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We revisit one of Zilber's early results in model-theoretic algebra, viz., definability in Schur's lemma. This takes place in a broader context than the original version from the seventies.

La droite laisse couler du sable.
Toutes les transformations sont possibles.
Paul Éluard

The present contribution discusses and proves a linearisation result originating in Zilber's early work. Let us note to begin:

(1) o-minimal dimension and Borovik–Morley–Poizat rank are examples of finite dimensions.

(2) All necessary definitions are in Section 2.1.

(3) I have preferred not to conflate T with $\mathbb{K}$ in the statement.

(4) There are classical corollaries in Section 2.4.

(5) The result bears no relationship to indecomposable generation discussed in Section 2.5.

Theorem (Zilber's skew-field lemma). *Work in a finite-dimensional theory. Let V be a definable, connected, abelian group and $S, T \le \operatorname{DefEnd}(V)$ be two invariant rings of definable endomorphisms such that*

- *V is irreducible as an S-module*;
- *$C(S) = T$ and $C(T) = S$, with centralisers taken in* $\operatorname{DefEnd}(V)$;
- *S and T are infinite*;
- *S **or** T is unbounded.*

Then there is a definable skew-field $\mathbb{K}$ such that $V \in \mathbb{K}\text{-}\mathbf{Vect}_{<\aleph_0}$; moreover, $S \simeq \operatorname{End}(V : \mathbb{K}\text{-}\mathbf{Vect})$ and $T \simeq \mathbb{K}\operatorname{Id}_V$ are definable.

The present exposition contains results stemming from more general research pursued with Frank O. Wagner [Deloro and Wagner ≥ 2024].

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Section 1 provides context. Section 2 discusses the statement, and gives all definitions. The proof is in Section 3.

1. Introduction

Section 1.1 explains the relation to Schur's Lemma. Section 1.2 makes some historical remarks. Section 1.3 discusses a more famous corollary on fields in abstract groups.

1.1. ***Schur's lemma.*** Among the early work of Zilber are a couple of gems in model-theoretic algebra. (More on Zilber's early work is in [Hodges 2024] in the present volume.) This article deals with one of the phenomena he discovered: *many $\aleph_1$-categorical groups interpret infinite fields*. The result, or the method, or the general line of thought, is often called *Zilber's field theorem*. It stems from Schur's lemma in representation theory:

Lemma (Schur's lemma). *Let R be a ring and V be a simple R-module. Then the covariance ring $\mathbb{F} = C_{\mathrm{End}(V)}(R)$ is a skew-field, V is a vector space over $\mathbb{F}$, and $R \hookrightarrow \mathrm{End}_R(V)$.*

Zilber's deep observation is simple:

in many model-theoretically relevant cases, $\mathbb{F}$ is definable.

A precise and modern form of the latter statement, given as Corollary 1 in Section 2.4, is a straightforward consequence of the main theorem above. (One should remember that every module is actually a bimodule by introducing Schur's covariance ring.) I shall henceforth call it (in long form) the *Schur–Zilber skew-field lemma*, hoping that Boris will not mind being in good company. Far be it from me to minimise its significance by dubbing it a lemma instead of a theorem; quite the opposite as lemmas are versatile devices — methods.

1.2. ***Editorial fortune of the lemma.*** This subsection is a layman's attempt at providing historical remarks. I apologise for misconceptions.

- As one learns from [Curtis 1999, p. 139], Schur's lemma itself appears in [Schur 1904, §2, I.] with comment: "*der auch in der Burnside'schen Darstellung der Theorie eine wichtige Rolle spielt*".

- Before Zilber's result was known, Cherlin [1979, §4.2, Theorem 1] found a definable field independently. There interpretation is obtained by hand (and seemingly by miracle), without a general method. Cherlin heard about Zilber's work after completing his own; [Cherlin 1979, §1.4] is very informative.

• The lemma itself seems not to have drawn as much attention as its corollary on soluble groups (Section 1.3). There are few traces of the lemma as a stand-alone statement.

• All sources discussing the topic [Zilber 1977; 1984; Thomas 1983; Nesin 1989a; 1989b; Poizat 1987; Loveys and Wagner 1993; Borovik and Nesin 1994; Macpherson and Pillay 1995] rely on indecomposable generation (however, see Section 2.5).

• This is different in the *o*-minimal context, but [Peterzil et al. 2000, Theorem 2.6] has its own techniques. (The earlier [Nesin et al. 1991, Proposition 2.4], which bears no reference to Zilber, resembles the coordinatisation by hand of [Cherlin 1979].) This and the above item may have given the impression that the Schur–Zilber lemma is a finite Morley rank gadget; *the present contribution shows that it isn't*.

• Most sources focus on the ring *generated* by the action instead of going to the centraliser; exceptions are [Nesin 1989a; Macpherson and Pillay 1995]. Only the under-cited [Nesin 1989a] discusses rings and makes the connection with Schur's lemma, while [Macpherson and Pillay 1995, p. 487] notices resemblances between various linearisation results but concludes:

> *There appear to be no immediate implications between this and the results recorded here, though it looks similar to Theorem 1.2.*

The present contribution elucidates the desired relations.

• My own interest in the topic started when I read [Nesin 1989a] while preparing [Deloro 2016]. This resulted in a very partial version of the theorem, in finite Morley rank and using indecomposability. After I gave a talk on generalising "Zilber's field theorem" in Lyon in January 2016, Wagner shared numerous ideas, which will bear all their fruits in the collaboration [Deloro and Wagner $\geq$ 2024].

1.3. ***Fields in soluble groups.*** To some extent, the Schur–Zilber lemma is the poor relation of the following theorem [Zilber 1984, Corollary, p. 175] (currently undergoing generalisation by Wagner):

> *connected, nonnilpotent, soluble groups of finite Morley rank interpret infinite fields.*

I believe the significance of the latter principle has been exaggerated for three reasons.

(1) In the local analysis of simple groups of finite Morley rank, different soluble subquotients may interpret nonisomorphic fields. Since there are strongly minimal structures interpreting *different* infinite fields [Hrushovski 1992], any field structure could be a false lead. (For more on how experts approach the algebraicity conjecture on simple groups of finite Morley rank, and the influence of finite group theory instead of pure model theory, see [Cherlin 2024; Poizat 2024].)

(2) Fields obtained by this method can have "bad" properties, typically nonminimal multiplicative group [Baudisch et al. 2009].

(3) The corollary focused on abstract groups and distracted us from doing representation theory (see the remarkable [Borovik 2024]).

2. The theorem

Section 2.1 contains all necessary definitions. Section 2.2 justifies the structure of the statement. Section 2.3 discusses optimality, Section 2.4 gives corollaries, and Section 2.5 considers the relation to "indecomposable generation".

The general version of the skew-field lemma is a double-centraliser theorem, repeated below. Alternative names could have been "bimodule theorem" or "double-centraliser linearisation".

Theorem. *Work in a finite-dimensional theory. Let V be a definable, connected, abelian group and $S, T \leq \mathrm{DefEnd}(V)$ be two invariant rings of definable endomorphisms such that*

- *V is irreducible as an S-module (viz., in the definable, connected category);*
- *$C(S) = T$ and $C(T) = S$, with centralisers taken in* $\mathrm{DefEnd}(V)$;
- *S and T are infinite;*
- *S **or** T is unbounded.*

Then there is a definable skew-field $\mathbb{K}$ such that $V \in \mathbb{K}\text{-}\mathbf{Vect}_{<\aleph_0}$; moreover, $S \simeq$ $\mathrm{End}(V : \mathbb{K}\text{-}\mathbf{Vect})$ *and $T \simeq \mathbb{K}\,\mathrm{Id}_V$ are definable.*

It would be interesting to recast this kind of double-centraliser result in the abstract ring $S \otimes T$, with no reference to V. (This is not planned in [Deloro and Wagner $\geq$ 2024].)

2.1. *Definitions.*

• Connected: with no definable proper subgroup of finite index. (Since the context does not provide a DCC, not all definable groups have a connected component.)

• Bounded: which does not grow larger when taking larger models. (The algebraist may fix a saturated model with inaccessible cardinality and argue there; bounded then means small. Also see [Halevi and Kaplan 2023].)

• Type-definable: a bounded intersection of definable sets.

• Invariant: a bounded union of type-definable sets. (The name comes from the action of the Galois group of a "large" model. Section 2.2 gives reasons for considering the invariant category instead of the definable one.)

• Irreducible: no nontrivial proper submodule — a submodule being definable *and connected*. (This is weaker than usual algebraic simplicity, which would also exclude finite submodules. Model theory will handle those in its own way.)

• Finite-dimensional: which bears a reasonable dimension on interpretable sets. Here [Wagner 2020] would say *fine*, *integer-valued*, *finite-dimensional*. The definition is as follows.

Definition [Wagner 2020]. A theory T is [fine, integer-valued] finite-dimensional if there is a dimension function dim from the collection of all interpretable sets in models of T to $\mathbb{N} \cup \{-\infty\}$, satisfying the following for a formula $\varphi(x, y)$ and interpretable sets X and Y:

- Invariance: If $a \equiv a'$ then $\dim(\varphi(x, a)) = \dim(\varphi(x, a'))$.
- Algebraicity: X is finite nonempty if and only if $\dim(X) = 0$, and $\dim(\varnothing) = -\infty$.
- Union: $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
- Fibration: If $f : X \to Y$ is an interpretable map such that $\dim(f^{-1}(y)) \geq d$ for all $y \in Y$, then $\dim(X) \geq \dim(Y) + d$.

The dimension extends to type-definable, and then to invariant sets; of course one should no longer expect nice additivity properties.

Except for a key "field definability lemma" (Section 2.5) we shall use little from [Wagner 2020]. There is an ACC and a DCC on definable, *connected* subgroups.

2.2. *Explaining the statement.* Our statement deviates from traditional versions in several respects, and we make three cases for three notions.

Skew-fields rather than fields. Schur's lemma produces a skew-field, and so does Zilber's model-theoretic version.

• This went first unnoticed since $\aleph_1$-categorical skew-fields are commutative (answering a question of Macintyre's, proved by Cherlin and Shelah — see note on [Borovik and Nesin 1994, p. 139] — and independently by Zilber [1977].)

• It is easy to construct, in tame geometry, so-called "quaternionic representations", where the Schur field is the skew-field of quaternions.

• Also, the subring $\langle A \rangle \leq \mathrm{End}(V)$ generated by a *commutative* group action can be smaller than its Schur skew-field $C_{\mathrm{End}(V)}(A)$: classical focus on the former (as in most sources) captures only partial geometric information.

So skew-fields are naturally unavoidable. (There remains the question of which skew-fields can arise in a finite-dimensional theory. Skew-fields abound in number theory, but arguably number theory is far from tame. One can also doubt that the more exotic objects constructed in [Cohn 1995] will be finite-dimensional. The bold would conjecture that infinite skew-fields in finite-dimensional theories are commutative and real closed, commutative and algebraically closed, or quaternionic over a commutative real closed field. The more reasonable may be content with conjecturing that they are finite extensions of their centres. Either of these claims, if true, would have an impact on their stability-theoretic properties.)

Rings rather than groups. Let V be an abelian group; then $\mathrm{End}(V)$ is a ring. This accounts for studying representations of *rings*.

- If $G \leq \mathrm{Aut}(V)$ is a definable acting group, the subring of $\mathrm{End}(V)$ it generates need not be definable (see "invariance" below). This may have baffled pioneers in the topic.

- Rings were long neglected after the seminal [Zilber 1977] (a remarkable exception being [Nesin 1989a]). Going to the enveloping ring, however, gives powerful results, inaccessible to group-theoretic reasoning; see [Borovik 2024].

Invariance rather than definability. Leaving definability may have stopped first investigators of the matter; it is however salutary.

- If $G \leq \mathrm{Aut}(V)$ is a definable group, then the generated subring $\langle G\rangle \leq \mathrm{End}(V)$ is $\bigvee$-definable; this is closer to definability than invariance is. However (see "skew-fields" above), $\langle G\rangle$ does not capture enough geometric information. The double-centraliser $C(C(G)) \geq \langle G\rangle$ is more adapted to Schur-style arguments.

- So let $R \leq \mathrm{End}(V)$ be a definable ring. Then Schur's covariance ring $C_{\mathrm{DefEnd}(V)}(R)$ need not be definable, but it is invariant. And if R itself is invariant, $C_{\mathrm{DefEnd}(V)}(R)$ is too.

So model-theoretic invariance arises as naturally as centralisers do.

2.3. ***Optimality.***

- Both S and T must be infinite.

  Let $\mathbb{K}$ be a pure algebraically closed field of positive characteristic p and $V = \mathbb{K}_+$, which is definably minimal. Now $\mathrm{DefEnd}(V)$ consists of quasi-p-polynomials, viz., of all maps $x \mapsto \sum_{k=-n}^{n} a_{p^k}\,\mathrm{Fr}_{p^k}$, where Fr is the Frobenius automorphism of relevant power, and $a_{p^k} \in \mathbb{K}$; there is no bound on n. Only the action of $\mathbb{F}_p$ commutes to all these. We then let $S = \mathrm{DefEnd}(V)$ and $T = \mathbb{F}_p$ (or vice-versa). The first is not definable.

- At least one must be unbounded.

  For the same V, now let S be the ring of all quasi-p-polynomials *with coefficients in* $\mathbb{F}_p$, viz., the subring of $\mathrm{DefEnd}(V)$ generated by Fr_p and its inverse. Then one easily sees that $C(S) = S$ is countable, and not definable.

On the other hand, it so happens that S-irreducibility can be relaxed to irreducibility as an (S, T)-bimodule [Deloro and Wagner $\geq$ 2024]. So in retrospect, the main theorem can be retrieved as a corollary to [Deloro and Wagner $\geq$ 2024, Theorem 2].

2.4. ***Corollaries.*** I give three corollaries, proved in Section 3.5. The first relates the main, "double-centraliser" theorem to Schur's lemma. The second retrieves what is called "Zilber's field theorem" in sources such as [Borovik and Nesin 1994]. The third is a variation coming from Nesin's work and isolated by Poizat.

Corollary 1 (Schur–Zilber, one-sided form). *Work in a finite-dimensional theory. Let V be a definable, connected, abelian group and $S \leq \mathrm{DefEnd}(V)$ be an invariant, unbounded ring of definable endomorphisms. Suppose that V is irreducible as an S-module. Then $C_{\mathrm{DefEnd}(V)}(S)$ is a definable skew-field.*

Corollary 1 is, however, not equivalent to our main result, which also covers the case of unbounded T and infinite S.

Corollary 2 (see [Deloro 2016, Théorème IV.1]). *Work in a finite-dimensional theory. Let V be a definable, connected, abelian group and $G \leq \mathrm{DefAut}(V)$ be a definable group such that V is irreducible as a G-module* ***and*** *$C_{\mathrm{DefEnd}(V)}(G)$ is infinite. Then $T = C_{\mathrm{DefEnd}(V)}(G)$ is a definable skew-field (so the action of G is linear).*

Corollary 2 (or a minor variation) unifies and should replace various results such as [Zilber 1984, Lemma 2; Loveys and Wagner 1993, Theorem 4; Nesin 1989a, Lemma 12; Macpherson and Pillay 1995, Theorem 1.2(b); Deloro 2016, Théorème IV.1; Peterzil et al. 2000, Theorem 2.6; Macpherson et al. 2000, Proposition 4.1]. However, there are no claims on finite generation.

Corollary 3 (after Nesin and Poizat). *Work in a finite-dimensional theory. Let V be a definable, connected, abelian group and $R \leq \mathrm{DefEnd}(V)$ be an invariant, unbounded, commutative ring of definable endomorphisms. Suppose there is an invariant group $G \leq \mathrm{DefAut}(V)$ such that*

- *V is irreducible as a G-module;*
- *G normalises R;*
- *G is connected.*

Then there is a definable skew-field $\mathbb{K}$ such that $V \in \mathbb{K}\text{-}\mathbf{Vect}_{<\aleph_0}$; moreover, $R \hookrightarrow \mathbb{K}\,\mathrm{Id}_V$ and $G \hookrightarrow \mathrm{GL}(V : \mathbb{K}\text{-}\mathbf{Vect})$.

It would be interesting to relax the assumption on commutativity of R. Further generalisations are expected using endogenies instead of endomorphisms [Deloro and Wagner $\geq$ 2024].

2.5. ***Indecomposable generation (and how to avoid it).*** Contrary to widespread belief, the Schur–Zilber lemma has nothing to do with another celebrated result from Boris' early work: the "indecomposability theorem" [Zilber 1977, Theorem 3.3], which by analogy with the algebraic case I prefer to call the *Chevalley–Zilber* *generation lemma* (again with hope that Boris will not mind being in good company). For more on the topic, see [Poizat 2024, §8].

Both results are often presented jointly, which serves neither clarity nor purity of methods. In contrast, the proof given here relies on another phenomenon.

Lemma (field definability; extracted from [Wagner 2020, Proposition 3.6]). *Work in a finite-dimensional theory. Let* $\mathbb{K}$ *be an invariant skew-field such that*

- *there is an upper bound on dimensions of type-definable subsets of* $\mathbb{K}$;
- $\mathbb{K}$ *contains an invariant*, *unbounded subset.*

Then $\mathbb{K}$ *is definable.*

The first clause is satisfied if there is a definable $\mathbb{K}$-vector space of finite $\mathbb{K}$-linear dimension.

3. The proofs

The corollaries are derived in Section 3.5. Let V, S, T be as in the theorem. The proof is a series of claims arranged in propositions.

Proof of Zilber's skew-field lemma. It is convenient to let T act from the right and treat V as an (S, T)-bimodule.

Proposition. (i) *T is a domain acting by surjections with finite kernels*; *for $t \in T \setminus \{0\}$ one has $Vt = V$.*

This will later be reinforced in (x).

Proof. (i) Let $t \in T \setminus \{0\}$. Then $0 < Vt$ is S-invariant, definable, and connected; by S-irreducibility $Vt = V$, so t is onto. In particular, T is a domain. Finally, $\dim \ker t = \dim V - \dim Vt = 0$, so $\ker t$ is finite. □

The global behaviour is difficult to control, so we go down to a more "local" scale with a suitable notion of lines.

3.1. *Lines.*

Notation. Let $\delta = \min\{\dim sV : s \in S \setminus \{0\}\}$ and $\Lambda = \{sV : \dim sV = \delta\}$ be the set of *lines*.

Proposition. (ii) *Every line is T-invariant.*

(iii) *If $L \in \Lambda$ and $s \in S$ are such that $sL \neq 0$, then $sL \in \Lambda$*; *in particular*, *$L \cap \ker s$ is finite.*

(iv) *V is a finite sum of lines.*

(v) *S is transitive on Λ.*

Items (iii) and (iv) will later be reinforced in (vi) and (ix), respectively.

Proof. (ii) This is obvious since S and T commute.

(iii) Say $L = s_0 V$. If $sL \neq 0$, then $0 < \dim sL = \dim((ss_0)V) \leq \dim(s_0 V) = \delta$, so by minimality of δ one has $sL \in \Lambda$. This also implies $\dim(L \cap \ker s) = \dim \ker s_{|L} = \dim L - \dim sL = 0$, and $L \cap \ker s$ is finite.

(iv) The subgroup $0 < \sum \Lambda \leq V$ is definable, connected, and S-invariant; by S-irreducibility, it equals V. Since dimension is finite, it is a finite sum.

(v) Let $L_1, L_2 \in \Lambda$, say $L_i = s_i V$. Now as above, $V = \sum_S sL_1 \not\leq \ker s_2$, so there is $s \in S$ such that $s_2 s L_1 \neq 0$. But then $0 < s_2 s L_1 = s_2 s s_1 V \leq s_2 V = L_2$, and equality holds. □

3.2. ***Linearising lines.***

Proposition. (vi) *If $L \in \Lambda$ and $s \in S$ are such that $sL \neq 0$, then $L \cap \ker s = 0$.*

(vii) *T acts by automorphisms on every line.*

The proof is different depending on whether S or T is unbounded.

Proof if T is unbounded. (vi) Suppose $sL \neq 0$; we show $L \cap \ker s = 0$. By (v), S is transitive on Λ, so there is $s' \in S$ with $s'sL = L$. Now $L \cap \ker s \leq L \cap \ker(s's)$, so we may assume that $sL = L$. Recall that $\ker s_{|L} = L \cap \ker s$ is finite by (iii). Considering $s^2_{|L} : L \to L$, which is onto, we inductively find $|\ker s^n_{|L}| = |\ker s_{|L}|^n$, so $K = \sum_{n \in \mathbb{N}} \ker s^n_{|L}$ is either trivial or countably infinite. *Since T is unbounded*, there is $t \in T \setminus \{0\}$ annihilating K. But t has a finite kernel by (i), so $K = 0$, as desired.

(vii) Let $t \in T$. Then $\ker t$ is finite and S-invariant, while S is infinite; so there is $s_0 \in S \setminus \{0\}$ with $s_0(\ker t) = 0$.

Since $s_0 \neq 0$ and $V = \sum \Lambda$ by (iv), there is $L_0 \in \Lambda$ such that $s_0 L_0 \neq 0$. Then $s_0(L_0 \cap \ker t) = 0$ so $L_0 \cap \ker t \leq L_0 \cap \ker s_0$ by (vi).

Now if L is any other line, then there is $s \in S$ with $sL = L_0$ by (v). Therefore $s(L \cap \ker t) \leq L_0 \cap \ker t = 0$, and $L \cap \ker t \leq L \cap \ker s = 0$ by (vi) again.

So $\ker t$ intersects each line trivially. □

Proof if S is unbounded. The strategy is different here and we first prove weakened versions in reverse order.

Weak (vii)′: We first prove that T acts by automorphisms on *some* line. By (iv), $V = \sum \Lambda$ is a finite sum, so there are $L_1, \ldots, L_n$ such that $\bigcap_{i=1}^n \operatorname{Ann}_S(L_i) = 0$. In particular $(S, +) \hookrightarrow \prod_i S/\operatorname{Ann}_S(L_i)$ as abelian groups. *Since S is unbounded*, there exists some line L such that the quotient group $\Sigma = S/\operatorname{Ann}_S(L)$ is unbounded. Let $t \in T \setminus \{0\}$. Then $K = \sum_{n \in \mathbb{N}} \ker t^n_{|L}$ is either trivial or countably infinite. Since Σ is unbounded, there is $\sigma \in \Sigma \setminus \{0\}$ annihilating K, i.e., there is $s \in S$ annihilating K but not L. By (iii) this shows $K = 0$, as desired.

Weak (vi)′: We next prove: *if T acts by automorphisms on L*, then for $s \in S$ with $sL \neq 0$ one has $L \cap \ker s = 0$. Indeed, $L \cap \ker s$ is finite by (iii). Since T is infinite there is $t \in T \setminus \{0\}$ with $(L \cap \ker s)t = 0$, but t induces an automorphism of L. This proves (vi), but only for lines on which T acts by automorphisms.

(vii) and (vi): By (vii)′, let L be a line on which T acts by automorphisms and L' be another line. Then by transitivity (v), there is $s \in S$ with $sL = L'$. Suppose $w \in L' \cap \ker t$. Then there is $v \in L$ with $sv = w$. Now $s(vt) = (sv)t = wt = 0$, so $vt \in L \cap \ker s = 0$. Since T acts by automorphisms on L, (vi)′ implies $v = 0$ and $w = 0$, as desired. □

Since it is unclear at this stage whether every element belongs to a line, we cannot immediately conclude that T acts by automorphisms; this requires writing V as a direct sum.

3.3. *Globalising local geometries.* Instead of *morphism of T-modules*, we simply say *T-covariant* map. We tend to reserve it for definable maps, even implicitly.

Proposition. (viii) *Lines are complemented as T-modules, viz., for $L \in \Lambda$ there is a definable, connected, T-invariant $H \le V$ with $V = L \oplus H$.*

(ix) *V is a finite, **direct** sum of lines.*

(x) *T is a skew-field acting by automorphisms.*

Proof. (viii) Say $L = s_0 V$. Since $V = \sum_S sL$ by (iv) and (v), there is $s \in S$ with $s_0 s L \neq 0$, so $0 < s_0 s L = s_0 s s_0 V \le L$. Let $s_1 = s_0 s$, so that $L = s_1 V = s_1 L$. Then for $v \in V$ there is $\ell \in L$ with $s_1 v = s_1 \ell$; in particular, $v = \ell + (v - \ell)$ with $\ell \in L$ and $v - \ell \in \ker s_1$. Therefore $H = \ker s_1$ is such that $V = L + H$; it also is T-invariant as S and T commute. Now $L \cap H = L \cap \ker s_1 = 0$ by (vi), so actually $V = L \oplus H$. Connectedness of H follows.

Since $V = L \oplus H$ is a direct decomposition as a T-module, the associated projections are T-covariant (viz., morphisms of T-modules).

(ix) As long as possible, we recursively construct lines $L_1, \dots, L_i$ with direct complements H_j (as definable, connected T-modules) satisfying

> for $j \le i$, one has $L_j \le \bigcap_{k<j} H_k$ (viz., each new line is contained in all previous complements).

The construction starts by (viii). Now suppose $L_1, \dots, L_i$ and $H_1, \dots, H_i$ are as claimed. A quick induction yields:

$$V = \left(\bigoplus_{j=1}^{i} L_j \right) \oplus \left(\bigcap_{j=1}^{i} H_j \right).$$

Let q project V onto $\bigcap_{j=1}^{i} H_i$ with kernel $\bigoplus_{j=1}^{i} L_j$. Then q is T-covariant, so $q \in C(T) = S$. If $\bigoplus_{j=1}^{i} L_j < V$, then $q \neq 0$. Now $V = \sum \Lambda$ so there is $L' \in \Lambda$

such that $qL' \neq 0$. Then let $L_{i+1} = qL' \in \Lambda$; it satisfies $L_{i+1} \leq \bigcap_{j=1}^{i} H_j$. Picking a complement as in (viii), we have reached stage $i+1$.

However the process must terminate because $\dim \bigoplus_{j=1}^{i} L_j = \delta \cdot i$ remains bounded by $\dim V$. So at some stage one obtains $\bigoplus_{j=1}^{i} L_j = V$, as wanted.

(x) Say $V = \bigoplus_{i=1}^{n} L_i$ by (ix). Then for $t \in T$ one has $\ker t = \bigoplus_{i=1}^{n}(L_i \cap \ker t) = 0$ by (vii). □

Hence T is a skew-field and $V \in T$-**Vect**, but we still fall short of definability.

3.4. *Definability.* We return to lines. The next result is of a purely auxiliary nature.

Proposition. (xi) *Let $L_1, L_2 \in \Lambda$. If $\sigma : L_1 \simeq L_2$ is definable and T-covariant, then there is an **invertible** $s \in S^\times$ inducing σ.*

Proof. (xi) Using (viii), write $V = L_1 \oplus H_1$ for some $\pi_1 \in S$ with $L_1 = \operatorname{im} \pi_1$ and $H_1 = \ker \pi_1$.

If $L_2 \cap H_1 = 0$, then H_1 is a common direct complement for L_1 and L_2. Glue $\sigma : L_1 \to L_2$ with Id_H to produce a T-covariant map, viz., an element of $C_{\mathrm{DefEnd}(V)}(T) = S$, inducing σ. It clearly is invertible.

If $L_2 \leq H_1$, then the process proving (ix) enables us to take L_1 and L_2 as the first two lines in a direct sum decomposition. Consider the map given on L_1 by σ, on L_2 by σ^{-1}, and on the remaining sum by 1. It is T-covariant and bijective, hence invertible in S; it induces σ.

The case $0 < L_2 \cap H_1 < L_2$ cannot happen, for then $\ker \pi_{1|L_2} \geq L_2 \cap H_1 > 0$ so by definition of lines, $\pi_1 L_2 = 0$ and $L_2 \leq H_1$. □

Notation. For $L \in \Lambda$, by (viii) there exists a definable, connected, T-invariant H such that $V = L \oplus H$.

- Let π_L be the relevant projection and $S_L = \pi_L S \pi_L$.
- Also let $T_L \leq \mathrm{DefEnd}(L)$ be the image of T.

In full rigour, S_L also depends on the complement chosen; we omit it from the notation. This will not create difficulties.

Proposition. (xii) *S_L and T_L are skew-fields contained in* $\mathrm{DefEnd}(L)$.

(xiii) *Inside* $\mathrm{DefEnd}(L)$ *one has $C(S_L) = T_L$ and $C(T_L) = S_L$.*

(xiv) *T is definable.*

Proof. In case T is unbounded, one may directly jump to (xiv).

(xii) Keep in mind that S_L is an additive subgroup of S closed under multiplication but it need not contain 1. (Sometimes S_L is called a *subrng*, for "subring without identity".) However, S_L per se is a ring with identity π_L, as the latter acts on L as Id_L. Moreover, if $\pi_L s \pi_L$ annihilates L, then since it annihilates the chosen direct complement, it is 0 as an endomorphism of V, viz., $\pi_L s \pi_L = 0$ in S. So S_L can be viewed as a subring of $\mathrm{DefEnd}(L)$, and it is exactly the subring of restrictions-corestrictions $\{s_{|L}^{|L} : s \in \mathrm{Stab}_S(L)\}$. (This explains why the complement plays no role in our construction. It is however useful to have both points of view on S_L.)

Let $s \in S_L \setminus \{0\}$. Then $sL = L$, so by (vi) and since S and T commute, it induces some T-covariant automorphism σ of L; by (xi) there is $s' \in S^\times$ inducing σ. Now $\pi_L s'^{-1} \pi_L$ is a two-sided inverse of s in S_L. This proves that S_L is a skew-field. So is T by (x); now the restriction map $T \to T_L$, which is onto by definition, is injective since T acts by automorphisms. Therefore T_L is a skew-field as well.

(xiii) One of them is easy. Let $f : L \to L$ be a definable, T_L-covariant morphism, viz., $f \in C_{\mathrm{DefEnd}(L)}(T_L)$. By definition, f commutes with the action of T. Take any T-invariant direct complement H and set $\hat{f} = 0$ on H. Then $\hat{f} : V \to V$ is T-covariant. Hence $\hat{f} \in C(T) = S$ and $\pi_L \hat{f} \pi_L = f \in S_L$.

Now let $g : L \to L$ be definable and S_L-covariant, viz., $g \in C_{\mathrm{DefEnd}(L)}(S_L)$. We aim at extending g to an S-covariant endomorphism of V.

For $M \in \Lambda$ first use transitivity (v) to choose $s \in S$ with $sL = M$. By (xi) we may assume $s \in S^\times$. Notice that sgs^{-1} leaves M invariant, and let $g_M \in \mathrm{DefEnd}(M)$ be the induced map. We claim that this does not depend on the choice of s. Indeed let s' be another invertible choice, giving rise to g'_M. Then $s^{-1}s'$ induces an element of S_L, so g commutes with it and we find $g_M = g'_M$.

We deduce as follows that $g_M \in C(S_M)$. For if $\eta \in S_M$ then we may assume $\eta \neq 0$ so by (xi) it is induced by an invertible element $h \in S^\times$ normalising M. Then $s' = hs$ is another invertible element taking L to M. By the preceding paragraph, $s' g s'^{-1} = h g_M h^{-1}$ and $sgs^{-1} = g_M$ agree on M, so g_M commutes with η in the ring S_M.

We even prove: if $s \in S$ induces $\sigma : M \simeq N$, then $g_N \sigma = \sigma g_M$. Both are maps from M to N. By (xi), we freely suppose s invertible and pick invertible s_M, s_N inducing $L \simeq M, N$. Then $s'_M = s^{-1} s_N \in S$ takes L to M, so $s'_M g s'^{-1}_M$ agrees with $s_M g s_M^{-1} = g_M$ on M. Thus for arbitrary $m \in M$ we find

$$\begin{aligned} g_N \sigma(m) &= ss^{-1} \cdot s_N g s_N^{-1} \cdot s(m) \\ &= s \cdot (s^{-1} s_N) g (s_N^{-1} s)(m) = s g_M(m) = \sigma g_M(m). \end{aligned}$$

Therefore $g_N \sigma = \sigma g_M$, as claimed.

Finally take a direct sum $V = \bigoplus L_i$ as in (ix) and let $\hat{g}\left(\sum \ell_i\right) = \sum g_{L_i}(\ell_i)$, which is definable, well-defined, and extends g. We want to show $\hat{g} \in C(S)$. Let $s \in S$; also let $s_i = \pi_i s$. It is enough to show that $\hat{g}$ commutes with each s_i, and it is enough to show that they commute on each L_j. We have thus reduced to checking that $\hat{g}$ and $\sigma : L_j \simeq L_i$ induced by an element of S commute. But this is the previous paragraph.

Hence $\hat{g} \in C(S) = T$ and therefore $g = \hat{g}_{|L} \in T_L$.

(xiv) Recall that T is a skew-field by (x). If T is unbounded we directly apply the field definability lemma from Section 2.5 (in that case, (xii) and (xiii) are not necessary). So we suppose that S is unbounded.

We first prove that there is L such that S_L is unbounded. By (ix) take any decomposition $V = \bigoplus_{i=1}^n L_i$ and form projections π_i onto L_i with kernels $\bigoplus_{j \neq i} L_j$. Let $S_{i,j} = \pi_i S \pi_j$, an additive subgroup of S. We contend that one of them is unbounded. Indeed, the additive group homomorphism

$$S \to \prod_{i,j} S_{i,j}, \quad s \mapsto (\pi_i s \pi_j)_{i,j},$$

is injective since $\sum_k \pi_k = 1$. Now if $S_{L,M}$ and $S_{L',M'}$ are defined as the $S_{i,j}$, one easily sees $S_{L,M} \simeq S_{L',M'}$ definably; so all rings S_L are unbounded.

A caveat: because S_L and T_L are mutual centralisers only in $\mathrm{DefEnd}(L)$ and not in $\mathrm{End}(L)$, the following paragraph cannot be made more trivial.

Therefore S_L is an unbounded skew-field by (xii). By field definability of Section 2.5, S_L is definable; now $\dim S_L > 0$ and $\dim L$ is finite, so $L \in S_L\text{-}\mathbf{Vect}_{<\aleph_0}$. In particular, all S_L-endomorphisms of L are definable, so by (xiii) one has $T_L = \mathrm{End}(L : S_L\text{-}\mathbf{Vect})$. This is a skew-field by (xii), so the *linear* dimension over S_L is 1 and $T \simeq T_L \simeq S_L^{\mathrm{op}}$ is unbounded as well. □

By field definability, the skew-field T is definable and infinite, so $\dim T > 0$; now $\dim V$ is finite so $V \in T\text{-}\mathbf{Vect}_{<\aleph_0}$. Finally $S = C(T) = \mathrm{End}(V : T\text{-}\mathbf{Vect})$. Lines in our sense now coincide with 1-dimensional T-subspaces of V. This completes the proof of Zilber's skew-field lemma. □

3.5. ***Proofs of corollaries.*** We repeat the statements already given in Section 2.4.

Corollary 1 (Schur–Zilber, one-sided form). *Work in a finite-dimensional theory. Let V be a definable, connected, abelian group and $S \leq \mathrm{DefEnd}(V)$ be an invariant, unbounded ring of definable endomorphisms. Suppose that V is irreducible as an S-module. Then $C_{\mathrm{DefEnd}(V)}(S)$ is a definable skew-field.*

Proof. Let $T = C_{\mathrm{DefEnd}(V)}(S)$. Notice that T acts by surjective endomorphisms, so it is a domain. If it is finite, then it is a definable field. Otherwise we wish to apply our theorem, but it is unclear whether $S = C_{\mathrm{DefEnd}(V)}(T)$. It actually does not matter. Let $\hat{S} = C_{\mathrm{DefEnd}(V)}(T) \geq S$, which is invariant and unbounded. Moreover, $C_{\mathrm{DefEnd}(V)}(\hat{S}) = T$ as a "triple centraliser", and V remains $\hat{S}$-minimal. So we apply the theorem with $(\hat{S}, T)$ and get definability of the skew-field $C_{\mathrm{DefEnd}(V)}(\hat{S}) = T$. □

Corollary 2. *Work in a finite-dimensional theory. Let V be a definable, connected, abelian group and $G \leq \mathrm{DefAut}(V)$ be a definable group such that V is irreducible* *as a G-module* **and** $C_{\mathrm{DefEnd}(V)}(G)$ *is infinite. Then* $T = C_{\mathrm{DefEnd}(V)}(G)$ *is a definable skew-field* (*so the action of* G *is linear*).

Proof. Let $T = C_{\mathrm{DefEnd}(V)}(G)$ and $S = C_{\mathrm{DefEnd}(V)}(T) \supseteq G$. Apply the theorem. □

Corollary 3 (after Nesin and Poizat). *Work in a finite-dimensional theory. Let* V *be a definable*, *connected*, *abelian group and* $R \leq \mathrm{DefEnd}(V)$ *be an invariant*, *unbounded*, *commutative ring of definable endomorphisms. Suppose there is an invariant group* $G \leq \mathrm{DefAut}(V)$ *such that*

- V *is irreducible as a* G*-module*;
- G *normalises* R;
- G *is connected.*

Then there is a definable skew-field $\mathbb{K}$ *such that* $V \in \mathbb{K}$-$\mathbf{Vect}_{<\aleph_0}$; *moreover*, $R \hookrightarrow \mathbb{K}\,\mathrm{Id}_V$ *and* $G \hookrightarrow \mathrm{GL}(V : \mathbb{K}\text{-}\mathbf{Vect})$.

Proof. Let V, R, G be as in the statement. The proof follows that of [Poizat 1987, Théorème 3.8] closely. Let $W \leq V$ be R-irreducible, viz., minimal as a definable, connected, R-submodule; this exists by the DCC on definable, connected subgroups. Let $\mathfrak{p} = \mathrm{Ann}_R(W)$, a relatively definable ideal of R.

For $g \in G$, the definable, connected subgroup $gW \leq V$ is R-invariant, and hence an R-submodule. Clearly $\mathrm{Ann}_R(gW) = g\mathfrak{p}g^{-1}$. Moreover, $R/\mathfrak{p} \simeq R/(g\mathfrak{p}g^{-1})$.

Now, by G-irreducibility, $V = \sum_G gW$. So there are $g_1, \ldots, g_n \in G$ such that $V = \sum_{i=1}^n g_i W$. In particular, $\bigcap_{i=1}^n \mathrm{Ann}_R(g_i W) = 0$, and $R \hookrightarrow \prod R/(g_i \mathfrak{p} g_i^{-1})$. We just saw that all terms have the same cardinality. They are therefore unbounded.

Hence, the unbounded, commutative ring $R/\mathfrak{p}$ acts faithfully on the $R/\mathfrak{p}$-irreducible module W. Notice that $R/\mathfrak{p} \leq C_{\mathrm{DefEnd}(W)}(R/\mathfrak{p})$. By the theorem, the action of $R/\mathfrak{p}$ on W is linearisable, and $R/\mathfrak{p}$ acts by scalars. The problem is to make this linear structure global without losing the action of G. But we know that $\mathfrak{p}$ is a prime ideal of R.

Now consider the set of prime ideals $P = \{h\mathfrak{p}h^{-1} : h \in G\}$. Suppose $\mathfrak{p}_1, \ldots, \mathfrak{p}_k \in P$ are distinct, say $\mathfrak{p}_i = h_i \mathfrak{p} h_i^{-1}$. By prime avoidance, there are elements $r_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$. Then taking products, there are elements $r_i' \in \bigcap_{j \neq i} \mathfrak{p}_j \setminus \mathfrak{p}_i$. These are used to show that the sum $\sum_{i=1}^k h_i W$ is direct. In particular, $k \leq \dim V$ and P is finite.

Since G is connected and transitive on the finite set P, the latter is a singleton, namely $P = \{\mathfrak{p}\}$. But by faithfulness one had $\bigcap P = 0$, so $\mathfrak{p} = 0$.

Now let $r \in R \setminus \{0\}$. Then $r \notin \mathfrak{p}$ acts on W as a nonzero scalar, so $W \leq \mathrm{im}\, r$. Since r was arbitrary, for any $g \in G$, one has $gW \leq \mathrm{im}\, r$. Summing, $\mathrm{im}\, r = V$; this implies that $\ker r$ is finite. Then $K = \sum_{n \in \mathbb{N}} \ker r^n$ is either trivial or countably infinite. But by commutativity, it is R-invariant. Since R is unbounded, there is $r_0 \in R \setminus \{0\}$ annihilating K. Since r_0 has a finite kernel in V, we see $K = 0$. Thus the domain R acts by automorphisms on V.

Hence $\mathbb{F} = \mathrm{Frac}(R)$ is naturally a subring of $\mathrm{DefEnd}(V)$. By field definability, it is definable. Now G normalises $\mathbb{F}$ and centralises it [Wagner 2020, §3.3]. In particular, G centralises R. Therefore, $S = C_{\mathrm{DefEnd}(V)}(R)$, which contains R by commutativity, also contains G. It follows that V is S-irreducible and we apply the theorem globally to conclude. □

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