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This expository paper is written in celebration of Boris Zilber's 75th birthday. We discuss Ax–Schanuel type statements focusing on the case of positive characteristic.

1. Introduction

During the Spring 2005 Isaac Newton Institute program "Model Theory and Applications to Algebra and Analysis" in Cambridge, I learnt that I would be a MODNET postdoc with Boris Zilber in Oxford for the academic year 2005/06. Still in Cambridge, Boris suggested that I start thinking on "positive characteristic versions of Ax's theorem". In this expository paper, I will describe what has happened next.

It may be a good moment for a general disclaimer. This is an expository paper representing my experience with respect to Boris's suggestion above and I do not claim that this paper describes adequately the state of the art in the vast area of Ax–Schanuel type problems. In particular, comparatively very little will be said about the amazing developments of Jonathan Pila (and many others) regarding the modular version of Ax–Schanuel and its applications to diophantine problems, most notably the André–Oort conjecture. I will write more about it in Section 2.

This paper is organized as follows. In Section 2, we describe the history of this circle of problems in the case of characteristic 0. In Section 3, we focus on the positive characteristic case and present some of the results I obtained following Boris's suggestion. In Section 4, we speculate on some recent ideas regarding general forms of Ax–Schanuel and its Hasse–Schmidt differential versions.

2. Characteristic zero

In this section, we summarize the characteristic 0 situation regarding the Ax–Schanuel problems. The disclaimer from the introduction applies mostly here.

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2A. *Results.* In the 1960s, Schanuel formulated two conjectures [Lang 1966, pages 30–31]: one about transcendence of complex numbers [Ax 1971, (S)] and one about transcendence of power series [Ax 1971, (SP)]. We state them below.

Schanuel's conjecture (complex numbers). *Let $x_1, \dots, x_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then*

$$\operatorname{trdeg}_{\mathbb{Q}}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) \geqslant n.$$

Schanuel's conjecture (power series). *Let $x_1, \dots, x_n \in X\mathbb{C}[\![X]\!]$ be linearly independent over $\mathbb{Q}$. Then*

$$\operatorname{trdeg}_{\mathbb{C}(X)}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) \geqslant n.$$

The conjecture on the complex numbers is open even for $n = 2$, since (using Euler's identity $e^{i\pi} + 1 = 0$) it covers the open problem of algebraic independence of π and e and it is even still unknown whether $\pi + e$ is irrational (it is named a "candidate for the most embarrassing transcendence question in characteristic zero" in [Brownawell 1998])! Schanuel's conjecture for power series was proved in [Ax 1971, (SP)].

Ax [1971] also showed the following differential version of the power series conjecture, which he actually used to show the other statements from [Ax 1971].

Differential Ax–Schanuel theorem [Ax 1971, (SD)]. *Let (K, ∂) be a differential field of characteristic 0 and C be its field of constants. For $x_1, \dots, x_n \in K$ and $y_1, \dots, y_n \in K^*$, if*

$$\partial x_1 = \frac{\partial y_1}{y_1}, \dots, \partial x_n = \frac{\partial y_n}{y_n}$$

and $\partial x_1, \dots, \partial x_n$ are $\mathbb{Q}$-linearly independent, then

$$\operatorname{trdeg}_C(x_1, \dots, x_n, y_1, \dots, y_n) \geqslant n + 1.$$

Remark 2.1. There are the following passages between the power series and the differential version of Ax's theorem above.

(1) Since the ring of power series has a natural differential structure, the differential version implies the power series version.

(2) Going the other way is more subtle. Seidenberg's embedding theorem [1958] says that any finitely generated differential field of characteristic 0 differentially embeds into the differential field of meromorphic functions on an open subset of $\mathbb{C}$. Using this theorem, one can reduce the differential version of Ax's theorem to the power series one (this is explained in detail around [Freitag and Scanlon 2018, Theorem 4.1] and in [Pila and Tsimerman 2016, Section 2.5]).

Similar passages apply to the more complicated cases of analytic (or formal) Ax–Schanuel statements versus the differential ones as well. Such more complicated cases are described below.

In a subsequent paper written one year later, Ax [1972] proved the following general geometric result about the dimension of intersections of algebraic subvarieties of complex algebraic groups with analytic subgroups.

Ax's theorem on the dimension of intersections [Ax 1972, Theorem 1]. *Let G be an algebraic group over the field of complex numbers $\mathbb{C}$. Let $\mathcal{A}$ be a complex analytic subgroup of $G(\mathbb{C})$ and V be an irreducible algebraic subvariety of G over $\mathbb{C}$. We assume that $\mathcal{K} := \mathcal{A} \cap V(\mathbb{C})$ is Zariski dense in $V(\mathbb{C})$. Then there is an analytic subgroup $\mathcal{B} \subseteq G(\mathbb{C})$ containing $V(\mathbb{C})$ and $\mathcal{A}$ such that*

$$\dim(\mathcal{B}) \leqslant \dim(\mathcal{A}) + \dim(V) - \dim(\mathcal{K}).$$

This theorem implies Schanuel's conjecture on power series by taking:

- G as the product of the vector group $\mathbb{G}_{\mathrm{a}}^n$ and the torus $\mathbb{G}_{\mathrm{m}}^n$,
- $\mathcal{A}$ as the n-th Cartesian power of the graph of the exponential map,
- V as the algebraic locus of the tuple $(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$.

Ax's theorem on the dimension of intersections applies also (more generally) to the case of the exponential map on a semiabelian variety [Ax 1972, Theorem 3]. The consequences of Ax's theorem on the dimension of intersections go beyond the case of the exponential map; for example, this theorem applies to the case of analytic maps between the multiplicative group and an elliptic curve. We state it precisely below, since this statement is amenable for a possible transfer to the positive characteristic case (see Remark 2.4).

Theorem 2.2. *Let*

$$\gamma : \mathbb{G}_{\mathrm{m}}(\mathbb{C}) \to E(\mathbb{C})$$

be an analytic epimorphism, where E is an elliptic curve. Let

$$x_1, \ldots, x_n \in 1 + X\mathbb{C}[\![X]\!]$$

be multiplicatively independent. Then

$$\mathrm{trdeg}_{\mathbb{C}(X)}(x_1, \ldots, x_n, \gamma(x_1), \ldots, \gamma(x_n)) \geqslant n.$$

After Ax's work in 1970s, Brownawell and Kubota [1977] proved a version of the differential Ax's theorem in the case of elliptic curves, and then Kirby [2009] generalized it to arbitrary semiabelian varieties. These results were not included in [Ax 1972], however they are closely related using the "passages" from Remark 2.1. Bertrand [2008] extended [Ax 1972, Theorem 3] to commutative algebraic groups not having vector quotients (e.g., maximal nonsplit vectorial extensions of a semiabelian variety).

The differential Ax's theorem [Ax 1971, (SD)] is generalized further to "very nonalgebraic formal maps" in [Kowalski 2008, Theorem 5.5]. This generalization includes a differential version of Bertrand's result and a differential Ax–Schanuel type result about raising to nonalgebraic powers on an algebraic torus [Kowalski 2008, Theorem 6.12]. We state it below in the power series case (see Remark 2.1), since this statement has a positive characteristic interpretation (see Remark 2.4). Before the statement, we note that for $x \in 1 + X\mathbb{C}[[X]]$ and $\alpha \in \mathbb{C}$, we define

$$x^{\alpha} := \exp(\alpha \log(x)),$$

where $\exp, \log \in \mathbb{Q}[[X]]$ are the standard formal power series corresponding to the exponential and the logarithmic maps.

Theorem 2.3. *Suppose that $\alpha \in \mathbb{C}$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}] > n$. Let $x_1, \dots, x_n \in 1+X\mathbb{C}[[X]]$ be multiplicatively independent. Then*

$$\operatorname{trdeg}_{\mathbb{C}(X)}(x_1, \dots, x_n, x_1^{\alpha}, \dots, x_n^{\alpha}) \geqslant n.$$

We now briefly describe modular analogues of Ax's theorem. Our disclaimer from the introduction applies very much here. Ax–Schanuel statements may go beyond the context of group homomorphisms: the first example here is the j-function map

$$j : \mathbb{H} \to \mathbb{C},$$

where $\mathbb{H}$ is the upper half plane. The linear independence assumption from Ax's theorem is replaced with *modular independence*. Pila's notes [2015] contain an excellent comprehensive survey of the state of the art in this field up to 2013. Such results have very important diophantine applications such as

- another proof of the Manin–Mumford conjecture [Pila and Zannier 2008];
- the first unconditional proof of the André–Oort conjecture for $\mathbb{C}^n$ [Pila 2011];
- a recent proof of the full André–Oort conjecture for Shimura varieties [Pila et al. 2021].

Following a suggestion by the referee, we would like to point out that only the Ax–Lindemann–Weierstrass type of results are needed in Manin–Mumford and André–Oort, while Ax–Schanuel (in fact, a weak form of it) is used in Zilber–Pink type problems.

In [Casale et al. 2020], the Ax–Lindemann–Weierstrass theorem with derivatives for the uniformizing functions of genus zero Fuchsian groups of the first kind is shown. This result is used in [Casale et al. 2020] to answer a question of Painlevé from 1895.

Remark 2.4. We analyze now which statements of the Ax–Schanuel results discussed above are transferable to the positive characteristic case. We would like to mention that all the analytic Ax–Schanuel type results over $\mathbb{C}$ may be replaced with their formal counterparts over an arbitrary field C, which was already done by Ax: the reader is advised to compare Ax's theorem on the dimension of intersections with its formal counterpart [Ax 1972, Theorem 3], which will be stated in a more general form in Section 3. Let us recall the setup first.

Definition 2.5 [Bochner 1946]. An n-dimensional *formal group* (law) over C is a tuple of power series $F \in C[\![X, Y]\!]^{\times n}$ ($|X| = |Y| = |Z| = n$) satisfying

$$F(0, X) = F(X, 0) = X,$$

$$F(X, F(Y, Z)) = F(F(X, Y), Z).$$

A *morphism* from an n-dimensional formal group G into an m-dimensional formal group F is a tuple of power series $f \in C[\![X]\!]^{\times m}$ such that

$$F(f(X), f(Y)) = f(G(X, Y)).$$

There is a well-known formalization functor $G \mapsto \widehat{G}$ (see pages 5 and 13 in [Manin 1963]) from the category of algebraic groups to the category of formal groups.

Such characteristic 0 formal statements seem to be transferrable to the positive characteristic context in the cases when the corresponding formal maps exist.

(1) The very original version of Ax–Schanuel does not look transferable, since there are no reasonable exponential maps in positive characteristic (we will briefly touch on the Drinfeld context at the end of Section 3).

(2) Therefore, other analytic maps need to be considered. "Analytic" may be replaced with "formal" (as mentioned above) and then the closest one to the exponential map which survives in the case of positive characteristic seems to be the formal isomorphism between the multiplicative group and an ordinary elliptic curve.

(3) The other types of such maps come from raising to powers in the multiplicative group.

Items (2) and (3) above will be discussed in the positive characteristic context in Section 3.

2B. ***Motivations and applications.*** Zilber [2002] used the differential Ax's theorem to prove *weak CIT*, which is a weak version of the *conjecture on intersection with tori* (CIT), which was also stated in [Zilber 2002]. CIT is a finiteness statement about intersections of subtori of a given torus with certain subvarieties of this torus. Weak CIT was used in [Baudisch et al. 2009] to produce a characteristic 0 *bad field*.

The existence of such a field was an open problem in model theory for almost 20 years.

Regarding the positive characteristic case, weak CIT does not hold and Zilber formulated a conjectural statement in (the very last statement of) [Zilber 2005]. It is still open whether a bad field in the positive characteristic case exists, however, Wagner [2003] showed that its existence in the case of characteristic $p > 0$ implies the existence of infinitely many *p-Mersenne primes*, which is an open problem in number theory — but it is widely believed that there are finitely many of them (for each individual prime p). Therefore, the existence of bad fields in positive characteristic looks very unlikely. However, pursuing the following path of research still looks interesting:

(1) prove positive characteristic versions of Ax–Schanuel;

(2) show a version of weak CIT in positive characteristic using (1);

(3) construct a version of a bad field in positive characteristic using (2);

(4) check the possible number-theoretic consequences of results obtained in (3).

As was mentioned in the previous subsection, Jonathan Pila and others used Ax–Schanuel type results to show different versions of the André–Oort conjecture; see, e.g., [Pila 2011; Tsimerman 2018; Casale et al. 2020; Pila et al. 2021].

There are also model-theoretic consequences of results of Ax–Schanuel type and we would like to point out some of them.

- Kirby [2009] used his version of an Ax–Schanuel statement to obtain the complete first-order theories of the exponential differential equations of semi-abelian varieties which arise from an amalgamation construction in the style of Hrushovski.
- Aslanyan [2022] did a version of the above for the j-function in place of the exponential function on semiabelian varieties.
- Freitag and Scanlon [2018] used Ax–Lindemann–Weierstrass to establish strong minimality and triviality of the differential equation of the j-function. This was generalized in [Aslanyan 2021] to a more general and formal setting.
- In [Casale et al. 2020] and [Blázquez-Sanz et al. 2021], the authors go in a quite opposite way: they first establish strong minimality using differential Galois theory, then use Zilber's trichotomy to prove triviality, then use that to establish Ax–Lindemann–Weierstrass and later Ax–Schanuel. That is, they give a new proof to Ax–Schanuel for the j-function and in fact for all Fuchsian automorphic functions.

3. Positive characteristic

The first (to my knowledge) positive characteristic Ax–Schanuel result concerns additive power series. Interestingly, it is not included in the cases considered in Remark 2.4, because such formal maps have no counterpart in the characteristic 0 case, since any additive formal power series in characteristic 0 is linear, so it is "not interesting". This positive characteristic additive Ax–Schanuel result is explained in detail below.

For any commutative algebraic group G, we have the following two (usually noncommutative) rings:

(1) the ring of *algebraic endomorphisms* (that is, endomorphisms of G in the original category of algebraic groups), denoted $\mathrm{End}_{\mathrm{algebraic}}(G)$;

(2) the ring of *formal endomorphisms* (that is, endomorphisms of the formalization of G, as below Definition 2.5, in the category of formal groups), denoted $\mathrm{End}_{\mathrm{formal}}(G)$.

Let C be a field of characteristic $p > 0$ and $\mathbb{G}_{\mathrm{a}}$ denote the additive group scheme over C. We consider the following two rings.

- The ring of additive polynomials over C (with composition), which we denote by $C[\mathrm{Fr}]$. This is also the skew polynomial ring over C and we have the ring isomorphism
$$\mathrm{End}_{\mathrm{algebraic}}(\mathbb{G}_{\mathrm{a}}) \cong C[\mathrm{Fr}].$$
- The ring of additive power series over C (with composition), which we denote by $C[\![\mathrm{Fr}]\!]$. We have the ring isomorphism
$$\mathrm{End}_{\mathrm{formal}}(\mathbb{G}_{\mathrm{a}}) \cong C[\![\mathrm{Fr}]\!].$$

These rings are commutative if and only if $C = \mathbb{F}_p$ and then they are also domains (isomorphic to the rings of polynomials or the ring of power series). We denote the fraction field of $\mathbb{F}_p[\mathrm{Fr}]$ by $\mathbb{F}_p(\mathrm{Fr})$. We state below the main theorem of [Kowalski 2012].

Ax–Schanuel for additive power series [Kowalski 2012, Theorem 1.1]. *Let F be an additive power series over $\mathbb{F}_p$ and assume that*

$$[\mathbb{F}_p(\mathrm{Fr})(F) : \mathbb{F}_p(\mathrm{Fr})] > n.$$

Let $x_1, \ldots, x_n \in t\mathbb{F}_p[\![t]\!]$ be linearly independent over $\mathbb{F}_p[\mathrm{Fr}]$. *Then we have*

$$\mathrm{trdeg}_{\mathbb{F}_p}(x_1, \ldots, x_n, F(x_1), \ldots, F(x_n)) \geqslant n + 1.$$

We describe a general Ax–Schanuel result from [Kowalski 2019], which is valid in all characteristics. We need two technical assumptions. Before stating them, we try to motivate them. One of the crucial properties (used in the proofs in [Ax 1972]) of analytic homomorphisms between algebraic groups is that they take invariant algebraic differential forms into invariant algebraic differential forms. The first technical assumption below, which is absolutely necessary, is both formalizing and generalizing this crucial property. Regarding the second assumption, the exponential map gives a formal isomorphism between any commutative algebraic (and even formal) group in the case of characteristic 0 and a Cartesian power of the additive group. This is false in the positive characteristic case, for example there is no formal isomorphism between the additive and the multiplicative group (no exponential map in positive characteristic!). To make the proofs work, we still need to impose an additional assumption in the positive characteristic case, to mimic the above characteristic 0 situation. The 1-dimensional group H in this assumption plays the role of $\mathbb{G}_a$ and we need to put some extra conditions on H which are true for $\mathbb{G}_a$. We would prefer to avoid this second assumption, but we were unable to do so in [Kowalski 2019].

(1) We define a *special* formal map as one which "resembles a homomorphism" in the sense that it takes invariant differential forms into the "usual" differential forms (before taking the completion; see [Kowalski 2019, Definition 3.10]). In the positive characteristic case, the notion of differential forms has to be replaced by Vojta's notion [2007] of *higher differential forms*; see [Kowalski 2019, Remark 5.18(3)].

(2) We say that a commutative algebraic group A defined over the field C of characteristic p is "good" (see [Kowalski 2019, Definition 3.4]) if there is a one-dimensional algebraic group H over C such that we have the following (in the case of $p = 0$, we drop (c)):
   (a) $\widehat{A} \cong \widehat{H^n}$.
   (b) The map $\mathrm{End}(\widehat{H}) \to \mathrm{End}_C(\Omega_H^{\mathrm{inv}})$ $(= C)$ is onto.
   (c) H is $\mathbb{F}_p$*-isotrivial*, i.e., $H \cong H^{\mathrm{Fr}}$.

To motivate the next result and give a general feeling regarding "what is it about", we quote from [Kowalski 2019] the following, where "the main theorem of this paper" refers to Theorem 3.1.

> A continuous map between Hausdorff spaces which is constant on a dense set is constant everywhere. The same principle applies to an algebraic map between algebraic varieties and to the Zariski topology (which is not Hausdorff). However, if we mix categories there is no reason for this principle to hold, e.g., there are nonconstant *analytic* maps between *algebraic* varieties which are constant on a Zariski dense subset. The main theorem of this paper roughly says that the principle above can be saved for certain formal maps (resembling homomorphisms) between

an algebraic variety and an algebraic group at the cost of replacing the range of the map with its quotient by a formal subgroup of the controlled dimension.

Theorem 3.1. *Let V be an algebraic variety, $\mathcal{K}$ a Zariski dense formal subvariety of V, A a "good" commutative algebraic group and $\mathcal{F} : \widehat{V} \to \widehat{A}$ a special formal map. Assume $\mathcal{F}$ vanishes on $\mathcal{K}$. Then there is a formal subgroup $\mathcal{C} \leqslant \widehat{A}$ such that $\mathcal{F}(\widehat{V}) \subseteq \mathcal{C}$ and*

$$\dim(\mathcal{C}) \leqslant \dim(V) - \dim(\mathcal{K}).$$

As a consequence of Theorem 3.1, we obtained in [Kowalski 2019] a result which is parallel to Ax–Schanuel for additive power series, where an additive power series (that is, a formal endomorphism of the additive group) is replaced with a "multiplicative" power series (that is, a formal endomorphism of the multiplicative group). Let $\mathbb{Z}_p$ denote the ring of p-adic integers. By [Hazewinkel 1978, Theorem 20.2.13(i)], we have the ring isomorphism

$$\mathrm{End}_{\mathrm{formal}}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}_p.$$

We obtain an interesting positive characteristic version (see Example 4.15(3) and Theorem 4.16 in [Kowalski 2019]) of raising to powers Ax–Schanuel (see Theorem 2.3). For $x \in 1 + XC[\![X]\!]$ and $\alpha \in \mathbb{Z}_p$ ($\mathrm{char}(C) = p > 0$), we represent α as $\sum_{i=0}^{\infty} \alpha_i p^i$ for some $\alpha_i \in \{0, 1, \ldots, p-1\}$ and we have

$$x^{\alpha} := \lim_{n\to\infty} \prod_{i=0}^{n} x^{\alpha_i p^i}.$$

Theorem 3.2. *Suppose that $\alpha \in \mathbb{Z}_p$ and $[\mathbb{Q}(\alpha) : \mathbb{Q}] > n$. Let $x_1, \ldots, x_n \in 1 + XC[\![X]\!]$ be multiplicatively independent. Then*

$$\mathrm{trdeg}_{C(X)}(x_1, \ldots, x_n, x_1^{\alpha}, \ldots, x_n^{\alpha}) \geqslant n.$$

There is a general setup including the additive and multiplicative cases, which we describe below following [Kowalski 2019]. Let us fix a positive integer n and a one-dimensional algebraic group H over C. We introduce the following notation from [Kowalski 2019].

- Let $\boldsymbol{R} := \mathrm{End}_{\mathrm{algebraic}}(H)$ and $\boldsymbol{S} := \mathrm{End}_{\mathrm{formal}}(H)$.
- We restrict our attention to algebraic groups H such that $\boldsymbol{S}$ is a commutative domain. We regard $\boldsymbol{R}$ as a subring of $\boldsymbol{S}$.
- Let $\boldsymbol{K}$ denote the field of fractions of $\boldsymbol{R}$ and $\boldsymbol{L}$ be the field of fractions of $\boldsymbol{S}$. We regard $\boldsymbol{K}$ as a subfield of $\boldsymbol{L}$.

Example 3.3. In the characteristic 0 case, we always have $\boldsymbol{S} = C$, so the commutativity assumption is satisfied and we can consider any one-dimensional algebraic group as H. We give some examples below.

(1) If $H = \mathbb{G}_{\mathrm{a}}$ and the characteristic is 0, then $\boldsymbol{R} = \boldsymbol{S} = C$.

(2) If $H = \mathbb{G}_{\mathrm{a}}$ and the characteristic is $p > 0$, then $\boldsymbol{R} = C[\mathrm{Fr}]$ and $\boldsymbol{S} = C[\![\mathrm{Fr}]\!]$ (see the notation introduced in the beginning of this section). This is why we needed to take $C = \mathbb{F}_p$ to ensure that $\boldsymbol{S}$ is commutative.

(3) If $H = \mathbb{G}_{\mathrm{m}}$ and the characteristic is 0, then $\boldsymbol{R} = \mathbb{Z}$. In the case of characteristic $p > 0$, we have $\boldsymbol{S} = \mathbb{Z}_p$ as mentioned above.

Below is our transcendental statement about formal endomorphisms (see [Kowalski 2019, Theorem 4.16.]). We need to introduce the following notions from [Kowalski 2019]. Let A be a commutative algebraic group over the field C of characteristic $p > 0$.

- A formal map into $\widehat{A}$ is an A*-limit map* if it can be "strongly approximated" by a sequence of polynomial maps $(f_n)_n$ in the sense that the differences $f_{n+1} - f_n$ are in the image of the n-th power of the appropriate Frobenius map. For example, any formal endomorphism of $\widehat{\mathbb{G}_{\mathrm{a}}}$ is a $\mathbb{G}_{\mathrm{a}}$-limit map (approximated by additive polynomials), and any formal endomorphism of $\widehat{\mathbb{G}_{\mathrm{m}}}$ is a $\mathbb{G}_{\mathrm{m}}$-limit map (approximated by multiplicative polynomials appearing in the description of x^{α} before the statement of Theorem 3.2).
- We fix a complete C-algebra $\mathcal{R}$ with the residue field C such that $\mathcal{R}$ is linearly disjoint from C^{alg} over C and in the case of characteristic p such that $L^{p^{\infty}} = C$, where L is the fraction field of $\mathcal{R}$ (e.g., $\mathcal{R}$ may be the power series algebra).
- For $x \in A(\mathcal{R})$, we call x *subgroup independent* if for any proper algebraic subgroup $A_0 < A$ defined over C, we have $x \notin A_0(\mathcal{R})$.
- The formal locus of $x \in A(\mathcal{R})$ over C is defined as the formal subscheme of $\widehat{A}$ corresponding to the image of the map $\widehat{\mathcal{O}}_{A,0} \to \mathcal{R}$.
- The number $\operatorname{andeg}(x)$ denotes the dimension of the formal locus of x over C.

Theorem 3.4. *Take $\gamma \in \boldsymbol{S}$ such that $[\boldsymbol{K}[\gamma] : \boldsymbol{K}] > n$ and $\gamma : \widehat{H} \to \widehat{H}$ is an H-limit map. Let $\mathcal{E} : \widehat{A} \to \widehat{A}$ be the n-th cartesian power of γ, where $A = H^n$. Then for any subgroup independent $x \in A(\mathcal{R})_*$ we have*

$$\operatorname{trdeg}_C(x, \mathcal{E}_K(x)) \geqslant n + \operatorname{andeg}_C(x).$$

We showed in [Kowalski 2019] that an unproved version of Theorem 3.1 without the "goodness" assumptions implies the following conjecture. This conjecture is important for the following reasons.

- If the field C has characteristic 0, then this conjecture is a theorem of Ax [1972, Theorem 1F].
- Ax [1972, Section 3] showed that in the case of characteristic 0 (Ax did not consider the positive characteristic case), [Ax 1972, Theorem 1F] implies the

Ax–Schanuel statements regarding the differential equation of the “appropriate” formal/analytic homomorphisms between algebraic groups (Ax focused on the exponential maps on semiabelian varieties). The corresponding implication holds in the positive characteristic case as well.

Main conjecture (arbitrary characteristic). *Let G be an algebraic group over a field C of arbitrary characteristic, $\widehat{G}$ the formalization of G at the origin and $\mathcal{A}$ a formal subgroup of $\widehat{G}$. Let $\mathcal{K}$ be a formal subscheme of $\mathcal{A}$ and let V be the Zariski closure of $\mathcal{K}$ in G. Then there is a formal subgroup $\mathcal{B}$ of $\widehat{G}$ which contains $\mathcal{A}$ and $\widehat{V}$ such that*

$$\dim(\mathcal{B}) \leqslant \dim(V) + \dim(\mathcal{A}) - \dim(\mathcal{K}).$$

We formulate below a specific statement which would follow from this main conjecture.

Specific conjecture. *Suppose that* $\operatorname{char}(C) = p > 0$ *and let* $\gamma : \widehat{\mathbb{G}_{\mathrm{m}}} \to \widehat{E}$ *be a formal isomorphism, where E is an ordinary elliptic curve. Let $x_1, \ldots, x_n \in 1 + XC[\![X]\!]$ be multiplicatively independent. Then*

$$\operatorname{trdeg}_{C(X)}(x_1, \ldots, x_n, \gamma(x_1), \ldots, \gamma(x_n)) \geqslant n.$$

This case seems to be related to the “interesting research paths (1)–(4)” from Section 2B. More precisely, the formal map appearing in the specific conjecture looks “closest” to the exponential map from the original Ax’s theorem, which was used by Zilber to show weak CIT (see Section 2B).

We finish this section with a brief discussion of the case of the Drinfeld modules. Drinfeld [1974] introduced elliptic modules, which are now called *Drinfeld modules*. Drinfeld modules can be understood as certain homomorphisms between $\mathbb{F}_q[X]$ and $K[\mathrm{Fr}]$, where q is a power of p and $K = \mathbb{F}_q((\theta))$ is the non-Archimedean field of Laurent series over $\mathbb{F}_q$. An additive power series over K is associated to each Drinfeld module and this series is entire on K. A number of transcendence results for such additive power series was obtained; see, e.g., [Yu 1986]. To the best of my knowledge, such results never include a version of the full Ax–Schanuel statement. For a survey of this theory, we refer the reader to [Brownawell 1998]. Before the invention of Drinfeld modules, a special case of such a series was introduced by Carlitz, which is now called the *Carlitz exponential* and has the form

$$\exp_C = X + \sum_{i=1}^{\infty} \frac{X^{p^i}}{(\theta^{p^i} - \theta)(\theta^{p^i} - \theta^{p}) \cdots (\theta^{p^i} - \theta^{p^{i-1}})}.$$

There are several Schanuel type results for the Carlitz exponential (see [Denis 1995]) and a Carlitz exponential version of the (still open) conjecture on algebraic independence of logarithms of algebraic numbers was proved in [Papanikolas 2008, Theorem 1.2.6]. The power series we consider do not fit in the Drinfeld module framework, since we consider power series with constant coefficients, that is, there is no transcendental element θ in the coefficients of our additive power series.

4. Recent ideas and speculations

In this section, we describe some recent early stage developments concerning Ax–Schanuel type problems. One of them regards combining the results from [Blázquez-Sanz et al. 2021] with Ax's theorem on the dimension of intersections. The other one is about differential versions of Ax–Schanuel in positive characteristic.

4A. ***Towards a general statement of Ax–Schanuel.*** Ax–Schanuel statements for analytic/formal homomorphisms in the case of characteristic 0 have one "umbrella statement" from which they all follow, which is Ax's theorem on the dimension of intersections from Section 3. No such "umbrella statement" was known for Ax–Schanuel statements for the maps like the j-invariant map until the recent preprint [Blázquez-Sanz et al. 2021], where a general form of an Ax–Schanuel type result is given (see [Blázquez-Sanz et al. 2021, Theorem A]). In this statement, the algebraic group G is again back in the picture (e.g., $G = \mathrm{PGL}_2(\mathbb{C})$ in the case of the j-invariant map), but the statement is quite technical and it is phrased in terms of leaves of flat connections on G-principal bundles, where such a leaf plays the role of the analytic subgroup $\mathcal{A}$ from Ax's theorem on the dimension of intersections from Section 3.

Connection version of Ax–Schanuel [Blázquez-Sanz et al. 2021, Theorem A]. *Let ∇ be a G-principal flat connection on the algebraic bundle $P \to Y$ such that*

- *the algebraic group G is sparse*;
- *the Galois group of ∇ coincides with G.*

Let V be an algebraic subvariety of P and $\mathcal{L}$ be a horizontal leaf of ∇. If

$$\dim V < \dim(V \cap \mathcal{L}) + \dim G$$

then the projection of $V \cap \mathcal{L}$ in Y is contained in a ∇-special subvariety of Y.

Sparsity of the algebraic group G above means that there are no proper Zariski dense complex analytic subgroups of G. The notion of a "∇-special" is more technical; it is phrased in terms of the Galois group of a connection (see [Blázquez-Sanz et al. 2021, Definition 2.4]).

Unlike in the case of [Ax 1972, Theorem 1], no analytic subgroup appears in [Blázquez-Sanz et al. 2021, Theorem A], so this theorem does not generalize [Ax 1972, Theorem 1]. We propose such a generalization which encompasses both the connection version of Ax–Schanuel and [Ax 1972, Theorem 1]. It will appear in [Gogolok and Kowalski ≥ 2024].

Connection and subgroup Ax–Schanuel. *Let ∇ be a G-principal flat connection on the algebraic bundle $P \to Y$ such that the Galois group of ∇ coincides with G and*

- *V is an algebraic subvariety of P,*
- *$\mathcal{A}$ is an analytic subgroup of G,*
- *$\mathcal{L}$ is a horizontal leaf of ∇.*

Suppose that $\mathcal{V}$ is an analytic submanifold of $\mathcal{A}$ which is Zariski dense in V. If

$$\dim V < \dim(V \cap \mathcal{L}) + \dim G$$

then there is an analytic subgroup $\mathcal{H}$ of G such that

$$\dim \mathcal{H} < \dim(V) - \dim(\mathcal{V})$$

and $V \subseteq \mathcal{A}\mathcal{H}$.

The results mentioned above concern the case of characteristic 0. In the "main conjecture" from Section 3, the notion of "analytic" is replaced with the notion of "formal" (see Remark 2.4), which makes sense in the case of arbitrary characteristic. The connection version of Ax–Schanuel [Blázquez-Sanz et al. 2021, Theorem A] mentioned above has not been considered in the positive characteristic case before, since it requires an appropriate version of the notion of a connection in positive characteristic. This is work in progress [Gogolok and Kowalski ≥ 2024].

4B. ***Hasse–Schmidt differential Ax–Schanuel.*** Positive characteristic versions of the differential Ax's theorem have not been studied yet. It is clear that we cannot consider the usual derivations anymore, since the constants of differential fields of positive characteristic contain the image of the Frobenius map, and hence there is no room for any transcendence. It looks natural in this case to replace the derivations with iterative Hasse–Schmidt derivations and the field of constants with the field of absolute constants. We give the necessary definitions below.

- A sequence $\partial = (\partial_n : R \to R)_{n \in \mathbb{N}}$ of additive maps on a ring is called an *HS-derivation* if ∂_0 is the identity map, and for all $n \in \mathbb{N}$ and $x, y \in R$, we have
$$\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y).$$
- An HS-derivation ∂ is called *iterative* if for all $i, j \in \mathbb{N}$ we have
$$\partial_i \circ \partial_j = \binom{i+j}{i} \partial_{i+j}.$$

- If (K, ∂) is a field with a Hasse–Schmidt derivation, then its field of absolute constants is the intersection
$$\bigcap_{i=1}^{\infty} \ker(\partial_i).$$

The passages between the differential Ax–Schanuel and the power series Ax–Schanuel (described in Remark 2.1) work only one way for the positive characteristic case, since the power series ring has a natural iterative Hasse–Schmidt derivation on it. However, it is not clear how to proceed in the opposite way, so Hasse–Schmidt differential Ax–Schanuel type results need to be proved separately. This is work in progress [Gogolok and Kowalski ≥ 2024].

We state below two such results which will appear in [Gogolok and Kowalski ≥ 2024] to give a flavour of these kinds of Ax–Schanuel conditions. Assume that (K, ∂) is a field of characteristic $p > 0$ with a Hasse–Schmidt derivation and C is a field contained in the field of absolute constants of (K, ∂).

Additive Hasse–Schmidt differential Ax–Schanuel. *Let*
$$F = \sum_{m=0}^{\infty} c_m X^{p^m} \in \mathbb{F}_p[\![\mathrm{Fr}]\!]$$
and suppose that the algebraic degree of F over $\mathbb{F}_p(\mathrm{Fr})$ is greater than n. Take $x_1, \dots, x_n, y_1, \dots, y_n \in K$ such that $x_1, \dots, x_n$ are linearly independent over $\mathbb{F}_p[\mathrm{Fr}]$ and for all $i \in \{1, \dots, n\}$,
$$\begin{aligned} D_1(y_i - c_0 x_i) &= 0, \\ D_p(y_i - c_0 x_i - c_1 x_i^p) &= 0, \\ &\vdots \\ D_{p^m}(y_i - c_0 x_i - c_1 x_i^p - \cdots - c_m x^{p^m}) &= 0, \\ &\vdots \end{aligned}$$
Then we have
$$\mathrm{trdeg}_{\mathbb{F}_p}(x_1, y_1, \dots, x_n, y_n) \geqslant n + 1.$$

Multiplicative Hasse–Schmidt differential Ax–Schanuel. *Let*
$$\gamma = \sum c_i p^i \in \mathbb{Z}_p$$
and suppose that the algebraic degree of γ over $\mathbb{Q}$ is greater than n. Take $x_1, \dots, x_n, y_1, \dots, y_n \in K$ such that $x_1, \dots, x_n$ are multiplicatively independent and for all $i \in \{1, \dots, n\}$,

$$
\begin{aligned}
D_1(y_i x_i^{-c_0}) &= 0,\\
D_p(y_i x_i^{-c_0-c_1 p}) &= 0,\\
&\vdots\\
D_{p^m}(y_i x_i^{-c_0-c_1 p-\cdots-c_m p^m}) &= 0,\\
&\vdots
\end{aligned}
$$

Then we have

$$
\operatorname{trdeg}_C(x_1, y_1, \ldots, x_n, y_n) \geqslant n+1.
$$

References

[Aslanyan 2021] V. Aslanyan, "Ax–Schanuel and strong minimality for the j-function", *Ann. Pure Appl. Logic* **172**:1 (2021), art. id. 102871. MR Zbl

[Aslanyan 2022] V. Aslanyan, "Adequate predimension inequalities in differential fields", *Ann. Pure Appl. Logic* **173**:1 (2022), art. id. 103030. MR Zbl

[Ax 1971] J. Ax, "On Schanuel's conjectures", *Ann. of Math.* (2) **93** (1971), 252–268. MR Zbl

[Ax 1972] J. Ax, "Some topics in differential algebraic geometry, I: Analytic subgroups of algebraic groups", *Amer. J. Math.* **94** (1972), 1195–1204. MR Zbl

[Baudisch et al. 2009] A. Baudisch, M. Hils, A. Martin-Pizarro, and F. O. Wagner, "Die böse Farbe", *J. Inst. Math. Jussieu* **8**:3 (2009), 415–443. MR Zbl

[Bertrand 2008] D. Bertrand, "Schanuel's conjecture for non-isoconstant elliptic curves over function fields", pp. 41–62 in *Model theory with applications to algebra and analysis*, vol. 1, edited by Z. Chatzidakis et al., London Math. Soc. Lecture Note Ser. **349**, Cambridge Univ. Press, 2008. MR Zbl

[Blázquez-Sanz et al. 2021] D. Blázquez-Sanz, G. Casale, J. Freitag, and J. Nagloo, "A differential approach to Ax–Schanuel, I", preprint, 2021. arXiv 2102.03384

[Bochner 1946] S. Bochner, "Formal Lie groups", *Ann. of Math.* (2) **47** (1946), 192–201. MR Zbl

[Brownawell 1998] W. D. Brownawell, "Transcendence in positive characteristic", pp. 317–332 in *Number theory* (Tiruchirapalli, India, 1996), edited by V. K. Murty and M. Waldschmidt, Contemp. Math. **210**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Brownawell and Kubota 1977] W. D. Brownawell and K. K. Kubota, "The algebraic independence of Weierstrass functions and some related numbers", *Acta Arith.* **33**:2 (1977), 111–149. MR Zbl

[Casale et al. 2020] G. Casale, J. Freitag, and J. Nagloo, "Ax–Lindemann–Weierstrass with derivatives and the genus 0 Fuchsian groups", *Ann. of Math.* (2) **192**:3 (2020), 721–765. MR Zbl

[Denis 1995] L. Denis, "Indépendance algébrique et exponentielle de Carlitz", *Acta Arith.* **69**:1 (1995), 75–89. MR Zbl

[Drinfeld 1974] V. G. Drinfeld, "Elliptic modules", *Mat. Sb.* (*N.S.*) **94(136)** (1974), 594–627. In Russian; translated in *Math. USSR-Sb.* **23**:4 (1976), 561–592. MR Zbl

[Freitag and Scanlon 2018] J. Freitag and T. Scanlon, "Strong minimality and the j-function", *J. Eur. Math. Soc.* (*JEMS*) **20**:1 (2018), 119–136. MR Zbl

[Gogolok and Kowalski ≥ 2024] J. Gogolok and P. Kowalski, "Connections and differential Ax–Schanuel in arbitrary characteristic", in preparation.

[Hazewinkel 1978] M. Hazewinkel, *Formal groups and applications*, Pure Appl. Math. **78**, Academic Press, New York, 1978. MR Zbl

[Kirby 2009] J. Kirby, "The theory of the exponential differential equations of semiabelian varieties", *Selecta Math.* (*N.S.*) **15**:3 (2009), 445–486. MR Zbl

[Kowalski 2008] P. Kowalski, "A note on a theorem of Ax", *Ann. Pure Appl. Logic* **156**:1 (2008), 96–109. MR Zbl

[Kowalski 2012] P. Kowalski, "Schanuel property for additive power series", *Israel J. Math.* **190** (2012), 349–363. MR Zbl

[Kowalski 2019] P. Kowalski, "Ax–Schanuel condition in arbitrary characteristic", *J. Inst. Math. Jussieu* **18**:6 (2019), 1157–1213. MR Zbl

[Lang 1966] S. Lang, *Introduction to transcendental numbers*, Addison-Wesley, Reading, MA, 1966. MR Zbl

[Manin 1963] J. I. Manin, "Theory of commutative formal groups over fields of finite characteristic", *Uspekhi Mat. Nauk* **18**:6(114) (1963), 3–90. In Russian; translated in *Russ. Math. Surv.* **18**:6 (1963), 1–83. MR Zbl

[Papanikolas 2008] M. A. Papanikolas, "Tannakian duality for Anderson–Drinfeld motives and algebraic independence of Carlitz logarithms", *Invent. Math.* **171**:1 (2008), 123–174. MR Zbl

[Pila 2011] J. Pila, "O-minimality and the André–Oort conjecture for $\mathbb{C}^n$", *Ann. of Math.* (2) **173**:3 (2011), 1779–1840. MR Zbl

[Pila 2015] J. Pila, "Functional transcendence via o-minimality", pp. 66–99 in *O-minimality and diophantine geometry*, edited by G. O. Jones and A. J. Wilkie, London Math. Soc. Lecture Note Ser. **421**, Cambridge Univ. Press, 2015. MR Zbl

[Pila and Tsimerman 2016] J. Pila and J. Tsimerman, "Ax–Schanuel for the j-function", *Duke Math. J.* **165**:13 (2016), 2587–2605. MR Zbl

[Pila and Zannier 2008] J. Pila and U. Zannier, "Rational points in periodic analytic sets and the Manin–Mumford conjecture", *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **19**:2 (2008), 149–162. MR Zbl

[Pila et al. 2021] J. Pila, A. N. Shankar, J. Tsimerman, H. Esnault, and M. Groechenig, "Canonical heights on Shimura varieties and the André–Oort conjecture", preprint, 2021. arXiv 2109.08788

[Seidenberg 1958] A. Seidenberg, "Abstract differential algebra and the analytic case", *Proc. Amer. Math. Soc.* **9** (1958), 159–164. MR Zbl

[Tsimerman 2018] J. Tsimerman, "The André–Oort conjecture for $\mathcal{A}_g$", *Ann. of Math.* (2) **187**:2 (2018), 379–390. MR Zbl

[Vojta 2007] P. Vojta, "Jets via Hasse–Schmidt derivations", pp. 335–361 in *Diophantine geometry* (Pisa, 2005), edited by U. Zannier, CRM Series **4**, Ed. Norm., Pisa, 2007. MR Zbl

[Wagner 2003] F. O. Wagner, "Bad fields in positive characteristic", *Bull. London Math. Soc.* **35**:4 (2003), 499–502. MR Zbl

[Yu 1986] J. Yu, "Transcendence and Drinfeld modules", *Invent. Math.* **83**:3 (1986), 507–517. MR Zbl

[Zilber 2002] B. Zilber, "Exponential sums equations and the Schanuel conjecture", *J. London Math. Soc.* (2) **65**:1 (2002), 27–44. MR Zbl

[Zilber 2005] B. Zilber, "Analytic and pseudo-analytic structures", pp. 392–408 in *Logic Colloquium 2000* (Paris, 2000), edited by R. Cori et al., Lect. Notes Log. **19**, Assoc. Symbol. Logic, Urbana, IL, 2005. MR Zbl

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