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Around the algebraicity problem in odd type

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We discuss the algebraicity conjecture in odd type, with particular attention to some unfinished business involving work of Jeffrey Burdges.

The notion of strongly minimal set was known to Vaught. He, and probably others, knew the Steinitz theorem could be generalized.

Bill Marsh, 1966 [43]

Introduction

According to my recollection, I first encountered the internal geometry of strongly minimal sets in Marsh's 1966 thesis. On looking back at that thesis, I find that Marsh indulges in very little speculation about that geometry, but at the time it seemed suggestive. Fortunately, the matter was not left there, and once the dust had settled and the mists had cleared, we found ourselves with a robust geometrical stability theory which supports applications. As a result, the distinction between pure and applied model theory has become less fraught than it once was. As

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Zilber anticipated, this development involves both the internal geometry of strongly minimal sets and a study of the groups interpretable therein.

Nonetheless, the algebraicity conjecture (or problem) remains unsettled: *are the simple groups of finite Morley rank algebraic*? At this point one leans toward the expectation that counterexamples exist — possibly, as Zilber has suggested, coming naturally from the general direction of analysis. Leaving all of that aside, I will address some unfinished business which is connected with a portion of the *Borovik program*. This has become a highly developed subject with a great deal of technical material inspired in part by finite group theory, in part by algebraic group theory, and occasionally by developments in pure model theory. The various sections of the glossary in Appendix B should be helpful as we get into the details, and may merit an early glance as well.

In another direction, there has been considerable progress in the direction of "linearization", which we touch on at the end — this is dealt with comprehensively in the contributions of Borovik and of Deloro to this volume [13; 29].

The Borovik program aims to do what can be done on the positive side of the problem with existing techniques, notably those modeled on methods of finite group theory, and to identify specific problematic configurations which resist such an analysis. This program has undergone three waves of development, as the power of existing techniques has been refined and their scope enlarged.

In the first instance, an extraneous "tameness" hypothesis[1] was liberally employed, in the manner of "stone soup". This amounts to listing *bad fields* as one of the known problematic configurations. In the second wave, the stone was removed from the soup and the focus turned to the group theoretic configurations associated with a hypothetical *minimal* counterexample. This is the K^* theory, described in detail below. In a third wave, we aim at somewhat more. This is the L^* theory.

The most striking achievement of the L^* theory is the proof of the algebraicity conjecture for simple groups of finite Morley rank having *infinite* 2-rank (that is, when there is an infinite elementary abelian 2-group present) [3]. I will be discussing some classification results in *finite* 2-rank, and, notably, some unpublished work of Burdges. Namely, Burdges was actively pursuing some ideas about L^* theory in finite 2-rank of a more technical character when he became distracted by other matters. Eventually I thought I should try to do something about that, so in January 2016, at his wedding breakfast, I ransacked Jeff's computer and made off with the relevant files. At this point, it seems high time to document the state of affairs of this material.

So here we are. The subject could certainly use a more comprehensive and systematic account of what has been learned on the side of finite 2-rank (which

[1] I.e., no fields were injured in the production of the group; cf. [11] and see also [46; 6; 51].

would include the case in which there are no involutions), and the underlying theory of torsion elements in groups of finite Morley rank; this theory developed relatively late. But here we confine ourselves to situations where Sylow 2-subgroups are not too small.[2]

Some time ago, in [15], reference was made to the possibility of "eventually ... emancipating the odd type analysis from the K^*-hypothesis, a line of development which remains to be explored."[3] This was followed by the question ("Problem 1"):

> *Can one show that a simple group of finite Morley rank and degenerate type has no nontrivial involutory automorphism*?

We do not address this question, but we do take it seriously. We look in fact at how the theory would proceed in the presence of a positive solution to that problem (or a fair approximation to one).

We thank Adrien Deloro for very pertinent remarks concerning the contents of [28; 34].

A few remarks in the margin ... [4]

Without going much further into the history of the subject — which I think is very interesting, but not my own concern here — I'll note that I don't think the algebraicity conjecture was particularly central to Zilber's own concerns (and not precisely my own either, though more so). From his side the trichotomy conjecture[5] seems central and the algebraicity conjecture could be taken as one expression of it with the particular virtue of being accessible to mathematicians generally. On the other hand, for a time at least — a critical time, perhaps — we were both under the influence of Macintyre's striking paper on ω_1-categorical fields [42] (I was in fact obsessed by that paper, myself, for several years), and it certainly points the way.[6]

In my own case, while working with Macintyre and several others with similar interests, I came across what to me seemed an intriguing notion of "connectedness" in Kegel and Wehrfritz's informative [39, p. 97] while thinking about "totally categorical" groups, namely:

> If X is any subgroup of G, put $X^0 = \bigcap \mathbf{C}_X Y$, where the intersection is taken over all subsets Y of G for which the index $|X : \mathbf{C}_X Y|$

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[2] Cf. [44, 4th heading].

[3] Repeated in the notes to §II.6 of [3] in the following terms: "... *everything* we do depends on [L^*-theory]. A major open problem is to develop a parallel theory in odd type groups, at a comparable level of generality."

[4] As the referee remarks, this section has rather a large number of footnotes. We apologize in advance.

[5] Or perhaps more properly, dichotomy; cf. [38].

[6] Wikipedia as of November 2022: "... very influential in the development of geometric stability theory.[citation needed]"

> is finite. Clearly for every subgroup X of G the index $|X : X^0|$ is finite since G is an $\mathfrak{M}_c$-group. Call the subgroup X *connected* (*in* G) if $X = X^0$. Obviously A, $\mathbf{Z}G$ and G are all connected.

Much as in the case of Marsh's thesis, I suppose, this prompted some reveries that were perhaps not in the text.

And at some point fairly early on I became aware, one way or another, of Zilber's remarkable "ladder theorem", to the effect that all uncountably categorical structures are built from a strongly minimal set, the operation of algebraic closure, and some group actions by definable groups. This result would suggest that the algebraicity conjecture might be a major ingredient in the classification of the possible structures. The relatively recent work on permutation groups of finite Morley rank (noted by Borovik elsewhere in this volume) aims to address this to some extent. Around 1980, I became particularly interested in some possibilities for using the projected classification of the finite simple groups in model theory, which is really a rather different subject from the algebraicity conjecture, in terms of its aims and content — though compatible with the Borovik program. As the first algebra course I took was given by Walter Feit, and the first Janko group came along while I was an undergraduate — and as I eventually found myself employed at what was at the time the world headquarters of that classification program (or its management) — this was a natural line to fall in with.

Fortunately, the reader looking for a more balanced and informative account of developments around the algebraicity conjecture may consult the historical survey by Poizat in this volume [47] for a coherent account of the subject that shows quite precisely how (though perhaps not why) the subject emerged into print, as far as both Zilber and I were concerned. This should be supplemented by Hodges' account in [37] of how some of us in the "West[7]" became aware of some important aspects of Zilber's thinking. That account enters into some detail concerning what was taking place either prior to publication or independently of publication.[8] From my own perspective Zilber's "VINITI" report [53] that Hodges mentions was particularly central, and I regret that I have not seen it in the last 40 years — at some point, as I have recently realized, I lost track of my own copy. It would be good, I think, to locate a copy of that report and put it in the public record. The VINITI report had a wealth of material,[9] some of which I lectured on in the model theory year in

[7] Including Vancouver.

[8] In addition the concise historical remarks at the end of the introduction to [3] may become more illuminating when combined with these two accounts.

[9] According to my current recollection, for what it is worth, 83 pages (though this is perhaps a subjective reaction to the fact that it was in Russian).

Jerusalem in 1980–1981.[10] To acquire that document, one had to write directly to a certain address in Moscow, and in due course one received one's personal copy. But I have now drifted entirely away from my present subject, to which I return.[11]

1. Classification in finite 2-rank: the K^* and L^* theories in a top-down approach

This section lays out our current subject matter with more precision, specifying the results and problems we will be focusing on here. That is, we describe two flavors of the Borovik program (K^* and L^*) and the "top-down" approach which has been taken over — greatly simplified — from the practice of finite group theorists. The reader may find a preliminary quick review of the material of Section B2 helpful at this point, before entering into the substance of the discussion here.

1A. ***K^* and L^*.*** Following the finite group theorists, we call a group of finite Morley rank a *K-group* if all of its definable connected simple sections are algebraic groups.[12] And we call a group a K^*-group if the same applies to its *proper* definable connected simple sections.

A K^*-group is either a K-group or a minimal counterexample to the algebraicity conjecture. That conjecture can be phrased in these terms as follows: all K^*-groups are K-groups. One particular version of the Borovik program aims at bounding the 2-rank of any exceptions. This can be done, which is satisfying in its own way, but this does not limit the complexity of an arbitrary counterexample to the conjecture. It means only that any counterexample to the algebraicity conjecture involves a counterexample of low complexity as a definable section.

It would be very valuable to have *absolute bounds* on the complexity of counterexamples. For that matter, even in the finite case one might well ask for a more qualitative proof that the number of sporadic finite simple groups is finite, or at least that their 2-ranks are bounded, without passing through an explicit classification of the exceptions.

The Borovik program (as such) focused on K^*-groups, sometimes with additional constraints. Altınel suggested a broader notion suitable for analyzing simple groups of finite Morley rank which have infinite 2-rank, essentially by relativizing the definition of K^* to this class.

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[10] For some reason, what was actually on my own mind at the time was the problem of finiteness of Morley rank in the $\aleph_0$-categorical case, and the ideas of [41]. I had previously spent some effort in spring 1978 on the construction of an $\aleph_0$-categorical pseudoplane, without much success, a project I then abandoned.

[11] And I have managed to say not a word about the Soviet Union.

[12] This means, concretely, that these sections are Chevalley groups over algebraically closed fields — possibly with additional structure — and one will think mainly in those terms if one enters into the details. See Section B5.

More precisely, one first divides the groups of finite Morley rank of infinite 2-rank further into the following two classes:

(1) *even type* groups, where there is a bound on the exponents of 2-elements;

(2) *mixed type* groups, where there is no such bound, and in fact there is also a nontrivial *divisible* abelian 2-subgroup.

All algebraic groups of infinite 2-rank are of even type. Groups of mixed type and finite Morley rank do occur naturally, as products of algebraic groups (over different fields), but they do not occur as algebraic groups. In fact the finite Morley rank category is closed under finite direct products.

Definition 1.1 [3, Definition II.6.1]. A group of finite Morley rank and even type is an *L-group* if all of its definable simple sections of even type are algebraic.

L^*-groups (of even type) are defined correspondingly.

Here "L" stands for the first letter after "K", nothing more. The main theorems are the following, which we phrase as two independent results. We give references to [3], where one can also find historical notes and further bibliographic references; these results build on a large body of work with many contributors (among whom I feel one should take particular note of Éric Jaligot).

Proposition 1.2 [3, Proposition VIII.6.2]. *A simple L^*-group of finite Morley rank of even type is a Chevalley group.*

Proposition 1.3 [3, Mixed Type Theorem, Chapter V]. *If all simple groups of finite Morley rank and even type are algebraic*, *then there are no simple groups of finite Morley rank of mixed type.*

Putting the two propositions together, one has the following.

Theorem 1.4 [3, Main Theorem, Chapter X]. *A simple group of finite Morley rank and infinite* 2*-rank is algebraic.*

The striking and unexpected point is that Proposition 1.2 can be proved even though it makes no assumption on the degenerate type sections of the group in question. One proceeds largely by ignoring such sections — and, in particular, one does not worry at all about whether or not one might even enrich the structure of an algebraic group so as to make such a section appear. What makes this kind of analysis feasible is a sort of orthogonality principle.

Lemma 1.5 (Altınel's lemma, cf. [3, Proposition I.10.13]). *If a connected elementary abelian* 2*-group acts definably as a group of automorphisms of a connected group of finite Morley rank and finite* 2*-rank*, *then it acts trivially.*

This is the main reason to expect the K^* theory to go over in some form to an L^* theory. Lemma 1.5 can also be expressed in a structural form, as follows. The following is a slightly simplified formulation of [3, Proposition II.6.2].

Lemma 1.6. *Let G be an L-group of finite Morley rank and even type. Let $U_2(G)$ be the subgroup of G generated by its definable connected 2-subgroups. Then $U_2(G)$ is a K-group.*

Proofs in the L^* context are often more laborious than those in the K^* context, involving more exotic configurations, but the general approach taken is much the same under either hypothesis. The K^* project was still very much underway in the even type context when the L^* project came along, but after retracing its initial steps at this level of generality — with a more elaborate treatment of some uniqueness cases (strong embedding, weak embedding) — we switched over to that greater level of generality and finished the version of the classification project appropriate to infinite 2-rank in that setting [3]. As one might expect, another feature of that proof that it shares with the finite case is that it was the product of a community (including some members of the community that dealt with the finite case, who pointed out relevant strategic options not always leaping to the eye in the literature).

In view of Theorem 1.4, we may turn our attention to the case of finite 2-rank.

1B. *Finite 2-rank: Prüfer and normal 2-ranks.* In addition to the ordinary 2-rank, we have the important notion of *Prüfer* 2*-rank*.

Definition 1.7. Let G be a group of finite Morley rank. The 2-rank of a maximal 2-torus[13] in G is called its *Prüfer* 2*-rank*.

By a conjugacy theorem, this notion is well-defined. The Prüfer 2-rank corresponds to the Lie rank in the relevant groups (where the base field is algebraically closed and the characteristic is not 2). The Prüfer 2-rank is essential for our purposes. Groups of finite Morley rank and finite 2-rank are divided into the following types, again following the lead and terminology of the finite group theorists (though not, in this instance, their definitions).

- Prüfer 2-rank 0: *degenerate type*.
- Prüfer 2-rank 1: *thin type*.
- Prüfer 2-rank 2: *quasithin type*.
- Prüfer 2-rank at least 3: *generic type*.

The experience of finite group theorists suggests that the high end of the problem should be the most amenable to systematic treatment, and that the complexity and the general weirdness of the analysis will increase as one moves downward from the top.

Groups of degenerate type have 2-rank 0 (i.e., no involutions at all). In fact, in a group of finite Morley rank and finite 2-rank, any involution belongs to some 2-torus [20]. The nondegenerate type groups of finite Morley rank and finite 2-rank

[13] Divisible abelian 2-subgroup; a product of "Prüfer 2-groups," which are 2-tori of 2-rank 1.

are said to be of *odd type*, since the algebraic ones have base fields of characteristic not two (so even 0 is "odd", oddly enough). And all nonzero Prüfer 2-ranks (i.e., Lie ranks) certainly occur.

There are other ways of organizing this family of groups that do not match up neatly with the above taxonomy — at least, not a priori. One encounters the theoretical possibility of *minimal connected simple* groups regardless of Prüfer rank, and notably the so-called *uniqueness cases* which will be of considerable importance. But we will come back to that later.

Our focus here will be on the quasithin and generic type cases, that is, the Prüfer 2-rank is taken to be at least 2.

There is yet another notion of 2-rank which plays a role similar to that of the Prüfer 2-rank in the finite case, namely the normal 2-rank. This is defined as follows.

Definition 1.8. The maximal rank of a normal elementary abelian 2-subgroup of a Sylow 2-subgroup of G is called its *normal* 2*-rank*.

Again, by a conjugacy theorem, this is well-defined. In the cases of interest to us here, this parameter actually agrees with the Prüfer 2-rank. This took some time to be noticed, and is nontrivial.

Fact 1.9 [28, Lemma 1]. *Let G be a connected group of finite Morley rank and finite* 2*-rank. Then the Prüfer* 2*-rank and normal* 2*-rank of G coincide.*

1C. *L* revisited.* What happens if we relativize the notion of K^*-group to the class of *odd type* groups, rather than even type?

Definition 1.10. A group of odd type is called an *L-group* if all of its definable connected simple sections of odd type are Chevalley groups. We define *L^*-groups* of odd type correspondingly.

There is an immediate obstacle to the development of the theory: the lack of a known analog for Altınel's lemma in this context. Such a lemma would control actions of 2-tori on degenerate sections of groups of odd type. The natural action of the multiplicative group of a field of characteristic zero on its additive group is just such an action, so a blanket prohibition on them is out of the question in this context.

Burdges suggested, nonetheless, pursuing the version of the theory in odd type that imposes both the L^* hypothesis and a suitable analog of Altınel's lemma. We will adopt the following terminology.

Definition 1.11. A group of odd type is said to be NTA_2 if whenever a definable section H_1 acts definably on a definable simple section H_2 of degenerate type, then any 2-torus in H_1 acts trivially on H_2. In other words, if the action is required to be faithful then H_1 must be of degenerate type.

One may read "NTA_2" as "no definable 2-toral actions".

When we refer to the L^* theory in odd type we will generally be taking NTA_2 as an assumption as well. Conditions of this kind, and more generally the study of involutory automorphisms of simple groups of degenerate type, are certainly of interest, and have been considered in the literature.[14]

My sense — after working with the concept for a while — is that the combination of L^* and NTA_2 in odd type does correspond quite neatly to the point of departure for the L^* theory in even type, and that to the extent that one can work in that setting, the results are more informative than the results of the K^* theory.

That is, in odd type, the L^* hypothesis gives us a useful dividing line, separating issues belonging properly to the study of simple groups of degenerate type from those bearing directly on odd type. And while the weight of opinion no doubt favors the existence of degenerate type simple groups, the question of 2-toral actions on such groups appears to be more delicate.

As a technical remark on the definitions, Burdges noted that in addition to the critical property NTA_2, there could be significant issues with the Glauberman Z^* theorem.[15] But experience to date suggests configurations of this type can be eliminated by close analysis on an ad hoc basis. We will next review and compare the status (in the odd type setting) of the K^* theory on the one hand, and the L^* theory with NTA_2 on the other.

1D. ***K* theory in odd type: results.*** As far as published results are concerned, one has mainly the *generic case for K^*-groups.*

Theorem 1.12 [18]. *Let G be a simple K^*-group of finite Morley rank, finite 2-rank, and of generic type. Then G is algebraic.*

The proof follows the template developed in [7] or [8], with some technical improvements.

This proof forks at a very early stage, with one branch leading to the desired identification and the other branch leading off in an entirely different direction, arriving eventually at a contradiction. This point is of central importance, so we give more detail.

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[14]Cf. [34], [30, §1.3], and (as previously mentioned) [15, Problem 1]. From a different direction, similar questions arise in connection with questions about actions of finite groups which arise in the theory of permutation groups of finite Morley rank. In particular a conjecture from [27] concerning actions of Alt_n or Sym_n leads fairly rapidly to consideration of such actions on simple groups of degenerate type, as discussed in [2]. This direction might also provide some tightly constrained "extremal configurations" deserving close attention.

[15]See, for example, https://en.wikipedia.org/w/index.php?title=Z∗_theorem&oldid=1095862232. The original proof uses character theory. A proof which may well be of more use in the setting of finite Morley rank is in [52]; I have not looked into that, but it seems well worth looking at. (One can run into difficulties with arguments of an elementary nature as well; for example, the easy group theoretic proof that a group of order $2m$ with m odd has a subgroup of index 2 does not go over very readily to the finite Morley rank context.)

Definition 1.13. Let G be a group of finite Morley rank and S a 2-subgroup. Then $\Gamma_{S,2}(G)$ denotes the smallest definable subgroup of G containing the normalizer of every elementary abelian subgroup of S of rank 2.

The 2-*generated core* of G is $\Gamma_{S,2}(G)$ with S a Sylow 2-subgroup, which is well-defined up to conjugacy.

The case division that concerns us is whether or not the 2-generated core of G is proper. If it is, we are in a somewhat exceptional situation of the type referred to generally as a "uniqueness" case or "black hole," where it is hard to get at the whole of G using local analysis. In these cases we aim to push the configuration steadily to more extreme forms, the most extreme such case (short of a contradiction) being *strong embedding*: here some proper definable subgroup with an involution contains the normalizers of all of its nontrivial 2-subgroups.

On the main line of the proof, Theorem 1.12 takes the form that one has either a proper 2-generated core or one arrives at the desired conclusion.

On the uniqueness branch of the analysis, one has the following.

Theorem 1.14. *Let G be a simple K^*-group of finite Morley rank and odd type with a proper* 2*-generated core $M = \Gamma_{S,2}(G)$.*

(1) *If G has Prüfer* 2*-rank at least* 2, *and normal* 2*-rank at least* 3, *then M is strongly embedded and G is a minimal connected simple group.*

(2) *If G is minimal connected simple*, *then G has Prüfer* 2*-rank* 1.

These two points are dealt with in [17] and [26], respectively, and jointly eliminate this branch of the analysis in the generic setting, though the second point is of continuing interest. Burdges' involvement here indicates, among other things, that this is (at least) the second iteration of the Borovik program, and that once again there are major precursors to these results, under less general conditions.

Turning to the case of Prüfer rank 2, we have three types of algebraic groups to identify, of 2-rank at most 4, as shown in Table 1.

Burdges' work on this problem is unpublished even in the K^* setting, and is thoroughly entangled with the development of the L^* theory, so we will discuss it in that context. His treatment of 2-rank at least 4 was complete (pending further review of the details) and we will say more about that below. The treatment of 2-rank 3 led to an interesting exotic configuration similar to G_2 in characteristic 3 and known to the finite group theorists as worthy of separate consideration. In the finite case, they eliminated this exotic configuration via character theory (at first

type	A_2 (PSL_3)	G_2	B_2 (PSp_4)
2-*rank*	2	3	4

Table 1. Algebraic groups of Lie rank 2.

using modular character theory, and later by ordinary character theory). There is also prior work in 2-rank 2 by Altseimer and there were hopes of returning to that, but we are still dealing with the more accessible cases of higher 2-rank.

1E. *L^* theory in odd type: results.* Recall that here we will always be assuming the condition NTA_2. We turn to mostly unpublished results. There are some useful partial results that did make it into print at this level of generality, and some hints of the general program can be seen.[16] Here I give my views of what is currently known.

One result that is in print concerns the case of strong embedding (Theorem 2.2 below). A simplified version of that is the following.

Fact 1.15. *Let G be a simple L^*-group of finite Morley rank and finite 2-rank, with a strongly embedded subgroup, and Prüfer 2-rank at least 2. Then the 2-rank of G is its Prüfer 2-rank.*

This result (and the full version given later) is less satisfactory than what we have in the K^* case, where the Prüfer rank is reduced to 1, and the situation merits further exploration. Thus the uniqueness branch of the L^* theory remains open in general, and our statements of classification results reflect that.

In particular, for the generic case, the classification result reads as follows.

Theorem 1.16 (Burdges, cf. [21]). *A simple NTA_2 L^*-group of finite Morley rank and Prüfer 2-rank at least 3 is either a Chevalley group or has a strongly embedded subgroup (so in the latter case, it has 2-rank equal to its Prüfer rank).*

Coming to the case of Prüfer rank 2, the results currently focus on the cases with 2-rank at least 3. Here the uniqueness case does not pose difficulties (if one uses the full force of Theorem 2.2). But other problems arise in the case of 2-rank 3.

Theorem 1.17 [22; 23]. *A simple NTA_2 L^*-group of finite Morley rank, Prüfer 2-rank 2, and 2-rank at least 4 is PSp_4.*

For the case of 2-rank 3 the result claimed is as follows.

Theorem 1.18 [22; 23; 24]. *A simple NTA_2 L^*-group G of finite Morley rank, Prüfer 2-rank 2, and 2-rank 3 satisfies one of the following conditions.*

(1) *$G \simeq G_2$ over an algebraically closed field.*

(2) *All involutions of G are conjugate; for i an involution of G, $C(i)$ has the form $SL_3 * SL_3$ (possibly over different fields); the characteristics of both base fields are 3, and their ranks are equal; and if $\mathbb{X}$ is the product of two root groups, one from each factor, then*

$$N_G^\circ(\mathbb{X}) \le C(i). \qquad (*)$$

[16] In [19, §0.5] one can find some thoughts about an L^* theory (with no mention of NTA_2). Some results on L^* groups in the same vein as the K^* results are given there.

Condition (∗) is the exotic condition. This seems to be the most elaborate configuration currently known which may possibly occur in a minimal counterexample to the algebraicity conjecture.

The preprints [21; 22; 23; 24] can be found (in some form) at my website and should become available at arXiv after some additional review and editing. As Burdges has become occupied with other matters it may take some time for that to advance.[17]

In principle one could try to push the analysis further to the level of Prüfer 2-rank 2 and 2-rank 2, but one loses the signalizer functor theory, and with it, the last general method of component analysis in centralizers of involutions. There is some prior unpublished work by Altseimer in the K^* setting. Beyond this point, one approaches the theories of minimal simple groups and groups of degenerate type. At that point, while one is not necessarily limited to the K^* setting, the L^* theory does not provide a useful point of view.

We spend the rest of the present article filling out our discussion of the results mentioned, with one further note at the end. Accordingly, our usual notation and hypotheses will be as follows.

Hypothesis 1.19. *G is a connected simple group of finite Morley rank and finite 2-rank. It is an L^* group and satisfies condition NTA_2.*

Of course, before dealing with the L^* case one also develops some theory for L-groups with NTA_2, mainly in the connected case.

2. Uniqueness cases

In this section and the next we return to a discussion of the first main branch in the various analyses under discussion, namely splitting off the so-called uniqueness cases. The reader who prefers to follow the other branch can pass on to Section 4 with no loss of continuity.

The starting point for the uniqueness case is Theorem 2.2 below, which like everything in this line follows on several iterations of similar results which have been proved under varying hypotheses. It makes use of the following key notion, and several other technical terms which will be discussed further.

[17] I mention in passing that it has occurred to me that the sections devoted to ‘background material” in those preprints might be helpful in thinking about the content and focus of a comprehensive introduction to the theory of groups of finite Morley rank in odd type, for which it seems a book would be considerably more useful than a survey, as the material needs to be revisited, unified, and put at the level of generality most suited for potential applications. In the preprints, what can be quoted is quoted, and what needs elaboration or variation is elaborated on, or varied, by ad hoc arguments, with respect to the precise needs of the main argument.

Definition 2.1. Let G be a group of finite Morley rank and V an elementary abelian 2-subgroup. Then $\Gamma_V(G)$ is the subgroup generated by $C_G^\circ(E)$ with E varying over subgroups of V of index 2.

We are interested in Γ_V for V elementary abelian of rank 2, and thus Γ_V is generated by connected components of centralizers of involutions. Some terminology used in the following will be explained afterward.

Theorem 2.2 (generation theorem: [19, Theorem 6.6][18]). *Let G be a simple L^*-group of finite Morley rank and odd type with 2-rank at least 3, and V an elementary abelian 2-group of rank 2 with $\Gamma_V < G$. Then the following hold.*

(1) *The normalizer of Γ_V in G is a strongly embedded definable subgroup.*

(2) *G is a D^*-group.*

(3) *The Sylow 2-subgroups of G are 2-tori.*

(4) *The Weyl group W is nontrivial, and if r is the smallest prime divisor of the order of the Weyl group, then G contains a nontrivial unipotent r-subgroup.*

In particular, if one starts with any strongly embedded subgroup M then for any elementary abelian subgroup V of 2-rank at least 2 one has $\Gamma_V \leq M$ and thus one arrives at the same conclusions, with a strongly embedded subgroup which is definable (under the stated assumptions on G).

Now let us look at the rest of the terminology used above. First of all, in the L^* context the analogs of solvable group and minimal connected simple group are *D-group* and *D^*-group*, respectively.

Definition 2.3. A connected group of finite Morley rank is a *D-group* if all of its connected simple definable sections are of degenerate type. *D^*-groups* are defined similarly in terms of proper sections.

This is not to say that the class of D-groups is a *satisfactory* generalization of the class of solvable groups, but it is the class we are forced to deal with in this context.

The second point concerns the Weyl group, defined classically (for compact Lie groups, and later for algebraic groups) as $N(T)/T$ for a maximal torus (in the appropriate sense). But here the Weyl group is defined somewhat differently.

Definition 2.4. Let G be a group of finite Morley rank. A *decent torus* in G is a definable divisible abelian subgroup which is the definable hull of its torsion subgroup.

[18] Note on [19]: The statement of Theorem 1.2 given there is over-enthusiastic in its level of generality, replacing 2 by an arbitrary prime p, but overlooking a step where in fact p should have been 2. The situation, and the appropriate level of generality for the various results, is clarified by the remarks in the corresponding MathSciNet review, and in more detail in [28, Section 4].

The *Weyl group* associated with a maximal decent torus T is the finite group

$$W_T = N(T)/C(T).$$

By a conjugacy theorem, this is well-defined up to conjugacy in G.

One ingredient in the proof of Theorem 2.2 is a generation theorem for groups of degenerate type.

Theorem 2.5 [15, Theorem 5]. *Let G be a group of finite Morley rank of degenerate type and V a 4-group acting definably on G. Then $G = \Gamma_V(G)$; that is, G is generated by the connected centralizers*

$$C_G^\circ(a) \quad (a \text{ an involution in } V).$$

It was at this point in [15] that the question of emancipating oneself from the K^* hypothesis was raised.

Returning to the last point of Theorem 2.2, we can be more explicit about the nontriviality of the Weyl group. One begins with a Sylow 2-subgroup S contained in the strongly embedded subgroup M. At this point this is known to be a 2-torus. In this situation the involutions of S are conjugate in M and by a Frattini argument W_S acts transitively on this set. So W_S is nontrivial. Now take a maximal decent torus T containing S. By a Frattini argument W_T induces W_S on S.

3. Strong embedding

We discuss the proof of Theorem 1.14(2) from the point of view of the L^* setting. One would like to prove that a D^*-group of finite Morley rank and odd type satisfying the condition NTA_2, and having a definable strongly embedded subgroup M, must have Prüfer 2-rank 1. In the K^* case one has a choice of proofs, following either [26] or [4, Theorem 6.1]. Here we will be following the latter, but work in part with D^*-groups with NTA_2 rather than with minimal connected simple groups. There is a great deal of additional material which may be relevant, found in (at least) the papers [1; 4; 5; 17; 19; 18; 20; 26; 34].

We begin as follows. I am now discussing material for which there appears to be no formal reference at this level of generality; but see [4] for a highly relevant discussion — although the setting for that discussion was K^* theory, it very likely was also intended at the time to serve as a partial template for an approach to the L^* setting.

Proposition 3.1. *Let G be a D^*-group of finite Morley rank and odd type satisfying the condition NTA_2, and suppose G has a definable strongly embedded subgroup M. Set $B = M^\circ$ and let T be a Sylow 2-subgroup of M, F the Fitting subgroup of B (the largest definable connected nilpotent subgroup). Then the following hold.*

(1) $T \cap F = 1$.

(2) *If the Prüfer 2-rank is at least 2 then $M > B$ and $|M/B|$ is odd.*

(3) *If the Prüfer 2-rank is at least 2 then B is a maximal definable connected subgroup of G.*

(4) *B is a generous subgroup of G (i.e., a generic element of G belongs to a conjugate of B).*

There is a major case division in the proof according as the involutions of M lie inside the Fitting subgroup or outside it. Our first point above indicates that the first of these cases can be eliminated. The method in this case is to show that this would lead to two disjoint generic subsets of the connected group G. This line of argument is taken over from earlier analysis, but involves quite a bit of structural analysis. It is interesting to see how that plays out when we cannot immediately invoke the theory of solvable groups via the hypothesis of minimal connected simplicity.

The rest of the analysis above bears on the second case. The second point follows quickly from the first given that the involutions of T must be conjugate in M; we touched on this point earlier.

With Proposition 3.1 in hand, we return to the K^* context (and, specifically, to the last few lines of [4]). Then the group B is solvable, so it is a Borel subgroup of G, and one may apply the following theorem to conclude.

Theorem 3.2 [4, Theorem 3.12]. *Let G be a minimal connected simple group of finite Morley rank and B a nonnilpotent generous Borel subgroup. Then B is self-normalizing.*

This can usefully be broken down somewhat, as follows. We note that the Weyl group is variously defined as $N_G(T)/C_G(T)$ or $N_G(T)/C^\circ_G(T)$ for T a maximal decent torus of G, but for G connected the definitions agree, by [1, Theorem 1].

We will work with the following four results in the minimal connected simple case, and use them to give a direct treatment of the case of strong embedding.

Fact 3.3. *Let G be a minimal connected simple group of finite Morley rank with nontrivial Weyl group $W = W_T$ of odd order, p a prime divisor of the order, and let a be a p-element of G representing an element of order p in W.*

(1) *If p is a minimal prime divisor of the order of W then $C(a)$ contains a nontrivial p-unipotent subgroup* [20, Corollary 5.2].

(2) *If G is of degenerate type and $C(a)$ contains a nontrivial p-unipotent subgroup then G contains no divisible p-torsion* [25, Lemma 3.5].

(3) *Suppose that G is of degenerate type, that B_T is a Borel subgroup containing $C(T)$, that a normalizes B_T, and that T contains no p-torsion. Then $B_T = C(T)$ is nilpotent and $C_{B_T}(a) = 1$* [25, Proposition 3.10].

(4) *If B is a Borel subgroup containing a nontrivial unipotent p-subgroup then p does not divide $[N_G(B) : B]$* [1, Lemma 4.3].

Setting aside the degeneracy hypotheses occurring in the literature, for which the assumption that the Weyl group is of odd order should be sufficient, and coming to the case of groups of finite Morley rank of odd type with a strongly embedded subgroup M, and Prüfer rank at least 2 (to ensure a nontrivial Weyl group), one can apply these conditions successively to $B_T = M^\circ$, with p the smallest prime divisor of the order of the Weyl group; (1) and (2) permit the application of (3) to conclude that $B_T = F(B_T)$, contradicting Proposition 3.1(1).

We did not use (4) here, but it comes in to the proof of (3). We also did not use Theorem 3.2 as stated, either. The proof of that theorem makes use of the following.

Fact 3.4 [25, Corollary 3.6]. *Let G be a minimal connected simple group of degenerate type with a nontrivial Weyl group W and let p the smallest prime divisor of the order of the Weyl group. Then there is no p-divisible torsion in G.*

This in turn makes use of Fact 3.3(1).

There is a general discrepancy between the way these various principles are stated and the way they are applied. In the literature one separates out the degenerate type analysis from the odd type analysis rather sharply and in the case of odd type one uses, in particular, the existing theory of groups with strongly embedded subgroups. If one wants to redo that theory by other methods then one may want to borrow the theory of groups with nontrivial Weyl groups of odd order from its usual context of degenerate type. One must then check (and also, mention) that the results do not in fact require degeneracy.

This point only arises if one is attempting to redo (or generalize) the theory of strongly embedded subgroups without making use of results that rely on that result. In the existing literature, several intertwined articles appeared at the same time, with this point more relevant at some points than at others, and in particular [4] discusses the way its results relate to those of [25] and why it is necessary to avoid quoting certain prior results in their most general form.

Also noteworthy is the following result from [34], as well as its method of proof and the comments Frécon makes on that proof and its relation to prior work. (The paper [34] takes a rather different point of view — or point of departure — from the one adopted here, and in particular makes use of somewhat different definitions. But Frécon begins by discussing the relations among the various points of view available quite thoroughly, and in particular shows that the conventions adopted do not conflict. We pass over these issues here and refer the interested reader to Frécon's account.)

Fact 3.5 [34, Lemma 2.8]. *Let G be a minimal connected simple group of finite Morley rank, C a Carter subgroup and p the smallest prime divisor of the order of the Weyl group W_G. If C has a nontrivial p-element, then G is of odd type and W_G has even order.*

In particular, under these hypotheses, and supposing we have a nontrivial Weyl group of odd order, it follows that there is no nontrivial p-torus in G. This result was obtained about the same time, but a little later, than Fact 3.3, but as Frécon notes, the proof is more direct. We are presently in a context where this sort of nuance can be of great technical value. Frécon's rather concise discussion following his proof of Lemma 2.8 of some of the dependencies among different parts of the theory as they were developed is valuable and pertinent.

One point of interest, for us, is the way the theory of solvable groups comes in. This frequently involves one of the following two points: p-unipotent subgroups of solvable groups lie in the Fitting subgroups, and in the minimal connected simple case, each p-unipotent subgroup lies in a unique Borel. One can easily conceive of reasonable hypotheses on degenerate type groups which might allow this type of principle to be extended to D^*-groups.

On the other hand, the theory also makes some use on occasion of some other delicate points from the solvable theory: conjugacy of Carter subgroups, and the delicate Bender analysis, which amounts to a close study of the maximal intersections of pairs of Borel subgroups. The latter topic once more invokes the properties of p-unipotent subgroups, but also brings in the characteristic zero unipotence theory. There are some results of striking generality on the theory of Carter subgroups of general groups of finite Morley rank, something one does not have in the finite case, not limited to the K^* case, but these do not fully cover the degenerate case. (See Section B7.)

One interesting question (with a great number of reasonable variants) is whether the mere assumption that B is solvable allows a similar treatment of the L^* case (which, recall, is just the D^* case at this point). In this form, using the existing techniques, this does not seem very likely, at least not without considerable additional work. We will continue this discussion, which is largely a discussion of open questions, in an appendix of a more exploratory character.

One could ask quite similar questions about the theory of Carter subgroups in groups of degenerate type, but we have not taken this up.

4. High Prüfer 2-rank

If we set aside the strongly embedded case, the identification of L^*-groups with NTA_2 in Prüfer 2-rank at least 3 may be completed. This is Theorem 1.16. In [18] an axiomatic framework for the proof was set out which is sufficient for the application of the argument of [7]. This framework is the following.

Hypothesis 4.1. G is a connected simple group of finite Morley rank and odd type with Prüfer 2-rank at least 3.

T_2 is a maximal 2-torus of G.

Σ is a family of subgroups of G of type (P)SL_2. We suppose that Σ has the following properties.

(1) $\Sigma^g = \Sigma$.

(2) $\langle \bigcup \Sigma \rangle = G$.

(3) For K in Σ we have
   (a) K is normalized by T_2.
   (b) $C_K(T_2)$ is a maximal algebraic torus of K; this torus is denoted $\mathbb{T}_K$.
   (c) $K = E(C_G(C_{T_2}(K)))$.
   (d) K is a Zariski closed subgroup of any definable algebraic quasisimple subgroup of G which contains K, and which is normalized by T_2.

(4) For K_1, K_2 in Σ distinct, and $L = \langle K_1, K_2 \rangle$, we have
   (a) $C_T(K_1) \cap C_T(K_2) \neq 1$.
   (b) Either K_1 and K_2 commute or L is an algebraic group of type A_2, $B_2 = C_2$, or G_2, and in that case K_1 and K_2 are root SL_2-subgroups of L normalized by $\mathbb{T}_L$.
   (c) The maximal tori $\mathbb{T}_{K_1}$, $\mathbb{T}_{K_2}$ associated with K_1 and K_2 commute.
   (d) $T_2 \cap L = (T_2 \cap K_1) * (T_2 \cap K_2)$ is a Sylow° 2-subgroup of L.

Such a family Σ is called a family of *root* SL_2*-subgroups* with respect to the 2-torus T_2.

The key to the construction of a suitable family Σ in the context of Theorem 1.16 is the identification of suitable algebraic subgroups of $C_G(i)$ for involutions $i \in T_2$. More precisely, one works within algebraic components of $EC_G(i)$.[19] Much of the work goes into the proof of the existence of these components.

Once one has sufficiently many such subgroups (as expressed by condition (2)), the argument becomes comparatively formal. The failure of the generation condition (2) leads to a strongly embedded subgroup. After that, the rest of the analysis involves ordinary root subgroups inside algebraic components L, taken with respect to the algebraic torus $\mathbb{T}_L$. One occasionally invokes the assumption on high Prüfer rank, and induction (4(a)), to bypass what would otherwise be challenging technical issues.

To reach the point of departure, namely the existence of algebraic components in centralizers of involutions, one needs *signalizer functor theory*. In this area, one quickly encounters difficulties associated with bad fields, or more particularly, a possible embedding of the additive group of one field in characteristic zero into the multiplicative group of another such field. This is overcome using Burdges' unipotence theory, and involves consideration of the reduced ranks of the fields involved, taking into account the generation of algebraic groups by unipotent subgroups.

[19] See Section B4.

As far as the classification of groups of high Prüfer rank is concerned, it is primarily the adaptation of the signalizer functor theory to the L^* setting that is still unpublished. For this, see the preprint [21]. We mention two noteworthy points.

First, one can put the hypothesis NTA_2 into a more directly applicable form. What follows is a slightly specialized version of what is currently Proposition 3.10 of [21] (in §3.3, *Structure of L-groups with NTA*$_2$).

Fact 4.2. *Let H be a connected L-group of finite Morley rank and odd type, satisfying NTA*$_2$. *Suppose that*

$$OF(H) \le Z(H).$$

Then

$$H = \mathrm{E}_{\mathrm{alg}}(H) * K \textit{ where } K \textit{ is connected and } K/Z^\circ(K) \textit{ has degenerate type.}$$

Here $\mathrm{E}_{\mathrm{alg}}$ *is the product of the (individually) algebraic components of* $E(H)$.

Hence the Sylow 2-subgroup of K is central in H, and connected.

This result is certainly useful as stated, but for technical reasons it seems necessary to give a sharper version, replacing $OF(H)$ by the largest connected normal subgroup of $F(H)$ whose torsion subgroup has bounded exponent.

Our second point, below, is particularly technical, but it captures one of the main points. The statement highlights the role of *cotorality* as well as some rank conditions relating to the Burdges unipotence theory. Furthermore, as formulated, the following can also be applied in some situations in Prüfer rank 2 and 2-rank at least 3.

Fact 4.3 [21, Lemma 4.4]. *Let G be an L^*-group of finite Morley rank of odd type, satisfying NTA*$_2$. *Let i, j, k be three commuting involutions in G and let ρ be either a prime or a symbol $(0, r)$ satisfying the conditions*

$$r > r_{f,i}, \qquad r \ge r_{0,i}.$$

Suppose the following.

(1) *i and j are cotoral in G.*

(2) $\theta_\rho(k) \cap C_G(j) \le \theta_\rho(j)$.

Then

$$\theta_\rho(k) \cap C_G(i) \le \theta_\rho(i).$$

Here two involutions are said to be *cotoral* if they lie in some 2-torus of the group G. It is known that in a connected group of odd type, each involution lies in some 2-torus. The same theory casts some light on the cotorality condition as well [20]. The issue of cotorality becomes more delicate, and therefore more important, in the context of Prüfer rank 2. But for some purposes the condition that the Prüfer 2-rank is at least 3 can be replaced by the much less restrictive condition there is an elementary abelian 2-group A of 2-rank 3 such that each pair of involutions in A is cotoral — or even somewhat less.

The use of the symbol ρ as either a prime p or a symbol $(0, r)$ refers to the Burdges unipotence theory, which extends the notion of p-unipotence to include a family of notions of $(0, r)$-unipotence associated with the "prime" 0. In particular the notation $\theta_\rho(k)$ is defined using the unipotence theory as the unipotent radical, in the sense of ρ, of the group $O^\sigma C(k)$ (the largest connected normal solvable definable subgroup of $C(k)$ without involutions). This is the sort of subgroup one always considers in connection with signalizer functor theory, for reasons touched on further in Section B4.

Noteworthy here is the restriction on ρ in terms of two associated parameters $r_{f,i}$ and $r_{0,i}$ involving the structure of $C_G^\circ(i)$. We say a bit more about these.

The parameter $r_{0,i}$ is familiar from Burdges unipotence theory as the maximal reduced rank associated with the odd solvable radical of $C_G(i)$. The parameter $r_{f,i}$ on the other hand is less often met with. Here the notation "f" stands for *field*, and the parameter $r_{f,i}$ measures the maximum reduced rank of the multiplicative group of a field which occurs as the base field of a component of the group

$$\mathrm{E}_{\mathrm{alg}}\big(C_G(i)/O^\sigma(C_G(i))\big),$$

where as above, O^σ means "largest connected normal solvable definable subgroup without involutions".

This analysis is finely tuned, and a good deal of it is foreshadowed by Burdges' thesis; cf. [18].

5. Prüfer 2-rank 2, 2-Rank at least 3

We come to the case of Prüfer 2-rank 2 and 2-rank at least 3. Here the strongly embedded configuration cannot arise — that would involve a connected Sylow 2-subgroup, and in that case the 2-rank and Prüfer 2-rank would coincide (Fact 1.15). So one may aim outright at identification.

As we explained at the end of Section 1D, in Prüfer 2-rank 2, even the K^* version of the results is unpublished.[20] We now resume this discussion once more, at the level of the L^* theory. This material is the subject of [22–24].

In 2-rank at least 4 one arrives at the expected identification: the group is PSp_4. In 2-rank 3 the target is G_2. In the course of the analysis in 2-rank 3 two cases arise, one leading to G_2 as expected. The other branch leads to a configuration quite familiar in the finite case, but eliminated there by character theory — initially, by modular character theory, and later by ordinary character theory. This exotic case is associated with a "base field" of characteristic three. It also has some uniqueness properties with respect to unipotent subgroups.

[20] And elaborate; and not, as yet, particularly closely vetted. I give my current view of it.

Apart from the exotic case, the treatments of the case of 2-rank at least 4 and the case of 2-rank 3 have a great deal in common. In very general terms, the line of analysis in both cases is similar to the analysis in high Prüfer rank, and in particular the signalizer functor theory is brought to bear in the same way. The first stage involves the analysis of algebraic components of centralizers of involutions, and the second stage focuses on the structure of the Weyl group. However, the treatment is far less "axiomatic" and involves much more detail at the level of particular configurations, with a number of undesirable configurations requiring close attention prior to their elimination.

The axiomatic approach in higher Prüfer rank was based on the idea that the Dynkin diagram encodes the structure of certain Lie rank 2 subgroups and in Prüfer rank 3 or more the induction hypothesis already controls the possibilities for these.

In the case of Prüfer rank 2, there is considerable uniformity in the treatment of components. When one brings in the Weyl group the two cases divide and each is handled separately and quite explicitly. Rather than applying general theory, the general thrust of the analysis is a direct examination of the action of the Weyl group on root subgroups and the verification of a qualitative form of the Chevalley commutator formula on a case-by-case basis.

The target ultimately is to invoke the theory of BN-pairs of finite Morley rank [40]. As we have mentioned, the first phase, having to do with the existence and precise determination of the algebraic components in centralizers of involutions, involves some close analysis of potentially pathological configurations. Here one has recourse to some specialized topics borrowed from finite group theory, notably the Thompson $A \times B$ theorem.

As far as the case of Prüfer rank 2 and 2-rank 2 is concerned, where one aims at identification of PSL_3, the terrain is largely unexplored, with the exception of early unpublished work by Altseimer. Here one should focus initially on the K^* context. In that case, one can at least rule out the strongly embedded configuration at the start.

6. Lately: linearization theorems

From the very beginning, understanding the representation theory of algebraic groups in the finite Morley rank category has been a major challenge, though partial results on the topic have already played a major role in such topics as the classification problem and the theory of permutation actions of finite Morley rank.

A recent milestone in this area is the following, a long-standing conjecture.[21]

[21] The genesis of this result, as a byproduct of measures taken in the recent (and as of this writing, ongoing) pandemic, is discussed in [12]. The author remarks "My triumph would be better deserved if it was someone else's conjecture; unfortunately, it was my own." This I take to be as much a sociological remark as a philosophical one.

Theorem 6.1 [13, Theorem 1.4]. *Let K be an algebraically closed field of characteristic $p > 0$ and G the group of points over K of a simple algebraic group defined over K. Assume that G acts definably and irreducibly on an elementary abelian p-group V of finite Morley rank. Then V has a structure of a finite dimensional K-vector space V_K, compatible with the action of G.*

The paper [13] uses enveloping algebra techniques in ways not previously seen in the subject. This result clearly has implications for the general study of definable actions of groups of finite Morley rank on abelian groups, and hence in the theory of permutation groups generally. See [10].

The theorem could well be of some relevance also in looking into some of the more recalcitrant configurations associated with the classification problem, such as the strongly embedded configuration in the context of L^* theory in odd type.

A comprehensive account will be found in Borovik's contribution to this volume [13]. In addition to the linearization of actions of algebraic groups, there is a more general form of linearization in the finite Morley rank context, which forces a field into existence. This "Schur–Zilber" approach is discussed by Deloro [29], in the context of a broad generalization of the result, and a novel approach.

Appendix A: More on strong embedding

As we have indicated, a satisfactory treatment of strong embedding relies on the hypothesis of minimal connected simplicity, that is, the assumption that all proper definable connected subgroups are solvable, and then on the theory of solvable groups of finite Morley rank, which is a rich subject that can be exploited in a variety of ways, notably via Carter subgroup theory.

More than one route has been taken to the treatment of this case in the K^* context. We indicated one such in the discussion around Fact 3.3.

Here we add a few comments about what is known in this direction more generally, and consider weakenings of minimal connected simplicity that suggest themselves as contexts for a broader treatment by the same methods.

A1. *A few general results.*

Fact A.1 [19, Theorem 6.6]. *Let G be a simple L^*-group of finite Morley rank of odd type, with $m_2(G) \geq 2$. Let V be an elementary abelian 2-group of rank 2 with $\Gamma_V < G$.*

If p is the least prime divisor of M/M°, then G contains a nontrivial unipotent q-subgroup for some prime $q \leq p$.

(One may rephrase the hypothesis on Γ_V as stating that G has a strongly embedded subgroup.)

Fact A.2 [15, Proposition 1.1, Theorem 4]. *Let G be a connected, nontrivial group of finite Morley rank and $g \in G$. Then the centralizer $C_G(g)$ is infinite. If g is a p-element for some prime p, then $C(g)$ contains an infinite abelian p-subgroup.*

Lemma A.3. *Let G be a D^*-group of finite Morley rank with a definable strongly embedded subgroup M, and $B = M^\circ$. Suppose also condition NTA_2.*

Let $w \in M \backslash B$. Then $C_B(w)$ is of unipotent type (i.e., contains no nontrivial divisible torsion).[22]

Proof. Suppose S is a nontrivial divisible abelian torsion subgroup in $C_B(w)$. Then $C_B(S)$ contains a maximal decent torus T of M and hence $C(S) \le M$. But $C(S)$ is connected [1, Theorem 1], and so $C(S) \le B$. Thus $w \in C(S) \le B$, a contradiction. □

Taking p to be a divisor of the order of the Weyl group in this setting, it follows that $U_p F(B) = 1$.

From the solvable theory, the following is key for our purposes. and stands apart from much of the rest of the theory.

Fact A.4 [25, Lemma 6.6 (Frécon)]. *Let G be a connected solvable group of finite Morley rank, and let $H < G$ be a definable connected subgroup of G such that $N_G^\circ(H) = H$. Then $N_G(H) = H$.*

It would be helpful to have something that can be put to similar use in the context of D-groups.

A2. ***Generalizations of minimal connected simplicity.*** The condition of minimal connected simplicity states that proper connected definable subgroups are solvable. One can generalize this condition either by restricting attention to some definable subgroups — notably $N^\circ(A)$ for suitable definable subgroups A — or by weakening the solvability condition, as in the D^*-condition.

Since there are substantial obstacles to the extension of the K^* theory to the D^* context it seems worthwhile to consider natural extensions of the K^* context which support the existing techniques. In that setting, one anticipates that the few points where the theory of solvable groups enters in a serious way might pose further problems, and if so it would be good to identify them.

We discuss some formal aspects of this. The following conditions are very natural.

Definition A.5. Let G be a group of finite Morley rank, p a prime, and H a definable subgroup.

[22] As remarked in [16], among all possible definitions of unipotence, this was "the broadest one we can imagine". Fortunately — given that — it is also a nontrivial condition that can be applied.

(1) The group G is *U_p-trivial* if for every degenerate type simple definable section L of G, any definable action of a unipotent p-group on L is trivial. In particular, L contains no nontrivial p-unipotent subgroup.

(2) H is *U_p-solvable* if the subgroup $U_p(H)$ generated by p-unipotent subgroups is solvable (equivalently, $U_p(H) \le F(H)$).

(3) A definable subgroup H of G is a *U_p-uniqueness* subgroup if for every nontrivial p-unipotent subgroup U of H, we have $N_H^\circ(U) \le H$; and we suppose also that H does in fact contain some nontrivial p-unipotent subgroup.

(4) G is *U_p-minimal* if every proper connected definable subgroup is U_p-solvable.

(5) G is an *$N^\circ_{U_p}$-group* if for every nontrivial abelian subgroup A the connected normalizer $N_G^\circ(A)$ is U_p-solvable.

Here condition (1) is a reasonable form of "tameness" to impose, allowing one to explore the configurations remaining when the more extreme configurations are eliminated. However one would not expect to work directly with that type of hypothesis, but rather with more abstract conditions of the type of (4) or (5). Here condition (3) is an expression of "Bender principle" used by Jaligot and developed further by Burdges, which one would expect to play a major role.

In particular solvable groups are U_p-solvable, minimal connected simple groups are U_p-minimal, and U_p-minimal connected simple groups are $N^\circ_{U_p}$-groups. And if a Borel subgroup B of a minimal connected simple group G contains a nontrivial p-unipotent subgroup, then it is a U_p-uniqueness subgroup.

We would also prefer to put more emphasis on the particular subgroup $B = M^\circ$, to the extent possible, and on the connected centralizers of Weyl group representatives, but not on the first pass.

Lemma A.6. *Let H be a connected D-group of finite Morley rank. If H is U_p-trivial then it is U_p-solvable.*

Proof. Let $U = U_p(H)$. By assumption $U \le C(EH)$.

We first treat the case in which

$$U_p F(H) = 1.$$

Then $U \le C(F(H))$ and hence $U \le C(F(H)E(H)) = Z(F(H))$, so $U = 1$ in this case.

For the general case, let $\overline{H} = H/U_pF(H)$. Then $U_pF(\overline{H}) = 1$ and $\overline{U} \le U_p(\overline{H}) = 1$, so $U \le U_pF(H)$ as claimed. □

Thus a connected simple D^*-group of finite Morley rank which is U_p-trivial is U_p-minimal.

Lemma A.7. *Let G be a connected simple D^*-group of finite Morley rank which is an $N^\circ_{U_p}$-group. Then any maximal connected definable U_p-solvable subgroup H with $U_p(H) > 1$ is a U_p-uniqueness subgroup.*

In other words, if we call a maximal connected definable U_p-solvable subgroup a U_p*-Borel* subgroup, then each U_p-local subgroup is contained in a unique U_p-Borel subgroup.

This is the usual argument.

Proof. Assuming the contrary we can find a nontrivial p-unipotent subgroup U of H and another maximal connected definable U_p-solvable subgroup H_1 containing it.

We may suppose further that the pair (H, H_1) is chosen to maximize U. If $U = U_p(H) = U_p(H_1)$ then $H = H_1$ for a contradiction, so we may suppose $U_p(H) > U$. Then $N^\circ_{U_p(H)}(U) > U$ and by maximality $N^\circ_G(U) \le H$. It follows that $U_p(H_1) \le H$, so $U = U_p(H_1)$ and $H_1 \le N^\circ_G(U) \le H$, for a contradiction. □

Appendix B: Glossary

Here we review some of the technical notions that occur at various points in the discussion, to simplify navigation. Many of these notions are not explicitly defined above, but in such cases we indicate the ideas behind them and the roles they play.

It may be helpful to have some of this collected together in one place. We also include some introductory remarks which are less directly pertinent to the technical discussion.

B1. ***Group theoretic terminology.*** We use terminology coming both from finite group theory and algebraic group theory. The one place where there is a notable terminological conflict between these two subjects is the use of the term "simple" in algebraic group theory in the sense of "quasisimple" in finite group theory. We use the term "simple" in its more literal sense.

Notions from finite group theory (or abstract group theory in general) can typically be taken over directly into our subject; notions from algebraic group theory may inspire similar notions with less geometrical definitions, which should be equivalent to the original definitions in the context of algebraic groups over algebraically closed fields *which carry no additional definable structure.* These notions are thoroughly covered by [14], and again, with some additions, in [3], where the bulk of Chapter I concerns various topics that fall under this heading.

We tend to work with *definable* subgroups, and definable sections (quotients of a definable subgroup by a definable normal subgroup). On the other hand, some very important subgroups that come into play tend not to be definable — notably, 2-Sylow subgroups — and accordingly when subgroups are meant to be definable, this is always specified. One has, in general, the *definable hull* of any subgroup —

or any subset — that is, the smallest definable group containing the given set. (One should avoid using the possibly more natural expression *definable closure* in this sense, as it has another meaning in model theory, of a very different character.)

It is a theorem of algebraic group theory that simple algebraic groups are linear groups, and the classification of the simple algebraic groups makes use of this point. We tend to identify these groups (which are functors) with the actual groups of rational points over an algebraically closed field. This is in some ways similar to talking about structures — or a particular structure — rather than theories. In a similar vein we tend to assume that our groups are saturated, though in the context of algebraic groups, there is occasionally a point to considering what happens over the algebraic closure of the prime subfield. None of this will be visible in the discussion in this paper; it lurks in the background.

Any algebraic group is a group of finite Morley rank, when realized concretely as a group over some fixed algebraically closed field, and the most striking applications of the theory to classical problems of mathematics, to date, actually come in the context of abelian varieties, and hence lie more or less at the opposite pole from the algebraicity conjecture. On the other hand, those applications pass in some cases through differential algebra, and in that context one has also a rich Galois theory and structural issues in the simple case, so in that respect at least, the subjects are not entirely foreign to one another. In this connection I would point to [35; 36].

Less concretely relevant, but I think of some importance, is the fact that the theory sits within the broader subject of *stable group theory*, which provides possibly the most satisfactory framework for thinking about the model theoretic issues that arise. For this the main point of entry remains [45], or its English translation.

In Table 2 (at the end) we list some more or less standard group theoretic operators whose definitions may vary a bit in the setting of groups of finite Morley rank, depending on whether or not issues of definability or connectedness arise. In most cases one proves definability under standard definitions. Notational conventions may vary, and we follow the preprints [21; 22] here, but most of this is found in [14].

Our convention here is that there is an ambient group G, and that H is one of the customary notations for a subgroup (more often than not, a definable subgroup). We mention that one occasionally takes connected components of nondefinable subgroups, using relatively definable subgroups (i.e., suitable intersections).

In the finite case the use of $O(G)$ is based on the Feit–Thompson theorem and it presumably corresponds more closely to $O^\sigma(G)$. The most immediate analog of “odd order” in the context of finite Morley rank is “without involutions”. It is awkward to try to work with a very direct analog of the operator $O(G)$ as used in the finite case — consider for example an algebraic torus — so we pass directly to the connected analog. Most of the time it is used in a context like $OF(H)$ where it is already solvable, and one has the solvable version O^σ available when it is more appropriate.

B2. ***Basic notions.*** Simple groups of finite Morley rank are divided into *degenerate*, *odd*, *even*, and *mixed* types (pp. 511, 510).

The degenerate and odd types have finite 2-*rank* (zero or positive, respectively), while even and mixed type have infinite 2-rank (pp. 506, 511, 510). When the 2-rank is finite, a critical parameter is the *Prüfer* 2-rank, which corresponds in the algebraic setting to the Lie rank (p. 511).

The focus of the *Borovik program* (p. 506) has been on K^*-*groups*, but it turns out that in even and mixed type, by extending the theory to the so-called L^*-*groups* one can get a proof of the algebraicity conjecture for even and mixed type (p. 509).

A variant L^*-theory for odd type presents more difficulties. In this case the natural definition has to be supplemented by a condition denoted NTA_2 which is parallel to Altınel's Lemma 1.5 in the even or mixed type setting. The condition NTA_2 remains conjectural and appears to have roughly the level of difficulty of a full classification in the case of simple groups of Prüfer rank 1 (Definition 1.11).

In the L^*-theory one has also the more technical notions of *D-groups* and D^*-*groups*, which play much the same role in that context as solvable groups and minimal connected simple groups do in the K^* context (Definition 2.3).

We subdivide odd type correspondingly into *thin*, *quasithin*, and *generic* type, corresponding to Prüfer 2-rank 1, 2, or higher (Definition 1.7).

We also must consider some notions of groups of *uniqueness* type, notably the case of *strong embedding*. One hopes that these groups will have Prüfer 2-rank 1. This is known in the K^* context (Theorems 1.14 and 2.2).

To get one's bearings in the technical side of the subject it is helpful to go to [18], which among other things provides a guide to a substantial body of prior work.

B3. ***Torsion and Weyl groups.*** A Π-*torus* is a divisible abelian torsion group and a p-*torus* is a divisible abelian p-group. A *decent torus* is the definable hull of a Π-torus. One has conjugacy theorems for the maximal p-tori, Π-tori, or decent tori. The *Weyl group* of a group G of finite Morley rank is the finite group $N_G(T)/C_G(T)$ where T is a maximal decent torus (or a maximal Π-torus). See Definition 2.4.

The study of torsion in groups of finite Morley rank leads into the close study of Weyl groups in exotic configurations and is of particular importance in groups which are small in the sense of uniqueness type or which are minimal connected simple (p. 519).

B4. ***Classification techniques.*** We have made rather cavalier use of the notation $EC(i)$, beginning with p. 522. This permeates the classification theory for finite groups as well as the L^* theory as discussed here, So we elaborate.

Here i is an involution, and $C(i)$ its centralizer. In an odd type group an involution plays the role of a semisimple element, and the conjecture we aim to prove predicts the structure of this group very precisely. The subgroup $EC(i)$ is the largest normal subgroup of $C(i)$ which is a direct product of quasisimple groups (that is, groups which are simple modulo their center, and perfect). For our purposes only the algebraic factors of $EC(i)$ are useful; the degenerate factors will be ignored. If one has enough algebraic factors then one hopes to reconstruct the entire group from them. Here the odd solvable radical $O^\sigma C_G^\circ(i)$ represents a potential obstacle to this.

The key ingredient in the analysis of $EC(i)$ (more specifically, for the control of $O^\sigma C(i)$) is the highly technical *signalizer functor* theory (p. 522). This leads eventually to the desired algebraic components of $EC(i)$.

This theory makes extensive use of the *Burdges unipotence theory*, which provides a characteristic zero analog of the *p-unipotence theory* (p. 523). The theory also requires a good understanding of *torality* and *cotorality* of involutions (p. 523), when one comes to the case of Prüfer 2-rank 2.

For our purposes, the most important point of the Burdges unipotence theory is that the additive group of a field is "more unipotent" than its multiplicative group (and also, that simple algebraic groups are generated by copies of additive groups of the base field). This feeds into the signalizer functor theory via a study of "sufficiently unipotent" base fields, in the case in which simple algebraic sections of a given group involve more than one base field. The precise measure of this is given by two parameters denoted $r_{f,i}$ and $r_{0,i}$ associated with an involution i, where the subscript "f" refers to base fields and the subscript "0" refers to the general unipotence theory in characteristic zero. We do not give further details here.

Another important feature of the unipotence theory is a notion of unipotent radical. Given that there are several notions of unipotence in play, there are several associated notions of unipotent radical, and not all are well-behaved. Subscripts as in θ_ρ tend to make (oblique) references to such notions.

B5. ***Simple algebraic groups.*** We rely in a certain sense on the classification of the simple algebraic groups as Chevalley groups over algebraically closed fields. The point here is that we have no hope of classifying the possible theories of these groups in an arbitrary language. There is presently a very rich supply of theories of algebraically closed fields of finite Morley rank, inspired by Hrushovski's refutation of the original formulation of Zilber trichotomy. We aim only to classify the groups obtained as abstract groups, and for this it is very natural to work toward some explicit presentation of the group; some such approach remains necessary to establish the existence of these groups, in fact, a point which remained an oddly open question for half a century between the construction of the smallest algebraic group of exceptional type by Dickson and Chevalley's explanation of how to use a suitably chosen basis for the associated Lie algebra as to allow for a sufficiently well-defined exponential map over an arbitrary base field. Standard references for this would be Steinberg's notes and Carter's book, both of which work very directly with explicit generators and relations. Chevalley had also asked for a more geometrical theory, and in addition to the general theory of algebraic groups, the Tits theory of buildings (and its later refinement to Moufang buildings) provides a satisfactory approach. On a more technical level, Tits' more ad hoc theory of BN-pairs presents an efficient way of reaching the Chevalley–Steinberg relations. Both of these approaches of Tits have been taken over to the context of groups of finite Morley rank and provide essential tools for the efficient recognition of simple groups as algebraic groups once a sufficient amount of group theoretic structural analysis has been carried out.

It will be noticed that in this context we are never actually in a position to use the classification of the simple algebraic groups, but it tells us what we are aiming for. The same is true in the context of finite group theory, where one has also to make one's way past 26 sporadic groups (and several phantoms of other, nonexistent, exceptions) and allow for the so-called "twisted forms" which occur when the base field is not algebraically closed, but using the same underlying theories.

The Borovik program, in its various incarnations, accepts that we are studying what is ultimately an algebraic problem involving the pure group language and that there does not appear to be a more purely model theoretic route toward significant structural results. At the same time we have learned a good deal more about what pure model theory has to contribute, notably from the direction of model theory of fields (this is also the case in the study of permutation groups of finite Morley rank).

We will come back to all of this in a more concrete spirit below, in terms of how this works out in the context of the theory described in this paper. (The material of [3] relies for the most part on fundamentally different methods which arose much later, within the specific context of the classification of the finite simple group, and which were in fact still undergoing development as that project came to an end.)

B6. ***Identification theorems.*** More concretely, the developments just mentioned provide the methods used in the study of simple groups of finite Morley rank in odd type to identify the groups in favorable cases.

In high Prüfer 2-rank one can use a method of *Curtis–Tits–Phan* as a way of recovering the Steinberg–Chevalley presentation efficiently, after sufficient structural analysis, and in Prüfer 2-rank 2 one may use the theory of *BN-pairs* of finite Morley rank for the same purpose, the latter generally requiring a more detailed structural analysis. Here, as generally, see [3] for a detailed review of how that actually works.

The model for the treatment in high Prüfer rank is [7][23] and the corresponding axiomatization in Section 4 can be taken as the definition, for our purposes, of the Curtis–Tits–Phan approach.

[23] Or possibly the sharper [8], which retains a K^* hypothesis but in a more limited form. This is published only on arXiv, but finds application in [9].

Our discussion of the case of Prüfer 2-rank 2 (and 2-rank at least 3) in Section 5 does not say much about the final identification or the underlying theory of BN-pairs. Our discussion here has tended to focus on 2-tori as an approximation to algebraic tori. The general thrust of the theory of BN-pairs is to identify a "Borel" subgroup B and the "normalizer N of a maximal torus in B" with properties sufficient to characterize the ambient group.

We work with the group $N = N(T_2)$ and with a certain subgroup $B = TU$ of G, where the construction of U is one of the main difficulties (this is blocked by one bad configuration in the case in which the target group is G_2).

The relevant identification theorem for our purposes is supplied by [48; 49] and once one has identified suitable groups B, N, the structural information required to apply the identification theorem comes down to a verification that the root subgroups of U with respect to T (and their opposites) can be labeled so as to give the expected action of the Weyl group (as well as some loose structural information of the sort given in an explicit form by the Chevalley commutator formula).

In [3] we discussed these two approaches to identification in detail in Sections 6, 7, and 10 of Chapter III, which was devoted to a number of "Specialized Topics" under the broad heading of "Methods". Section 9 of that chapter discusses the signalizer functor theory, which is a considerably more specialized topic that lies more or less at the center of the technical concerns of the present discussion (see above). It is in fact one of the main tools for actually bringing the structural analysis to the point where the standard approaches to identification can be applied.

B7. ***Solvable group theory; Carter subgroups, unipotence theory.*** The point of view taken by L^* theory in odd type makes only limited use of solvable group theory, when compared to the prior K^* theory, which makes very good use of it, notably the parts that go beyond the "basic theory". Of particular importance in that context are the *Borel subgroups*, which as usual are maximal connected solvable subgroups. In the L^* setting one makes good use of the Fitting subgroup and, occasionally, a slightly larger solvable subgroup of momentary interest, but the formal analog of "solvable group" would be "D-group", for which there is not much of a theory in existence, or expected.

From the basic part of the theory comes, in particular, the theory of Hall subgroups (and, in particular, Sylow theory in full generality), the Fitting subgroup, and a good structure theory for nilpotent subgroups. At a more sophisticated level one has *Carter subgroup theory.* Carter subgroups are classically defined in the context of finite solvable groups as self-normalizing nilpotent subgroups, and provide a kind of analog of maximal tori in a general setting. In particular, they are conjugate.

In the context of groups of finite Morley rank, Carter subgroups are taken rather to be definable nilpotent almost-selfnormalizing subgroups (that is, of finite index

Symbol	*Finite version*	*Our version*	*See*
$C_G(X)$	centralizer	(same)	Section B4
$E(H)$	$\langle$quasisimple subnormal components$\rangle$	(same)	Section B4
$F(H)$	Fitting subgroup	(same)	Proposition 3.1
$N_G(H)$	normalizer	(same)	
$O(H)$	odd order radical	*connected* degenerate radical	Fact 4.2
$O^\sigma(H)$	not used	$O(\sigma) = \sigma(O)$	p. 524
$\sigma(H)$	solvable radical	(same)	
U_p	not used	$\langle p$-unipotent subgroups$\rangle$	Lemma 1.6
$U_{(0,r)}$	inconceivable	characteristic 0 unipotence	Section B4
W_T, W_G	not used	Weyl group	Definition 2.4
H°	not used	connected component	[14]
$C^\circ_G, N^\circ_G, \ldots$	not used	connected component of $C_G, N_G, \ldots$	e.g., Theorem 1.18
$\Gamma_{S,2}$	$\langle N_G(A) \mid m_2(A) = 2\rangle$	similar (definable hull)	Definition 1.13
$\Gamma_V(G)$	$\langle C_G(E) \mid [V : E] = 2\rangle$	$\langle C^\circ_G(E) \mid [V : E = 2]\rangle$	Definition 2.1
θ	signalizer functors (ad hoc)	signalizer functors (nilpotent)	Section B4

Table 2. Group theoretic operators.

in their normalizers). In this context, one does not require solvability to prove existence, and in the solvable case one is able to recover a fully satisfactory analog of the classical (finite) theory. The theory also provides a possible point of departure for a truly geometrical approach to the classification problem and issues around the algebraicity conjecture, not fully realized, but playing a very significant role in the development of the theory over the last two decades.

At this point it seems appropriate to simply quote a large portion of [3, pp. 108–109], which refers to §I.8 (*Solvable groups*) and more specifically to §I.8.4 (*Carter subgroups*).

> The Carter subgroup was treated first by Wagner in [50], and a full theory given by Frécon in a series of papers beginning with his thesis [31] [cf. [32]]...general and extensive.... The detailed theory of solvable groups is particularly relevant to the study of minimal simple connected groups, ...which comes into its own in the treatment of odd type, where problems are often reduced to the minimal simple case and then handled by close analysis there.
>
> The theory of Carter subgroups is very powerful.

Also noteworthy, though not much exploited, is Frécon's work on the conjugacy problem in general, one of the tours de force of the subject [33].

The unipotence theory, on the other hand, continues to play a strong role in the theory and consequently has been discussed above. It tends to come into the picture in connection with Fitting subgroups of not necessarily solvable groups (either as subgroups, or with respect to the action of abelian subgroups on the Fitting subgroup). The results are rather scattered in the literature and this is one of a number of points that would be worth revisiting in a comprehensive text relating to the methods and results of theory of odd type groups.

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