dim(P) - e$.

Since R is Noetherian, $I \cap R$ is finitely generated. Any finite set of elements of $I \cap R$ must already belong to $P \cap R(n) + Q \cap R(n)$ for some n. Since $R(n)$ is integral over R, and the components of $P \cap R(n) + Q \cap R(n)$ have dimension $\geq \dim(P) - e$, it follows that every minimal prime of $I \cap R$ has dimension $\geq \dim(P) - e$. (The image of an irreducible variety under a morphism with finite fibers is an irreducible variety of the same dimension.)

In particular, I has dimension $\delta \geq \dim(P) - e$. By Lemma 2.7 some periodic prime ideal P' containing I has dimension δ; by Remark 2.3, δ as well as $\dim(P)$ must be a multiple of e. We saw that $\delta < \dim(P)$, so the only choice is $\delta = \dim(P) - e$. Thus $I \cap R$ is pure of dimension $\dim(P) - e$. Hence Lemma 2.8 applies, and shows that the components $V(P_1), \ldots, V(P_n)$ of $V(I)$ all have dimension exactly $d - e$. □

Proposition 2.10. *Let P and Q be periodic prime ideals of R_σ. Then every irreducible component of $V(P) \cap V(Q)$ has dimension* $\geq \dim(P) + \dim(Q) - \dim(0)$; *it is determined by a periodic prime ideal of R_σ intersecting R in minimal prime ideals over* $(P \cap R) + (Q \cap R)$.

Proof. This can be deduced from Lemma 2.9 by reduction to an intersection with the diagonal Δ (identifying $V(P) \cap V(Q)$ with $P \times Q \cap \Delta$). □

The general case. The results in the cumulative case extend easily to the general case, in most cases simply replacing equality of ideals by the equivalence relation $\sim$. The fact that we consider also coordinate rings of semibasic types makes things a little more complicated, but Lemma 2.2 is of use. Also, Proposition 2.1 allows us to juggle between periodic and virtual ideals. Recall our assumptions:

> (R, R_σ) is a tensor product of coordinate rings of semibasic types, and all associated basic types have virtual dimension e.

Lemma 2.11. *Let I be an ideal of R, of dimension d. Then, up to $\sim$, there are only finitely many virtual prime ideals of R_σ which contain I and are of virtual dimension d.*

Proof. We may assume that R_σ satisfies (ALG1). Then a prime ideal of R_σ which contains I and is of virtual dimension d must extend a prime ideal P of R of dimension d containing I. As R is Noetherian, there are only finitely many such prime ideals, and we may therefore assume that $I = P$ is prime, and extends to a virtual prime ideal of R_σ.

Let us first assume that the semibasic types involved in R_σ are all basic. Then Proposition 3.10 of [7], together with Proposition 2.1, gives us the result.

Let us now do the general case. We consider the rings R^i introduced in Section 1.4. Recall that $R^1 \subseteq R^2 \subseteq R^3 \supseteq R$. As R^3_σ is integral algebraic over R_σ, and satisfies (ALG1), Lemma 2.2 tells us that any virtual prime ideal of R_σ extends to a virtual prime ideal of R^3_σ. On the other hand, there are only finitely many prime ideals of R^3 which extend P, so we may assume that $R = R^3$, $R_\sigma = R^3_\sigma$.

The first case gives us that $P \cap R^1$ extends to finitely many prime virtual ideals of R^1_σ, up to $\sim$, and by Proposition 2.1, we may assume they are periodic. As R^2 and R^2_σ are localizations of R^1 and R^1_σ, respectively, a periodic prime ideal of R^1_σ extends to at most one (periodic) prime ideal of R^2_σ. Say Q is a prime σ^ℓ-ideal of $R^2_{\sigma^\ell}$ which extends $P \cap R^2$. Then there are only finitely many prime ideals of $R^2_{\sigma^\ell}[R^3]$ which extend Q, and by Lemma 3.9 of [7], to each of these corresponds at most one (up to $\sim$) virtual ideal of R^3_σ. Hence, up to $\sim$, there are only finitely many virtual ideals of R^3_σ extending P. □

Corollary 2.12. *Let I be an ideal of R_σ of virtual dimension d. Then, up to $\sim$, there are only finitely many virtual prime ideals of R_σ of virtual dimension d and which contain $I \cap R_{\sigma^m}$ for some $m > 0$.*

Proof. Such an ideal contains in particular $I \cap R$. The result follows from Lemma 2.11. □

Corollary 2.13. *Let I be an ideal of R_σ of virtual dimension d. Then there are periodic prime ideals $P_1, \dots, P_s$ of R_σ of virtual dimension d, and a finite subset*

F of I, such that if P is a periodic prime ideal which contains F and is of virtual dimension d, then $V(P) = V(P_i)$ for some i.

Proof. By Lemma 2.11, if F is a finite subset of R_σ which generates an ideal of dimension d and per(F) denotes the set of prime periodic ideals of R_σ containing F and of dimension d, then per$(F)/\sim$ is finite. Take a sufficiently large finite F such that per$(F)/\sim =$ per$(I)/\sim$. □

2.14. *Warning.* This set F is not necessarily contained in R, nor in $\bigcap_m R_{\sigma^m}$, unless R_σ is cumulative.

We will need a version of Lemma 2.8 without the purity assumption. We claim a weaker conclusion, namely that $V(I)$ is contained in some $V(P_i)$ of maximal dimension.

Lemma 2.15. *Let I be a virtual ideal of R_σ of virtual dimension d. Then there are $m \geq 1$ and a prime σ^m-ideal of R_{σ^m} of dimension d which contains $I \cap R_{\sigma^m}$.*

Proof. We may assume that $I = \sigma(I)$, and that R_σ satisfies (ALG1). Let $F \subset I$ be given by Corollary 2.13. Let X be the set of prime ideals of R_σ of dimension d containing I, and for $n \in \mathbb{N}$, let $R(n)$ be the subring of R_σ generated by $\sigma^i(R)$, $-n \leq i \leq n$, and X_n be the set of prime ideals of $R(n)$ containing $I \cap R(n)$ and of dimension d. Each X_n is finite, nonempty, and we have natural maps $X \to \prod_{n\in\mathbb{N}} X_n$ and $X_{n+1} \to X_n$. The automorphism σ acts continuously on the compact set X, and therefore has a recurrent point Q. Let n be such that $R(n)$ contains F. Then for some $m > 0$, we have $Q \cap R(n) = \sigma^m(Q) \cap R(n)$. By Remarks 1.3(2), there is a prime σ^m-ideal Q' of $R(n)_{\sigma^m}$ which extends $Q \cap R(n)[\sigma^{-m}(R(n))]$. Applying Proposition 2.1 to $R(n)_{\sigma^m}$, we obtain a prime σ^ℓ-ideal Q'' of R_σ which extends Q'; then Q'' contains F and has dimension d. □

Lemma 2.16 (correct version of Lemma 3.7 in [7]). *Let R be a domain which is integrally closed, k a subfield of R, and k_1 an algebraic extension of k. Let $S = k_1 \otimes_k R$. Let Q be a prime ideal of S.*

(1) *There is a unique prime ideal of S which intersect R in* (0) *and is contained in Q.*

(2) *If P' is a prime ideal of S which intersects R in* (0) *and if k_1 is separably algebraic over k, then S/P' is integrally closed.*

Proof. For both (1) and (2), we may assume that S is finitely generated over R, i.e., that k_1 is a finite extension of k. Furthermore, observe that if $b \in S$, then b^{p^n} belongs to the subring $(k_1 \cap k^s) \otimes_k R$ of S for some n, and that a prime ideal P of S contains b if and only if its intersection with $(k_1 \cap k^s) \otimes_k R$ contains b^{p^n}, i.e., the restriction map $\operatorname{Spec}(S) \to \operatorname{Spec}((k_1 \cap k^s) \otimes_k R)$ is a bijection. We may therefore assume that k_1 is separably algebraic over k of the form $k[a]$ for some $a \in k_1$.

Let $f(T)$ be the minimal monic polynomial of a over k and consider its factorization $\prod_{i=1}^{m} g_i(T)$ over Frac(R) into monic irreducible polynomials. Because R is integrally closed, all $g_i(T)$ are in $R[T]$ (see, e.g., Theorem 4, Chapter V, §3 in [14]). Moreover, since f is separable, their coefficients are actually in the subfield $R \cap k^s$ of R, and if $i \neq j$, then $(g_i(T), g_j(T)) = (1)$. Thus any prime ideal of S, and in particular Q, contains one and only one of the elements $g_i(a)$, and the ideal of S generated by $g_i(a)$ is prime. (For this last assertion, use the fact that $g_i(T)$ is irreducible over Frac(R), and that $S \simeq R[T]/f(T)$). This shows (1).

For (2), viewing R as the coordinate ring of an affine variety V over k, we know that V is normal. A minimal prime ideal of S corresponds therefore to an irreducible component of the (nonirreducible) variety V_{k_1}, and as the property of normality is a local property, each component of V_{k_1} is normal, i.e., with P' as above, S/P' is integrally closed. Here we are using the fact that k_1/k is separable, so that the map Spec$(k_1) \to$ Spec(k) is étale and if k_1/k is finite, then S is a product of domains.

The fact that R is not necessarily finitely generated over K is not important: it is a union of finitely generated K-algebras which are integrally closed. □

Proposition 2.17. *Let* (R, R_σ) *be a pair of coordinate rings associated to semibasic types satisfying* (ALG1). *Then* (R, R_σ) *satisfies the following*: *if* Q *is a prime ideal of* R_σ *and if* P *is a prime ideal of* R *which is contained in* $Q \cap R$, *then there are only finitely many prime ideals of* R_σ *which extend* P *and are contained in* Q.

Proof. Let $Q \subset R_\sigma = S$ be a prime ideal, and let P be a prime ideal of R such that $P \subseteq Q \cap R$. Let us first assume that R/P is integrally closed. Let $(x_1, \ldots, x_n)$ be the coordinates corresponding to R, i.e., $R = K\{x_1\} \otimes_K \cdots \otimes_K K\{x_n\}$ and $K\{x_i\} = k_0(x_i) \otimes_{k_0} K$. Then

$$S = \big(\cdots((R \otimes_{K\{x_1\}} K\{x_1\}_\sigma) \otimes_{K\{x_2\}} K\{x_2\}_\sigma) \cdots \otimes_{K\{x_n\}} K\{x_n\}_\sigma\big).$$

We know that each $K\{x_i\}_\sigma$ is integral algebraic over $K\{x_i\}$ (by (ALG1)). However, it may not be separably integral algebraic. So, we consider instead the ring

$$S' = \big(\cdots(R \otimes_{K\{x_1\}} (K\{x_1\}_\sigma \cap K\{x_1\}^s)) \otimes_{K\{x_2\}} \cdots \otimes_{K\{x_n\}} (K\{x_n\}_\sigma \cap K\{x_n\}^s)\big).$$

If $b \in S$, some p^m-th power of b lies in S', so that any prime ideal of S' extends uniquely to a prime ideal of S. It therefore suffices to prove the result for S'.

Applying Lemma 2.16 to $k = K\{x_1\}$ and $S_1 = R \otimes_{K\{x_1\}} (K\{x_1\}_\sigma \cap K\{x_1\}^s)$, we obtain that there is a unique prime ideal P_1 of S_1 which extends P and is contained in $Q \cap S_1$. Furthermore, S_1/P_1 is integrally closed. Iterate the reasoning to obtain that there is a unique prime ideal P_n of S' which extends P and is contained in Q (and furthermore, S'/P_n is integrally closed).

In the general case, let A be the integral closure of R/P. Because R/P is a localization of a finitely generated K-algebra, it follows that A is a finite R/P-module (see Theorem 9, Chapter V, §4 of [14]; observe also that a localization of an integrally closed domain is integrally closed), and is integral algebraic over R/P. So the map $\mathrm{Spec}(A) \to \mathrm{Spec}(R/P)$ is finite, with fibers of size at most g for some g. Hence, the prime ideal Q/PS of S/PS has exactly s extensions $Q_1, \dots, Q_s$ to $\widetilde{S} = (S/PS) \otimes_{R/P} A$, for some s with $1 \le s \le g$. Let P' be a prime ideal of S extending P and contained in Q; then P' contains PS, and therefore P'/PS extends to a prime ideal Q' of $\widetilde{S}$. This Q' must be contained in one of the Q_i. By the first case, this determines Q' uniquely, and therefore also P'. Hence P has at most s extensions to prime ideals of R_σ which are contained in Q. □

Lemma 2.18. *Let I be a virtual perfect ideal of R_σ, with $I \cap R$ pure of dimension d. Then there are periodic prime ideals $P_1, \dots, P_s$ of virtual dimension d such that $V(I) = V(P_1) \cup \cdots \cup V(P_s)$.*

Proof. We already know, by Lemma 2.11, that $V(I)$ has only finitely many irreducible components of dimension d. It therefore suffices to show that every point of $V(I)$ is in one of these components. Let $a \in V(I)$, and $m \ge 1$ such that R_σ satisfies (ALGm), $I \cap R_{\sigma^m}$ is a perfect σ^m-ideal and $Q = (x - a)_{\sigma^m} \supseteq I \cap R_{\sigma^m}$. We work in R_{σ^m}, so without loss of generality, $m = 1$. For $n \in \mathbb{N}$, let $R(n)$ be the subring of R_σ generated by the rings $\sigma^i(R)$, $-n \le i \le n$. Then for each $n \in \mathbb{N}$, the ideal $I \cap R(n)$ is pure of dimension d, and therefore, the set X_n of prime ideals P of $R(n)$ of dimension d containing $I \cap R(n)$ and contained in Q is finite and nonempty. Moreover, if $P \in X_{n+1}$, then $P \cap R(n) \in X_n$. Hence, the compact subset $X = \varprojlim X_n$ of $\mathrm{Spec}(R_\sigma)$ is nonempty. It is the set of prime ideals of R_σ of dimension d, containing I and contained in Q. If $P \in X$, then $P \cap R$ belongs to the finite set X_0; hence, by Lemma 2.16, X is finite. On the other hand, X is stable under the (continuous) action of σ, because I and Q are σ-ideals. Hence, for some ℓ, σ^ℓ is the identity on X, i.e., all ideals in X are prime σ^ℓ-ideals. □

Proposition 2.19 (Proposition 2.6 in [7]). *Let $(R, R_\sigma) \in \mathcal{R}$ be a pair of coordinate rings, and let P_1, P_2 be two virtual prime ideals of R_σ. Then $V(P_1) \cap V(P_2) = V(I)$ for some virtual perfect ideal I. The irreducible components of $V(P_1) \cap V(P_2)$ correspond to virtual prime ideals Q_i with $Q_i \cap R$ minimal prime containing $P_1 \cap R + P_2 \cap R$.*

Proof. We may assume that R_σ satisfies (ALG1), and that P_1 and P_2 are prime σ-ideals. (In fact, at every stage of the proof, we allow ourselves to replace R_σ by R_{σ^m} so that our ideals remain σ-ideals, and without explicitly saying so). For the first assertion, it suffices to show that $V(P_1) \cap V(P_2)$ has only finitely many irreducible components: if these are of the form $V(Q_i)$, $i = 1, \dots, s$, for Q_i a prime σ^m-ideal of R_{σ^m}, then one takes $I = \bigcap_{i=1}^s Q_i$, a perfect σ^m-ideal of R_{σ^m} (which contains $P_1 \cap R_{\sigma^m} + P_2 \cap R_{\sigma^m}$).

If $V(P_1) \cap V(P_2) = \varnothing$ then there is nothing to prove, so we assume it is nonempty. The elements of $V(P_1) \cap V(P_2)$ are in correspondence with the elements of $(V(P_1) \times V(P_2)) \cap \Delta$, where the corresponding pair of coordinate rings is $(R_\sigma \otimes_K R_\sigma, R \otimes_K R)$, and Δ denotes the diagonal of the underlying ambient set $V(0) \times V(0)$. The same observation holds at the level of the Zariski closures. We therefore replace P_1 by the ideal P of $R_\sigma \otimes_K R_\sigma$ generated by $P_1 \otimes 1 + 1 \otimes P_2$, and P_2 by the ideal corresponding to Δ, i.e., the ideal $I(\Delta)$ of $R_\sigma \otimes_K R_\sigma$ generated by all $a \otimes 1 - 1 \otimes a$, for $a \in R_\sigma$. Write R_σ as the tensor product over K of the rings $K\{x_i\}_\sigma$, $i = 1, \ldots, n$, with $K\{x_i\}$ associated to the semibasic type q_i. Then $\Delta = \bigcap \Delta_i$, where $\Delta_i \subset V(0) \times V(0)$ is defined by $x_i = x_i'$ inside

$$S_\sigma = (K\{x_1\}_\sigma \otimes_K \cdots \otimes_K K\{x_n\}_\sigma) \otimes_K (K\{x_1'\}_\sigma \otimes_K \cdots \otimes_K K\{x_n'\}_\sigma).$$

It then suffices to show the result for $P + I(\Delta_1)$, then for each $P' + I(\Delta_2)$ where P' is a prime periodic ideal minimal containing $P + I(\Delta_1)$, etc.

Let us first assume that q_i is basic and that P does not contain $I(\Delta_i)$. The proof is very similar to the proof of Lemma 2.9, with small changes. Let $S = R \otimes_K R$, $S_\sigma = R_\sigma \otimes_K R_\sigma$, and $S(n) \subset S_\sigma$ the subring generated by $\sigma^i(S)$, $-n \leq i \leq n$, for $n \in \mathbb{N}$. Reasoning as in the proof of Lemma 2.9, all minimal prime ideals over $P + I(\Delta_i)$ have dimension $\geq \dim(P) - e$. By Lemma 2.15, $P + I(\Delta_i)$ is contained in a prime periodic ideal P' of dimension $\dim(P + I(\Delta_i))$. By Remark 2.3, $\dim(P + I(\Delta_i))$ must be a multiple of e, and this implies it must equal $\dim(P) - e$. Hence all irreducible components of $V(P + I(\Delta_i))$ have dimension $\dim(P) - e$.

Note that the minimal virtual prime ideals containing $P + I(\Delta_i)$ do indeed extend minimal prime ideals over $P \cap S + I(\Delta_i) \cap S$, since they have the same dimension.

We now do the general case. As R_σ^3 is integral algebraic over R_σ, we may assume that $R_{q_i} = R_{q_i}^3$, $R_{q_i,\sigma} = R_{q_i,\sigma}^3$, by Lemma 2.2. Write the variables of q_i as $(y, y_1, \ldots, y_r)$. Then $I(\Delta_i)$ is the intersection of the r σ-ideals

$$(y_1 - y_1')_\sigma, (y_2 - y_2')_\sigma, \ldots, (y_r - y_r', y - y')_\sigma.$$

The first $r - 1$ of these ideals have dimension $\text{tr.deg}_K(S) - e$ in S_σ; for the last one, work inside $S_\sigma/(y_1 - y_1', y_2 - y_2', \ldots, y_{r-1} - y_{r-1}')_\sigma$. Then the minimal prime σ-ideals over $I(\Delta_i)/(y_1 - y_1', y_2 - y_2', \ldots, y_{r-1} - y_{r-1}')_\sigma$ all have dimension $\text{tr.deg}_K(R_\sigma)$. Apply the first case to these ideals to conclude. □

Corollary 2.20 (the dimension theorem [7, (4.16)]). *Let P_1 and P_2 be virtual prime ideals of R_σ, and let n be the* evSU*-rank of $V(0)$ (i.e., there are exactly n basic types which are associated to R_σ). Then all nonempty irreducible components of $V(P_1) \cap V(P_2)$ have evSU-rank* $\geq (\dim(P_1) + \dim(P_2))/e - n$.

3. Going through Sections 2, 3 and 4 of [7]

We describe which of the results of these three sections remain true without changes, which ones are false or unnecessary, and which ones need to be repaired. Note that while our coordinate rings are not "friendly" (because they do not satisfy $(*1)$), the assumption we make on the semibasic types considered are usually slightly stronger than those made in the paper. Unless preceded by "the present", references are to results in [7].

Section 2. We gave up on the idea of finding a general setting (a modified version of friendliness satisfied by our coordinate rings) in which one would be able to prove the dichotomy theorem, and so in all the results, the hypotheses of friendliness should be replaced by our hypotheses on semibasic types: the associated basic types all have dimension e.

Notation and definitions are given in more details in (2.1) and (2.2), as well as some examples. Proposition 2.4 states the basic results on the duality between sets $V(I)$ and virtual ideals.

Proposition 2.6 is the present Proposition 2.19. The proof of Proposition 2.8 goes through verbatim.

Section 3. Paragraphs (3.1) to (3.6) are definitions and notation.

Lemma 3.7 is *false*. The correct version is given by the present Lemma 2.16(1), but it is not enough to prove $(*1)$ for our coordinate rings. Thus Proposition 3.8 is false as well.

However, the proofs of Lemma 3.9 and Proposition 3.10 go through without change (except for a typo on line 4 of the proof of 3.10: it should be the inverse image of $Q \cap K[x_1, \dots, x_r]_\sigma$).

Theorem 3.11 is implied by the present Corollary 2.12.

Proposition 3.12 goes through verbatim (note that the claim is the present Remark 2.3). Note also that once more, Proposition 2.6 (the present Proposition 2.19) is instrumental.

Section 4. Paragraph (4.1) consists of definitions and notation.

Proposition 4.2 remains true, but the proof needs to be slightly modified (as it appeals to the false Lemma 3.7) towards the end. The modification is as follows: we are in the situation of R_σ satisfying (ALG1), have chosen $a_1, \dots, a_n, a \in V(P)$ such that the field of definition of the ideal $P \cap R$ is contained in $k_0(a_1, \dots, a_n)$, and a is generic over $k_0(a_1, \dots, a_n)$. By (ALG1) and the way our coordinate rings are defined, we know that the ideal I of R_σ generated by $P \cap R$ is pure of dimension $\dim(P)$. As $V(I)$ has finitely many irreducible components and by genericity of a, a is in only one irreducible component of $V(I)$, and that component must be $V(P)$. Hence, for any ℓ, $P \cap R_{\sigma^\ell}$ is defined over $\mathrm{cl}_{\sigma^\ell}(k_0, a, a_1, \dots, a_n)$.

Corollary 4.2 and Propositions 4.3, 4.4 and 4.5 go through without change, except in the proof of 4.3, $(*1)$ should be replaced by the present Proposition 2.17.

In (4.6), we slightly strengthen the requirements and only consider 0-closed sets defined by *virtual perfect* ideals. This is to ensure that they have only finitely many irreducible components.

Proposition 4.7 remains true, with a slight change at the end of the proof, similar to the one given for 4.3.

Proposition 4.9 and Lemma 4.10 go through without change. Note the following consequences of Lemma 4.10 of [7], which while not needed for the main theorems, are quite useful in applications. We assume the hypotheses of 4.10.

Corollaries (of Lemma 4.10 of [7]). (1) *Let d_1 and d_2 be tuples of realizations of basic types among $\{p_1, \dots, p_n\}$. Then $\mathrm{acl}(d_1) \cap \mathrm{acl}(d_2) = \mathrm{acl}(e)$, where e consists of realizations of types in $\{p_1, \dots, p_n\}$.*

(2) *Let b realize a tuple of semibasic types, and $a \in \mathrm{acl}(b)$ be such that $\mathrm{qftp}(a/k_0)$ satisfies* (ALGm) *for some m. Then $\mathrm{qftp}(a/k_0)$ is semibasic.*

Proof. Choosing c in 4.10 to be the empty sequence, (1) follows from the conclusion.

(2) Indeed, without loss of generality b consists of realizations of basic types; take b' realizing $\mathrm{qftp}(b/a)$ and independent from b over a. Then $a = \mathrm{acl}(b) \cap \mathrm{acl}(b')$ and we may apply (1). □

Let us now discuss Theorem 4.11. The set $\mathcal{Y}$ needs to be modified in the following manner:

- Condition (i) stays the same: for any semibasic type q, $\mathcal{X}_q(K) \subset \mathcal{Y}(K)$ or $\mathcal{X}_q(K) \cap \mathcal{Y}(K) = \varnothing$;
- Condition (ii) becomes: if $b \in \mathcal{Y}(K)^n$ for some n, and $a \in \mathrm{acl}(k_0 b)$ is such that $q = \mathrm{qftp}(a/k_0)$ satisfies (ALGm) for some m, then $\mathcal{X}_q(K) \subset \mathcal{Y}(K)$.

(The set $\mathcal{Y}$ was in fact incorrectly defined in [7], and the current definition is the one which was used in the proof.) In the cumulative case, we furthermore impose that all our semibasic types are cumulative.

Once this change is done, the proof goes through, although one needs to pay attention to a clash of notation: the tuple d which appears on line 13 of page 283 has nothing to do with the one discussed earlier in the proof; it consists of realizations of basic types, and is independent from c over k_0.

Proposition 4.12 of [7] goes through verbatim, as well as Remark 4.14, Proposition 4.15 and the verification of the axioms for Zariski geometries given in (4.16), for the set $\mathcal{Y}_b(K) = \bigcup_{p \text{ basic of dimension } e} \mathcal{X}_p(K)$. Note that the present Corollary 2.20 gives us Corollary 4.16 of [7] for semibasic types.

4. Using the Zariski geometry to get the trichotomy

The first paragraphs of Chapter 5 of [7] introduce Robinson theories and universal domains. The real work starts with Lemma 5.10 of [7], which out of a group configuration produces a quantifier-free definable subgroup of an algebraic group, in some reduct $\Omega[m]$. Note that in the cumulative case, the subgroup G_1 can be chosen so that its generic type is cumulative, by Proposition 1.15 of [5]. Then all results of [7] up to Proposition 5.14 go through without change.

Paragraph (5.15) is the statement of the trichotomy theorem:

Theorem 5.15. *Let p be a basic type, and assume that $\mathcal{X}_p(K)$ is not modular. Then $\mathcal{X}_p(K)$ interprets an algebraically closed field of rank* 1.

The proof given in [7] goes through, as it is just an adaptation of the proof of [11] to our particular case.

We now come to the main result of the paper, given at the beginning of Section 6:

Theorem. *Let $K \models$ ACFA, let $E = \mathrm{acl}_\sigma(E) \subseteq K$, and let p be a type over E with $\mathrm{SU}(p) = 1$. Then p is not modular if and only if p is nonorthogonal to the formula $\sigma^m(x) = x^{p^n}$ for some relatively prime $m, n \in \mathbb{Z}$ with $m \neq 0$.*

Proof. The proof goes through verbatim, to show that for some $m > 0$ (passing maybe to a larger E), if a realizes p, then there is some $a' \in \mathrm{acl}_\sigma(Ea)$ such that $\mathrm{evSU}(a'/E) = \mathrm{SU}(a'/E)[m] = 1$, and $\mathrm{qftp}(a'/E)[m]$ is nonorthogonal to the formula $(\sigma^m)^r(x) = \mathrm{Frob}^n(x)$ for some integers $r \neq 0$ and n, with $(n, r) = 1$ (and in fact, $r = 1$). The proof is now routine, using Lemma 1.12 of [2]: let b, c be tuples such that, in $\Omega[m]$, c is independent from $\mathrm{acl}_\sigma(Ea) = \mathrm{acl}_\sigma(Ea')$ over E, b satisfies $(\sigma^m)^r(x) = \mathrm{Frob}^n(x)$ and belongs to $E_0 = \mathrm{acl}_{\sigma^m}(Ea'c)$. The proof of Lemma 1.12 of [2] then gives us an $\mathrm{acl}_\sigma(Ea)$-σ^m-embedding φ of $F_0 = \mathrm{acl}_\sigma(Ea)E_0$ into $\Omega[m]$, such that the fields $\sigma^i\varphi(F_0)$, $i = 0, \ldots, m-1$, are linearly disjoint over $\mathrm{acl}_\sigma(Ea)$. It then follows that $\varphi(c)$ is independent from a over E (in Ω), and therefore p is nonorthogonal to $\sigma^{mr}(x) = \mathrm{Frob}^n(x)$. □

The proofs of the results of Section 7 are also unchanged.

We have proved one part of the trichotomy, namely the dichotomy between modularity and a field structure. The second leg is proved in all characteristics in (5.12) of [2]: if p is modular but has nontrivial algebraic closure geometry, then p is nonorthogonal to an SU-rank one definable subgroup of an algebraic group, indeed of the additive or multiplicative group, or a simple abelian variety.

Additional information concerning the nonorthogonality is available in [3; 4]. The internal structure of modular subgroups of semiabelian varieties is fully understood; see [6]. In the additive case, a bilinear map is definable in some cases; describing the full induced structure remains open.

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