Model Theory

no. 3 vol. 3 2024

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Model Theory
Vol. 3, No. 3, 2024
https://doi.org/10.2140/mt.2024.3.883

Correction to the article Kim-independence in positive logic

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Volume 1:1 (2022), 55–113

The proof of the independence theorem for Kim-independence in positive thick NSOP_1 theories by Dobrowolski and Kamsma (*Model Theory* **1** (2022), 55–113) contains a gap. The theorem is still true, and in this corrigendum we give a different proof.

1. Introduction

The proof of the independence theorem for Kim-independence in thick NSOP_1 theories [Dobrowolski and Kamsma 2022, Theorem 7.7] contains a gap. Everything in that proof is fine up to the point where it is argued how the theorem follows from what is called "Claim 2" (at the bottom of page 88). By compactness, an M-indiscernible sequence $(g_i h_i g'_i h'_i g''_i h''_i)_{i \in \mathbb{Z}}$ is extracted from the data from Claim 2. However, it may be that the properties $(h''_i g''_{i+1})_{i \in \mathbb{Z}} \models (q'|_{z_0, y})^{\otimes \mathbb{Z}}|_M$ and $h_i g_{i+1} \equiv_{M h_{>i} g_{>i+1} h''_{>i} g''_{i+1}} h''_i g''_{i+1}$ are not carried over.

The theorem, as stated, is still true, and in this corrigendum we give a different proof. We assume familiarity with [Dobrowolski and Kamsma 2022].

2. Technical tools

We reformulate the chain condition in a form that will be useful to us.

Lemma 2.1 (chain condition). *Let T be a thick $NSOP_1$ theory. Suppose that $a \mathop{\smile\!\!\!\!\!\;|}^K_M b$ and that $(b_i)_{i<\omega}$ is a Morley sequence in some global M-Ls-invariant type with $b_0 = b$. Then, writing $p(x, b) = \mathrm{tp}(a/Mb)$, we have that*

$$\bigcup_{i<\omega} p(x, b_i)$$

does not Kim-divide over M.

Kamsma is supported by the EPSRC grant EP/X018997/1 .
MSC2020: primary 03C45; secondary 03C10.
Keywords: Kim-independence, Kim-dividing, positive logic, NSOP1 theory, independence theorem.

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Proof. Let $q(x)$ be the global M-Ls-invariant type in which $(b_i)_{i<\omega}$ is a Morley sequence. As $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$ we have by [Dobrowolski and Kamsma 2022, Proposition 4.2] that there is an Ma-indiscernible $(b'_i)_{i<\omega} \models q^{\otimes\omega}|_M$ with $b'_0 = b$. So we have $(b'_i)_{i<\omega} \equiv_M (b_i)_{i<\omega}$ and we let a^* be such that $a(b'_i)_{i<\omega} \equiv_M a^*(b_i)_{i<\omega}$. Then $(b_i)_{i<\omega}$ is Ma^*-indiscernible, and so $a^* \mathop{\smile\!\!\!\!\!\!|}^K_M (b_i)_{i<\omega}$ by [loc. cit., Lemma 6.1]. We conclude by noting that $a^*b_i \equiv_M a^*b_0 \equiv_M ab'_0 = ab$ for all $i < \omega$. □

Proposition 2.2 (being Ls-invariant is type-definable). *Let T be a thick theory. Let C be some parameter set and let $N \supseteq C$ be $(2^{|C|+\lambda_T})^+$-saturated (possibly N is the monster). Define $\Sigma(x)$ to be the following partial type over N*

$$\bigcup\{\mathrm{d}_C(xb, xb') \leq 2 : b, b' \in N \textit{ are finite tuples such that } \mathrm{d}_C(b, b') \leq 1\}.$$

Then a type $q(x)$ over N is C-Ls-invariant if and only if $\Sigma(\alpha)$ for $\alpha \models q$.

Proof. Let $q(x)$ be a C-Ls-invariant type over N and let $\alpha \models q$. Let $b, b' \in N$ be finite tuples such that $\mathrm{d}_C(b, b') \leq 1$. Then there is a C-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0b_1 = bb'$, which we may assume to be in N by saturation. Using saturation again, we find a λ_T-saturated $C \subseteq M \subseteq N$ such that $(b_i)_{i<\omega}$ is M-indiscernible. In particular this means that $bM \equiv^{\mathrm{Ls}}_C b'M$ and so $\alpha bM \equiv^{\mathrm{Ls}}_C \alpha b'M$. It follows that $\alpha b \equiv^{\mathrm{Ls}}_M \alpha b'$ and thus by our choice of M we get $\mathrm{d}_C(\alpha b, \alpha b') \leq 2$. As b, b' were arbitrary, we conclude that $\models \Sigma(\alpha)$.

For the other direction we let $q(x)$ be a type over N such that for $\alpha \models q$ we have $\models \Sigma(\alpha)$. Now let $d, d' \in N$ be (potentially infinite tuples) such that $d \equiv^{\mathrm{Ls}}_C d'$. Let $n < \omega$ be such that $\mathrm{d}_C(d, d') \leq n$, we claim that $\mathrm{d}_C(\alpha d, \alpha d') \leq 2n$, which implies the required $\alpha d \equiv^{\mathrm{Ls}}_C \alpha d'$. By thickness we have that the condition $\mathrm{d}_C(\alpha d, \alpha d') \leq 2n$ is given by

$$\bigcup\{\mathrm{d}_C(\alpha b, \alpha b') \leq 2n : b \subseteq d \text{ and } b' \subseteq d' \text{ are finite matching tuples}\}.$$

So we have reduced the problem to the case where d and d' are finite. By saturation then there are $d = d_0, d_1, \ldots, d_n = d'$ in N such that $\mathrm{d}_C(d_i, d_{i+1}) \leq 1$ for all $0 \leq i < n$. By assumption we thus have that $\mathrm{d}_C(\alpha d_i, \alpha d_{i+1}) \leq 2$ for all $0 \leq i < n$. We conclude that $\mathrm{d}_C(\alpha d, \alpha d') \leq 2n$, as required. □

Proposition 2.3 (extending Ls-invariant types). *Let T be a thick theory. Let $N \supseteq C$ be $(2^{|C|+\lambda_T})^+$-saturated. Suppose that $p(x) = \mathrm{tp}(a/N)$ is a C-Ls-invariant type, then $p(x)$ extends to a unique global C-Ls-invariant type $q(x)$.*

Proof. Let $\Sigma(x)$ be the global partial type from Proposition 2.2 expressing C-Ls-invariance. We will show that $p(x) \cup \Sigma(x)$ is finitely satisfiable. So let $\varphi(x, e) \in p(x)$, where e is a tuple of parameters from N, and let $\Sigma_0(x) \subseteq \Sigma(x)$ be finite. Let $b_1, \ldots, b_n$ and $b'_1, \ldots, b'_n$ be the finite tuples that occur in $\Sigma_0(x)$, so $\mathrm{d}_C(b_i, b'_i) \leq 1$ for all $1 \leq i \leq n$. By saturation of N we find $d_1, \ldots, d_n, d'_1, \ldots, d'_n \in N$ such that

$d_1 \cdots d_n d_1' \cdots d_n' \equiv_{Ce} b_1 \cdots b_n b_1' \cdots b_n'$. So for all $1 \leq i \leq n$ we have $\mathrm{d}_C(d_i, d_i') \leq 1$, and hence $\mathrm{d}_C(ad_i, ad_i') \leq 2$ by Proposition 2.2 applied to $p(x)$. Now let a^* be such that $ad_1 \cdots d_n d_1' \cdots d_n' \equiv_{Ce} a^* b_1 \cdots b_n b_1' \cdots b_n'$. Then by construction we have that $\models \varphi(a^*, e)$ and $\models \Sigma_0(a^*)$, which proves finite satisfiability of $p(x) \cup \Sigma(x)$. By compactness we then find a realisation α of $p(x) \cup \Sigma(x)$, so that $q(x) = \mathrm{tp}(\alpha/\mathfrak{M})$ is our desired C-Ls-invariant type. The uniqueness claim follows from [Dobrowolski and Kamsma 2022, Fact 7.6]. □

We recall from [loc. cit., Definition 3.12] that $a \mathop{\smile\!\!\!\!\!\!\mid}_C^{iLs} b$ means that $\mathrm{tp}(a/Cb)$ extends to a global C-Ls-invariant type.

Proposition 2.4. *Let T be a thick theory. If $(a_i)_{i<\omega}$ is a C-indiscernible sequence such that $a_i \mathop{\smile\!\!\!\!\!\!\mid}_C^{iLs} a_{<i}$ for all $i < \omega$ then $(a_i)_{i<\omega}$ is a Morley sequence in some global C-Ls-invariant type.*

Proof. By compactness we find a_ω such that $(a_i)_{i \leq \omega}$ is C-indiscernible. Set $p(x) = \mathrm{tp}(a_\omega / Ca_{<\omega})$ and let $\Sigma(x)$ be the global partial type from Proposition 2.2. We claim that $p(x) \cup \Sigma(x)$ is consistent. Indeed, for any finite $p'(x) \subseteq p(x)$ there is some $i < \omega$ so that $p'(x)$ only contains parameters in $Ca_{<i}$, and so $\models p'(a_i)$ by C-indiscernibility. As $a_i \mathop{\smile\!\!\!\!\!\!\mid}_C^{iLs} a_{<i}$ we then have that $p'(x)$ extends to a global C-Ls-invariant type $q'(x)$, and any realisation of $q'(x)$ will then be a realisation of $p'(x) \cup \Sigma(x)$. So $p(x) \cup \Sigma(x)$ is finitely satisfiable and hence consistent.

Let α^* be a realisation of $p(x) \cup \Sigma(x)$ and set $q^*(x) = \mathrm{tp}(\alpha^*/\mathfrak{M})$, so $q^*(x)$ is global C-Ls-invariant. Let $a^* \equiv^{\mathrm{Ls}}_{Ca_{<\omega}} \alpha^*$, then there is $f \in \mathrm{Aut}(\mathfrak{M}/Ca_{<\omega})$ such that $f(a^*) = a_\omega$. Set $q = f(q^*)$, so $q(x)$ is global C-Ls-invariant by [loc. cit., Lemma 3.8(i)] with $p(x) \subseteq q(x)$ and, letting α be a realisation of q, we have $\alpha \equiv^{\mathrm{Ls}}_{Ca_{<\omega}} a_\omega$.

For any $i < \omega$ we thus have $a_i \equiv^{\mathrm{Ls}}_{Ca_{<i}} a_\omega \equiv^{\mathrm{Ls}}_{Ca_{<i}} \alpha$. We therefore have $a_{<i} \models q^{\otimes i}|_C$ for all $i < \omega$ and so $(a_i)_{i<\omega} \models q^{\otimes \omega}|_C$. So $(a_i)_{i<\omega}$ is the automorphic image over C of a Morley sequence over C, hence it is itself a Morley sequence in a (potentially different) global C-Ls-invariant type. □

3. Spread out trees

We recall various definitions concerning trees and trees of parameters (which we will from now on also simply call trees) from [Kaplan and Ramsey 2020]. In particular, we will work with the ill-founded trees $\mathcal{T}_\alpha$ from [loc. cit., Definition 5.1] and we use the same notation, so we assume familiarity with those definitions. We refer to [Kamsma 2024] for the definitions and terminology involving s-indiscernibility, str-indiscernibility and generalised EM-types. We slightly adjust [Kaplan and Ramsey 2020, Definition 5.7] to fit our situation.

Definition 3.1. Let $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ be a tree and let M be an e.c. model:

(i) We call $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ *spread out over* M if for all $\eta \in \mathcal{T}_\alpha$ with $\mathrm{dom}(\eta) = [\beta+1, \alpha)$ for some $\beta < \alpha$, there is a global M-Ls-invariant type $q_\eta \supseteq \mathrm{tp}(a_{\trianglerighteq \eta^\frown\langle 0\rangle}/M)$ such that $(a_{\trianglerighteq \eta^\frown\langle i\rangle})_{i<\omega}$ is a Morley sequence in q_η over M.

(ii) A *Morley tree over* M is an str-indiscernible and spread out tree over M.

(iii) A *tree Morley sequence over* M is a branch in an infinite height Morley tree over M.

Lemma 3.2. *Suppose that $(a_i)_{i<\omega}$ is a tree Morley sequence over M*:

(i) *If $b_i \subseteq a_i$ for each $i < \omega$, of matching length and position, then $(b_i)_{i<\omega}$ is a tree Morley sequence over M.*

(ii) *Fix $1 \leq n < \omega$ and define $d_i = (a_{ni}, \dots, a_{ni+n-1})$ for all $i < \omega$. Then $(d_i)_{i<\omega}$ is a tree Morley sequence over M.*

Proof. This is essentially [Kaplan and Ramsey 2020, Lemma 5.9], but we work with slightly different definitions, so we go through the proof here. Part (i) is clear, because being a Morley tree is preserved under taking subtuples. For (ii) we let $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ be a Morley tree such that $(a_i)_{i<\omega}$ is a branch in $(b_\eta)_{\eta\in\mathcal{T}_\omega}$. We may assume that $(a_i)_{i<\omega}$ is the branch indexed by the constant zero functions. We define $j : \mathcal{T}_\omega \to \mathcal{T}_\omega$ so that for $\eta \in \mathcal{T}_\omega$ with $\mathrm{dom}(\eta) = [k, \omega)$ we have $\mathrm{dom}(j(\eta)) = [nk+n-1, \omega)$ and

$$j(\eta)(m) = \begin{cases} \eta((m-(n-1))/n) & \text{if } n \mid (m-(n-1)), \\ 0 & \text{otherwise,} \end{cases}$$

for all $m \in [nk+n-1, \omega)$. We define $(c_\eta)_{\eta\in\mathcal{T}_\omega}$ by $c_\eta = (b_{j(\eta)}, \dots, b_{j(\eta)^\frown\langle 0\rangle^{n-1}})$. This corresponds to the n-fold elongation of $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ from [Chernikov and Ramsey 2016]. One then straightforwardly verifies that $(c_\eta)_{\eta\in\mathcal{T}_\omega}$ is a Morley tree over M, so $(c_{\zeta_i})_{i<\omega}$ is a tree Morley sequence over M. For $i < \omega$ we have

$$c_{\zeta_i} = (b_{\zeta_{ni+n-1}}, \dots, b_{\zeta_{ni}}) = (a_{ni+n-1}, \dots, a_{ni}),$$

so by reversing the order of the tuples we see that $(d_i)_{i<\omega}$ is a tree Morley sequence over M. □

Lemma 3.3 (Kim's lemma for tree Morley sequences). *Let T be a thick NSOP$_1$ theory. Let M be an e.c. model and let $\Sigma(x, b)$ be a partial type over M. Then the following are equivalent*:

(i) *$\Sigma(x, b)$ Kim-divides over M.*

(ii) *For some tree Morley sequence $(b_i)_{i<\omega}$ over M with $b_0 = b$ we have that $\bigcup_{i<\omega} \Sigma(x, b_i)$ is inconsistent.*

(iii) *For every tree Morley sequence* $(b_i)_{i<\omega}$ *over* M *with* $b_0 = b$ *we have that* $\bigcup_{i<\omega} \Sigma(x, b_i)$ *is inconsistent.*

Proof. This is [Kaplan and Ramsey 2020, Corollary 5.14], whose proof is really found in [loc. cit., Proposition 5.13]. Our setting requires some minor extra verifications, which we will do below, but the proof is essentially the same.

Given the existence of tree Morley sequences starting with b (Lemma 3.10), the equivalence of these three statements reduces to proving that for any tree Morley sequence $(b_i)_{i<\omega}$ over M with $b_0 = b$ we have that $\Sigma(x, b)$ Kim-divides if and only if $\bigcup_{i<\omega} \Sigma(x, b_i)$ is inconsistent.

Let $(c_\eta)_{\eta\in\mathcal{T}_\omega}$ be a Morley tree over M such that $(b_i)_{i<\omega}$ is a branch in that tree, which we may assume to be the constant zero branch. For $i < \omega$ define $\eta_i \in \mathcal{T}_\omega$ to be the function with domain $[i, \omega)$ such that

$$\eta_i(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

By str-indiscernibility, the sequences $(c_{\zeta_i})_{i<\omega}$ and $(c_{\eta_i})_{i<\omega}$ are M-indiscernible. We claim that $(c_{\eta_i})_{i<\omega}$ is a Morley sequence over M in a global M-Ls-invariant type. Indeed, because $(c_\eta)_{\eta\in\mathcal{T}_\omega}$ is spread out over M we have that $c_{\eta_i} \mathop{\smile\!\!\!\!\!\!|}\,^{iLs}_M (c_{\eta_j})_{j<i}$ for all $i < \omega$. So the claim follows from Proposition 2.4. By str-indiscernibility we also have for all $i < \omega$ that c_{ζ_i}, c_{η_i} starts an $M(c_{\zeta_j}, c_{\eta_j})_{j>i}$-indiscernible sequence. So since T is NSOP_1 we can apply [Dobrowolski and Kamsma 2022, Lemma 5.10] to conclude that $\bigcup_{i<\omega} \Sigma(x, c_{\zeta_i})$ is inconsistent if and only if $\bigcup_{i<\omega} \Sigma(x, c_{\eta_i})$ is inconsistent. The former is just $\bigcup_{i<\omega} \Sigma(x, b_i)$, and the latter is inconsistent if and only if $\Sigma(x, b)$ Kim-divides by Kim's lemma for NSOP_1 theories [loc. cit., Proposition 4.4], which concludes the proof. □

Fact 3.4 (tree modelling theorems). *Let* T *be a thick theory*:

(i) *Let* $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ *be a tree of tuples and let* C *be any set of parameters, then there is a tree* $(b_\eta)_{\eta\in\mathcal{T}_\alpha}$ *that is s-indiscernible over* C *and* EM_s*-based on* $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ *over* C.

(ii) *Let* C *be any parameter set*, κ *any cardinal, and let* $\lambda = \beth_{(2^{|T|+|C|+\kappa})^+}$. *Given any tree* $(a_\eta)_{\eta\in\mathcal{T}_\lambda}$ *of* κ*-tuples that is s-indiscernible over* C, *there is a tree* $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ *that is str-indiscernible over* C *str-based on* $(a_\eta)_{\eta\in\mathcal{T}_\lambda}$ *over* C. *The latter means that for any finite tuple* $\bar{\eta} \in \mathcal{T}_\omega$ *there is* $\bar{\nu} \in \mathcal{T}_\lambda$ *such that* $\bar{\eta}$ *and* $\bar{\nu}$ *have the same str-quantifier-free type and* $b_{\bar{\eta}} \equiv_C a_{\bar{\nu}}$.

Proof. Part (i) is [Kamsma 2024, Theorem 4.6], which is essentially just compactness applied to [Dobrowolski and Kamsma 2022, Proposition 5.8]. Part (ii) is [Kamsma 2024, Theorem 4.8], which is technically stated for well-founded trees, but its proof applies to the ill-founded trees we are interested in here. □

Lemma 3.5. *Let T be a thick theory. Suppose that $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is s-indiscernible and spread out over M and that $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ is str-based on $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ over M, then $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ is spread out over M.*

Proof. Let $\eta\in\mathcal{T}_\omega$, we have to show that $(b_{\trianglerighteq\eta^\frown\langle i\rangle})_{i<\omega}$ is a Morley sequence in some global M-Ls-invariant type. We claim that $b_{\trianglerighteq\eta^\frown\langle i\rangle}\mathop{\smile\!\!\!\!\!\!\vert}\,^{iLs}_M(b_{\trianglerighteq\eta^\frown\langle j\rangle})_{j<i}$ for all $i<\omega$. This is indeed enough, because $(b_{\trianglerighteq\eta^\frown\langle i\rangle})_{i<\omega}$ is M-indiscernible by str-indiscernibility over M, and so the result follows by Proposition 2.4.

We prove the claim by showing that for all $i<\omega$ and all finite $b\subseteq b_{\trianglerighteq\eta^\frown\langle i\rangle}$ and $b'\subseteq(b_{\trianglerighteq\eta^\frown\langle j\rangle})_{j<i}$ we have $b\mathop{\smile\!\!\!\!\!\!\vert}\,^{iLs}_M b'$, which is enough by Proposition 2.2. Let $\bar\nu_{i_1},\dots,\bar\nu_{i_n}$ be finite tuples in $\mathcal{T}_\omega$ such that $i_1<\dots<i_n<\omega$ and $\bigwedge\bar\nu_{i_k}\trianglerighteq\eta^\frown\langle i_k\rangle$ for all $1\le k\le n$. By str-basing there are $\gamma,\bar\mu_{i_1},\dots,\bar\mu_{i_n}$ in $\mathcal{T}_\alpha$ such that $\gamma\bar\mu_{i_1}\cdots\bar\mu_{i_n}$ has the same str-quantifier-free type as $\eta\bar\nu_{i_1}\cdots\bar\nu_{i_n}$ and $b_\eta b_{\bar\nu_{i_1}}\cdots b_{\bar\nu_{i_n}}\equiv_M a_\gamma a_{\bar\mu_{i_1}}\cdots a_{\bar\mu_{i_n}}$. We now have reduced the problem to showing that $a_{\bar\mu_{i_n}}\mathop{\smile\!\!\!\!\!\!\vert}\,^{iLs}_M a_{\bar\mu_{i_1}}\cdots a_{\bar\mu_{i_{n-1}}}$. As $\gamma\triangleleft\bigwedge\bar\mu_{i_n}$, there must be some $m<\omega$ such that $\bigwedge\bar\mu_{i_n}\trianglerighteq\gamma^\frown\langle m\rangle$. Furthermore, we have for every $1\le k<n$ that $\gamma\triangleleft\bigwedge\bar\mu_{i_k}$ and $\bigwedge\bar\mu_{i_k}<_{\text{lex}}\bigwedge\bar\mu_{i_n}$, and so $\bigwedge\bar\mu_{i_k}\trianglerighteq\gamma^\frown\langle j\rangle$ for some $j<m$. Because $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is spread out over M we have $a_{\trianglerighteq\gamma^\frown\langle m\rangle}\mathop{\smile\!\!\!\!\!\!\vert}\,^{iLs}_M(a_{\trianglerighteq\gamma^\frown\langle j\rangle})_{j<m}$, and so $a_{\bar\mu_{i_n}}\mathop{\smile\!\!\!\!\!\!\vert}\,^{iLs}_M a_{\bar\mu_{i_1}}\cdots a_{\bar\mu_{i_{n-1}}}$, as required. □

Corollary 3.6. *Let T be a thick theory, and let C be some parameter set and κ some cardinal. Set $\lambda=\beth_{(2^{\kappa+2^{\lambda_T+|C|}})^+}$. Given a tree $(a_\eta)_{\eta\in\mathcal{T}_\lambda}$ of κ-tupes that is s-indiscernible and spread out over C, there is a Morley tree $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ over C that is str-Ls-based on $(a_\eta)_{\eta\in\mathcal{T}_\lambda}$ over C. The latter means that for any finite tuple $\bar\eta\in\mathcal{T}_\omega$ there is $\bar\nu\in\mathcal{T}_\lambda$ such that $\bar\eta$ and $\bar\nu$ have the same str-quantifier-free type and $b_{\bar\eta}\equiv^{\text{Ls}}_C a_{\bar\nu}$.*

Proof. By [Dobrowolski and Kamsma 2022, Fact 2.12] there is λ_T-saturated $M\supseteq C$ with $|M|\le 2^{\lambda_T+|C|}$. As $\kappa+|T|+|M|\le\kappa+|T|+2^{\lambda_T+|C|}=\kappa+2^{\lambda_T+|C|}$, we can use Fact 3.4(ii) to find a tree $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ that is str-indiscernible over M and str-based on $(a_\eta)_{\eta\in\mathcal{T}_\lambda}$ over M. In particular $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ is str-based on $(a_\eta)_{\eta\in\mathcal{T}_\lambda}$ over C, so it is spread out over C by Lemma 3.5 and hence it is a Morley tree over C. Finally, by str-basing, we have that for any finite tuple $\bar\eta\in\mathcal{T}_\omega$ there is $\bar\nu\in\mathcal{T}_\lambda$ such that $\bar\eta$ and $\bar\nu$ have the same str-quantifier-free type and $b_{\bar\eta}\equiv_M a_{\bar\nu}$. By our choice of M this implies $b_{\bar\eta}\equiv^{\text{Ls}}_C a_{\bar\nu}$, as required. □

The following key lemma in constructing spread out trees is due to N. Ramsey, for which we take terminology from [Chernikov et al. 2023, Definition 1.14].

Definition 3.7. We call a sequence of trees $((a^i_\eta)_{\eta\in\mathcal{T}_\alpha})_{i<\omega}$ *mutually s-indiscernible over C* if $(a^i_\eta)_{\eta\in\mathcal{T}_\alpha}$ is s-indiscernible over $C((a^j_\eta)_{\eta\in\mathcal{T}_\alpha})_{j\neq i,j<\omega}$ for all $i<\omega$.

Lemma 3.8. *Let T be a thick theory and let $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ be a tree that is s-indiscernible over M. Then there is a Morley sequence $((a^i_\eta)_{\eta\in\mathcal{T}_\alpha})_{i<\omega}$ in some global M-Ls-invariant type with $(a^0_\eta)_{\eta\in\mathcal{T}_\alpha}=(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ that is mutually s-indiscernible over M.*

Proof. Let $q((x_\eta)_{\eta \in \mathcal{T}_\alpha}) \supseteq \text{tp}((a_\eta)_{\eta \in \mathcal{T}_\alpha}/M)$ be a global M-Ls-invariant type. Let $N \supseteq M$ be $(2^{|M|+\lambda_T})^+$-saturated, and let $(a'_\eta)_{\eta \in \mathcal{T}_\alpha} \models q|_N$. Apply the s-modelling theorem (Fact 3.4(i)) to find a tree $(a''_\eta)_{\eta \in \mathcal{T}_\alpha}$ that is s-indiscernible over N and EM_s-based on $(a'_\eta)_{\eta \in \mathcal{T}_\alpha}$ over N.

Claim 3.8.1. *The type* $\text{tp}((a''_\eta)_{\eta \in \mathcal{T}_\alpha}/N)$ *is M-Ls-invariant.*

Proof of claim. By Proposition 2.2 it is enough to show that for any finite $b, b' \in N$ with $\text{d}_M(b, b') \leq 1$ we have $\text{d}_M((x_\eta)_{\eta \in \mathcal{T}_\alpha} b, (x_\eta)_{\eta \in \mathcal{T}_\alpha} b') \leq 2 \subseteq \text{tp}((a''_\eta)_{\eta \in \mathcal{T}_\alpha}/N)$. By thickness we have that $\text{d}_M((x_\eta)_{\eta \in \mathcal{T}_\alpha} b, (x_\eta)_{\eta \in \mathcal{T}_\alpha} b') \leq 2$ is given by

$$\bigcup \{\text{d}_M(x_{\bar{\eta}} b, x_{\bar{\eta}} b') \leq 2 : \bar{\eta} \text{ is a finite tuple in } \mathcal{T}_\alpha\}.$$

Let $\bar{\eta}$ be any finite tuple in $\mathcal{T}_\alpha$. For any $\bar{\nu}$ that has the same s-quantifier-free type as $\bar{\eta}$ we have that $\text{d}_M(x_{\bar{\nu}} b, x_{\bar{\nu}} b') \leq 2 \subseteq \text{tp}((a'_\eta)_{\eta \in \mathcal{T}_\alpha}/N)$ by Proposition 2.2, because $\text{tp}((a'_\eta)_{\eta \in \mathcal{T}_\alpha}/N) = q|_N$ is M-Ls-invariant. We thus see that $\text{d}_M(x_{\bar{\eta}} b, x_{\bar{\eta}} b') \leq 2 \subseteq \text{EM}_s((a'_\eta)_{\eta \in \mathcal{T}_\alpha}/N) \subseteq \text{tp}((a''_\eta)_{\eta \in \mathcal{T}_\alpha}/N)$, which concludes the proof of the claim. □

By Claim 3.8.1, Proposition 2.3 and our choice of N there is a unique global M-Ls-invariant type $q''((x_\eta)_{\eta \in \mathcal{T}_\alpha}) \supseteq \text{tp}((a''_\eta)_{\eta \in \mathcal{T}_\alpha}/N)$. Let $((b^i_\eta)_{\eta \in \mathcal{T}_\alpha})_{i<\omega}$ be a Morley sequence in q'' over N.

Claim 3.8.2. *The sequence* $(b^i_\eta)_{\eta \in \mathcal{T}_\alpha}$ *is mutually s-indiscernible over N.*

Proof of claim. Fix $i < \omega$. We prove by induction on $k \geq i$ that $(b^i_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over $N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j \neq i, j < k}$.

For the base case $k = i$ we need to prove that $(b^i_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over $N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j<i}$. Let $\bar{\eta}, \bar{\nu} \in \mathcal{T}_\alpha$ be finite tuples with the same s-quantifier-free type. As $(b^i_\eta)_{\eta \in \mathcal{T}_\alpha} \equiv_N (a''_\eta)_{\eta \in \mathcal{T}_\alpha}$, we have that it is s-indiscernible over N. So there is a single type (after renaming variables) $p(y) = \text{tp}(b^i_{\bar{\eta}}/N) = \text{tp}(b^i_{\bar{\nu}}/N)$, which is M-Ls-invariant by Claim 3.8.1. Since $q''(x_{\bar{\eta}})$ and $q''(x_{\bar{\nu}})$ are both global M-Ls-invariant extensions of $p(y)$ we have that $q''(x_{\bar{\eta}}) = q''(x_{\bar{\nu}})$, after renaming variables. By construction $b^i_{\bar{\eta}} \models q''(x_{\bar{\eta}})|_{N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j<i}}$ and $b^i_{\bar{\nu}} \models q''(x_{\bar{\nu}})|_{N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j<i}}$, so $b^i_{\bar{\eta}} \equiv_{N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j<i}} b^i_{\bar{\nu}}$ follows, as required.

For the successor step we have $k > i$ and we assume that $(b^i_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over $N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j \neq i, j<k}$. Let $\bar{\eta}, \bar{\nu} \in \mathcal{T}_\alpha$ be finite tuples with the same s-quantifier-free type. By the induction hypothesis we have

$$b^i_{\bar{\eta}} \equiv^{\text{Ls}}_{N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j \neq i, j<k}} b^i_{\bar{\nu}},$$

where we get equivalence of Lascar-strong types instead of just normal types from s-indiscernibility; see e.g., [Kamsma 2024, Proposition 4.5]. As $(b^k_\eta)_{\eta \in \mathcal{T}_\alpha}$ realises an M-Ls-invariant type over $N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j<k}$ and $N \supseteq M$ we get

$$(b^k_\eta)_{\eta \in \mathcal{T}_\alpha} b^i_{\bar{\eta}} \equiv^{\text{Ls}}_{N((b^j_\eta)_{\eta \in \mathcal{T}_\alpha})_{j \neq i, j<k}} (b^k_\eta)_{\eta \in \mathcal{T}_\alpha} b^i_{\bar{\nu}},$$

which completes the induction step and thus the proof of the claim. □

We have $(b^0_\eta)_{\eta \in \mathcal{T}_\alpha} \equiv_M (a''_\eta)_{\eta \in \mathcal{T}_\alpha} \equiv_M (a'_\eta)_{\eta \in \mathcal{T}_\alpha} \equiv_M (a_\eta)_{\eta \in \mathcal{T}_\alpha}$, where the middle equality of types follows because $(a'_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over M and so its EM_s-type over M is maximal (i.e., is the same as its type over M) and $(a''_\eta)_{\eta \in \mathcal{T}_\alpha}$ is in particular EM_s-based on $(a'_\eta)_{\eta \in \mathcal{T}_\alpha}$ over M. So by an automorphism we find $((a^i_\eta)_{\eta \in \mathcal{T}_\alpha})_{i<\omega} \equiv_M ((b^i_\eta)_{\eta \in \mathcal{T}_\alpha})_{i<\omega}$, with $(a^0_\eta)_{\eta \in \mathcal{T}_\alpha} = (a_\eta)_{\eta \in \mathcal{T}_\alpha}$, which is then as required by construction of $((b^i_\eta)_{\eta \in \mathcal{T}_\alpha})_{i<\omega}$ and Claim 3.8.2. □

Remark 3.9. Lemma 3.8 is in fact a missing ingredient in [Kaplan and Ramsey 2020], in particular in the inductive steps in their Lemmas 5.11 and 6.4. There they replace some spread out tree A by an s-indiscernible tree B locally based on A (in our terminology: EM_s-based). However, this process might not preserve the property of being spread out. By replacing the inductive step by Lemma 3.8, the argument can be fixed.

In existing work on Kim-independence over arbitrary sets there is the same issue, as discussed in [Chernikov et al. 2023, page 7]. This can be fixed in a similar manner: [loc. cit., Lemma 1.15] is a variant of Lemma 3.8 over arbitrary sets (in full first-order logic), and can then be used in the inductive steps in the same way.

We also remark that this is not an issue in [Dobrowolski and Kamsma 2022], because the proofs there make use of a different notion called "q-spread-out". The point of this notion is that it is type-definable, so it can be captured by the EM_s-type. The gap in the proof of the Independence Theorem that this corrigendum addresses is of a different nature.

The following lemma illustrates the use of Lemma 3.8 and completes the proof of Lemma 3.3.

Lemma 3.10. *Let T be a thick theory. For any a and M there is a tree Morley sequence $(a_i)_{i<\omega}$ over M with $a_0 = a$.*

Proof. Let λ be the cardinal from Corollary 3.6, where M and $|a|$ take the respective roles of C and κ there. By induction on $\alpha \leq \lambda$ we will construct trees $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$, such that:

(1) For all $\eta \in \mathcal{T}_\alpha$ we have $a^\alpha_\eta \equiv_M a$.

(2) The tree $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ is spread out and s-indiscernible over M.

(3) For all $\beta < \alpha$ we have $a^\alpha_{\iota_{\beta\alpha}(\eta)} = a^\beta_\eta$ for all $\eta \in \mathcal{T}_\beta$.

We start by setting $a^0_\varnothing = a$. For a limit stage ℓ, we set $a^\ell_{\iota_{\beta\ell}(\eta)} = a^\beta_\eta$, where β ranges over all ordinals $< \ell$ and η ranges over all elements in $\mathcal{T}_\beta$. This is well-defined by property (3), and properties (1) and (2) follow immediately from the induction hypothesis.

For the successor step we suppose $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ has been constructed. By Lemma 3.8 we find a Morley sequence $((a^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha})_{i<\omega}$ in some global M-Ls-invariant type with $(a^\alpha_{\eta,0})_{\eta \in \mathcal{T}_\alpha} = (a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ that is mutually s-indiscernible over M. Define a tree $(b_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ by setting $b_\varnothing = a$ and $b_{\langle i \rangle ^\frown \eta} = a^\alpha_{\eta,i}$ for all $\eta \in \mathcal{T}_\alpha$ and $i < \omega$. The EM_s-type of $(b_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ over M satisfies the following properties:

(i) It contains $\mathrm{tp}((b_\eta)_{\eta \in \mathcal{T}_{\alpha+1} \setminus \{\varnothing\}}/M)$. This is because $(b_{\trianglerighteq \langle i \rangle})_{i<\omega}$ forms an M-indiscernible sequence of trees that is mutually s-indiscernible over M.

(ii) The EM_s-type specifies that the type of the root is $\mathrm{tp}(a/M)$.

We apply Fact 3.4(i) to find an s-indiscernible tree $(a^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ over M that is EM_s-based over M on $(b_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$. By an automorphism and (i) we may assume that $a^{\alpha+1}_{\langle i \rangle ^\frown \eta} = b_{\langle i \rangle ^\frown \eta} = a^\alpha_{\eta,i}$ for all $\eta \in \mathcal{T}_\alpha$ and $i < \omega$, and so (3) is satisfied. This then also implies that (2) is satisfied and (1) is satisfied by (ii), completing the inductive construction.

We thus have constructed a tree $(a^\lambda_\eta)_{\eta \in \mathcal{T}_\lambda}$ that is spread out and s-indiscernible over M with $a^\lambda_\eta \equiv_M a$ for all $\eta \in \mathcal{T}_\lambda$. We can now apply Corollary 3.6 to find a Morley tree $(a_\eta)_{\eta \in \mathcal{T}_\omega}$ that is str-Ls-based on $(a^\lambda_\eta)_{\eta \in \mathcal{T}_\lambda}$ over M. In particular $a_\eta \equiv_M a$ for all $\eta \in \mathcal{T}_\omega$, and so by an automorphism we may assume $a_{\zeta_0} = a$. Then setting $a_i = a_{\zeta_i}$ for all $i < \omega$ we obtain the required tree Morley sequence $(a_i)_{i<\omega}$. □

4. The independence theorem

We now give a new proof of the independence theorem [Dobrowolski and Kamsma 2022, Theorem 7.7]. The statement remains exactly the same. The proof is essentially that of [Kaplan and Ramsey 2020, Theorem 6.5], with Lemma 3.8 mixed in.

Lemma 4.1. *Let T be a thick NSOP$_1$ theory. Suppose that $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ and fix some cardinal κ. Suppose that $q(x, y) = \mathrm{tp}(N/\mathfrak{M})$ is a global M-Ls-invariant type such that $q|_x$ extends* $\mathrm{Lstp}(b/M)$, *where $N \supseteq M$ is $\beth_\omega(\lambda_T + |Mab| + |\mathcal{T}_\kappa|)$-saturated and the x variable matches b. If $(b_\eta)_{\eta \in \mathcal{T}_\alpha}$, with $\alpha \leq \kappa$, is a tree that is spread out over M, such that for all $\eta \in \mathcal{T}_\alpha$ we have $b_\eta \equiv^{\mathrm{Ls}}_M b$ and $b_\eta \models (q|_x)|_{Mb_{\triangleright \eta}}$, then, writing $p(x, b) = \mathrm{tp}(a/Mb)$,*

$$\bigcup_{\eta \in \mathcal{T}_\alpha} p(x, b_\eta)$$

does not Kim-divide over M.

Proof. We follow the proof of [Kaplan and Ramsey 2020, Lemma 6.2], replacing their use of [loc. cit., Proposition 6.1] by [Dobrowolski and Kamsma 2022, Proposition 7.5]. The proof is by induction on α. For $\alpha = 0$ there is nothing to do, and limit stages follow from the induction hypothesis by finite character. Now suppose that $(b_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ is as in the statement. By the induction hypothesis we have that

$$\bigcup_{\eta \trianglerighteq \langle 0 \rangle} p(x, b_\eta)$$

does not Kim-divide over M. Because $(b_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ is spread out we have that $(b_{\trianglerighteq\langle i\rangle})_{i<\omega}$ is a Morley sequence in some global M-Ls-invariant type. By the chain condition Lemma 2.1 we then have that

$$\bigcup_{i<\omega}\bigcup_{\eta\trianglerighteq\langle i\rangle} p(x, b_\eta)$$

does not Kim-divide over M. At the same time we have $b_\varnothing \models (q|_x)|_{Mb_{\triangleright\varnothing}}$ and so by our assumptions on q we have $b_\varnothing \mathop{\smile\!\!\!\!\!\!|}^{*}_{M} b_{\triangleright\varnothing}$; see [Dobrowolski and Kamsma 2022, Definition 7.3]. Using that $p(x, b_\varnothing)$ does not Kim-divide (because $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b$), we can apply the weak independence theorem [loc. cit., Proposition 7.5] to see that

$$p(x, b_\varnothing) \cup \bigcup_{i<\omega}\bigcup_{\eta\trianglerighteq\langle i\rangle} p(x, b_\eta)$$

does not Kim-divide (here we implicitly used the assumption that $b_\eta \equiv^{\mathrm{Ls}}_M b$ for all $\eta\in\mathcal{T}_{\alpha+1}$). Unfolding definitions, this is exactly saying that

$$\bigcup_{\eta\in\mathcal{T}_{\alpha+1}} p(x, b_\eta),$$

does not Kim-divide, completing the induction step and thereby the proof. $\square$

Lemma 4.2 (zig-zag lemma). *Let T be a thick NSOP$_1$ theory. Suppose that $b \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} c$. Then there is a global M-Ls-invariant type $q(x, y) = \mathrm{tp}(N/\mathfrak{M})$, where $N \supseteq M$ is some $\beth_\omega(\lambda_T + |Mbc|)$-saturated model and $q|_x$ extends $\mathrm{tp}(b/M)$, and a tree Morley sequence $(b_i, c_i)_{i<\omega}$ over M such that*:

(i) *If $i \le j$ then $b_i c_j \equiv_M bc$.*

(ii) *If $i > j$ then $b_i \models (q|_x)|_{Mc_j}$.*

Proof. We basically verify that the proof of [Kaplan and Ramsey 2020, Lemma 6.4] goes through, while fixing a gap by mixing in a use of Lemma 3.8 (see also Remark 3.9).

Let λ be the cardinal from Corollary 3.6, where the C and κ are M and $|bc|$ respectively. Let $N \supseteq Mb$ be $\beth_\omega(|\mathcal{T}_\lambda|)$-saturated (note that $|\mathcal{T}_\lambda| \ge \lambda_T + |Mbc|$). Let $q(x, y)$ be a global M-Ls-invariant extension of $\mathrm{Lstp}(N/M)$, where the x variable matches b. In particular, for $\beta \models q|_x$ we have $\beta \equiv^{\mathrm{Ls}}_M b$. We write $p(z, b) = \mathrm{tp}(c/Mb)$. By induction on $\alpha \le \lambda$ we will construct trees $(b^\alpha_\eta, c^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$, such that:

(1) For all $\eta\in\mathcal{T}_\alpha$ we have $b^\alpha_\eta \models (q|_x)|_{Mb^\alpha_{\triangleright\eta}c^\alpha_{\triangleright\eta}}$ and $b_\eta \equiv^{\mathrm{Ls}}_M b$.

(2) For all $\eta\in\mathcal{T}_\alpha$ we have $c^\alpha_\eta \models \bigcup_{\nu\trianglerighteq\eta} p(z, b^\alpha_\nu)$.

(3) The tree $(b^\alpha_\eta, c^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$ is spread out and s-indiscernible over M.

(4) For all $\beta < \alpha$ we have $b^\alpha_{\iota_{\beta\alpha}(\eta)} c^\alpha_{\iota_{\beta\alpha}(\eta)} = b^\beta_\eta c^\beta_\eta$ for all $\eta\in\mathcal{T}_\beta$.

We start by setting $b^0_\varnothing c^0_\varnothing = bc$. For a limit stage ℓ, we set $b^\ell_{\iota_{\beta\ell}(\eta)} c^\ell_{\iota_{\beta\ell}(\eta)} = b^\beta_\eta c^\beta_\eta$, where β ranges over all ordinals $< \ell$ and η ranges over all elements in $\mathcal{T}_\beta$. This is well-defined by property (4), and properties (1)–(3) then follow immediately from the induction hypothesis.

For the successor step we suppose $(b^\alpha_\eta, c^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ has been constructed. Using Lemma 3.8 we find a Morley sequence $((b^\alpha_{\eta,i}, c^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha})_{i<\omega}$ in some global M-Ls-invariant type with $(b^\alpha_{\eta,0}, c^\alpha_{\eta,0})_{\eta \in \mathcal{T}_\alpha} = (b^\alpha_\eta, c^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ that is mutually s-indiscernible over M. Define a tree $(d_\eta, e_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ by setting $d_{\langle i \rangle ^\frown \eta} e_{\langle i \rangle ^\frown \eta} = b^\alpha_{\eta,i} c^\alpha_{\eta,i}$ for all $\eta \in \mathcal{T}_\alpha$ and $i < \omega$. This leaves us to define $d_\varnothing$ and $e_\varnothing$. Let $\beta \models q|_x$ and pick $d_\varnothing$ such that

$$d_\varnothing \equiv^{\mathrm{Ls}}_{M d_{\rhd \varnothing} e_{\rhd \varnothing}} \beta.$$

We can then apply Lemma 4.1 to the tree $(d_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ to see that

$$\bigcup_{\eta \in \mathcal{T}_{\alpha+1}} p(z, d_\eta)$$

does not Kim-divide over M. In particular, this set is consistent and so we can let $e_\varnothing$ be a realisation of this set. The EM_s-type of $(d_\eta, e_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ over M satisfies the following properties:

(i) It contains $\mathrm{tp}((d_\eta, e_\eta)_{\eta \in \mathcal{T}_{\alpha+1} \setminus \{\varnothing\}}/M)$. This is because $(d_{\trianglerighteq \langle i \rangle}, e_{\trianglerighteq \langle i \rangle})_{i<\omega}$ forms an M-indiscernible sequence sequence of trees that is mutually s-indiscernible over M.

(ii) It contains the type $r(x_\varnothing, (x_\eta)_{\eta \rhd \varnothing}, (z_\eta)_{\eta \rhd \varnothing}) = \mathrm{tp}(d_\varnothing, d_{\rhd \varnothing}, e_{\rhd \varnothing}/M)$, and note that by construction $r(x, d_{\rhd \varnothing}, e_{\rhd \varnothing}/M) = (q|_x)|_{M d_{\rhd \varnothing} e_{\rhd \varnothing}}$. Indeed, let $\bar{\eta}$ and $\bar{\nu}$ be two finite tuples in $\mathcal{T}_{\alpha+1}$ with the same s-quantifier free type that do not contain the root. Then we have $d_{\bar{\eta}} e_{\bar{\eta}} \equiv^{\mathrm{Ls}}_M d_{\bar{\nu}} e_{\bar{\nu}}$, see (i) for the justification. The claim then follows from M-Ls-invariance of q.

(iii) It captures that $d_\varnothing \equiv^{\mathrm{Ls}}_M d_{\langle i \rangle}$ for all $i < \omega$. By construction we have $\mathrm{d}_M(d_\varnothing, d_{\langle 0 \rangle}) \leq n$ for some $n < \omega$, so $\mathrm{d}_M(d_\varnothing, d_{\langle i \rangle}) \leq n + 1$ for all $i < \omega$. By thickness $\mathrm{d}_M(x_\varnothing, x_{\langle i \rangle}) \leq n + 1$ is type-definable over M, and this partial type is thus contained in the EM_s-type.

(iv) It captures that $e_\varnothing \models \bigcup_{\nu \trianglerighteq \varnothing} p(z, d_\nu)$.

We apply Fact 3.4(i) to find an s-indiscernible tree $(b^{\alpha+1}_\eta, c^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ over M that is EM_s-based over M on $(d_\eta, e_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$. By an automorphism and (i) we may assume that $b^{\alpha+1}_{\langle i \rangle ^\frown \eta} c^{\alpha+1}_{\langle i \rangle ^\frown \eta} = d_{\langle i \rangle ^\frown \eta} e_{\langle i \rangle ^\frown \eta} = b^\alpha_{\eta,i} c^\alpha_{\eta,i}$ for all $\eta \in \mathcal{T}_\alpha$ and $i < \omega$, and so (4) is satisfied. This then also implies that (3) is satisfied. Finally, (1) is satisfied because of (ii) and (iii) and (2) is satisfied because of (iv), in both cases combined with the induction hypothesis. This completes the inductive construction.

We thus have constructed a tree $(b^\lambda_\eta, c^\lambda_\eta)_{\eta \in \mathcal{T}_\lambda}$ satisfying (1)–(3). We now apply Corollary 3.6 to find a Morley tree $(b_\eta, c_\eta)_{\eta \in \mathcal{T}_\omega}$ over M str-Ls-based on $(b^\lambda_\eta, c^\lambda_\eta)_{\eta \in \mathcal{T}_\lambda}$

over M. Property (2) is clearly preserved under str-Ls-basing. To see that property (1) is preserved under str-Ls-basing we show that, for any $\eta \in \mathcal{T}_\omega$ and finite tuple $\bar{\nu}$ in $\mathcal{T}_\omega$, we have $b_\eta \models (q|_x)|_{M b_{\bar{\nu}} c_{\bar{\nu}}}$. Indeed, by str-Ls-basing we find $\gamma, \bar{\mu} \in \mathcal{T}_\omega$ such that $\gamma\bar{\mu}$ has the same str-quantifier-free type as $\eta\bar{\nu}$ and $b_\eta b_{\bar{\nu}} c_{\bar{\nu}} \equiv^{\mathrm{Ls}}_M b^\lambda_\gamma b^\lambda_{\bar{\mu}} c^\lambda_{\bar{\mu}}$. Let $\beta \models q|_x$, then we have by M-Ls-invariance of $q|_x$ that

$$b_\eta b_{\bar{\nu}} c_{\bar{\nu}} \equiv^{\mathrm{Ls}}_M b^\lambda_\gamma b^\lambda_{\bar{\mu}} c^\lambda_{\bar{\mu}} \equiv \beta b^\lambda_{\bar{\mu}} c^\lambda_{\bar{\mu}} \equiv^{\mathrm{Ls}}_M \beta b_{\bar{\nu}} c_{\bar{\nu}},$$

as required. So setting $(b_i, c_i) = (b_{\zeta_i}, c_{\zeta_i})$ for all $i < \omega$ we find our desired tree Morley sequence. □

Theorem 4.3 (independence theorem). *Let T be a thick NSOP$_1$ theory. Suppose that $a \equiv^{\mathrm{Ls}}_M a'$, $a \mathop{\smile\!\!\!\!\!\!\;\;|}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!\;\;|}^K_M c$ and $b \mathop{\smile\!\!\!\!\!\!\;\;|}^K_M c$. Then there is a'' with $a'' \equiv^{\mathrm{Ls}}_{Mb} a$, $a'' \equiv^{\mathrm{Ls}}_{Mc} a'$ and $a'' \mathop{\smile\!\!\!\!\!\!\;\;|}^K_M bc$.*

Proof. We now have all the tools in place to follow the proof of [Kaplan and Ramsey 2020, Theorem 6.5]. To get our conclusion about Lascar strong types, we apply the same trick as at the start of [Dobrowolski and Kamsma 2022, Theorem 7.7]: as described there we may assume b and c to enumerate λ_T-saturated e.c. models containing M. So we have reduced our goal to proving that, for $p_0(x, b) = \mathrm{tp}(a/Mb)$ and $p_1(x, c) = \mathrm{tp}(a'/Mc)$, the partial type $p_0(x, b) \cup p_1(x, c)$ does not Kim-divide over M.

Let $(b_i, c_i)_{i<\omega}$ and $q(x, y)$ be as in Lemma 4.2, and we may assume $b_1 c_1 = bc$. Let a'' be such that $a'' c_0 \equiv^{\mathrm{Ls}}_M a' c$, which can be done because $c = c_1 \equiv^{\mathrm{Ls}}_M c_0$. We then have $a \equiv^{\mathrm{Ls}}_M a''$ as well as $a \mathop{\smile\!\!\!\!\!\!\;\;|}^K_M b_1$, $a'' \mathop{\smile\!\!\!\!\!\!\;\;|}^K_M c_0$ and $b_1 \mathop{\smile\!\!\!\!\!\!\;\;|}^*_M c_0$, because $b_1 \models (q|_x)|_{Mc_0}$, so by [Dobrowolski and Kamsma 2022, Proposition 7.5] we have that $p_0(x, b_1) \cup p_1(x, c_0)$ does not Kim-divide over M. Since $(b_i, c_i)_{i<\omega}$ is a tree Morley sequence over M, we can apply both parts of Lemma 3.2 to see that $(b_{2i+1}, c_{2i})_{i<\omega}$ is a tree Morley sequence over M. Hence by Kim's lemma for tree Morley sequences (Lemma 3.3) we have that

$$\bigcup_{i<\omega} p_0(x, b_{2i+1}) \cup p_1(x, c_{2i})$$

is consistent. Thus

$$\bigcup_{i<\omega} p_0(x, b_{2i+1}) \cup p_1(x, c_{2i+2})$$

is consistent, as this is contained in the above set. Again, by Lemma 3.2, we have that $(b_{2i+1}, c_{2i+2})_{i<\omega}$ is a tree Morley sequence over M. Since $b_1 c_2 \equiv_M bc$ we thus have by Kim's lemma for tree Morley sequences (Lemma 3.3) again that $p_0(x, b) \cup p_1(x, c)$ does not Kim-divide over M, which finishes the proof. □

Acknowledgements

We would like to thank W. Kho [2023], who found the gap while working on his Master's thesis. We would also like to thank I. Kaplan and N. Ramsey for discussions leading to this proof.

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Received 15 Jan 2024. Revised 27 Mar 2024.

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Model Theory (ISSN 2832-904X electronic, 2832-9058 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

MT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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nonprofit scientific publishing

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Model Theory

no. 3 vol. 3 2024