Model Theory

New simple theories from hypergraph sequences

Maryanthe Malliaris and Saharon Shelah

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Dedicated to Boris Zilber on the occasion of his 75th birthday.

We develop a family of simple rank one theories built over quite arbitrary sequences of finite hypergraphs. (This extends an idea from the recent proof that Keisler's order has continuum many classes, however, the construction does not require familiarity with the earlier proof.) We prove a model-completion and quantifier-elimination result for theories in this family and develop a combinatorial property which they share. We invoke regular ultrafilters to show the strength of this property, showing that any flexible ultrafilter which is good for the random graph is able to saturate such theories.

It is our pleasure to dedicate this to Boris for all the wonderful discoveries in model theory and its interaction with the rest of mainstream mathematics.

Recently, we proved that Keisler's order has continuum many pairwise incomparable classes, within the simple rank one theories [7]. A surprising point of that proof is that the theories built to obtain the continuum many incomparable classes can be very well understood, and are close to the random graph in various precise ways. So we can analyze carefully how their types are realized and omitted; this understanding helps in proving incomparability. Briefly, those theories were built over template sequences of growing finite graphs, and aspects of the combinatorics of the template graphs such as edge densities played a role in the behavior of types in the associated theories. This was a very nice interaction of the finite and the infinite, where the role of graphs seemed central; we should ask whether this understanding applies to a larger, significant family of simple theories. See also [2] and [9, Chapter VI] for context on Keisler's order.

In the present paper, we indeed find a way to extend ideas from the construction of the theories in [7] to build a nontrivial family of theories close to the random graph. Informally, the previous idea of using templates of sequences of growing finite graphs can be extended to templates of sequences of growing finite hypergraphs of

any arity. We also indicate modifications of the construction involving equivalence relations rather than trees. Although we have found these theories in the context of investigating Keisler's order, indications are that they may be of general interest. Hence we have taken care to present them in a hopefully easily accessible way.

Meanwhile, an interesting aspect of Keisler's order on simple unstable theories is that it seems to be pointing the way towards isolating and analyzing an interesting family of theories "near" the random graph, which includes the incomparable theories of [7], and now the more general family developed here. We do not yet have indications whether this is *the* family. We do intend to look at whether the incomparability via ultraproducts can be carried out at the generality of these theories, and to consider other related questions in a future manuscript.

1. Templates and theories

To define our theories we first need to define a template, which is a growing sequence of finite hypergraphs, all of the same fixed arity k, satisfying certain mild conditions on the number of nodes and of edges. Our main case is $k > 2$, but the construction also makes sense for $k = 2$ (graphs) and so generalizes a slight variant[1] of the construction from [7]. The construction a priori makes sense without the conditions in Definition 1.4, but the model completion and quantifier elimination arguments use them. Given any such template, we then build a theory in a natural way.

Definition 1.1. Given a hypergraph (H, E), where E is a relation of arity k, say that k is the *arity* of the hypergraph.

Definition 1.2. Call a hypergraph (H, E) of arity k a *k-full hypergraph* if we can partition $E = E^* \cup E^{<k}$ such that (H, E^*) is a k-uniform hypergraph, meaning the edge relation is symmetric and irreflexive and holds only on tuples of k distinct elements, and $E^{<k}$ holds on *all* tuples with $< k$ distinct elements.

Informally, k-full hypergraphs are those obtained by starting with a k-uniform hypergraph, where the edge is symmetric and irreflexive and holds only on tuples of k distinct elements, and then extending it by setting the edge relation to hold on *all* tuples with repetition. (This is a technical help since nonedges in template hypergraphs indicate inconsistency in the related theory.) Note that it still is well defined to call k the arity of the hypergraph.

Definition 1.3. Given a hypergraph (H, E) of arity k, a *k-full-clique* is a set[2] $A \subseteq H$ where every sequence of k elements of A belongs to E, and a *k-independent set* is

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[1]The reader familiar with the earlier paper will remember that the theories there were built on bipartite graphs, which had certain advantages for the ultrapower analysis. In order to extend to hypergraphs, rather than solving the problem of extending the bipartition to a multipartition, the problem was solved in a more satisfying way by eliminating the bipartition; then the extension to higher arities is even more natural.

a set $A \subseteq H$ with $\geq k$ members such that no sequence of k distinct elements of A belongs to E.

Definition 1.4. A *template* of arity k, $2 \leq k < \omega$, consists of a sequence $\mathcal{H} = \bar{h} = \langle \boldsymbol{h}_n : n < \omega \rangle$ and a function $f_{\mathcal{H}} : \omega \to \omega \setminus \{0\}$ such that:

(0) $\lim_{n \to \infty} f_{\mathcal{H}}(n) = \infty$, meaning that for every $N < \omega$ there is $n < \omega$ such that $m \geq n \implies f_{\mathcal{H}}(m) \geq N$.

(1) for all $n < \omega$, $\boldsymbol{h}_n = (H_n, E_n)$ is a finite k-full hypergraph, $H_n = \|\boldsymbol{h}_n\|$ is a finite cardinal and so we identify the set of vertices H_n with the set $\{0, \ldots, H_n - 1\}$.

Moreover, for all $n < \omega$:

(2) $f_{\mathcal{H}}(n) \leq H_n < \aleph_0$.

(3) *(extension)* Let $t = f_{\mathcal{H}}(n)$. For every $i_0^0, \ldots, i_{k-2}^0, \ldots, i_0^{t-1}, \ldots, i_{k-2}^{t-1}$ from H_n, there exists $s \in H_n$ such that $\langle s, i_0^\ell, \ldots, i_{k-2}^\ell \rangle \in E_n$ for all $\ell < t$.

We say $\mathcal{H}$ is a template if $(\mathcal{H}, f)$ is for some f.

Remark 1.5. For notational simplicity in Definition 1.4, we fix k. We could also have defined a parameter $\boldsymbol{k}_n$ for each n measuring the fullness.

Definition 1.6. A *template* is a template of arity k for some $k < \omega$.

For example, the sequence of hypergraphs given by $H_n = n + 1$ and $E_n = {}^k H_n$ is a template of arity k. For a more interesting example, choose the $\boldsymbol{h}_n$ to be a sequence of finite random hypergraphs, with size and edge probability sufficient to give the extension condition Definition 1.4(3). For a similar sufficient calculation in the original case of graphs, see [7, §6].

As the next definition suggests, it will be useful to think of trees naturally associated to paths through the template hypergraphs.

Definition 1.7. Given a template $\mathcal{H}$, and recalling H_n from Definition 1.4, define

$$X_{\mathcal{H}} := \{\rho : \rho \in {}^{\omega >}\omega, 0 \leq \rho(n) < H_n \text{ for all } n < \lg n(\rho)\}$$

to be, informally, the set of finite sequences of choices of vertices from initial segments of our hypergraph sequence, naturally forming a tree. Define

$$\text{leaves}(X_{\mathcal{H}}) = \{\rho \in {}^{\omega}\omega : \eta_i \upharpoonright n \in X_{\mathcal{H}} \text{ for all } n < \omega\}$$

to be the "limit points" of this set.

Definition 1.8. We define a theory $T_0 = T_0(\mathcal{H})$ based on the template $\mathcal{H}$ to be the following universal theory in the following language.

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[2] In the interesting case, a set with $\geq k$ members, but this hypothesis is not strictly needed as the sequences can contain repetitions. In the case of the independent set, we need $|A| \geq k$ and could have asked $|A| > k$.

(1) $\mathcal{L} = \mathcal{L}_{\mathcal{H}}$ contains equality, a k-place relation R, and countably many unary predicates

$$\{Q_\eta : \eta \in X_{\mathcal{H}}\}.$$

(2) T_0 contains universal axioms stating that R is a symmetric k-uniform hypergraph, i.e., R holds only on distinct k-tuples and if it holds on some k-tuple it holds on all its permutations.[3]

(3) If $\eta \trianglelefteq \nu \in X_{\mathcal{H}}$ then T_0 contains the axiom

$$(\forall x)(Q_{\langle\rangle}(x)) \wedge (\forall x)(Q_\nu(x) \Longrightarrow Q_\eta(x))$$

saying that $Q_{\langle\rangle}$ names everything, and Q_ν refines Q_η.

(4) If $\eta \in X_{\mathcal{H}}$, $\mathrm{lgn}(\eta) = m$ and $i \neq j < \|\boldsymbol{h}_m\|$ then T_0 contains the axiom

$$(\forall x)\big(\neg(Q_{\eta^\frown\langle i\rangle}(x) \wedge Q_{\eta^\frown\langle j\rangle}(x))\big).$$

Moreover, T_0 contains the axiom $(\forall x)\big(Q_\eta(x) \Longrightarrow \bigvee_i Q_{\eta^\frown\langle i\rangle}(x)\big)$, so the predicates $\langle Q_{\eta^\frown\langle i\rangle} : i < \|\boldsymbol{h}_m\|\rangle$ partition Q_η.

(5) For every $\eta_0, \ldots, \eta_{k-1}$ from $X_{\mathcal{H}}$ and $n < \min\{\mathrm{lgn}(\eta_0), \ldots, \mathrm{lgn}(\eta_{k-1})\}$, if $\langle \eta_0(n), \ldots, \eta_{k-1}(n)\rangle \notin E_n$ then T_0 contains the axiom

$$(\forall x_0, \ldots, x_{k-1})(P_{\eta_0}(x_0) \wedge \cdots \wedge P_{\eta_{k-1}}(x_{k-1}) \Longrightarrow \neg R(x_0, \ldots, x_{k-1}))$$

forbidding any edges across these predicates.

Discussion 1.9. Informally, the unary predicates give a model $M \models T_0$ the (hard-coded) structure of a tree. We have $\forall x \, Q_{\langle\rangle}(x)$. The model is first partitioned into predicates $Q_{\langle i\rangle}$ for $i < \|\boldsymbol{h}_0\|$. By induction on $m \geq 1$, each predicate Q_η (where $\mathrm{lgn}(\eta) = m$, i.e., η is a function with domain $\{0, \ldots, m-1\}$) is partitioned into $\|\boldsymbol{h}_m\|$ disjoint pieces, the $Q_{\eta^\frown\langle i\rangle}$. So any $a \in M$ will be in some concentric sequence of predicates $\langle Q_{\rho\restriction n} : \rho \in \mathrm{leaves}(X_{\mathcal{H}}), n < \omega\rangle$. Call ρ the *leaf* of a (see Definition 1.10). Note that we have arranged our indexing so that, in this notation, if $\rho(n) = i$ we have

$$a \in Q_{(\rho\restriction n)^\frown\langle i\rangle},$$

in other words, that its predicate at level n corresponds to the i-th element of H_n. The final condition on edges amounts to the following. Given $a_0, \ldots, a_{k-1}$ in a model $M \models T_0$, each element a_i belongs to some leaf ρ_i, and an edge R cannot occur on $\langle a_0, \ldots, a_{k-1}\rangle$ *unless* for every $n < \omega$, $\langle \rho_i(n) : i < k\rangle$ is an edge in E_n. (Since T_0 is a universal theory, of course, it records here just what is forbidden, and remains agnostic about whether edges do occur if permitted; a model completion, such as we shall construct soon, would have more information.)

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[3]Note that R, the hypergraph relation in the theory, is symmetric irreflexive, while E_n, the hypergraph relations in the templates, need not be irreflexive by the definition of "k-full".

Notice the "sparsification" of edges, or rather the accumulation of rules forbidding edges, as we go deeper into the "tree". If $\eta_0, \ldots, \eta_{k-1}$ are elements of $X_{\mathcal{H}}$ of length $m + 1$, and $\langle \eta_0(m), \ldots, \eta_{k-1}(m) \rangle \notin E_m$, then in M we know there can be no R-edges spanning elements chosen from the predicates $Q_{\eta_0}, \ldots, Q_{\eta_{k-1}}$ regardless of how these elements sit in subsequent predicates. If on the other hand $\langle \eta_0(\ell), \ldots, \eta_{k-1}(\ell) \rangle \in E_\ell$ for $\ell \leq m$, then a priori there may be edges spanning *some* elements from the predicates $Q_{\eta_0}, \ldots, Q_{\eta_{k-1}}$, but it may depend a priori on how those elements sit in subsequent predicates and what the templates say there.

The following auxiliary objects may clarify the picture.

Definition 1.10. Fix a template $\mathcal{H}$ of arity k. Let $T_0 = T_0(\mathcal{H})$ and let $M \models T_0$.

(1) For $a \in M$, define $\mathrm{leaf}(a)$ to be the unique $\rho \in \mathrm{leaves}(X_{\mathcal{H}})$ such that

$$M \models a \in Q_{\rho \restriction n} \quad \text{for all } n < \omega.$$

(2) Let $\boldsymbol{h}_\infty$ be the k-uniform hypergraph with vertex set $H_\infty := \mathrm{leaves}(X_{\mathcal{H}})$ and with edge relation E_∞ given by

$$\langle \rho_0, \ldots, \rho_{k-1} \rangle \in E_\infty \quad \Longleftrightarrow \quad \langle \rho_0(n), \ldots, \rho_{k-1}(n) \rangle \in E_n \text{ for all } n < \omega.$$

Of course $\boldsymbol{h}_\infty = \boldsymbol{h}_\infty(\mathcal{H})$.

Observation 1.11. Definition 1.8(5) implies that if $M \models T_0$, $a_0, \ldots, a_{k-1} \in M$, we can have $M \models R(a_0, \ldots, a_{k-1})$ only if $\langle \mathrm{leaf}(a_0), \ldots, \mathrm{leaf}(a_{k-1}) \rangle \in E_\infty$.

Example 1.12. Suppose that $k = 3$, $\langle 0, 1, 2 \rangle \in E_0$ and $\langle 3, 4, 5 \rangle \in E_1$. Then R-edges are not a priori forbidden in T_0 between $Q_{\langle 03 \rangle}, Q_{\langle 14 \rangle}, Q_{\langle 25 \rangle}$, nor between $Q_{\langle 04 \rangle}, Q_{\langle 15 \rangle}, Q_{\langle 23 \rangle}$ remembering symmetry of E_1, nor between $Q_{\langle 01 \rangle}, Q_{\langle 10 \rangle}, Q_{\langle 20 \rangle}$ remembering E_1 is k-full.

2. Model completion and quantifier elimination

Convention 2.1. For the entirety of this section, fix a template $\mathcal{H}$, $f_{\mathcal{H}}$ of arity $k \geq 2$, and thus $\boldsymbol{h}_\infty$ and T_0 as in Definitions 1.10 and 1.8, respectively.

Claim 2.2. *For any $\rho \in H_\infty$ there are continuum many tuples $\rho_0, \ldots, \rho_{k-2} \in H_\infty$ such that $\langle \rho, \rho_0, \ldots, \rho_{k-2} \rangle \in E_\infty$, i.e., each leaf in this graph is contained in continuum many edges.*

Proof. By extension (Definition 1.4(3)). $\qquad \square$

The next claim is a key use of Definition 1.4(3): in some sense, it shows that consistency in the template at large enough finite levels can be extended to full consistency.

Claim 2.3 (completion to a type). *For any $1 \leq t < \omega$ and any choice of t k-tuples $\rho_0^0, \ldots, \rho_{k-2}^0, \ldots, \rho_0^{t-1}, \ldots, \rho_{k-2}^{t-1}$ from H_∞, **if** there exists $v \in X_\mathcal{H}$ such that $\lgn(v) > \min\{m : f(n) \geq t \text{ for all } n \geq m\}$ and*

$$\langle v(\ell), \rho_0^i(\ell), \ldots, \rho_{k-2}^i(\ell) \rangle \in E_\ell \quad \text{for all } i < m \text{ and } \ell < \lgn(v),$$

then *we can choose v_* such that $v \trianglelefteq v_* \in H_\infty$ and*

$$\langle v_*(\ell), \rho_0^i(\ell), \ldots, \rho_{k-2}^i(\ell) \rangle \in E_\infty \quad \text{for all } i < m \text{ and } \ell < \omega.$$

Proof. Let $n := \lgn(v)$. By induction on $r < \omega$ let us prove that we can find $v_r \in X_\mathcal{H}$ of length $n + r$ such that $v \trianglelefteq v_r$ and

$$\langle v_r(\ell), \rho_0^i(\ell), \ldots, \rho_{k-2}^i(\ell) \rangle \in E_\ell \quad \text{for all } i < m \text{ and } \ell < n + r.$$

For $\ell = 0$ take $v_t = v$. For $\ell > 0$, apply extension (Definition 1.4(3)) to the tuples

$$\rho_0^0(n+r-1), \ldots, \rho_{k-2}^0(n+r-1), \ldots, \rho_0^{t-1}(n+r-1), \ldots, \rho_{k-2}^{t-1}(n+r-1)$$

in the hypergraph $\boldsymbol{h}_{n+r-1}$ and let b be the appropriate element of H_{n+r-1} returned by that axiom. Then $v_r := v_{r-1}^\frown \langle a \rangle$ fits the bill. $\square$

Definition 2.4. For any $m < \omega$, define T_0^m to be the restriction of T_0 to the language with equality, a k-place relation R, and unary predicates

$$\{Q_\eta : \eta \in X_\mathcal{H}, \lgn(\eta) \leq m\}.$$

Claim 2.5. *For each $m < \omega$, the model completion T^m of T_0^m exists.*

Proof. Just as in the case of graphs [7, Observation 2.16], each T_0^m is a universal theory in a finite relational language. The class of its models has the joint embedding property JEP for any two M_1, M_2 with $|M_1| \cap |M_2| = \varnothing$, and the amalgamation property AP when we have models M_1, M_2 and M_0 with $M_0 \models T_0^m$ and $M_0 \subseteq M_\ell$ for $\ell = 1, 2$ and $|M_1| \cap |M_2| = |M_0|$. To see this in both cases, the model N whose domain is $|M_1| \cup |M_2|$, such that $Q^N = Q^{M_1} \cup Q^{M_2}$ for each unary predicate Q and $R^N = R^{M_1} \cup R^{M_2}$ for the edge relation R, will be a model of T_0^m. Thus T^m exists. $\square$

Remark 2.6. Regarding the model completion, if $M \models T^m$, then M is infinite, and indeed for each unary predicate $Q \in \tau(T^m)$, Q^M is infinite. Moreover,[4] for any $\eta_0, \ldots, \eta_{k-1} \in X_\mathcal{H}$ with $\lgn(\eta_\ell) = m$ for $\ell < k$:

(a) If $(\eta_0(i), \ldots, \eta_{k-1}(i)) \in E_i$ for all $i < m$, then R^M on $Q_{\eta_0}^M \times \cdots \times Q_{\eta_{k-1}}^M$ "is a random hypergraph" in the sense of first-order logic, meaning that *if $A \subseteq Q_{\eta_1}^M \times \cdots \times Q_{\eta_{k-1}}^M$ and $B \subseteq |M|^k$ and[5] "$A \cap B = \varnothing$" in the strong sense*

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[4]We can extend case (a) to $\{\eta_0, \ldots, \eta_{\ell-1}\}$ for some larger finite ℓ which form a k-full-clique in the same strong hereditary sense.

[5]We could have asked that $B \subseteq Q_{\eta_1}^M \times \cdots \times Q_{\eta_{k-1}}^M$, but the stronger statement is true.

that no permutation of any $(a_1, \ldots, a_{k-1}) \in A$ belongs to B, *then* the set of formulas

$$p(x) = \{R(x, a_1, \ldots, a_{k-1}) : (a_1, \ldots, a_{k-1}) \in A\}$$
$$\cup \{\neg R(a, b_1, \ldots, b_{k-1}) : (b_1, \ldots, b_{k-1}) \in B\}$$

is a partial type in M, so in particular is realized if A, B are both finite.[6]

(b) If not, then $(Q^M_{\eta_0} \times \cdots \times Q^M_{\eta_{k-1}}) \cap R^M = \varnothing$.

Next we upgrade [7, Claim 2.17] to the context of hypergraphs.

Notation 2.7. For an ordered set X, let $\mathrm{inc}_\ell(X)$ be the set of strictly increasing ℓ-tuples of elements of X.

Definition 2.8. Given T^m for some $m < \omega$ and $M, N \models T^m$, recall that

(1) $a_1, \ldots, a_n \in |M|$ and $b_1, \ldots, b_n \in |N|$ have the same quantifier-free $\tau(T^m)$-type when they agree on equality, instances of R, and predicates Q_η up to $\mathrm{lgn}(\eta) = m$.

(2) $\varphi(x, y_1, \ldots, y_n)$ is a *complete* quantifier-free formula of $\tau(T^m)$ when
  (a) for every unary predicate $Q \in \tau(T^m)$ and variable $z \in \{x, y_1, \ldots, y_n\}$, either $\varphi \vdash Q(z)$ or $\varphi \vdash \neg Q(z)$;
  (b) for every z_0, z_1 from $\{x, y_1, \ldots, y_n\}$, either $\varphi \vdash z_0 = z_1$ or $\varphi \vdash z_0 \neq z_1$;
  (c) for every $z_0, \ldots, z_{k-1}$ from $\{x, y_1, \ldots, y_n\}$, either $\varphi \vdash R(z_0, \ldots, z_{k-1})$ or $\varphi \vdash \neg R(z_0, \ldots, z_{k-1})$.

  Recall that the language is finite so this is well defined.

Our next lemma says that for each m, the truth of sentences of $\tau(T^m_0)$ of length $\leq m$ soon stabilizes in the sequence of theories T^k as k goes to infinity.

Lemma 2.9. *For every $m < \omega$, the following holds. Let*

$$m_* \geq \min\{n : n' \geq n \Longrightarrow f_{\mathcal{H}}(n') \geq m\}.$$

If $M \models T^{m_*}$, $N \models T^{m_*+1}$ and φ is a sentence of $\tau(T^m)$ of length $\leq m$, then $M \models \varphi \Longleftrightarrow N \models \varphi$.

Proof. To prove the lemma by induction on complexity of formulas, it suffices to show the following:

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[6]Note that by our assumption of the template hypergraphs being "k-full", we are in case (a) whenever $|\{\eta_0, \ldots, \eta_{k-1}\}| < k$. The hypergraph edge R is a k-uniform hypergraph in M, of course, so any $(a_0, \ldots, a_{k-1}) \in R^M$ will be a tuple of distinct elements, but fullness of the template hypergraphs means some of the elements in such a tuple are a priori allowed to come from the same predicate at any given level. In particular, for each $\eta \in X_{\mathcal{H}}$ with $\mathrm{lgn}(\eta) = m$, $(Q^M_\eta, R^M \upharpoonright Q^M_\eta)$ is a random k-ary hypergraph in the usual sense of first-order logic.

> Suppose $\varphi(x, y_1, \ldots, y_n)$ is a complete quantifier-free formula of
> $\tau(T^m)$ of length $\leq m$, so note $n < m$. Suppose $a_1, \ldots, a_n \in |M|$
> and $b_1, \ldots, b_n \in |N|$ have the same quantifier-free $\tau(T^m)$-type. Then $\quad (\oplus)$
> there exists $a \in |M|$ such that $M \models \varphi(a, a_1, \ldots, a_n)$ *if and only if* there
> exists $b \in |N|$ such that $N \models \varphi(b, b_1, \ldots, b_n)$.

Without loss of generality, the sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are without repetition.

For left to right, suppose that $a \in |M|$ exists and $a \in \{a_1, \ldots, a_n\}$; otherwise it is trivial. We will need notation to record edges and nonedges made by a. For $\bar{\iota}$ any sequence of elements of $\{1, \ldots, n\}$, denote by $\bar{a}_{\bar{\iota}}$ the sequence $\langle a_{\bar{\iota}(\ell)} : \ell < \mathrm{lgn}(\bar{\iota}) \rangle$. Let

$$C = \{\bar{\iota} = \langle i_0, \ldots, i_{k-2} \rangle : \bar{\iota} \in \mathrm{inc}_{k-1}(\{1, \ldots, n\}), \langle a \rangle^\frown \bar{a}_{\bar{\iota}} \in R^M\}$$

represent the set of R-edges made by a to $\{a_1, \ldots, a_{n-1}\}$. Note that $|C| < n^k$. Correspondingly, let

$$D = \mathrm{inc}_{k-1}(\{1, \ldots, n\}) \setminus C$$

represent the set of non-R-edges made by a to $\{a_1, \ldots, a_{n-1}\}$. If $C = \varnothing$ finding a corresponding b is immediate, so assume $C \neq \varnothing$.

Each element c of M belongs to a unique predicate Q_η with $\mathrm{lgn}(\eta) = m_*$; call it "the m_*-leaf of c" and write $\mathrm{leaf}_{m_*}(c) = \eta$. Let $\rho = \mathrm{leaf}_{m_*}(a)$ and let $\rho_i = \mathrm{leaf}_{m_*}(a_i)$ for $i = 1, \ldots, n$. The definition of $T_0^{m_*}$ and the existence of a tell us that necessarily

$$\text{for every } \iota = \langle i_0, \ldots, i_{k-2} \rangle \in C, \text{ for every } \ell < m_*,$$
$$\langle \rho(\ell), \rho_{i_0}(\ell), \ldots, \rho_{i_{k-2}}(\ell) \rangle \in E_\ell.$$

Meanwhile, each element d of N belongs to a unique predicate Q_η with $\mathrm{lgn}(\eta) = m_* + 1$; write $\mathrm{leaf}_{m_*+1}(d) = \eta$. So let $\nu_i = \mathrm{leaf}_{m_*+1}(b_i)$ for $i = 1, \ldots, n$. Note that leaf_{m_*} and leaf_{m_*+1} a priori depend on the models M and N, but by our assumption that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ have the same quantifier-free $\tau(T_m)$-type, necessarily $\nu_i \restriction m_* = \rho_i$ for $i = 1, \ldots, n$. Apply extension (Definition 1.4(3)) to the set of $(k-1)$-tuples

$$\{\langle \nu_{i_0}(\ell), \ldots, \nu_{i_{k-2}}(\ell) \rangle : \bar{\iota} = \langle i_0, \ldots, i_{k-2} \rangle \in C\},$$

recalling our choice of m_*, and let s be the element of H_{m_*} returned. Define $\nu = \eta^\frown \langle s \rangle$. Now we have that

$$\text{for every } \iota = \langle i_0, \ldots, i_{k-2} \rangle \in C, \text{ for every } \ell < m_* + 1,$$
$$\langle \nu(\ell), \nu_{i_0}(\ell), \ldots, \nu_{i_{k-2}}(\ell) \rangle \in E_\ell.$$

So by definition of $T_0^{m_*+1}$, $\varphi(x, b_1, \ldots, b_n)$ is consistent with N, and b exists because N is model complete.

The other direction, right to left, is simpler. Suppose that $b \in |N|$ exists and $b \notin \{b_1, \ldots, b_n\}$. As before, define C to be the set of representatives of edges. Suppose $\text{leaf}_{m_*+1}(b) = v$ and $\text{leaf}_{m_*+1}(b_i) = v_i$. Then since b exists and N is a model of $T_0^{m_*+1}$, necessarily

$$\text{for every } \iota = \langle i_0, \ldots, i_{k-2}\rangle \in C, \text{ for every } \ell < m_* + 1,$$
$$\langle v(\ell), v_{i_0}(\ell), \ldots, v_{i_{k-2}}(\ell)\rangle \in E_\ell.$$

A fortiori, then,

$$\text{for every } \iota = \langle i_0, \ldots, i_{k-2}\rangle \in C, \text{ for every } \ell < m_*,$$
$$\langle v(\ell), v_{i_0}(\ell), \ldots, v_{i_{k-2}}(\ell)\rangle \in E_\ell,$$

so by definition of $T_0^{m_*}$, $\varphi(x, a_1, \ldots, a_n)$ is consistent with M, and since it is complete $\varphi \vdash Q_{(v \restriction m_*)}(x)$, and a exists because M is model complete. □

Corollary 2.10. *"The limit theory of $\langle T^m : m < \omega\rangle$ is well defined and is a complete, model complete theory which extends T_0." For every $m < \omega$ and every formula φ of $\tau(T^m)$ in at least one free variable,[7] for some quantifier-free formula ψ of $\tau(T^m)$, for every n large enough, we have that*

$$(\forall \bar{x})(\varphi(\bar{x}) \equiv \psi(\bar{x})) \in T^n.$$

Lemma 2.11. *The theory T is simple rank 1.*

Proof. Assume for a contradiction that $\langle \bar{a}_i : i < \kappa\rangle$, $\kappa = \text{cof}(\kappa) \geq (2^{\aleph_0})^+$ witnesses that some formula $\varphi(\bar{x}, \bar{y})$ n-divides, in a large κ-saturated model $M \models T$. Without loss of generality, possibly adding dummy variables, $\text{lgn}(\bar{x}) = \text{lgn}(\bar{y}) =: m$.

For each $i < \kappa$, let $\bar{b}_i$ be such that $M \models \varphi[\bar{b}_i, \bar{a}_i]$. Since κ is large enough (i.e., since $\text{cof}(\kappa) > 2^{\aleph_0}$), for some $\mathcal{U} \in [\kappa]^\kappa$, for each $\ell < m$ there is $\nu_\ell \in H_\infty$ such that $\text{leaf}(\bar{b}_{i,\ell})$ is constantly equal to ν_ℓ, and there is $\rho_\ell \in H_\infty$ such that $\text{leaf}(\bar{a}_{i,\ell})$ is constantly equal to ρ_ℓ.

Let φ' be an extension of φ which is complete for $\{=, R\}$ (it will obviously only contain information about unary predicates up to some finite level) such that $M \models \varphi'[\bar{b}_i, \bar{a}_i]$ for $i \in \mathcal{V} \in [\mathcal{U}]^\kappa$. We may assume φ' is quantifier-free. Without loss of generality, φ' does not imply any instances of equality among the x's or between the x's and the y's. In what follows, replace φ by φ' and $\langle \bar{a}_i : i < \kappa\rangle$ by $\langle \bar{a}_i : i \in \mathcal{V}\rangle$.

We would like to show that

$$\Sigma(\bar{x}) = \{\varphi(\bar{x}, \bar{a}_i) : i < \kappa\} \text{ is consistent.}$$

It suffices by induction on $j < m$ to choose elements b_j so that b_j realizes the set of formulas $\Sigma^j(b_0, \ldots, b_{j-1}, x_j)$, where Σ^j is the restriction of Σ to the

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[7]Since we do not have constants in the language.

variables $x_0, \ldots, x_j$. In the case $\ell(\bar{x}) = 1$, write $\nu = \text{leaf}(x)$, and this case follows from three simple observations:

- φ is without loss of generality quantifier-free; we assumed no instances of equality between the x's, and our theory has no algebraicity.

- The template hypergraphs contribute no restriction to this set of formulas, since if $R(x, a_{j_0}, \ldots, a_{j_{k-2}})$ is implied by Σ then we know by our construction that $(\nu, \rho_{j_0}, \ldots, \rho_{j_{k-2}}) \in E_\infty$.

- The indiscernibility of $\langle \bar{a}_i : i < \kappa \rangle$, transitivity of equality, and consistency of each instance $\varphi(x, \bar{a}_i)$ together mean that if $R(x, a_{j_0}, \ldots, a_{j_{k-2}})$ is implied by Σ and $\neg R(x, a_{\ell_0}, \ldots, a_{\ell_{k-2}})$ is implied by Σ, then no permutation of $\langle a_{j_0}, \ldots, a_{j_{k-2}} \rangle$ is equal to $\langle a_{\ell_0}, \ldots, a_{\ell_{k-2}} \rangle$ (so the "positive" and "negative" edges required by Σ cause no explicit contradiction).

Observe that the inductive step, since we will have already chosen the earlier values b_ℓ ($\ell < j$), will reduce to the case $\text{lgn}(\bar{x}) = 1$ (using $\text{lgn}(\bar{y}) = m + j$). This is enough to deduce the consistency of Σ, so there is no dividing. $\qquad \square$

Conclusion 2.12. *Given any template $\mathcal{H}$, the universal theory $T_0 = T_0(\mathcal{H})$ has a model completion $T = T(\mathcal{H})$ which is well defined, eliminates quantifiers, is simple rank 1, and is equal to the limit of $\langle T^m : m < \omega \rangle$.*

Discussion 2.13. We could have defined the theory to be "based on" predicates naming classes of crosscutting finite equivalence relations, rather than levels of trees, in the natural way. Alternatively, we could make E_n be a k-place relation on $\prod_{\ell \leq n} H_\ell$.

3. A combinatorial property

In this section we give Definition 3.1, which is supposed to capture what is simple about the theories of Section 1, not necessarily what is complicated about them. In Section 4 we shall use this to give a sufficient condition for ultrafilters to saturate such theories. First let us motivate the property.

Suppose, with no assumptions on T or φ, we have a sequence of instances of φ

$$\varphi(\bar{x}, \bar{a}_0), \ldots, \varphi(\bar{x}, \bar{a}_{s-1})$$

forming a partial type, and suppose we replace each $\bar{a}_i$ by a sequence $\bar{b}_i$ having the same type over the empty set. (We don't ask that $\bar{a}_i$ and $\bar{a}_j$ have the same type for $i \neq j$, just that $\bar{a}_i$ and $\bar{b}_i$ have the same type for each i.) Then a priori,

$$\varphi(\bar{x}, \bar{b}_0), \ldots, \varphi(\bar{x}, \bar{b}_{s-1})$$

need not remain a partial type. An example is $\varphi(x; y_0, y_1) = y_0 < x < y_1$ in the theory of dense linear orders: any two pairs of increasing elements have the same

type over the empty set, but we can choose the $\bar{a}$'s to be a sequence of intervals which are concentric, and the $\bar{b}$'s a sequence which are disjoint. Similar examples arise whenever we have a tuple beginning two indiscernible sequences, one which witnesses dividing of φ and one which does not.

An example of (T, φ) where such a substitution *does* remain a partial type, for trivial reasons, is the theory $T = T_{\mathbf{rg}}$ of the random graph, and $\varphi(x, y) = R(x, y)$, using only the positive instance. Note that $\varphi(x; y, z) = R(x, y) \wedge \neg R(x, z)$ would not work, however, since in changing from $\bar{a}$'s to $\bar{b}$'s we could introduce collisions among the parameters. A less trivial example is the positive instance of the edge relation in the theories of Section 1, which in fact satisfy a stronger condition, (as does the random graph), as we shall now see.

Among the examples of (T, φ) where this *does* work, we can ask just how much of each type we need to preserve when changing the parameters from $\bar{a}_i$'s to $\bar{b}_i$'s. Rather than preserving all formulas, perhaps it would be sufficient to enumerate some formulas of the type of each parameter in some coherent way, and then preserve some finite initial segment of each of these lists. It is reasonable that the length of the initial segment needed would depend on s, the number of instances we are dealing with. This is essentially what the next definition says.[8]

Definition 3.1. We say that $(T, \varphi(\bar{x}, \bar{y}))$ has the *pseudo-nfcp* when T is countable and we can assign to each type $p \in \mathcal{P}$, where

$$\mathcal{P} := \{p : p \in S_{\ell(\bar{y})}(\varnothing) \text{ and } p \text{ contains the formula } \exists \bar{x}\, \varphi(\bar{x}, \bar{y})\},$$

a function $f_p : \omega \to \omega$ such that

(1) (continuity) for each $m < \omega$, if $f_p(m) = r$, then for some $\psi(\bar{y}) \in p$, for any other $q \in \mathcal{P}$, if $\psi \in q$, then $f_q(m) = r$.

(2) For notational convenience, if $p = \mathrm{tp}(\bar{a}) \in \mathcal{P}$, we may write $f_{\bar{a}}$ for f_p.

(3) For every $s \geq 1$ there is $n < \omega$ such that whenever $\bar{a}_0, \ldots, \bar{a}_{s-1}, \bar{b}_0, \ldots, \bar{b}_{s-1}$ are sequences from $\mathfrak{C}_T$, hence each realizing types in $\mathcal{P}$, and

$$f_{\bar{a}_\ell} \upharpoonright n = f_{\bar{b}_\ell} \upharpoonright n \quad \text{for all } \ell < s$$

and $\{\varphi(\bar{x}, \bar{a}_\ell) : \ell < s\}$ is a partial type, then $\{\varphi(\bar{x}, \bar{b}_\ell) : \ell < s\}$ is also a partial type. In the proofs that follow, we will refer to this by saying "(T, φ) is (s, n)-compact."

Discussion 3.2. (1) So Definition 3.1 is a kind of compactness demand, that is, given $(T, \varphi(\bar{x}, \bar{y}))$, to know if $\mathfrak{C}_T \models (\exists \bar{x}) \bigwedge_{\ell < s} \varphi(\bar{x}, \bar{b}_\ell)$ we need to know just finite approximations to the type of each $\bar{b}_\ell$ (not of $\bar{b}_0 ^\frown \cdots \bar{b}_{s-1}$!) and the size of "finite", represented here by n, depends just on s (and on T and φ).

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[8]The provisional name is because it captures a key property of theories from [7].

(2) We could have defined the range of each function f to be finite subsets of ω, as would be convenient in Claim 3.4, or a more complicated set (of bounded, say countable, size); or we could have used $\{0, 1\}$.

(3) We could extend the definition to uncountable theories with more work.

Remark 3.3. In the context of Definition 3.1, when that definition is satisfied, we may define two functions F and G as follows.

(a) Define $F : \omega \to \omega$ by

$$s \mapsto \min\{n < \omega : (T, \varphi) \text{ is } (s, n)\text{-compact}\}$$

which expresses that in order for s instances to remain consistent, their functions f must be preserved at least up to $F(s)$. This is well defined since we assume the definition is satisfied. There are two cases:

(1) $\lim_{s \to \infty} F(s) \to \infty$.

(2) $\lim_{s \to \infty} F(s) = N < \infty$.

(b) Define $G : \omega \to \omega \cup \{\infty\}$ by $n \mapsto \infty$ if (T, φ) is (s, n)-compact for all $n < \omega$, and otherwise by $n \mapsto \max\{s < \omega : (T, \varphi) \text{ is } (s, n)\text{-compact}\}$, which expresses that if the functions f are preserved up to n then $G(n)$ instances can safely remain consistent. Here

$$\lim_{n \to \infty} G(n) = \infty, \qquad\qquad (\star)$$

possibly attaining the limit already at some finite n.

Claim 3.4. *Let T be one of the theories from Section 1, built from $\mathcal{H}$, $f_{\mathcal{H}}$ of arity k. Let $\varphi(x, y_0, \ldots, y_{k-2}) = R(x; y_0, \ldots, y_{k-2})$. Then (T, φ) has the pseudo-nfcp.*

Proof. In this context, by quantifier elimination, the set of 1-types over the empty set are the set of "leaves", that is, each 1-type is specified by choosing some $\eta \in \text{leaves}(X_{\mathcal{H}})$ and considering $\{Q_{\eta \restriction n} : n < \omega\}$.

If $k = 2$, this also specifies $\mathcal{P}$. Otherwise, specifying a type $p(y_0, \ldots, y_{k-2}) \in \mathcal{P}$ involves specifying the leaf of each y_i, and if two elements share the same leaf, whether they are equal.

Consider any enumeration $\langle \psi_i : 1 \le i < \omega \rangle$ of the predicates $Q_\eta(y)$ of $\tau(T)$ which enumerates in nondecreasing order of $\text{lgn}(\bar{\eta})$. Fix also in advance an enumeration of the subsets of $(k-2) \times (k-2)$, and of the subsets of $k-2$. For each $p \in \mathcal{P}$, let $f(0)$ code the instances of equality among $y_0, \ldots, y_{k-2}$, and for $1 \le m < \omega$, let $f(m)$ code which subset of $\{y_0, \ldots, y_{k-2}\}$ has the m-th predicate as part of their type. (Alternately, we could have enumerated the predicates with different variables $Q_0(y_0), Q_0(y_1), \ldots$, and let f take values in $\{0, 1\}$.)

Now, if we preserve initial segments of f, we clearly hold constant the types of the parameters up to some level k in our hard-coded tree. Lemma 2.9 tells us that m exists as a function of s, as desired.

Unless $\mathcal{H}$ is very uncomplicated (for example, cliques all the way up) the theory will normally be in case $(a)(1)$ of Remark 3.3. For a question relating to the pseudo-nfcp, see [8, 3.1.5]. $\qquad\square$

4. A separation via flexibility

The theories built above are simple rank one (Lemma 2.11 above), and thus they are low. In this section, we consider flexible ultrafilters, those which Kunen called "OK", which are necessary to saturate any nonlow theory in Keisler's order (see [3]).

Definition 4.1. Recall that the ultrafilter $\mathcal{D}$ on I, $|I| = \lambda$, is *flexible* if it has a regularizing family below any nonstandard integer, that is, for every sequence of natural numbers $\langle n_i : i \in I \rangle$ such that $\prod_{i \in I} n_i / \mathcal{D} > \aleph_0$, there is $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{D}$ such that for all $i \in I$,

$$|\{\alpha < \lambda : i \in X_\alpha\}| \le n_i.$$

Definition 4.2. Recall that a necessary and sufficient condition for a regular ultrafilter $\mathcal{D}$ on I, $|I| = \lambda$, to be *good for the random graph* is that for any infinite M and any $A, B \subseteq M^I / \mathcal{D}$ such that $|A| + |B| \le \lambda$ and $A \cap B = \varnothing$, there is an internal predicate P such that $A \subseteq P$, whereas $B \cap P = \varnothing$.

Theorem 4.3. *Suppose $\mathcal{D}$ is a regular ultrafilter on I, $|I| = \lambda$, which is flexible and good for the random graph. Suppose (T, φ) has the pseudo-nfcp and $M \models T$. Then $M^I / \mathcal{D}$ is λ^+-saturated for positive φ-types.*

Proof. Let $M \models T$ and let $N = M^I / \mathcal{D}$. Consider a positive φ-type $p(x)$, where $\varphi = \varphi(\bar{x}, \bar{y})$. Enumerate the type as $\langle \varphi_\alpha(\bar{x}, \bar{a}_\alpha) : \alpha < \lambda \rangle$. Fix $i_* = \langle i_t : t \in I \rangle / \mathcal{D}$ a nonstandard integer (so that "max" will be well defined). For a finite tuple $\bar{a}$ from N, let $f_{\bar{a}}$ mean $f_{\text{tp}(\bar{a}, \varnothing, N)}$ and given in addition an index $t \in I$, let $f_{\bar{a}[t]}$ mean $f_{\text{tp}(\bar{a}[t], \varnothing, M)}$. For each $\alpha < \lambda$ and each $t \in I$ (i.e., for each formula and each index), define

- $n(\alpha, t)$ to be the largest $n \le i_t$ such that for all $\ell < m$, the type of $\bar{a}_\alpha[t]$ aligns with that of $\bar{a}_\alpha$ up to level n as measured by f, that is,

$$n(\alpha, t) := \max\{n \le i_t : f_{\bar{a}_\alpha[t]} \upharpoonright n = f_{\bar{a}_\alpha} \upharpoonright n\}.$$

- $s(\alpha, t) := G(n(\alpha, t))$, using the notation of Remark 3.3.

The first is well defined since the condition is trivially true for 0. By Łoś' theorem, since the f's reflect formulas for each $n < \omega$ and each $\alpha < \lambda$,

$$\{t \in I : n < n(\alpha, t)\} \in \mathcal{D}.$$

Hence, for each $\alpha < \lambda$, $n_\alpha := \prod_t n(\alpha, t) / \mathcal{D}$ is a nonstandard integer. It follows from Remark 3.3(b)($\star$) that for each $\alpha < \lambda$, $s_\alpha := \prod_t s(\alpha, t) / \mathcal{D}$ is either "∞" on a large set, or a nonstandard integer.

Since $\mathcal{D}$ is good for the random graph, $\mathrm{lcf}(\omega, \mathcal{D}) \geq \lambda^+$, so there is a nonstandard integer $s = \langle s[t] : t \in I \rangle / \mathcal{D}$ such that for each $\alpha < \lambda$, $s < s_\alpha$ mod $\mathcal{D}$. Since $\mathcal{D}$ is flexible and s is a nonstandard integer, we may choose $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{D}$ regularizing $\mathcal{D}$ and with the property that for each $t \in I$,

$$|\{\alpha < \lambda : t \in X_\alpha\}| \leq s[t].$$

Define a map $d : [\lambda]^1 \to \mathcal{D}$ by

$$\{\alpha\} \mapsto \{t \in I : s[t] < s(\alpha, t)\} \cap X_\alpha.$$

That is, we assign α to an index set where we can be sure that the type of each $\bar{a}_\alpha[t]$ is "correct" up to the level needed to handle $s(\alpha, t) = G(n(\alpha, t))$ instances, thus a fortiori $s[t]$ instances. The intersection with X_α ensures, for each $t \in I$, the set $U(t) := \{\alpha : t \in d(\{\alpha\})\}$ of instances assigned to index t has size $\leq s[t]$.

Now for each $t \in I$, in the ultrapower N, $\{\varphi(\bar{x}, \bar{a}_\alpha) : \alpha \in U_t\}$ is a set of no more than $s[t]$ positive instances of φ, and by definition is a partial type. Also by our definition, for each α, and in particular for each $\alpha \in U_t$,

$$f_{\bar{a}_\alpha} \upharpoonright n(\alpha, t) = f_{\bar{a}_\alpha[t]} \upharpoonright n(\alpha, t).$$

It follows that $\{\varphi(\bar{x}, \bar{a}_\alpha[t]) : \alpha \in U_t\}$ remains a partial type in the index model M. So we can realize the type at each index under this distribution, and thus in the ultrapower N. $\square$

Corollary 4.4. *If T is a theory from Section 1, $M \models T$, and $\mathcal{D}$ is a regular ultrafilter on I, $|I| = \lambda$, which is flexible and good for the random graph, then $M^I / \mathcal{D}$ is λ^+-saturated.*

Proof. We argue almost identically to [7, Definition 4.7, Claim 4.8, Fact 5.2 and Conclusion 5.7] (changing just the arity of the edge relation, and eliminating the bipartition from the case of graphs) that in regular ultrapowers which are good for the theory of the random graph, for λ^+-saturation it suffices to consider partial types of the form

$$p(x) = \{Q_\nu(x)\} \cup \{R(x, \bar{a}) : \bar{a} \in {}^{k-2}A\}$$

for $\mathrm{lgn}(\nu) < \omega$. (Briefly, those definitions and claims note that any regular ultrapower has a certain weak saturation, for instance leaves are large, and instances of equality in types can be safely ignored. Now use quantifier elimination to get a simple normal form for types by specifying the leaf of x, a set of tuples it connects to, and a disjoint set of tuples it does not connect to. Since saturation of ultrapowers reduces to saturation of φ-types, it is sufficient to deal with only a finite amount of information on the leaf of x. Finally, since "goodness for the random graph" allows us to internally separate sets of size $\leq \lambda$, it suffices to handle the positive part of the type.) $\square$

Definition 4.5. For the purposes of the next corollaries, call a theory T a pseudo-nfcp theory if there is a set Σ of formulas of the language such that

(a) (T, φ) has the pseudo-nfcp for each $\varphi \in \Sigma$, and

(b) given any regular ultrafilter $\mathcal{D}$ over λ and $M \models T$, whether $M^\lambda/\mathcal{D}$ is λ^+-saturated depends only on λ^+-saturation for positive φ-types for $\varphi \in \Sigma$.

Corollary 4.6. *Let T be a pseudo-nfcp theory, and let $\trianglelefteq$ denote Keisler's order.*

(a) *Let T_* be any nonlow simple theory. Then $T \trianglelefteq T_*$.*

(b) *$T \trianglelefteq T_{\mathrm{feq}}$.*

Thus, if T is a pseudo-nfcp theory and T_* is any nonlow or nonsimple theory, $T \trianglelefteq T_*$. In particular, this is true for all the theories of Section 1 above.

Proof. Any regular ultrafilter on $\lambda \geq \aleph_0$ which is good for some unstable theory is necessarily good for the random graph, as the random graph is the $\trianglelefteq$-minimum unstable theory. Any regular ultrafilter which is good for T_{feq} is flexible [3, Lemma 8.8], and indeed any regular ultrafilter $\mathcal{D}$ which is good for some nonlow simple theory is flexible [3, Lemma 8.7]. The last line of the corollary now follows from the fact that T_{feq} is the Keisler-minimum nonsimple theory [4, Theorem 13.1] (as T_{feq} is minimum among theories with TP_2, whereas SOP_2 implies maximality). □

Discussion 4.7. The current instances of incomparability in Keisler's order mostly use one of two main ideas. The first is to say on one hand, changing the distance in the alephs between λ and some smaller μ (the size of a maximal antichain in a certain Boolean algebra used in building the ultrafilter) affects for which values of k the theories $T_{k+1,k}$ are saturated, and on the other, the "canonical simple nonlow theory" (see appendix to [5]) requires the ultrafilter to be flexible; under large cardinal assumptions, these two indicators can be varied independently; see [10; 5]. In ZFC, this phenomenon can be scaled down to see an incomparability between the $T_{k+1,k}$'s and a certain theory based on trees, which is low [6]. A second, much larger scale of incomparability was produced in [7], with continuum many simple rank one theories, the graph precursors of the hypergraph theories built here. As this discussion suggests, and as the proofs of this section show, once the ultrafilter becomes flexible, the noise of any differences in the present theories is drowned out by the huge power of the regularizing families available. Do there exist incomparable simple nonlow theories? Is incomparability mainly visible in the absence of forking?

We also record that, as an interesting immediate consequence of earlier arguments [10; 5], the theories built in Section 2 are (assuming a large cardinal) distinguishable in Keisler's order from the theories $T_{k+1,k}$, the higher analogues of the triangle-free random graph from [1]. That is:

Conclusion 4.8. *Assuming a supercompact cardinal, for arbitrarily large λ and any $\ell < \omega$ there is a regular ultrafilter $\mathcal{D}$ on λ which is flexible and good for the random graph,* ***thus*** *good for theories of Section 2, but not good for $T_{k+1,k}$ for any $2 \leq k < \ell$.*

Proof. Claim 10.32 in [5] gives the existence of the needed ultrafilter and in clause (a) shows it is not good for $T_{k+1,k}$ for $k < \ell$. Claim 10.30 in [5] shows this ultrafilter is flexible and good for the random graph. So by Theorem 4.3 above it can handle the theories of Section 2. □

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