Model Theory

no. 3 vol. 3 2024

M ⊨ T

Noncommutative algebraic geometry I: Monomial equations with a single variable

Zlil Sela

msp

Model Theory
Vol. 3, No. 3, 2024
https://doi.org/10.2140/mt.2024.3.733

Noncommutative algebraic geometry
I: Monomial equations with a single variable

Zlil Sela

This paper is the first in a sequence on the structure of sets of solutions to systems of equations over a free associative algebra. We start by constructing a Makanin–Razborov diagram that encodes all the homogeneous solutions to a homogeneous monomial system of equations. Then we analyze the set of solutions to monomial systems of equations with a single variable.

Algebraic geometry studies the structure of sets of solutions to systems of equations usually over fields or commutative rings. The developments and the considerable abstraction that currently exist in the study of varieties over commutative rings still resists application to the study of varieties over nonabelian rings or over other nonabelian algebraic structures.

Since 1960 ring theorists such as P. M. Cohn [1971], G. M. Bergman [1969] and others have tried to study varieties over nonabelian rings, notably free associative algebras (and other free rings). However, the pathologies that they tackled and the lack of unique factorization that they study in detail [Cohn 1971, Chapters 3–4] prevented any attempt to prove or even speculate what can be the structure of varieties over free associative algebras.

In this sequence of papers we suggest studying varieties over free associative algebras using techniques and analogies of structural results from the study of varieties over free groups and semigroups. Over free groups and semigroups geometric techniques as well as low-dimensional topology play an essential role in the structure of varieties. These include Makanin's algorithm for solving equations, Razborov's analysis of sets of solutions over a free group, the concepts and techniques that were used to construct and analyze the JSJ decomposition, and the applicability of the JSJ machinery to study varieties over free groups and semigroups [Sela 2001; 2016]. Our main goal is to demonstrate that these techniques and concepts can be modified to be applicable over free associative algebras as well.

Furthermore, we believe that the concepts and techniques that proved to be successful over free groups and semigroups can be adapted to analyze varieties over

Partially supported by an Israel Academy of Sciences fellowship.
MSC2020: primary 16S10; secondary 03C68, 20F65.
Keywords: varieties, systems of equations, associative algebra, Makanin–Razborov diagram.

free objects in other noncommutative and at least "partially" associative algebraic structures. In that respect, we hope that it will be possible to use or even axiomatize the properties of varieties over the free objects in these algebraic structures, in order to set dividing lines between noncommutative algebraic structures, in analogy with classification theory (of first-order theories) in model theory [Shelah 1990].

We start the analysis of systems of equations over a free associative algebra with what we call monomial systems of equations. These are systems of equations over a free associative algebra in which every polynomial in the system contains two monomials. In Section 1 we analyze the case of homogeneous solutions to homogeneous monomial systems of equations. In this case it is possible to apply the techniques that were used in analyzing varieties over free semigroups [Sela 2016], and associate a Makanin–Razborov diagram that encodes all the homogeneous solutions to a homogeneous monomial system of equations.

In Section 2 we introduce *limit algebras*, which are a natural analogue of a *limit group*, and prove that such algebras are always embedded in (limit) division algebras (in analogy with the embeddings of limit semigroups in limit groups, that we termed *limit pairs* in [Sela 2016]). The automorphism (modular) groups of these division algebras are what is needed in the sequel in order to modify and shorten solutions to monomial systems of equations.

In Section 3 we present a combinatorial approach to (cases of) the celebrated Bergman's centralizer theorem [1969]. Finally, in the fourth section we use this combinatorial approach to analyze the set of solutions to a monomial system of equations with a single variable. The results that we obtain are analogous to the well known structure of the set of solutions to systems of equations with a single variable over a free group or semigroup. We prove all our results under the assumption that the top homogeneous parts of the coefficients in the equations are monomials with no periodicity, in order to simplify our arguments, but we believe that eventually this assumption can be dropped.

In the next paper in the sequence we use the techniques that are presented in this paper to analyze monomial systems of equations that have more than a single variable, but have no quadratic (or surface) parts. In the third paper in the sequence we analyze the quadratic parts of monomial systems of equations. Eventually, we hope to use our analysis of sets of solutions to monomial systems of equations to the analysis of general varieties.

1. Homogeneous solutions of monomial equations

For simplicity, we will always assume that the free algebras that we consider are over the field with two elements GF_2. Let FA be a free associative algebra over GF_2: $FA = GF_2\langle a_1, \dots, a_k\rangle$. In order to study the structure of general varieties over the associative algebra FA, we start with varieties that are defined by monomial systems of equations. A system of equations Φ is called *monomial* if it is defined using a finite set of unknowns $x_1, \dots, x_n$, and a finite set of equations

$$u_1(c_1, \dots, c_\ell, x_1, \dots, x_n) = v_1(c_1, \dots, c_\ell, x_1, \dots, x_n),$$
$$\vdots$$
$$u_s(c_1, \dots, c_\ell, x_1, \dots, x_n) = v_s(c_1, \dots, c_\ell, x_1, \dots, x_n),$$

where the words u_i and v_i are monomials in the free algebra generated by the variables $x_1, \dots, x_n$ and coefficients $c_1, \dots, c_\ell$ from the algebra FA, i.e., a word in the free semigroup generated by these elements (note that the coefficients $c_1, \dots, c_\ell$ are general elements and not necessarily monomials). A monomial system of equations is called *homogeneous* if all the coefficients $c_1, \dots, c_\ell$ in the system are homogeneous elements in the free associative algebra FA.

We start by analyzing all the homogeneous solutions of a homogeneous monomial system, i.e., all the assignments of homogeneous elements in FA to the variables $x_1, \dots, x_n$ such that the equalities in a homogeneous monomial system of equations are valid.

Let $x_1^0, \dots, x_n^0$ be a homogeneous solution of the monomial system Φ. Substituting the elements $x_1^0, \dots, x_n^0$ in the monomials u_i and v_i, $1 \leq i \leq s$, we get a finite set of equalities in the free algebra FA. Since all the elements that appear in each of these equalities are homogeneous, for each index i we can associate a segment J_i of length that is equal to the degree of u_i and v_i after the substitution of $x_1^0, \dots, x_n^0$. We further add notation on the segment J_i for the beginnings and the ends of each of the elements $x_1^0, \dots, x_n^0$ and the coefficients $c_1, \dots, c_\ell$ of the system.

With the segments $J_1, \dots, J_s$, and the notation for the beginnings and ends of $x_1^0, \dots, x_n^0$ and $c_1, \dots, c_\ell$, we can naturally associate a generalized equation as in [Makanin 1977; Casals-Ruiz and Kazachkov 2011], or alternatively a band complex (bands are added for different appearances of the same variable) as it appears in [Bestvina and Feighn 1995]. All the lengths that appear in the band complex are integers, so the band complex must be simplicial. Note that all the operations that are used in the Rips machine, or in the Makanin procedure, to transfer the original complex into a standard band complex are valid in our context, i.e., it is possible to cut the elements $x_1^0, \dots, x_n^0$ and $c_1, \dots, c_\ell$ and represent them as multiplication of new elements according to the operations that are performed in modifying the band complexes (or the generalized equation) along the procedure.

To clarify the applicability of the Makanin moves, one can look at the band complex or the corresponding Makanin generalized equation differently. Given the homogeneous solution $x_1^0, \dots, x_n^0$, and substituting it in the homogeneous monomial system of equations, we can naturally associate with each side of a monomial equation a homogeneous tree. Since each of the trees is composed from homogeneous elements, there are no cancellations between paths (monomials) in each separate tree, so the monomial equation implies that the homogeneous trees that are associated with the two sides of the equation are identical.

Now, the identical trees that are associated with the two sides of a monomial equation admit two product structures that are associated with the two sides of the equation. Therefore, the tree that is associated with a monomial equation admits a product structure which is the common refinement of the product structure coming from the two sides of the equation. Each band in the band complex, or alternatively each pair of bases in the Makanin generalized equation that is associated with the system, indicates that a certain part in this refined product structure of the tree that is associated with one monomial equation is identical to another part in the product structure of a tree that is associated with another (possibly the same) monomial equation. Alternatively, homogeneous elements in a free associative algebra have the unique factorization property. Hence, given two factorizations of a homogeneous element, there is a common refinement of the two factorizations.

Furthermore, each of the basic Makanin moves that can be performed on generalized equations can be performed in an identical way on the homogeneous trees that are associated with homogeneous monomial equations using their refined product structure. This means that the entire Makanin process to analyze solutions to systems of equations over a free semigroup, which is composed from sequences of basic moves, can be applied to the product structures of homogeneous trees that are associated with homogeneous monomial systems of equations.

The ability to apply the Makanin basic moves to the generalized equation or the band complex that is associated with a homogeneous system of monomial equations implies that it is possible to associate with such a system of equations a Makanin–Razborov diagram, using the construction of such a diagram for a system of equations over a free semigroup as it appears in [Sela 2016]. As in a free semigroup, the constructed diagram encodes all the homogeneous solutions to the homogeneous system of equations in the free algebra *FA*.

Let G_Φ be the semigroup that is generated by copies of $x_1, \dots, x_n$ and the coefficients $c_1, \dots, c_\ell$ modulo the relations

$$u_i(c_1, \dots, c_\ell, x_1, \dots, x_n) = v_i(c_1, \dots, c_\ell, x_1, \dots, x_n)$$

for $1 \leq i \leq s$, where the monomials u_i and v_i are interpreted as words in a free semigroup. With a (homogeneous) solution of the system Φ it is possible to associate a homomorphism from G_Φ into the free semigroup that is generated by a free generating set of *FA*.

Conversely, given a semigroup homomorphism of G_Φ into a free semigroup that fits with a decomposition of the constants $c_1, \dots, c_\ell$ into a product of homogeneous elements (there are finitely many possible ways to represent each of the coefficients $c_1, \dots, c_\ell$ as such a product), it is possible to associate with such a product a family of solutions of the systems Φ.

Therefore, the study of homogeneous solutions of a homogeneous monomial system of equations over an associative free algebra is reduced to the study of a collection of semigroup homomorphisms from a given f.g. semigroup into a free semigroup. By [Sela 2016] with this collection of semigroup homomorphisms it is possible to associate canonically a finite collection of pairs $(S_1, L_1), \dots, (S_m, L_m)$, where each of the groups L_j is a limit group, and each of the semigroups S_j is a f.g. subsemigroup that generates L_j. Furthermore, with G_Φ and its collection of homomorphisms it is possible to associate (noncanonically) a Makanin–Razborov diagram that encodes all its homomorphisms into free semigroups. By our observation, this Makanin–Razborov diagram of pairs encodes all the homogeneous solutions of the system Φ in the algebra *FA*.

Theorem 1.1. *With a homogeneous monomial system of equations over the free associative algebra FA it is possible to associate* (*noncanonically*) *a Makanin–Razborov diagram that encodes all its homogeneous solutions.*

As a corollary of the encoding of homogeneous solutions of a system of homogeneous monomial equations by pairs of limit groups and their subsemigroups we get the following.

Corollary 1.2. *The collections of sets of homogeneous solutions to homogeneous monomial systems of equations is Noetherian, i.e., every descending sequence of such sets terminates after a finite time.*

Proof. Follows immediately from the descending chain condition for limit groups [Sela 2001], or the Noetherianity of varieties over free groups and semigroups [Guba 1986]. □

Theorem 1.1 associates a Makanin–Razborov diagram with the set of homogeneous solutions to a homogeneous monomial systems of equations. Our main goal in this sequence of papers is to associate a Makanin–Razborov diagram with the set of (not necessarily homogeneous) solutions of a general monomial system of equations, at least in the minimal rank case, i.e., in the case in which the Makanin–Razborov diagram that is associated with the homogeneous system that is associated with top homogeneous part of the nonhomogeneous system contains no free products.

2. Limit algebras, their division algebras and modular groups

The construction of the Makanin–Razborov diagram of a system of equations over a free group uses extensively the (modular) automorphism groups of the limit groups that are associated with its nodes. These modular groups, defined in [Sela 2001, Definition 5.2], enable one to proceed from a limit group to maximal shortening quotients of it that are always proper quotients.

The semigroups that appear in the construction of the Makanin–Razborov diagram of a system of equations over a free semigroup do not have a large automorphism group in general, e.g., a finitely generated free semigroup has a finite automorphism group. Hence, to study homomorphisms from a given f.g. semigroup S to the free semigroup FS_k we did the following in [Sela 2016].

Given a f.g. semigroup S we can naturally associate a group with it. Given a presentation of S as a semigroup, we set the f.g. group $\mathrm{Gr}(S)$ to be the group with the presentation of S interpreted as a presentation of a group. Clearly, the semigroup S is naturally mapped to the group $\mathrm{Gr}(S)$ and the image of S in $\mathrm{Gr}(S)$ generates $\mathrm{Gr}(S)$. We set $\eta_S : S \to \mathrm{Gr}(S)$ to be this natural homomorphism of semigroups.

The free semigroup FS_k naturally embeds into a free group F_k. By the construction of the group $\mathrm{Gr}(S)$, every homomorphism of semigroups $h : S \to FS_k$ extends to a unique homomorphism of groups $h_G : \mathrm{Gr}(S) \to F_k$ such that $h = h_G \circ \eta_S$.

By construction, every homomorphism (of semigroups) $h : S \to FS_k$ extends to a homomorphism (of groups) $h_G : \mathrm{Gr}(S) \to F_k$. Therefore, the study of the structure of $\mathrm{Hom}(S, FS_k)$ is equivalent to the study of the structure of the collection of homomorphisms of groups $\mathrm{Hom}(\mathrm{Gr}(S), F_k)$ that restrict to homomorphisms of (the semigroup) S into the free semigroup (the *positive cone*) FS_k.

By (canonically) associating a finite collection of maximal limit quotients with the set of homomorphisms $\mathrm{Hom}(\mathrm{Gr}(S), F_k)$ that restrict to (semigroup) homomorphisms from S to FS_k, we are able to (canonically) replace the pair $(S, \mathrm{Gr}(S))$ with a finite collection of limit quotients $(S_1, L_1), \dots, (S_m, L_m)$, where each of the groups L_i is a limit group. Limit groups have rich modular groups, and these are later used to proceed to the next levels of the Makanin–Razborov diagram of the given system of equations over the free semigroup FS_k.

In studying sets of solutions to systems of equations over a free associative algebra, we need to study homomorphisms: $h : A \to FA_k$, where A is a f.p. algebra and FA_k is the free associative algebra of rank k. As in the case of groups and semigroups, to study such homomorphisms we pass to convergent sequences of homomorphisms $\{h_n : A \to FA_k\}$, and look at the *limit algebras* LA that are associated with such convergent sequences. Algebras, and in particular limit algebras, have automorphisms, but these are not the automorphisms that will be needed in the sequel to modify and shorten homomorphisms.

By a classical construction of [Malcev 1948; Neumann 1949], and by different constructions of Amitsur [1966] and others, the free associative algebra FA_k can be embedded into a division algebra $\mathrm{Div}(FA_k)$ (note that there are various different division algebras into which FA_k embeds). Given a convergent sequence

$\{h_n : A \to FA_k\}$ with an associated limit algebra LA, it is straightforward to get an embedding $LA \to \mathrm{Div}(LA)$, where $\mathrm{Div}(LA)$ is a division algebra that is also obtained from the convergent sequence and from the embedding $FA_k \to \mathrm{Div}(FA_k)$.

In the sequel we will use (a subgroup of) the group of automorphisms of the division algebra $\mathrm{Div}(LA)$ in order to modify (shorten) the homomorphisms $h : A \to FA_k$ that we need to study. These will be the modular groups that are associated with limit algebras that appear along the nodes of the Makanin–Razborov diagrams of the given systems of equations over the free associative algebra FA_k.

An important example is (a special case of) what we call *surface* (or *quadratic*) *algebra*:

$$SA = \langle x_1, \dots, x_n \mid x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)} \rangle$$

for an appropriate permutation $\sigma \in S_n$. Such a surface algebra is a limit algebra. Hence, it is embedded in a division algebra $\mathrm{Div}(SA)$. For appropriate convergent sequences, the modular group of $\mathrm{Div}(SA)$ contains the automorphism group of a corresponding surface. Therefore, we call the modular group of $\mathrm{Div}(SA)$ the *Bergman modular group* of a surface algebra, since it contains (or is generated by) generalized Dehn twists that are inspired by Bergman's centralizer theorem [1969]. These modular groups generalize the mapping class groups of surfaces; they will be defined in the sequel, and they play an essential role in constructing Makanin–Razborov diagrams for monomial systems of equations over a free associative algebra.

3. A combinatorial approach to Bergman's theorem

In the first section we studied homogeneous solutions to homogeneous monomial systems of equations. In this section we start the study of nonhomogeneous solutions to arbitrary monomial systems of equations. We start by studying the centralizers of elements in a free associative algebra, i.e., we give combinatorial proof to Bergman's theorem, and then use these techniques to study related systems of equations. We start with the following theorem, which can be proved easily by a direct induction, but we also present a proof that uses techniques that we will use in the sequel.

Theorem 3.1. *Let $u \in FA$ be an element for which its top degree homogeneous part is a monomial, and suppose that this top degree monomial has no nontrivial roots. Then the centralizer of u in FA is precisely the elements in the* (*one variable*) *algebra that is generated by u.*

Proof. Suppose that x is a (nontrivial) element that satisfies $xu = ux$. By our analysis of homogeneous elements, the top degree homogeneous part of x must be a monomial which is a power of the top degree monomial in u. Hence, the top degree monomial of x has to be identical to the top degree monomial of u^m for some m. Therefore, $\deg(x+u^m) < \deg(x)$ and $u(x+u^m) = (x+u^m)u$, so the theorem follows by induction on the degree of x. □

For later applications we present a different proof.

Proof. First, note that $xu = ux$ if and only if $x(u+1) = (u+1)x$. Hence, we may assume that the monomials in u do not include the one corresponding to the identity.

In the sequel we denote by G^m the (additive) abelian group that is generated by all the monomials of degree at most m in FA. Given $x, y \in FA$, we write $x = y \mod G^m$ if x and y have the same monomials of degree bigger than m. If $x \in FA$, we say that a monomial w is a monomial of codegree m in x if $\deg(w)+m = \deg(x)$, and w is in the support of x (i.e., w appears in writing x as a sum of monomials).

Lemma 3.2. *Suppose that* $\deg(u) \geq 2$, *the top degree homogeneous part of u is a monomial and has no periodicity, and that* $\deg(x) \geq 2\deg(u)$. *There exists an element* $w \in FA$, *such that*

$$x = uw = wu \qquad \mod G^{\deg(x)-2},$$
$$ux = xu = uwu \qquad \mod G^{\deg(ux)-2}.$$

Proof. We analyze the codegree 1 monomials in the two sides of the equation $xu = ux$. Let u_0, x_0 be the top monomials, and let u_1, x_1 be the codegree 1 element in u, x. Clearly, $x_0u_1 + x_1u_0$ is the codegree 1 element in xu, and $u_0x_1 + u_1x_0$ is the codegree 1 element in ux.

Suppose that there are no cancellations between the codegree 1 monomials (that are obtained using the distributive law) in each of the two sides of the equation. In that case, since u_0 was assumed to have no periodicity, monomials in x_0u_1 cannot be monomials in u_1x_0, since otherwise x_0 overlaps with itself in a shift of a single place, and x_0 has a period which is u_0 that has degree at least 2 by the assumption of the lemma. Hence, monomials in x_0u_1 have to be monomials in u_0x_1. Similarly, monomials in u_1x_0 have to be monomials in x_1u_0.

Hence, if $x_0 = u_0^m$ and $w_0 = u_0^{m-1}$, then from the right side of the equation $xu = ux$, $x_1 = u_1w_0 + u_0\hat{w}_1$. From the left side of the equation, $x_1 = w_0u_1 + \hat{w}_2u_0$. So if we consider elements of codegree at most 1, $x = uw_1 = w_2u$. From the equation $xu = ux$, we get that for elements of codegree at most 1, $uw_1u = uw_2u$, so $u(w_1 - w_2)u$=0, so $w_1 = w_2$, and $x = wu = uw$, for elements of codegree at most 1 (for some element w of degree $\deg(w) = \deg(x) - \deg(u)$).

Suppose that there are cancellations between codegree 1 monomials in the left-hand side xu. In that case monomials in x_0u_1 cancel with monomials in x_1u_0. Let v_1 be the codegree 1 suffix of u_0, and y_1 be the unique monomial in x_1 for which $x_0v_1 = y_1u_0$. In that case $y_1 = u_0\widetilde{w}$. Hence, the monomial y_1 has a product structure which is similar with the other codegree 1 monomials in x_1. Therefore, as in the previous case, when analyzing monomials of codegree at most 1 in x, x can be described both as w_1u and uw_2 (from the two sides of the equation), and the same argument that was used in case there are no cancellations works. □

We continue the proof of Theorem 3.1, by iteratively uncovering the homogeneous parts in an element x that is in the centralizer of u from top to bottom. Since $x = t_1u \mod G^{\deg(x)-2}$, if $\deg(x) \geq 2\deg(u)$, it follows that $xu = t_1u^2$ and $ux = ut_1u$ mod $G^{\deg(xu)-2}$. Hence, if $\deg(t_1) \geq 2\deg(u)$, then $t_1 = t_2u = ut_2 \mod G^{\deg(t_1)-2}$. Applying these arguments iteratively we get that $x = tu^m \mod G^{\deg(x)-2}$, for some t that satisfies $\deg(t) = \deg(u)$.

Therefore, $xu = tu^m u = ux = utu^m \mod G^{\deg(xu)-2}$, which means that $tu = ut$ mod $G^{\deg(tu)-2}$ and $\deg(t) = \deg(u)$.

In this case, in which $tu = ut$ and $\det(t) = \deg(u)$, the top degree monomials of t and u are identical, and we denote this monomial u_0. Suppose that s, v are monomials of codegree 1 in either t or u, and suppose that $su_0 = u_0v$. In that case v is the suffix of u_0 and s is the prefix of u_0. Since u_0 is not a proper power, vu_0 cannot be presented as u_0w for any codegree 1 monomial w, and u_0s cannot be presented as wu_0 for any codegree 1 monomial w.

Hence, if s, v are codegree 1 monomials in either t or u, and $su_0 = u_0v$, then both u_0s and vu_0 can be presented uniquely in each of the two products tu and ut, which implies that s and v must be codegree 1 monomials in both u and t. Therefore, the codegree 1 monomials of t and u must be identical, so $t = u \mod G^{\deg(t)-2}$, and $x = u^{m+1} \mod G^{\deg(x)-2}$ for some nonnegative integer m.

We use (a finite) induction and assume that $x = u^{m+1} \mod G^{\deg(x)-c}$, for some positive integer $c < \deg(u)$, i.e., we assume that the equality holds for all the monomials in x and u of codegree smaller than c. To complete the proof of the theorem, we need to prove the same equality for all the monomials of codegree at most c.

By the inductive hypothesis, $x = u^{m+1} \mod G^{\deg(x)-c}$. Hence, $x = x_{c-1} + v$, where x_{c-1} is the sum of all the monomials of codegree smaller than c in x and $\deg(v) \leq \deg(x) - c$. Furthermore, x_{c-1} is precisely the sum of all the monomials of codegree smaller than c in u^{m+1}.

Let u_{c-1} be the sum of the monomials of codegree less than c in u. We set s_c to be the sum of all the monomials of codegree c in u_{c-1}^{m+1}. By construction, $u_{c-1}(x_{c-1} + s_c) = (x_{c-1} + s_c)u_{c-1} = u_{c-1}^{m+2} \mod G^{\deg(xu)-(c+1)}$, i.e., the monomials of codegree at most c are identical for the three different products.

Recall that $x = x_{c-1} + v$, where $\deg(v) \leq \deg(x) - c$. We set $x = x_{c-1} + s_c + r$, where $\deg(s_c) = \deg(x) - c$ and $\deg(r) \leq \deg(x) - c$. Let q_c be the sum of the monomials of codegree c in u. Then

$$ux = (u_{c-1}+q_c)(x_{c-1}+s_c+r) = xu = (x_{c-1}+s_c+r)(u_{c-1}+q_c) \mod G^{\deg(xu)-(c+1)}.$$

Since $u_{c-1}(x_{c-1}+s_c)=(x_{c-1}+s_c)u_{c-1} \mod G^{\deg(xu)-(c+1)}$, it follows that

$$u_{\text{top}}r+q_c x_{\text{top}}=r u_{\text{top}}+x_{\text{top}}q_c \mod G^{\deg(xu)-(c+1)},$$

where u_0 and x_0 are the top monomials in u and x in correspondence. Therefore, all these monomials are products of a top degree monomial with a codegree c monomial, and these can be broken precisely as in the codegree 1 case, assuming $\deg(x)\geq 2\deg(u)$.

We are left with the case in which $\deg(x)=\deg(u)$. In that case we write $u=u_{c-1}+q_c$ and $x=u_{c-1}+r_c \mod G^{\deg(x)-(c+1)}$, where q_c and r_c are the codegree c monomials in u and x in correspondence. Since the contributions of products of monomials of codegree smaller than c in xu and in ux are identical, we need to look only at the equation $u_0 r_c+q_c x_0=x_0 q_c+r_c u_0$ for the monomials of codegree c, where $x_0=u_0$ are identical monomials. By the argument that was used in the codegree 1 case (when $\deg(x)=\deg(u)$), it follows that $q_c=r_c$, and the general step of the induction is proved.

So far we may conclude that $x=u^{m+1} \mod G^{\deg(x)-\deg(u)}$. Thus, $x+u^{m+1}$ commutes with u and $\deg(x+u^{m+1})\leq\deg(x)-\deg(u)$, and the theorem follows. □

So far we assumed that the top homogeneous element of u is a monomial and that its top monomial doesn't have a proper root. We continue by allowing u to be a proper power.

Theorem 3.3. *Let $u\in FA$ be an element for which its top degree homogeneous part is a monomial, and suppose that $u=p(v)$ and the top degree monomial of v does not have a proper root. Then the centralizer of u in FA is precisely the elements in the algebra that are generated by v.*

Proof. Suppose that x is a (nontrivial) element that satisfies $xp(v)=p(v)x$. First, note that like Theorem 3.1, Theorem 3.3 can be proved easily by replacing x by $x+v^m$ for an appropriate m such that $\deg(x+v^m)<\deg(x)$ and $(x+v^m)p(v)=p(v)(x+v^m)$. However, as in the proof of Theorem 3.1 and for future purposes, we prefer to present a different proof. For that proof we assume that $\deg(v)>1$.

As in Theorem 3.1, by our analysis of homogeneous elements, the top degree homogeneous part of x must be a monomial, which is a power of the top degree monomial in v.

As in the proof of Theorem 3.1, if $\deg(x)>\deg(u)$, then the arguments that were used in the proof of Lemma 3.2, which remain valid under the assumptions of the theorem, enable us to analyze the codegree 1 monomials in x. In that case, as in the proof of Theorem 3.1, there exists an element t_1 that contains a top degree monomial and a homogeneous part of codegree 1 such that $x=ut_1$ and $x=t_1u \mod G^{\deg(x)-2}$.

Applying these arguments iteratively, as in the proof of Theorem 3.1, we get that $x=tu^m \mod G^{\deg(x)-2}$ for some t that satisfies $\deg(t)\leq\deg(u)$, which means that $tu = ut \mod G^{\deg(tu)-2}$. In particular, the top degree monomial of t must be a power of the top degree monomial of v.

In case $\deg(t) \leq \deg(u) \mod G^{\deg(tu)-2}$, we apply the same argument that we used in case $\deg(t) = \deg(u)$ in the proof of Theorem 3.1. By these arguments, if $u = v^b \mod G^{\deg(u)-2}$, then $t = v^s \mod G^{\deg(t)-2}$, where s is an integer, $1 \leq s \leq b$. This implies that $x = v^\ell \mod G^{\deg(x)-2}$ for some positive integer ℓ.

We continue in the same way as we did in proving Theorem 3.1. We use (a finite) induction and assume that $x = v^\ell \mod G^{\deg(x)-c}$ for some positive integer $c < \deg(v)$, i.e., we assume that the equality holds for all the monomials in x and v^ℓ of codegree smaller than c. To complete the proof of the theorem, we need to prove the same equality for all the monomials of codegree at most c.

By the inductive hypothesis, $x = v^\ell \mod G^{\deg(x)-c}$. Hence, $x = x_{c-1} + h$, where x_{c-1} is the sum of all the monomials of codegree smaller than c in x and $\deg(h) \leq \deg(x) - c$. Furthermore, x_{c-1} is precisely the sum of all the monomials of codegree smaller than c in v^ℓ.

Let u_{c-1} be the sum of the monomials of codegree less than c in u, and let v_{c-1} be the sum of the monomials of codegree less than c in v. We set s_c to be the sum of all the monomials of codegree c in v_{c-1}^ℓ.

We have $u = p(v)$, so we set d_c to be the sum of all the codegree c monomials in $p(v_{c-1})$. By construction: $(u_{c-1} + d_c)(x_{c-1} + s_c) = (x_{c-1} + s_c)(u_{c-1} + d_c) = (v_{c-1})^{\ell+b} \mod G^{\deg(ux)-(c+1)}$, i.e., the monomials of codegree at most c are identical for the three different products.

Recall $x = x_{c-1} + h$, where $\deg(h) \leq \deg(x) - c$. We set $x = x_{c-1} + s_c + r$, where $\deg(s_c) = \deg(x) - c$ and $\deg(r) \leq \deg(x) - c$. Similarly, we set $u = u_{c-1} + d_c + q$, where $\deg(q) \leq \deg(u) - c$. Then

$$
\begin{aligned}
ux &= (u_{c-1} + d_c + q)(x_{c-1} + s_c + r) \\
&= xu = (x_{c-1} + s_c + r)(u_{c-1} + d_c + q) \mod G^{\deg(xu)-(c+1)}.
\end{aligned}
$$

Since $(u_{c-1} + d_c)(x_{c-1} + s_c) = (x_{c-1} + s_c)(u_{c-1} + d_c)$ modulo the same group, it follows that $u_{\text{top}} r + q x_0 = r u_0 + x_0 q \mod G^{\deg(xu)-(c+1)}$, where u_0 and x_0 are the top monomials in u and x in correspondence. Therefore, all these monomials are products of a top degree monomial with a codegree c monomial, and these can be broken precisely as in the codegree 1 case, assuming $\deg(x) > \deg(u)$.

As in the codegree 1 case, we are left with the case in which $\deg(x) \leq \deg(u)$. In that case we write $u = u_{c-1} + d_c + q_c$ and $x = x_{c-1} + s_c + r_c$ as above. By the same argument that was used in that case in analyzing the codegree 1 monomials, the monomials of codegree c in x that are contained in r_c are precisely the monomials of codegree c in $v^\ell + s_c$, and the induction follows for $c \leq \deg(v)$. Hence, $x = v^\ell \mod G^{\deg(x)-\deg(v)}$. Since both x and v^ℓ commute with u, the sum $x + v^\ell$ commutes with u, and $\deg(x + v^\ell) < \deg(x)$, the theorem follows. □

It is possible to use the techniques that we used in this section to analyze centralizers of general elements with monomial top homogeneous part, and centralizers of general elements, but we won't need to apply these techniques in this generality in the sequel, so we omit these generalizations.

4. Equations with a single variable

In the previous section we gave combinatorial proofs to special cases of Bergman's theorem on the structure of centralizers in free associative algebras. Such combinatorial proofs are needed in order to study the set of solutions to related systems of equations that play a central role in understanding the set of solutions to a general monomial system of equations.

In this section we study the set of solutions to monomial systems of equations with a single variable. As will be demonstrated in the next paper in this sequence, the techniques that are used in this section play an essential role in studying monomial equations with no quadratic nor free parts. Note that in the general analysis of the set of homogeneous solutions to the homogeneous system of equations that is associated with the top level of a monomial system of equations, the Makanin–Razborov diagram of such a homogeneous system of equations, as it appears in the first section of this paper (that is based on [Sela 2016]), may contain quadratic and free parts.

Recall that over free groups and semigroups equations with a single variable were analyzed in [Lorenc 1963; Appel 1968] long before the analysis of general systems of equations. The approach that we use combines the technique and results for studying equations with a single variable over a free group and semigroup with the combinatorial approach that we used in analyzing centralizers.

Lemma 4.1. *Let $u, v \in FA$ and suppose that the top homogeneous parts of u and v are monomials with no periodicity (i.e., the top monomials in u and v contain no subword α^2 for some nontrivial word α).*

If the equation $ux = xv$ has a nontrivial solution, then the set of solutions to the equation $ux = xv$ is a set $\{wp(v)\}$, where $uw = wv$ and p is an arbitrary polynomial in a single variable. Furthermore, the element w, which is the solution of minimal degree of the equation, is unique.

Proof. Recall that in a free semigroup, if u and v are nontrivial and have no periodicity, then the set of solutions to the equation $ux = xv$ is $\{w_0 v^m = u^m w_0\}$, where w_0 is a fixed element, m is an arbitrary nonnegative integer, and $\text{length}(w_0) \leq \text{length}(u)$.

Also, note that since we assumed that the top homogeneous parts of u and v are monomials, then the homogeneous equation that is associated with the highest degree parts in u, x, v implies that the highest degree part of x is a monomial that satisfies the same equation in the corresponding free semigroup.

The set of solutions of the equation $ux = xv$ is a linear subspace of *FA*. If w_1 and w_2 are solutions to the equation $ux = xv$, and they are of the same degree, then their top homogeneous monomials are identical. Hence, $w_1 + w_2$, which is also a solution of that equation, has strictly smaller degree than w_1 and w_2. Therefore, if the equation $ux = xv$ has a solution, then it has a unique solution of minimal degree that we denote w.

If x_0 is an arbitrary solution of $ux = xv$, then there exists some nonnegative integer b such that wv^b and x_0 have the same top monomial. Since both x_0 and wv^b are solutions of the equation $ux = xv$, the sum $x_0 + wv^b$ is a solution of this equation and $\deg(x_0 + wv^b) < \deg(x_0)$. Hence, the proof of the lemma follows by induction on the degree of the solution x_0. □

Unlike the case of free groups or semigroups, the equation $ux = xv$ may have a solution, and still it can be that there are no solutions with $\deg(x) \leq \deg(u) = \deg(v)$.

Let t, μ, and ρ be arbitrary elements in the algebra *FA*. Let $w = t\mu t\rho t\mu t$, $v = (\rho t\mu + \mu t\rho t\mu)t$ and $u = t(\mu t\rho + \mu t\rho t\mu)$. Then $uw = wv$, and in general there is no element $y \in FA$ such that $\deg(y) \leq \deg(u) = \deg(v)$ and $uy = yv$.

To bound the degree of a minimal degree solution we need the following lemma.

Lemma 4.2. *Let FA be the free associative algebra over GF_2 that is freely generated by k elements. Let $u, v \in FA$ be as in Lemma 4.1, and suppose that the equation $ux = xv$ has a nontrivial solution. Then there exists a solution w, $uw = wv$, with* $\deg(w) \leq \deg(u) \cdot (k^{\deg(u)} + 2)$.

Proof. Suppose that $x_1 \neq 0$ satisfies $ux_1 = x_1v$. If $\deg(x_1) \leq \deg(u) \cdot (k^{\deg(u)} + 2)$ the lemma follows. Hence, we may assume that $\deg(x_1) > \deg(u) \cdot (k^{\deg(u)} + 2)$.

We use the analysis that was applied in analyzing centralizers in the previous section. By the analysis of homogeneous elements, the top degree homogeneous part of x_1 must be a monomial. Let u_0, v_0, and x_0 be the top monomials of u, v, and x_1. Then they must satisfy $u_0x_0 = x_0v_0$. Therefore, there exists a monomial z_0 such that $x_0 = u_0z_0 = z_0v_0$.

As in analyzing centralizers, we continue the analysis of x_1 by analyzing its codegree 1 monomials. We examine the codegree 1 monomials in the products ux_1 and x_1v. By the proof of Lemma 3.2 we get an element z such that $x_1 = zv = uz \mod G^{\deg(x_1)-2}$.

We continue iteratively by analyzing products of codegree 2 in the equality $ux_1 = x_1v$, using the equality $x_1 = uz = zv$ for the top and codegree 1 parts.

Note that monomials of codegree 2 in the equality $ux_1 = x_1v$ that are products of monomials of codegree 0 and 1 of u, v and z, that correspond to codegree 1 monomials of u and v and codegree 1 monomials of x_1 (from the two sides of the equation), cancel in pairs. The other codegree 2 monomials in the two sides of the equation are products that are obtained as one of the following:

(1) A product of a codegree 1 monomial of x_1 with a codegree 1 monomial of v.

(2) A product of a codegree 1 monomial of u with a codegree 1 monomial of x_1. In parts (1) and (2) we need to consider only monomials that do not cancel as products of top and codegree 1 monomials of u, v and z from the two sides of the equation.

(3) A product of a codegree 2 monomial of x_1 with the top monomial of v.

(4) A product of the top monomial of u with a codegree 2 monomial of x_1.

(5) A product of a codegree 2 monomial of u with the top monomial of x_1.

(6) A product of the top monomial of x_1 with a codegree 2 monomial of v.

Note that because of the equation, for each codegree 2 monomial in the products in the two sides of the equation, either 2, 4 or all 6 options occur. However, because the top monomials of u and v have no periodicity, possibilities (5) and (6) cannot occur together, so only 2 or 4 possibilities can occur.

Suppose that (1) occurs for some codegree 2 monomial. In that case we can assume that the codegree 1 monomial of x_1 is a product of the top monomial in u with a codegree 1 monomial of z, since otherwise such a product cancels with a product of type (2) by our analysis of codegree 1 products.

If in addition to (1) only (2) occurs for that codegree 2 monomial, we add a codegree 2 monomial to z that is obtained from the given codegree 2 monomial in the product by cutting a prefix which is equal to the top monomial of u, and a suffix which is the top monomial of v.

If only (3) occurs (in addition to (1)) for the given codegree 2 monomial, we also add a codegree 2 monomial to z that is identical to the one we added in case only (1) and (2) occur. If only (4) occurs, we do not add anything. If (5) occurs we add a codegree 2 monomial to z (the same codegree 2 monomial as in the previous cases).

If (6) occurs, we do not add anything.

Suppose that (2) occurs for some codegree 2 monomial. In that case we can assume that the codegree 1 monomial of x_1 is a product of a codegree 1 monomial of z with the top monomial of v. Hence, this can be dealt with precisely as what we did in (1).

Suppose that (3) occurs. If in addition only (4) occurs, we add a codegree 2 monomial to z. If only (5) occurs, we do not add anything. If only (6) occurs, we add a codegree 2 monomial to z. Suppose that (4) occurs. This can be dealt with precisely as the case in which (3) occurs. Again, since the top monomials of u and v do not have periodicity, (5) and (6) cannot occur together. Hence, we are only left with cases in which four of the possibilities occur.

Suppose that (1) and (2) occur for some codegree 2 monomial. In this case we can assume that (1) occurs as a product of a codegree 1 monomial of z and a codegree 1 of v, and (2) as a product of a codegree 1 monomial of u and codegree 1 monomial of z (otherwise (1) and (2) cancel from our analysis of codegree 1 monomials in x_1). If in addition only (3) and (4) occur, we do not add anything. If (3) and (5) occur, we add a codegree 2 monomial to z. If (3) and (6) occur, we do not add anything. The cases in which in addition to (1) and (2), cases (4) and (5) occur or cases (4) and (6) occur, are symmetric to (3) and (5) or (3) and (6).

Suppose that (1) occurs and (2) does not. Again, we may assume that in (1), it is a product of a codegree 1 monomial of z with a codegree 1 monomial of v. If in addition (3), (4) and (5) occur, we do not add anything. If (3), (4) and (6) occur, then we add a codegree 2 monomial to z. The case in which (1) does not occur and (2) occurs is symmetric.

After possibly adding codegree 2 monomials to z, the equation that was valid for codegree 1 products is now valid for codegree 2 products, i.e., $x_1 = uz = zv \mod G^{\deg(x_1)-3}$.

We continue iteratively to construct the element z by adding higher codegree monomials, so that the constructed element z satisfies the equation $x_1 = uz = zv$ for products of higher and higher codegree. Suppose that $x_1 = uz = zv \mod G^{\deg(x_1)-d}$, i.e., that the equation holds for all the products of codegree at most $d-1$, where d is a positive integer with $d \leq \deg(u)$. We iteratively add codegree d monomials to z so that the equalities hold for all codegree d products as well.

As in analyzing codegree 2 products, products of codegree d that include monomials of codegree smaller than d of u, v and z that correspond to smaller codegree monomials of x_1 (from the two sides of the equation) cancel in pairs.

The various cases are straightforward generalizations of the cases in analyzing codegree 2 products. Suppose that a codegree d product can be presented as either

(1) an odd number of products of the top monomial of u with codegree m_i monomials of z and codegree ℓ_i monomials of v, for some subset of tuples (m_i, ℓ_i), where $m_i + \ell_i = d$ and m_i, ℓ_i are positive integers, for every index i;

(2) an odd number of products of codegree s_j monomials of u with codegree t_j monomials of z and with the top monomial of v, for some subset of tuples (s_j, t_j), where $s_j + t_j = d$ and s_j, t_j are positive integers, for every index j;

(3) a product of a codegree d monomial of x_1 with the top monomial of v;

(4) a product of the top monomial of u with a codegree d monomial of x_1;

(5) a product of a codegree d monomial of u with the top monomial of x_1;

(6) a product of the top monomial of x_1 with a codegree d monomial of v.

Note that since we assumed that the top monomials of u and v do not have periodicity, cases (5) and (6) cannot occur together unless $d = \deg(u) = \deg(v)$. In case both (5) and (6) occur for $d = \deg(u)$, it must be that both u and v contain the constant monomial 1. We have $ux_1 = x_1 v$ if and only if $(u+1)x_1 = x_1(v+1)$. Hence, in case both u and v contain the constant monomial 1, we replace them by $u+1$ and $v+1$. This does not change the set of solutions, and after the change, cases (5) and (6) do not occur together for all $2 \le d \le \deg(u)$.

The analysis of codegree d products, according to the various possibilities of subsets of the six cases, is identical to the analysis that was used to analyze codegree 2 products. According to the analysis we decide what codegree d monomials to add to the element z.

After possibly adding these codegree d monomials to z, the equation that was valid for all products up to codegree $d-1$ is now valid for codegree d products, i.e., $x_1 = uz = zv \mod G^{\deg(x_1)-(d+1)}$.

Finally, we get an element z that satisfies $x_1 = uz = zv \mod G^{\deg(x_1)-\deg(u)-1}$.

After possibly changing the elements u and v so that not both of them contain the constant monomial 1, we continue the analysis of codegree d products in the two sides of the equation $ux_1 = x_1 v$, for all d, $\deg(u)+1 \le d \le \deg(x_1) - \deg(u)$, precisely as we analyzed codegree d products for $2 \le d \le \deg(u)$, and iteratively add codegree d monomials to the element z. Finally, we get an element z that satisfies $x_1 = uz = zv \mod G^{\deg(u)-1}$, i.e., the equalities hold for all products up to degree $\deg(u) = \deg(v)$.

We continue by looking at the equality $uz = zv \mod G^{\deg(u)-1}$. Repeating the same argument we can find an element z_2 such that $z = uz_2 = z_2 v \mod G^{\deg(u)-1}$. Continuing inductively, we get an element z_{r+1} such that

$$z_r = uz_{r+1} = z_{r+1} v \mod G^{\deg(u)-1}.$$

We are working over the free associative algebra FA_k, i.e., the algebra is over GF_2 and it is freely generated by k elements. Hence, $G^{\deg(u)-1}$ as a vector space over GF_2 has dimension bounded by $k^{\deg(u)}$. Therefore, there exist elements of distinct degrees $\{s_m \mid m = 1, \dots, k^{\deg(u)}+1\}$ (that are the elements z_r that were constructed iteratively from a given long solution) such that $\deg(s_m) \le (1+m)\deg(u)$ and $us_m = s_m v \mod G^{\deg(u)-1}$.

By a simple pigeonhole argument, there exists a subcollection of the indices $1 \le i_1 < \cdots < i_f \le k^{\deg(u)}+1$ such that $s = s_{i_1} + \cdots + s_{i_f}$ and $us = sv$. Hence, s is a solution of the given equation, and $\deg(s) \le \deg(u) \cdot (2 + k^{\deg(u)})$. □

So far we assumed that the top degree elements of u and v are monomials that are not proper powers. First, we omit the periodicity assumption, and allow the top degree monomials of u and v to have nontrivial roots.

Lemma 4.3. *With the notation of Lemma 4.2, let $u, v \in FA$ and suppose that the top homogeneous parts of u and v are monomials. Suppose that the top degree monomial of u has nontrivial roots of degree bounded by q.*

Suppose that the equation $ux = xv$ has a nontrivial solution. Then there exists elements $w_1, \dots, w_d$, $1 \le d \le q$, such that the set of solutions to the equation $ux = xv$ is a set of the form $\{w_1 p_1(v) + \cdots + w_d p_d(v)\}$, where $uw_i = w_i v$ and $p_1, \dots, p_d$ are arbitrary polynomials in v.

Proof. Let u_0 and v_0 be the top monomials of u and v, and let x_0 be the top monomial of a solution x. Let t_0 be a primitive root of v_0. Then there exists some fixed element s_0, $\deg(s_0) < \deg(t_0)$, such that $x_0 = s_0 t_0^m$ for some nonnegative integer m. Note that the element s_0 is fixed and does not depend on the solution x, since we assumed that u_0 and v_0 are proper powers of t_0, which is primitive, and if s_0 is not fixed, t_0 must have a proper root.

Suppose that $t_0^q = v_0$. The top monomial of a solution x is of the form $x_0 = s_0 t_0^m$, so we can divide the solutions x_1 according to the residue classes of the nonpositive integers m modulo q. For each residue class for which there is a solution, we fix one of the shortest solutions in the class. We denote these shortest solutions, $w_1, \dots, w_d$, for some d, $1 \le d \le q$.

Let x be a solution. x must be in the same class as one of the fixed shortest solutions w_i. Hence, for some nonnegative integer b, both x and $w_i v^b$ are solutions and they have the same top monomial. Therefore, if $x \ne w_i$, $x + w_i v^b$ is a nontrivial solution and $\deg(x + w_i v^b) < \deg(x)$. By a finite induction, $x = w_1 p_1(v) + \cdots + w_d p_d(v)$ for some polynomials $p_1, \dots, p_d$. ☐

So far we analyzed the equation $ux = xv$. We use similar methods to analyze the more general equation $u_1 x u_2 = v_1 x v_2$.

Theorem 4.4. *Let $u_1, u_2, v_1, v_2 \in FA$ and suppose that the top homogeneous parts of u_i and v_i are monomials with no periodicity (i.e., the top monomials in u_i and v_i contain no subwords α^2 for some nontrivial word α), and that $\deg(u_1) > \deg(v_1)$.*

Suppose that the equation $u_1 x u_2 = v_1 x v_2$ has a solution of degree bigger than $2(\deg(u_1) + \deg(v_2))^2$.

(1) *There exist elements $s, t \in FA$ such that $u_1 = v_1 s$ and $v_2 = t u_2$.*

(2) *An element $x \in FA$ is a solution to the equation $u_1 x u_2 = v_1 x v_2$ if and only if it is a solution of the equation $sx = xt$.*

Proof. First, note that if (1) is true and x satisfies $u_1 x u_2 = v_1 x v_2$, then we have $v_1 s x u_2 = v_1 x t u_2$. Hence, $sx = xt$. Conversely, every solution of the equation $sx = xt$ satisfies $u_1 x u_2 = v_1 x v_2$, so (2) is true.

As we did in analyzing centralizers and analyzing the equation $ux = xv$, we analyze the homogeneous parts in x and in u_i and v_i going from top to bottom.

Let u_i^0 and v_i^0, $i = 1, 2$, be the top monomials in u_i and v_i. Let x_1 be a solution of the equation $u_1 x u_2 = v_1 x v_2$, and suppose that $\deg(x_1) > \max(\deg(u_1), \deg(v_2)) + 2(\deg(u_1) - \deg(v_1))$. By our analysis of homogeneous solutions, the top homogeneous part of the solution x_1 must be a monomial as well, which we denote x_0.

Since $u_1^0 x_0 u_2^0 = v_1^0 x_0 v_2^0$, there exists monomials s_0, t_0, $\deg(s_0) = \deg(t_0)$, such that $u_1^0 = v_1^0 s_0$, $v_2^0 = t_0 u_2^0$ and $x_0 = f_0 t_0^b = s_0^b e_0$, for some positive integer b, and $\deg(f_0) = \deg(e_0) < \deg(s_0)$.

We continue by analyzing monomials of codegree 1 in u_i, v_i and x_1. By the same analysis that was used in analyzing centralizers and in Lemma 4.2, there exist elements s, t with top monomials s_0 and t_0, and an element w with top monomial w_0, $w_0 t_0 = s_0 w_0 = x_0$, such that

(i) $u_1 = v_1 s \mod G^{\deg(u_1)-2}$.

(ii) $v_2 = t u_2 \mod G^{\deg(v_2)-2}$.

(iii) $x_1 = sw = wt \mod G^{\deg(x_1)-2}$.

By iteratively applying the same construction, the above three equalities imply that $w = s^m f = e t^m \mod G^{\deg(w)-2}$, for some positive integer m and elements e, f with $\deg(s) \le \deg(e) = \deg(f) < 2\deg(s)$.

We continue by analyzing products of codegree 2. First, note that as in analyzing centralizers, if we look at codegree 2 products that involve only top monomials and codegree 1 monomials from s, t, u_i, v_i and w such that the products restrict to codegree 0 or 1 monomials of x_1, u_i and v_i, then such codegree 2 products cancel in pairs from the two sides of the equation.

We further look at codegree 2 products that contain a codegree 1 monomial of u_2. If the codegree 2 product contains the top monomial of t, then such a codegree 2 product cancels with a corresponding codegree 2 product from the other side of the equation, since all the corresponding monomials of u_i, v_i and x_1 (from the two sides of the equation) are either codegree 0 or codegree 1. Hence, we look at codegree 2 products that contain codegree 1 monomials of t and u_2, and, therefore, top monomials of u_1 and w. Such a codegree 2 product, which is a product of the top monomial of u_1, a codegree 1 monomial of x_1 and a codegree 1 monomial of u_2, cancels with either

(1) a product of the top monomial of v_1, the top monomial of x_1 and a codegree 2 monomial of v_2;

(2) a product of the top monomial of v_1, a codegree 1 monomial of x_1 and a codegree 1 monomial of v_2;

(3) a product of the top monomial of v_1, a codegree 2 monomial of x_1 and the top monomial of v_2;

(4) a product of the top monomial of u_1, a codegree 2 monomial of x_1 and the top monomial of u_2;

(5) a product of the top monomial of u_1, the top monomial of x_1 and a codegree 2 monomial of u_2;

(6) a product of a codegree 1 monomial of u_1, a codegree 1 monomial of x_1 and the top monomial of u_2.

If the given codegree 2 product cancels only with a product of type (1) we don't add anything to w nor to t. Suppose that the given codegree 2 product cancels only with a product of type (2). If the codegree 1 monomial of v_2 equals the product of the top monomial of t with a codegree 1 monomial of u_2, then the codegree 2 product of type (2) cancels with a codegree 2 product from the other side of the equation that contains only codegree 0 and 1 monomials of u_1, x_1 and u_2. Hence, in case (2) we can assume that the codegree 1 monomial of v_2 is a product of a codegree 1 monomial of t with the top monomial of u_2. In that case we add a codegree 2 monomial to t and leave w unchanged.

If the given codegree 2 product cancels only with a codegree 2 product of type (3), we add a codegree 2 monomial to t and a codegree 2 monomial to w. If the given codegree 2 product cancels only with a product of type (4) we add a codegree 2 monomial to t. In case the given product cancels only with a codegree 2 product of type (5) we don't add anything (apart from the codegree 2 monomial of u_2). If it cancels only with a codegree product of type (6) we add a codegree 2 monomial to t and a codegree 2 monomial to w.

Because the top monomial of v_2 does not have periodicity, a product of type (5) cannot cancel with a product of type (3) nor (6). Hence, if five possibilities occur in addition to the given one, it must be (1)–(4) and (6). In that case, we do not add anything.

Hence, the only left possibilities are a collection of products of three different types that cancel with the given codegree 2 product. We list the various possibilities for the collections of codegree 2 products of three different types that cancel with the given codegree 2 product and indicate what we add in each possibility:

(i) Products (1)–(3) cancel. We add a codegree 2 monomial to w, apart from an existing codegree 2 monomial of v_2 (that is equal to the products of the codegree 1 monomials of t and u_2 in the given codegree 2 product).

(ii) Products (1), (2) and (4) cancel. In that case we don't add anything to w and t. A monomial of codegree 2 already appears in v_2, and is equal to the product of the given codegree 1 monomials of t and u_2.

(iii) Products (1), (2) and (5) cancel. We add a codegree 2 monomial to t, in addition to the codegree 2 monomials that already appear in u_2 and v_2.

(iv) Products (1), (2) and (6) cancel. Product (1) cancels with the given codegree 2 product. We add a codegree 2 monomial to w.

(v) Products (1), (3) and (4) cancel. We add a codegree 2 monomial to w, and the existing codegree 2 monomial to v_2.

(vi) Products (1), (3) and (6) cancel. Products (3) and (6) cancel, so this is identical to the case that only (1) occurs in addition to the given codegree 2 product.

(vii) Products (2), (3) and (4) cancel. We add a codegree 2 monomial to w and a codegree 2 monomial to t.

(viii) Products (2), (3) and (6) cancel. Like (vi), (3) and (6) cancel.

(ix) Products (1), (4) and (5) cancel. We add a codegree 2 monomial to t, and the existing codegree 2 monomials to u_2 and v_2.

(x) Products (1), (4) and (6) cancel. Product (1) cancels with the given codegree 2 product. We add a codegree 2 monomial to w.

(xi) Products (2), (4) and (5) cancel. We just add the existing codegree 2 monomial to u_2.

(xii) Products (2), (4) and (6) cancel. Like (vi), (2) and (4) cancel.

(xiii) Products (3), (4) and (6) cancel. Like (vi), (3) and (6) cancel.

So far we analyze codegree 2 products that cancel with a given codegree 2 product that is a product of the top monomial of u_1 and w and codegree 1 monomials of t and u_2. The same analysis is valid for codegree 2 products that cancel with a codegree 2 product of type (1), and the analogous cases from the left side of the equation.

We continue by analyzing case (2), i.e., those codegree 2 products that cancel with a given codegree 2 product of the top monomial of v_1, a codegree 1 monomial of x_1 that is a product of the top monomial of s and a codegree 1 monomial of w, and a codegree 1 monomial of v_2 that is equal to a product of a codegree 1 monomial of t with the top monomial of u_2. Such a given codegree 2 product can cancel with either

(1) a product of the top monomial of v_1, a codegree 2 monomial of x_1 and the top monomial of v_2;

(2) a product of the top monomial of v_1, the top monomial of x_1 and a codegree 2 monomial of v_2;

(3) a product of the top monomial of u_1, a codegree 2 monomial of x_1 and the top monomial of u_2;

(4) a product of the top monomial of u_1, the top monomial of x_1 and a codegree 2 monomial of u_2;

(5) a product of the top monomial of u_1, a codegree 1 monomial of x_1 and a codegree 1 monomial of u_2;

(6) a product of a codegree 1 monomial of u_1, which is a product of the top monomial of v_1 with a codegree 1 monomial of s, with a codegree 1 monomial of x_1, which is the product of a codegree 1 monomial of w with the top monomial of t, and with the top monomial of u_2.

If the given codegree 2 product equals only to a codegree 2 product of type (1), we add a codegree 2 monomial to w. If it equals only to a codegree 2 product of type (2), we add a codegree 2 monomial to t, apart from the existing codegree 2 monomial of v_2. If it equals only to a codegree 2 product of type (3), we do not add anything. If it equals only to a codegree 2 product of type (4) we add a codegree 2 monomial to t, apart from the existing codegree 2 monomial of u_2. We analyzed case (5) with all the codegree 2 products that it cancels with previously. If it equals only to a product of type (6) we add a codegree 2 monomial to w.

Case (5) was analyzed previously, so we can assume it does not occur. A codegree 2 product of types (1) or (6) cannot cancel with a codegree 2 product of type (4). A monomial of type (5) that cancels with a monomial of type (6) is a product of lower codegree monomials of u_i, v_i and x_1 from the two sides of the equation, so we omit this case. Hence, there are 5 cases left:

(i) Products (1), (2) and (3) cancel with the given codegree 2 product. In that case we add a codegree 2 monomial to w and a codegree 2 monomial to t, apart from the existing codegree 2 monomial of v_2.

(ii) Products (1), (2) and (6) cancel. We add a codegree 2 monomial to t.

(iii) Products (1), (3) and (6) cancel. In that case we do not add anything.

(iv) Products (2), (3) and (4) cancel. In that case we only add the already existing codegree 2 monomials of u_2 and v_2.

(v) Products (2), (3) and (6) cancel. We add a codegree 2 monomial to w and t.

Codegree 2 products that contain codegree 1 monomials of v_1 or u_1 are treated exactly in the same way. Hence, we are left with sets of codegree 2 products that cancel, and each of these codegree 2 products is a product of top monomials with codegree 2 monomials of one of the u_i, v_i or x_1. These are analyzed precisely as they are treated in the proof of Theorem 4.4 and in analyzing codegree 1 products, and in each such cancellation codegree 2 monomials may be added to either s, t or w, apart from existing codegree 2 monomials of u_i and v_i. Finally, we (possibly) added codegree 2 monomials to s, w and t, such that

(i) $u_1 = v_1 s \mod G^{\deg(u_1)-3}$.

(ii) $v_2 = t u_2 \mod G^{\deg(v_2)-3}$.

(iii) $x_1 = sw = wt \mod G^{\deg(x_1)-3}$.

We continue iteratively with products with higher codegree. Let

$$d = \min(\deg(v_1), \deg(u_2), \deg(u_1) - \deg(v_1)).$$

Let $r \leq d-1$ and suppose that we added codegree r monomials to s, w and t such that the equations above hold for all products of codegree bounded by $r-1$.

We analyze codegree r products in the same way we analyzed codegree 2 products. First, note that if a codegree r product is a product of monomials of u_i, v_i, s, t and w that correspond to products of monomials of codegree smaller than r of u_i, v_i and x_1 from the two sides of the equation, then such codegree r products cancel in pairs.

Suppose that a codegree r product is a product of the top monomials of u_1 and w, and monomials of codegree q_i of t and codegree m_i of u_2, such that $q_i + m_i = r$ and q_i, m_i are positive integers, and there are odd number of such pairs (q_i, m_i). We treat this case in the same way we treated the case of a codegree 2 product that includes a codegree 1 monomial of t and a codegree 1 monomial of u_2. This odd set of codegree r products (that are all equal) cancels with either

(1) a product of the top monomial of v_1, the top monomial of x_1 and a codegree r monomial of v_2;

(2) an odd set of codegree r products of the top monomial of v_1, codegree e_j monomial of x_1 and codegree p_j monomial of v_2, for some positive set of pairs (e_j, p_j) that satisfy $e_j + p_j = r$, and such that the codegree p_j monomial of v_2 is a product of a codegree p_j monomial of t with the top monomial of u_2;

(3) a product of the top monomial of v_1, a codegree r monomial of x_1 and the top monomial of v_2;

(4) a product of the top monomial of u_1, a codegree r monomial of x_1 and the top monomial of u_2;

(5) a product of the top monomial of u_1, the top monomial of x_1 and a codegree r monomial of u_2;

(6) an odd set of codegree r products of a codegree a_j monomial of u_1 and a codegree b_j monomial of x_1 with the top monomial of u_2, for some positive set of pairs (a_j, b_j) that satisfy $a_j + b_j = r$, and such that the codegree a_j monomial of u_1 is a product of the top monomial of v_1 and a codegree a_j monomial of s.

The treatment of the various cases is identical to what we did in analyzing codegree 2 products (cases (i)–(xiii)), just that instead of adding codegree 2 monomials to the various elements we add codegree r monomials. The other cases of codegree r

products are treated exactly as we treated codegree 2 products. Therefore, we constructed elements s, t, w for which

(i) $u_1 = v_1 s \mod G^{\deg(u_1)-d}$.

(ii) $v_2 = t u_2 \mod G^{\deg(v_2)-d}$.

(iii) $x_1 = sw = wt \mod G^{\deg(x_1)-d}$.

We divide the continuation according to the minimum between $\deg(v_1)$, $\deg(u_2)$ and $\deg(u_1) - \deg(v_1)$. First we assume that

$$d = \min\bigl(\deg(v_1), \deg(u_2), \deg(u_1) - \deg(v_1)\bigr) = \deg(u_1) - \deg(v_1).$$

In analyzing codegree d products, there are special codegree d products that we need to single out and treat separately, as they may involve cancellations between codegree d products that contain codegree d monomials of u_1 or v_1 and those that contain codegree d monomials of u_2 or v_2.

As in analyzing smaller codegree products, note that codegree d products that are products of smaller codegree monomials of the u_i, v_i, s, w and t, and correspond to smaller codegree monomials of u_i, v_i and x_1 from the two sides of the equation cancel in pairs.

We continue by analyzing codegree d products that are products of top degree monomials of u_1 and w, codegree q_i monomials of t and codegree m_i monomials of u_2, such that q_i and m_i are positive and $q_i + m_i = d$, there are odd number of such pairs (q_i, m_i), and the product of these monomials of t and u_2 is not equal to u_2^0, the top monomial of u_2.

Such codegree d products are analyzed exactly in the same way they were analyzed in codegree r products for $r < d$. Similarly, we analyze codegree d products that are obtained an odd number of times as the product of the top monomials of v_1 and s_1, a codegree e_j monomial of w and a codegree p_j monomial of v_2, such that the product of a codegree e_j monomial of w and a codegree p_j monomial of v_2 is not $w_0 u_2^0$, i.e., the product of the top monomials of w and u_2.

In a similar way we analyze codegree d products that are products of smaller codegree monomials of u_1 and w and the top monomials of t and u_2, and products of smaller codegree monomials of v_1 and s and the top monomials of w and v_2, assuming the products of these smaller degree monomials are not equal to $v_1^0 w_0$ or to v_1^0.

We continue by analyzing canceling pairs of codegree d products that are products of top monomials of v_i, u_i and x_1, with one codegree d monomial of these elements, such that this codegree d monomial of u_1 is not v_1^0, the codegree d monomial of x_1 is not w_0 and the codegree d monomial of v_2 is not u_2^0. These codegree d products are analyzed in the same way they were analyzed for smaller codegree products.

We are left with codegree d products that are either

(1) a product of a codegree d monomial of u_1 that is equal to v_1^0, with top monomials of x_1 and u_2;

(2) a product of the top monomial of v_1 with top monomials of x_1 and a codegree d monomial of v_2 that is equal to u_2^0;

(3) a product of the top monomial of u_1 with a codegree d monomial of x_1 that is equal to the top monomial of w, with the top monomial of u_2;

(4) a product of the top monomial of v_1 with a codegree d monomial of x_1 that is equal to the top monomial of w, with the top monomial of v_2;

(5) an odd set of codegree d products of the top monomial of v_1, codegree e_j monomial of x_1 and codegree p_j monomial of v_2, for some positive pairs (e_j, p_j) that satisfy $e_j + p_j = d$, such that the monomial of x_1 is a product of the top monomial of s with a codegree e_j monomial of w, and the product of each codegree e_j monomial of x_1 with a codegree p_j monomial of v_2 is equal to $w_0 v_2^0$ (the product of the top monomials of w and v_2);

(6) an odd set of codegree d products of the top monomial of u_1, codegree q_i monomials of x_1 that are products of the top monomial of w with a codegree q_i monomial of t, with codegrees m_i monomials of u_2, for some positive set of pairs (q_i, m_i) that satisfy $q_i + m_i = d$, such that the product of each codegree q_i monomial of t with a codegree m_i monomial of u_2 is equal to u_2^0;

(7) an odd set of codegree d products of codegree f_i monomials of u_1 and codegree g_i monomials of x_1 with the top monomial of u_2, for some positive pairs (f_i, g_i) that satisfy $f_i + g_i = d$, such that the monomial of x_1 is a product of a codegree g_i monomial of w with the top monomial of t, and the product of each codegree f_i monomial of u_1 with a codegree g_i monomial of x_1 is equal to $u_1^0 w_0$ (the product of the top monomials of u_1 and w);

(8) an odd set of codegree d products of codegree h_j monomials of v_1 and codegree k_j monomials of x_1 that are products of a codegree k_j monomial of s with the top monomial of w, with the top monomial of v_2, for some positive pairs (h_j, k_j) that satisfy $h_j + k_j = d$, such that the product of each codegree h_j monomial of v_1 with a codegree k_j monomial of s is equal to v_1^0.

First note that (3) exists if and only if (4) exists and they cancel each other. If (3) and (4) are the only existing possibilities, we add a codegree d monomial to w, which is the codegree d prefix or suffix of the top monomial of w. Also note that if cases (1) or (2) exist, codegree d monomials that already appear in u_1 or v_2 are added to them. Suppose that only two of the possibilities (1), (2) and (5)–(8) exist, possibly in addition to (3) and (4). We go over the various alternatives:

(i) If only (1) and (2) exist, we add the constant element 1 to s and t, and the codegree d prefix of w_0 to w, where w_0 is the top monomial of w. If (3) and (4) exists as well, we only add 1 to s and t.

(ii) If only (5) and (6) exist, we just add 1 to t. If (3) and (4) exist as well, we add the codegree d prefix of w_0 to w. The case in which only (7) and (8) exist is identical.

(iii) If only (5) and (8) exist, we add 1 to s and the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we only add 1 to s. The case in which only (6) and (7) exist is treated identically.

(iv) If only (5) and (7) exist, we add the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we do not add anything to any of the variables.

(v) If only (6) and (8) exist, we add 1 to s and t and the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we only add 1 to s and t.

(vi) If only (1) and (8) exist, we add 1 to s and the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we only add 1 to s.

(vii) If only (1) and (7) exist, we do the same as in (v), adding 1 to s and the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we only add 1 to s.

(viii) If only (1) and (6) exist, we add 1 to s and t, and the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we only add 1 to s and t.

(ix) If only (1) and (5) exist, we add 1 to s and the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we only add 1 to s.

The cases in which only case (2) and one of the cases (5)–(8) exist are treated according to cases (vi)–(ix). Suppose that exactly four of the cases (1), (2) and (5)–(8) exist, possibly in addition to (3) and (4). We go over the alternatives:

(i) If only (1), (2), (5) and (6) exist, we add 1 to s and the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we just add 1 to s. The case in which only (1), (2), (7) and (8) exist is identical.

(ii) If only (1), (2), (5) and (8) exist, we add 1 to t. If (3) and (4) exist as well, we add 1 to t and the codegree d prefix of w_0 to w. The case in which only (1), (2), (6) and (7) exist is identical.

(iii) If only (1), (2), (5) and (7) exist, we add 1 to s and t. If (3) and (4) exist as well, we add 1 to s and t and the codegree d prefix of w_0 to w.

(iv) If only (1), (2), (6) and (8) exist, we do not change any of the variables. If (3) and (4) exist as well, we add the codegree d prefix of w_0 to w.

(v) If only (5), (6), (7) and (8) exist, we add 1 to s and t. If (3) and (4) exist as well, we add 1 to s and t and the codegree d prefix of w_0 to w.

(vi) If only (1), (5), (6) and (7) exist, we add 1 to s and t. If (3) and (4) exist as well, we add 1 to s and t and the codegree d prefix of w_0 to w.

(vii) If only (1), (5), (6) and (8) exist, we add 1 to t. If (3) and (4) exist as well, we add 1 to t and the codegree d prefix of w_0 to w.

(viii) If only (1), (5), (7) and (8) exist, we add the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we do not add anything to any of the variables.

(ix) If only (1), (6), (7) and (8) exist, we add 1 to t and the codegree d prefix of w_0 to w. If (3) and (4) exist as well, we just add 1 to t.

The cases in which only case (2) and three of the cases (5)–(8) exist are treated according to cases (vi)–(ix). Suppose that cases (1), (2) and (5)–(8) exist. In that case we add the codegree d prefix of w_0 to w. If cases (3) and (4) exist as well, we do not add anything to any of the variables.

This completes the analysis of codegree d products. We continue with the analysis of codegree $d+1$ products. First, as in analyzing smaller codegree products, codegree $d+1$ products that are products of smaller codegree monomials of u_i, v_i, s, t, w, that correspond to products of smaller degree monomials of u_i, v_i and x_1 from the two sides of the equation, cancel in pairs.

Lemma 4.5. *Suppose that a codegree $d+1$ product is a product of the top monomials of u_1 and w, and monomials of codegree q of t and codegree m of u_2, such that $q \geq 0$, $m > 0$ and $q+m = d+1$. Such a codegree $d+1$ product cannot be*

(1) *a product of the top monomial of v_1, a codegree e monomial of x_1 and a codegree f monomial of v_2, for $e > 0$ and $f \geq 0$, that satisfy $e+f = d+1$, and the codegree f monomial of v_2 is a product of a codegree f monomial of t with the top monomial u_2^0 of u_2;*

(2) *a product of the top monomial of u_1, a codegree $d+1$ monomial of x_1 and the top monomial of u_2.*

Proof. If such a codegree $d+1$ product can be presented as a product in the forms (1) or (2), u_2^0 has a prefix which is a suffix of t_0. Hence, v_2^0 has nontrivial periodicity that contradicts our assumptions. □

Suppose that a codegree $d+1$ product can be presented as a product of the top monomials of u_1 and w, and monomials of codegrees q_i of t and codegree m_i of u_2, such that $q_i \geq 0$, $m_i > 0$ and $q_i + m_i = d+1$, and there are an odd number of such pairs (q_i, m_i).

By Lemma 4.5, the same codegree $d+1$ product is the product of the top monomial of v_1, the top monomial of x_1 and a codegree $d+1$ monomial of v_2.

Furthermore, by the same argument that was used in the proof of Lemma 4.5, if a codegree $d+1$ product is the product of the top monomials of v_1 and x_1 and a codegree $d+1$ monomial of v_2, then it must be the product of an odd number of products of the top monomials of u_1 and w, and monomials of codegrees q_i of t and codegrees m_i of u_2, such that $q_i \geq 0$, $m_i > 0$ and $q_i + m_i = d+1$.

Lemma 4.6. *Suppose that a codegree $d+1$ product can be presented in an odd number of ways as products of the top monomial of v_1, a codegree e_j monomial of x_1, which is the product of the top monomial of s with a codegree e_j monomial of w, and a codegree f_j monomial of v_2, for positive e_j and f_j and $e_j + f_j = d+1$, and the codegree f_j monomials of v_2 are products of codegree f_j monomials of t with the top monomial u_2^0 of u_2.*

(1) *Suppose that this codegree $d+1$ product cannot be presented in an odd number of ways as products of a codegree g_j monomial of u_1, and a codegree h_j monomial of x_1 with the top monomial of u_2, where the codegree h_j monomial of x_1 is a codegree h_j monomial of w with the top monomial of t, for positive g_j and h_j and $g_j + h_j = d+1$, and the codegree g_j monomials of u_1 are products of v_1^0 with codegree g_j monomials of t.*

Then the same codegree $d+1$ product is either the product of the top monomial of u_1, a codegree $d+1$ monomial of x_1, and the top monomial of u_2, or the product of the top monomial of v_1, a codegree $d+1$ monomial of x_1 and the top monomial of v_2, and exactly one of the two occurs.

(2) *Suppose that this codegree $d+1$ product can be presented in an odd number of ways as products of a codegree g_j monomial of u_1, and a codegree h_j monomial of x_1 with the top monomial of u_2, and the codegree h_j monomial of x_1 is a codegree h_j monomial of w with the top monomial of t, for positive g_j and h_j and $g_j + h_j = d+1$, and the codegree g_j monomials of u_1 are products of v_1^0 with codegree g_j monomials of t.*

Then either the same codegree $d+1$ product is both the product of the top monomial of u_1, a codegree $d+1$ monomial of x_1, and the top monomial of u_2, and the product of the top monomial of v_1, a codegree $d+1$ monomial of x_1 and the top monomial of v_2, or none of these two possibilities occur.

Proof. Such a codegree $d+1$ product does not cancel only with the product of monomials of codegree less than d of u_i, v_i and x_1, from the two sides of the equation. By Lemma 4.5 such codegree $d+1$ products cannot be equal to the following products:

(1) top degree monomials of u_1 and w with monomials of t and u_2;

(2) top degree monomials of v_1 and x_1 and a codegree $d+1$ monomial of v_2;

(3) monomials of v_1 and s with top degree monomials of w and v_2;

(4) codegree $d+1$ monomials of u_1 with top degree monomials of x_1 and of u_2.

Therefore, such a codegree $d+1$ product must be equal to an odd number of products in the forms that are listed in the statement of the lemma. □

Lemmas 4.5 and 4.6 enable us to treat codegree $d+1$ products in a similar way to the analysis of codegree r products for $r<d$.

Suppose that a codegree $d+1$ product is obtained in an odd number of ways as the product of the top monomials of u_1 and x_1 with a codegree q_i monomial of t, and a codegree m_i monomial of u_2, such that $q_i \geq 0$, $m_i > 0$ and $q_i+m_i=d+1$. By Lemma 4.5, such a product must be equal to a product of the top monomials of v_1 and x_1 and a codegree $d+1$ monomial of v_2.

An analogous conclusion holds if a codegree $d+1$ product is obtained in an odd number of ways as the product of a codegree m_i monomial of v_1 with a codegree q_i of s with the top monomials of x_1 and v_2, such that $q_i \geq 0$, $m_i > 0$ and $q_i+m_i=d+1$.

Suppose that a codegree $d+1$ product can be presented in an odd number of ways as products of the top monomial of v_1, a codegree e_j monomial of x_1, which is the product of the top monomial of s with a codegree e_j monomial of w, and a codegree f_j monomial of v_2, for positive e_j and f_j and $e_j+p_j=d+1$, and the codegree f_j monomials of v_2 are products of codegree f_j monomials of t with the top monomial u_2^0 of u_2.

Suppose that this codegree $d+1$ product cannot be presented in an odd number of ways as products of a codegree g_j monomial of u_1, and a codegree h_j monomial of x_1, which is a codegree h_j monomial of w with the top monomial of t, for positive g_j and h_j and $g_j+h_j=d+1$, and the codegree g_j monomials of u_1 are products of v_1^0 with codegree g_j monomials of t.

If the same codegree $d+1$ product is the product of the top monomial of u_1, a codegree $d+1$ monomial of x_1, and the top monomial of u_2, we do not add anything. If it is the product of the top monomial of v_1, a codegree $d+1$ monomial of x_1 and the top monomial of v_2, we add a codegree $d+1$ monomial to w.

Suppose that this codegree $d+1$ product can be presented in an odd number of ways as products of a codegree g_j monomial of u_1, and a codegree h_j monomial of x_1, which is a codegree h_j monomial of w with the top monomial of t, for positive g_j and h_j and $g_j+h_j=d+1$, and the codegree g_j monomials of u_1 are products of v_1^0 with codegree g_j monomials of t.

If the same codegree $d+1$ product is both the product of the top monomial of u_1, a codegree $d+1$ monomial of x_1, and the top monomial of u_2, and the product of the top monomial of v_1, a codegree $d+1$ monomial of x_1 and the top monomial of v_2, then we do not add anything. If none of these two possibilities occur, we add a codegree $d+1$ monomial to w (by Lemma 4.6 either both or none occur).

Suppose that a codegree $d+1$ product can be presented only as the product of the top monomial of u_1, a codegree $d+1$ monomial of x_1 and the top monomial of u_2, and as the product of the top monomial of v_1, a codegree $d+1$ monomial of x_1, and the top monomial of v_2. In that case we add a codegree $d+1$ monomial to w.

This concludes the analysis of codegree $d+1$ products. The analysis of codegree $d+r$ products, $r<d$, is identical to the analysis of codegree $d+1$ products. Hence, we (possibly) finally add codegree $d+r$ monomials to w, and the existing codegree $d+r$ monomials to u_i and v_i for $1 \leq r < d$, and do not change s and t, such that

(i) $u_1 = v_1 s \mod G^{\deg(u_1)-2d}$.

(ii) $v_2 = t u_2 \mod G^{\deg(v_2)-2d}$.

(iii) $x_1 = sw = wt \mod G^{\deg(x_1)-2d}$.

In analyzing codegree $2d$ products, as in analyzing codegree d products, there are special codegree $2d$ monomials that we need to single out and treat separately, as they may involve cancellations between codegree $2d$ products that contain codegree d or $2d$ monomials of u_1 or v_1 and those that contain codegree d or $2d$ monomials of u_2 or v_2.

As in analyzing smaller codegree products, note that codegree $2d$ products that are products of smaller codegree monomials of the u_i, v_i, s, w and t, and correspond to smaller codegree monomials of u_i, v_i and x_1 from the two sides of the equation cancel in pairs.

As in analyzing codegree d products, we continue by analyzing codegree $2d$ products that are products of top degree monomials of u_1 and w, codegree q_i monomials of t and codegree m_i monomials of u_2, such that $q_i + m_i = 2d$, there are an odd number of such pairs (q_i, m_i), and the product of these monomials of t and u_2 is not equal to a codegree d suffix of u_2^0, the top monomial of u_2 (which is a codegree $2d$ suffix of v_0^2, the top monomial of v_2). Such codegree $2d$ products must cancel with the product of the top monomials of v_1 and x_1 and a codegree $2d$ monomial of v_2. In this case we only add the already existing codegree $2d$ monomial to v_2.

Similarly, we analyze codegree $2d$ products that are obtained in an odd number of ways as the product of the top monomials of v_1 and s_1, a codegree e_j monomial of w and a codegree p_j monomial of v_2, such that the product of a codegree e_j monomial of w and a codegree p_j monomial of v_2 does not have a suffix which is the codegree d suffix of u_2^0. We analyze codegree $2d$ products that contain similar monomials of v_1, u_1 and x_1 in a similar way.

Suppose that a codegree $2d$ product is obtained in an odd number of ways as the product of the top monomials of u_1 and x_1, a codegree q_i monomial of t and a codegree m_i monomial of u_2, such that q_i and m_i are positive and $q_i + m_i = 2d$, and the product of the monomial of t and the monomial of u_2 is a codegree d monomial of u_2^0.

Because we assumed that the coefficients do not have any periodicity, such a codegree $2d$ product must cancel with either a product of the top monomials of u_1 and x_1 and a codegree $2d$ monomial of u_2, or a product of the top monomials of v_1 and x_1 and a codegree $2d$ monomial of v_2. In both of these cases we only add the already existing codegree $2d$ monomials to u_2 or v_2.

If x_1 contains a monomial that is equal to the $2d$ prefix (or suffix) of x_1^0, then the codegree $2d$ product that contains the top monomials of the u_i and this codegree $2d$ monomial of x_1 cancels with the codegree $2d$ product of the top monomials of the v_i with that codegree $2d$ monomial of x_1.

As in analyzing codegree d products, we continue by analyzing canceling pairs of codegree $2d$ products that are products of top monomials of v_i, u_i and x_1, with one codegree $2d$ monomials of these elements, such that this codegree $2d$ monomial of u_1 is not the codegree d prefix of v_1^0, the codegree $2d$ monomial of x_1 is not the codegree d prefix (or suffix) of w_0, and the codegree $2d$ monomial of v_2 is not the codegree d suffix of u_2^0. These codegree $2d$ products are analyzed in the same way they were analyzed for smaller codegree products.

We are left with codegree $2d$ products that are either

(1) a product of the top monomial of u_1 with a codegree $2d$ monomial of x_1 that is equal to a codegree d prefix (or suffix) of the top monomial of w with the top monomial of u_2;

(2) a product of the top monomial of v_1 with a codegree $2d$ monomial of x_1 that is equal to the codegree d prefix of the top monomial of w with the top monomial of v_2;

(3) an odd set of codegree $2d$ products of the top monomial of v_1, codegree e_j monomial of x_1 and codegree p_j monomial of v_2, for some positive pairs (e_j, p_j) that satisfy $e_j + p_j = 2d$, such that the codegree e_j monomial of x_1 is the product of the top monomial of s with a codegree e_j monomial of w, and the product of each codegree e_j monomial of x_1 with a codegree p_j monomial of v_2 is equal to the product of a codegree d prefix of w_0 with v_2^0;

(4) an odd set of codegree $2d$ products of codegree f_i monomials of u_1 and codegree g_i monomials of x_1 with the top monomial of u_2, for some positive pairs (f_i, g_i) that satisfy $f_i + g_i = 2d$, such that the codegree g_i monomial of x_1 is the product of a codegree g_i monomial of w with the top monomial of t, and the product of each codegree f_i monomial of u_1 with a codegree g_i monomial of x_1 is equal to the product of u_1^0 with the codegree d prefix of w_0.

Note that (1) exists if and only if (2) exists and they cancel each other. If (1) and (2) are the only existing possibilities, we add a codegree $2d$ monomial to w, which is the codegree d prefix or suffix of the top monomial of w.

If only possibilities (3) and (4) exist, we add the codegree $2d$ prefix of w_0 to w. If (1)–(4) do all exist, w remains unchanged.

This completes the analysis of codegree $2d$ products. Codegree $2d+r$ products, for $1 \leq r < d$, are treated in the same way we treated codegree $d+r$ products. Codegree $3d$ products are treated in the same we treated $2d$ products, and so on. Finally, in case $d = \deg(u_1) - \deg(v_1) = \deg(v_2) - \deg(u_2)$, we obtained the conclusion of the theorem.

Suppose that $d = \min(\deg(v_1), \deg(u_2), \deg(u_1) - \deg(v_1)) = \deg(v_1)$. In that case we continue the analysis of codegree r homogeneous parts in u_i, v_i and x_1, $d \leq r < \deg(u_1) - \deg(v_1)$, precisely as we analyzed the codegree r homogeneous parts for $1 \leq r \leq d-1$. For $r = \deg(u_1) - \deg(v_1)$, we use the same analysis that we apply for codegree d products in case $d = \deg(u_1) - \deg(v_1)$. For $r > \deg(u_1) - \deg(v_1)$, we continue the analysis of codegree r homogeneous parts according to the analysis of codegree higher than d in case $d = \deg(u_1) - \deg(v_1)$. The analysis in the case $d = \min(\deg(v_1), \deg(u_2), \deg(u_1) - \deg(v_1)) = \deg(u_2)$ is identical. $\square$

Theorem 4.4 reduces the analysis of solutions to the equation $u_1 x u_2 = v_1 x v_2$ to the equation $xt = sx$, in case the equation $u_1 x u_2 = v_1 x v_2$ has a long enough solution, and the coefficients have no periodicity. The same techniques allow one to reduce a general equation with one variable, in case the coefficients have no periodicity.

Theorem 4.7. *Let FA be the free associative algebra over GF_2 that is freely generated by k elements. Let $u_1, \dots, u_n, v_1, \dots, v_n \in FA$ and suppose that the top homogeneous parts of u_i and v_i are monomials with no periodicity, and that for at least one index i, $1 \leq i \leq n$, $u_i \neq v_i$. Suppose that the equation*

$$u_1 x u_2 x u_3 \cdots u_{n-1} x u_n = v_1 x v_2 x v_3 \cdots v_{n-1} x v_n$$

has a solution x_1 of degree bigger than $2(\deg(u_1) + \cdots + \deg(u_n))^2$.

By Section 1, the top homogeneous part of the solution x_1 has to be a monomial x_1^0, and x_1^0 has to satisfy an equation in a free semigroup

$$u_1^0 x_1^0 u_2^0 x_1^0 u_3^0 \cdots u_{n-1}^0 x_1^0 u_n^0 = v_1^0 x_1^0 v_2^0 x_1^0 v_3^0 \cdots v_{n-1}^0 x_1^0 v_n^0,$$

where u_i^0 and v_i^0 are the top monomials in u_i and v_i. Every solution of this equation is semiperiodic, i.e., has to be of the form $r_0 w_0^m$, where $\operatorname{length}(r_0) < \operatorname{length}(w_0)$ *and w_0 is primitive. We say that w_0 is the period of x_0, and we further assume that* $\operatorname{length}(w_0) > 1$.

Suppose further that $\deg(u_i), \deg(v_i) > \operatorname{length}(w_0)$ *for every $i = 1, \dots, n$, and that the period of the top monomial of x_1 contains no periodicity, and that in addition the top monomials from the two sides of the equation that are obtained* *from the two sides of the equation after substituting the solution* x_1, *contain no periodicity except the one in the top monomial of* x_1 (*this translates to a condition on the coefficients* u_i *and* v_i, $1 \leq i \leq n$, *in the equation*).

Then there exist some elements $s, t \in FA$, $\deg(s) = \deg(t) \leq \max \deg(u_i)$, *such that*

(1) *every solution of the equation* $sx = xt$ *is a solution of the given equation*;

(2) *every solution* x_2 *of the given equation that satisfies*

$$\deg(x_2) > 2(2 + k^{\deg(s)+2}) + 2(\deg(u_1) + \cdots + \deg(u_n))^2$$

is a solution of the equation $sx = xt$.

Proof. Let x_1 be a solution of the given equation that satisfies

$$\deg(x_1) > 2(\deg(u_1) + \cdots + \deg(u_n)).$$

We start by looking at the top homogeneous part of x_1, which we denote x_1^0. Clearly x_1^0 satisfies the homogeneous equation

$$u_1^0 x_1^0 u_2^0 x_1^0 u_3^0 \cdots u_{n-1}^0 x_1^0 u_n^0 = v_1^0 x_1^0 v_2^0 x_1^0 v_3^0 \cdots v_{n-1}^0 x_1^0 v_n^0,$$

where u_i^0 and v_i^0 are the top monomials in u_i and v_i.

We start the analysis of the given equation under the assumption that there exists an index i for which $\deg(u_i) \neq \deg(v_i)$. In that case there is a shift between the appearances of some of the (homogeneous) elements x_1^0 in the two sides of the (homogeneous) equation. Let i_1 be the first index i for which $\deg(u_i) \neq \deg(v_i)$. The next appearances of x_1^0 in the two sides of the equation must have a shift of $|\deg(u_{i_1}) - \deg(v_{i_1})|$. Since the top homogeneous parts of u_i and v_i are monomials, it follows that the top homogeneous part of x_1 is a monomial as well. We keep the notation x_1^0 for the top monomial of x_1.

Let d be the minimum positive shift between pairs of appearances of x_1^0 in the top degree equation. Then $x_1^0 = e_0(t_0)^b = (s_0)^b f_0$, for some elements t_0, s_0 in the semigroup generated by the free generators $a_1, \ldots, a_k$ of the algebra FA. Note that $\deg(s_0) = \deg(t_0) = d$, e_0 is a prefix of s_0 and suffix of t_0 and f_0 is a suffix of t_0 and prefix of s_0. Since the top monomial u_i^0 and v_i^0 have no periodicity, t_0 and s_0 have no periodicity as well.

Since we assumed that the length of x_1^0 is bigger than the sum of the lengths of the degrees $\deg(u_i)$, an appearance of x_1^0 in the product that is associated with the top monomial in the left side of the equation overlaps with the corresponding appearance of x_1^0 in the right side of the equation, and may overlap with the previous or the next appearance of x_1^0 of the right side of the equation as well. Our assumptions that $\deg(u_i), \deg(v_i) > d$ together with the assumption that the coefficients have no periodicity imply that an appearance of x_1^0 in the product that is associated with the top monomial in one side of the equation may overlap only with the corresponding appearance of x_1^0 in the other side of the equation (and not with the previous or the next one).

Let $1 \le i_1 < \cdots < i_\ell \le n-1$ be the indices for which there is a (nontrivial) shift between the appearances of x_1^0 in the two sides of the equation. Let $1 \le j_1 < \cdots < j_{n-1-\ell} \le n-1$ be the complementary indices, i.e., those indices for which there is no shift between the corresponding appearances of the monomials x_1^0 in the two sides of the equation.

We start by analyzing the codegree 1 monomials in the products that are associated with the two sides of the equation. We further assumed that the length of the period in x_1^0, i.e., $d = |\deg(u_{i_1}) - \deg(v_{i_1})| > 1$. Note that any codegree 1 monomial in the two products is a product of top monomials with a single codegree 1 monomial from one of the u_i, v_i or one of the appearances of x_1.

Suppose that $i_1 > 1$. Then x_1^0 is quasiperiodic (or rather a fractional power), its period is of length at least 2 and x_1^0 contains at least 2 periods. Hence, a codegree 1 product that contains a codegree 1 monomial of u_1 can cancel with either a codegree 1 product that contains a codegree 1 monomial in v_1 or a codegree 1 product that contains a codegree 1 monomial in the first appearance of x_1.

If the two canceling codegree 1 products contain codegree monomials of u_1 and v_1, then these two codegree 1 monomials must be equal. Otherwise, the codegree 1 product that contains a codegree 1 monomial of u_1 cancels with a codegree 1 product that contains a codegree 1 monomial of the first appearance of x_1. Now, this last codegree 1 monomial appears in the other side of the equation as well, and it can cancel only with a codegree 1 product that contains a codegree 1 monomial of v_1 that must be identical to the codegree 1 monomial of u_1 that we started with. Therefore, the codegree 1 homogeneous parts of v_1 and u_1 are equal. Continuing with the same argument iteratively, the codegree 1 homogeneous parts of the elements u_i and v_i are equal for all $i < i_1$ and $i > i_\ell$.

Let j_s be one of the indices for which there is no shift between the corresponding appearances of x_1^0 in the two sides of the equation. We look at the codegree 1 products in the two sides of the equation. Each such codegree 1 product is a product of a single codegree 1 monomial from a single appearance of x_1^0 or exactly one of the coefficients u_i or v_i, with top degree monomials. Note that the codegree 1 products that involve codegree 1 monomials of the j_s appearance of x_1 in the two sides of the equation (and top degree monomials from all the coefficients and the other appearances of x_1), are precisely the same codegree 1 products in the two sides of the equation. Hence, these do cancel. All the other codegree 1 products in the two sides of the equation contain x_1^0 in the j_s appearance of x_1. Since x_1^0 is periodic, and the length of the period is bigger than 1, a codegree 1 product that includes a codegree 1 monomial to the left of the j_s appearance of x_1 cannot be equal to a codegree 1 product that contains a codegree 1 monomial to the right of the j_s appearance of x_1.

Therefore, the left codegree 1 products (with respect to the j_s appearance of x_1) from the two sides of the equation have to cancel and the right codegree 1 products have to cancel as well. In particular, if for some index i, both the $i-1$ and the i appearances of x_1 in the two sides of the equation have no shift, then u_i and v_i have the same codegree 1 homogeneous parts.

At this point we need to examine the appearances of the variables x_1 in which there is a shift between the two sides of the equation, i.e., in places $i_1, \dots, i_\ell$, and the coefficients u_i and v_i that are connected to these appearances. To do that we break the appearances of the variables x_1, and the coefficients u_i, v_i in the two sides of the equations into regions (or intervals).

We look at the top monomial in the two sides of the equation. For each index i we add a breakpoint at the left point of the pair u_i, v_i, and to the right of that pair. We denote the variable that is associated with the region (interval) between the right point of the pair u_i, v_i and the left point of the pair u_{i+1}, v_{i+1} by w_i. The top monomial of w_i is a prefix or a suffix of the top monomial x_1^0 of x_1. We denote by q_i the variable that is associated with the region between w_{i-1} and w_i. Note that the region that is associated with q_i contains the support of u_i and v_i. If the region that is associated with q_i contains the right part of the $i-1$ appearance of x_1, we denote the variable that is associated with that right part by t_{i-1}. If it contains the left part of the i appearance of x_1, we denote the variable that is associated with that left part by s_i.

As in our previous arguments, we intend to break the solution x_1, so that $x_1 = s_i w_i = w_i t_i$, whenever the variables w_i, s_i, t_i are defined and in appropriate abelian (quotient) groups. Furthermore, each of the elements q_i can be broken according to the two sides of the equation. Hence, we intend to show that $q_i = t_{i-1} u_i s_i$, or $q_i = t_{i-1} u_i$, or $q_i = u_i s_i$, or $q_i = u_i$, and correspondingly for the elements v_i (instead of the u_i), depending on the way the elements q_i are broken in the two sides of the equation.

Because of the periodicity of x_1^0, and since we assume that the length of the period of x_1^0 is bigger than 1, a codegree 1 product that contains a codegree 1 monomial of v_i cannot cancel with a codegree 1 product that contains a codegree 1 monomial in $v_{i'}$ or $u_{i'}$ for $i \neq i'$, and likewise for the u_i.

Suppose that $q_i = v_i = t_{i-1} u_i s_i$ for the top monomials. In that case two codegree 1 products that contain codegree 1 monomials of the $i-1$ and i appearances of x_1 that are both from the v_i side, or both from the u_i side, cannot cancel. Furthermore, two codegree 1 products that cancel and belong to the two sides of the equation cannot contain codegree 1 monomials from both appearances $i-1$ and i of x_1.

Hence, in that case a pair of canceling codegree 1 products may be either

(1) codegree 1 monomials of the same appearance of x_1 from the two sides of the equation;

(2) a codegree 1 monomial of either the i or $i-1$ appearance of x_1 for one product, and a codegree 1 monomial of u_i for the second product;

(3) a codegree 1 monomial of either the i or $i-1$ appearance of x_1 for one product, and a codegree 1 monomial of v_i for the second product;

(4) a codegree 1 monomial of u_i in one product, and a codegree 1 monomial of v_i in the second product.

If case (1) occurs we add a codegree 1 monomial to w_i or w_{i-1} (depending on the appearance of x_1). In case (2) we add a codegree 1 monomial to t_{i-1} or s_i, and the existing one to u_i. In case (3) we add the existing codegree 1 monomial to v_i and a codegree 1 monomial to t_{i-1} or s_i (depending on the appearance of x_1). In case (4) we add only the existing codegree 1 monomials to v_i and u_i.

Suppose that $q_i = t_{i-1}v_i = u_is_i$ for the top monomials. Two codegree 1 products that contain codegree 1 monomials of the $i-1$ and i appearances of x_1 from the same side of the equation cannot be equal. Furthermore, a codegree 1 product that contains a codegree 1 monomial of the $i-1$ appearance of x_1 in the u_i side cannot cancel with a codegree 1 product that contains a codegree 1 monomial in the i appearance of x_1 from the v_i side. Since we assumed that $\deg(u_i), \deg(v_i) > d$, and the coefficients have no periodicity, a codegree 1 product that contains a codegree 1 monomial of the $i-1$ appearance of x_1 in the v_i side cannot cancel with a codegree 1 product that contains a codegree 1 monomial in the i appearance of x_1 from the u_i side.

Like in the case $q_i = v_i = t_{i-1}u_is_i$, in that case a pair of canceling codegree 1 products may be either

(1) codegree 1 monomials of the same appearance of x_1 from the two sides of the equation;

(2) a codegree 1 monomial of either the i or $i-1$ appearance of x_1 for one product, and a codegree 1 monomial of u_i for the second product;

(3) a codegree 1 monomial of either the i or $i-1$ appearance of x_1 for one product, and a codegree 1 monomial of v_i for the second product;

(4) a codegree 1 monomial of u_i in one product, and a codegree 1 monomial of v_i in the second product.

If case (1) occurs we add a codegree 1 monomial to w_i or w_{i-1} (depending on the appearance of x_1). In case (2), if a codegree 1 of the $i-1$ appearance of x_1 is part of the canceling pair, we add a codegree 1 monomial to w_{i-1} (only if the codegree 1 product is from the u_i side of the equation), a codegree 1 monomial to t_{i-1}, and the existing codegree 1 monomial to u_i. If a codegree 1 of the i appearance of x_1 is part of the cancelling pair, we add a codegree 1 monomial to s_i, and the existing codegree 1 monomial to u_i. In case (3) we do the equivalent additions for v_i, w_i, t_{i-1} and s_i. In case (4) we just add the codegree 1 existing monomials to u_i and v_i.

So far we have constructed elements w_i, t_i and s_i such that the equations $x_1 = w_i t_i$, $x_1 = s_i w_i$, $q_i = u_i s_i$ or $q_i = t_{i-1} u_i$, or $q_i = t_{i-1} u_i s_i$ or $q_i = u_i$ (and correspondingly for the v_i) hold for products of codegree at most 1. We continue by analyzing products of codegree r, $r < d$, assuming that we analyzed all the products of smaller codegree, constructed the elements w_i, s_i and t_i, and they satisfy the last equations for products of codegree at most $r-1$.

We analyze codegree r products in a similar way to their analysis in the proof of Theorem 4.4. First, note that if a codegree r product is a product of monomials of u_i, v_i, s_i, t_i and w_i that correspond to products of monomials of codegree smaller than r of u_i, v_i and all the appearances of x_1 from the two sides of the equation, then such codegree r products cancel in pairs.

Let i be an index for which $\deg(u_i) = \deg(v_i)$, and there is no shift between the $i-1$ and i appearances of x_1. By our analysis of codegree 1 monomials, the top monomials, and the codegree 1 homogeneous parts of u_i and v_i are identical. Codegree r products from one side of the equation that contain codegree r monomials of the i or $i-1$ appearances of x_1 cancel with corresponding codegree r products from the other side of the equation. Hence, a codegree r product that contains a codegree r monomial of u_i can cancel only with a codegree r product that contains a codegree r monomial of v_i. Therefore, the codegree r homogeneous part of u_i is identical to the codegree r homogeneous part of v_i. Furthermore, for the purpose of analyzing codegree r products, the given equation can be broken into finitely many equations by taking out such pairs of coefficients u_i, v_i, and the appearances of the solution x_1 that are adjacent to them.

Suppose that for some index i there is no shift between the appearances of x_1 in the two sides of the equation. In that case codegree r products that contain codegree r monomials of the i appearance of x_1 from one side of the equation cancel with codegree r products that contain codegree r monomials of that i appearance of x_1 from the other side of the equation. Hence, for the purpose of analyzing codegree r products, the given equation breaks into several equations, by taking out all the appearances of x_1 that have no shift. Therefore, for the continuation of the analysis of codegree r products, we may assume that there are no appearances of x_1 with no shift.

Since we assumed that the equation does not contain appearances of x_1 in the two sides of the equation with no shift between them, the analysis of codegree r products that contain positive codegree monomials of either u_1 or v_1, or positive codegree monomials of either u_n or v_n, is identical to the analysis of codegree r monomials in Theorem 4.4, i.e., in the equation $u_1xu_2 = v_1xv_2$. Hence, we only need to analyze codegree r products that contain positive codegree monomials from some element $q_i = u_is_i = t_{i-1}v_i$ or from an element $q_i = t_{i-1}u_is_i = v_i$.

Let $q_i = u_is_i = t_{i-1}v_i$. Since u_i and v_i have no periodicity, a codegree r product that contains a codegree r monomial of x_1 in its $i-1$ appearance cannot cancel with a codegree r from the same side of the equation that contains a codegree r monomial of x_1 in its i appearance. Furthermore, a codegree r product that contains a codegree r monomial of x_1 in its $i-1$ appearance from the u_i side of the equation, cannot cancel with a codegree r product that contains a codegree r monomial of x_1 in its i appearance from the v_i side of the equation.

Suppose that a codegree r product can be expressed as products of codegree q_j monomials of t_{i-1} and codegree m_j monomials of v_i with top monomials of the other elements in the v_i side of the equation such that $q_j \geq 0$ and m_j is positive and $q_j + m_j = r$, in an odd number of ways. Such codegree r products can cancel with either

(1) an odd number of products of codegree f_j monomials of u_i and codegree g_j monomials of s_i with top monomials of the other elements in the u_i side of the equation, such that $g_j \geq 0$ and f_j is positive and $f_j + g_j = r$;

(2) a product of a codegree r monomial of x_1 in its $i-1$ appearance with other top monomials in the v_i side of the equation;

(3) a product of a codegree r monomial of x_1 in its i appearance with other top monomials in the v_i side of the equation;

(4) a product of a codegree r monomial of x_1 in its $i-1$ appearance with other top monomials in the u_i side of the equation;

(5) a product of a codegree r monomial of x_1 in its i appearance with other top monomials in the u_i side of the equation;

(6) an odd number of products of a codegree b_j monomial of w_{i-1} with a codegree a_j monomial of t_{i-1} for positive a_j, b_j, $a_j + b_j = r$, with top monomials of the other elements from the v_i side;

(7) an odd number of products of a codegree c_j monomial of w_i with a codegree h_j monomial of s_i for positive c_j, h_j, $c_j + h_j = r$, with top monomials of the other elements from the u_i side.

If only case (1) occurs we don't add anything to any of the elements except the existing codegree r monomials of u_i and v_i. If only case (2) occurs we add a codegree r monomial to t_{i-1}. If only case (3) occurs we add a codegree r monomial to w_i and a codegree r monomial to s_i. If only case (4) occurs we add a codegree

r monomial to w_{i-1} and to t_{i-1}. If only case (5) occurs we add a codegree r monomial to s_i.

Cases (2) and (3) cannot occur together, nor cases (4) and (5), nor cases (3) and (4). If only cases (1), (2) and (4) occur, we add a codegree r monomial to w_{i-1}. If only cases (1), (2) and (5) occur, we add codegree r monomials to t_{i-1} and s_i. If only (1), (3) and (5) occur, we add a codegree r monomial to w_i.

We still need to treat cases (6) and (7). Note that the existence of these cases means that codegree r products, which were supposed to exist given the smaller codegree monomials of the various elements, may or may not exist, depending on the existence of codegree r monomials in the various appearances of the element x_1. Also, note that case (6) cannot occur with case (3), and case (7) cannot occur with case (4).

If only case (6) appears, we add a codegree r monomial to t_{i-1}. If only case (7) appears we add a codegree r monomial to s_i. If only cases (1), (2) and (6) appear, we do not add anything. If only cases (1), (2) and (7) appear, we add a codegree r monomial to t_{i-1} and to s_i. If only (1), (3) and (7) appear, we do not add anything. If only (1), (4) and (6) appear, we add a codegree r monomial to w_{i-1}. If only (1), (5) and (6) appear, we add codegree r monomials to t_{i-1} and to s_i. If only (1), (5) and (7) appear, we don't add anything.

If only (2), (4) and (6) appear, we add codegree r monomials to w_{i-1} and to t_{i-1}. If only (2), (5) and (6) appear, we add a codegree r monomial to s_i. If only (2), (5) and (7) appear, we add a codegree r monomial to t_{i-1}. If only (3), (5) and (7) appear, we add codegree r monomials to w_i and to s_i.

If only (1), (6) and (7) appear, we add codegree r monomials to t_{i-1} and to s_i. If only (2), (6) and (7) appear, we add a codegree r monomial to s_i. If only (5), (6) and (7) appear, we add a codegree r monomial to t_{i-1}. If only (1), (2), (5), (6) and (7) appear, we do not add anything.

The case in which the codegree r product is a product of case (1) is dealt with in a symmetric way. Hence, suppose that the codegree r product is not a product of case (1) and cannot be expressed in an odd number of ways as products of codegree q_j monomials of t_{i-1} and codegree m_j monomials of v_i with top monomials of the other elements in the v_i side of the equation, such that $q_j \geq 0$ and m_j is positive and $q_j + m_j = r$.

If only (2) and (4) appear, we add a codegree r monomial to w_{i-1}. If only (2) and (5) appear, we add codegree r monomials to t_{i-1} and s_i. If only (2) and (6) appear, we do not add anything. If only (2) and (7) appear, we add codegree r monomials to t_{i-1} and s_i. If only (3) and (5) appear, we add a codegree r monomial to w_i. If only (3) and (7) appear, we add a codegree r monomial to w_i. If only (4) and (6) appear, we add a codegree r monomial to w_{i-1}. If only (5) and (6) appear, we add codegree r monomials to t_{i-1} and s_i. If only (5) and (7) appear, we do not add anything. If only (6) and (7) appear, we add codegree r monomials to t_{i-1} and s_i. Finally, if (2), (5), (6) and (7) appear, we do not add anything.

As in the proof of Theorem 4.4, it can still be that a codegree r product is of type (7) and can also be presented in an odd number of ways as products of a codegree b_j monomial of w_i with a codegree a_j monomial of t_i for positive a_j, b_j, $a_j + b_j = r$, with top monomials of the other elements from the v_i side.

In that case it can either be presented only in these two forms or also in both forms (3) and (5). If it can be presented in forms (3) and (5) we do not add anything. If it cannot we add a codegree r monomial to w_i.

This concludes the construction of the elements s_i, t_i, w_i for codegree r products that involve $q_i = u_i s_i = t_{i-1} v_i$. Suppose that $q_i = v_i = t_{i-1} u_i s_i$. Since u_i and v_i have no periodicity, a codegree r product that contains a codegree r monomial of x_1 in its $i-1$ appearance cannot cancel with a codegree r product that contains a codegree r monomial of x_1 in its i appearance.

Suppose that a codegree r product can be expressed as products of codegree q_j monomials of t_{i-1}, codegree m_j monomials of u_i and codegree p_j monomials of s_i with top monomials of the other elements in the u_i side of the equation, such that $q_j, m_j, p_j \geq 0$, either $m_j > 0$ or $q_j, p_j > 0$, and $q_j + m_j + p_j = r$, in an odd number of ways. Such codegree r products can cancel with either

(1) a product of a codegree r monomial of v_i with other top monomials;
(2) a product of a codegree r monomial of x_1 in its $i-1$ appearance with other top monomials in the u_i side of the equation;
(3) a product of a codegree r monomial of x_1 in its i appearance with other top monomials in the u_i side of the equation;
(4) a product of a codegree r monomial of x_1 in its $i-1$ appearance with other top monomials in the v_i side of the equation;
(5) a product of a codegree r monomial of x_1 in its i appearance with other top monomials in the v_i side of the equation;
(6) an odd number of products of a codegree b_j monomial of w_{i-1} with a codegree a_j monomial of t_{i-1} for positive a_j, b_j, $a_j + b_j = r$, with top monomials of the other elements from the u_i side;
(7) an odd number of products of a codegree c_j monomial of w_i with a codegree h_j monomial of s_i for positive c_j, h_j, $c_j + h_j = r$, with top monomials of the other elements from the u_i side.

According to the various cases, we add monomials to the variables t_i, s_i, w_i, in a similar way to what we did in case $q_i = u_i s_i = t_{i-1} v_i$. If only case (1) occurs we don't add anything to any of the elements except the existing codegree r monomials of u_i and v_i. If only case (2) occurs we add a codegree r monomial to t_{i-1}. If only case (3) occurs we add a codegree r monomial to s_i. If only case (4) occurs we add a codegree r monomial to w_{i-1} and to t_{i-1}. If only case (5) occurs we add a codegree r monomials to w_i and to s_i.

Cases (2) and (3) cannot occur together, nor cases (4) and (5), nor cases (3) and (4), nor (2) and (5). If only cases (1), (2) and (4) occur, we add a codegree r monomial to w_{i-1}. If only (1), (3) and (5) occur, we add a codegree r monomial to w_i.

As in the case in which $q_i = u_i s_i = t_{i-1} v_i$, the existence of cases (6) and (7) means that codegree r products that were supposed to exist given the smaller codegree monomials of the various elements, may or may not exist, depending on the existence of codegree r monomials in the various appearances of the element x_1. Also, note that case (6) cannot occur with cases (3) or (5), and case (7) cannot occur with cases (2) or (4).

If only case (6) appears, we add a codegree r monomial to t_{i-1}. If only case (7) appears we add a codegree r monomial to s_i. If only cases (1), (2) and (6) appear, we do not add anything. If only (1), (3) and (7) appear, we do not add anything. If only (1), (4) and (6) appear, we add a codegree r monomial to w_{i-1}. If only (1), (5) and (7) appear, we add a codegree r monomial to w_i.

If only (2), (4) and (6) appear, we add codegree r monomials to w_{i-1} and to t_{i-1}. If only (3), (5) and (7) appear, we add codegree r monomials to w_i and to s_i.

The case in which case (1) occurs is dealt with in an analogous way. Hence, suppose that the codegree r product is not a product of case (1) and cannot be expressed in an odd number of ways as products of codegree q_j monomials of t_{i-1}, codegree m_j monomials of u_i and codegree p_j monomials of s_i with top monomials of the other elements in the u_i side of the equation, such that $q_j, m_j, p_j \geq 0$, either m_j is positive or both q_j, p_j are positive and $q_j + m_j + p_j = r$.

If only (2) and (4) appear, we add a codegree r monomial to w_{i-1}. If only (2) and (6) appear, we do not add anything. If only (3) and (5) appear, we add a codegree r monomial to w_i. If only (3) and (7) appear, we do not add anything. If only (4) and (6) appear, we add a codegree r monomial to w_{i-1}. If only (5) and (7) appear, we add a codegree r monomial to w_i.

It can still be that a codegree r product is of type (7) and can also be presented in an odd number of ways as products of a codegree b_j monomial of w_i with a codegree a_j monomial of t_i for positive $a_j, b_j, a_j + b_j = r$, with top monomials of the other elements from the v_i side. We treat this case precisely as we treated it in the case $q_i = u_i s_i = t_{i-1} v_i$.

This concludes the construction of the elements s_i, t_i, w_i for codegree r products when $r < d$. The elements w_i, t_i, and s_i that we constructed so far satisfy the equations $x_1 = w_i t_i$, $x_1 = s_i w_i$, $q_i = u_i s_i$ or $q_i = t_{i-1} u_i$, or $q_i = t_{i-1} u_i s_i$ or $q_i = u_i$ (and correspondingly for the v_i) for products of codegree smaller than d.

To continue we need to analyze products of codegree d and higher. For presentation purposes we start this analysis under the additional assumption that all the appearances of x_1 in the two sides of the equation have nontrivial shifts, i.e., the appearances of the top monomial of the solution x_1^0 in the two sides of the equality for the top monomials are shifted. This assumption enables us to analyze the higher codegree products using the arguments that were used in the proof of Theorem 4.4 and in analyzing smaller codegree products. Afterwards we drop this assumption.

As in Theorem 4.4, in analyzing codegree d products, there are special codegree d products that we need to single out and treat separately, as they may involve cancellations between codegree d products that contain codegree d monomials of u_i or v_i and those that contain codegree d monomials of u_{i+1} or v_{i+1}.

As in analyzing smaller codegree products, note that codegree d products that are products of smaller codegree monomials of the u_i, v_i, s_i, w_i and t_i, and correspond to smaller codegree monomials of u_i, v_i and x_1 from the two sides of the equation cancel in pairs.

In analyzing codegree r products for $r < d$, there is no interaction between elements in q_i and q_j for $i \neq j$. As in the proof of Theorem 4.4, in analyzing codegree d products such interaction may happen if i and j are consecutive indices. Hence, in analyzing codegree d products we need to go over the various possibilities for q_i and q_{i+1}.

Suppose that $q_i = u_i s_i = t_{i-1} v_i$. Suppose that a codegree d product can be expressed as products of codegree q_j monomials of t_{i-1} and codegree m_j monomials of v_i with top monomials of the other elements in the v_i side of the equation, such that $q_j \geq 0$ and m_j is positive and $q_j + m_j = d$, in an odd number of ways. If the q_i part of such a product is not equal to u_i^0 nor to v_i^0 (the top monomials of u_i and v_i), such codegree r products are analyzed exactly in the same way they were analyzed in codegree r products for $r < d$.

We have $u_i^0 \neq v_i^0$ because we assumed that the top monomials of the coefficients have no periodicity. If the q_i part of such a product equals v_i^0, the codegree d product may be equal to a codegree d product that contains positive codegree monomials in q_{i-1}. If the q_i part of such a product equals u_i^0, the codegree d product may be equal to a codegree d product that contains positive codegree monomials in q_{i+1}.

Suppose that the q_i part of the codegree d product equals u_i^0. Suppose further that $q_{i+1} = u_{i+1} s_{i+1} = t_i v_{i+1}$. In that case such a codegree d product can cancel with codegree d products that are either a subset of those analyzed for products of smaller codegree, or products that include positive codegree monomials of q_{i+1}:

(1) an odd number of products of codegree f_j monomials of u_i and codegree g_j monomials of s_i with top monomials of the other elements in the u_i side of the equation, such that $g_j \geq 0$ and f_j is positive and $f_j + g_j = d$;

(2) a product of a codegree d monomial of x_1 in its i appearance with other top monomials in the v_i side of the equation;

(3) a product of a codegree d monomial of x_1 in its i appearance with other top monomials in the u_i side of the equation;

(4) an odd number of products of codegree q_j monomials of t_i and codegree m_j monomials of v_{i+1} with top monomials of the other elements in the v_i side of the equation, such that $q_j \geq 0$ and m_j is positive and the product of the monomial of t_i with the monomial of v_{i+1} is v_{i+1}^0;

(5) an odd number of products of codegree f_j monomials of u_{i+1} and codegree g_j monomials of s_{i+1} with top monomials of the other elements in the u_i side of the equation, such that $g_j \geq 0$ and f_j is positive and the product of the monomial of u_{i+1} with the monomial of s_{i+1} is v_{i+1}^0;

(6) an odd number of products of a codegree c_j monomial of w_i with a codegree h_j monomial of s_i for positive $c_j, h_j, c_j + h_j = d$, with top monomials of the other elements from the u_i side;

(7) an odd number of products of a codegree b_j monomial of w_i with a codegree a_j monomial of t_i for positive $a_j, b_j, a_j + b_j = d$, with top monomials of the other elements from the v_i side.

Note that case (2) occurs if and only if case (3) occurs. If only one of the cases (1) or (6) occurs, we treat them as they were treated in analyzing codegree r products for $r < d$. If only case (4) or only case (5) occurs we add 1 (the identity) to s_i and t_i, and the codegree d prefix of w_i^0 to w_i. If only case (7) occurs we add 1 to s_i and the codegree d prefix of w_i^0 to w_i.

If only cases (1)–(3) occur, or only cases (2), (3) and (6) occur, we treat them as they were treated for codegree r products, $r < d$. If only cases (2) and (3) in addition to one of the cases (4) or (5) occur, we add 1 to s_i and t_i. If only cases (2), (3) and (7) occur, we add 1 to s_i. If only (1), (4) and (5) occur, we don't add anything. If only (1), (6) and one of (4) or (5) occur, we add 1 to t_i and the codegree d prefix of w_i^0 to w_i. If only (1), (7) and one of (4) or (5) occur, we add 1 to t_i. If only (1), (6) and (7) occur, we add the codegree d prefix of w_i^0 to w_i. If only (6), (7) and one of (4) or (5) occur, we add 1 to s_i and t_i and the codegree d prefix of w_i^0 to w_i. If only (4), (5) and (6) occur, we add 1 to s_i. If only (4), (5) and (7) occur, we add 1 to s_i and the codegree d prefix of w_i^0 to w_i.

If only (1)–(5) occur, we add the codegree d prefix of w_i^0 to w_i. If only (1)–(3) and (6)–(7) occur, we do not add anything. If only (1)–(3), (6) and one of (4) or (5) occur, we add 1 to t_i. If only (1)–(3), (7) and one of (4) or (5) occur, we add 1 to t_i and the codegree d prefix of w_i^0 to w_i. If only (1) and (4)–(7) occur, we add the codegree d prefix of w_i^0 to w_i.

If only (2)–(6) occur, we add 1 to s_i and the prefix of codegree d of w_i^0 to w_i. If only (2)–(5) and (7) occur, we add 1 to s_i. If only (2)–(3), (6)–(7) and one of (4) or (5) occur, we add 1 to s_i and t_i and the codegree d prefix of w_i^0 to w_i. If all the possibilities (1)–(7) occur, we do not add anything.

Suppose that a codegree d product can be expressed as a product in case (1), and cannot be expressed as products of codegree q_j monomials of t_{i-1} and codegree m_j monomials of v_i with top monomials of the other elements in the v_i side of the equation, such that $q_j \geq 0$ and m_j is positive and the q_i part of the product is u_i^0 in an even number (possibly none) ways. In that case the analysis of such a product and the monomials that are added to the elements t_i, s_i and w_i are analogous to the analysis described above.

Suppose that such a codegree d product cannot be expressed as a product in case (1), but it can be expressed as a product in case (6). If only (6) and (7) occur, we add the codegree d prefix of w_i^0 to w_i. If only (6) and one of (4) or (5) occur, we add 1 to t_i and the prefix of codegree d of w_i^0 to w_i. If only (4)–(7) occur, we add the codegree d prefix of w_i^0 to w_i. If only (2)–(3) and (6)–(7) occur, we do not add anything. If only (2)–(3), (6) and one of (4) or (5) occur, we add 1 to t_i. If only (2)–(7) occur, we do not add anything.

This concludes the analysis of such codegree d products in the case that $q_{i+1} = u_{i+1}s_{i+1} = t_i v_{i+1}$. Suppose that $q_{i+1} = u_{i+1} = t_i v_{i+1} s_{i+1}$. As before, such a codegree d product can cancel with codegree d products that are either a subset of the ones that were analyzed for products of smaller codegree, or products that include positive codegree monomials of q_{i+1}:

(1) an odd number of products of codegree f_j monomials of u_i and codegree g_j monomials of s_i with top monomials of the other elements in the u_i side of the equation, such that $g_j \geq 0$ and f_j is positive and $f_j + g_j = d$;

(2) a product of a codegree d monomial of x_1 in its i appearance with other top monomials in the v_i side of the equation;

(3) a product of a codegree d monomial of x_1 in its i appearance with other top monomials in the u_i side of the equation;

(4) an odd number of products of codegree q_j monomials of t_i, codegree m_j monomials of v_{i+1} and codegree p_j monomials of s_{i+1} with top monomials of the other elements in the v_i side of the equation, such that $q_j, m_j, p_j \geq 0$, either $m_j > 0$ or $q_j, p_j > 0$, and $q_j + m_j + p_j = d$, and the product of the corresponding monomials of t_i, v_{i+1} and s_{i+1} is the codegree d suffix of u_{i+1}^0;

(5) a product of a monomial of u_{i+1}, which is the codegree d suffix of u_{i+1}^0, with the top monomials of the all the other elements from the u_i side of the equation;

(6) an odd number of products of a codegree c_j monomial of w_i with a codegree h_j monomial of s_i for positive $c_j, h_j, c_j + h_j = d$, with top monomials of the other elements from the u_i side;

(7) an odd number of products of a codegree b_j monomial of w_i with a codegree a_j monomial of t_i for positive $a_j, b_j, a_j + b_j = d$, with top monomials of the other elements from the v_i side.

Analyzing the various possibilities in this case is identical to the case $q_{i+1} = u_{i+1}s_{i+1} = t_i v_{i+1}$.

Recall that we assumed that $q_i = u_i s_i = t_{i-1} v_i$. In addition suppose that a codegree d product can be expressed as products of codegree f_j monomials of u_i and codegree g_j monomials of s_i with top monomials of the other elements in the u_i side of the equation, such that $f_j \geq 0$ and g_j is positive and $f_j + g_j = d$, in an odd number of ways, and such that the product of the monomial of u_i with the monomial of s_i is v_i^0. In that case, the codegree d product may be equal to a codegree d product that contains positive codegree monomials in q_{i-1}. Such a codegree d product can cancel with codegree d products that are either a subset of the ones that were analyzed for products of smaller codegree, or products that include positive codegree monomials of q_{i-1}:

(1) an odd number of products of codegree q_j monomials of t_{i-1} and codegree m_j monomials of v_i with top monomials of the other elements in the v_i side of the equation, such that $q_j \geq 0$ and m_j is positive and $q_j + m_j = d$;

(2) a product of a codegree d monomial of x_1 in its $i-1$ appearance with other top monomials in the u_i side of the equation;

(3) a product of a codegree d monomial of x_1 in its $i-1$ appearance with other top monomials in the v_i side of the equation;

(4) an odd number of products of codegree f_j monomials of u_{i-1} and codegree g_j monomials of s_{i-1} with top monomials of the other elements in the u_i side of the equation, such that $g_j \geq 0$ and f_j is positive and the product of the monomial of u_{i-1} with the monomial of s_{i-1} is u_{i-1}^0;

(5) an odd number of products of q_j monomials of t_{i-2} and codegree m_j monomials of v_{i-1} with top monomials of the other elements in the v_i side of the equation, such that $q_j \geq 0$ and m_j is positive and the product of the monomial of t_{i-2} with the monomial of v_{i-1} is u_{i-1}^0;

(6) an odd number of products of a codegree c_j monomial of w_{i-1} with a codegree h_j monomial of t_{i-1} for positive $c_j, h_j, c_j + h_j = d$, with top monomials of the other elements from the v_i side;

(7) an odd number of products of a codegree a_j monomial of s_{i-1} with a codegree b_j monomial of w_{i-1} for positive $a_j, b_j, a_j + b_j = d$, with top monomials of the other elements from the u_i side.

The analysis of this case is identical to the case in which the q_i part of a codegree d product is v_i^0, and there is a possible cancellation with codegree d products that contain positive codegree monomials of q_{i+1}. An identical analysis applies also when $q_{i-1} = v_{i-1} = t_{i-2}u_{i-1}s_{i-1}$.

Suppose that $q_i = u_i = t_{i-1}v_is_i$ and $q_{i+1} = v_{i+1} = t_iu_{i+1}s_{i+1}$. Suppose that a codegree d product can be presented in an odd number of ways as products of codegree q_j monomials of t_{i-1}, codegree m_j monomials of v_i and codegree p_j monomials of s_i with top monomials of the other elements in the v_i side of the equation, such that $q_j, m_j, p_j \geq 0$, either $m_j > 0$ or $q_j, p_j > 0$, and $q_j + m_j + p_j = d$, and the product of the corresponding monomials of t_{i-1}, v_i and s_i is the codegree d prefix of u_i^0.

Such a codegree d product can cancel with codegree d products that are either a subset of the ones that were analyzed for products of smaller codegree, or products that include positive codegree monomials of q_{i+1}:

(1) a product of a monomial of u_i which is the codegree d prefix of u_i^0 with the top monomials of all the other elements from the u_i side of the equation;

(2) a product of a codegree d monomial of x_1 in its i appearance with other top monomials in the v_i side of the equation;

(3) a product of a codegree d monomial of x_1 in its i appearance with other top monomials in the u_i side of the equation;

(4) a product of a monomial of v_{i+1} which is the codegree d suffix of v_{i+1}^0 with the top monomials of the all the other elements from the v_i side of the equation;

(5) an odd number of products of codegree q_j monomials of t_i, codegree m_j monomials of u_{i+1} and codegree p_j monomials of s_{i+1} with top monomials of the other elements in the u_i side of the equation, such that $q_j, m_j, p_j \geq 0$, either $m_j > 0$ or $q_j, p_j > 0$, and $q_j + m_j + p_j = d$, and the product of the corresponding monomials of t_i, v_{i+1} and s_{i+1} is the codegree d suffix of v_{i+1}^0;

(6) an odd number of products of a codegree c_j monomial of w_i with a codegree h_j monomial of s_i for positive $c_j, h_j, c_j + h_j = d$, with top monomials of the other elements from the v_i side;

(7) an odd number of products of a codegree b_j monomial of w_i with a codegree a_j monomial of t_i for positive $a_j, b_j, a_j + b_j = d$, with top monomials of the other elements from the u_i side.

Analyzing the various possibilities in this case is identical to the case $q_i = t_{i-1}v_i = u_i s_i$. The analysis of the remaining case, in which $q_i = u_i = t_{i-1}v_i s_i$ and $q_{i-1} = v_{i-1} = t_{i-2}u_{i-1}s_{i-1}$, is identical to the previous cases as well.

This concludes the construction of the elements s_i, t_i, w_i for codegree r products when $r \leq d$, in case all the pairs of appearances of the top monomial of the solution x_1 in the two sides of the equation have nontrivial shifts. The elements w_i, t_i and s_i that we constructed so far satisfy the equations $x_1 = w_i t_i$, $x_1 = s_i w_i$, $q_i = u_i s_i$ or $q_i = t_{i-1}u_i$, or $q_i = t_{i-1}u_i s_i$ or $q_i = u_i$ (and correspondingly for the v_i) for products of codegree smaller or equal to d.

As in the proof of Theorem 4.4, we continue with the analysis of codegree $d+r$ products for $r < d$. First, as in analyzing smaller codegree products, codegree $d+r$ products that are products of smaller codegree monomials of u_i, v_i, s_i, t_i and w_i, that correspond to products of smaller codegree monomials of u_i, v_i and x_1 (in all its appearances) from the two sides of the equation, cancel in pairs. We start with two lemmas that are the analogues of Lemmas 4.5 and 4.6.

Lemma 4.8. *Suppose that a codegree $d+r$ product can be presented both as*

(1) *a product of a codegree h monomial of s_i with a codegree c monomial of w_i, for positive $c, h, c+h = d+r$, with top monomials of the other elements from the u_i side*;

(2) *a product of a codegree b monomial of w_i with a codegree a monomial of t_i for positive $a, b, a+b = d+r$, with top monomials of the other elements from the v_i side.*

Such a codegree $d+r$ product may only be presented as a product of smaller codegree monomials or (*only*) *in one of the following two products*:

(i) *a product of a codegree $d+r$ monomial of x_1 in its i appearance with other top monomials in the v_i side of the equation*;

(ii) *a product of a codegree $d+r$ monomial of x_1 in its i appearance with other top monomials in the u_i side of the equation.*

Proof. In case it can be presented as another product of a codegree $d+r$ monomial with top degree monomials, either the top monomial of s_i or the top monomial of t_i overlap with themselves with a cyclic shift. Hence they must be periodic, a contradiction to the assumption that the coefficients do not have nontrivial periodicity. □

Lemma 4.9. *With the notation of Lemma 4.8, if a codegree $d+r$ product can be presented in an odd number of ways as a product in the form* (1) *and in an even number of ways as a product of form* (2), *then such a product can be presented precisely in one of the forms* (i) *or* (ii). *If a codegree $d+r$ product can be presented precisely in one of the forms* (i) *or* (ii), *then it can be presented precisely in one of the forms* (1) *or* (2) *in an odd number of ways.*

If a codegree $d+r$ product can be presented in an odd number of ways in both forms (1) *and* (2), *then it can either be presented in both forms* (i) *and* (ii) *or in neither of them. If a codegree $d+r$ product can be presented in both forms* (i) *and* (ii) *then it can either be presented in both forms* (1) *or* (2) *in an odd number of ways*, *or in both of them in an even number of ways.*

Proof. If a codegree $d+r$ product can be presented in both forms (1) and (2) (odd or even number of times), the conclusion follows from Lemma 4.8. Suppose that it can be presented in an odd number of ways in form (1) and none in form (2). If it can also be presented as a codegree $d+r$ product that involves positive codegree monomials of u_j, v_j, s_j, t_j or x_j, for $j > i$, the top monomial of u_{i+1} must have nontrivial periodicity, a contradiction. If it can be also presented as a codegree $d+r$ product from the u_i sides of the equation that involves monomials of positive codegree monomials of u_j, s_j, t_j or x_j, for $j < i$, the top monomial of u_i must have nontrivial periodicity, a contradiction.

Suppose that the given codegree $d+r$ product can also be presented as a product of either

(1) a codegree q of t_{i-1} and a codegree m of v_i with other top monomials from the v_i side of the equation;

(2) a codegree f of u_i and a codegree g of s_i with other top monomials from the u_i side of the equation;

(3) a codegree $d+r$ product from the v_i side of the equation that involves monomials of positive codegree monomials of v_j, s_j, t_j or x_j for $j < i$.

In all these cases the suffix of length r of the top monomial of u_i is identical to the prefix of length r of the period of x. If $r \leq \deg(v_i) - d$, then v_i has nontrivial periodicity, a contradiction. Otherwise, the top monomial in the two sides of the equation contains periodicity that is not part of the periodicity of the solution x, a contradiction to our assumptions. □

Suppose that $q_i = u_i s_i = t_{i-1} v_i$, and let r be an integer, $0 < r < d$. By Lemma 4.9 if a codegree $d+r$ product can be presented in an odd number of ways in the form (1) of Lemma 4.8 then either

(1) it can be also presented in an odd number of ways as in form (2) of Lemma 4.8 and either in both forms (i) and (ii) in Lemma 4.8 or in neither of them;

(2) it can be presented in an even or no ways in form (2) of Lemma 4.8, and it can also be presented precisely in one of the forms (i) or (ii) in Lemma 4.8.

By Lemma 4.9, if a codegree $d+r$ product can be presented in form (i) of Lemma 4.8, and in even or no ways in forms (1) or (2) of that lemma, then it can also be presented in form (ii) of Lemma 4.8.

Hence, if a codegree $d+r$ product can be presented in an odd number of ways in one of the forms (1), (2), (i) or (ii), then the appearances of the codegree $d+r$ products in these forms cancel in pairs. If it appears in an odd number of ways in forms (1) and (2), and in forms (i) and (ii), we do not add anything. If it appears in an odd number of ways in forms (1) and (2) and not in the forms (i) nor (ii), we add a codegree $d+r$ monomial to w_i. If it appears in an odd number of ways in the form (1), in an even number of or no ways in the form (2), and appears in the form (i) we add a codegree $d+r$ monomial to w_i. If it appears in an odd number of ways in the form (1), in an even number of or no ways in the form (2), and in the form (ii), we do not add anything. If it appears in an even number of or no ways in the forms (1) and (2), and in both form (i) and (ii), we add a codegree $d+r$ monomial to w_i.

Therefore, if a codegree $d+r$ product can be presented in an odd number of ways as products of codegree q_j monomials of t_{i-1} and codegree m_j monomials of v_i with top monomials of the other elements in the v_i side of the equation, such that $q_j \geq 0$ and m_j is positive and $q_j + m_j = d+r$, then it must be presented in an odd number of ways as products of codegree f_j monomials of u_i and codegree g_j monomials of s_i with top monomials of the other elements in the u_i side of the equation, such that $g_j \geq 0$ and f_j is positive and $f_j + g_j = r$.

This concludes the construction of the elements s_i, t_i, w_i in case $q_i = t_{i-1}v_i = u_i s_i$ (note that the elements s_i, t_i did not change), to ensure that the equalities they are supposed to satisfy hold for products up to codegree $d+r$.

Suppose that $q_i = u_i = t_{i-1}v_i s_i$. Lemmas 4.8 and 4.9 and their proofs remain valid in this case. Hence, a codegree $d+r$ product can be expressed in an odd number of ways as products of codegree q_j monomials of t_{i-1}, codegree m_j monomials of v_i, and codegree p_j monomials of s_i with top monomials of the other elements in the v_i side of the equation, such that $q_j, m_j, p_j \geq 0$, either $m_j > 0$ or $q_j, p_j > 0$, and $q_j + m_j + p_j = d+r$, if and only if it is equal to a codegree $d+r$ monomial of u_i.

This concludes our treatment of codegree $d+r$ products for $r < d$. We continue by analyzing codegree $2d$ products. Lemmas 4.8 and 4.9 remain valid for codegree $2d$ products. Hence, the analysis of codegree $2d$ products is identical to the analysis of codegree $d+r$ products for $r < d$. The analysis of higher codegree products, for codegree up to twice the maximal degree of the elements u_i, v_i, is identical as well.

Hence, in case $\deg(u_i), \deg(v_i) > d$ and all the appearances of the elements x_1 in the two sides of the equation have nontrivial shifts, we finally constructed elements s_i, t_i, w_i that satisfy the equations

(i) $q_i = u_i s_i = t_{i-1} v_i$ or $q_i = v_i = t_{i-1} u_i s_i$ or with exchanging the appearances of u_i and v_i in the second equation;

(ii) $x_1 = s_i w_i = w_i t_i \mod G^{\deg(x_1) - 2(\deg(s_i))}$.

Therefore, s_1 and t_{n-1} are uniquely defined, and (given x_1) w_1 and w_{n-1} are uniquely defined mod $G^{\deg(w_1)-2(\deg(s_i))}$. Hence, t_1 and s_2 are uniquely defined, and w_2 is uniquely defined mod $G^{\deg(w_2)-2(\deg(s_i))}$. Continuing iteratively, all the elements s_i, t_i are uniquely defined, and the elements w_i are uniquely defined mod $G^{\deg(w_i)-2(\deg(s_i))}$.

Since $s_i w_i = w_i t_i$, it follows that $s_i x_1 = x_1 t_i$ mod $G^{\deg(s_i x_1)-2(\deg(s_i))}$. This implies that for every pair i, j, $1 \le i, j \le n$, we have $(s_i + s_j)x_1 = x_1(t_i + t_j)$ mod $G^{\deg(s_i x_1)-2(\deg(s_i))}$, so for every pair i, j either $s_i = s_j$ and $t_i = t_j$ or $s_i = s_j + 1$ and $t_i = t_j + 1$.

Since every pair (s_i, t_i) is either (s_1, t_1) or $(s_1 + 1, t_1 + 1)$, it follows that every element $\hat{x}$ that satisfies $s_1\hat{x} = \hat{x}t_1$ is a solution of the given equation. It remains to prove that every long enough solution of the given equation is a solution of the equation $s_1 x = x t_1$.

Let x_2 be a solution of the given equation that satisfies

$$\deg(x_2) > 2(2 + k^{\deg(s)+2}) + (2(\deg(u_1) + \cdots + \deg(u_n))^2.$$

By continuing the analysis of higher codegree monomials of the solution x_2, we get that there exist elements w_i such that for every index i, $1 \le i \le n$, we have $s_i w_i = w_i t_i = x_2$ mod $G^{\deg(s_1)-1}$. By the argument that was used to prove Lemma 4.2, it follows that there exists a solution $\hat{x}$ to the equation $s_1 x = x t_1$.

Note that x_2 satisfies $s_1 x_2 = x_2 t_1$ mod $G^{2\deg(s_1)-1}$. Hence, there exists an element $\hat{x}_2$ which is a solution of the equation $s_1 x = x t_1$, and $x_2 + \hat{x}_2 = r$, where $\deg(r) \le 2 + k^{\deg(s_1)+2}$.

Suppose the given equation is $v_1 x v_2 x v_3 = u_1 x u_2 x u_3$, where $\deg(v_1) < \deg(u_1)$ and $\deg(v_2) = \deg(u_2)$. In this case, $u_1 = v_1 s_1$, $t_1 u_2 = v_1 s_2$ and $v_3 = t_2 u_3$. Hence, $(\hat{x}_2 + r)v_2(\hat{x}_2 + r)t_2 = s_1(\hat{x}_2 + r)u_2(\hat{x}_2 + r)$. Since $\hat{x}_2$ is a solution to the equation $s_1 x = x t_1$, it is a solution to the given equation. Therefore

$$\hat{x}_2 v_2 r t_2 + r v_2 \hat{x}_2 t_2 = s_1 \hat{x}_2 u_2 r + s_1 r u_2 \hat{x}_2 \text{ mod } G^{\deg(r v_2 r t_2)}.$$

Hence

$$\hat{x}_2(v_2 r t_2 + t_1 u_2 r) = (r v_2 s_2 + s_1 r u_2)\hat{x}_2 \text{ mod } G^{\deg(r v_2 r t_2)}.$$

Since $s_1\hat{x}_2 = \hat{x}_2 t_1$ it follows that for any polynomial p, $p(s_1)\hat{x}_2 = \hat{x}_2 p(t_1)$. This implies $v_2 r t_2 + t_1 u_2 r = p(t_1)$ and $r v_2 s_2 + s_1 r u_2 = p(s_1)$ mod $G^{\deg(v_2 r t_2)+\deg(r)-\deg(x_2)}$ for the same polynomial p.

We have $t_1 u_2 = v_2 s_2$, so $v_2(r t_2 + s_2 r) = p(t_1)$ mod $G^{\deg(v_2 r t_2)+\deg(r)-\deg(x_2)}$. By our assumption on $\deg(x_2)$ it follows that $v_2(r t_2 + s_2 r) = p(t_1)$. Similarly, $(r t_1 + s_1 r)u_2 = p(s_1)$. Hence, $p(s_1)$ is either 0 or its leading term is of degree at least 2.

Since (s_1, t_1) equals (s_2, t_2) or $(s_2 + 1, t_2 + 1)$, we get that $v_2(r t_1 + s_1 r)u_2 = v_2 p(t_1) = p(s_1)u_2$. We look at the leading term in the two sides of the last equality.

Since we assumed that the top monomials of u_2 and v_2 are not periodic, the top monomial of u_2 must be βs_0, and the top monomial of v_2 must be $t_0\beta$, where β is a prefix of t_0 and a suffix of s_0. Hence, $t_0=\beta\alpha$ and $s_0=\alpha\beta$. But this is a contradiction, since we assumed that the periodicity in the top monomials in the two sides of the given equation is contained in the solution x_2. Therefore, $s_1r+rt_1=0$, so r is a solution of the equation $s_1x=xt_1$, which means that $x_2=\hat{x}_2+r$ is a solution to $s_1x=xt_1$ as well.

If the equation is $u_1xu_2xu_3=v_1xv_2xv_3$, and $\deg(v_1)>\deg(u_1)$, $\deg(v_3)>\deg(u_3)$, then by the same arguments we get that r (the remainder) has to satisfy the equation

$$(rt_1+s_1r)v_2s_2\hat{x}_2=\hat{x}_2t_1v_2(rt_2+s_2r).$$

That implies that if $rt_1+s_1r\neq 0$, u_2 must contain periodicity, a contradiction to our assumptions. Therefore, $rt_1+s_1r=0$, and both r and x_2 are solutions of the equations $s_1x=xt_1$.

Suppose that the length of the equation is bigger. Then x_2 is a long solution, and $x_2=\hat{x}_2+r$, where $\hat{x}_2$ is a solution of the equation $s_1x=xt_1$, and $\deg(r)\leq 2+k^{\deg(s)+2}$. In that case we get the equality

$$(\hat{x}_2+r)v_2(\hat{x}_2+r)v_3\cdots v_{n-1}(\hat{x}_2+r)t_{n-1}=s_1(\hat{x}_2+r)u_2(\hat{x}_2+r)u_3\cdots u_{n-1}(\hat{x}_2+r),$$

and since $\hat{x}_2$ is a solution of the equation $s_1x=xt_1$, we get the equality

$$\begin{aligned} &rv_2\hat{x}_2v_3\cdots v_{n-1}\hat{x}_2t_{n-1}+\cdots+\hat{x}_2v_2\hat{x}_2v_3\cdots\hat{x}_2v_{n-1}rt_{n-1}\\ &\qquad=s_1ru_2\hat{x}_2u_3\cdots u_{n-1}\hat{x}_2+\cdots+s_1\hat{x}_2u_2\hat{x}_2u_3\cdots\hat{x}_2u_{n-1}r \mod G^{m_2},\end{aligned}$$

where $m_1=\deg(s_1\hat{x}_2u_2\hat{x}_2u_3\cdots\hat{x}_2u_{n-1}r)$, and $m_2=m_1-\deg(\hat{x}_2)+\deg(r)$.

That implies the equality

$$\begin{aligned} &(s_1r+rt_1)u_2\hat{x}_2u_3\cdots u_{n-1}\hat{x}_2+\hat{x}_2v_2(s_2r+rt_2)u_3\hat{x}_2u_4\cdots\hat{x}_2u_n\\ &\qquad+\hat{x}_2v_2\cdots\hat{x}_2v_{n-2}(s_{n-2}r+rt_{n-2})u_{n-1}\hat{x}_2u_n\\ &\qquad\qquad+\hat{x}_2v_2\cdots\hat{x}_2v_{n-1}(s_{n-1}r+rt_{n-1})=0 \mod G^{m_2}.\end{aligned}$$

Suppose that there exists an index j, $1\leq j\leq n-1$, for which $s_jr+rt_j\neq 0$. We set j_0 to be the minimal index for which s_jr+rt_j has maximal degree. We look at the top degree homogeneous part in $s_{j_0}r+rt_{j_0}$. The monomials in this homogeneous part of $s_{j_0}r+rt_{j_0}$ contribute to top degree monomials in the j_0-th product in the sum above. These top degree monomials cancel with top degree monomials from other summands that contain part of the top monomial of $\hat{x}_2$ in place of the top monomial of $s_{j_0}r+rt_{j_0}$. Hence, the top degree homogeneous part of $s_{j_0}r+rt_{j_0}$ has to be a monomial as well.

Furthermore, as for an equation of length 3, this cancellation of the top monomials implies that the top monomials of u_{j_0} and v_{j_0} contain parts of the top monomial of $\hat{x}_2$, that by our assumption is bigger than the length of the period of the top monomial of $\hat{x}_2$. Hence, as for equation of length 3, when we substitute $\hat{x}_2$ in the equation, the top monomial has periodicity that is not contained in one of the appearances of $\hat{x}_2$, a contradiction to one of the assumptions of Theorem 4.7. Therefore, for every j, $s_j r + r t_j = 0$, so r is a solution of the equation $s_1 x = x t_1$, and so is x_2.

This concludes the proof of Theorem 4.7 in case all the appearances of the top monomial of a solution x_1 in the two monomials that are the top products in the two sides of the given equation have nontrivial shifts. We still need to complete the proof in the cases in which there are appearances of the top monomial of a solution x_1 with zero shifts.

Lemma 4.10. *Let $u_1, u_2, v_1, v_2 \in FA$ satisfy $u_1 \neq v_1$, $\deg(u_i) = \deg(v_i)$, $i = 1, 2$, and suppose that the top homogeneous parts of u_i and v_i are monomials (for $i = 1, 2$) with no nontrivial periodicity. Then, if there exists a solution x_1 to the equation $u_1 x u_2 = v_1 x v_2$, and $\deg(x_1) > 2(\deg(u_1) + \deg(u_2))$, then there exist elements $s, t \in FA$ such that x is a solution of the equation $u_1 x u_2 = v_1 x v_2$ if and only if it is a solution of the equation $sx = xt$.*

Proof. The top monomials of u_1 and v_1, and of u_2 and v_2, have to be equal. We set $u_1 = v_1 + \mu_1$, $v_2 = u_2 + \mu_2$, $\deg(\mu_1) < \deg(v_1)$ and $\deg(\mu_2) < \deg(u_2)$. Hence, $(v_1 + \mu_1) x u_2 = v_1 x (u_2 + \mu_2)$, which implies $\mu_1 x u_2 = v_1 x \mu_2$. Since the top homogeneous parts of v_1 and u_2 are monomials with no periodicity, so are the top homogeneous parts of μ_1 and μ_2. Since $\deg(\mu_1) < \deg(v_1)$ and $\deg(\mu_2) < \deg(v_2)$, the conclusion of the lemma follows from Theorem 4.4. ☐

Proposition 4.11. *Let $u_1, u_2, u_3, v_1, v_2, v_3 \in FA$ satisfy $u_1 \neq v_1$, $u_3 \neq v_3$, $\deg(u_i) = \deg(v_i)$, $i = 1, 2, 3$, and suppose that the top homogeneous parts of u_i and v_i are monomials (for $i = 1, 2, 3$) with no nontrivial periodicity. Then, if there exists a solution x_1 to the equation $u_1 x u_2 x u_3 = v_1 x v_2 x v_3$, and the only nontrivial periodicity in the top monomials of the two sides of the equation is contained in the top monomials of the solution x_1, and $\deg(x_1) > 2(\deg(u_1) + \deg(u_2) + \deg(u_3))$, then there exist elements $s, t \in FA$ such that up to a swap between the u's and the v's:*

1. *There exists μ_1 for which $u_1 = \mu_1(s+1)$ and $v_1 = \mu_1 s$.*
2. *There exists μ_2 and τ_2 for which $t\mu_2 = \tau_2 s$. Furthermore, $u_2 = \tau_2(s+1)$ and $v_2 = (t+1)\mu_2$.*
3. *There exists μ_3 for which $u_3 = t\mu_3$ and $v_3 = (t+1)\mu_3$.*

As in the conclusion of Theorem 4.7, every solution of the equation $sx = xt$ is a solution of the given equation $u_1 x u_2 x u_3 = v_1 x v_2 x v_3$. Every solution x_2 of the given equation $u_1 x u_2 x u_3 = v_1 x v_2 x v_3$ for which $\deg(x_2) > 2(2 + k^{\deg(s_1)+2} + \deg(u_1) + \deg(u_2) + \deg(u_3))$ is a solution of the equation $sx = xt$.

Proof. The top homogeneous parts of the u_i and v_i are monomials, and the equation forces these monomials to be equal. Hence, $v_i = u_i + \rho_i$, where $\deg(\rho_i) < \deg(u_i)$, $i = 1, 2, 3$. Let ρ_i^0 be the top homogeneous part in ρ_i. We start the proof by arguing that the top homogeneous part of a solution x_1 with $\deg(x_1) > 2(\deg(u_1) + \deg(u_2) + \deg(u_3))$ has to be a monomial as well.

Suppose that $\deg(\rho_1) < \max(\deg(\rho_2), \deg(\rho_3))$. In that case the top homogeneous parts have to satisfy $\rho_2^0 x_1^0 v_3^0 = u_2^0 x_1^0 \rho_3^0$. Since u_2^0 and v_3^0 are monomials, it follows that x_1^0 is a monomial, and so are ρ_2^0 and ρ_3^0. If $\deg(\rho_1^0) \geq \max(\deg(\rho_2^0), \deg(\rho_3^0))$, then ρ_1^0 has to be a monomial. This forces x_1^0 to be a monomial as well.

We look at the highest degree for which for some index i, $u_i \neq v_i$. This cannot occur for a single index i. If $u_2 = v_2$ at that highest degree, then the top monomial in u_2 (and v_2) must have periodicity, a contradiction to our assumptions. Let d be the codegree of that degree, and suppose that up to this codegree $u_3 = v_3$. In that case, the equation for codegree d products reduces to the equation $u_1 x u_2 = v_1 x v_2$. If we set $u_i = v_i + \mu_i$, $i = 1, 2, 3$, then for the codegree d products, we get the equation $\mu_1 x u_2 = v_1 x \mu_2$. This implies that the top part of μ_1 and μ_2 are monomials that are the codegree d prefix and suffix of the top monomials of v_1 and v_2 in correspondence, and that the top monomial of x_1 has a period of length d.

In that case, it must be that $u_3 = v_3$ for all the homogeneous parts of codegree less than $2d$, and hence, $\mu_1 x u_2 = v_1 x \mu_2$ for all the products up to codegree d. Therefore, there exists an element s, and an element t, such that $v_1 = u_1 = \mu_1 s \mod G^{\deg(u_1)-d}$ and $v_2 = u_2 = t\mu_2 \mod G^{\deg(u_2)-d}$.

Since $u_3 = v_3$ for all the homogeneous parts of codegree less than $2d$, and the top monomial of u_3 (and v_3) do not have nontrivial periodicity, it follows that $u_3 = v_3$. Hence, $\mu_1 x u_2 = v_1 x \mu_2$, and the conclusion follows from Theorem 4.4 in this case (note that in the statement of the proposition we assumed that $u_i \neq v_i$, $i = 1, 3$).

Suppose that for the codegree d homogeneous parts $u_i \neq v_i$ for $i = 1, 2, 3$. In that case, we get the equation

$$(v_1 + \mu_1) x u_2 x u_3 = v_1 x v_2 x (u_3 + \mu_3),$$

and $u_i = v_i$, $i = 1, 2, 3$, for all the homogeneous parts of codegree smaller than d. Hence, the top homogeneous parts of μ_1 and μ_3 are monomials, which are the codegree d prefix and suffix of the top monomials of u_1 and u_3 in correspondence. The top monomial of x_1 (the given solution to the given equation) has to be quasiperiodic (or rather fractional periodic), with a period of length d. Furthermore, $v_2 = b_2 + \mu_2$ and $u_2 = b_2 + \tau_2 \mod G^{\deg(u_2)-(d+1)}$, where the top homogeneous parts of μ_2 and τ_2 are the codegree d prefix and suffix of the top monomial of u_2 (and v_2).

We continue by looking at products of codegree $d + 1$. Every such product that contains monomials in u_i that appear also in v_i, for $i = 1, 2, 3$, cancels with a similar product from the other side of the equation. Hence, to analyze cancellations, we need to consider codegree $d+1$ products that contain monomials from μ_1 or μ_3, or monomials of codegree d and $d+1$ of u_2 and v_2 that do not appear in both.

Suppose that a codegree $d+1$ product contains a codegree $d+1$ monomial from μ_1, i.e., a codegree $d+1$ monomial in u_1 that is not in v_1. Such a codegree $d+1$ product must contain the top monomial of x_1 in its two appearances, and the top monomial of u_2 and u_3. Since the top monomial of v_1 doesn't have nontrivial periodicity, such a codegree $d+1$ product cannot cancel with a codegree $d+1$ product that contains the top monomial of v_1 (since otherwise the suffix of the top monomial of v_1 equals a shift by 1 of itself, which implies that the suffix of v_1 contains periodicity). Therefore, a codegree $d+1$ product that cancels with it must contain a codegree 1 monomial of either u_1 or v_1, or the top monomial of μ_1. Since the top monomial of v_2 contains no periodicity, if this codegree $d+1$ product contains a codegree 1 monomial of u_1 or v_1 it must contain the top monomial of μ_2. Hence, this codegree $d+1$ product has to be from the v_i side of the equation, and the codegree $d+1$ monomial of μ_1 is the codegree d prefix of a codegree 1 monomial in v_1, times the (prefix) period of the top monomial of x_1, which is the degree d suffix of v_1. If such a codegree $d+1$ product cancels with a codegree $d+1$ product that contains the top monomial of μ_1, then it must contain a codegree 1 monomial of x_1.

By the techniques that we used in the proofs of Theorem 4.4 and in the first part of Theorem 4.7, there exists an element s_1, $\deg(s_1)=d$, with a top monomial μ_1, such that $\mu_1 s_1 = u_1 = v_1 \mod G^{\deg(u_1)-2}$.

Suppose that a codegree $d+1$ product contains a codegree $d+1$ monomial of u_2 or v_2. Since the top monomial of u_2 (and v_2) contains no periodicity, such a product can cancel only with either

(1) a codegree $d+1$ product that contains a codegree 1 monomial of u_2 or v_2 and the top monomial of either μ_1 or μ_3;

(2) a codegree $d+1$ product that contains the top monomial of μ_2, and a codegree 1 monomial in the second appearance of x_1, and the top monomial of μ_1, the top monomial of u_2, and the same codegree 1 monomial in the second appearance of x_1;

(3) a codegree $d+1$ product that contains the top monomial of τ_2, and a codegree 1 monomial in the first appearance of x_1, and the top monomial of μ_3, the top monomial of v_2, and the same codegree 1 monomial in the first appearance of x_1.

Note that the two products that appear in possibilities (2) and (3) cancel each other. Hence, a codegree $d+1$ product that contains a codegree $d+1$ product that appears in u_2 or v_2, but not both, must cancel with a unique codegree $d+1$ product that is described in (1).

Suppose that a codegree $d+1$ product contains the top monomial of μ_1 and a codegree 1 monomial of u_2. Since the top monomial of u_2 (and v_2) has no periodicity, it can cancel only with a codegree $d+1$ product that contains either

(1) a codegree 1 monomial of v_2 and the top monomial of μ_3;

(2) a codegree $d+1$ monomial of u_2 or v_2;

(3) a codegree 1 monomial of the first appearance of x_1 and the top monomial of μ_2.

Similarly, suppose that a codegree $d+1$ product contains the top monomial of μ_3 and a codegree 1 monomial of v_2. It can cancel only with a codegree $d+1$ product that contains either

(1) a codegree 1 monomial of u_2 and the top monomial of μ_1;

(2) a codegree $d+1$ monomial of u_2 or v_2;

(3) a codegree 1 monomial of the second appearance of x_1 and the top monomial of τ_2.

Furthermore, a codegree $d+1$ product that contains the top monomial of μ_2 cannot cancel with a codegree $d+1$ product that contains the top monomial of τ_2.

Hence, we can look at the collection of codegree $d+1$ products that contain the top monomial of μ_1 and the entire collection of codegree 1 monomials of u_2. Each such product cancels with precisely one product that contains either a codegree $d+1$ monomial of u_2 or v_2, or a codegree 1 monomial of the first appearance of x_1 and the top monomial of μ_2, or a codegree 1 monomial of v_2 and the top monomial of μ_3. A similar statement holds for codegree $d+1$ products that contain a codegree 1 monomial of v_2 and the top monomial of μ_3.

Therefore, there exist elements $t_1, s_2, b, w_1, w_2, \tau_2, \mu_2$ such that

(1) $t_1\mu_2 = v_2$ and $\tau_2 s_2 = v_2 \bmod G^{\deg(u_2)-2}$, $\deg(s_2) = \deg(t_1) = d$.

(2) $b+\tau_2 = u_2$ and $b+\mu_2 = v_2 \bmod G^{\deg(u_2)-(d+2)}$.

(3) $x_1 = s_1w_1 = w_1t_1 = s_2w_2 = w_2t_2 \bmod G^{\deg(x_1)-2}$.

We continue by induction for $1 \le r \le d$, and assume that for $r < d$ there exist elements $t_1, s_2, b, w_1, w_2, \tau_2, \mu_2$ such that the equalities that were true for the top 2 homogeneous parts and codegree d and codegree $d+1$ monomials hold for the top r monomials, and for codegree $d+r-1$ monomials,

(1) $t_1\mu_2 = v_2$ and $\tau_2 s_2 = v_2 \bmod G^{\deg(u_2)-r}$, $\deg(s_2) = \deg(t_1) = d$.

(2) $b+\tau_2 = u_2$ and $b+\mu_2 = v_2 \bmod G^{\deg(u_2)-(d+r)}$.

(3) $x_1 = s_1w_1 = w_1t_1 = s_2w_2 = w_2t_2 \bmod G^{\deg(x_1)-r}$.

We continue by studying codegree $d+r$ products. All such products that involve only monomials of codegree less than d of the u_i, v_i, $1 \le i \le 3$, cancel in pairs. All such products that involve only monomials of codegree less than $d+r$ of the u_i, v_i, $1 \le i \le 3$, and codegree less than r of x_1 (in its two appearances from both sides of the equation) cancel in pairs by the induction hypothesis.

Hence, to analyze the structure of u_1 and v_1 (and hence, of μ_1 and s_1) we only need to consider codegree $d+r$ products that contain

(i) a codegree $d+r$ monomial of u_1 that does not appear in v_1 and vice versa;

(ii) a codegree r monomial of v_1 and the top monomial of μ_2;

(iii) a codegree $d+q_j$ monomial of μ_1, $q_j < r$, and a codegree $r-q_j$ monomial of the first appearance of x_1;

(iv) a codegree p_j monomial of v_1, $p_j < r$, and a codegree $r-p_j$ monomial of the first appearance of x_1 and the top monomial of μ_2.

A product of type (iv) that cancels with products of type (i) or (ii) must cancel with a corresponding product of type (iii) by our induction hypothesis. A product of type (iii) that cancels with a product of type (i) or (ii) and in which q_j is positive, and the codegree $r-q_j$ monomial of the first appearance of x_1 is obtained as a product of a codegree $r-m_j$ monomial of s_1 with a codegree m_j-q_j monomial of w_1, for $q_j < m_j < r$, cancels with a product of type (iv).

Therefore, to analyze the structure of u_1, v_1, s_1 and w_1, we consider only those codegree $d+r$ products that can be presented either in form (i) or (ii), that we denote (1) and (2) in the sequel, or in the form

(3) a product of the top monomial of μ_1, and a codegree r monomial of the first appearance of x_1.

A codegree $d+r$ product that can be presented in one of the forms (1)–(3) can cancel with either

(4) an odd number of products of a codegree $d+q_j$ monomial of μ_1, $0 < q_j < r$, and a codegree $r-q_j$ monomial of the first appearance of x_1;

(5) an odd number of products of a codegree $d+q_j$ monomial of μ_1, $0 < q_j < r$, and a product of a codegree $r-q_j$ monomial of s_1 with the top monomial of w_1;

(6) an odd number of products of a codegree $d+q_j$ monomial of μ_1, $q_j < r$, and a product of a codegree $r-m_j$ monomial of s_1 with a codegree m_j-q_j monomial of w_1, where $q_j < m_j < r$;

(7) an odd number of products of a codegree p_j monomial of v_1, $0 < p_j < r$, and a codegree $r-p_j$ monomial of the first appearance of x_1 and the top monomial of μ_2;

(8) a product of the top monomial of v_1, a codegree r monomial of the first appearance of x_1 and the top monomial of μ_2;

(9) an odd number of products of a codegree p_j monomial of v_1, $0 < p_j < r$, and a codegree m_j monomial of the first appearance of x_1, $0 < m_j$, $p_j + m_j < r$ and a codegree $d + r - p_j - m_j$ monomial of μ_2;

(10) an odd number of products of a codegree $d + q_j$ monomial of μ_1, a codegree m_j monomial of the first appearance of x_1, $0 < m_j$, $q_j + m_j < r$, and a codegree $r - m_j - q_j$ monomial of u_2.

If (1) or (2) occur, (8) cannot occur, and (6) occurs if and only if (7) occurs as well. If (1) occurs, (3) cannot occur. Suppose that (1) occurs. If in addition only (2) occurs, we add a codegree $d + r$ monomial to μ_1. If in addition to (1) only (4) and (5) occur, we also add a codegree $d + r$ monomial to μ_1. If in addition to (1) only (5), (6) and (7) occur, we add a codegree $d + r$ monomial to μ_1. If (1) occurs, (9) and (10) cannot occur.

Suppose that (2) occurs. If in addition only (3) occurs (and in addition possibly (4), (6) and (7)) we add a codegree r monomial to s_1. If in addition to (2) only (4) and (5) occur, we do not add anything. If in addition to (2) only (5), (6) and (7) occur, we do the same. If (2) occurs, (8)–(10) cannot occur.

Suppose that (3) occurs. The codegree r monomial of x_1 cannot be presented both as a product of the top monomial of s_1 with a codegree r monomial of w_1, and as a codegree r monomial of s_1 with the top monomial of w_1. We look at all the possible ways to present the codegree r monomial of x_1 as a product of a codegree q_j monomial of s_1 with a codegree $r - q_j$ monomial of w_1, for $0 < q_j < r$. If the number of such products is odd we don't add anything. If the number is even, we either add a codegree r monomial to s_1 or a codegree r monomial to w_1 (but not both). The validity of this addition of a codegree r monomial to either s_1 or w_1 can be verified by going over the possible cancellation of the given codegree $d + r$ product with all the other possible forms of such a product.

This concludes the adaptation of s_1, μ_1 and w_1 to include codegree r monomials. The same adaptation works for t_2, μ_3 and w_2. It is still left to analyze u_2 and v_2 in order to add codegree r monomials to μ_2 and τ_2 such that the equalities that by induction hold for the top codegree $r - 1$ parts of these elements will hold for the top codegree r part.

To analyze the structure of u_2 and v_2 (and hence, of μ_2, τ_2, t_1 and s_2) we start by observing the following:

(i) The codegree $d + r$ products that contain either a positive codegree monomial of u_1 or a positive codegree monomial of the first appearance of x_1, a codegree $d + q_j$ monomial of τ_2, $q_j < r$, a monomial of the second appearance of x_1, and a monomial of u_3, cancel with codegree $d + r$ products that contain either a positive codegree monomial of v_1 or a positive codegree monomial of the first appearance of x_1, a codegree p_j monomial of v_2, $p_j < r$, a monomial of the second appearance of x_1, and a monomial of μ_3.

(ii) The codegree $d+r$ products that contain a monomial of v_1, a monomial of the first appearance of x_1, a codegree $d+q_j$ monomial of μ_2, $q_j < r$, and either a positive codegree monomial of the second appearance of x_1, or a positive codegree monomial of v_3, cancel with codegree $d+r$ products that contain a monomial of μ_1, a monomial of the first appearance of x_1, a codegree p_j monomial of u_2, $p_j < r$, either a positive codegree monomial of the second appearance of x_1, or a positive codegree monomial of u_3.

Hence, to analyze the structure of u_2 and v_2 we only need to consider codegree $d+r$ products that contain

(i) a codegree $d+r$ monomial of u_2 or of v_2;

(ii) the top monomial of μ_1 and a codegree r monomial of u_2 or a codegree r monomial of v_2 and the top monomial of μ_3;

(iii) a codegree $d+q_j$ monomial of τ_2, $q_j < r$, and a codegree $r-q_j$ monomial of the second appearance of x_1 or a codegree $r-q_j$ of the first appearance of x_1 and a codegree $d+q_j$ monomial of μ_2, $q_j < r$;

(iv) the top monomial of μ_1, a codegree $r-p_j$ monomial of the first appearance of x_1 and a codegree p_j monomial of u_2, $p_j < r$, or a codegree p_j monomial of v_2, $p_j < r$ and a codegree $r-p_j$ monomial of the second appearance of x_1 and the top monomial of μ_3.

If there are two products of codegree $d+r$ of type (i), they cancel each other, and we can ignore them in analyzing codegree $d+r$ products. Therefore, to analyze the structure of u_2, v_2, s_2, t_1, μ_2 and τ_2, we consider only those codegree $d+r$ products that can be presented either in form (i) or (ii), that we denote (1) and (2) in the sequel, or in codegree $d+r$ products in the form

(3) a product of the top monomial of τ_2, and a codegree r monomial of the second appearance of x_1;

(4) a codegree r monomial of the first appearance of x_1, and the top monomial of μ_2;

(5) an odd number of products of a codegree $d+q_j$ monomial of τ_2, $0 < q_j < r$, and a codegree $r-q_j$ monomial of the second appearance of x_1;

(6) an odd number of products of a codegree $d+q_j$ monomial of τ_2, $0 < q_j < r$, and a product of a codegree $r-q_j$ monomial of s_2 with the top monomial of w_2;

(7) an odd number of products of a codegree $d+q_j$ monomial of τ_2, $q_j<r$, and a product of a codegree $r-m_j$ monomial of s_2 with a codegree m_j-q_j monomial of w_2, where $q_j<m_j<r$;

(8) an odd number of products of a codegree p_j monomial of v_2, $0<p_j<r$, and a codegree $r-p_j$ monomial of the second appearance of x_1 and the top monomial of μ_3;

(9) a product of the top monomial of v_2, a codegree r monomial of the second appearance of x_1 and the top monomial of μ_3.

And similarly, from the other sides of the equation,

(10) an odd number of products of a codegree $r-q_j$ monomial of the first appearance of x_1, and a codegree $d+q_j$ monomial of μ_2, $0<q_j<r$;

(11) an odd number of products of the top monomial of w_1, a codegree $r-q_j$ monomial of t_1, and a codegree $d+q_j$ monomial of μ_2, $0<q_j<r$;

(12) an odd number of products of a codegree m_j-q_j monomial of w_1, a codegree $r-m_j$ monomial of t_1, a codegree $d+q_j$ monomial of μ_2, $q_j<r$, $q_j<m_j<r$;

(13) an odd number of products of the top monomial of μ_1, a codegree $r-p_j$ monomial of the first appearance of x_1, and a codegree p_j monomial of u_2, $0<p_j<r$;

(14) a product of the top monomial of μ_1, a codegree r monomial of the first appearance of x_1, and the top monomial of u_2.

Suppose that (1) occurs. If only one of the possibilities in (2) occurs, we add a codegree $d+r$ monomial to μ_2 or τ_2, depending which of the two possibilities in (2) occurs. If (1) occurs, (3) and (4) cannot occur. If in addition to (1) only (5) occurs, then (6) or (7) must occur and not both. If only (5) and (6) occur, we add a codegree $d+r$ monomial to τ_2. If in addition to (1), (5) and (7) occur, then (8) must occur as well, and hence at least an additional possibility must occur. If in addition to (1), (8) occurs, then (5) and (7) must occur as well, so an additional possibility must occur. If (1) occurs, (9) cannot occur. The possibilities (10)–(14) are parallel to (5)–(9) and are dealt with accordingly.

Suppose that (1) and the two possibilities in (2) occur. If in addition only (5) and (6) occur, we add a codegree $d+r$ monomial only to μ_2, and if only (10) and (11) occur, we add a codegree $d+r$ monomial to τ_2. Suppose that (1) and only one of the products in the form (2) occur, without loss of generality the product from the v_i side, i.e., the one that contains μ_3. If in addition (5), (6), (10) and (11) occur, we add a codegree $d+r$ monomial to μ_2.

Suppose that one of the possibilities in (2) occurs, without loss of generality the one from the v_i side. If the only additional product that cancels with it is also a product in form (2) from the u_i side of the equation, we add a codegree $d+r$ monomial to both τ_2 and μ_2. If in addition to the form (2) only possibility (3) occurs, we add a codegree r monomial to s_2. Form (4) cannot occur. If only (5) and (6) occur, we do not add anything. If (5) and (7) occur, (8) must occur as well. Form (9) cannot occur. If in addition (10) and (11) occur, we add a codegree $d+r$ monomial to both τ_2 and μ_2. If only (3), (5), (6), (10) and (11) occur, we add a codegree r monomial to s_2, and a codegree $d+r$ monomial to both μ_2 and τ_2.

Suppose that the two possibilities in part (2) occur. In that case (3) cannot occur. If in addition (5), (6), (11) and (12) occur, we do not add anything. Suppose that (3) occurs. In that case (4) cannot occur. If in addition only (5) and (6) occur, we add a codegree r monomial to s_2. If in addition to (3) only (9) occurs, we add a codegree r monomial to w_2. If (3) occurs, then (10)–(14) cannot occur. If (4) occurs the analysis is analogous to the case in which (3) occurs.

Suppose that (5) and (6) occur. In that case (9) cannot occur. If (10) and (11) occur as well, we add a codegree $d+r$ monomial to both τ_2 and μ_2.

This concludes our treatment of codegree $d+r$ products for $r<d$. So far we proved that

(1) $\mu_1 s_1 = u_1 = v_1 \mod G^{\deg(u_1)-d}$, $\deg(s_1)=d$, $u_1 = v_1+\mu_1 \mod G^{\deg(u_1)-2d}$.

(2) $t_1\mu_2 = v_2$ and $\tau_2 s_2 = v_2 \mod G^{\deg(u_2)-d}$, $\deg(s_2)=\deg(t_1)=d$, $\deg(\mu_2)=\deg(\tau_2)=\deg(u_2)-d$.

(3) $b_2+\tau_2 = u_2$ and $b_2+\mu_2 = v_2 \mod G^{\deg(u_2)-(2d)}$.

(4) $x_1 = s_1 w_1 = w_1 t_1 = s_2 w_2 = w_2 t_2 \mod G^{\deg(x_1)-d}$.

We continue by analyzing codegree $2d$ products. The analysis of codegree $2d$ products is similar to the analysis of codegree $d+r$ products for $r<d$. In their analysis we use the following observations:

(i) All the codegree $2d$ products that contain monomials of codegree smaller than d from the elements u_i, v_i and x in its two appearances cancel in pairs.

(ii) All the codegree $2d$ products that contain a monomial of codegree bigger than d, from b_1, b_2 or b_3, cancel in pairs.

Hence, we need to analyze only those codegree $2d$ products that contain monomials from either $\mu_1, \mu_2, \tau_2, \mu_3$, or monomials of codegree d from b_1, b_2, b_3. To analyze the elements $u_1, v_1, b_1, \mu_1, s_1$ and w_1, we need to analyze codegree $2d$ products that contain one of the following:

(i) a codegree $2d$ monomial of u_1 that does not appear in v_1 and vice versa;

(ii) a codegree d monomial of v_1 and the top monomial of μ_2;

(iii) a codegree $d+q_j$ monomial of μ_1, $q_j<d$, and a codegree $d-q_j$ monomial of the first appearance of x_1;

(iv) a codegree p_j monomial of v_1, $p_j < d$, and a codegree $d - p_j$ monomial of the first appearance of x_1 and the top monomial of μ_2.

(v) Note that the codegree $2d$ product that contains the top monomials of μ_1 and τ_2 cancels with the product that contains the top monomials of μ_2 and μ_3. Also the codegree $2d$ products that contain a codegree d monomial of u_1 which is from b_1 (i.e., also a monomial of v_1), and the top monomial of τ_2, cancel with the products that contain the same codegree d monomial from v_1, and the top monomial of μ_3.

Because of (v), the analysis of codegree $2d$ monomials of u_1 and v_1 is identical to the analysis of codegree $d+r$ monomials of these elements. This concludes the construction of the element s_1, and adds codegree $2d$ monomials to μ_1, and codegree d monomials to w_1. The analysis of the elements u_3, v_3, b_3, μ_3 and w_2 is identical.

We continue by analyzing the codegree $2d$ monomials in u_2, v_2, τ_2 and μ_2. The observations (i) and (ii) that we used in analyzing the codegree $d+r$ monomials of these elements for $r < d$ remain valid for codegree $2d$ monomials. In addition by part (v) in the analysis of codegree $2d$ monomials of u_1 and v_1, it follows that the codegree $2d$ product that contains the top monomials of μ_1 and τ_2 cancels with the product that contains the top monomials of μ_2 and μ_3. Hence, the rest of the analysis of codegree $2d$ monomials of u_2 and v_2 is identical to the analysis of codegree $d+r$ monomials of these elements for $r < d$.

We continue by analyzing higher codegree products and monomials. We assume inductively for $r > 0$ that

(1) $\mu_1(s_1+1) = u_1$ and $\mu_1 s_1 = v_1 \mod G^{\deg(u_1)-(d+r)}$, $\deg(s_1) = d$, $u_1 = v_1 + \mu_1 \mod G^{\deg(u_1)-(2d+r)}$.

(2) $(t_1+1)\mu_2 = v_2$, $\tau_2(s_2+1) = u_2$ and $t_1\mu_2 = \tau_2 s_2 \mod G^{\deg(u_2)-(d+r)}$, $\deg(s_2) = \deg(t_1) = d$, $\deg(\mu_2) = \deg(\tau_2) = \deg(u_2) - d$.

(3) $b_2 + \tau_2 = u_2$ and $b_2 + \mu_2 = v_2 \mod G^{\deg(u_2)-(2d+r)}$.

(4) $x_1 = s_1 w_1 = w_1 t_1 = s_2 w_2 = w_2 t_2 \mod G^{\deg(x_1)-(d+r)}$.

And we continue by analyzing codegree $2d+r$ products. The analysis is similar to the analysis of codegree $d+r$ and codegree $2d$ products. We use the following observations:

(i) All the codegree $2d+r$ products that contain monomials of u_i, v_i, $i = 1, 2, 3$, that are all of codegree smaller than d cancel in pairs. In particular, all the codegree $2d+r$ products that contain a monomial of x of codegree bigger than $d+r$, in one of its two appearances, cancel in pairs.

(ii) All the codegree $2d+r$ products that contain monomials from all b_1, b_2 and b_3 cancel in pairs.

(iii) A codegree $2d+r$ product that contains a monomial from μ_1 of codegree more than d, and a monomial from the first appearance of x, such that the sum of their codegrees is less than $2d+r$, an element from b_2 and an element from b_3, cancels with a product that contains an element from b_1, an element from the first appearance of x, an element from μ_2 and the same element from b_3. The same holds for products that contain monomials from b_1, b_2 and μ_3 with parallel restrictions.

(iv) A codegree $2d+r$ product that contains a monomial from b_1, a monomial from τ_2 of codegree bigger than d, and a monomial from b_3, such that the sum of the codegrees of the monomial from τ_2 and the monomial from the second appearance of x is smaller than $2d+r$, cancels with a product that contains the same monomials of b_1 and the first appearance of x, a monomial from b_2 and a monomial from μ_3. The same holds for products that contain monomials from b_1, μ_2 and b_3 with parallel restrictions.

(v) A codegree $2d+r$ product that contains a monomial from μ_1 of codegree bigger than d, a monomial from τ_2, and a monomial from b_3, cancels with a product that contains a monomial from b_1, a monomial from μ_2, and a monomial from μ_3. The same holds for products that contain monomials from b_1, μ_2 and μ_3 with parallel restrictions.

Hence, like in the analysis of codegree $2d$ products, to analyze the elements u_1, v_1, b_1, μ_1 and w_1, we need to analyze codegree $2d+r$ products that contain one of the following:

(i) a codegree $2d+r$ monomial of u_1 that does not appear in v_1 and vice versa;

(ii) a codegree $d+r$ monomial of v_1 (which is a monomial of b_1) and the top monomial of μ_2;

(iii) a codegree $d+q_j$ monomial of μ_1, $q_j < d+r$, and a codegree $d+r-q_j$ monomial of the first appearance of x_1;

(iv) a codegree p_j monomial of v_1 (which is a monomial of b_1), $p_j < d+r$, and a codegree $d+r-p_j$ monomial of the first appearance of x_1 and the top monomial of μ_2.

Hence, the analysis of codegree $2d+r$ monomials of u_1 and v_1 is identical to the analysis of codegree $d+r$ and $2d$ monomials of these elements. Note that in analyzing products of codegree greater than $2d+r$, the element s_1 is already fixed, and we only add codegree $2d+r$ monomials to μ_1 and b_1, and codegree $d+r$ monomials to w_1. The analysis of the elements u_3, v_3, b_3, μ_3 and w_2 is identical.

We continue by analyzing the codegree $2d$ monomials in u_2, v_2, τ_2 and μ_2. The observations (i)–(v) that we used in analyzing the codegree $2d+r$ monomials of b_1 and μ_1 imply that analyzing codegree $2d+r$ monomials of b_2, τ_2 and μ_2 is similar to the analysis of the codegree $d+r$ monomials of these elements. Hence, we can finally deduce that

(1) $\mu_1(s_1+1)=u_1$ and $\mu_1 s_1=v_1$, $\deg(s_1)=d$ and $u_1=v_1+\mu_1$.
(2) $(t_1+1)\mu_2=v_2$, $\tau_2(s_2+1)=u_2$ and $t_1\mu_2=\tau_2 s_2$, $\deg(s_2)=\deg(t_1)=d$ and $\deg(\mu_2)=\deg(\tau_2)=\deg(u_2)-d$.
(3) $b_2+\tau_2=u_2$ and $b_2+\mu_2=v_2$.
(4) $x_1=s_1w_1=w_1t_1=s_2w_2=w_2t_2 \mod G^{\deg(x_1)-\deg(u_1u_2u_3)}$.

This proves the structure of the coefficients in the statement of Proposition 4.11. Suppose that there exists a solution x_2 to the given equation, and

$$\deg(x_2)>2(2+k^{\deg(s)+2}+\deg(u_1)+\deg(u_2)+\deg(u_3)).$$

As in the analysis of the same equation in case there are shifts between the appearances of the element x_2, we can continue the analysis of higher codegree monomials of the solution x_2, and get that there exist elements w_i, $i=1,2$, that satisfy $s_iw_i=w_it_i=x_1 \mod G^{\deg(s_1)-1}$. By the argument that was used to prove Lemma 4.2, it follows that there exists a solution $\hat{x}$ to the equations $s_ix=xt_i$, $i=1,2$.

The element x_2 satisfies $s_1x_2=x_2t_1 \mod G^{2\deg(s_1)-1}$. Hence, there exists an element $\hat{x}_2$, which is a solution of the equation $s_1x=xt_1$, and $x_2+\hat{x}_2=r$, where $\deg(r)\le 2+k^{\deg(s_1)+2}$.

Also, x_2 is a solution to the equation $v_1xv_2xv_3=u_1xu_2xu_3$, where $v_1=\tau_1s_1$, $u_1=\tau_1(s_1+1)$, $v_2=(t_1+1)\mu_2$, $u_2=\tau_2(s_2+1)$, $v_3=(t_2+1)\mu_3$, $u_3=t_2\mu_3$, and $\tau_2s_2=t_1\mu_2$. Hence

$$\tau_1(s_1+1)(\hat{x}_2+r)\tau_2(s_2+1)(\hat{x}_2+r)t_2\mu_3=\tau_1s_1(\hat{x}_2+r)(t_1+1)\mu_2(\hat{x}_2+r)(t_2+1)\mu_3.$$

Therefore

$$(s_1+1)r\tau_2(s_2+1)\hat{x}_2t_2+(s_1+1)\hat{x}_2\tau_2(s_2+1)rt_2 \\ =s_1r(t_1+1)\mu_2\hat{x}_2(t_2+1)+s_1\hat{x}_2(t_1+1)\mu_2r(t_2+1) \mod G^{\deg(s_1r\tau_2s_2rt_2)}.$$

Since $s_1\hat{x}_2=\hat{x}_2t_1$, this implies

$$((s_1+1)r\tau_2(s_2+1)s_2+s_1r(t_1+1)\mu_2(s_2+1))\hat{x}_2 \\ =\hat{x}_2((t_1+1)\tau_2(s_2+1)rt_2+t_1(t_1+1)\mu_2r(t_2+1)) \mod G^{\deg(s_1r\tau_2s_2rt_2)}.$$

Therefore,

$$(s_1+1)r\tau_2(s_2+1)s_2+s_1r(t_1+1)\mu_2(s_2+1)=p(s_1),$$
$$(t_1+1)\tau_2(s_2+1)rt_2+t_1(t_1+1)\mu_2r(t_2+1)=p(t_1)$$

for some polynomial p.

This implies that $r\tau_2 s_2 + s_1 r\mu_2$ is a polynomial in s_1, and $\tau_2 r t_2 + t_1\mu_2 r$ is a polynomial in t_1. Hence, $(rt_1 + s_1 r)\mu_2$ is a polynomial in s_1, and $\tau_2(rt_2 + s_2 r)$ is a polynomial in t_1.

Since we assumed that the top monomials of the coefficient do not contain periodicity, it cannot be that the top monomials of τ_2 and μ_2 are equal, and equal to the top monomials of t_1 and s_1. Hence, $rt_1 + s_1 r = rt_2 + s_2 r \neq 1$.

If $\deg(\tau_2) = \deg(s_1)$, then the top monomials of s_1 and t_1 are equal, and the top monomials of u_2 and v_2 have periodicity, a contradiction. The top monomial of τ_2 has no periodicity, so $\deg(\tau_2) < 2\deg(s_1)$. If $\deg(\tau_2) > \deg(s_1)$, then necessarily the top monomials of u_2 and v_2 contain periodicity, a contradiction.

Suppose that $\deg(\tau_2) < \deg(s_1)$. If the top monomial of τ_2 is the same as the top monomial of μ_2, then the top monomials of the two sides of the equation contain periodicity, a contradiction. If the top monomials of μ_2 and τ_2 are distinct, then the top monomials of u_2 and v_2 contain periodicity, a contradiction.

Therefore, $rt_1 + s_1 r = 0$, so r is a solution of the equation $s_1 x = xt_1$ and so is $x_2 = \hat{x}_2 + r$, and the conclusion of Proposition 4.11 follows. □

Proposition 4.11 and its proof enable us to prove Theorem 4.7 in case there are no shifts, i.e., in case the degrees of the elements u_i, v_i satisfy $\deg(u_i) = \deg(v_i)$ for all indices i.

Proposition 4.12. *Let $u_1, \dots, u_n, v_1, \dots, v_n \in FA$, where FA is the free associative algebra over GF_2 that is generated by k elements, and suppose that the equation*

$$u_1 x u_2 x u_3 \cdots u_{n-1} x u_n = v_1 x v_2 x v_3 \cdots v_{n-1} x v_n$$

has a solution x_1 of degree bigger than $2(\deg(u_1)+\cdots+\deg(u_n))^2$. Suppose further that

(1) *For every index i, $1 \le i \le n$, $\deg(u_i) = \deg(v_i)$.*

(2) *The top homogeneous parts of u_i and v_i are monomials with no periodicity.*

(3) *For some index i, $u_i \neq v_i$.*

(4) *All the periodicity in the top monomials that are associated with the top monomials of the two sides of the equation after substituting the solution x_1 is contained in the periodicity of the top monomial of the solution x_1.*

Then there exist some elements $s, t \in FA$, $\deg(s) = \deg(t) < \min\deg(u_i)$, such that

(1) *Every solution of the equation $sx = xt$ is a solution of the given equation.*

(2) *Every solution x_2 of the given equation that satisfies*

$$\deg(x_2) > 2(2 + k^{\deg(s)+2} + \deg(u_1) + \cdots + \deg(u_n))$$

is also a solution of the equation $sx = xt$.

(3) *For every index* i, $1 \le i \le r$, *for which* $u_i \neq v_i$, *there exist elements* τ_i, μ_i *such that the elements* u_i, v_i *are either* $\tau_i(s_i+1)$ *or* $(t_{i-1}+1)\mu_i$ *or* $\tau_i s_i$ *or* $t_{i-1}\mu_i$, *where the elements* s_i *are either* s *or* $s+1$, *and the elements* t_i *are either* t *or* $t+1$, *and* $t_{i-1}\mu_i = \tau_i s_i$.

Proof. The proof of the structure of the coefficients is similar to the proof of Proposition 4.11. Given the structure of the coefficients, it is clear that every solution of the equation $sx = xt$ is a solution of the given equation. It is left to prove that every long enough solution of the given equation is a solution of the equation $sx = xt$.

Suppose that x_2 is a solution of the given equation that satisfies

$$\deg(x_2) > 2(2 + k^{\deg(s)+2} + \deg(u_1) + \cdots + \deg(u_n)).$$

By the argument that we used in Proposition 4.11, it follows that the equation $sx = xt$ has a solution, and that $x_2 = \hat{x}_2 + r$, where $\hat{x}_2$ is a solution to the equation $sx = xt$, and $\deg(r) \le 2 + k^{\deg(s)+2}$.

In that case we get the equality

$$\begin{aligned}\tau_1(s_1+1)(\hat{x}_2+r)\tau_2(s_2+1)(\hat{x}_2+r)\cdots\tau_{n-1}(s_{n-1}+1)(\hat{x}_2+r)t_{n-1}\mu_n \\ = \tau_1 s_1(\hat{x}_2+r)(t_1+1)\mu_2(\hat{x}_2+r)\cdots(t_{n-2}+1)\mu_{n-1}(\hat{x}_2+r)(t_{n-1}+1)\mu_n,\end{aligned}$$

and since $\hat{x}_2$ is a solution of the equation $s_1 x = x t_1$, we get the equality

$$\begin{aligned}&(s_1+1)r\tau_2(s_2+1)\hat{x}_2\cdots\tau_{n-1}(s_{n-1}+1)\hat{x}_2 t_{n-1} \\ &\qquad + \cdots + (s_1+1)\hat{x}_2\tau_2(s_2+1)\hat{x}_2\cdots\tau_{n-1}(s_{n-1}+1)r t_{n-1} \\ &\quad = s_1 r(t_1+1)\mu_2\hat{x}_2\cdots(t_{n-2}+1)\mu_{n-1}\hat{x}_2(t_{n-1}+1) \\ &\qquad + \cdots + s_1\hat{x}_2(t_1+1)\mu_2\hat{x}_2\cdots(t_{n-2}+1)\mu_{n-1}r(t_{n-1}+1) \mod G^{m_2},\end{aligned}$$

where

$$m_1 = \deg((s_1+1)r\tau_2(s_2+1)\hat{x}_2\cdots\tau_{n-1}(s_{n-1}+1)\hat{x}_2 t_{n-1})$$

and $m_2 = m_1 - \deg(\hat{x}_2) + \deg(r)$.

By the same argument that we used in the proof of Proposition 4.11, since the top monomials of the coefficients u_i, v_i, $i = 1, \ldots, n$, do not have periodicity, and since the top monomial in the two sides of the equation after substituting the solution x_2 has no periodicity, except the one that is contained in the appearances of the top monomial of x_2, it follows that for some i, $1 \le i \le n-1$, $s_i r = r t_i$. Hence, r is a solution to the equation, $sx = xt$, and so is $x_2 = \hat{x}_2 + r$, since both $\hat{x}_2$ and r are solutions to this equation. □

At this point we need to consider equations in which some of the appearances of the elements x are shifted, and some are not.

Lemma 4.13. *Let* $u_1, u_2, u_3, v_1, v_2, v_3 \in FA$ *satisfy* $u_1 \neq v_1$, $\deg(u_1) = \deg(v_1)$, $\deg(u_2) > \deg(v_2)$, $\deg(v_3) > \deg(u_3)$, *where FA is the free associative algebra over* GF_2 *that is generated by k elements.*

Suppose that the top homogeneous parts of u_i *and* v_i *are monomials (for* $i = 1, 2, 3$*) with no nontrivial periodicity. If there exists a solution* x_1 *to the equation* $u_1 x u_2 x u_3 = v_1 x v_2 x v_3$, *and the only nontrivial periodicity in the top monomials of the two sides of the equation after substituting* x_1 *is contained in the top monomial of the solution* x_1 *(this translates to a condition on the top monomials of the coefficients), and* $\deg(x_1) > 2(\deg(u_1)+\deg(u_2)+\deg(u_3))^2$, *then there exist elements* $s, t \in FA$, *such that either*

(1) *There exists* μ_1 *for which* $u_1 = \mu_1(s+1)$ *and* $v_1 = \mu_1 s$.

(2) *There exist* μ_2 *and* s_2, t_2 *for which* $(t+1)\mu_2 = v_2$ *and* $t\mu_2 s_2 = u_2$. *Furthermore,* $v_3 = t_2 u_3$ *and the pair* (s_2, t_2) *is either* (s, t) *or* $(s+1, t+1)$.

or

(1) *There exists* μ_1 *for which* $u_1 = \mu_1 s$ *and* $v_1 = \mu_1(s+1)$.

(2) *There exist* μ_2 *and* s_2, t_2 *for which* $(t+1)\mu_2 = u_2$ *and* $v_2 s_2 = t\mu_2$. *Furthermore,* $v_3 = t_2 u_3$ *and the pair* (s_2, t_2) *is either* (s, t) *or* $(s+1, t+1)$.

As in the conclusion of Theorem 4.7, every solution of the equation $sx = xt$ *is a solution of the given equation* $u_1 x u_2 x u_3 = v_1 x v_2 x v_3$. *Every solution* x_2 *of the given equation* $u_1 x u_2 x u_3 = v_1 x v_2 x v_3$ *that satisfies*

$$\deg(x_2) > 2(2 + k^{\deg(s_1)+2} + \deg(u_1) + \deg(u_2) + \deg(u_3))$$

is also a solution of the equation $sx = xt$.

Proof. The proof is similar to the proof of Proposition 4.11. □

At this point we can complete the proof of Theorem 4.7. We already analyzed the case in which there are nontrivial shifts between (the top monomials of) pairs of appearances of the variable x in the two sides of the equation. Propositions 4.11 and 4.12 analyze the case in which there are no shifts between pairs of appearances of the variable x in the two sides of the equation, and Lemma 4.13 analyzes the case $n = 3$ in which there is a pair with no shift and a pair with a shift.

By the techniques that were used in proving Proposition 4.11 and in analyzing the case in which there are nontrivial shifts between pairs of appearances of the variable x, if there is a pair of coefficients, u_i, v_i such that $u_i = v_i$ and the $i-1$ (hence, also the i-th) pair of appearances of the variable x has no shift, then the equation breaks into two equations, the first contains the coefficients $u_1, \dots, u_{i-1}, v_1, \dots, v_{i-1}$, and the second contains the coefficients $u_{i+1}, \dots, u_n$, $v_{i+1}, \dots, v_n$. Therefore, in the sequel we may assume that there is no such pair of coefficients u_i, v_i.

Then there exist some elements $s, t \in FA$, $\deg(s) = \deg(t) < \min \deg(u_i)$, and elements $s_1, \dots, s_{n-1}, t_1, \dots, t_{n-1}$, such that

(1) For every index i, the pair (s_i, t_i) is either (s, t) or $(s+1, t+1)$.

(2) For every pair of coefficients, u_i, v_i for which the two pairs of appearances of the variable x from the two sides of the pair of coefficients have no nontrivial shift, either $u_i = v_i$ or there exist elements τ_i and μ_i such that either $u_i = \tau_i s_i$ and $v_i = \tau_i(s_i + 1)$ (or vice versa), or $u_i = t_{i-1}\mu_i$ and $v_i = (t_{i-1}+1)\mu_i$ (or vice versa), or $u_i = (t_{i-1}+1)\mu_i$ and $v_i = \tau_i(s_i+1)$ (or vice versa).

(3) If $\deg(u_1) = \deg(v_1)$, either $u_1 = v_1$ or there exists τ_1 such that $u_1 = \tau_1 s_1$ and $v_1 = \tau_1(s_1+1)$ (or vice versa). If $\deg(u_n) = \deg(v_n)$, either $u_n = v_n$ or there exists an element μ_n such that $u_n = t_{n-1}\mu_n$ and $v_n = (t_{n-1}+1)\mu_n$ (or vice versa).

(4) For every pair u_i, v_i for which the two pairs of appearances of the variable x from the two sides of the pair of coefficients have nontrivial shifts, $u_i s_i = t_{i-1} v_i$ (or vice versa), or $u_i = t_{i-1} v_i s_i$ (or vice versa).

(5) If $\deg(u_1) \neq \deg(v_1)$, then $u_1 = v_1 s_1$ or vice versa. If $\deg(u_n) \neq \deg(v_n)$, then $u_n = t_{n-1} v_n$ or vice versa.

(6) Suppose that $\deg(u_i) \neq \deg(v_i)$, $1 < i < n$, there is no shift between the $i-1$ appearances of the variable x, and there is a nontrivial shift between the i-th appearances of the variable x from the two sides of the equation. Then either $u_i s_i = v_i$ or vice versa, in which case the original equation can be broken into two equations, the first contains the first $i-1$ pairs of coefficients, and the second contains the last $n+1-i$ pairs of coefficients, or $v_i = (t_{i-1}+1)\mu_i$ and $u_i s_i = t_{i-1}\mu_i$ (or vice versa), or $u_i = (t_{i-1}+1)\mu_i$ and $v_i = t_{i-1}\mu_i s_i$ (or vice versa).

(7) Suppose that $\deg(u_i) \neq \deg(v_i)$, $1 < i < n$, there is no shift between the i-th appearances of the variable x, and there is a nontrivial shift between the $i-1$ appearances of the variable x from the two sides of the equation. Then either $t_{i-1} u_i = v_i$ or vice versa, in which case the original equation can be broken into two equations, the first contains the first i pairs of coefficients, and the second contains the last $n-i$ pairs of coefficients, or $v_i = \tau_i(s_i+1)$ and $t_{i-1}u_i = \tau_i s_i$ (or vice versa), or $u_i = \tau_i(s_i+1)$ and $v_i = t_{i-1}\tau_i s_i$ (or vice versa).

This description of the coefficients in a general equation with one variable, in which the coefficients have no periodicity, and the top homogeneous parts of the coefficients are monomials, finally implies:

(1) Every solution of the equation $sx = xt$ is a solution of the given equation.

(2) Every solution x_2 of the given equation that satisfies

$$\deg(x_2) > 2(2 + k^{\deg(s)+2} + \deg(u_1) + \cdots + \deg(u_n))$$

is also a solution of the equation $sx = xt$.

The proof of (1) follows from the structure of the coefficients, and the proof of (2) follows by the argument that was used to prove (2) for the case in which there are no shifts between the various appearances of the top monomial of the solution x_2 in the two sides of the given equation in Proposition 4.12.

This concludes the proof of Theorem 4.7. □

Acknowledgments

I would like to thank Eliyahu Rips, who inspired me years ago to try and study varieties over noncommutative algebras, and David Kazhdan for his continuing interest. I am grateful to the referee, who went carefully over the text that is not easy to follow and verify, and suggested quite a few changes that improved the paper.

References

[Amitsur 1966] S. A. Amitsur, "Rational identities and applications to algebra and geometry", *J. Algebra* **3** (1966), 304–359. MR Zbl

[Appel 1968] K. I. Appel, "One-variable equations in free groups", *Proc. Amer. Math. Soc.* **19** (1968), 912–918. MR Zbl

[Bergman 1969] G. M. Bergman, "Centralizers in free associative algebras", *Trans. Amer. Math. Soc.* **137** (1969), 327–344. MR Zbl

[Bestvina and Feighn 1995] M. Bestvina and M. Feighn, "Stable actions of groups on real trees", *Invent. Math.* **121**:2 (1995), 287–321. MR Zbl

[Casals-Ruiz and Kazachkov 2011] M. Casals-Ruiz and I. V. Kazachkov, "On systems of equations over free products of groups", *J. Algebra* **333** (2011), 368–426. MR Zbl

[Cohn 1971] P. M. Cohn, *Free rings and their relations*, London Math. Soc. Monographs **2**, Academic Press, London, 1971. MR Zbl

[Guba 1986] V. S. Guba, "Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems", *Mat. Zametki* **40**:3 (1986), 321–324. In Russian; translated in *Math. Notes* **40**:3 (1986), 688–690. MR Zbl

[Lorenc 1963] A. A. Lorenc, "Solution of systems of equations in one unknown in free groups", *Dokl. Akad. Nauk SSSR* **148** (1963), 1253–1256. In Russian; translated in *Soviet Math. Dokl.* **4** (1963), 262–265. MR Zbl

[Makanin 1977] G. S. Makanin, "The problem of the solvability of equations in a free semigroup", *Mat. Sb.* (*N.S.*) **103**:2 (1977), 147–236. In Russian; translated in *Math. USSR-Sb.* **32**:2 (1977), 129–198. MR Zbl

[Malcev 1948] A. I. Malcev, "On the embedding of group algebras in division algebras", *Doklady Akad. Nauk SSSR* (*N.S.*) **60** (1948), 1499–1501. MR Zbl

[Neumann 1949] B. H. Neumann, "On ordered division rings", *Trans. Amer. Math. Soc.* **66** (1949), 202–252. MR Zbl

[Sela 2001] Z. Sela, "Diophantine geometry over groups, I: Makanin–Razborov diagrams", *Publ. Math. Inst. Hautes Études Sci.* 93 (2001), 31–105. MR Zbl

[Sela 2016] Z. Sela, "Word equations, I: Pairs and their Makanin–Razborov diagrams", preprint, 2016. arXiv 1607.05431

[Shelah 1990] S. Shelah, *Classification theory and the number of nonisomorphic models*, 2nd ed., Studies Logic Found. Math. **92**, North-Holland Publishing Co., Amsterdam, 1990. MR Zbl

Received 20 Mar 2022. Revised 28 Mar 2024.

ZLIL SELA:

zlil.sela@mail.huji.ac.il
Mathematics Department, Hebrew University, Jerusalem, Israel

Model Theory

msp.org/mt

See inside back cover or msp.org/mt for submission instructions.

Model Theory (ISSN 2832-904X electronic, 2832-9058 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

MT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
https://msp.org/

Model Theory

no. 3 vol. 3 2024