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The Zilber–Pink conjecture pertains to the "finiteness of unlikely intersections" and falls within the realms of logic, algebraic, and arithmetic geometry. Smooth parametrization involves dividing mathematical objects into simple pieces and then representing each piece parametrically while maintaining control over high-order derivatives. Originally, such parametrizations emerged and were predominantly utilized in applications of real algebraic geometry in smooth dynamics.

The paper comprises two parts. The first part provides informal insights into certain basic results and observations in the field, aimed at elucidating the recent convergence of the seemingly disparate topics mentioned above. The second part offers a retrospective account spanning from 1964 to 1974. During that period, Boris and I studied at the same places, initially in Tashkent and later in Novosibirsk Akademgorodok.

1. Introduction

The author first encountered Boris Zilber at the 110th Tashkent Physics-Mathematics School in 1964. From then until 1974, we shared the same academic journey, studying mathematics in Tashkent, and later in Novosibirsk Akademgorodok. While we engaged in numerous discussions about mathematics, our paths diverged in terms of specialization: Boris delved into mathematical logic and model theory, while I pursued analysis and differential topology. Initially, the gap seemed immense. However, mathematics is a unified discipline! It is one! After many years, my favorite topic, *smooth parametrization*, emerged as an important tool in the recent remarkable progress in the Zilber–Pink conjecture.

In Sections 2 to 4 below, I attempt to explain, in a very informal manner, the connections between these seemingly distinct topics. I am grateful to have received insights from some of the most active participants in the modern research towards the Zilber–Pink conjecture. Their explanations were indispensable to me. I hope that my brief presentation below can be of assistance to some readers.

Finally, in Section 5, I share some recollections from the Tashkent and Akademgorodok period, from 1964 to 1974, which Boris and I experienced together.

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2. The Zilber–Pink conjecture, and how one can prove it

The Zilber–Pink conjecture ([7; 20; 28], see also [17; 27]) concerns "unlikely intersections". The intersection of two algebraic subvarieties in a variety V is deemed unlikely if its dimension is larger than expected. The conjecture asserts that under some conditions the sets of unlikely intersections are finite. The conjecture offers a uniform framework for various classical problems in Algebraic and Arithmetic Geometry, along with other significant consequences. Recently, there has been dramatic progress in several classical problems, directly related to the Zilber–Pink conjecture: [2; 4; 19; 21] is a very small sample.

Various forms of "smooth parametrization" have played an important role in this progress. Before delving into this topic in Section 4, let's provide a highly informal and intuitive overview of the Pila–Zannier approach that has recently enabled the proof of some very important specific versions of the Zilber–Pink conjecture.

In fact, in the applications of the Pila–Zannier strategy to special point or unlikely intersection problems, the points v which Condition A below concerns, live in a certain preimage $\tilde{V}$ of the algebraic V, rather than in V itself.

In many cases a "height" can be associated to the objects $v \in \tilde{V}$ we aim to count in Zilber–Pink. For a rational number $r = p/q$ the height $H(r)$ is defined as $\max\{|p|, |q|\}$. Similarly for torsion points on pseudo-abelian varieties, and so on. Suppose the number of $v \in \tilde{V}$ with $H(v) \leq H$ is finite for each $H < \infty$, and let $N(\tilde{V}, H)$ denote the cardinality of the set $v \in \tilde{V}$, with $H(v) \leq H$. We then assume the following Condition A: *For a transcendental $\tilde{V}$ and for each $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that*

$$N(\tilde{V}, H) \leq c(\epsilon) H^{\epsilon}, \quad H > 0. \tag{2.1}$$

Results of this nature are now available for counting rational points on transcendental varieties, and in many other cases, starting with the fundamental works of Bombieri and Pila [6] and Pila and Wilkie [18]. Smooth parametrizations appeared, in the context of Diophantine geometry, essentially, in [6]. We discuss them, in somewhat more detail, in Section 4.

Now let us make an additional assumption, called Condition B: *There exist $\epsilon_0 > 0$ and $C > 0$ such that for any $H > 0$ if there are v of height H then, in fact,*

$$N(\tilde{V}, H) \geq C H^{\epsilon_0}. \tag{2.2}$$

In some important cases this second assumption is also satisfied (for instance, due to the Galois group action on the v's; see [4; 21]). Now, if Conditions A and B are satisfied, we get the required finiteness. Indeed, fix $\epsilon < \epsilon_0$ in (2.1). If there exist v's with an arbitrarily big height H, we get a contradiction with the asymptotic bound, for $H \to \infty$, provided by (2.2). We conclude that the height of v is bounded, and hence, so is the total number of v's. This completes the sketch of the Pila–Zannier approach to the Zilber–Pink conjecture.

3. Bombieri–Pila and Pila–Wilkie

In this section we provide a very informal overview of some results and approaches presented in the foundational papers by Bombieri–Pila [6] and Pila–Wilkie [18]. These papers establish, among other things, Condition A for counting rational points on certain transcendental analytic varieties. In [6], the focus is on curves, while [18] extends the results to varieties of higher dimension, definable in a certain o-minimal structure.

The approach involves considering rational points v of a given height H on the graph Γ of a C^k-function Ψ with explicitly bounded derivatives up to order k. The objective is to demonstrate that all these rational points lie on a “small” number of algebraic hypersurfaces W of a certain degree d, which depends on the dimension, k, and H. Later, a form of Bézout's theorem, if available, is utilized to bound the intersections $\Gamma \cap W$, which contain our rational points v.

The key step in this approach is to derive an upper bound on certain Vandermonde-type determinants *VdM*, whose vanishing indicates that the points v lie on an algebraic hypersurface W of degree d. Here the k-smoothness of Ψ and Γ comes into play: the entries of the *VdM* are represented via Taylor expansion, leading to significant cancellations, and ultimately, to the required upper bound.

On the other hand, since the entries of the *VdM* are rational points v of the given height H, the determinants *VdM* are themselves rational numbers with the height, explicitly bounded by a certain D, which depends on H, the number of points, and the dimensions. Therefore, if we can show that $|VdM| < \frac{1}{D}$, we conclude that, indeed, $VdM = 0$, implying that our points lie on an algebraic hypersurface W of degree d. Orchestrating the interrelations between H, k, d, and other parameters is a highly nontrivial task, but ultimately successful.

This concludes the process of counting rational points of a given height H on the graph Γ of a C^k-function Ψ with explicitly bounded derivatives up to order k.

The paper [16] was useful to the author in better understanding (especially in several variables) this part of the approach of [6].

To extend the result from a smooth piece Γ to counting rational points on a transcendental analytic variety V, it becomes necessary to cover V with the graphs Γ_j of C^k-functions Ψ_j, with explicitly bounded derivatives up to order k. Such a covering is what we refer to as a *smooth parametrization*. The existence of such smooth parametrizations for V — a bounded semialgebraic set — was demonstrated in [22] and [14]. While this result sufficed for applications in dynamics, for which it was initially intended, it needed to be extended to analytic varieties for applications in counting rational points.

This extension of smooth parametrizations to sets V definable in a certain o-minimal structure was achieved in [18], together with proving condition A for such V: for each $\epsilon > 0$,

$$N(V^{\mathrm{tr}}, H) \le c(\epsilon) H^{\epsilon}, \quad H > 0,$$

where V^{tr} denotes the "transcendental part" of V. Moreover, the very important *Wilkie conjecture* was posed in [18]: if V is definable in the o-minimal structure, generated by the exponential function, then, in fact, $N(V^{\mathrm{tr}}, H)$ is bounded by a polynomial in $\log H$.

Some special cases of the Wilkie conjecture were settled before a restricted case of this conjecture was proved in [1]. Finally, the full conjecture was confirmed in [5]. New important developments in *o*-minimal structures and in smooth parametrizations were achieved in [5], a paper that also offers an excellent introduction to smooth parametrizations.

4. Smooth parametrizations

In this section we discuss smooth parametrizations in somewhat more detail. We include a short and informal discussion of the striking recent work [2], where a powerful new class of analytic parametrizations was defined.

"Parametrization" is a change of variables that simplifies the understanding of a mathematical structure under investigation. The most important example in the realm of algebraic and analytic geometry is provided by the resolution of singularities, in its various versions. In many problems of dynamics, analysis, Diophantine and computational geometry it is crucial to maintain control over high-order derivatives while performing a change of variables. Parametrizations of this type are referred to as "smooth parametrizations".

An illustrative example is provided by the C^k*-parametrization* of a semi-algebraic set A. This can be seen as a high-order quantitative version of the well-known result on the existence of a triangulation of such sets A, with the number of simplices bounded in terms of the combinatorial data (the *degree*) of A. In a C^k-version we additionally require that each simplex S_j in the triangulation be an image of the standard simplex Δ, under the parametrization mapping Ψ_j, with all the derivatives of Ψ_j up to order k uniformly bounded.

To state the C^k-parametrization theorem of [14; 23; 22] more precisely, let's recall the definition of semi-algebraic sets. Semi-algebraic sets in $\mathbb{R}^n$ are defined by a finite number of real polynomial equations and inequalities, plus set-theoretic operations.

Given a semi-algebraic set $A \subset \mathbb{R}^n$, the diagram $D(A)$ of the set A comprises the "discrete" data of A — the ambient dimension n, the degrees and the number of the equations and inequalities, and the set-theoretic formula defining A. Hence, $D(A)$ does not depend on specific values of the coefficients of the polynomials involved.

Theorem 4.1. *For any natural k and for any compact semi-algebraic set A inside the cube I^n in $\mathbb{R}^n$, there exists a C^k-parametrization of A, with the number $N(A, k)$ of the C^k-charts Ψ_j, depending only on k and on the diagram $D(A)$ of A.*

The bound on $N(A, k)$ obtained in [14; 23; 22; 8] was explicit but high (initially doubly exponential in k). See also [3]. While this sufficed for the intended applications in dynamics (the "entropy conjecture" for C^∞ maps), it soon became apparent that controlling questions like the semicontinuity modulus of the entropy required polynomial growth of $N(A, k)$ in k. This remained an open problem for a long time, along with some dynamical consequences.

To circumvent these difficulties, *analytic parametrizations* were introduced in [24]. Here, we require the above parametrization mappings Ψ_j to be real analytic, extendible to a complex neighborhood of Δ_j of a controlled size, and explicitly bounded there. This worked in dimensions 1 (and also 2, for diffeomorphisms), but faced challenges in higher dimensions. The primary issue was that typically, an infinite number of analytic charts Ψ_j was required to cover A, because of the hyperbolic geometry of the problem. This direction was further developed in [24; 25; 11; 12; 13], but the finiteness problem remained unsolved.

Let us mention also [15], where some initial steps towards applications of smooth parametrization in computational geometry were provided, and [26], which gave an overview of different types of smooth parametrization and their possible applications (up to 2015).

As for newer advances, let us mention, besides [2], a very recent development by D. Burguet [9] of smooth parametrization techniques for dynamics of curves. It led to the solution of long-standing open problems in smooth dynamics.

Finally, in [2] a new type of analytic parametrization was introduced and termed *complex cellular structures*. The key distinction between complex cellular structures and analytic parametrizations is that the domain of the parametrization mappings Ψ_j (in complex dimension one) is either the unit disk, as before, *or an annulus with a prescribed ratio between radii*. In higher dimensions the domains of the parametrization mappings Ψ_j are constructed inductively, combining the two one-dimensional models. The construction and proofs heavily rely on complex hyperbolic geometry.

Complex cellular structures not only restored the finiteness of parametrizations but achieved much more. In particular, a polynomial bound on the grows of $N(A, k)$ with k in Theorem 4.1 was established, thus proving several longstanding conjectures in smooth dynamics (in combination with [10]). Complex cellular structures provide significant advances also in Diophantine geometry.

Anticipated further progress in these areas is highly promising, and likely to address also open questions regarding various types of smooth and analytic parametrizations, including those raised in [24; 25; 11; 12; 13].

5. The old stories

I first met Boris at the 110th Tashkent Physics-Mathematics School in 1964, if I remember correctly. This school was a remarkable place to learn, to hope, and to dream. Mathematics, in the form of problems to solve, books, lectures, and discussions, was omnipresent. One of our schoolmates once proposed a solution to the Fermat problem ($x^k + y^k = z^k$), and I (like many others, I believe) could not sleep until finding a flaw in the colleague's arguments. Then there were mathematical Olympiads, starting with the school level, then advancing to the city level, and finally reaching the All-Siberian Olympiad in 1965, held at Novosibirsk State University in the famous Akademgorodok near Novosibirsk. Both Boris and I were among the winners of the lower level Olympiads and were invited to the All-Siberian Olympiad.

However, before we arrived (in August 1965), an unusual incident occurred. On the first of May 1965, as on any other May Day, there was a mass demonstration organized by the authorities at the central square of Tashkent, the capital of Uzbekistan, then a part of the USSR. All the glory and power of Uzbekistan's authorities were showcased at the central podium. We, at our 110th school, were compelled to participate in this mass demonstration. As per tradition, when the columns of participants passed near the central podium, the loudspeakers usually announced congratulations and greetings to the Communist Party, the people of the Soviet Union, or other similarly grand entities. Sometimes, however, the congratulations were more specific, such as to the workers of a particular industrial plant currently passing near the podium. And as we passed, a miracle occurred: the loudspeakers congratulated the winners of the preliminary tour of the All-Siberian Olympiad, explicitly mentioning our humble names!

We, in our small group, were elated and proud, but this was not the end of the story. Comrade Rashidov, the first secretary of the Uzbek Communist Party, who was present at the central podium, immediately noticed that among the six explicitly mentioned names of the winners, there was no clear Uzbek name! Perhaps one of our good friends had a partial Uzbek heritage. However, even this winner, upon investigation, turned out to be only half Uzbek and half Tatar. Comrade Rashidov promptly demanded correction of this egregious error. The next day, as usual, the central Tashkent Russian-language newspaper *Pravda Vostoka* (something like “The Truth of the East”) published a detailed report on the May Day 1965 mass demonstration in Tashkent. Included in the newspaper were the names of the winners of the preliminary tour of the All-Siberian Olympiad: six distinctly Uzbek names that I was hearing for the first time. Our small group was a little apprehensive — were we still to go to the All-Siberian Olympiad? But no specific instructions to the contrary followed, so we decided to proceed as if everything were in order.

Comrade Rashidov was, of course, not the first to correct, in line with Party directives, minor personal matters. There was a similar incident involving Stalin. Once, he was quite displeased with certain verbal statements made by Lenin’s widow, Nadezhda Krupskaya. Stalin ordered his subordinates: “Tell this fool that if she does not cease, we will find another widow for Comrade Lenin.” But you see, by 1965, Stalin’s era had firmly passed! It was our original group that eventually made it to the All-Siberian Olympiad.

In total, six of us were invited, all from the 110th Tashkent school. We embarked on the three-day train journey from Tashkent to Novosibirsk, accompanied by our math teacher, Tamara Vladimirovna Reshetnikova.

The three weeks at the Summer School in Akademgorodok, which included the final stage of the All-Siberian Olympiad, were truly exhilarating! Both Boris and I were among the winners of the final stage, granting us an opportunity to enroll in Novosibirsk Physics-Mathematics School. I decided to seize this exciting opportunity, while Boris opted to return to the Tashkent 110th school. However, a year later, he returned to Akademgorodok to participate in the entrance examinations to Novosibirsk State University. I was also there; despite finishing with top grades at the Novosibirsk Physics-Mathematics School, I gained no advantage at the entrance examinations. A fair rule indeed! It was challenging, but we both succeeded, becoming first-year students at Novosibirsk State University.

I won’t delve into our student years here. While it was an exhilarating experience for us, from an outside perspective, things were probably quite ordinary. However, as we approached completing our M.A. theses and especially entering the Ph.D. study, we found ourselves in an entirely new reality.

Now, I am compelled to recount a sordid tale of antisemitic persecution in Akademgorodok, beginning in 1968 and culminating in 1971, the year of our graduation. I cherished my life in Akademgorodok and am grateful to the kind individuals (some of whose names I will mention below) who assisted Boris and me, as well as many others, in navigating through those difficult times. However, omitting this part of our lives would be impossible; it was crucially significant for me, and likely for Boris as well.

From 1965, when I arrived there, until 1968, I observed no signs of antisemitism in Akademgorodok. Perhaps it existed among the higher social echelons, but not among the students. The entrance examinations to Novosibirsk State University were utterly fair! While I cannot provide documented evidence, I believe that around 30 percent of the new mathematics students in 1966 were Jews from various parts of the USSR. They knew that in Novosibirsk, there was a fair chance! However, in 1968 and 1969, the situation underwent a dramatic shift. Some of our Russian student acquaintances now served on the new entrance examination committees, intentionally created to hinder the chances of Jewish applicants. Occasionally, they confided in us, maintaining the open traditions of our old friendship, about what transpired during these entrance examinations. They recounted the now well-known tales of exceptionally challenging mathematical problems posed to Jewish candidates, among other tactics.

We were also informed that the authorities' decision was to make Akademgorodok "judenrein" — free of Jews. I am uncertain whether this German term was used by the Akademgorodok Party Committee, but this is what was communicated to us. While all this was disconcerting, it did not directly affect us; we were veterans of 3–4 years, not the unfortunate new entrants.

However, in the spring of 1971, the year of our graduation, our turn arrived. Suddenly (for us, as we had not taken the earlier warning signs seriously), two-thirds of our top Jewish graduates received insufficient grades in their final exams. They could no longer aspire to continue their doctoral studies at Novosibirsk State University, or anywhere in Akademgorodok. This was a devastating blow! Both Boris and I weathered this challenging experience successfully (mostly due to the efforts of our mentors)! We could carry on! And at that moment, I still did not fully comprehend what was happening! This purge expelled our best and closest friends, and I merely participated in their farewell graduation celebrations, still hopeful for a bright future.

Allow me to digress briefly about myself — it's a rather amusing anecdote! Thanks to the vigorous efforts of my advisor, Vladimir Ivanovich Kuzminov, I was to get a starting research position (as a so-called "stager") at the Institute of Mathematics. This position was deemed secure: theoretically it could withstand even mediocre grades in the final exams. However, I harbored no illusions — I was certain they could find fault even with the stager position. Indeed, in April 1971, I was promptly selected for immediate military service, a fate usually reserved before 1971 for relatively weak graduates. This time, the list of potential servicemen included 13 highly accomplished graduates, among them 10 Jews, including myself (but not Boris), and 3 Russians. The absence of Boris's name from this glorious list may be understood from what is explained below: the main target was not him but his advisor Taitslin.

Naturally, my stager position at the Institute of Mathematics was revoked. We, the new servicemen, were slated to serve in the Moscow rocket defense. If I indeed entered this service, I could not entertain the notion of emigrating to Israel for at least 20 years, due to the secrecy restrictions. Moreover, even *perestroika*, as we now know, would not have provided much respite: mass emigration to Israel commenced only in 1991, precisely 20 years after 1971.

Eventually, after some introspection, I wholeheartedly accepted this shift in my fate. I commenced a series of farewell gatherings. Surprisingly, I quickly discovered that this was a far simpler existence than pursuing mathematics or preparing for Ph.D. entrance exams. Particularly since we, the Jewish candidates, were uncertain where we might face persecution: in mathematics exams, in Communist philosophy, or elsewhere.

Thankfully, Marshal Grechko, the defense minister of the USSR, struck out all the Jewish names from the Novosibirsk list of potential servicemen. Marshal Grechko surely had to personally intervene in this minor issue solely due to the global significance of the Moscow rocket defense.

Now I returned to my tribulations, and Boris and I found ourselves facing the oral Ph.D. entrance exams. The first was in mathematics. Remarkably, the math exams were conducted rather transparently: all the Jews who had survived the purge at the graduation exams (around 70 such Jews, as I recall, from different faculties of Novosibirsk State University) were assigned to a single examination committee. Its head was Academician Yanenko, and after the first day of exams, we all knew precisely what was happening. Typically, Yanenko would interrupt the examinee after three minutes, declaring, “No, this is not it!” If the examinee persisted, Yanenko would repeat this scenario more frequently and forcefully until, in 8–10 minutes, he rendered the verdict: “You may know something, but at best to the level of 3.” A grade 3 in mathematics did not qualify for doctoral studies.

This scheme operated flawlessly. As far as I am aware, only three Jews out of the 70 managed to breach this absolute defense: Boris, myself, and Grisha Soifer. My success story was brief and straightforward: during my examination, at the fourth minute, Academician Yanenko abruptly departed — perhaps for a place even academicians have to reach on foot. The rest of the committee promptly awarded me the highest grade, 5, and then, with great insistence, urged me to leave the room. I complied most happily. I am uncertain how Grisha Soifer achieved it! As for Boris, it seems his admission was a move in a campaign against his unofficial advisor, Michael Abramovich Taitslin. Well before our doctoral study entrance exams described above, Taitslin had apparently been warned that he had more than enough Jewish students and could not accept any more. Now, for Boris, the pivotal events occurred on the last possible day to apply to the Ph.D. entrance exams. Among the required application documents was a written agreement from the prospective advisor. In Boris’s case, his official advisor was Yu. L. Ershov. Ershov was expected to agree, but on this final day he let Boris know that he could not serve as his advisor due to several significant administrative and scientific reasons. Boris's only hope of continuing his Ph.D. studies in Akademgorodok was if Taitslin agreed to be his advisor in the remaining few hours. And he did agree, despite the warnings. Boris passed Yanenko's scrutiny without difficulty, but as a result of his generosity in taking on Boris, Taitslin was expelled from the Institute of Mathematics and later from Akademgorodok altogether.

The remaining Ph.D. entrance exams, including Communist philosophy, proceeded smoothly. And now, after all these trials, there began a joyous season for me and, I presume, for Boris. Three years of unfettered scientific research and study, with scarcely a thought of the entrance battles. This period was profoundly significant for all our future endeavors! I am grateful to the circumstances for this felicitous interlude.

I wish to express my gratitude to another outstanding individual who greatly aided many of us during difficult times: Alexei Andreevich Lyapunov. Sometime during my three-year Ph.D. study, the Novosibirsk Energy Institute, where I was slated to work upon completing my Ph.D., announced that they were no longer interested in my services. Presumably, this was due to the burgeoning phenomenon of Jewish emigration to Israel, and they wished to avoid potential entanglements with me. I was apprehensive that my Ph.D. study might be affected, but nothing of the sort occurred. A few weeks after the announcement from the Novosibirsk Energy Institute, a young and athletic-looking individual knocked on my door. He relayed that he had been tasked by Lyapunov, one of the prominent scientists in Akademgorodok, to contact me. Lyapunov's message to me was that he might wield some influence at the Novosibirsk Energy Institute and, if I were interested, he could exert his best efforts to reinstate my position. I was deeply appreciative but requested him not to intervene. By then, I was already seriously contemplating emigrating to Israel.

I previously mentioned my advisor Vladimir Ivanovich Kuzminov. Allow me to also acknowledge Igor Aleksandrovich Shvedov. During those trying times, they both did what they could to assist their Jewish students. And, of course, once again, I express my gratitude to Alexei Andreevich Lyapunov.

One more pleasant recollection from those halcyon days. One autumn (probably, of 1973) Boris and I both served as group leaders in the obligatory autumn student agricultural service — in our case, this involved harvesting potatoes. My group consistently ranked last in the daily ratings, and I felt dejected by my failure. However, a couple of years later, I was appointed group leader of the combined group of female students from the Tashkent Polytechnical Institute. This time, the obligatory autumn student agricultural service involved picking cotton. Again, my group languished at the bottom of the daily ratings, but this time, I was not disheartened by my failure. As far as I know, Boris missed out on this opportunity to lead students in picking cotton.

In 1974, I left Novosibirsk for Barnaul, and after three months there, I moved to Tashkent. Finally, in 1978, I immigrated to Israel. Boris relocated to Kemerovo, and we only crossed paths again in the late 1990s. Since then, we have been meeting more or less regularly.

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