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Zilber–Pink for raising to the power i

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To Boris Zilber on the occasion of his 75th birthday.

We consider the multivalued raising-to-the-power-i function through the Schanuel–Ax–Zilber lens. We formulate and prove an analogue of the Zilber–Pink conjecture.

1. Introduction

The purpose of this paper is to consider the (multivalued) function $w = z^i$ through the Schanuel–Ax–Zilber lens [Ax 1971; Zilber 2002], and in particular to formulate and prove an analogue of the Zilber–Pink conjecture [Zilber 2002; Bombieri et al. 2007; Pink 2005]. We follow the path taken by Zilber leading to his formulation of the Zilber–Pink conjecture for semiabelian varieties: beginning with an analogue of Schanuel's Conjecture 3.1, we consider a "uniform" version (Conjecture 4.1), and formulate a Zilber–Pink-type statement (Conjecture 4.2) connecting the classical Schanuel conjecture (SC) with the uniform version for z^i. Our Schanuel variant is equivalent to a formulation of Zilber [2003a].

We then prove the Zilber–Pink-type statement, in the more general form in Theorem 1.3 below. The connection with SC is explicated in Sections 2 and 3. Theorem 1.3 is a somewhat exotic variant of the Zilber–Pink conjecture for even powers of $\mathbb{G}_{\mathrm{m}} = \mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathbb{C}^\times$, in which key difficulties disappear thanks to the Gelfond–Schneider theorem.

We take $w = z^i$ to be the predicate $\Gamma \subset \mathbb{G}_{\mathrm{m}}^2$ defined by

$$(z, w) \in \Gamma \iff \exists u \in \mathbb{C}\,[\exp(u) = z \wedge \exp(iu) = w].$$

We let $\Gamma_n = \Gamma^n$ denote the cartesian power of this predicate on $\mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$.

To formulate our theorem, we recall that the Zilber–Pink conjecture (ZP) for subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ can be framed in terms of *optimal subvarieties* for $V \subseteq \mathbb{G}_{\mathrm{m}}^n$; see [Habegger and Pila 2016]. These are subvarieties $W \subseteq V$ which cannot be enlarged inside V without increasing their *defect* (which is the difference between their dimension and the dimension of the smallest *torsion coset* of $\mathbb{G}_{\mathrm{m}}^n$ which contains them). ZP is equivalent to the statement that a subvariety $V \subset \mathbb{G}_{\mathrm{m}}^n$ contains only

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finitely many such optimal subvarieties. Here torsion cosets (which are cosets of subtori by torsion points) are the "special subvarieties" of $\mathbb{G}_{\mathrm{m}}^n$.

In treating $w = z^i$, the appropriate "special subvarieties" are subtori of $\mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ of a special form.

Definition 1.1. A *plu-torus* $T \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ is a subtorus whose lattice of exponent vectors $\Lambda(T) \subset \mathbb{Z}^n \times \mathbb{Z}^n$ is closed under the operation $(q, r) \mapsto (-r, q)$.

If $A \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ is a subvariety which meets Γ_n then there is a smallest plu-torus containing A (see Section 2), denoted $((A))$. We define the *plu-defect* of A to be

$$\delta(A) = \dim((A)) - \dim A.$$

Definition 1.2. Let $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$. A subvariety $A \subset V$ is called *plu-optimal* for V if $A \cap \Gamma_n \neq \varnothing$, and if $A \subset B \subset V$ and $\delta(B) \le \delta(A)$ imply $B = A$.

Theorem 1.3. *Let $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$. Then V contains only finitely many plu-optimal subvarieties.*

To motivate the above definitions, we consider implications of SC for $w = z^i$. Zilber [2003a; 2015] studied Schanuel-type conjectures for raising to powers in algebraically closed fields, and model-theoretic properties of the fields satisfying them. A version of the present result that SC implies a certain uniformity in the corresponding conjecture for raising to the power i is obtained there. Zilber uses a two-sorted setup, and the conjectures are framed using the corresponding logarithms, whereas our statement involves only the relation Γ. However, the two structures are biinterpretable, and the raising-to-the power-i Schanuel conjectures are equivalent (I thank J. Kirby for explaining these points to me). Structures with a predimension with similar shape to that considered here are considered in [Caycedo and Zilber 2014; Zilber 2003b]; see also related structures in the context of "pseudoexponentiation" discussed in [Bays and Kirby 2018].

We first observe that the pair $(z, w) \in \Gamma$ "knows" which branch of log connects them: if $u = \log z$ has the required property, any other u' would need to satisfy

$$u' - u \in 2\pi i\mathbb{Z}, \qquad iu' - iu \in 2\pi i\mathbb{Z}$$

and the intersection of $2\pi i\mathbb{Z}$ and $2\pi\mathbb{Z}$ consists of $\{0\}$ only.

Applying SC to

$$u_1, \ldots, u_n, iu_1, \ldots, iu_n, x_1, \ldots, x_n, y_1, \ldots, y_n,$$

where $\exp(u_i) = x_i$, $\exp(iu_i) = y_i$, and eliminating the u_i, iu_i, gives the following statement, in which $\mathrm{t.d.}(A)$ denotes $\mathrm{tr.deg.}\, \mathbb{Q}(A)/\mathbb{Q}$:

Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{C}^\times$ with $(x_j, y_j) \in \Gamma$, $j = 1, \ldots, n$. Then

$$\mathrm{t.d.}(x_1, \ldots, x_n, y_1, \ldots, y_n) \ge n$$

unless $x_1, \ldots, x_n, y_1, \ldots, y_n$ are multiplicatively dependent.

Indeed, SC asserts that $\text{t.d.}(u_j, iu_j, x_j, y_j : 1 \le j \le n) \ge 2n$ unless u_j, iu_j are linearly dependent over $\mathbb{Q}$. Now $\text{t.d.}(u_j, iu_j : 1 \le j \le n) \le n$, while if the u_j, iu_j are linearly dependent over $\mathbb{Q}$ then the x_j, y_j are multiplicatively dependent: if, say, $\sum_j q_j u_j + i \sum_j r_j u_j = 0$, where $q_j, r_j \in \mathbb{Z}$, not all zero, then we get

$$\prod_j x_j^{q_j} \prod_j y_j^{r_j} = 1.$$

However, multiplicative dependence of x_j, y_j might hold even when $\mathbb{Q}$-linear dependence of u_j, iu_j does not, so that the above statement seems to lose some information. For example, if $u_1 = \log 2$, $u_2 = 2\pi i$, giving

$$x_1 = 2, \quad x_2 = 1, \quad y_1 = 2^i, \quad y_2 = e^{-2\pi}$$

then the above conjecture does not predict $\text{t.d.}(x_1, x_2, y_1, y_2) \ge 2$, but SC does, as does Conjecture 3.1 (or see the provisional version below).

Suppose $\sum_j q_j u_j + i \sum_j r_j u_j = 0$ as above. Then, upon multiplying by i, we find that $-\sum_j r_j u_j + i \sum_j q_j u_j = 0$ and we get a second multiplicative relation

$$\prod_j x_j^{-r_j} \prod_j y_j^{q_j} = 1.$$

The claim is that a pair of such multiplicative relations (which is easily seen to never be dependent) forces the underlying u_j, iu_j to be linearly dependent over $\mathbb{Q}$. Indeed, from the first, we find that

$$\delta = \sum_j q_j u_j + i \sum_j r_j u_j \in 2\pi i \mathbb{Z}$$

but then the second relation implies that

$$i\delta = -\sum_j r_j u_j + i \sum_j q_j u_j \in 2\pi i \mathbb{Z},$$

and so $\delta = 0$.

Thus our "exceptional" point $(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ lies in a subtorus of codimension at least 2 and of rather specific form: a plu-torus. This leads us to the following provisional formulation of Schanuel's conjecture for z^i:

Let $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{C}^\times$ with $(x_j, y_j) \in \Gamma$, $j = 1, \dots, n$. Then

$$\text{t.d.}(x_1, \dots, x_n, y_1, \dots, y_n) \ge n$$

unless there exist integers q_j, r_j, not all zero, such that

$$\prod_j x_j^{q_j} \prod_j y_j^{r_j} = 1 = \prod_j x_j^{-r_j} \prod_j y_j^{q_j}.$$

Thus, SC for z^i leads naturally to the consideration of plu-tori. The corresponding Zilber–Pink analogue arises from considering a uniform version of SC.

In the next section we investigate more fully the notion of plu-tori, and the related cosets (weakly special subvarieties) with their underlying dimension notion. This enables us to give a more refined version of Schanuel's conjecture for z^i in Section 3. The uniform version and corresponding Zilber–Pink-type statement are set out in Section 4, where we arrive at the formulation of Theorem 1.3. The subsequent sections are devoted to proving Theorem 1.3 and related statements. We gather some Ax–Schanuel-type statements in Section 5, and then finally in Sections 6–9 we gather the ingredients required to prove Theorem 1.3, first for $V/\overline{\mathbb{Q}}$ and then in a uniform version for families of subvarieties, from which the general case follows.

It has been my privilege over many years now to have Boris Zilber as a colleague, to discuss mathematics with him, and in particular to hear at first-hand his unique and inspirational approach to mathematical structures. I dedicate this paper to Boris and look forward to many further conversations.

2. Plu-tori

We introduce some dimension notions involving pairs of linear relations. We need this notion in the first instance for pairs $(z, w) \in \mathbb{C}^2$, but we need it also for coordinate functions defining linear spaces.

Let

$$D_n = \{(z_1, \ldots, z_n, iz_1, \ldots, iz_n) : z_1, \ldots, z_n \in \mathbb{C}\} \subset \mathbb{C}^n \times \mathbb{C}^n,$$

and let

$$\exp : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$$

be the coordinatewise exponential map. Then $\Gamma_n = \exp(D_n)$.

Definition 2.1. Let V be a finite-dimensional complex vector space. A finite set $(z_1, w_1), \ldots, (z_n, w_n)$ of elements of V^2 is called *plu-linearly independent* if they do not satisfy any pair of nontrivial $\mathbb{Q}$-linear equations of the form

$$\sum_j q_j z_j + \sum_j r_j w_j = 0, \tag{1}$$

$$-\sum_j r_j z_j + \sum_j q_j w_j = 0. \tag{1^i}$$

Nontrivial means that $\sum (q_j^2 + r_j^2) \neq 0$. If they do satisfy such a "plu-pair" of equations they are called *plu-linearly dependent*.

Let $B=\{(z_1,w_1),\dots,(z_n,w_n)\}$. We say that (z_0,w_0) is plu-linearly dependent on B if there is a plu-pair

$$q_0z_0+\sum_j q_jz_j+\sum_j r_jw_j+r_0w_0=0,$$
$$-r_0z_0-\sum_j r_jz_j+\sum_j q_jw_j+q_0w_0=0,$$

in which q_0 or r_0 is nonzero.

If say $q_1\neq 0$ in the plu-pair (1), (1^i), then as above we may use (1) to eliminate z_1 but not w_1 from (1^i), and we may use (1^i) to eliminate w_1 but not z_1 from (1) to get a new plu-pair of equations

$$\sum_j (r_1q_j-q_1r_j)z_j+\sum_j (q_1q_j+r_1r_j)w_j=0, \qquad (r_1(1)+q_1(1^i))$$
$$\sum_j (-q_1q_j-r_1r_j)z_j+\sum_j (r_1q_j-q_1r_j)w_j=0. \qquad (-q_1(1)+r_1(1^i))$$

We now show that plu-linear dependence leads to a well-defined dimension: the cardinality of a maximal plu-independent subset, which we call a *plu-basis*. For this we of course need the exchange property.

Proposition 2.2. *Let B be as above. If (z_0,w_0) is plu-dependent on B and (z_*,w_*) is plu-dependent on $B\cup\{(z_0,w_0)\}$ then (z_*,w_*) is plu-dependent on B.*

Proof. We can assume that the plu-pair for the dependence of (z_0,w_0) on B has the form

$$z_0+\sum_j q_jz_j+\sum_j r_jw_j=0,\quad -\sum_j r_jz_j+\sum_j q_jw_j+w_0=0.$$

We use these to eliminate z_0, w_0 from the dependence of (z_*,w_*) on $B\cup\{(z_0,w_0)\}$, which remains a plu-pair. □

Proposition 2.3. *Any two plu-bases have the same cardinality.*

Proof. Let B, B' be two maximal plu-linearly independent subsets. If $B=B'$ we are done; otherwise, say $(z_i,w_i)\in B'\setminus B$. By the maximality, (z_i,w_i) is plu-linearly dependent over B. But since B' is plu-linearly independent the plu-pair must have a nonzero coefficient for some $(z_j,w_j)\in B\setminus B'$.

The claim is that B^* with (z_i,w_i) replacing (z_j,w_j) in B is again a maximal plu-linearly independent subset. First, it is plu-linearly independent. Otherwise, we have (z_i,w_i) plu-linearly dependent on $B\setminus\{(z_j,w_j)\}$. But then by Definition 2.1 we would have (z_j,w_j) dependent on $B\setminus\{(z_j,w_j)\}$, a contradiction. But also by Definition 2.1 we see that it "spans".

This shows that $\#B\geq \#B'$. We symmetrically get the other inequality. □

Definition 2.4. A $\mathbb{Q}$-linear subspace $L \subset \mathbb{C}^n \times \mathbb{C}^n$ is called *plu-linear* if the set of $\mathbb{Q}$-linear forms defining it is closed under the operation $(q, r) \mapsto (-r, q)$. A plu-linear $\mathbb{Q}$-subspace is also called a *plu-subspace*.

We observe that plu-linear subspaces have even dimension as linear subspaces. Let L be a plu-subspace. We consider the complex vector space of pairs of complex linear forms

$$\Big(\sum_j c_j z_j, \sum_j d_j w_j\Big), \quad c_j, d_j \in \mathbb{C},$$

as functions on L. If the coordinate functions z_j, w_j are plu-linearly independent as functions on L then there are no equations and $L = \mathbb{C}^n \times \mathbb{C}^n$. Otherwise, we have a basis of some dimension m and then as a $\mathbb{Q}$-subspace we have $\dim L = 2m$.

The intersection of two (or more) plu-subspaces is a plu-subspace. If $A \subset \mathbb{C}^n \times \mathbb{C}^n$ then there is a smallest plu-subspace containing A, denoted $\langle\langle A \rangle\rangle$.

If $A \subset D_n$ then the smallest $\mathbb{Q}$-linear subspace of $\mathbb{C}^n \times \mathbb{C}^n$ containing A is a plu-subspace, because the "conjugate" of a given equation follows from multiplying it through by i.

A plu-subspace of dimension $2m$ (as a $\mathbb{Q}$-linear subspace) intersects D_n in a $\mathbb{Q}$-subspace of D_n of dimension at least dimension m since (as the "conjugate" of any given equation holds automatically) the intersection is equal to the intersection of D_n with a $\mathbb{Q}$-subspace defined by $2n - (n + m) = n - m$ independent linear equations, whence has dimension at least $n + (n + m) - 2n = m$. But it is also at most this dimension, as each such equation (with its "conjugate") eliminates one variable. Thus, the intersection of D_n with a plu-subspace of dimension $2m$ is a $\mathbb{Q}$-subspace of D_n of dimension m.

Definition 2.5. A torus $T \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ is called a *plu-torus* if it is the image under exp of some plu-linear $\mathbb{Q}$-subspace $L \subset \mathbb{C}^n \times \mathbb{C}^n$.

The set of exponent vectors $(q, r) \in \mathbb{Z}^n \times \mathbb{Z}^n$ defining a plu-torus is closed under the operation $(q, r) \mapsto (-r, q)$; this is an equivalent condition to the one in Definition 1.1. Each plu-torus $T = \exp(L)$, where L is a plu-subspace of dimension $2m$, contains the image of $\exp(L \cap D_n)$, which we denote $\Gamma_T = \Gamma_n \cap T$.

The intersection of tori is not in general a torus. However, the intersection of two tori contains among its components a unique torus. And if the two tori are plu-tori so is the torus component of their intersection.

Consider a subvariety (i.e., irreducible) $A \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$. Suppose that $A \cap \Gamma_n \neq \varnothing$ and that $A \subset T$ and $A \subset T'$, where T, T' are plu-tori. Then $A \subset T \cap T'$, and hence is contained in one of its components. These are disjoint from each other. The unique preimage of $(x, y) \in A \cap \Gamma_n$ is in D_n and lies in the intersection of the plu-linear subspaces L, L' corresponding to T, T'. We thus see that A is contained in the unique plu-torus component of the intersection.

Thus for $A \subset \mathbb{G}_m^n \times \mathbb{G}_m^n$ with $A \cap \Gamma_n \neq \varnothing$ there is a unique smallest plu-torus containing A, which we have denoted $((A))$. In particular, for $(x, y) \in \Gamma_n$ there is a smallest plu-torus $((x, y))$ containing (x, y). And plu-tori have even dimension (as complex subvarieties).

Remark 2.6. More generally, if X is a quasiprojective variety (or even more generally a connected open semialgebraic subset of one; see [Pila 2022, Chapter 14]) we can define a *designated collection* on X to be a collection $\mathcal{S}$ of subvarieties (relatively closed and irreducible) of X such that (i) $X \in \mathcal{S}$ and (ii) $\mathcal{S}$ is closed under taking irreducible components of intersections. This is somewhat more general than the notion of "prespecial structure" considered in [Klingler et al. 2018] (we do not insist that special points be Zariski-dense in special subvarieties), and the still more elaborate setting of "distinguished categories" [Barroero and Dill 2021], but still gives a well-defined notion of "smallest special subvariety containing A" for any $A \subset X$. If $\Omega \subset X$ is some complex analytic subset then one can consider a *designated collection on X meeting* Ω to be a collection of subvarieties of X which have nonempty intersection with Ω, such that (i) $X \in \mathcal{S}$ and (ii) if $Y, Z \in \mathcal{S}$ and W is a component of $Y \cap Z$ which meets Ω then $W \in \mathcal{S}$. Then, as above, one has a well-defined "smallest special subvariety" containing A for any $A \subset X$ which has a nonempty intersection with Ω. This notion arises, as here, naturally in considering ZP-type formulations relevant to certain Schanuel-type statements.

We also want the corresponding "weakly special subvarieties". These come from considering pairs of linear equations modulo some suitable constants.

Definition 2.7. Let V be a finite-dimensional complex vector space. A finite set $(z_1, w_1), \ldots, (z_n, w_n)$ of elements of V^2 is called *strictly plu-linearly independent modulo* $\mathbb{C}$ if there is no nontrivial pair of equations

$$\sum_j q_j z_j + \sum_j r_j w_j = c, \quad -\sum_j r_j z_j + \sum_j q_j w_j = ic$$

with $q_j, r_j \in \mathbb{Q}$ (not all zero), and $c \in \mathbb{C}$.

There is a well-defined notion of *strict plu-mod* $\mathbb{C}$ *basis*, the cardinality of a maximal strictly plu-linearly independent modulo $\mathbb{C}$ subset.

Definition 2.8. A linear subvariety $L \subset \mathbb{C}^n \times \mathbb{C}^n$ is called a *strict plu-linear subvariety* if it is defined by linear equations which are closed under the operation

$$\sum_j q_j z_j + \sum_j r_j w_j = c \mapsto -\sum_j r_j z_j + \sum_j q_j w_j = ic.$$

A strict plu-linear subvariety L has even dimension $2m$. It intersects D_n in a subspace of dimension at least m.

The intersection of two strict plu-linear subspaces is a strict plu-linear subspace. Given $A \subset \mathbb{C}^n \times \mathbb{C}^n$, there is a smallest strict plu-linear subvariety containing it, which we denote $\langle\langle A \rangle\rangle_{\mathrm{SPL}}$.

Definition 2.9. A torus coset $T \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ is called a *strict plu-coset* if it is the image under exp of a strict plu-linear subvariety.

Equivalently, a strict plu-coset is a coset T of a torus defined by equations with the property that if $x^q y^r = c$ on T then $x^{-r} y^q = d$ on T for some d with $(c, d) \in \Gamma$.

Consider a subvariety $A \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$. Suppose that $A \cap \Gamma_n \neq \varnothing$ and that $A \subset T$ and $A \subset T'$, where T, T' are strict plu-cosets. Then $A \subset T \cap T'$, and hence is contained in one of its components. These are disjoint from each other. The unique preimage of $(x, y) \in A \cap \Gamma_n$ is in D_n and lies in the intersection of the strict plu-linear subvarieties L, L' corresponding to T, T'. We thus see that A is contained in the unique strict plu-coset component of the intersection.

Thus for $A \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ with $A \cap \Gamma_n \neq \varnothing$ there is a unique smallest strict plu-coset containing A, which we denote

$$((A))_{\mathrm{SPC}}.$$

There is also a weaker notion of "plu-linear dependence modulo $\mathbb{C}$" in which the pair of constants do not need to be related by multiplication by i. There are corresponding "plu-linear subvarieties" and "plu-cosets", their images under exp.

For $A \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ there is a unique smallest plu-coset containing A, denoted

$$((A))_{\mathrm{PC}}.$$

3. Schanuel's conjecture for z^i

We can now state a more precise analogue of Schanuel's conjecture for z^i.

Conjecture 3.1 (Schanuel's conjecture for z^i (z^iSC)). Suppose that $(x_i, y_i) \in \Gamma$, $i = 1, \ldots, n$. Then

$$\mathrm{t.d.}(x_1, \ldots, x_n, y_1, \ldots, y_n) \geq \tfrac{1}{2} \dim ((x_1, \ldots, x_n, y_1, \ldots, y_n)).$$

Here and throughout, "dim" denotes the complex dimension of an algebraic variety. For $n = 1$ the statement reduces to the Gelfond–Schneider theorem and so is true: for if $\mathrm{tr.deg.}(x, y) = 0$ we must have $x^i = y$ with $x, y \in \overline{\mathbb{Q}}^\times$. But this is impossible unless $x = 1$ by Gelfond–Schneider, and since we then also have $y^i = 1/x$ we must have $y = 1$ as well, and then they are skew-multiplicatively dependent and $\dim ((x, y)) = 0$.

Remarks 3.2. (1) As a reduct of complex exponentiation, $(\mathbb{C}; +, \times, 0, 1, \Gamma)$ is conjecturally "tame" [Zilber 2005]; in unpublished work, Wilkie has proved it is quasiminimal. (Quasiminimality of the corresponding structure including predicates for *all* complex powers has recently been proven in [Gallinaro and Kirby 2023].)

As an expansion of the real field the structure $(\mathbb{R}, +, \times, \Gamma)$ is also tame, though "d-minimal" (not o-minimal); see [Miller 2005].

(2) Another approach to formulating SC for z^i is to use an equivalent formulation of SC in terms of $q(z) = \exp(2\pi i z)$ rather than $\exp(z)$.

Observe that if $x_1, \dots, x_n$ are multiplicatively independent algebraic numbers then, under SC, their logarithms $u_1, \dots, u_n$, under any determination, are algebraically independent. Hence these u_j, iu_j are certainly linearly independent over $\mathbb{Q}$, and we get a conjectural analogue of Lindemann's theorem (often called the Lindemann–Weierstrass theorem).

Conjecture 3.3 (z^i-Lindemann–Weierstrass conjecture (z^iLW)). Suppose that the algebraic numbers $x_1, \dots, x_n \in \overline{\mathbb{Q}}^\times$ are multiplicatively independent and that $(x_j, y_j) \in \Gamma$, $j = 1, \dots, n$. Then $y_1, \dots, y_n$ are algebraically independent.

In fact this statement already follows from z^iSC, which would seem much weaker than SC.

Proposition 3.4. *z^iSC implies z^iLW.*

Proof. Assume z^iSC. Suppose $x_1, \dots, x_n$ are algebraic and multiplicatively independent. Then their logarithms (under any determination) are linearly independent over $\mathbb{Q}$. Then, by Baker's theorem [Baker 1975], they are linearly independent over $\overline{\mathbb{Q}}$. Then x_j, y_j are plu-multiplicatively independent and so, by z^iSC, $\text{t.d.}(x_i, y_i) = n$. Thus $y_1, \dots, y_n$ are algebraically independent. □

Remark 3.5. Note that if $(z, w) \in \Gamma$ then also $(w, z^{-1}) \in \Gamma$. Thus the analogue of "algebraic independence of logarithms" for z^i, which we might call z^iAIL, is in fact equivalent to z^iLW: *if $y_1, \dots, y_n$ are algebraic and multiplicatively independent and $(x_j, y_j) \in \Gamma$, $j = 1, \dots, n$, then $x_1, \dots, x_n$ are algebraically independent.* It seems interesting to consider other "z^i analogues" of consequences of SC.

4. Uniform Schanuel conjecture and Zilber–Pink conjecture for z^i

We can rephrase z^iSC as follows:

> *Let $V \subset \mathbb{G}_m^n \times \mathbb{G}_m^n$ be defined over $\overline{\mathbb{Q}}$ and with* $\dim V < n$. *If $(x, y) \in V \cap \Gamma$ then (x, y) are plu-multiplicatively dependent.*

More generally, if $T \subset \mathbb{G}_m^n \times \mathbb{G}_m^n$ is a $2m$-dimensional plu-torus, $V \subset T$ is defined over $\overline{\mathbb{Q}}$ with $\dim V < m$, and $(x, y) \in V \cap \Gamma$, then (x, y) belongs to a proper plu-subtorus of T.

The uniform version, following [Zilber 2002], asserts that, given such T and V, *finitely many* proper plu-subtori of T account for all such (x, y).

Conjecture 4.1 (uniform Schanuel conjecture for z^i (z^iUSC)). Let $T \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ be a plu-torus of dimension $2m$ and $V \subset T$ an algebraic subvariety, defined over $\overline{\mathbb{Q}}$ with $\dim V < m$. There is a finite set $\mathcal{U}$ of proper plu-subtori of T such that if $(x, y) \in V \cap \Gamma_T$ then $(x, y) \in U$ for some $U \in \mathcal{U}$.

Now let $L \subset \mathbb{C}^n \times \mathbb{C}^n$ be the $\mathbb{Q}$ subspace associated to T. We have $W = (D_n \cap L) \times V \subset L \times T$ of dimension $\dim W < 2m$; the ambient $L \times T$ has dimension $4m$. Therefore, any point in the intersection $V \cap \Gamma_T$ is a point in W on the graph of exp restricted to L. If we assume SC (ideologically speaking assuming z^iSC should be enough, but this is unclear) then, as shown in [Zilber 2002], any point $(x, y) \in V \cap \Gamma$ is in an atypical intersection of V with some plu-subtorus. Thus to get from SC to z^iUSC we need the following Zilber–Pink-type statement, in analogy with "CIT" of [Zilber 2002].

We state the conjecture for $V/\mathbb{C}$ and without dimension restrictions although for the purposes of connecting SC and z^iUSC, only $V/\overline{\mathbb{Q}}$, $2 \dim V < \dim T$ is required.

Conjecture 4.2 (z^iZP). Let $T \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ be a plu-torus. Let $V \subset T$. There is a finite set $\mathcal{U}$ of proper plu-subtori of T with the following property. If $S \subset T$ is a plu-subtorus and $A \subset_{\mathrm{cpt}} V \cap S$ is atypical in dimension with $A \cap \Gamma_T \neq \varnothing$, then there exists $U \in \mathcal{U}$ such that $A \subset U$.

Now given $A \subset T$ with $A \cap \Gamma_n \neq \varnothing$, we have seen that there is a smallest plu-torus containing A, denoted $((A))$, and defined the *plu-defect* $\delta(A) = \dim((A)) - \dim A$, and corresponding notion of *plu-optimal* subvariety.

As in [Habegger and Pila 2016], Conjecture 4.2 is then formally equivalent to the statement formulated as Theorem 1.3.

Conjecture 4.3. Let $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$. Then there are only finitely many plu-optimal subvarieties of V.

The reason that we can prove this statement, while multiplicative ZP remains seemingly far out of reach, is the following. An atypical intersection $V \cap S$ is typically a point, and any intersection point is atypical provided $\dim V + \dim S < \dim T$. In [Habegger and Pila 2016] it is shown that the full Zilber–Pink conjecture (in the modular and abelian settings) reduces to finiteness of optimal points in general. Now if we consider "plu-optimal points" they must be algebraic, on the one hand, if $V/\overline{\mathbb{Q}}$, since tori are defined over $\overline{\mathbb{Q}}$, but they must also belong to Γ. But the Gelfond–Schneider theorem (see, e.g., [Baker 1975]) implies that the only such point is $(\mathbf{1}, \mathbf{1})$.

For example, consider the case of a curve $V \subset T$, defined over $\overline{\mathbb{Q}}$. Suppose S is a plu-subtorus and $(x, y) \in V \cap S$. If (x, y) is an isolated intersection then it is algebraic, and consequently $(x, y) \in \Gamma_n$ if and only if $(x, y) = (\mathbf{1}, \mathbf{1})$. So if V intersects atypically in a component meeting Γ_n then this component is either $(\mathbf{1}, \mathbf{1})$

or all of V, in which case V is contained in a proper plu-subtorus. Thus we see that Conjecture 4.3 holds for V.

Our strategy, following [Habegger and Pila 2016], is to apply Ax–Schanuel to reduce to looking for plu-optimal points in the translate spaces of finitely many families, and then the above argument is decisive in showing that there is at most one plu-optimal point, namely the one corresponding to $(\mathbf{1}, \mathbf{1})$, in each such family. We go from $V/\overline{\mathbb{Q}}$ to $V/\mathbb{C}$ via a uniform version. The context for this argument is described further in Section 9 where it is presented.

Implementing this strategy requires the analogous "optimal" notions for strict and general plu-cosets. The analogous notion for weakly special subvarieties in the multiplicative setting is "geodesic optimal" (see [Habegger and Pila 2016]), which appeared earlier in [Poizat 2001], and then elsewhere, as "cd-maximal". We define the corresponding defects:

$$\delta_{\mathrm{SPC}}(A) = \dim((A))_{\mathrm{SPC}} - \dim A, \quad \delta_{\mathrm{PC}}(A) = \dim((A))_{\mathrm{PC}} - \dim A.$$

Definition 4.4. Let $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$. We say that $A \subset V$ is *strictly plu-geodesic optimal* for V if it is maximal for δ_{SPC} among subvarieties of V containing A and meeting Γ, and *plu-geodesic optimal* if it is maximal for δ_{PC} among subvarieties of V containing A.

5. Ax–Schanuel for z^i

Let K be a differential field with $\mathbb{Q} \subset \mathbb{Q}(i) \subset C \subset K$ with commuting derivations D_j and constant field C. The following is a special case of "Ax–Schanuel" [Ax 1971, Theorem 3].

Proposition 5.1. *Let $u_1, \dots, u_n, x_1, \dots, x_n, y_1, \dots, y_n \in K^\times$ with*

$$D_j x_k = x_k D_j u_k, \quad D_j y_k = i y_k D_j u_k \quad \textit{for all } j, k.$$

Then

$$\mathrm{tr.deg.}_C(u_1, \dots, u_n, x_1, \dots, x_n, y_1, \dots, y_n) \ge 2n + \mathrm{rank}_K(D_j u_k)$$

unless the u_k, iu_k are linearly dependent over $\mathbb{Q}$ modulo C.

Suppose that the u_j, iu_j are linearly dependent over $\mathbb{Q}$ modulo C, say

$$\sum_j q_j u_j + i \sum_j r_j u_j = c \in C \tag{2}$$

with $q_j, r_j \in \mathbb{Z}$ not all zero. Then we find that

$$\prod_j x_j^{q_j} \prod_j y_j^{r_j} = c' \in C,$$

as it is in the kernel of all the derivations. Multiplying (2) through by i we get a second relation (1^i), and a second multiplicative relation

$$\prod_j x_j^{-r_j} \prod_j y_j^{q_j} = c'' \in C.$$

Now morally one wants to say that $c' = \exp(c)$, $c'' = \exp(ic)$ so that $c'' = c'^{\,i}$, but the differential field setting has no interpretation of this.

Conversely, if we are given $x_1, \dots, x_n, y_1, \dots, y_n$ satisfying the differential relations

$$\frac{D_j y_k}{y_k} = i \frac{D_j x_k}{x_k} \quad \text{for all } j, k$$

then if $u_1, \dots, u_n$ satisfy $D_j x_k = x_k D_j u_k$ for all j, k then they also satisfy the equations $D_j y_k = i y_k D_j u_k$, and if the x_j, y_j satisfy multiplicative relations mod C then the u_k, iu_k satisfy linear relations over $\mathbb{Q}$ modulo C.

And if we have a linear relation $\sum_j q_j u_j + i \sum_j r_j u_j = c$ then the "conjugate" relation indeed has constant ic.

Finally we note that with any u_j, x_j, y_j as in Proposition 5.1 we have

$$\text{rank}_K(D_j u_k) = \text{rank}_K(D_j x_k) = \text{rank}_K(D_j y_k) = \text{rank}_K(D[x], D[y]),$$

where $D[x] = (D_j x_i)$ and $D[y] = (D_j y_i)$.

Corollary 5.2. *Let* $x_1, \dots, x_n, y_1, \dots, y_n \in K^\times$ *with*

$$\frac{D_j y_k}{y_k} = i \frac{D_j x_k}{x_k} \quad \textit{for all } j, k.$$

Then

$$\text{tr.deg.}_C(x_1, \dots, x_n, y_1, \dots, y_n) \geq \tfrac{1}{2} \dim ((x_1, \dots, x_n, y_1, \dots, y_n))_{\text{PC}} + \text{rank}_K(D_j x_k),$$

where $((\dots))_{\text{PC}}$ *is defined using plu-multiplicative dependence modulo* C.

Proof. Given x_j, y_j, $j = 1, \dots, n$ satisfying these equations then some suitable u_j exist (perhaps in some extension differential field), and then the statement follows from the above discussion. □

We next state "weak CIT" in this setting, following [Zilber 2002]. Given a variety (or family of varieties) V then there is some finite set of linear dependencies which, up to translations, accounts for all deficiencies in transcendence degree.

Suppose that $V \subset \mathbb{G}_m^n \times \mathbb{C}^k$ is a family of algebraic varieties of generic dimension k, parameterized by $t \in W \subset \mathbb{C}^k$, with the fibre having $\dim V_t = k$ provided $t \notin W'$, where W' is a proper subvariety of W.

Proposition 5.3. *Let $V \subset \mathbb{G}_m^n \times \mathbb{G}_m^n \times \mathbb{C}^m$ be a family of algebraic varieties, parameterized by points of $W \subset \mathbb{C}^m$, of generic dimension k outside W'.*

Then there exists a finite set Σ of integer vectors $(q, r) \in \mathbb{Z}^{2n} \backslash \{0\}$ with the following property. Suppose $t \in W \backslash W'$, and $A \subset_{\mathrm{cpt}} V_t \cap \Gamma$ with

$$\dim A > k - n.$$

Then there exists $(q, r) \in \Sigma$ and $c, c' \in \mathbb{C}$ such that

$$\prod_j x_j^{q_j} \prod_j y_j^{r_j} = c \quad \text{and} \quad \prod_j x_j^{-r_j} \prod_j y_j^{q_j} = c'$$

for any point $(x, y) \in A$.

Proof. This is essentially a special case of Proposition 8 of [Zilber 2002], though we consider families of general dimension, not necessarily $k < n$. It is an application of the compactness theorem of first-order logic. Suppose, towards a contradiction, that no such finite set exists. Certainly if $\dim A = 0$ then this property is satisfied for any such tuple. Therefore, for some $\ell > 0$ we have the property that for any finite set Σ of tuples there exists $t \in W \backslash W'$ and a component $A \subset_{\mathrm{cpt}} V_t \cap \Gamma$ of dimension ℓ with $\ell > k - n$. Then it is consistent to have a differential field with ℓ derivations, of rank ℓ on some set $\{x_1, \dots, x_n\}$ of functions and to have also functions $y_1, \dots, y_n$ satisfying the required equations but with the x_j, y_j, $j = 1, \dots, n$ not plu-multiplicatively dependent modulo constants, giving the contradiction. □

By repeating this on the families of intersections (the parameter now being the constants $c', c'' \in \mathbb{C}$), we find that some finite collection of families of plu-cosets accounts for all plu-geodesic optimal intersections with varieties in the family V.

6. The defect condition

Given a plu-torus T and a subvariety $A \subset T$ meeting Γ_T we have three defects: the first with respect to plu-tori, the second with respect to strict plu-cosets, and the third with respect to (general) plu-cosets:

$$\delta(A) = \dim\,(\langle A \rangle) - \dim A,$$
$$\delta_{\mathrm{SPC}}(A) = \dim\,(\langle A \rangle)_{\mathrm{SPC}} - \dim A,$$
$$\delta_{\mathrm{PC}}(A) = \dim\,(\langle A \rangle)_{\mathrm{PC}} - \dim A.$$

Evidently

$$\delta_{\mathrm{PC}} \le \delta_{\mathrm{SPC}}(A) \le \delta(A).$$

Suppose that (x_i, y_i) are a basis of coordinate pairs on A with respect to plu-multiplicative dependence. The difference between the defect measures to what extent these functions are strictly plu-dependent mod $\mathbb{C}$, and then the extent to which the remaining strictly plu-dependent mod $\mathbb{C}$ ones are plu-multiplicatively dependent mod $\mathbb{C}$.

Suppose $A \subset B$. Then strict plu-multiplicative relations modulo constants on B remain strict plu-multiplicative relations modulo constants on A, and if the constant pairs (c_k, d_k) in these relations are plu-multiplicatively independent on B they remain so on A. We therefore see that the *defect condition* holds between δ and δ_{SPC}, namely

$$\delta(B) - \delta_{\mathrm{SPC}}(B) \le \delta(A) - \delta_{\mathrm{SPC}}(A).$$

Similarly, the defect condition holds between δ_{PC} and δ_{SPC}.

Proposition 6.1. *Fix $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$.*

(i) *A plu-optimal subvariety for V is strictly plu-geodesic optimal.*

(ii) *A strictly plu-geodesic optimal subvariety is plu-geodesically optimal.*

Proof. Suppose $A \subset V$ is plu-optimal and $A \subset B \subset V$. Suppose $\delta_{\mathrm{SPC}}(B) \le \delta_{\mathrm{SPC}}(A)$. Then, by the defect condition,

$$\delta(B) \le \delta(A) - \delta_{\mathrm{SPC}}(A) + \delta_{\mathrm{SPC}}(B) \le \delta(A).$$

Since A is optimal, we have $B = A$, and so A is strictly plu-geodesically optimal. This proves (i). The proof of (ii) is similar. □

7. Families of plu-cosets

In this section we introduce some terminology and notation that will be needed in the proofs of the main results. For general properties of linear tori see [Bombieri and Gubler 2006, Chapter 3.1]. A family of (general) plu-cosets of a plu-torus X is determined by a finite set of plu-multiplicative pairs of equations

$$x^{q^{(k)}} y^{r^{(k)}} = 1, \quad x^{-r^{(k)}} y^{q^{(k)}} = 1 \quad \text{for } k = 1, \dots, K$$

which define X, and a finite set of exponent vectors, independent of those above, for some further equations which determine the cosets in the family:

$$x^{q^{(k)}} y^{r^{(k)}} = c_k, \quad x^{-r^{(k)}} y^{q^{(k)}} = d_k \quad \text{for } k = K+1, \dots, K+L,$$

such that the fibres are torus cosets in X (i.e., the exponent vectors generate a primitive lattice).

The plu-cosets are parameterized by the coordinates

$$(c, d) = (c_{K+1}, \dots, c_{K+L}, d_{K+1}, \dots, d_{K+L}) \in \mathbb{G}_{\mathrm{m}}^L \times \mathbb{G}_{\mathrm{m}}^L.$$

We denote the parameter space by $X_S = \mathbb{G}_{\mathrm{m}}^L \times \mathbb{G}_{\mathrm{m}}^L$. The cosets are then the fibres of the map $\pi : X \to X_S$ given by

$$\pi(x, y) = \big(x^{q^{(K+1)}} y^{r^{(K+1)}}, \dots, x^{q^{(K+L)}} y^{r^{(K+L)}}, x^{-r^{(K+1)}} y^{q^{(K+1)}}, \dots, x^{-r^{(K+L)}} y^{q^{(K+L)}}\big).$$

Such a family we denote S^D, where D is the data (exponent vectors for the equations of T and for the additional equations of the cosets in the family). The fibre over $(c, d) \in X_S$ is denoted $S^D_{c,d}$. The fibre $S^D_{c,d}$ is a strict plu-coset just if $(c, d) \in \Gamma_L$. The union X of the plu-cosets over the family we call the *envelope* of S^D and denote it $[S^D]$.

We observe that the preimage under π of a plu-torus is a plu-torus, and likewise for strict and general plu-cosets. To see this, consider a condition of the form

$$c^Q d^R = \gamma, \quad c^{-R} d^Q = \delta,$$

where $Q = (Q_1, \dots, Q_L)$, $R = (R_1, \dots, R_L)$ are tuples of integers. The preimage in X is determined, in addition to the equations for X, by

$$x^{\sum Q_j q^{(j)} - \sum R_j r^{(j)}} y^{\sum Q_j r^{(j)} + \sum R_j q^{(j)}} = \gamma,$$
$$x^{-\sum R_j q^{(j)} - \sum Q_j r^{(j)}} y^{-\sum R_j r^{(j)} + \sum Q_j q^{(j)}} = \delta,$$

which is a plu-pair.

8. Proof of Theorem 1.3 for $V/\overline{\mathbb{Q}}$

We can now prove Theorem 1.3 (Conjecture 4.3) for $V/\overline{\mathbb{Q}}$ following the first part of the proof of [Habegger and Pila 2016, Theorem 10.1], using the fact that Γ has just one algebraic point (a consequence of the Gelfond–Schneider theorem).

Theorem 8.1. *Let $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ with $V/\overline{\mathbb{Q}}$. Then there are only finitely many plu-optimal subvarieties of V.*

Proof. Let $A \subset V$ be a plu-optimal component which meets Γ_n.

We observe that if $A \subset S$ for some plu-coset S then this coset must be strict. For suppose $(x, y) \in A \cap \Gamma_n$, so $(x, y) \in S$. Let (u, iu) be the tuple of logarithms. Say that $x^q y^r = c$, $x^{-r} y^q = d$ is a pair of equations defining S. So we have

$$\sum q_j u_j + i \sum r_j u_j = \gamma$$

with $\exp(\gamma) = c$. But then, multiplying by i, we also have

$$-\sum r_j u_j + i \sum q_j u_j = i\gamma,$$

whence $d = \exp(i\gamma)$.

Plu-optimal subvarieties are plu-geodesic optimal. The plu-geodesic-optimal subvarieties of V arise from intersections with finitely many families S^D of plu-cosets. Fix one of these families S and denote the parameter space X_S. Let $\tau = \dim S_{c,d}$ be the dimension of the plu-cosets in the family.

We have a projection $\pi : [S] \to X_S$ whose fibres are the $S_{c,d}$. The intersections $V \cap S_{c,d}$ are the fibres of the restriction of this projection to V, whose image in X_S we denote V_S, and we have $V_S/\overline{\mathbb{Q}}$. There is a Zariski-open subvariety $V' \subset V$ in which the fibre over the image has the generic fibre dimension $\nu = \dim V - \dim V_S$.

Suppose $A \subset V \cap S_{c,d}$ is plu-optimal and meets Γ_n. If $A \subset V \backslash V'$ then it is certainly plu-optimal for the component of $V \backslash V'$ it is in. The proof is then concluded by induction on $\dim V$ (the base case $\dim V = 1$ was dealt with in the second paragraph following the statement of Conjecture 4.3).

So we assume that $A \cap V' \neq \varnothing$ and then $\dim A = \nu$ and $\pi(A) = (c, d)$. Since $A \cap \Gamma_n \neq \varnothing$ we have some $(x, y) \in A \cap \Gamma_n$. Thus each component $(x_i, y_i) \in \Gamma_1$, with logarithm u_i. And now if

$$\sum_j q_j u_j + i \sum_j r_j u_j = \gamma$$

then this implies

$$-\sum_j r_j u_j + i \sum_j q_j u_j = i\gamma.$$

Then $c = \exp(\gamma)$, $d = \exp(i\gamma)$ and we have $(c, d) \in \Gamma$.

The claim is that $\{(c, d)\}$ is a plu-optimal point component of V_S. We can assume that we have already dealt with any family of smaller plu-cosets that might have given rise to A, i.e., we can assume that

$$S_{c,d} = ((A))_{\mathrm{PC}}.$$

Then

$$\dim ((A)) = \dim S_{c,d} + \dim ((c, d)),$$

whence

$$\delta(A) = \dim ((A)) - \dim A = \dim ((c, d)) + \tau - \nu.$$

Suppose that $\{(c, d)\} \subset B$, $\{(c, d)\} \neq B$ with $\delta(B) \leq \delta(c, d) = \dim ((c, d))$. Let C be the component of the preimage of B in V' containing A. Then

$$\delta(C) = \dim ((C)) - \dim C \leq \dim ((B)) + \tau - (\dim B + \nu) \leq \dim ((c, d)) + \tau - \nu = \delta(A).$$

Then $C = A$ by the plu-optimality of A and so $B = \{(c, d)\}$ is optimal.

But then (c, d) is algebraic, and since it belongs to Γ_n we must have $(c, d) = (\mathbf{1}, \mathbf{1})$. So we get at most one plu-optimal subvariety in each family. □

Remarks 8.2. (1) Can one effectively determine the finitely many families of plu-cosets? (For the general multiplicative setting [Bombieri et al. 2007] gives an effective argument for this.) For curves this seems clearly possible.

(2) This shows that SC would imply a uniform z^iSC. Does z^iSC itself imply some uniformity, and if so what is the intervening “ZP” statement?

9. Uniformity and proof of Theorem 1.3 for $V/\mathbb{C}$

Here we prove that Conjecture 4.3 holds uniformly for varieties in families, in the sense of [Scanlon 2004]: the formal sum of the optimal subvarieties is bounded as a cycle, which we make precise in Conjecture 9.2. This uses the fact established in [Habegger and Pila 2016], already exploited here, that ZP is equivalent to showing that the number of optimal points on any subvariety is bounded. Here we need to upgrade this to show that the number of optimal points is uniformly bounded on a family of varieties. We do this following the argument sketched in [Zannier 2012], which we have fully worked out [Pila 2022, Chapter 24] for ZP in the modular and multiplicative settings. As a by-product, we establish that Conjecture 4.3 holds for $V/\mathbb{C}$.

Let $X = \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$. A *family of subvarieties* of X means a subvariety $V \subset X \times P$ for some constructible set P, considered as the family of fibres $V_p \subset X$, $p \in P$. The *fibre dimension* of a family is the maximum dimension of a fibre.

If V is a family of subvarieties of X and h is a positive integer then we have the incidence variety

$$\mathrm{Inc}^h(V) = \{(z_1, \ldots, z_h) \in X^h : \exists p \in P : z_j \in V_p, \, j = 1, \ldots, h\}.$$

Since P is only assumed constructible $\mathrm{Inc}^h(V)$ may not be Zariski closed, and we denote by $V^{\langle h \rangle}$ its Zariski closure. In particular $V^{\langle 1 \rangle}$ is the Zariski closure of the union of all the fibres, which we call the *envelope* of the family and denote also by $[V]$.

We have already seen that plu-cosets of a plu-torus $T \subset X$ come in families. Such a family is a family $S \subset X \times X_S$ (in the above sense). The envelope $[S]$ is a plu-torus.

Theorem 9.1. *Let $V \subseteq X \times P$ be a family of subsets of X defined over $\overline{\mathbb{Q}}$. Then there is a uniform bound on the number of optimal points of V_p, $p \in P$.*

Proof. We prove the theorem by induction (first) on the dimension of the parameter space P. The case $\dim P = 0$ is addressed by Theorem 8.1. We may then assume that the Zariski closure of P is irreducible, that the fibre dimension $v = \dim V_p$ is constant and equal to the generic fibre dimension, and that there is a single family $S = S^D$ of plu-cosets such that every fibre V_p in the family has $((V_p))_{PC} = S_{c,d}$ for some $(c, d) \in X_S$. Thus $[V] \subset [S]$. We adopt the notation of Section 7 for this family.

For given $\dim P$ we may assume that the theorem holds for all families $W \subset X \times Q$ (i.e., of any dimension of the parameter space Q) such that each fibre has $((W_q))_{PC} = S'_{c,d}$ for some fibre of a family S' of plu-cosets, and for which either $\dim[S'] < \dim[S]$ or $\dim[S'] = \dim[S]$ and $\dim S'_{c,d} < \dim S_{c,d}$; the base cases are trivial.

Now we take a positive integer h, to be specified below, and consider $V^{\langle h\rangle} \subset [S]^h$. Then in fact $V^{\langle h\rangle} \subset S^{\langle h\rangle}$, and the latter is a plu-torus: in addition to the equations for $[S]^h$ it is defined by the plu-pairs of equations

$$(x^{(j)})^{q^{(\ell)}}(y^{(j)})^{r^{(\ell)}} = (x^{(k)})^{q^{(\ell)}}(y^{(k)})^{r^{(\ell)}}, \quad (x^{(j)})^{-r^{(\ell)}}(y^{(j)})^{q^{(\ell)}} = (x^{(k)})^{-r^{(\ell)}}(y^{(k)})^{q^{(\ell)}}$$

for $j \neq k$ and $\ell = K+1, \ldots, K+L$. Thus

$$\dim S^{\langle h\rangle} = h\dim[S] - L(h-1).$$

Now suppose that V_p is a fibre of V which contains h optimal points $(x_0^{(j)}, y_0^{(j)})$, $j = 1, \ldots, h$. Then they are atypical as point subvarieties of V_p (unless $\dim V_p = 0$ in which case the conclusion is trivial for V). Thus there are plu-tori T_j, $j = 1, \ldots, h$ such that $(x_0^{(j)}, y_0^{(j)}) \in T_j \cap \Gamma_X$, $j = 1, \ldots, h$ and

$$\dim T_j + \dim V_p < \dim[S].$$

Consider the plu-torus

$$T = T_1 \times \cdots \times T_h.$$

Since the equations defining $S^{\langle h\rangle}$ are between different groups of variables, they are independent of the equations defining each T_j, and we have

$$\dim T \cap S^{\langle h\rangle} = \dim T - L(h-1).$$

Then

$$(\boldsymbol{x}_0, \boldsymbol{y}_0) = \big((x_0^{(1)}, \ldots, x_0^{(h)}, y_0^{(1)}, \ldots, y_0^{(1)})\big) \in T \cap S^{\langle h\rangle} \cap \Gamma_T$$

and is atypical for $V^{\langle h\rangle}$ as a subvariety of $S^{\langle h\rangle}$ provided that

$$\dim V^{\langle h\rangle} + \dim T \cap S^{\langle h\rangle} < \dim S^{\langle h\rangle}.$$

Thus we find that $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is atypical provided

$$\dim P + h\dim v + h(\dim[S] - v - 1) - L(h-1) < h\dim[S] - L(h-1),$$

that is, provided $h > \dim P$. We now assume this (but the choice of h needs to be on the basis of some combinatorial principles further below).

We can now apply Theorem 8.1 to $V^{\langle h\rangle}$, which is defined over $\overline{\mathbb{Q}}$, to conclude that atypical points are contained in one of finitely many proper plu-subtori $U \subset S^{\langle h\rangle}$. Each such U is determined by at least one plu-pair of equations

$$\prod_j (x^{(j)})^{s^{(j)}}(y^{(j)})^{t^{(j)}} = 1, \quad \prod_j (x^{(j)})^{-t^{(j)}}(y^{(j)})^{s^{(j)}} = 1.$$

Consider the exponent vector pair $(s(\ell), t(\ell))$ on a particular set of variables. If this is not in the lattice $\Lambda(S)$ generated by the equations defining S (the fixed ones and the variable one), and if we have some points $(x^{(j)}, y^{(j)})$, $j \neq \ell$ and sufficiently many points $(x^{(\ell)}, y^{(\ell)})$ such that, for each of the $(x^{(\ell)}, y^{(\ell)})$, the relation

$$(x^{(1)})^{s^{(1)}} \cdots (x^{(\ell)})^{s(\ell)} \cdots (x^{(h)})^{s^{(h)}} (y^{(1)})^{t^{(1)}} \cdots (y^{(\ell)})^{t(\ell)} \cdots (y^{(h)})^{t^{(h)}} = 1,$$

and the companion relation

$$(x^{(1)})^{-t^{(1)}} \cdots (x^{(\ell)})^{-t(\ell)} \cdots (x^{(h)})^{-t^{(h)}} (y^{(1)})^{s^{(1)}} \cdots (y^{(\ell)})^{s(\ell)} \cdots (y^{(h)})^{s^{(h)}} = 1,$$

hold, then we get many points $(x^{(\ell)}, y^{(\ell)}) \in V_p$ satisfying an additional plu-relation mod $\mathbb{C}$ with the exponent-pair $(s(\ell), t(\ell))$. The corresponding family S' of plu-cosets has smaller fibre dimension than S.

If $(s(\ell), t(\ell))$ is in $\Lambda(S)$, then using the relations defining $S^{\langle h\rangle}$ we can replace this plu-pair by an equivalent pair of equations with trivial exponent pair $(s(\ell), t(\ell))$. It may be that, for some U, every $(s(\ell), t(\ell)) \in \Lambda(S)$. But since the equations defining U define a proper plu-subtorus of $S^{\langle h\rangle}$, when the relation is shifted to a single set of variables it must be one that does not hold identically on $[S]$, but gives a proper plu-subtorus S_U.

We thus have that, for any h optimal points on some fibre V_p, we get one of finitely many possibilities: that some designated coordinate lies in one of the S_U, or the h-tuple of tuples satisfies one of finitely many relations involving exponent pairs that, wherever they are nontrivial on a group of variables, do not belong to $\Lambda(S)$.

If we now choose a much larger $\mathcal{H}$ and have plu-optimal $(x_0^{(j)}, y_0^{(j)})$, $j = 1, \ldots, \mathcal{H}$ in some order, then by the hypergraph Ramsey theorem we can be assured that there is some subset of H of them for which all choices (in order) of h satisfy the same one of these conditions. We thus find that we have many points on some family of plu-cosets of smaller dimension, or many points in some smaller plu-torus S_U. We can therefore complete the proof by induction. □

A rephrasing of Conjecture 4.3 is that, for $V \subset \mathbb{G}_m^n \times \mathbb{G}_m^n$, the formal sum of plu-optimal subvarieties is a cycle V^{opt}. We now want to frame a uniform version that, over families V, the plu-optimal cycle is uniformly bounded as a cycle. We formulate this following [Scanlon 2004].

It should be borne in mind that z^iZP makes a nontrivial statement only for subvarieties that meet Γ_n. If V does not meet Γ_n then, by definition, V has no plu-optimal subvarieties and V^{opt} is empty, while if V does meet Γ_n then V itself is plu-optimal for V and V^{opt} is nonempty.

Conjecture 9.2 (Uz^iZP). Let $V \subset \mathbb{G}_m^n \times \mathbb{G}_m^n \times P$ be a family of subvarieties. Then there is a family $W \subset \mathbb{G}_m^n \times \mathbb{G}_m^n \times Q$ such that, for every $p \in P$ for which V_p meets Γ_n, there exists $q \in Q$ such that $V_p^{\text{opt}} = W_q$.

Theorem 9.3. *Uz^iZP holds for families V defined over $\overline{\mathbb{Q}}$.*

Proof. We replay the proof of Theorem 8.1. The plu-geodesic optimal subvarieties of all the V_p come in finitely many families. For each such family, the plu-optimal subvarieties on the fibres of V correspond to plu-optimal points on the family of fibres of the projections. For each such family there is a uniformly bounded number of plu-optimal points on a fibre by Theorem 9.1. □

Corollary 9.4 (Theorem 1.3). *Conjecture 4.3 holds for* $V/\mathbb{C}$.

Proof. Every such V is a fibre in a family defined over $\overline{\mathbb{Q}}$. □

In a similar way, Theorem 9.3 holds for families defined over $\mathbb{C}$ as every such family $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n \times P$ is a subfamily (meaning its fibres are a subset of the fibres) of a larger family $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n \times Q$ defined over $\overline{\mathbb{Q}}$.

The present results generalize suitably to other algebraic powers, as will be shown in forthcoming work of Cassani. It seems also interesting to consider modular analogues.

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