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We show that for any $n \geq 3$ the theory of open generalized n-gons is complete, decidable and strictly stable, yielding a new class of examples in the zoo of stable theories.

1. Introduction

Generalized polygons were introduced by Tits in order to give geometric interpretations of the groups of Lie type in rank 2, in the same way that projective planes correspond to groups of type A_2. In fact, generalized polygons are the rank 2 case of spherical buildings. A generalized n-gon is a bipartite graph with diameter n (i.e., any two vertices have distance at most n), girth $2n$ (i.e., the smallest cycles have length $2n$) and such that all vertices have valency at least 3. Clearly, for $n = 2$ such a graph is simply a complete bipartite graph and in what follows we always assume $n \geq 3$. Thinking of the bipartition as corresponding to points and lines, we see that the case $n = 3$ is simply a different way of phrasing the axioms of a projective plane, namely, any two points lie on a unique line, any two lines intersect in a unique point and every line contains at least three points. (It then easily follows that every point has at least three lines passing through it.) Remarkably, if the graph is finite, then by a fundamental result of Feit and Higman [1964] the only possible values for n are 3, 4, 6 and 8. Similar restrictions hold for other well-behaved or *tame* categories of generalized polygons, e.g., if one assumes that the underlying sets of vertices are compact, or algebraic, one obtains the same restrictions. Since we tend to think of finite Morley rank as a rather strong tameness assumption it might be remarkable that this restriction does not hold in this setting; see [Tent 2000].

In fact, it is easy to see that infinite generalized n-gons exist for any $n \geq 3$: starting with a finite bipartite configuration that does not contain any $2k$-cycles for $k < n$, one can easily complete this by freely adding enough paths in order to make sure

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that the graph has diameter n (see Definition 2.4 below). In fact, such constructions yield the only known examples of generalized n-gons for $n \neq 3, 4, 6, 8$.

Free projective planes were studied by M. Hall [1943], Siebenmann [1965], and Kopeikina [1945] and their model theory was studied in [Tent 2011; Hyttinen and Paolini 2021; Tent and Zilber 2015]. The theory of the free projective planes is strictly stable by [Hyttinen and Paolini 2021] and the notion of independence in the sense of stability agrees with the one studied in [Tent 2014; Müller and Tent 2019]. In this note we extend the results from [Tent 2011; Hyttinen and Paolini 2021] to open generalized polygons, using the methods developed in [Tent 2000; 2014; Müller and Tent 2019]. In particular, it was shown in [Hyttinen and Paolini 2021] that the theory of open projective planes is complete, strictly stable, does not have a prime model and has uncountably many nonisomorphic countable models.

2. Generalized polygons

We first recall some graph-theoretic notions. For a and b in A, the *distance* $d(a, b)$ between a and b is the smallest number m for which there is a path

$$a = a_0, a_1, \ldots, a_m = b$$

with a_i in A, where a_i and a_{i+1} are incident for $0 \leq i < m$. We may write $d_A(a, b)$ to emphasize the dependence on the graph A.

The *girth* of a graph A is the length of a shortest cycle in A. The *diameter* of a graph A is the maximal distance between two elements in A. We say that a subgraph A of a graph B is *isometrically embedded* into B if for all $a, b \in A$ we have $d_A(a, b) = d_B(a, b)$. For a vertex $a \in A$ we write $D_1(a)$ for the set of neighbours of a. Then $|D_1(a)|$ is called the valency of a in A.

From now on we fix $n \geq 3$.

Definition 2.1. A *weak generalized n-gon* Γ is a bipartite graph with diameter n and girth $2n$. If Γ is *thick*, i.e., if each vertex has valency at least 3, then Γ is a *generalized* n-gon.

A *partial* n-gon is a connected bipartite graph of girth at least $2n$.

A (partial) n-gon Γ_0 is *nondegenerate* if Γ_0 contains a cycle of length at least $2n + 2$ or a path $\gamma = (x_0, \ldots, x_{n+3})$ with $d_{\Gamma_0}(x_0, x_{n+3}) = n + 3$.

A (generalized) n-gon Γ_0 contained as a subgraph in a generalized n-gon Γ is called a (generalized) sub-n-gon of Γ.

Remark 2.2. Note that every thick generalized n-gon is nondegenerate.

The assumption that a partial n-gon is connected is not strictly necessary (and it is not required in [Hall 1943] for $n = 3$). Note that for $n = 3$ any two distinct points have distance 2 (and similarly for lines). This is not true anymore for $n > 3$, so the requirement that the graph is connected prevents ambiguities.

Definition 2.3. Let $(x = x_0, \ldots, x_{n-1} = y)$ be a path in Γ. If every x_i, $1 \le i \le n-2$, has valency 2 in Γ, then $(x_1, \ldots, x_{n-2})$ is called a clean arc in Γ (with endpoints x, y). A *loose end* is a vertex of valency at most 1.

A *hat-rack* of length $k \ge n+3$ is a path $(x_0, \ldots, x_k)$ together with subsets of $D_1(x_i)$ for $1 \le i \le k-1$.

The following definition is due to Tits [1977], who first introduced free extensions for generalized polygons, expanding earlier definitions by M. Hall and Siebenmann [Hall 1943; Siebenmann 1965].

Definition 2.4. Let Γ_0 be a partial n-gon. We define the free completion of Γ_0 by induction on $i < \omega$ as follows:

For $i \ge 0$ we obtain Γ_{i+1} from Γ_i by adding a clean arc between every two elements of Γ_i which have distance $n+1$ in Γ_i. Then $\Gamma = F(\Gamma_0) = \bigcup_{i<\omega} \Gamma_i$ is called the *free n-completion* of Γ_0. We say that Γ is *freely generated* over Γ_0.

Note that if Γ_0 does not contain vertices at distance $\ge n+1$, then $F(\Gamma_0) = \Gamma_0$. Also note that by adding a clean arc between vertices of distance $n+1$ we are creating a new cycle of length $2n$.

Remark 2.5. If two elements in a generalized n-gon Γ have distance less than n, there is a unique shortest path in Γ connecting them (otherwise we would obtain a short cycle).

A weak generalized n-gon which contains a $2(n+1)$-cycle is a generalized n-gon (see [van Maldeghem 1998, Section 1.3]). Hence if Γ_0 is a partial, nondegenerate n-gon, then $F(\Gamma_0) = \Gamma$ contains a $2(n+1)$-cycle and in fact, Γ is an infinite generalized n-gon [van Maldeghem 1998] and every vertex z in $F(\Gamma_0)$ has infinite valency.

We also note the following for future reference:

Remark 2.6. Let Γ be a generalized n-gon and let $\gamma \subset \Gamma$ be a $2n$-cycle. Then for any $x \in \gamma$ there is a unique $x' \in \gamma$ with $d(x, x') = n$ (x' is called the opposite of x in γ), and for any $y \in D_1(x) \setminus \gamma$ there is a unique $y' \in D_1(x')$ such that $d(y, y') = n-2$. Note that the result of adding a clean arc to $\gamma \cup \{y\}$ is the same as adding a clean arc to $\gamma \cup \{y'\}$.

Definition 2.7. Let Γ be a generalized n-gon and $A \subset \Gamma$. Then $\langle A\rangle_\Gamma$ denotes the intersection of all generalized sub-n-gons of Γ containing A. For $\Gamma_0 \subset \Gamma$ we put $\langle A\rangle_{\Gamma_0} = \langle A\rangle_\Gamma \cap \Gamma_0$.

Remark 2.8. If $A \subseteq \Gamma_0 \subseteq \Gamma$, then $\langle A\rangle_{\Gamma_0}$ is the intersection of all $B \supset A$, $B \subseteq \Gamma_0$, such that A is isometrically contained in B. If A is nondegenerate, then $\langle A\rangle_\Gamma$ is a generalized sub-n-gon of Γ, the n-gon (not necessarily freely) generated by A in Γ. Since shortest paths between vertices at distance $n-1$ are unique, clearly $\langle A\rangle_\Gamma \subseteq \mathrm{acl}(A)_\Gamma$. If $\Gamma = F(A)$, then $\langle A\rangle_\Gamma = \Gamma$.

We note the following useful observations:

Lemma 2.9. *Let Γ_0 be a nondegenerate partial n-gon, and let $\Gamma = F(\Gamma_0) = \bigcup \Gamma_i$ be as in Definition 2.4.*

(i) *If $A \subset \Gamma_k \setminus \Gamma_i$ is isometrically embedded into Γ_k, then $\langle A \rangle_\Gamma$ does not intersect Γ_i and $\langle A \rangle_\Gamma = F(A)$.*

(ii) *If $A \subset \Gamma_0$ is such that $\Gamma_k \setminus A$ is isometrically embedded into Γ_k, then $\langle \Gamma_k \setminus A \rangle_\Gamma$ does not intersect A.*

(iii) *Any automorphism of Γ_0 extends to an automorphism of Γ.*

Proof. All parts follow directly from the construction: e.g., for (i) it suffices to show inductively that $\Gamma_1(A)$ is isometrically embedded into $\Gamma_{k+1} \setminus \Gamma_i$. Then (i) follows by induction. Let $\gamma \subset \Gamma_{k+1} \setminus \Gamma_i$ be a clean arc connecting $a, b \in A$ with $d_A(a, b) = n + 1$. Any $c \in \gamma$ has valency 2, so any path from c to an element in A passes through a or b. Since A is isometrically embedded in Γ_k, the claim follows. The proof for part (ii) is similar and part (iii) is clear. □

Now we can state the main definition of this note, extending the definition of free and open projective planes from [Hall 1943] to generalized n-gons.

Definition 2.10. A (partial) generalized n-gon Γ is *open* if every finite subgraph contains a loose end or a clean arc.

We call a generalized n-gon Γ *free* if it is the free n-completion of a hat-rack of length at least $n + 3$. In particular, we let Γ^k denote the free n-completion of the path $\gamma_k = (x_0, \dots, x_k)$ for $k \geq n + 3$.

Note that Γ^k is a free generalized n-gon for $k \geq n + 3$.

Remark 2.11. Clearly, every free generalized n-gon is open. Beware, however, that the converse is not true in general (see Proposition 3.16), but holds for finitely generated generalized n-gons (see Proposition 4.1).

Clearly, as observed by [Hyttinen and Paolini 2021] for the case $n = 3$ being an open generalized n-gon is a first-order property. We can therefore define:

Definition 2.12. Let T_n denote the theory of open generalized n-gons in the language of graphs expanded by predicates for the bipartition.

Note that T_n is $\forall\exists$-axiomatizable. We start with some easy observations:

Remark 2.13. It follows immediately from Remark 2.5 and the definition of an open generalized n-gon that for $M \models T_n$ and a nondegenerate subgraph $A \subseteq M$ we have $\operatorname{acl}(A) \models T_n$. In other words, every algebraically closed nondegenerate subset of a model of T_n is itself a model of T_n. Clearly, $\operatorname{acl}(A)$ is prime over A [Tent and Ziegler 2012, Section 5.3].

Remark 2.14. Let $T_{n,\gamma}$ be the theory T_n expanded by constants for the vertices of a path $\gamma = (a_0, \dots, a_{n+3})$. Then $F(\gamma)$ is the prime model of $T_{n,\gamma}$ since $F(\gamma)$ is algebraic over γ, hence countable and atomic, hence prime (see [Tent and Ziegler 2012, Theorem 4.5.2]).

This is similar to the situation in free groups described in Sela's seminal results, but obviously much easier to prove in the current setting: both theories are strictly stable, and only the "natural embeddings" are elementary. Namely, we will see later that $\Gamma^k \preccurlyeq \Gamma^m$ if and only if $k \leq m$ and the embedding is the natural one.

Adapting[1] Siebenmann's definition for the case $n = 3$ [Siebenmann 1965] we define:

Definition 2.15. If A is a partial n-gon, a *hyperfree minimal extension* of A is an extension by a clean arc between two elements $a, b \in A$ with $d_A(a, b) = n+1$ or by a loose end.

Let Γ and Γ' be partial n-gons. We say that Γ is *HF-constructible* from Γ' (or over Γ') if there is an ordinal α and a sequence $(\Gamma_\beta)_{\beta<\alpha}$ of partial n-gons such that

(i) $\Gamma_0 = \Gamma'$;

(ii) if $\beta = \gamma + 1$, then Γ_β is a hyperfree minimal extension of Γ_γ;

(iii) if β is a limit ordinal, then $\Gamma_\beta = \bigcup_{\gamma<\beta} \Gamma_\gamma$;

(iv) $\Gamma = \bigcup_{\beta<\alpha} \Gamma_\beta$.

Clearly, any free completion of a partial n-gon Γ_0 is HF-constructible from Γ_0.

As in [Hyttinen and Paolini 2021] one can show that any open generalized n-gon has an HF-ordering, but since we will not be using this ordering, we omit the details.

Definition 2.16. Let $A, B \subseteq M \models T_n$, $A \cap B = \varnothing$. We call B *closed over* A if B contains neither a clean arc with endpoints in $A \cup B$ nor a loose end. We say that B is *open over* A if B contains no finite set closed over A and in this case we write $A \leq_o A \cup B$. We write $\widehat{A}_M = A \cup \bigcup\{B_0 \subset M \mid B_0 \text{ finite and closed over } A\}$.

Remark 2.17. Note that if B_1, B_2 are closed over A, then so is $B_1 \cup B_2$.

Lemma 2.18. *If B is open over A and $B \setminus A$ is finite, then B is HF-constructible over A. In particular, if A is a finite open partial n-gon, then A is HF-constructible from the empty set. Moreover, if $A \leq_o \Gamma$, where Γ is a generalized n-gon, then $F(A) \cong \langle A \rangle_\Gamma \subseteq \Gamma$.*

Proof. If $B \setminus A$ is a minimal counterexample, then it cannot contain either a loose end or a clean arc, contradicting the assumption of B being open over A. $\square$

[1] Note that Siebenmann also allows adding vertices of valency 0.

Now consider the class $\mathcal{K}$ of finite open partial n-gons (in the language of bipartite graphs) with strong embeddings given by $\leq_o$. Note that $\mathcal{K}$ is contained both in the class of partial n-gons considered in [Tent 2011] as well as in the class of partial n-gons considered in [Tent 2000] (see Lemma 3.12).

Definition 2.19. For graphs $A \subseteq B, C$, let $B \otimes_A C$ denote the free amalgam of B and C over A.

Let $A \leq_o B, C$ be open partial n-gons (contained in some generalized n-gon Γ) with $\langle A\rangle_B = \langle A\rangle_C = A$. Then we call $B \oplus_A C := F(B \otimes_A C)$ the *canonical amalgam* of B and C over A.

The canonical amalgam was used in [Tent 2011] (and for $n = 3$ in [Hyttinen and Paolini 2021]).

Remark 2.20. (i) If $A \leq_o B, C$ are open partial n-gons with $\langle A\rangle_B = \langle A\rangle_C = A$, then $B, C \leq_o B \otimes_A C \leq_o B \oplus_A C$. If $B \otimes_A C$ is nondegenerate, then $B \oplus_A C$ is an open generalized n-gon.

(ii) If $B \cap C = A$ and $B \cup C \leq_o \Gamma$ for some generalized n-gon Γ, then $B \cup C \cong B \otimes_A C$ and $\langle B \cup C\rangle_\Gamma \cong B \oplus_A C$.

The following is as in [Tent 2000; 2011; Hyttinen and Paolini 2021]:

Proposition 2.21. *Let $\mathcal{K}$ be the class of finite connected open partial n-gons. Then $(\mathcal{K}, \leq_o)$ satisfies*

(i) <u>*amalgamation*</u>: *if $A, B_1, B_2 \in \mathcal{K}$ such that $\iota_i : A \to B_i$ and $\iota_i(A) \leq_o B_i$, $i = 1, 2$, then there exist $C \in \mathcal{K}$ and $\kappa_i : B_i \to C$, $i = 1, 2$ such that $\kappa_i(B_i) \leq_o C$, $i = 1, 2$, and $\kappa_1(\iota_1(a)) = \kappa_2(\iota_2(a))$ for all $a \in A$.*

(ii) <u>*joint embedding*</u>: *for any two graphs $A, B \in \mathcal{K}$ there is some $C \in \mathcal{K}$ such that A, B can be strongly embedded (in the sense of $\leq_o$) into C.*

Hence the limit $\Gamma_{\mathcal{K}}$ exists and is an open generalized n-gon.

Proof. Since $\varnothing \in \mathcal{K}$, it suffices to verify the amalgamation property. Inductively we may assume that B is a minimal hyperfree extension of A, so either a clean arc or a loose end. If C does not contain a copy of B over A, then $B \otimes_A C \in \mathcal{K}$ and this is enough. □

Note that the class $(\mathcal{K}, \leq_o)$ is unbounded in the sense that for any $A \in \mathcal{K}$ there exists some $B \in \mathcal{K}$ with $A \neq B$ and $A \leq_o B$.

Definition 2.22. Let $M \models T_n$. Then we say that M is $\mathcal{K}$-saturated if for all finite sets $A, B \in \mathcal{K}$ with $A \leq_o B$ and any copy A' of A strongly embedded into M there is a strong embedding of B over A' into M.

Note that by construction, $\Gamma_{\mathcal{K}}$ is $\mathcal{K}$-saturated and that (as in any such Hrushovski construction) every $\mathcal{K}$-saturated structure is $\mathcal{K}$-homogeneous in the sense that any partial automorphism between strongly embedded substructures extends to an automorphism.

Theorem 2.23. *For any $n \geq 3$, the theory T_n of open generalized n-gons is complete and hence decidable.*

Proof. Let $M \models T_n$. It suffices to show that M is elementarily equivalent to $\Gamma_{\mathcal{K}}$. Clearly we may assume that M is ω-saturated and we claim that any ω-saturated M is $\mathcal{K}$-saturated: Let $A \leq_o B$ be from $\mathcal{K}$ and assume that $A \leq_o M$ (via some strong embedding). We have to show that we can find an embedding B' of B into M such that there does not exist a finite set closed over B' in M. This is clear if B is an extension of A by a clean arc since such paths are unique. If B is an extension of A by a loose end b, then the type of b over A expressing that there is no finite set D closed over $A \cup \{b\}$ is realized in $\Gamma_{\mathcal{K}}$, so it is consistent and therefore realized in M by ω-saturation. Now both M and $\Gamma_{\mathcal{K}}$ are $\mathcal{K}$-saturated from which it follows (by standard back-and-forth) that they are partially isomorphic and hence elementarily equivalent. □

We say that a set B neighbours a set A if every $a \in A$ has a neighbour in $B \setminus A$.

Lemma 2.24. *Let $M \models T_n$, $A \subset M$ finite. Then M does not contain three disjoint sets B_1, B_2, B_3 each closed over A and neighbouring A. In particular, if B is closed over A, then B is algebraic over A.*

Proof. Consider $C = A \cup B_1 \cup B_2 \cup B_3 \subset M$. Then every vertex in A has valency at least 3 in C and C contains no clean arc. It follows that C is not open, contradicting $M \models T_n$.

Now suppose B is minimally closed over A and not algebraic over A with $|B \setminus A|$ minimal. Since B is not algebraic over A, we find disjoint copies B_1, B_2, B_3 of B over A, contradicting the first part of the lemma. □

Lemma 2.24 directly implies:

Corollary 2.25. *If $M \models T_n$ and $A \subseteq M$, then $\operatorname{acl}(A)_M \leq_o M$.*

Definition 2.26. Let $B \models T_n$ and let A be a subgraph of B. We put

(i) $\mathrm{Cl}_0(A)_B = A$;

(ii) $\mathrm{Cl}_{i+1}(A)_B = \langle \widehat{\mathrm{Cl}(A)_i} \rangle_B$ (see Definition 2.16);

(iii) $\mathrm{Cl}_B(A) = \bigcup_{i<\omega} \mathrm{Cl}_i(A)_B$.

In other words, $\mathrm{Cl}(A)_B$ is the limit obtained from alternating between adding all closed finite subsets, and completing the partial n-gons in B.

Remark 2.27. For any subset A of $B \models T_n$ we have $\mathrm{Cl}_B(A) \leq_o B$ and by Lemma 2.24 $\mathrm{Cl}_B(A) \subseteq \mathrm{acl}_B(A)$.

Theorem 2.28. *Let $A, B \models T_n$ and $A \subseteq B$. The following are equivalent*:

(i) $A = \mathrm{acl}_B(A)$;

(ii) $A = \mathrm{Cl}_B(A)$;

(iii) $A \leq_o B$;

(iv) $A \preccurlyeq B$.

Proof. (i) implies (ii): This follows from Lemma 2.24.

(ii) implies (iii): This is by Remark 2.27.

(iii) implies (iv): By taking appropriate elementary extensions we may assume that A, B are ω_0-saturated and hence $\mathcal{K}$-saturated by the proof of Theorem 2.23. We use Tarski's test: Let $B \models \exists x\, \varphi(x, \bar{a})$ for some tuple $\bar{a} \subset A$ and let $b \in B$ such that $B \models \varphi(b, \bar{a})$. We find a countable set A_0 containing $\bar{a}$ such that $A_0 \leq_o A$ and similarly we find a countable set B_0 containing $A_0 \cup \{b\}$ such that $A_0 \leq_o B_0 \leq_o B$. Thus by $\mathcal{K}$-saturation we can embed B_0 over A_0 into A.

(iv) implies (i): This is also proved by Tarski's test. □

Corollary 2.29. *For $n+3 \leq k \leq m \leq \omega$ we have $\Gamma_k \preccurlyeq \Gamma_m$, i.e., the free generalized n-gons Γ^k form an elementary chain.*

The following lemma will be used in the proof of Theorem 2.32:

Lemma 2.30. *Let $M \models T_n$ and $A, C \subseteq M$, A finite and C algebraically closed. Then there exist $a \in A$ and $B_A = \{b_1, b_2\} \subset D_1(a)$ such that for any set B closed over $C \cup A$ and neighbouring A we have $B \cap B_A \neq \varnothing$.*

Proof. Suppose otherwise. Then by Remark 2.17 there is a set B closed over $C \cup A$ and neighbouring A such that for all $a \in A$ we have $|B \cap D_1(a)| \geq 3$. Since C is algebraically closed, we know that $B \cup A$ is open over C, so contains a loose end or a clean arc which is impossible since all $a \in A$ have valency at least 3 in $B \cup A$. □

Note that $B_A \subset \mathrm{acl}(AC)$ and B_A might be a singleton.

Exactly as in [Tent 2014] and [Müller and Tent 2019] we now define the following notion of independence (see also [Hyttinen and Paolini 2021]).

Definition 2.31. For any subsets A, B, C of the monster model $\mathfrak{M}$ of T_n, we call B and C independent over A, written $B \mathop{\smile\!\!\!\!\!\!\,|}^{*}_{A} C$, if

$$\mathrm{acl}(ABC) = \mathrm{acl}(AB) \oplus_{\mathrm{acl}(A)} \mathrm{acl}(AC).$$

Note that $B \mathop{\smile\!\!\!\!\!\!\,|}^{*}_{A} C$ implies $\mathrm{acl}(BA) \cup \mathrm{acl}(AC) \cong \mathrm{acl}(BA) \otimes_A \mathrm{acl}(CA)$.

We show that T_n is stable by establishing that $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^*$ satisfies the required properties of forking as in [Tent and Ziegler 2012, Theorem 8.5.10], where in the notation of that theorem, $B \mathop{\smile\!\!\!\!\!\!\mid}^*_A C$ should be read as $\mathrm{tp}(A/C) \sqsubseteq \mathrm{tp}(A/BC)$.

Theorem 2.32. *The theory T_n of open generalized n-gons is stable. In T_n, the notion $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^*$ satisfies the properties of stable forking*:

- *invariance*: *$\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^*$ is invariant under* $\mathrm{Aut}(\mathfrak{M})$.
- *local character*: *For all $A \subseteq \mathfrak{M}$ finite and $C \subseteq \mathfrak{M}$ arbitrary, there is some countable set $C_0 \subseteq C$ such that $A \mathop{\smile\!\!\!\!\!\!\mid}^*_{C_0} C$.*
- *weak boundedness*: *For all $B \subseteq \mathfrak{M}$ finite and $A \subseteq \mathfrak{M}$ arbitrary, there is some cardinal μ such that there are at most μ isomorphism types of $B' \subseteq \mathfrak{M}$ over C where $B' \cong_A B$ and $B' \mathop{\smile\!\!\!\!\!\!\mid}^*_A C$.*
- *existence*: *For all $B \subseteq \mathfrak{M}$ finite and $A \subseteq C \subseteq \mathfrak{M}$ arbitrary, there is some B' such that* $\mathrm{tp}(B/A) = \mathrm{tp}(B'/A)$ *and $B' \mathop{\smile\!\!\!\!\!\!\mid}^*_A C$.*
- *transitivity*: *If $B \mathop{\smile\!\!\!\!\!\!\mid}^*_A C$ and $B \mathop{\smile\!\!\!\!\!\!\mid}^*_{AC} D$ then $B \mathop{\smile\!\!\!\!\!\!\mid}^*_A CD$.*
- *weak monotonicity*: *If $B \mathop{\smile\!\!\!\!\!\!\mid}^*_A CD$, then $B \mathop{\smile\!\!\!\!\!\!\mid}^*_A C$.*

Proof. Invariance: Clearly $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^*$ is invariant under $\mathrm{Aut}(\mathfrak{M})$.

Local character: Let $A \subset \mathfrak{M}$ be finite and $C \subseteq \mathfrak{M}$ arbitrary. We construct a countable set $C_\infty \subset C$ such that $\mathrm{acl}(A \cup C_\infty) \cup C \leq_o \mathfrak{M}$. Then $B = \mathrm{acl}(A \cup C_\infty)$ is countable and by Remark 2.20(ii) we have $A \mathop{\smile\!\!\!\!\!\!\mid}^*_B C$. By Lemma 2.30 there is a finite set B_A which intersects any set B closed over $A \cup C$ and neighbouring A. Let $C_A \subset C$ be finite such that $B_A \subset \mathrm{acl}(A \cup C_A)$ and put $C_0 = C_A$, $B_0 = \mathrm{acl}(A \cup C_0)$. Suppose inductively that B_i, C_i have been defined, where B_i, C_i are countable. For a finite subset $X \subset B_i$ let B_X be the finite set intersecting any set D closed over $X \cup C$ and neighbouring X, and let $C_X \subset C$ be finite such that $B_X \subset \mathrm{acl}(C_i \cup C_X)$. Put $C_{i+1} = C_i \cup \bigcup\{C_X \mid X \subset B_i \text{ finite}\}$ and $B_{i+1} = \mathrm{acl}(A \cup C_{i+1})$. Note that C_{i+1}, B_{i+1} are again countable. Finally put $C_\infty = \bigcup_{i<\omega} C_i$.

We now claim that $\mathrm{acl}(A \cup C_\infty) \cup C \leq_o \mathfrak{M}$. Suppose otherwise and let D be a finite set closed over $\mathrm{acl}(A \cup C_\infty) \cup C$ (in particular, by the definition of being closed, $D \cap (\mathrm{acl}(A \cup C_\infty) \cup C) = \varnothing$). Let Z be the set of neighbours of D in $\mathrm{acl}(A \cup C_\infty)$. Since any element of D has at most one neighbour in $\mathrm{acl}(A \cup C_\infty)$ by Theorem 2.28, we have $|Z| \leq |D|$ and hence $Z \subseteq B_i$ for some $i < \omega$. Then by construction D is closed over $Z \cup C \subseteq \mathrm{acl}(A \cup C_\infty) \cup C$ and neighbours Z, so intersects the set B_Z nontrivially. But $B_Z \subset B_{i+1}$ by construction. Since D intersects B_Z nontrivially, this contradicts our assumption $D \cap (\mathrm{acl}(A \cup C_\infty) \cup C) = \varnothing$.

Weak boundedness: Let $B \subseteq \mathfrak{M}$ be finite and $A \subseteq C \subseteq \mathfrak{M}$ be arbitrary. If $B \subset \mathrm{acl}(A)$, the claim is obvious. So assume A, C are algebraically closed and $\mathrm{tp}(B_1/A) = \mathrm{tp}(B_2/A) = \mathrm{tp}(B/A)$, so $\mathrm{acl}(B_1 A) \cong \mathrm{acl}(B_2 A)$ and B_1 and B_2 are isometric over A.

Hence from $B_1, B_2 \downarrow^*_A C$, we have

$$\operatorname{acl}(B_1AC) \cong \operatorname{acl}(B_1A) \oplus_A C \cong \operatorname{acl}(B_2A) \oplus_A C \cong \operatorname{acl}(B_2AC).$$

In particular we have $B_1C \cong B_1 \otimes_A C \cong B_2C \cong B_2A \otimes_A C$ and so B_1 and B_2 are isometric over C. This isometry extends to an isometry from $\operatorname{acl}(B_1AC)$ to $\operatorname{acl}(B_2AC)$ fixing C and since $\operatorname{acl}(B_1AC)$ and $\operatorname{acl}(B_2C)$ are elementary substructures, this extends to an automorphism of $\mathfrak{M}$. Hence $\operatorname{tp}(B_1/C) = \operatorname{tp}(B_2/C)$.

Existence: Let $B \subseteq \mathfrak{M}$ be finite, $A \subset C \subseteq \mathfrak{M}$ arbitrary and $D = \operatorname{acl}(BA) \oplus_{\operatorname{acl}(A)} \operatorname{acl}(C)$. We may assume that C is nondegenerate and algebraically closed so that $C \preccurlyeq \mathfrak{M}$ and $C \preccurlyeq D$ by Theorem 2.28. By saturation and homogeneity we can embed D over C into $\mathfrak{M}$ in such a way that the image of D is an elementary substructure of $\mathfrak{M}$. Hence we find B' with $\operatorname{tp}(B/A) = \operatorname{tp}(B'/A)$ and $B' \downarrow^*_A C$.

Transitivity: Let $B \downarrow^*_A C$ and $B \downarrow^*_{AC} D$, so

$$\operatorname{acl}(ABC) = \operatorname{acl}(AB) \oplus_{\operatorname{acl}(A)} \operatorname{acl}(AC)$$

and

$$\begin{aligned}\operatorname{acl}(ABCD) &\cong \operatorname{acl}(ABC) \oplus_{\operatorname{acl}(AC)} \operatorname{acl}(ACD)\\ &\cong (\operatorname{acl}(AB) \oplus_{\operatorname{acl}(A)} \operatorname{acl}(AC)) \oplus_{\operatorname{acl}(AC)} \operatorname{acl}(ACD)\\ &\cong \operatorname{acl}(AB) \oplus_{\operatorname{acl}(A)} \operatorname{acl}(ACD),\end{aligned}$$

so $B \downarrow^*_A CD$.

Weak monotonicity: Let $B \downarrow^*_A CD$, so that

$$\Gamma = \operatorname{acl}(ABCD) \cong \operatorname{acl}(AB) \oplus_{\operatorname{acl}(A)} \operatorname{acl}(ACD).$$

Now $\operatorname{acl}(AB) \otimes_{\operatorname{acl}(A)} \operatorname{acl}(AC)$ embeds isometrically into $\operatorname{acl}(AB) \otimes_{\operatorname{acl}(A)} \operatorname{acl}(ACD)$ and hence by Lemma 2.9 we have

$$\begin{aligned}\langle \operatorname{acl}(AB) \otimes_{\operatorname{acl}(A)} \operatorname{acl}(AC)\rangle_\Gamma &= F(\operatorname{acl}(AB) \otimes_{\operatorname{acl}(A)} \operatorname{acl}(AC))\\ &= \operatorname{acl}(AB) \oplus_{\operatorname{acl}(A)} \operatorname{acl}(AC).\end{aligned}$$ □

As a corollary of the proof we obtain:

Theorem 2.33. *The theory T of open generalized n-gons is not superstable.*

Proof. It suffices to give an example of a finite set A and an algebraically closed set C such that there is no finite set $C_0 \subset C$ with $A \downarrow^*_{C_0} C$. Let

$$\Gamma_0 = \gamma_{n+3} = (x_0, \ldots, x_{n+3}) \leq_o \mathfrak{M}.$$

Then $\langle \Gamma_0 \rangle_{\mathfrak{M}} = \Gamma = \bigcup \Gamma_i$ is the free completion of Γ_0. For each $0 < i < \omega$ let $y_i \in \Gamma_i$ with $d(y_i, \Gamma_{i-1}) \geq \frac{n}{2} - 1$. Let $z_0 \neq x_{n+2}$ be a neighbour of x_{n+3} with $z_0 \downarrow^*_{x_{n+2}} \Gamma_0$ and let z_i, $0 < i < \omega$, be a neighbour of y_i with $z_i \downarrow^*_{y_i} \Gamma_0 z_0 \ldots z_{i-1}$. Finally connect z_i and z_{i-1} by a path λ_i of length $\geq n-1$ (depending on the parity).

Note that the resulting graph $\widetilde{\Gamma} = \Gamma \cup \bigcup_{i<\omega} \lambda_i$ is open with $\Gamma_0 \leq_o \widetilde{\Gamma}$, and hence we may assume $\widetilde{\Gamma} \leq_o \mathfrak{M}$.

Now put $A = \Gamma_0$ and $C = \operatorname{acl}(\{\lambda_i : i < \omega\})$. Then by construction there is no finite subset $C_0 \subset C$ such that $A \mathop{\smile\!\!\!\!\!\mid}^{*}_{C_0} C$. □

As in [Hyttinen and Paolini 2021] we can show that independence is not stationary:

Proposition 2.34. *In T_n we have* $\operatorname{acl} \neq \operatorname{dcl}$.

Proof. Let $\mathfrak{M}$ be an ω-saturated model of T_n.

If n is odd, let $A = (x_0, \ldots, x_{2n+2} = x_0) \leq_o \mathfrak{M}$ be an ordered $(2n+2)$-cycle in $\mathfrak{M}$. For $i = 0, \ldots, n$ let γ_i be the clean arc from x_i to x_{i+n+1} and let m_i denote the midpoint of γ_i. Let $C = A \cup \bigcup_{i=0,\ldots,n} \gamma_i$. By $\mathcal{K}$-homogeneity there is an automorphism of $\mathfrak{M}$ taking A to the ordered $(2n+2)$-cycle

$$A' = (x_{n+1}, \ldots, x_{2n+2} = x_0, \ldots, x_{n+1}).$$

This leaves the paths γ_i, $0 \leq i \leq n$, invariant and hence fixes each m_i. This shows that $A \not\subseteq \operatorname{dcl}(m_0, \ldots, m_n)$. On the other hand, C is closed over $\{m_i \mid i = 0, \ldots, n\}$ and hence $A \subset \operatorname{acl}(m_0, \ldots, m_{n-1})$ by Lemma 2.24.

If n is even, let $A = (x_0, \ldots, x_{2n} = x_0)$ be an ordered $2n$-cycle and for $i = 1, \ldots, n$ let $y_i \notin \{x_{i-1}, x_{i+1}\}$ be a neighbour of x_i and let z_i be the neighbour of x_{i+n} with $d(z_i, y_i) = n-2$. Let γ_i be the (unique) path of length $n-1$ from y_i to x_{i+n} and let m_i be its middle vertex. Note that $z_i \in \Gamma_i$. Let $D = A \cup \{y_1, \ldots, y_n\}$ and assume that $D \leq_o \mathfrak{M}$. Put $C = D \cup \bigcup_{i=1,\ldots,n} \gamma_i$. Then also $D' = A \cup \{z_1, \ldots, z_n\} \leq_o M$. By $\mathcal{K}$-homogeneity there is an automorphism of $\mathfrak{M}$ taking D to D'. This automorphism clearly leaves A and C invariant and fixes $m_1, \ldots, m_n$ pointwise, but does not fix any vertex in A. Thus as before we see that $A \not\subseteq \operatorname{dcl}(m_1, \ldots, m_n)$. Since C is closed over $\{m_i \mid i = 1, \ldots, n\}$ we have $A \subseteq C \subseteq \operatorname{acl}(m_1, \ldots, m_n)$ by Lemma 2.24. □

3. Elementary substructures

As noted in [Tent 2011, Section 2.2], if Γ_0 is isomorphic to Δ_0, their free n-completions are also isomorphic. The reverse is obviously not true: in a completion sequence, Γ_1 and Γ_0 are not isomorphic, but they clearly have the same free n-completion.

There is nevertheless a necessary criterion for the free n-completions to be isomorphic. This can be stated in terms of the rank function δ_n that was introduced in [Tent 2000] generalizing the rank function for projective planes introduced in [Hall 1943]. It was used again in [Tent 2011; Müller and Tent 2019].

Definition 3.1. (i) For a finite graph $\Gamma = (V, E)$ with vertex set V and edge set E, define $\delta_n(\Gamma) = (n-1) \cdot |V| - (n-2) \cdot |E|$.

(ii) A (possibly infinite) graph Γ_0 is n-strong in some graph Γ, written $\Gamma_0 \leq_n \Gamma$, if and only if for all finite subgraphs X of Γ we have

$$\delta_n(X/X \cap \Gamma_0) := \delta_n(X) - \delta_n(X \cap \Gamma_0) \geq 0.$$

Remark 3.2. Note that δ_n is submodular, i.e., if $A \leq_n B$ and $C \subseteq B$, then $A \cap C \leq_n C$. Let A and B be finite graphs and let $E(A, B)$ denote the edges between elements of A and elements of B. Then

$$\delta_n(A/B) = \delta_n(A \setminus B) - (n-2)|E(A, B)|.$$

Remark 3.3 (cf. [Tent 2000, Lemma 2.4]). Let B be a graph which arises from the graph A by successively adding clean arcs between elements of distance $n+1$. Then $A \leq_n B$, $\delta_n(A) = \delta_n(B)$ and hence if $A \subseteq B \subseteq \Delta$ for some graph Δ with $A \leq_n \Delta$, then $B \leq_n \Delta$. In particular, if Γ_0 is a finite partial n-gon and $\Gamma = F(\Gamma_0) = \bigcup \Gamma_i$ as in Definition 2.4, then $\delta_n(\Gamma_i) = \delta_n(\Gamma_0)$ for all $i < \omega$. Hence any finite subset A_0 of Γ is contained in a finite subset $A \subseteq \Gamma$ with $\delta_n(A) = \delta_n(\Gamma_0)$.

Lemma 3.4 [Tent 2011, Proposition 2.5]. *Let Γ be a generalized n-gon which is generated by the graph Γ_0. The following are equivalent*:

(i) $\Gamma_0 \leq_n \Gamma$.

(ii) $\Gamma = F(\Gamma_0)$.

Remark 3.5. Note that for $k \geq n+3$ any finite subset A_0 of Γ^k is contained in a finite subset $A \subset \Gamma^k$ such that $\delta_n(A) = n-1+k = \delta_n(\gamma_k)$ and that $n-1+k$ is minimal with that property. Hence, if A and B are finite partial n-gons such that $\Gamma(A) \cong \Gamma(B)$, then $\delta_n(A) = \delta_n(B)$. In particular, if $\Gamma^k \cong \Gamma^m$, then $k = m$.

Definition 3.6. If $A \leq_n B$ are finite graphs such that $\delta_n(B/A) = 0$ and there is no proper subgraph $A \subset B' \subset B$ with $A \leq_n B' \leq_n B$ then B is called a *minimal* 0-extension.

Remark 3.7. Recall that $\mathcal{K}$ is the class of finite connected open partial n-gons. If $B \in \mathcal{K}$ is a minimal 0-extension of A, then either B is an extension of A by a clean arc of length $n-2$ or B is closed over A in the sense of Definition 2.16.

Lemma 3.8. *Let M be a model of T_n and A a finite subset of M. If $A \subseteq B \subseteq M$ and $\delta_n(B/A) \leq 0$, then B is algebraic over A.*

Proof. If $\delta_n(B/A) < 0$, then B is not HF-constructible over A and hence algebraic over A by Lemma 2.24. Now suppose that $\delta_n(B/A) = 0$. By submodularity we can decompose the extension B over A into a finite series $A = B_0 \leq_n B_1 \leq_n \cdots \leq_n B_k = B$, where each B_i is a minimal 0-extension of B_{i-1}. Hence it suffices to prove the claim for minimal 0-extensions and for such extensions the claim follows from Remark 3.7 and Lemma 2.24. □

The previous lemma directly implies:

Corollary 3.9. *Let Γ be an open generalized n-gon. If $A \subset \Gamma$ is such that every finite set $B_0 \supset A$ is contained in a finite set B such that $\delta_n(B) = \delta_n(A)$, then $\Gamma \subseteq \mathrm{acl}(A)$. In particular, any elementary embedding of Γ^k, $k \geq n+3$, into itself is surjective.*

Corollary 3.10 (cf. [Hyttinen and Paolini 2021, Corollary 6.3]). *For $n+3 \leq k, m \leq \omega$ we have $\Gamma^k \preccurlyeq \Gamma^m$ if and only if $k \leq m$.*

Proof. The direction from right to left is contained in Corollary 2.29. For the direction from left to right suppose Γ^k embeds elementarily into Γ^m for $m < k$ via f, so $f(\Gamma^m) \preccurlyeq f(\Gamma^k) \preccurlyeq \Gamma^m$. By Corollary 3.9 and the direction from right to left, we have $f(\Gamma^m) = \Gamma^m$, contradicting the fact that $\Gamma^m \subsetneq \Gamma^k$. □

To see that T_n has no prime model, we use results from [Tent 2000]. Hence we recall the definition of the class $\mathfrak{K}$ considered in [Tent 2000]. We show below that $\mathcal{K} \subseteq \mathfrak{K}$ and hence the results from [Tent 2000] apply.

Definition 3.11. Let $\mathfrak{K}$ be the class of finite partial n-gons A such that if A contains a $2k$-cycle for some $k > n$, then $\delta_n(A) \geq 2n+2$.

The following was shown in [Tent 2000, Lemma 3.12] (unfortunately the statement there contains a typo):

Lemma 3.12. *Let $A \in \mathfrak{K}$ with $|A| \geq n+2$. Then $\delta_n(A) \geq 2n$. Moreover, we have in fact $\delta_n(A) \geq 2n+2$, unless $|A| = n+2$ or A is an ordinary n-gon with either a path with $n-1$ new elements or a loose end attached.*

Proposition 3.13. *If $A \in \mathcal{K}$ contains a $2k$-cycle for some $k > n$, then $\delta_n(A) \geq 2n+2$. Hence $\mathcal{K} \subseteq \mathfrak{K}$.*

Proof. Let A be a minimal counterexample, so A contains a $2k$-cycle for some $k > n$ and $\delta_n(A) < 2n+2$. By minimality, A cannot contain a loose end, so $A = A_0 \cup \gamma$ for some clean arc γ. Then $\delta_n(A) = \delta_n(A_0) < 2n+2$. By minimality A_0 does not contain any $2k$-cycle for $k > n$ and hence $A_0 \in \mathfrak{K}$. By Lemma 3.12 we know that $|A_0| = n+2$ or A_0 is an ordinary n-gon with either a path with $n-1$ new elements or a loose end attached. But then $A = A_0 \cup \gamma$ does not contain any $2k$-cycle for $k > n$, a contradiction. □

Corollary 3.14. *If Γ is a generalized n-gon such that every finite set A_0 is contained in a finite set A with $\delta(A) = 2n+2$, then Γ does not contain any proper elementary submodels. In particular, Γ^{n+3} is minimal.*

Proof. This follows directly from Corollary 3.9 and Proposition 3.13. □

Definition 3.15. Consider Γ^{n+3} and choose copies $(\Gamma_i : i < \omega)$ of Γ^{n+3} such that $\Gamma_i \subsetneq \Gamma_{i+1}$. Put $\Gamma' = \bigcup_{i<\omega} \Gamma_i$.

Note that $\Gamma' \models T_n$ since T_n is an $\forall\exists$-theory. Also, every finite subset A_0 of Γ' is contained in a finite set $A \subset \Gamma'$ with $\delta_n(A) = 2n+2$.

Proposition 3.16. *There exist open generalized n-gons which are not free. Specifically,* $\Gamma' = \bigcup \Gamma_i$ *is not free.*

Proof. Clearly Γ' is not finitely generated as any finite subset is contained in some Γ_i. So suppose towards a contradiction that Γ' is the free completion of an infinite hat-rack. Then for any $k \geq 2n+2$ there exists a subset X of Γ' with $\delta_n(X) \geq k$ and $X \leq_n \Gamma'$, a contradiction to the observation that every finite subset A_0 of Γ' is contained in a finite set $A \subset \Gamma'$ with $\delta_n(A) = 2n+2$. Thus Γ' is open and not free. □

Corollary 3.17. *The theory T_n of open generalized n-gons does not have a prime model.*

Proof. By Corollary 3.14, Γ^{n+3} and Γ' (as in Definition 3.15) have no proper elementary substructures. Since they are not isomorphic, this proves the claim. □

Since we can easily find (nonelementary) embeddings of Γ^m into Γ^k for $m \geq k$ we also obtain:

Corollary 3.18. *The theory of open generalized n-gons is not model complete and hence does not have quantifier elimination.*

Remark 3.19. Free ∞-gons are trees. Therefore the theory of free ∞-gons is in fact ω-stable as are their higher dimensional generalizations, right-angled buildings and free pseudospaces; see [Tent 2014].

Theorem 3.20 [Ammer 2022, Theorems 12.1 and 12.7]. *The theory T_n has weak elimination of imaginaries and is* 1*-ample, but not* 2*-ample.*

Furthermore, [Ammer 2022, Chapter 10] extends the proof from [Hyttinen and Paolini 2021] to obtain $2^{\aleph_0}$ many nonisomorphic countable open generalized n-gons for each n. Since T_n is not superstable, there are 2^κ many models of size κ for any uncountable κ.

4. Open vs. free

While we already saw in Proposition 3.16, that there are open generalized n-gons which are not free, we show in this final section that for finitely generated generalized n-gons the notions of open and free coincide. For $n = 3$ this was proved in [Hall 1943, Theorem 4.8].

Proposition 4.1. *Every finitely generated open generalized n-gon is free.*

For the proof we introduce the following concept:

Definition 4.2. We call partial n-gons A, B free-equivalent if $F(A) \cong F(B)$.

Lemma 4.3. *Let $\Gamma = F(A)$ be a generalized n-gon and suppose A is constructed from A_0 by first attaching a clean arc $\gamma = (x_1, \dots, x_{n-2})$ and then attaching loose ends $z_1, \dots, z_k$ whose respective (unique) neighbours belong to γ. Then there exist $z'_1, \dots, z'_k \in \Gamma \setminus A$ with unique neighbours in A_0 such that A is free-equivalent to $A_0 \cup \{z'_1, \dots, z'_k\}$.*

Proof. Let $\gamma' \subset A$ be a $2n$-cycle containing γ. Note that the opposites x'_i of x_i, $i = 1, \dots, n-2$, in γ belong to A_0. By Remark 2.6 we can replace $z_i \in D_1(x_j)$ by the appropriate neighbour z'_i of the opposite x'_j of x_j and remove γ. □

Lemma 4.4. *Let $\Gamma = F(A)$ be a generalized n-gon and suppose A does not contain any cycle. Then there is a hat-rack B free-equivalent to A.*

Proof. Let $\gamma = (x_0, \dots, x_k) \subset A$ be a simple path (i.e., without repetition of vertices) such that $k \geq n+3$ is maximal. The proof is by induction on the number of vertices of A not incident with γ. If A is a hat-rack, there is nothing to show. So let $a \in A$ have maximal distance from γ. If there is some $x_i \in \gamma$ such that $d(a, x_i) = n+1$, then let $a' \in \Gamma \setminus A$ be the unique neighbour of x_i with $d(a', a) = n-2$. Let A' be the graph obtained from A by replacing a by a'. Then $F(A') = F(A)$.

If there is no such vertex in γ, let $\gamma' \subset \Gamma$ be the clean arc connecting x_0 and x_{n+1}, so $F(A) = F(A \cup \gamma')$. There is some $y \in \gamma$ with $d(y, a) = n+1$. Let y' be the neighbour of y with $d(y', a) = n-2$. Then we replace A by $A' = (A \setminus \{a\}) \cup \gamma' \cup \{y'\}$. Thus $F(A) = F(A')$ and the claim follows from Lemma 4.3 and induction. □

We can now give the proof of Proposition 4.1:

Proof. Let Γ be an open generalized n-gon finitely generated over the finite partial n-gon A. We may assume that A is connected. If $\delta_n(A) = k$, then every finite set $A_0 \subset \Gamma$ is contained in a finite set A with $\delta(A) \leq k$. Hence we may assume that $A \leq_n \Gamma$ and so $\Gamma \cong F(A)$ by Lemma 3.4. Therefore it suffices to show that there is a finite hat-rack B free-equivalent to A.

By Lemma 2.18 consider a construction of A over the empty set. Clearly we may assume that the last construction step is the addition of a loose end. We now do induction over the number of steps adding a clean arc. If this number is zero, then A contains no cycles and the claim follows from Lemma 4.4. Now suppose A is obtained from A_0 by adding a clean arc γ and then adding a number of loose ends $z_1, \dots, z_k$ (where the loose ends may be attached consecutively at a previous loose end). If all loose ends are incident with γ, then we finish using Lemma 4.3. Otherwise, we inductively reduce the distance of the loose ends by replacing them by a loose end at smaller distance to A: if z_i is a loose end, there is some $x \in A$ with $d(z_i, x) = n+1$ and such that x is not a loose end in A. Now replace z_i by the unique $z'_i \in D_1(x)$ with $d(z_i, z'_i) = n-2$. In this way we reduce to the case in Lemma 4.3 and finish. □

Remark 4.5. Using similar arguments one can also show that for finitely generated $\Gamma, \Gamma' \models T_n$ we have $\Gamma \cong \Gamma'$ if and only if $\Gamma = F(A)$, $\Gamma' = F(B)$ for finite A, B such that $\delta_n(A) = \delta_n(B)$. We leave the proof as an exercise for the interested reader.

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