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The classification of dp-minimal integral domains

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We classify dp-minimal integral domains, building off the existing classification of dp-minimal fields and dp-minimal valuation rings. We show that if R is a dp-minimal integral domain, then R is a field or a valuation ring or arises from the following construction: there is a dp-minimal valuation overring $\mathcal{O} \supseteq R$, a proper ideal I in $\mathcal{O}$, and a finite subring $R_0 \subseteq \mathcal{O}/I$ such that R is the preimage of R_0 in $\mathcal{O}$.

1. Introduction

A dp-minimal domain is an integral domain whose first order theory is of dp-rank 1; see Section 2 for the definition. In [d'Elbée and Halevi 2021], the first two authors studied under which algebraic conditions dp-minimal domains are in fact valuation rings. After noting that any such ring R must be local, the answer was that R is a valuation ring if and only if its residue field is infinite or its residue field is finite and its maximal ideal is principal. The third author gave a complete classification of dp-minimal fields and valuation rings in [Johnson 2023a] (see Section 2.2 for the statements of these results), and a classification of dp-minimal Noetherian domains in [Johnson 2023b].

In this paper we continue the arguments of [d'Elbée and Halevi 2021; Johnson 2023a; 2023b], leading to a classification of dp-minimal integral domains.

Theorem. *Let R be an integral domain. Then R is dp-minimal if and only if one of the following holds*:

(1) *R is a dp-minimal field.*

(2) *R is a dp-minimal valuation ring.*

(3) *There is a valuation subring $\mathcal{O}$ of $K = \mathrm{Frac}(R)$, a proper ideal $I \lhd \mathcal{O}$, and a finite subring R_0 of $\mathcal{O}/I$ such that R is the preimage of R_0 under the quotient map $\mathcal{O} \to \mathcal{O}/I$, and the valuation ring $(\mathcal{O}, +, \cdot)$ is dp-minimal.*

Moreover, *in* (3) *the ring $\mathcal{O}$*, *ideal I*, *and subring R_0 can be chosen to be definable in R.*

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See Facts 2.2 and 2.3 for the classification of dp-minimal fields and valuation rings.

The "if" direction of this theorem — rings satisfying (3) are dp-minimal — is the relatively easy Proposition 3.2. The "only if" direction — any dp-minimal domain arises from one of the three constructions — is the much harder Theorem 6.19.

Here are two typical examples of the construction in (3):

Example 1.1. Let $K = \mathbb{Q}_3(i)$, where $i = \sqrt{-1}$. Let $\mathcal{O}$ be the natural valuation ring on K, namely $\mathcal{O} = \mathbb{Z}_3[i]$. Then $\mathcal{O}/27\mathcal{O}$ is a finite ring extending $\mathbb{Z}/27\mathbb{Z}$. Taking $R_0 = \mathbb{Z}/27\mathbb{Z}$, we get a dp-minimal integral domain $R \subseteq \mathcal{O}$, consisting of those $x \in \mathcal{O}$ which are congruent mod 27 to an element of $\{0, 1, 2, \ldots, 26\}$. One can describe R more directly as $R = \mathbb{Z} + 27\mathcal{O}$. Note that R is not a valuation ring.

Example 1.2. Let K be the field of Hahn series $\mathbb{F}_3^{\mathrm{alg}}((t^{\mathbb{Q}}))$, a model of $\mathrm{ACVF}_{3,3}$. Let $\mathcal{O}$ be the natural valuation ring on K, and let I be the (nondefinable) ideal $I = \{x \in \mathcal{O} : v(x) > \pi\}$. There is a finite subring $R_0 \subseteq \mathcal{O}/I$ of size 3^7 given by

$$R_0 = \{a + bt + ct^2 + dt^3 + I : a \in \mathbb{F}_3,\ b, c, d \in \mathbb{F}_9\}.$$

This yields a dp-minimal integral domain $R \subseteq \mathcal{O}$. Explicitly, R is the set of $x \in \mathcal{O}$ such that there are $a \in \mathbb{F}_3$ and $b, c, d \in \mathbb{F}_9$ with $v(x - (a + bt + ct^2 + dt^3)) > \pi$. Again, R is not a valuation ring.

During the proof of our main theorem, we prove (or observe) the following results which might be interesting in their own right:

Proposition (Proposition 4.9). *If $\mathcal{O}$ is a $\vee$-definable valuation subring of some sufficiently saturated field, then $\mathcal{O}$ is externally definable.*

Proposition (Proposition 6.20). *Let $(K, +, \cdot, A)$ be a dp-minimal expansion of a field K by an infinite, proper additive subgroup $(A, +) \subseteq (K, +)$. Then there is a nontrivial definable valuation ring on K.*

Observation (Fact 4.18). If R is an NIP commutative ring, then $R^{00} = R^0$.

We also make some preliminary observations about more general dp-minimal commutative rings (not domains), showing that each is the product of a finite ring and a dp-minimal henselian local ring, and in dp-minimal local rings, the prime ideals are linearly ordered (Proposition 7.1).

1.1. ***Sketch of the proof.*** In order to prove our main theorem, there are two things to check. First, we must show that the construction in part (3) of the theorem really does give dp-minimal integral domains. The proof is relatively easy, using Shelah expansions, and is given in Section 3.

The much more difficult direction is to show that if R is a dp-minimal integral domain other than a field or valuation ring, then R arises via the construction in part (3). The proof of this fact occupies Sections 4–6. We call such rings R *exceptional dp-minimal domains*. The fraction field $K = \mathrm{Frac}(R)$ turns out to be a dp-minimal field. Using the classification of dp-minimal fields, we divide into three cases: K is either *ACVF-like*, *pCF-like*, or *RCVF-like* (see Section 5).

Corollary 5.6 in [d'Elbée and Halevi 2021] essentially says that R must be comparable with any definable valuation ring on K (see Fact 4.3). Using this, we can quickly rule out the RCVF-like case, at least when R is exceptional (Proposition 6.2).

Proposition 3.14 in [d'Elbée and Halevi 2021] gives a dichotomy for the behavior of $\sqrt{\mathfrak{m}^{00}}$, the radical of the 00-connected component of the maximal ideal $\mathfrak{m}$ of R: it is either $\mathfrak{m}$ itself or the unique second-largest prime ideal in R (see Proposition 4.14 and Lemma 4.16). We show that these two cases correspond exactly to the ACVF-like case and the pCF-like case, respectively (Proposition 6.3). Combining this with [d'Elbée and Halevi 2021, Corollary 5.6] again, we finish the pCF-like case in Proposition 6.7.

The ACVF-like case is the hardest to deal with. Assuming that $\mathrm{Frac}(R)$ is ACVF-like, say that R is a *good domain* if $\mathfrak{m}^{00}$ is definable, and a *rogue domain* otherwise (Definition 6.8). Good domains are almost trivial to deal with (see the proof of Theorem 6.19); all the work goes into showing that *rogue domains do not exist* (Proposition 6.18). In Section 6.3, we analyze rogue domains, proving enough facts about their structure to get a contradiction. The proof is technical, but involves showing that the set $\mathcal{O} = \mathfrak{m}^{00} : \mathfrak{m}^{00} = \{x \in K : x\mathfrak{m}^{00} \subseteq \mathfrak{m}^{00}\}$ is the coarsest externally definable valuation ring on K, and studying the interactions between R and $\mathcal{O}$.

The proofs in Section 6 depend on a number of technical tools, which we collect in Section 4:

- In Section 4.1 we review some basic facts about NIP and dp-minimal domains from [d'Elbée and Halevi 2021; Johnson 2022].
- In Section 4.2 we show certain prime ideals in R are 00-connected.
- In Section 4.3 we show that $\vee$-definable valuation rings are externally definable, and use this to show that R^{00} is externally definable when R is a dp-minimal domain.
- In Section 4.4 we strengthen and improve the $\sqrt{\mathfrak{m}^{00}}$ dichotomy from [d'Elbée and Halevi 2021, Proposition 3.14], leveraging the earlier results on 00-connectedness and external definability.
- In Section 4.5 we use tools from topological ring theory to observe that $R^{00} = R^0$ and analyze the structure of R/R^{00} in certain cases using the classification of locally compact fields.
- In Section 4.6 we show that any dp-minimal domain R with an infinite definable additive proper subgroup also admits an infinite definable proper subring

(Proposition 4.22). We subsequently strengthen this to get an externally definable nontrivial valuation ring (Proposition 6.6) and ultimately a definable nontrivial valuation ring (Proposition 6.20).

2. Preliminaries

2.1. ***Notation.*** We employ fairly standard model theoretic notation and conventions; see [Tent and Ziegler 2012; Simon 2015].

We briefly review the definition of dp-rank and dp-minimality. If $\Sigma(x)$ is a partial type and κ is a cardinal, an *ict-pattern* of depth κ in $\Sigma(x)$ is an array of formulas $(\varphi_\alpha(x, b_{\alpha,i}) : \alpha < \kappa,\ i < \omega)$ over an elementary extension, such that for every function $\eta : \kappa \to \omega$, the type

$$\Sigma(x) \cup \{\varphi_\alpha(x, b_{\alpha,i}) : i = \eta(\alpha)\} \cup \{\neg\varphi_\alpha(x, b_{\alpha,i}) : i \neq \eta(\alpha)\}$$

is consistent. The *dp-rank of* $\Sigma(x)$, written dp-rk$(\Sigma(x))$, is the supremum of cardinals κ such that there is an ict-pattern of depth κ in $\Sigma(x)$. (This definition coincides with the alternative definition of dp-rank using indiscernible sequences; see [Kaplan et al. 2013, Lemma 2.5, Proposition 2.6].) The dp-rank of a definable set D, written dp-rk(D), is the dp-rank of the formula defining D. A one-sorted structure M is *dp-minimal* if dp-rk$(M) = 1$. In particular, dp-rk$(M) > 0$, so M must be infinite, with our definition of "dp-minimal". A one-sorted theory is dp-minimal if its models are dp-minimal.

By a ring we mean a (possibly trivial) commutative ring with identity. An ideal in R can be the improper ideal R. A "maximal ideal" means a maximal proper ideal. A local ring is a ring with a unique maximal ideal, possibly a field. Valuations can be trivial, so valuation rings can be fields.

If $M = (R, +, \cdot, \dots)$ is an expansion of an integral domain, then the structures M and $(\mathrm{Frac}(R), R, +, \cdot, \dots)$ are obviously interdefinable in M^{eq}, so for example saturation and definability pass from one to the other. We mostly do not distinguish between the two structures.

We assume the reader is familiar with valued fields. Given a valued field (K, v) we usually denote its value group by Γ and its residue field by k. Any cut in the value group Γ is of the form $\{x \in \Gamma : x > \gamma\}$ or $\{x \in \Gamma : x \geq \gamma\}$ for some $\gamma \in \Gamma' \succ \Gamma$. Thus any ideal of the valuation ring is of the form $\{x \in K : v(x) \,\square\, \gamma\}$, where $\square \in \{>, \geq\}$. An ideal I is principal if and only if $I = \{x \in K : v(x) \geq \gamma\}$ for some $\gamma \in \Gamma$.

We do not assume that we are working in a highly saturated monster model, unless stated otherwise.

2.2. ***Dp-minimal valuation rings and fields.*** We recall the classification of dp-minimal valuation rings and fields given in [Johnson 2023a]. First recall the following.

Fact 2.1 [Jahnke et al. 2017, Proposition 5.1]. *Let* $(\Gamma, \leq, +)$ *be a nontrivial ordered abelian group. Then* Γ *is dp-minimal if and only if* $|\Gamma/n\Gamma|$ *is finite for all* $n > 0$.

For example, $\mathbb{Z}$ and $\mathbb{R}$ are dp-minimal as ordered abelian groups.

Fact 2.2. *An infinite field* $(K, +, \cdot)$ *is dp-minimal if and only if there is a henselian defectless valuation ring* $\mathcal{O} \subseteq K$ *with maximal ideal* $\mathfrak{m}$ *such that*:

(1) *The value group* $\Gamma := K^\times/\mathcal{O}^\times$ *is trivial or dp-minimal as an ordered abelian group.*

(2) *The residue field* $k := \mathcal{O}/\mathfrak{m}$ *is algebraically closed*, *real closed*, *or* p*-adically closed for some prime* p.

(3) *If the residue field* k *is algebraically closed of characteristic* $p > 0$, *then the interval* $[-v(p), v(p)] \subseteq \Gamma$ *is* p*-divisible*, *where* $v(p) = +\infty$ *when* $\mathrm{char}(K) = p$.

In (2), we mean "p-adically closed" in the broad sense: K is p-adically closed if $K \equiv K'$ for some finite extension $K'/\mathbb{Q}_p$. Fact 2.2 is proved in [Johnson 2023a, Theorem 7.3], but only under the extra assumption that K is sufficiently saturated(!). For completeness, we have included a proof in the unsaturated case in the Appendix.

Next, dp-minimal valuation rings[1] are classified as follows:

Fact 2.3 [Johnson 2023a, Theorems 1.5 and 1.6]. *Let* $\mathcal{O}$ *be an infinite valuation ring with fraction field* K, *maximal ideal* $\mathfrak{m}$, *residue field* $k = \mathcal{O}/\mathfrak{m}$, *and value group* $\Gamma = K^\times/\mathcal{O}^\times$. *Then* $(\mathcal{O}, +, \cdot)$ *is dp-minimal if and only if the following conditions hold*:

(1) $(k, +, \cdot)$ *and* $(\Gamma, +, \leq)$ *are dp-minimal or finite.*

(2) $\mathcal{O}$ *is henselian and defectless.*

(3) *One of the following cases holds*:

- k *is finite of characteristic* $p > 0$, K *has characteristic* 0, *and the interval* $[-v(p), v(p)] \subseteq \Gamma$ *is finite.*
- k *is infinite of characteristic* $p > 0$, *and the interval* $[-v(p), v(p)] \subseteq \Gamma$ *is* p*-divisible.*
- k *has characteristic* 0.

For example, $\mathbb{Z}_p$ and $\mathbb{Q}_p[\![t]\!]$ are dp-minimal valuation rings, but $\mathbb{F}_p[\![t]\!]$ is not.

[1]The citation is to the classification of dp-minimal *valued fields* $(K, \mathcal{O})$ rather than dp-minimal *valuation rings* $\mathcal{O}$, but the distinction does not matter because the two structures $\mathcal{O}$ and $(K, \mathcal{O})$ have the same dp-rank by Fact 4.2 below. Therefore, a valuation ring $\mathcal{O}$ is dp-minimal if and only if the corresponding valued field $(K, \mathcal{O})$ is dp-minimal.

3. Constructing dp-minimal integral domains

Recall that if M is a structure, the *Shelah expansion* M^{Sh} is the expansion of M by all externally definable sets. If M is NIP, then the definable sets in M^{Sh} are precisely the externally definable sets in M, and M^{Sh} is also NIP [Shelah 2009; Simon 2015, Section 3.3].

The following fact is well known [Onshuus and Usvyatsov 2011, Observation 3.8], but we had trouble understanding the proof, so we include a more detailed proof for completeness:

Fact 3.1. *M^{Sh} has the same dp-rank as M, and in fact, if $D \subseteq M^n$ is definable, then D has the same dp-rank whether considered in M or in M^{Sh}.*

Proof. Let L and L^{Sh} be the languages of M and M^{Sh}. Let $\mathbb{M}^*$ be a monster model elementary extension of M^{Sh}, and let $\mathbb{M}$ be its reduct to L. (Note that $\mathbb{M}^*$ is probably not the Shelah expansion of $\mathbb{M}$.) Let $\psi(x, c)$ be the $L(M)$-formula defining D in M and M^{Sh}. Suppose dp-rk$(\psi(x, c)) \geq \kappa$ in $\mathbb{M}^*$. We must show dp-rk$(\psi(x, c)) \geq \kappa$ in $\mathbb{M}$. By definition of dp-rank, there are L^{Sh}-formulas $\varphi_\alpha(x, y)$, elements a_η in $\mathbb{M}$ for $\eta : \kappa \to \omega$, and elements $b_{\alpha,i}$ in $\mathbb{M}$ for $\alpha < \kappa$ and $i < \omega$, such that

$$\mathbb{M} \models \psi(a_\eta, c) \text{ for each } \eta,$$
$$\mathbb{M}^* \models \varphi_\alpha(a_\eta, b_{\alpha,i}) \iff \eta(\alpha) = i \text{ for each } \eta, \alpha, i.$$

Each L^{Sh}-formula $\varphi_\alpha(x, y)$ is equivalent *on the small set M* to an L-formula $\theta_\alpha(x, y, e_\alpha)$, with e_α in $\mathbb{M}$. Moving the e_α by an automorphism in Aut$(\mathbb{M}^*/M)$, we may assume that $\mathrm{tp}^{L^{\mathrm{Sh}}}(\bar{a}\bar{b}/M\bar{e})$ is finitely satisfiable in M. For any η, α, i, the L^{Sh}-formula

$$\neg(\theta_\alpha(x, y, e_\alpha) \leftrightarrow \varphi_\alpha(x, y))$$

is not satisfiable in M, so $(a_\eta, b_{\alpha,i})$ cannot satisfy it, and so

$$\mathbb{M}^* \models \theta_\alpha(a_\eta, b_{\alpha,i}, e_\alpha) \leftrightarrow \varphi_\alpha(a_\eta, b_{\alpha,i}).$$

Then

$$\mathbb{M} \models \theta_\alpha(a_\eta, b_{\alpha,i}, e_\alpha) \iff \eta(\alpha) = i,$$

and we have an ict-pattern of depth κ in D, in the original language L, showing that dp-rk$(D) \geq \kappa$ in M. □

In light of Fact 3.1, the following is nearly trivial:

Proposition 3.2. *Let $\mathcal{O}$ be a dp-minimal valuation ring, let $I \subseteq \mathcal{O}$ be a proper ideal (not necessarily definable), let R_0 be a finite subring of $\mathcal{O}/I$, and let $R \subseteq \mathcal{O}$ be the preimage of R_0 under the quotient map $\mathcal{O} \to \mathcal{O}/I$. Then R is a dp-minimal integral domain.*

Proof. Let K be the fraction field of $\mathcal{O}$, let Γ be the value group, and let $v : K \to \Gamma$ be the valuation. As an ideal in a valuation ring, I must have the form $\{x \in K : v(x) > \xi\}$ for some ξ in an elementary extension of Γ. The set $\{\gamma \in \Gamma : \gamma > \xi\}$ is externally definable, so it is definable in the Shelah expansion of $\mathcal{O}$. Then $\mathcal{O}/I$, R_0, and R are all interpretable in the Shelah expansion. It follows that R is externally definable in $\mathcal{O}$. Then $(\mathcal{O}, +, \cdot, R)$ is a reduct of the Shelah expansion $\mathcal{O}^{\mathrm{Sh}}$, so $(\mathcal{O}, +, \cdot, R)$ is dp-minimal, implying that $(R, +, \cdot)$ is dp-minimal (as $R \subseteq \mathcal{O}$). □

Remark 3.3. For which dp-minimal valuation rings $\mathcal{O}$ can we find an ideal I and a finite subring $R_0 \subseteq \mathcal{O}/I$? To begin with, $\mathcal{O}$ must have positive residue characteristic, or else $\mathcal{O}$ and $\mathcal{O}/I$ are $\mathbb{Q}$-algebras, which have no finite subrings. If the residue characteristic of $\mathcal{O}$ is $p > 0$, then there are two possibilities for $\mathcal{O}$, by Fact 2.3(3):

(1) The residue field of $\mathcal{O}$ is finite. Then $\mathcal{O}$ has mixed characteristic and is finitely ramified. The quotient $\mathcal{O}/I$ can only have positive characteristic if the ideal I contains $p^n\mathcal{O}$ for some n. Then R is the pullback of some subring of $\mathcal{O}/p^n\mathcal{O}$. Conversely, every subring of $\mathcal{O}/p^n\mathcal{O}$ gives a possibility for R, because $\mathcal{O}/p^n\mathcal{O}$ is finite. Example 1.1 is a typical example.

(2) The residue field of $\mathcal{O}$ can be infinite, such as a model of ACF_p. Example 1.2 is a typical example. When $\mathcal{O}$ has equicharacteristic p, there is at least one finite subring of $\mathcal{O}/I$ for any proper ideal I, namely $\mathbb{F}_p \subseteq \mathcal{O}/I$. When $\mathcal{O}$ has mixed characteristic, the ideal I must contain $p^n\mathcal{O}$ for some n, but there are many such ideals I because the convex hull of $\mathbb{Z} \cdot v(p)$ is p-divisible by Fact 2.3(3).

Over the course of the paper, we will see that these two cases behave very differently. For example, note that R has finite index in $\mathcal{O}$ in the first case, but infinite index in $\mathcal{O}$ in the second case (because $\mathcal{O}/I$ is infinite). In the first case, R is definable in the structure $(\mathcal{O}, +, \cdot)$, but in the second case this can fail (take I nondefinable). Later, we will see that R/R^{00} is infinite in the first case, but finite in the second case (Proposition 6.3).

4. Tools

We review some known and new results on NIP and dp-minimal rings and domains. Unless specified otherwise, any ring below is not assumed to be pure, i.e., there might be some more structure.

4.1. ***Basic facts about NIP and dp-minimal domains.*** Every NIP ring has finitely many maximal ideals [d'Elbée and Halevi 2021, Proposition 2.1] and if it is a dp-minimal domain then its prime ideals are linearly ordered by inclusion and so it is either a field or a local ring [d'Elbée and Halevi 2021, Corollaries 2.3 and 2.5].[2]

[2] Recall that in NIP theories dp-minimality and inp-minimality coincide.

Fact 4.1. *Any prime ideal $\mathfrak{p}$ in a ring R is externally definable.*

When R is NIP, this is [Johnson 2022, Proposition 2.13], but NIP turns out to be unnecessary.

Proof. Let $\Sigma(x)$ be the partial type $\{a \mid x : a \in R \setminus \mathfrak{p}\} \cup \{b \nmid x : b \in \mathfrak{p}\}$. Then $\Sigma(x)$ is finitely satisfiable — any finite subtype

$$\{a_1 \mid x,\ a_2 \mid x, \ldots,\ a_n \mid x,\ b_1 \nmid x,\ b_2 \nmid x, \ldots,\ b_m \nmid x\} \subseteq \Sigma(x)$$

is realized by the product $a_1 a_2 \cdots a_n$. If $R' \succ R$ is an elementary extension containing an element c realizing $\Sigma(x)$, then the formula $x \nmid c$ defines a set whose trace on R is $\mathfrak{p}$. □

As a consequence, if $(R, +, \cdot, \ldots)$ is an NIP ring and $\mathfrak{p}$ is any prime ideal, then the expansion $(R, +, \cdot, \ldots, \mathfrak{p})$ is NIP of the same dp-rank (Fact 3.1). The bi-interpretable structure $(R_\mathfrak{p}, R, +, \cdot, \ldots)$ is also NIP, and in fact has the same dp-rank as R by the following:

Fact 4.2. *If R is a definable ring and $\mathfrak{p}$ is a definable prime ideal in some structure, then the definable ring $R_\mathfrak{p}$ has the same dp-rank as R. In particular, if R is an integral domain then* $\mathrm{Frac}(R)$ *has the same dp-rank as R.*

This is essentially [d'Elbée and Halevi 2021, Proposition 2.8(2)] (purity of the structure R is inessential), or [Johnson 2020, Lemma 10.25] for the case of $\mathrm{Frac}(R)$.

Fact 4.3. *Let $M = (K, R, \mathcal{O}, +, \cdot, \ldots)$ be an expansion of a field together with a predicate for a subring and a valuation ring, with* $\mathrm{Frac}(R) = \mathrm{Frac}(\mathcal{O}) = K$. *If M is dp-minimal then either $R \subseteq \mathcal{O}$ or $\mathcal{O} \subseteq R$.*

This is essentially [d'Elbée and Halevi 2021, Corollary 5.6]. Again, purity of the structure R is inessential.

4.2. *00-connectedness of prime ideals.* For any type-definable group G in a sufficiently saturated NIP structure we let G^{00} be its 00-connected component [Simon 2015, Section 8.1.3].

Remark 4.4. Suppose G is an NIP group, possibly with additional structure. The following facts are well known:

- If G is sufficiently saturated, then G^{00} is definable if and only if G^{00} has finite index in G.
- If G is sufficiently saturated and $\widehat{G} \succ G$ is sufficiently saturated, then G^{00} is definable if and only if $(\widehat{G})^{00}$ is definable.
- We can talk about "G^{00} is definable" without assuming saturation, by going to a sufficiently saturated elementary extension. The choice of the sufficiently saturated elementary extension does not matter by the previous point.

- If $G \equiv H$, then G^{00} is definable if and only if H^{00} is definable.
- If G^{00} is definable and H is a reduct of G then H^{00} is definable. Note that the converse fails, though — it can happen that G^{00} is not definable, but becomes definable in a reduct. For example, if G is the circle group in RCF, then G^{00} is not definable, but it becomes definable in the pure group reduct.

More generally, analogous results hold when G is a definable group in an NIP structure M.

From now on, whenever G is some NIP expansion of a group, by "G^{00} is definable" we mean that $(\widehat{G})^{00}$ is definable for some/every sufficiently saturated elementary extension $\widehat{G} \succ G$.

Fact 4.5 [Johnson 2021, proof of Lemma 2.6]. (1) *Let R be a sufficiently saturated NIP integral domain and $I \trianglelefteq R$ a type-definable ideal. Then I^{00} is also an ideal of R.*

(2) *Let R be a sufficiently saturated NIP local domain with maximal ideal $\mathfrak{m}$. If $R/\mathfrak{m}$ is infinite then $\mathfrak{m} = \mathfrak{m}^{00}$ and more generally, $I = I^{00}$ for any definable ideal I.*

We prove a similar result for $\vee$-definable valuation rings.

Proposition 4.6. *Let K be a sufficiently saturated NIP field and $\mathcal{O}$ some $\vee$-definable valuation subring with maximal ideal $\mathfrak{m}$. Then $\mathfrak{m}$ is type-definable and unless $\mathfrak{m}$ is principal and $\mathcal{O}/\mathfrak{m}$ is finite, $\mathfrak{m}^{00} = \mathfrak{m}$.*

Proof. For nonzero $x \in K$, we have $x \in \mathfrak{m}$ if and only if $x^{-1} \notin \mathcal{O}$, so $\mathfrak{m}$ is type-definable and $\mathfrak{m}^{00}$ is an ideal in $\mathcal{O}$. Let v be the valuation on $K = \operatorname{Frac}(\mathcal{O})$ associated to $\mathcal{O}$.

Now, we have two cases:

Case 1: There is a minimal positive element in the value group of $\mathcal{O}$. Then $\mathfrak{m}$ is a principal ideal $t\mathcal{O}$, where t is a local uniformizer. The fact that $\mathfrak{m} = t\mathcal{O}$ means that $\mathfrak{m}$ is $\vee$-definable. It follows that $\mathcal{O}$ and $\mathfrak{m}$ are definable. As $\mathcal{O}/\mathfrak{m}$ is infinite in this situation, $\mathfrak{m} = \mathfrak{m}^{00}$ by Fact 4.5(2).

Case 2: There is no minimal positive element in the value group of $\mathcal{O}$. Suppose for the sake of contradiction that $\mathfrak{m} \neq \mathfrak{m}^{00}$. Take an element $a \in \mathfrak{m} \setminus \mathfrak{m}^{00}$, so $v(a) > 0$. If $x \in \mathfrak{m}^{00}$, then $x\mathcal{O} \subseteq \mathfrak{m}^{00}$ so $a \notin x\mathcal{O}$, implying that $x \in a\mathcal{O}$. This shows that $\mathfrak{m}^{00} \subseteq a\mathcal{O}$.

Because there is no minimal positive element in the value group, we can find a_1 with $0 < v(a_1) < v(a)$. Then find a_2 with $0 < v(a_2) < v(a_1)$. Continuing in this way, we can find a sequence

$$a = a_0, a_1, a_2, a_3, \ldots$$

with $v(a_0) > v(a_1) > v(a_2) > \cdots > 0$. Equivalently, there is a sequence $a_0, a_1, a_2, \ldots$ with

- $a_0 = a$,
- $a_i/a_j \in \mathfrak{m}$ for $i < j$,
- $a_i \in \mathfrak{m}$ for all i.

Since $\mathfrak{m}$ is type-definable, these conditions are type-definable, and we can use compactness to get a long sequence $\{a_i\}_{i<\kappa}$ satisfying the conditions above. We get a very long ascending chain of subgroups between $\mathfrak{m}^{00}$ and $\mathfrak{m}$:

$$\mathfrak{m}^{00} \subseteq a_0\mathcal{O} \subsetneq a_1\mathcal{O} \subsetneq a_2\mathcal{O} \subsetneq \cdots \subseteq \mathfrak{m}.$$

But the total number of abstract groups between $\mathfrak{m}^{00}$ and $\mathfrak{m}$ is $2^{|\mathfrak{m}/\mathfrak{m}^{00}|}$, so by taking $\kappa > 2^{|\mathfrak{m}/\mathfrak{m}^{00}|}$ we get a contradiction. □

Corollary 4.7. *Let R be a sufficiently saturated dp-minimal domain. For any nonmaximal type-definable prime ideal $\mathfrak{p}$, $\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{p}$, $R_{\mathfrak{p}}$ is a henselian valuation ring with maximal ideal $\mathfrak{p}$ and it satisfies $\mathfrak{p}^{00} = \mathfrak{p}$.*

Proof. The fact that $R_{\mathfrak{p}}$ is a henselian valuation ring with maximal ideal $\mathfrak{p}$ is [d'Elbée and Halevi 2021, Proposition 3.8, Theorem 3.9]. To show connectedness we use Proposition 4.6: $R_{\mathfrak{p}}$ is $\vee$-definable since $a \in R_{\mathfrak{p}}$ if and only if $a^{-1} \notin \mathfrak{p}$, so we only need to show that $R_{\mathfrak{p}}/\mathfrak{p}$ is not finite. Otherwise, $R/\mathfrak{p}$ injects into the field $R_{\mathfrak{p}}/\mathfrak{p}$, so the former is a finite integral domain, i.e., a field, contradicting nonmaximality of $\mathfrak{p}$. □

If R is dp-minimal and $a \in R$ is nonzero, then there is a unique maximal prime ideal P of R with the property $a \notin P$ [d'Elbée and Halevi 2021, Lemma 3.3]. We denote this prime ideal P_a.

Corollary 4.8. *If R is a sufficiently saturated dp-minimal domain and $a \in R$ is nonzero and not a unit, then P_a is a nonmaximal type-definable prime ideal, and $P_a^{00} = P_a$.*

Proof. Since a is not a unit, it is in the maximal ideal, so P_a must be nonmaximal. By [d'Elbée and Halevi 2021, Proposition 3.8 and Theorem 3.9], R_{P_a} is a valuation ring with maximal ideal P_a. By [d'Elbée and Halevi 2021, Remark 3.4], R_{P_a} is the localization $S^{-1}R$ with $S = \{a^n : n \in \mathbb{N}\}$. Then $R_{P_a} = \bigcup_{n=0}^{\infty} a^{-n}R$, so R_{P_a} is $\vee$-definable. The maximal ideal of a $\vee$-definable valuation ring is type-definable, so P_a is type-definable. The 00-connectedness of P_a then follows by Corollary 4.7. □

4.3. *External definability of $\vee$-definable valuation rings and R^{00}.*

Proposition 4.9. *Let $\mathcal{O}$ be a $\vee$-definable valuation subring of some sufficiently saturated field K. Then $\mathcal{O}$ is externally definable. If K is NIP, then every ideal of $\mathcal{O}$ is externally definable.*

Proof. As observed in the proof of Proposition 4.6, the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ is type-definable. By compactness, there is some definable set D with $\mathfrak{m} \subseteq D \subseteq \mathcal{O}$.

Let $\Sigma(x)$ be the set of formulas

$$\Sigma(x) = \{x \in aD : a \notin \mathcal{O}\} \cup \{x \notin bD : b \in \mathcal{O}\}.$$

First suppose that $\Sigma(x)$ is finitely satisfiable. Then there is an element c realizing $\Sigma(x)$ in an elementary extension N of the original model. Then

$$a \notin \mathcal{O} \implies c \in aD(N) \implies a \in cD(N)^{-1},$$
$$b \in \mathcal{O} \implies c \notin bD(N) \implies b \notin cD(N)^{-1}.$$

Then $\mathcal{O}$ is externally definable as the complement of $M \cap cD(N)^{-1}$.

So we may assume $\Sigma(x)$ is not finitely satisfiable. Then there are $a_1, \dots, a_n \notin \mathcal{O}$ and $b_1, \dots, b_m \in \mathcal{O}$ such that

$$\bigcap_{i=1}^{n} a_i D \subseteq \bigcup_{i=1}^{m} b_i D.$$

Note that $b_i D \subseteq \mathcal{O}$ for each i, because $b_i \in \mathcal{O}$ and $D \subseteq \mathcal{O}$. Conversely, for v the valuation associated to $\mathcal{O}$, if $x \in \mathcal{O}$ then $v(x) \geq 0$ and $v(a_i) < 0$ (for each i), so $v(x/a_i) > 0$ and $x/a_i \in \mathfrak{m} \subseteq D$, implying that $x \in a_i D$. So $\mathcal{O} \subseteq a_i D$ for each i. Then

$$\mathcal{O} \subseteq \bigcap_{i=1}^{n} a_i D \subseteq \bigcup_{i=1}^{m} b_i D \subseteq \mathcal{O}.$$

Then equality holds, so $\mathcal{O}$ equals the definable set $\bigcap_{i=1}^{n} a_i D$.

The final statement follows since expanding the structure by the externally definable set $\mathcal{O}$ does not change the set of externally definable sets, assuming NIP. Ideals of definable valuation rings are externally definable. □

Lemma 4.10. *Let R be a sufficiently saturated dp-minimal domain with fraction field K. Define*

$$\mathcal{O} = R^{00} : R^{00} := \{x \in K : xR^{00} \subseteq R^{00}\} \subseteq K.$$

Then $\mathcal{O}$ is a valuation ring. Further, if $a \in K$, then $a \in \mathcal{O}$ if and only if $aR/(aR \cap R)$ is finite. In particular, $\mathcal{O} = R^{00} : R^{00}$ is a $\vee$-definable and externally definable valuation ring.

Proof. The proof that $R^{00} : R^{00}$ is a valuation ring is similar to the proof of [d'Elbée and Halevi 2021, Proposition 3.14(1)].

Claim 4.11. *$R^{00} : R^{00}$ is a valuation ring.*

Proof. It is routine to show that $R^{00} : R^{00}$ is a ring, so it is sufficient to prove that it is a valuation ring, i.e., for all $a \in K$, $aR^{00} \subseteq R^{00}$ or $a^{-1}R^{00} \subseteq R^{00}$. Let $a \in K$. By

[Chernikov et al. 2015, Proposition 4.5], $aR^{00}/(R^{00} \cap aR^{00})$ or $R^{00}/(R^{00} \cap aR^{00})$ is small. Neither R^{00} nor aR^{00} has any type-definable subgroups of small index, so either $aR^{00} = R^{00} \cap aR^{00}$ or $R^{00} = R^{00} \cap aR^{00}$. Equivalently, $aR^{00} \subseteq R^{00}$ or $R^{00} \subseteq aR^{00}$, i.e., $a^{-1}R^{00} \subseteq R^{00}$. □

If $a \in \mathcal{O}$ then by definition $aR^{00} \subseteq R^{00}$. It follows that $aR^{00} \subseteq aR \cap R \subseteq aR$, so $aR \cap R$ has bounded, hence finite index in aR. For the other direction, for $a \in K$, if $aR \cap R$ has finite index in aR then $(aR)^{00} \subseteq aR \cap R \subseteq aR$. In particular, $(aR)^{00} \subseteq R$. However, $(aR)^{00} = aR^{00}$ so $aR^{00} \subseteq R$ and $aR^{00} = (aR^{00})^{00} \subseteq R^{00}$. It follows that $\mathcal{O}$ is defined by the disjunction of conditions of the form "$aR/(aR \cap R)$ is finite". By Proposition 4.9, $\mathcal{O}$ is externally definable. □

Corollary 4.12. *Let R be a sufficiently saturated dp-minimal domain with fraction field $K = \mathrm{Frac}(R)$ and let $\mathcal{O} = R^{00} : R^{00}$. Then R^{00} is an externally definable ideal of $\mathcal{O}$, and the ring R/R^{00} is either finite or dp-minimal.*

If, furthermore, R^{00} is a nonmaximal prime ideal of R, then R^{00} is the maximal ideal of $\mathcal{O}$ and $\mathcal{O}$ is equal to the localization of R at R^{00}.

Proof. If R is a field then $R = R^{00}$ and we have nothing to show; so assume otherwise. Then R^{00} is an ideal of the valuation ring $\mathcal{O}$, so by Lemma 4.10 and Proposition 4.9, it is also externally definable.

Thus, adding R^{00} as a predicate preserves dp-minimality (Fact 3.1), so R/R^{00} is either finite or dp-minimal as well.

If R^{00} is a nonmaximal prime ideal $\mathfrak{p}$ of R then by Corollary 4.7 the localization $R_{\mathfrak{p}}$ is a valuation ring with maximal ideal $\mathfrak{p}$. Note that $\mathcal{O}' = \mathfrak{m}' : \mathfrak{m}'$ for any valuation ring $(\mathcal{O}', \mathfrak{m}')$, and so

$$\mathcal{O} := R_{00} : R_{00} = \mathfrak{p} : \mathfrak{p} = R_{\mathfrak{p}}.$$

Thus $\mathcal{O}$ is the localization $R_{\mathfrak{p}}$ and R^{00} is the maximal ideal of $\mathcal{O}$. □

4.4. ***The $\sqrt{\mathfrak{m}^{00}}$ dichotomy.*** We observe that the proof of [d'Elbée and Halevi 2021, Proposition 3.14] had a gap[3], which we correct now.

Fact 4.13. *Let G be an abelian dp-minimal group in a structure M.*

(1) *If M is sufficiently saturated and X, Y are type-definable subgroups of G, then $X/(X \cap Y)$ or $Y/(X \cap Y)$ is small (relative to the degree of saturation).*

(2) *If X, Y are definable subgroups of G, then $X/(X \cap Y)$ or $Y/(X \cap Y)$ is finite.*

(3) *If X, Y are externally definable subgroups of G, then $X/(X \cap Y)$ or $Y/(X \cap Y)$ is finite.*

[3] In the proof of [d'Elbée and Halevi 2021, Proposition 3.14], we misunderstood the notation "$< \infty$" in [Kaplan and Shelah 2013, Proposition 3.12; Chernikov et al. 2015, Proposition 4.5(2)] to mean "finite", when it in fact meant "small".

Part (1) is [Chernikov et al. 2015, Proposition 4.5(2)]. It easily implies (2), which then implies (3) by passing to the Shelah expansion.

Proposition 4.14 [d'Elbée and Halevi 2021, Proposition 3.14]. *Let R be a sufficiently saturated dp-minimal domain with maximal ideal $\mathfrak{m}$ and fraction field K. Then*

(1) $\mathfrak{m}^{00} : \mathfrak{m}^{00}$ *is a valuation overring of R*;

(2) $\sqrt{\mathfrak{m}^{00}} = \{a \in \mathfrak{m} \mid R/aR \text{ is infinite}\}$;

(3) *for every $a \in \mathfrak{m}$ with R/aR finite*, $\sqrt{\mathfrak{m}^{00}} = P_a$.

In particular, *exactly one of the following holds*:

- *R/aR is infinite for all $a \in \mathfrak{m}$ and in this case $\sqrt{\mathfrak{m}^{00}} = \mathfrak{m}$.*
- *There is $a \in \mathfrak{m}$ with R/aR finite*, *and for any such a*, $\sqrt{\mathfrak{m}^{00}} = P_a$.

Proof. Before beginning the proof, we make a general observation about local rings: if $I \subseteq J \subsetneq R$ are proper ideals in a local ring R, and J/I is finite, then $J \subseteq \sqrt{I}$. Indeed, for any $b \in J$, the pigeonhole principle gives $b^n \equiv b^m \pmod{I}$ for some $n < m < \omega$. Then $b^n(1 - b^{m-n}) \in I$, implying that $b^n \in I$, because $1 - b^{m-n}$ is a unit. In particular, if $\mathfrak{p} \subsetneq \mathfrak{q}$ are prime ideals, then $\mathfrak{q}/\mathfrak{p}$ is infinite, or else $\mathfrak{q} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$. (Compare with [d'Elbée and Halevi 2021, Remark 3.7].)

(1) It is routine to show that $\mathfrak{m}^{00} : \mathfrak{m}^{00}$ is a ring, so it is sufficient to prove that it is a valuation ring, i.e., for all $a \in K$, $a\mathfrak{m}^{00} \subseteq \mathfrak{m}^{00}$ or $a^{-1}\mathfrak{m}^{00} \subseteq \mathfrak{m}^{00}$. This follows immediately by the same argument as in Claim 4.11.

(2) If $R/\mathfrak{m}$ is infinite then $\mathfrak{m} = \mathfrak{m}^{00}$ by Fact 4.5 and the equality is obvious. We assume that $R/\mathfrak{m}$ is finite and show that $a \in \mathfrak{m} \setminus \sqrt{\mathfrak{m}^{00}}$ if and only if R/aR is finite. When $R/\mathfrak{m}$ is finite, note that $\mathfrak{m}^{00} = R^{00}$, and so $\mathfrak{m}^{00}$ is externally definable by Corollary 4.12.

Let $a \in \mathfrak{m} \setminus \sqrt{\mathfrak{m}^{00}}$. By Fact 4.13(3) we have that $aR/(\mathfrak{m}^{00} \cap aR)$ is finite or $\mathfrak{m}^{00}/(\mathfrak{m}^{00} \cap aR)$ is finite. If it was the former, then by the remarks about local rings at the start of the proof,

$$a \in aR \subseteq \sqrt{\mathfrak{m}^{00} \cap aR} \subseteq \sqrt{\mathfrak{m}^{00}},$$

a contradiction. Consequently, $\mathfrak{m}^{00}/(\mathfrak{m}^{00} \cap aR)$ is finite. By the definition of $\mathfrak{m}^{00}$, we have $\mathfrak{m}^{00} \subseteq aR$ and so $|\mathfrak{m}/aR|$ is finite, which implies that $|R/aR| = |R/\mathfrak{m}|\,|\mathfrak{m}/aR|$ is finite as well.

Conversely, if R/aR is finite, then

$$\mathfrak{m}^{00} = R^{00} = (aR)^{00} = aR^{00} = a\mathfrak{m}^{00}.$$

Suppose for the sake of contradiction that $a \in \sqrt{\mathfrak{m}^{00}}$. Take n minimal such that $a^n \in \mathfrak{m}^{00}$. Then $a^n \in \mathfrak{m}^{00} = a\mathfrak{m}^{00}$, implying $a^{n-1} \in \mathfrak{m}^{00}$, which contradicts the choice of n.

(3) Assume that R/aR is finite. By (2), $a \notin \sqrt{\mathfrak{m}^{00}}$ so $a \notin \mathfrak{p}$ for some prime ideal $\mathfrak{p} \supseteq \mathfrak{m}^{00}$. By choice of P_a, we have $\sqrt{\mathfrak{m}^{00}} \subseteq \mathfrak{p} \subseteq P_a$. For the other inclusion, note that $P_a \subsetneq \mathfrak{m}$, and so $\mathfrak{m}/P_a$ is infinite by the remarks on local rings at the start of the proof. If $b \in P_a$, then $bR \subseteq P_a \subsetneq \mathfrak{m} \subseteq R$, so R/bR is infinite, and $b \in \sqrt{\mathfrak{m}^{00}}$ by (2). This shows the reverse inclusion $P_a \subseteq \sqrt{\mathfrak{m}^{00}}$. □

Corollary 4.15. *Let R be a sufficiently saturated dp-minimal domain with maximal ideal $\mathfrak{m}$ and fraction field K. Then $\mathfrak{m} = \mathfrak{m}^{00}$ if and only if $\mathfrak{m}^{00}$ is prime and R/aR is infinite for all $a \in \mathfrak{m}$.*

Proof. Assume that $\mathfrak{m} = \mathfrak{m}^{00}$; so it is prime. Assume further that R/aR is finite for some $a \in \mathfrak{m}$. By Proposition 4.14, $\mathfrak{m} = P_a$ so the latter is a maximal ideal which gives that a is a unit, contradicting the choice of a. So R/aR is infinite for any $a \in \mathfrak{m}$.

If $\mathfrak{m}^{00}$ is prime and R/aR is infinite for all $a \in \mathfrak{m}$ we conclude that $\mathfrak{m}^{00} = \mathfrak{m}$ by Proposition 4.14. □

Corollary 4.8 yields a strengthening of Proposition 4.14.

Lemma 4.16. *Let R be a sufficiently saturated dp-minimal domain with maximal ideal $\mathfrak{m}$.*

(1) *If R/aR is infinite for every $a \in \mathfrak{m}$, then $\sqrt{\mathfrak{m}^{00}} = \mathfrak{m}$.*
(2) *If $a \in \mathfrak{m}$ and R/aR is finite, then $\mathfrak{m}^{00} = P_a$, and $\mathfrak{m}^{00}$ is the largest nonmaximal prime ideal.*

Proof. Proposition 4.14 proves (1), and shows that in the setting of (2), we have $P_a = \sqrt{\mathfrak{m}^{00}}$. But $P_a = P_a^{00}$ by Corollary 4.8, and so

$$P_a = P_a^{00} \subseteq \mathfrak{m}^{00} \subseteq \sqrt{\mathfrak{m}^{00}} = P_a.$$

Equality must hold, and $\mathfrak{m}^{00} = P_a$. In particular, $\mathfrak{m}^{00}$ is a nonmaximal prime ideal. If $\mathfrak{m}^{00}$ fails to be the largest nonmaximal prime ideal, then there is a prime ideal $\mathfrak{p}$ with $\mathfrak{m}^{00} \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$. Take $b \in \mathfrak{m} \setminus \mathfrak{p}$. Enlarging $\mathfrak{p}$, we may assume $\mathfrak{p}$ is the largest prime ideal not containing b, i.e., $\mathfrak{p} = P_b$. Then

$$P_b = P_b^{00} \subseteq \mathfrak{m}^{00} \subsetneq P_b,$$

by Corollary 4.8 and the fact that $P_b \subseteq \mathfrak{m}$. This is absurd, so $\mathfrak{m}^{00}$ is truly the second-largest prime ideal. □

4.5. *The topological ring R/R^{00}.*

Let R be an NIP ring.

The "logic topology" on R/R^{00} is the topology where a set $X \subseteq R/R^{00}$ is closed if its preimage $\widetilde{X} \subseteq R$ is type-definable. The logic topology makes R/R^{00} into a compact Hausdorff topological ring; see [Simon 2015, Section 8.1].

In contrast to compact Hausdorff topological groups, the following holds:

Fact 4.17 [Ribes and Zalesskii 2010, Proposition 5.1]. *A topological ring is compact Hausdorff if and only if it is profinite.*

As a direct consequence:

Fact 4.18. *If R is an NIP ring then R/R^{00} is profinite and so $R^{00} = R^0$.*

The following should be well known, but we include a proof for completeness.

Fact 4.19. *Let A be an infinite compact Hausdorff integral domain such that every nontrivial principal ideal has finite index. Then $F = \mathrm{Frac}(A)$ is either isomorphic to a finite extension of $\mathbb{Q}_p$ or to $\mathbb{F}_q((t))$.*

Proof. We first make some observations.

(1) If $a \in A \setminus \{0\}$, then the principal ideal aA is closed because it is the image of the compact set A under the continuous function $f(x) = ax$. Since aA has finite index, aA is a clopen ideal.

(2) Every nonzero ideal I contains a principal ideal, and therefore is clopen and has finite index.

(3) Conversely, the zero ideal is not open, since A is infinite.

(4) The set of units $A^\times$ is closed, being the projection of the compact set

$$\{(x, y) \in A^2 : xy = 1\}$$

onto the first coordinate. If A is a field, this contradicts the previous point, so A is not a field.

(5) By [Ribes and Zalesskii 2010, Proposition 5.1.2], the open ideals form a neighborhood basis of 0. By the points above, the nonzero ideals of A form a neighborhood basis of 0, and the family

$$\{I + a : I \trianglelefteq A \text{ a nonzero ideal, } a \in A\}$$

is a basis for the topology.

(6) Every commutative profinite ring is a cartesian product of profinite local rings [Ribes and Zalesskii 2010, Exercise 5.1.3(2)] so as A is an integral domain it must be a local profinite ring.

Let $F = \mathrm{Frac}(A)$. By [Prestel and Ziegler 1978, Example 1.2], the family

$$\{I + a : I \trianglelefteq A \text{ a nonzero ideal, } a \in F\}$$

is a basis for a nondiscrete Hausdorff ring topology on F. Moreover, this ring topology is a field topology because A is local; see the proof of [Prestel and Ziegler 1978, Theorem 2.2(b)]. One can easily see that A is clopen in this topology. By the observations above, the induced topology on A is the original compact Hausdorff topology on A. Then F is a locally compact field, because it is covered by the compact open translates $A+b$.

As F is a locally compact field, it is isomorphic to $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or $\mathbb{F}_q((t))$ [Milne 2020, Remark 7.49]. Since $\mathbb{R}$ and $\mathbb{C}$ have no compact subrings it must be one of the latter two. □

Corollary 4.20. *Let R be a sufficiently saturated NIP integral domain with R^{00} a prime ideal satisfying $R^{00} = aR^{00}$ for any $a \in R \setminus R^{00}$. Then* $\mathrm{Frac}(R/R^{00})$ *is either finite or a finite extension of $\mathbb{Q}_p$.*

Proof. Since R^{00} is a prime ideal, R/R^{00} is a compact Hausdorff integral domain; assume it is infinite. Note that every nonzero principal ideal in R/R^{00} has finite index. Indeed if $a \in R \setminus R^{00}$ then $aR \supseteq (aR)^{00} = aR^{00} = R^{00}$, so aR has finite index in R. By Fact 4.19, $\mathrm{Frac}(R/R^{00})$ is either a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$. Since R^{00} is a prime ideal it is externally definable, by Fact 4.1. Then R/R^{00} and hence $\mathrm{Frac}(R/R^{00})$ is NIP. Since $\mathbb{F}_q((t))$ has IP [Kaplan et al. 2011, Theorem 4.3], $\mathrm{Frac}(R/R^{00})$ must instead be a finite extension of $\mathbb{Q}_p$. □

Corollary 4.21. *Let R be a sufficiently saturated dp-minimal integral domain with maximal ideal $\mathfrak{m}$. If $R/\mathfrak{m}$ is finite and R^{00} is a prime ideal then* $\mathrm{Frac}(R/R^{00})$ *is either finite or a finite extension of $\mathbb{Q}_p$.*

Proof. If R/R^{00} is finite then it is a finite integral domain, and hence a finite field. Assume that R/R^{00} is infinite.

In particular, $\mathfrak{p} = R^{00}$ is a nonmaximal prime ideal so it is the maximal ideal of the valuation overring $R_\mathfrak{p}$ of R (Corollary 4.7). This implies that $aR^{00} = R^{00}$ for any $a \in R_\mathfrak{p} \setminus R^{00}$, and hence in particular for all $a \in R \setminus R^{00}$. The result now follows from Corollary 4.20. □

4.6. ***Subrings from additive subgroups.*** In this section, we prove the existence of an infinite definable proper subring in dp-minimal structures $(K, +, \cdot, A)$, where $(K, +, \cdot)$ is a field and A is an infinite additive proper subgroup, without any saturation assumption. This will be used later.

Proposition 4.22. *Let $M = (K, +, \cdot, A)$ be a dp-minimal expansion of a field with A a proper infinite additive subgroup. Then $R = A : A = \{x \in K : xA \subseteq A\}$ is an infinite definable proper subring of K and* $\mathrm{Frac}(R) = K$.

Proof. There is no harm in assuming sufficient saturation of M. Also, we will later pass to the Shelah expansion; this too causes no harm.

The definable set A is neither finite nor cofinite, so $(K, +, \cdot, A)$ is not strongly minimal.

By [Johnson 2018, Theorem 1.3], M has a Hausdorff nondiscrete definable V-topology called the *canonical topology*. The canonical topology is characterized by the fact that

$$\{D - D : D \subseteq K \text{ definable and infinite}\}$$

is a neighborhood basis of 0, where $D - D = \{x - y : x, y \in D\}$. Taking $D = A$, we see that $A = A - A$ is a neighborhood of 0.

By saturation, there exists an externally definable valuation inducing this topology [Halevi et al. 2020, Proposition 3.5]. So after passing to the Shelah expansion, we may assume the existence of a definable valuation v inducing this topology. Let Γ be its value group.

Claim 4.23. *A is a bounded set with respect to the topology induced by v.*

Proof. Note that $(K, +)^0$ is an ideal in K of small index, so $(K, +)^0 = (K, +)$ and so K has no definable proper subgroups of finite index. Therefore K/A is infinite. Let $(c_i)_{i<\omega}$ be different representatives of distinct classes.

Assume towards a contradiction that A is not bounded, so for any $\gamma \in \Gamma$ there exists an element $a \in A$ with $v(a) < \gamma$. By saturation, and the fact that A is infinite, we can find a sequence of elements $(a_s)_{s<\omega}$ of A satisfying $v(a_s) < v(a_t) < v(c_i)$ for any $i < \omega$ and any $t < s < \omega$. Now consider the formulas $(x \in A + c_i)_{i<\omega}$ and $(v(x) = v(a_s))_{s<\omega}$; they contradict dp-minimality. □

The definable set R is certainly a definable subring of K, because A is an additive subgroup. Since A is a bounded neighborhood of 0, there is an open neighborhood $V \ni 0$ such that $V \cdot A \subseteq A$. Then R contains the infinite set V. The ring R is a proper subring because if we take $a \in A \setminus \{0\}$ and $b \notin A$ then $b/a \notin R$. Finally, $\operatorname{Frac}(R) = K$, since $(K, \operatorname{Frac}(R))$ is dp-minimal so $[K : \operatorname{Frac}(R)] = 1$. □

5. A trichotomy for dp-minimal fields

Definition 5.1. Let K be an infinite field.

(1) K is *ACVF-like* if there is a henselian valuation v on K with algebraically closed residue field.

(2) K is *pCF-like* if there is a henselian valuation v on K with finite residue field of characteristic p.

(3) K is *RCVF-like* if there is a henselian valuation v on K with real closed residue field.

We allow v to be trivial.

For example, $\mathbb{F}_p^{\text{alg}}$ and $\mathbb{C}((t))$ are ACVF-like, $\mathbb{R}$ and $\mathbb{R}((t))$ are RCVF-like, and $\mathbb{Q}_3(i)$ and $\mathbb{Q}_3((t))$ are 3CF-like.

If there is a henselian valuation w on K such that Kw is p-adically closed, then we can compose w with the canonical valuation on Kw, getting a henselian valuation v on K with Kv finite of characteristic p. Thus K is pCF-like.

Remark 5.2. If K is a dp-minimal field, then K is ACVF-like, pCF-like, or RCVF-like. In fact, if $\mathcal{O}_\infty$ is the intersection of all 0-definable valuation rings on K, then $\mathcal{O}_\infty$ is henselian and its residue field is algebraically closed, real closed, or finite. See [Johnson 2023a, Theorem 4.8] or Fact 2.2.

In fact, the three cases are mutually exclusive:

Proposition 5.3. *An infinite field K satisfies at most one of the following*:

(1) *K is ACVF-like.*

(2) *K is pCF-like.*

(3) *K is RCVF-like.*

In the second case, p is uniquely determined.

Proof. Otherwise, there are two henselian valuation rings $\mathcal{O}_1, \mathcal{O}_2$ whose residue fields k_1, k_2 are of a different nature. There are two cases:

(1) The two valuations are comparable, say, $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Then $\mathcal{O}_2$ is a coarsening of $\mathcal{O}_1$, so k_1 is the residue field of some henselian valuation w on k_2. If k_2 is finite, then w is trivial, so $k_1 = k_2$ and k_1 is finite of the same characteristic as k_1. If k_2 is algebraically closed or real closed, then (k_2, w) is a model of ACVF or RCVF, respectively, and so k_1 is algebraically closed or real closed. In each case, k_1 has the same nature as k_2.

(2) The two valuations are incomparable. Let $\mathcal{O}_3$ be the join $\mathcal{O}_1 \cdot \mathcal{O}_2$ and let k_3 be its residue field. Then $\mathcal{O}_1$ and $\mathcal{O}_2$ induce independent henselian valuations w_1, w_2 on k_3, and the residue fields of w_1 and w_2 are k_1 and k_2, respectively. Because k_3 has two independent nontrivial henselian valuations, it must be separably closed [Engler and Prestel 2005, Theorem 4.4.1], and then the two residue fields k_1 and k_2 must be algebraically closed [Engler and Prestel 2005, Theorem 3.2.11]. Then k_1 and k_2 again have the same nature. □

Moreover, everything is determined by the complete theory of $(K, +, \cdot)$:

Lemma 5.4. *Let K_1, K_2 be two dp-minimal fields. If $K_1 \equiv K_2$, then K_1 is ACVF-like (RCVF-like, pCF-like) if and only if K_2 is ACVF-like (RCVF-like, pCF-like).*

Proof. Let $\mathcal{O}_i$ be a henselian valuation ring on K_i whose residue field is algebraically closed, real closed, or finite. By Robinson joint consistency, there is a structure $(K_3, \mathcal{O}'_1, \mathcal{O}'_2)$ with elementary embeddings $(K_i, \mathcal{O}_i) \hookrightarrow (K_3, \mathcal{O}'_i)$ for $i = 1, 2$. In particular, $\mathcal{O}'_i$ is a henselian valuation ring on K_3 whose residue field has the same nature as the residue field of $\mathcal{O}_i$. By Proposition 5.3, the residue fields of $\mathcal{O}'_1$ and $\mathcal{O}'_2$ must have the same nature. Therefore, the residue fields of $\mathcal{O}_1$ and $\mathcal{O}_2$ must have the same nature. □

Remark 5.5. Let K be a sufficiently saturated dp-minimal field, possibly with extra structure. Suppose that K is pCF-like. Let $\mathcal{O}_{\mathrm{can}}$ be the intersection of all definable valuation rings on K. By [Johnson 2023a, Theorem 4.7], $\mathcal{O}_{\mathrm{can}}$ is henselian, definable, and has finite residue field, because K is pCF-like rather than ACVF-like or RCVF-like.[4] We claim that $\mathcal{O}_{\mathrm{can}}^{00}$ is the second largest prime ideal in $\mathcal{O}_{\mathrm{can}}$, and that $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ is a ring of characteristic 0.

The valuation ring $\mathcal{O}_{\mathrm{can}}$ is dp-minimal with finite residue field, so the interval $[-v(p), v(p)]$ in the value group of $\mathcal{O}_{\mathrm{can}}$ is finite by Fact 2.3. Then the value group is discretely ordered and so the maximal ideal $\mathfrak{m}_{\mathrm{can}}$ of $\mathcal{O}_{\mathrm{can}}$ is a principal ideal. Since $\mathcal{O}_{\mathrm{can}}/\mathfrak{m}_{\mathrm{can}}$ is finite, $\mathcal{O}_{\mathrm{can}}^{00} = \mathfrak{m}_{\mathrm{can}}^{00}$ is the second largest prime ideal in $\mathcal{O}_{\mathrm{can}}$ by Lemma 4.16.

Because $\mathcal{O}_{\mathrm{can}}^{00}$ is a nonmaximal prime ideal in $\mathcal{O}_{\mathrm{can}}$, the quotient $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ is an integral domain but not a field. In particular, it is infinite. By Corollary 4.21, the fraction field of $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ must be a finite extension of $\mathbb{Q}_p$, and must therefore have characteristic 0.

6. Proof of the main theorem

6.1. ***Exceptional dp-minimal domains.*** An integral domain which is neither a field nor a valuation ring is called *exceptional*. Our goal is to understand exceptional dp-minimal domains. By [d'Elbée and Halevi 2021, Lemma 3.6(1)], any exceptional dp-minimal domain R is a local domain with finite residue field $R/\mathfrak{m}$. In particular, for such domains, $R^{00} = \mathfrak{m}^{00}$. Note also that R is infinite.

Remark 6.1. Let R be an exceptional dp-minimal domain (possibly with extra structure). If $\mathcal{O}$ is an externally definable valuation ring on $\mathrm{Frac}(R)$, then $R \subseteq \mathcal{O}$. First, we may assume that $\mathcal{O}$ is definable by passing to R^{Sh}. Then Fact 4.3 shows that $R \subseteq \mathcal{O}$ or $\mathcal{O} \subseteq R$. The latter cannot happen as R is not a valuation ring.

Recall that every infinite dp-minimal field is either ACVF-like, RCVF-like or pCF-like, and only one of these properties holds (Remark 5.2 and Proposition 5.3).

Proposition 6.2. *If R is an exceptional dp-minimal domain, then* $\mathrm{Frac}(R)$ *is ACVF-like or pCF-like.*

Proof. We may assume that R is sufficiently saturated. Its fraction field $K = \mathrm{Frac}(R)$ is dp-minimal by Fact 4.2. Suppose for the sake of contradiction that K is RCVF-like. Then K admits a henselian valuation with real closed residue field. By Ax–Kochen–Ershov, K is elementarily equivalent to the field of Hahn series $L = \mathbb{R}((t^{\Gamma}))$ for some dp-minimal ordered abelian group Γ. This field L is orderable. By [Jahnke et al. 2017, proof of Theorem 6.2], any field-order on L is definable. As a result there is a definable field-ordering $\leq$ on K as well. Then $(K, R, \leq)$

[4]The notation reflects the fact that $\mathcal{O}_{\mathrm{can}}$ is the canonical henselian valuation ring on K.

is dp-minimal. Let $\mathcal{O}$ be the convex hull of $\mathbb{Z}$ in $(K, \leq)$. It is easily seen to be a valuation subring of K and since it is convex, it is externally definable. By Remark 6.1, $R \subseteq \mathcal{O}$. By saturation, $R \subseteq [-n, n]$ for some integer n, which is absurd. □

The next proposition shows that the two cases of Lemma 4.16 correspond to the case where $\mathrm{Frac}(R)$ is ACVF-like and pCF-like.

Proposition 6.3. *Let R be an exceptional dp-minimal domain with maximal ideal $\mathfrak{m}$.*

(1) *If R/aR is infinite for every $a \in \mathfrak{m}$, then* $\mathrm{Frac}(R)$ *is ACVF-like.*

(2) *If R/aR is finite for some $a \in \mathfrak{m}$, then* $\mathrm{Frac}(R)$ *is pCF-like.*

Proof. Since $\mathrm{Frac}(R)$ is either ACVF-like or pCF-like by Proposition 6.2, it suffices to show that $\mathrm{Frac}(R)$ is pCF-like if and only if R/aR is finite for some $a \in \mathfrak{m}$. There is no harm in assuming that R is sufficiently saturated.

First suppose that $\mathrm{Frac}(R)$ is pCF-like. Let $\mathcal{O}_{\mathrm{can}}$ be the canonical henselian valuation ring, as in Remark 5.5. Then $\mathcal{O}_{\mathrm{can}}$ is a definable henselian valuation ring with finite residue field, the second largest prime ideal of $\mathcal{O}_{\mathrm{can}}$ is $\mathcal{O}_{\mathrm{can}}^{00}$, and the quotient ring $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ has characteristic zero.

By Remark 6.1, as R is exceptional we have $R \subseteq \mathcal{O}_{\mathrm{can}}$ and hence $R^{00} \subseteq \mathcal{O}_{\mathrm{can}}^{00} \cap R$. Note that $\mathcal{O}_{\mathrm{can}}^{00} \cap R$ is a prime ideal of R. The quotient $R/(R \cap \mathcal{O}_{\mathrm{can}}^{00})$ injects into $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$, so it has characteristic zero. In contrast, $R/\mathfrak{m}$ is finite. Therefore $R \cap \mathcal{O}_{\mathrm{can}}^{00} \neq \mathfrak{m}$. Since $\mathfrak{m}^{00} = R^{00}$ is contained in the nonmaximal prime ideal $R \cap \mathcal{O}_{\mathrm{can}}^{00}$, we have $\sqrt{\mathfrak{m}^{00}} \subsetneq \mathfrak{m}$. By Lemma 4.16(1), R/aR is finite for some $a \in \mathfrak{m}$.

Conversely, suppose that R/aR is finite for some $a \in \mathfrak{m}$. By Lemma 4.16(2), $R^{00} = \mathfrak{m}^{00}$ is a nonmaximal prime ideal of R. In particular, R/R^{00} is infinite. Then $k = \mathrm{Frac}(R/R^{00})$ is a finite extension of $\mathbb{Q}_p$ by Corollary 4.21. Let $\mathcal{O}_p$ be the standard definable valuation ring of k.

Let $K = \mathrm{Frac}(R)$ and let $\mathcal{O} = R^{00} : R^{00}$; it is an externally definable valuation overring with maximal ideal R^{00} (Corollary 4.12), so $\mathcal{O}/R^{00}$ is its residue field. Now using the quotient map, the structure $(\mathcal{O}/R^{00}, R/R^{00})$ is dp-minimal, and hence so is $(\mathcal{O}/R^{00}, \mathrm{Frac}(R/R^{00}))$. As R/R^{00} is infinite, $\mathrm{Frac}(R/R^{00})$ is an infinite subfield of $\mathcal{O}/R^{00}$. Hence they must be equal by dp-minimality. Composing the valuation $\mathcal{O}_p$ on k with $\mathcal{O}$, we are supplied with an externally definable valuation on K with finite residue field; so $K = \mathrm{Frac}(R)$ is pCF-like. □

Combining with Lemma 4.16, we get the following dichotomy for exceptional dp-minimal domains:

Corollary 6.4. *Let R be an exceptional dp-minimal domain with maximal ideal $\mathfrak{m}$. Then exactly one of the following holds*:

(1) $\mathrm{Frac}(R)$ *is ACVF-like, R/aR is infinite for every $a \in \mathfrak{m}$, and $\sqrt{\mathfrak{m}^{00}} = \mathfrak{m}$.*

(2) $\operatorname{Frac}(R)$ *is pCF-like*, R/aR *is finite for some* $a \in \mathfrak{m}$, $\mathfrak{m}^{00}$ *is the second-largest prime ideal in* R, *and* $\mathfrak{m}^{00} = P_a$ *for any* $a \in \mathfrak{m}$ *such that* R/aR *is finite.*

Lemma 6.5. *Let* R_0 *be an exceptional dp-minimal integral domain with fraction field* $K_0 = \operatorname{Frac}(R_0)$. *If* K_0 *is ACVF-like, then there is an externally definable nontrivial valuation ring in the structure* $(K_0, R_0, +, \cdot)$.

Proof. Let $R \succ R_0$ be a sufficiently saturated elementary extension and $K = \operatorname{Frac}(R)$. Then R is exceptional and K is ACVF-like. Let $\mathcal{O} = R^{00} : R^{00}$ and $\mathcal{O}_0 = \mathcal{O} \cap K$ (see Lemma 4.10). Then $\mathcal{O}_0$ is an externally definable valuation ring on K_0. We only need to show that $\mathcal{O}_0$ is nontrivial.

Let $\mathfrak{m}_0$ and $\mathfrak{m}$ be the maximal ideals of R_0 and R. Take nonzero $a \in \mathfrak{m}_0 \subseteq \mathfrak{m}$. By Corollary 6.4, $\mathfrak{m} = \sqrt{\mathfrak{m}^{00}}$, and so $a^n \in \mathfrak{m}^{00} = R^{00}$ for some n. On the other hand, $1 \notin \mathfrak{m}^{00}$. Since $\mathfrak{m}^{00}$ is an ideal in $\mathcal{O}$, we must have $v(a^n) \neq v(1)$, where v is the valuation on K induced by $\mathcal{O}$. Then the restriction of v to K_0 is nontrivial, and $\mathcal{O}_0$ is nontrivial. □

By [Halevi et al. 2020, Proposition 3.5], every sufficiently saturated field with a definable V-topology admits a nontrivial externally definable valuation ring. The following gives the same conclusion by assuming the existence of a definable infinite proper additive subgroup, rather than assuming saturation.

Proposition 6.6. *Let* $(K, +, \cdot, A)$ *be a dp-minimal expansion of a field* K *by an infinite*, *proper additive subgroup* $(A, +) \subsetneq (K, +)$. *Then there is a nontrivial externally definable valuation ring on* K.

Proof. By Proposition 4.22, $R = A : A$ is an infinite definable proper subring of K and $\operatorname{Frac}(A) = K$. If it is a valuation ring, there is nothing to show so we assume it is exceptional. If $\operatorname{Frac}(R)$ is pCF-like, then it admits a definable valuation (the canonical valuation). So we assume that $\operatorname{Frac}(R)$ is ACVF-like. The result now follows from Lemma 6.5. □

After completing the classification, we will see that "externally definable" can be changed to "definable" (Proposition 6.20).

6.2. ***The pCF-like case.*** Let R be a dp-minimal domain with maximal ideal $\mathfrak{m}$, not necessarily pure, and $K = \operatorname{Frac}(R)$. In the following, we frequently use the fact that if R is exceptional then $R \subseteq \mathcal{O}_0$ for any externally definable valuation $\mathcal{O}_0$ of $K = \operatorname{Frac}(R)$ (Remark 6.1).

Proposition 6.7. *Let* R *be an exceptional dp-minimal domain such that* $K = \operatorname{Frac}(R)$ *is pCF-like. Let* $\mathcal{O}_{\text{can}}$ *be the canonical henselian valuation on* K *as in Remark 5.5.*

(1) *R has finite index in* $\mathcal{O}_{\text{can}}$.

(2) *In fact*, *there is some* n *such that* $\mathcal{O}_{\text{can}} \supseteq R \supseteq p^n \mathcal{O}_{\text{can}}$, *and each inclusion has finite index.*

In particular, R arises via the construction in Proposition 3.2 in a definable manner: $\mathcal{O}_{\mathrm{can}}$ is a definable valuation subring of K and R is the preimage of $R/p^n\mathcal{O}_{\mathrm{can}}$ under the definable map $\mathcal{O}_{\mathrm{can}} \to \mathcal{O}_{\mathrm{can}}/p^n\mathcal{O}_{\mathrm{can}}$.

Proof. Replacing R with an elementary extension, we may assume R is sufficiently saturated for both points.

(1) By Remark 6.1, $R \subseteq \mathcal{O}_{\mathrm{can}}$. As both rings are definable, it suffices to show $\mathcal{O}_{\mathrm{can}}/R$ is finite, or equivalently, small. Equivalently, we must show that $R^{00} = \mathcal{O}_{\mathrm{can}}^{00}$. Suppose not. Then $R^{00} \subsetneq \mathcal{O}_{\mathrm{can}}^{00} \subseteq \mathfrak{m}_{\mathrm{can}}$, where $\mathfrak{m}_{\mathrm{can}}$ is the maximal ideal of $\mathcal{O}_{\mathrm{can}}$.

Let $\mathfrak{p} = R^{00} = \mathfrak{m}^{00}$. By Corollary 6.4, $\mathfrak{p}$ is the second-largest prime ideal of R. By [d'Elbée and Halevi 2021, Proposition 3.8 and Theorem 3.9], the localization $R_{\mathfrak{p}}$ is a valuation ring with maximal ideal $\mathfrak{p}$. The fact that $\mathfrak{p} \subseteq \mathfrak{m}_{\mathrm{can}}$ implies the reverse inclusion on valuation rings: $\mathcal{O}_{\mathrm{can}} \subseteq R_{\mathfrak{p}}$. Then $\mathfrak{p}$ is closed under multiplication by elements of $\mathcal{O}_{\mathrm{can}}$, so $\mathfrak{p}$ is an ideal in $\mathcal{O}_{\mathrm{can}}$.

Let v be the valuation on K associated to $\mathcal{O}_{\mathrm{can}}$. Both $\mathfrak{p}$ and $\mathcal{O}_{\mathrm{can}}^{00}$ are ideals in $\mathcal{O}_{\mathrm{can}}$, so they are defined by cuts in the value group. The fact that $\mathfrak{p} \subsetneq \mathcal{O}_{\mathrm{can}}^{00}$ implies that there is some closed ball $B_{\geq\gamma}(0)$ (with respect to the valuation induced by $\mathcal{O}_{\mathrm{can}}$) separating the two:

$$\mathfrak{p} \subseteq B_{\geq\gamma}(0) \subseteq \mathcal{O}_{\mathrm{can}}^{00}.$$

By Fact 4.18, the compact Hausdorff ring R/R^{00} is profinite, which means that R^{00} is a filtered intersection of definable ideals $I \lhd R$ with finite index. The ball $B_{\geq\gamma}(0)$ is definable, so by saturation there is some finite-index definable ideal I such that $I \subseteq B_{\geq\gamma}(0) \subseteq \mathcal{O}_{\mathrm{can}}^{00}$. Then there is a ring homomorphism

$$R/I \to \mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}.$$

However, $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ has characteristic 0 (Remark 5.5) and R/I is finite, so this is absurd.

(2) If m is the index of R in $\mathcal{O}_{\mathrm{can}}$, then m annihilates the group $(\mathcal{O}_{\mathrm{can}}/R, +)$, meaning that $R \supseteq m\mathcal{O}_{\mathrm{can}}$. Write m as $m_0 p^n$, where m_0 is prime to p. Then m_0 is invertible in $\mathcal{O}_{\mathrm{can}}$, so

$$\mathcal{O}_{\mathrm{can}} \supseteq R \supseteq m\mathcal{O}_{\mathrm{can}} = p^n\mathcal{O}_{\mathrm{can}}.$$

Finally, $\mathcal{O}_{\mathrm{can}}/p^n\mathcal{O}_{\mathrm{can}}$ has finite index because $\mathcal{O}_{\mathrm{can}}$ has finite residue field and $v(p)$ is a finite multiple of the minimum positive element in the valuation group (Fact 2.3). □

6.3. *Rogue domains.* Let R be a dp-minimal domain with maximal ideal $\mathfrak{m}$, not necessarily pure, and $K = \mathrm{Frac}(R)$. Whenever R is sufficiently saturated we let $\mathcal{O}$ be the externally definable valuation ring $R^{00} : R^{00}$, $\mathfrak{m}_0$ its maximal ideal, v the corresponding valuation, Γ its value group, and $\boldsymbol{k}_0$ its residue field.

Definition 6.8. A *rogue domain* is an exceptional dp-minimal domain $(R, +, \cdot, \dots)$, possibly with extra structure, such that $\mathrm{Frac}(R)$ is ACVF-like and R^{00} is *not* definable.

We will eventually show that rogue domains do not exist.

Remark 6.9. Remark 4.4 has the following consequences.

(1) If R is a rogue domain and $R \equiv S$, then S is a rogue domain. In particular, elementary extensions of rogue domains are rogue.
(2) Any dp-minimal expansion of a rogue domain is a rogue domain. In particular, Shelah expansions of rogue domains are rogue.
(3) If R is a sufficiently saturated rogue domain, then R/R^{00} is infinite.

Lemma 6.10. *Let R be a sufficiently saturated rogue domain with fraction field K, and let $\mathcal{O} = R^{00} : R^{00}$. Then $\mathcal{O} \subsetneq \mathcal{O}'$ for any definable valuation ring $\mathcal{O}'$ on K.*

Proof. Otherwise, $\mathcal{O} \supseteq \mathcal{O}'$, because any two externally definable valuation rings are comparable (essentially by Fact 4.3). The definable valuation ring $\mathcal{O}'$ is henselian. If its residue field is finite, then K is pCF-like, contradicting the fact that K is ACVF-like. Thus $\mathcal{O}'$ has infinite residue field. By saturation, $\mathcal{O}'$ has residue field of large cardinality, as does the coarsening $\mathcal{O}$. In particular, the cardinality of the residue field $\boldsymbol{k}_0$ of $\mathcal{O}$ is much larger than that of R/R^{00}.

Let $S \subseteq R$ be a set of coset representatives of R^{00} in R, chosen so that $0 \in S$. Let v be the valuation associated with $\mathcal{O}$ and let res be the residue map. Let $\mathrm{rv} : K^\times \to K^\times/(1 + \mathfrak{m}_0)$ be the quotient map, where $\mathfrak{m}_0$ is the maximal ideal of $\mathcal{O}$.

Let
$$B = \{\mathrm{res}(x/y) : x, y \in S \setminus \{0\} \text{ and } v(x) = v(y)\} \subseteq \boldsymbol{k}_0.$$

Because $|R/R^{00}|$ is much smaller than $|\boldsymbol{k}_0|$, we have $B \subsetneq \boldsymbol{k}_0$. Take $c \in \mathcal{O}$ such that $\mathrm{res}(c) \in \boldsymbol{k}_0^\times \setminus B$. Note that $c \in \mathcal{O}^\times$ since $\mathrm{res}(c) \neq 0$. The fact that $c \in \mathcal{O}^\times$ implies that $cR^{00} = R^{00}$, by definition of $\mathcal{O} = R^{00} : R^{00}$. Then

$$R^{00} = c(R^{00}) = (cR)^{00} \subseteq cR,$$

so $R^{00} \subseteq cR \cap R$. We will arrive at a contradiction by showing that $R^{00} = cR \cap R$, and hence R^{00} is definable.

Otherwise, take $a_1 \in cR \cap R \setminus R^{00}$. Then $b_1 := a_1/c \in R$. If $b_1 \in R^{00}$, then $a_1 = cb_1 \in cR^{00} = R^{00}$, a contradiction. So $b_1 \notin R^{00}$.

By choice of S, there are some $a_2, b_2 \in S$ such that

$$a_2 \equiv a_1 \pmod{R^{00}}, \qquad b_2 \equiv b_1 \pmod{R^{00}}.$$

If $a_2 = 0$, then $a_1 \in R^{00}$, a contradiction. So $a_2 \in S \setminus \{0\}$. Similarly, $b_2 \in S \setminus \{0\}$ because $b_1 \notin R^{00}$.

Note that $a_1 - a_2 \in R^{00}$ but $a_1 \notin R^{00}$. As R^{00} is an ideal in $\mathcal{O}$ (so given by a cut), we have $v(a_1 - a_2) > v(a_1)$ and so $\operatorname{rv}(a_1) = \operatorname{rv}(a_2)$. Similarly, $v(b_1 - b_2) > v(b_1)$ and so also $\operatorname{rv}(b_1) = \operatorname{rv}(b_2)$. Consequently, since $c \in \mathcal{O}^\times$,

$$\operatorname{rv}(c) = \operatorname{rv}(a_1/b_1) = \operatorname{rv}(a_1)/\operatorname{rv}(b_1) = \operatorname{rv}(a_2)/\operatorname{rv}(b_2) = \operatorname{rv}(a_2/b_2).$$

In particular, $v(a_2/b_2) = v(c) = 0$, so $v(a_2) = v(b_2)$. By definition of B,

$$\operatorname{res}(c) = \operatorname{rv}(c) = \operatorname{rv}(a_2/b_2) = \operatorname{res}(a_2/b_2) \in B,$$

contradicting the choice of c. □

Corollary 6.11. *Let R be a rogue domain with fraction field K and let $\mathcal{O}'$ be a definable valuation ring on K. If $a \in K$ and $aR/(aR \cap R)$ is finite, then $a \in \mathcal{O}'$.*

Proof. We may replace R with a sufficiently saturated elementary extension. By Lemma 4.10, $a \in \mathcal{O} = R^{00} : R^{00}$ and so we conclude by Lemma 6.10. □

Lemma 6.12. *Let R be a sufficiently saturated rogue domain with fraction field K, and let $\mathcal{O} = R^{00} : R^{00}$. Then $\mathcal{O}$ is the finest externally definable valuation ring on K.*

Proof. The ring $\mathcal{O}$ is externally definable by Corollary 4.12. Fix an externally definable valuation ring $\mathcal{O}' \subseteq K$; it suffices to show that $\mathcal{O} \subseteq \mathcal{O}'$. By Lemma 4.10, we must show that if $a \in K$ and $aR/(aR \cap R)$ is finite, then $a \in \mathcal{O}'$. This holds by Corollary 6.11 applied to the Shelah expansion R^{Sh} of R, in which $\mathcal{O}'$ becomes a definable valuation ring (but saturation is lost). The Shelah expansion is still a rogue domain (Remark 6.9). □

Corollary 6.13. *Let R be a sufficiently saturated rogue domain with fraction field $K = \operatorname{Frac}(R)$, let $\mathcal{O}$ be the externally definable valuation ring $\mathcal{O} = R^{00} : R^{00} \subseteq K$, and let $\boldsymbol{k}_0$ be the residue field of $\mathcal{O}$.*

(1) *There are no externally definable nontrivial valuation rings on $\boldsymbol{k}_0$.*

(2) *If $A \subseteq \boldsymbol{k}_0$ is an externally definable additive subgroup, then A is finite or $A = \boldsymbol{k}_0$.*

Proof. The first point is a direct consequence of Lemma 6.12: if there were a nontrivial externally definable valuation on $\boldsymbol{k}_0$ then we could compose it with $\mathcal{O}$, getting a new externally definable valuation $\mathcal{O}' \subsetneq \mathcal{O}$, contradicting the lemma. The second statement is now a direct consequence of Proposition 6.6, noting that saturation was not assumed there. □

To a first approximation, the following lemma says that rogue domains cannot arise from Proposition 3.2.

Lemma 6.14. *Let R be a rogue domain. Let $\mathcal{O}'$ be an externally definable valuation ring on $K = \operatorname{Frac}(R)$ and let I be an ideal in $\mathcal{O}'$. If $I \subseteq R$, then the group R/I is infinite.*

Proof. Note that I is externally definable. Passing to the Shelah expansion, we may assume that $\mathcal{O}'$ and I are definable. Passing to an elementary extension, we may assume the structure is sufficiently saturated. Both changes are acceptable by Remark 6.9. Because K is ACVF-like, the residue field of $\mathcal{O}'$ must be infinite. Then every definable ideal is 00-connected by Fact 4.5(2), so $I = I^{00}$. If R/I is finite, then $R^{00} = I^{00} = I$, and R^{00} is definable, contradicting the fact that R is a rogue domain. □

Suppose the rogue domain R is sufficiently saturated. As R^{00} is an ideal in $\mathcal{O}$, it is defined by a cut Ξ in the value group:

$$R^{00} = \{x \in K : v(x) > \Xi\}.$$

The cut Ξ cannot be the cut $+\infty$ or $-\infty$, since R^{00} is neither 0 nor K.

Lemma 6.15. (1) *The value group Γ is densely ordered (not necessarily divisible).*

(2) *Ξ is not the cut γ^+ infinitesimally above some $\gamma \in \Gamma$. Equivalently, the set $\{\gamma \in \Gamma : \gamma < \Xi\}$ has no maximum.*

(3) *There is an increasing sequence $\gamma_0 < \gamma_1 < \gamma_2 < \cdots$ in Γ such that the following things hold:*

$$R/\{x \in R : v(x) \geq \gamma_i\} \text{ is finite for each } i,$$
$$\lim_{i\to\infty} \left|R/\{x \in R : v(x) \geq \gamma_i\}\right| = \infty.$$

(4) *If $\gamma_0 < \gamma_1 < \gamma_2 < \cdots$ is a sequence as in the previous point, then Ξ is the limit of the sequence, in the sense that*

$$\{x \in \Gamma : x > \Xi\} = \bigcap_{i=0}^{\infty} \{x \in \Gamma : x > \gamma_i\}.$$

Proof. Each closed ball $B_{\geq\gamma}(0) = \{x \in K : v(x) \geq \gamma\}$ is $\vee$-definable, because it is a scaled copy of the $\vee$-definable ring $\mathcal{O}$. Similarly, each open ball $B_{>\gamma}(0)$ is type-definable, being a scaled copy of the type-definable maximal ideal $\mathfrak{m} \lhd \mathcal{O}$.

If Γ is not densely ordered, then it is discretely ordered, and every closed ball is an open ball (of a different radius). In particular, $\mathcal{O}$ is type-definable, and therefore definable, contradicting Lemma 6.10, which says that $\mathcal{O}$ is strictly finer than any definable valuation ring. Therefore Γ is densely ordered, proving (1).

Claim 6.16. *If $\gamma > \Xi$, then $R/(R \cap B_{\geq\gamma}(0))$ is infinite, while if $\gamma < \Xi$, then $R/(R \cap B_{\geq\gamma}(0))$ is finite and $R \cap B_{\geq\gamma}(0)$ is definable.*

Proof. If $\gamma > \Xi$, then $B_{\geq\gamma}(0) \subseteq R^{00}$ because R^{00} is the "ball of radius Ξ", so to speak. Therefore $R \cap B_{\geq\gamma}(0) \subseteq R \cap R^{00} = R^{00}$, and

$$[R : R \cap B_{\geq\gamma}(0)] \geq [R : R^{00}].$$

Since R^{00} is not definable, $[R : R^{00}]$ is not finite.

Conversely, if $\gamma < \Xi$, then $B_{\geq\gamma}(0) \supseteq R^{00}$. The intersection $R \cap B_{\geq\gamma}(0)$ is $\vee$-definable and

$$[R : R \cap B_{\geq\gamma}(0)] \leq [R : R^{00}],$$

so $R \cap B_{\geq\gamma}(0)$ has small index in R. In general, $\vee$-definable subgroups of small index have finite index and are definable, by a compactness argument (analogous to the fact that open subgroups of a compact Hausdorff group are clopen of finite index). □

Now we can prove (2). Suppose for the sake of contradiction that $\Xi = \gamma^+$ for some $\gamma \in \Gamma$, so that

$$R^{00} = \{x \in K : x > \gamma\} = B_{>\gamma}(0).$$

Note that

$$R \supseteq R \cap B_{\geq\gamma}(0) \supseteq R \cap B_{>\gamma}(0) = B_{>\gamma}(0) = R^{00}.$$

The element γ is less than the cut γ^+, so $R/(R \cap B_{\geq\gamma}(0))$ is finite by the claim. On the other hand, $R/B_{>\gamma}(0) = R/R^{00}$ is infinite because R is a rogue domain. Therefore, $(R \cap B_{\geq\gamma}(0))/B_{>\gamma}(0)$ is infinite. It is an externally definable subgroup of $B_{\geq\gamma}(0)/B_{>\gamma}(0)$. Since the latter is (externally) definably isomorphic to the residue field of $\mathcal{O}$, by Corollary 6.13, we must have

$$\begin{aligned}(R \cap B_{\geq\gamma}(0))/B_{>\gamma}(0) &= B_{\geq\gamma}(0)/B_{>\gamma}(0),\\ R \cap B_{\geq\gamma}(0) &= B_{\geq\gamma}(0),\\ R &\supseteq B_{\geq\gamma}(0).\end{aligned}$$

As $B_{\geq\gamma}(0) \supseteq R^{00}$, $R/B_{\geq\gamma}(0)$ has finite index and $B_{\geq\gamma}(0)$ is an ideal in an externally definable valuation ring, contradicting Lemma 6.14. This proves (2). In particular, the set $\{\gamma \in \Gamma : \gamma < \Xi\}$ has no maximum.

For $\gamma \in \Gamma$, let

$$f(\gamma) = [R : R \cap B_{\geq\gamma}(0)].$$

Clearly, f is weakly increasing:

$$\gamma \leq \gamma' \implies R \cap B_{\geq\gamma}(0) \supseteq R \cap B_{\geq\gamma'}(0) \implies f(\gamma) \leq f(\gamma').$$

Moreover, $f(\gamma)$ is finite if and only if $\gamma < \Xi$, by Claim 6.16. To prove (3) and (4), it suffices to show that $f(\gamma)$ is unbounded as γ approaches Ξ from below.

Suppose not. Then there is some $\gamma_0 < \Xi$ and some n such that

$$\gamma_0 < \gamma < \Xi \implies f(\gamma) = n.$$

It follows that $R \cap B_{\geq\gamma}(0)$ must be some fixed subgroup G of R of index n, independent of $\gamma < \Xi$. Moreover, G is definable by Claim 6.16.

On the other hand, the fact that there is no maximum element below Ξ means that

$$\begin{aligned} R^{00} &= \{x \in K : v(x) > \Xi\} \\ &= \{x \in K : v(x) \geq \gamma \text{ for every } \gamma \in \Gamma \text{ with } \gamma < \Xi\} \\ &= \bigcap_{\gamma < \Xi} B_{\geq \gamma}(0) \end{aligned}$$

and so

$$R^{00} = R \cap R^{00} = \bigcap_{\gamma < \Xi} (R \cap B_{\geq \gamma}(0)) = G.$$

Thus R^{00} is definable, contradicting the fact that R is rogue. □

Lemma 6.17. *Let R be a rogue domain. Let $\mathcal{O}'$ be an externally definable valuation ring and let v' be the corresponding valuation. There cannot exist an ascending sequence $\gamma_0' < \gamma_1' < \gamma_2' < \cdots$ in the value group of v' such that*

$$R/\{x \in R : v'(x) \geq \gamma_i'\} \text{ is finite for each } i,$$
$$\lim_{i \to \infty} \left| R/\{x \in R : v'(x) \geq \gamma_i'\} \right| = \infty.$$

Proof. Replacing R with its Shelah expansion, we reduce to the case where $\mathcal{O}'$ is definable. Replacing R with an elementary extension, we may assume that R is sufficiently saturated. Both changes preserve the fact that R is rogue, by Remark 6.9. Furthermore, the existence of the ascending sequence is also preserved in the elementary extension.

Now let $\mathcal{O}$ be the usual valuation ring $R^{00} : R^{00}$, and let $v : K \to \Gamma$ be its corresponding valuation. By Lemma 6.10, $\mathcal{O} \subsetneq \mathcal{O}'$, meaning that v' is a strict coarsening of v. In other words, v' is the composition

$$K \xrightarrow{v} \Gamma \to \Gamma/\Delta$$

for some nontrivial convex subgroup $\Delta \subseteq \Gamma$.

Let γ_i be an element in Γ lifting $\gamma_i' \in \Gamma/\Delta$. Note that

$$\{x \in R : v'(x) \geq \gamma_{i+1}'\} \subseteq \{x \in R : v(x) \geq \gamma_i\} \subseteq \{x \in R : v'(x) \geq \gamma_i'\}$$

and so

$$R/\{x \in R : v(x) \geq \gamma_i\} \text{ is finite for each } i,$$
$$\lim_{i \to \infty} \left| R/\{x \in R : v(x) \geq \gamma_i\} \right| = \infty.$$

By Lemma 6.15, the cut Ξ defining R^{00} is the limit of the γ_i:

$$x \in R^{00} \iff v(x) > \Xi \iff (v(x) > \gamma_i \text{ for every } i) \iff (v(x) \geq \gamma_i \text{ for every } i).$$

For fixed i, we have

$$v'(x) > \gamma_i' \implies v(x) > \gamma_i \implies v(x) \geq \gamma_i \implies v'(x) \geq \gamma_i'.$$

Therefore, the following four conditions are sequentially weaker:

(1) $v'(x) > \gamma_i'$ for every i.

(2) $v(x) > \gamma_i$ for every i.

(3) $v(x) \geq \gamma_i$ for every i.

(4) $v'(x) \geq \gamma_i'$ for every i.

But the first and fourth are equivalent because the sequence $\gamma_0', \gamma_1', \ldots$ is strictly increasing. Thus all four conditions are equivalent to each other, and also to the condition $x \in R^{00}$. In summary,

$$x \in R^{00} \iff (v'(x) > \gamma_i' \text{ for all } i).$$

It follows that $\mathcal{O}' \cdot R^{00} \subseteq R^{00}$, or equivalently that $\mathcal{O}' \subseteq R^{00} : R^{00} = \mathcal{O}$, contradicting Lemma 6.10. □

Of course we can apply Lemma 6.17 to $\mathcal{O}' = \mathcal{O}$ itself, and the conclusion directly contradicts parts (3) and (4) of Lemma 6.15. Therefore, *rogue domains do not exist*, and we have proven the following:

Proposition 6.18. *Let R be a sufficiently saturated exceptional dp-minimal domain such that* $\mathrm{Frac}(R)$ *is ACVF-like. Then R^{00} is definable and has finite index in R. As a consequence, the valuation ring $\mathcal{O} = R^{00} : R^{00}$ is definable as well.*

With Propositions 6.2 and 6.7, we can now prove the main theorem of the paper.

Theorem 6.19. *Let R be a dp-minimal integral domain. If R is not a valuation ring, then there exists a valuation subring $\mathcal{O}$ of $K =$* $\mathrm{Frac}(R)$, *a proper ideal $I \trianglelefteq \mathcal{O}$, and a finite subring of R_0 of $\mathcal{O}/I$ such that R is the preimage of R_0 under the quotient map $\mathcal{O} \to \mathcal{O}/I$. The data $(\mathcal{O}, I, R_0)$ can be chosen to be definable in R.*

Proof. Let R be a dp-minimal integral domain which is not a valuation ring, i.e., it is exceptional. By Proposition 6.2, $\mathrm{Frac}(R)$ is either ACVF-like or pCF-like. If it is the latter, we conclude by Proposition 6.7, so assume that it is ACVF-like. There is no harm in assuming that R is sufficiently saturated. By Proposition 6.18, $I = R^{00}$ is a definable ideal in the dp-minimal definable valuation ring $\mathcal{O} = R^{00} : R^{00}$, and R^{00} has finite index in R. The desired conclusion follows easily: take

$$R_0 = R/R^{00} \subseteq \mathcal{O}/R^{00}.$$ □

Incidentally, we can now strengthen Proposition 6.6:

Proposition 6.20. *Let $(K, +, \cdot, A)$ be a dp-minimal expansion of a field K by an infinite, proper additive subgroup $(A, +) \subseteq (K, +)$. Then there is a nontrivial definable valuation ring on K.*

Proof. There is no harm in assuming sufficient saturation. By Proposition 4.22, the ring $R = A : A$ is an infinite definable proper subring of K with $\mathrm{Frac}(R) = K$. If R is a valuation ring, we are done. Otherwise, R is exceptional. If K is pCF-like, then the canonical valuation is a nontrivial definable valuation. If K is ACVF-like, Proposition 6.18 gives a nontrivial valuation ring $\mathcal{O} = R^{00} : R^{00}$. □

7. Remarks on dp-minimal commutative rings

Having classified dp-minimal integral domains, it is natural to ask what can be said about more general dp-minimal commutative rings. As a first step, we prove the following:

Proposition 7.1. (1) *Every dp-minimal (commutative) ring has the form $R \times S$, where R is a dp-minimal henselian local ring and S is a finite ring.*

(2) *If R is a dp-minimal local ring, then the prime ideals of R are linearly ordered. Consequently, there is a unique minimal prime ideal.*

(3) *If R is a dp-minimal local ring, then every prime ideal containing the zero divisors is comparable to any principal ideal.*

Proof. (1) If R is a dp-finite commutative ring, then R decomposes as a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a henselian local ring [Johnson 2023b, Theorem 1.3]. The decomposition is definable, and so

$$\text{dp-rk}(R) = \sum_{i=1}^{n} \text{dp-rk}(R_i).$$

In the case where R is dp-minimal, it follows that one of the R_i is a dp-minimal henselian local ring and the rest of the factors are finite.

(2) Otherwise, there are two nonmaximal prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ which are incomparable. Then $(\mathfrak{p}+\mathfrak{q})/\mathfrak{p}$ is a nonzero ideal in the integral domain $R/\mathfrak{p}$. Since $\mathfrak{p}$ is nonmaximal, $R/\mathfrak{p}$ is not a field, so $R/\mathfrak{p}$ is infinite and every nonzero ideal is infinite. Thus $(\mathfrak{p}+\mathfrak{q})/\mathfrak{p}$ is infinite. Similarly, $(\mathfrak{p}+\mathfrak{q})/\mathfrak{q}$ is infinite. Equivalently, $\mathfrak{p}/(\mathfrak{p}\cap\mathfrak{q})$ and $\mathfrak{q}/(\mathfrak{p}\cap\mathfrak{q})$ are infinite. As $\mathfrak{p}$, $\mathfrak{q}$ are externally definable, this contradicts dp-minimality (Fact 4.13).

(3) Let $\mathfrak{p}$ be a prime ideal of R. Recall that there is a ring homomorphism $\varphi : R \to R_{\mathfrak{p}}$ with

$$\ker\varphi = \{c \in R \mid \exists s \notin \mathfrak{p}\ cs = 0\}.$$

We denote by R^{φ}, $\mathfrak{p}^{\varphi}$, a^{φ}, etc. the images in $R_{\mathfrak{p}}$. Note that $\mathfrak{p}^{\varphi}$ is a prime ideal of R^{φ} and $\mathfrak{m}^{\varphi}$ is the maximal ideal of R^{φ}.

Claim 7.2. $$\mathfrak{p}^{\varphi} = \mathfrak{p}^{\varphi} R_{\mathfrak{p}}.$$

Proof. It is easy to check that $R^\varphi \cap \mathfrak{p}^\varphi R_\mathfrak{p} = \mathfrak{p}^\varphi$, so it is enough to prove that $R^\varphi = R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$. We may assume that $\mathfrak{p}^\varphi$ is nonmaximal in R^φ. By Facts 4.1 and 4.2, $(R_\mathfrak{p}, R^\varphi, \mathfrak{p}^\varphi, \dots)$ is dp-minimal, and hence by Fact 4.13, one of $R^\varphi/\mathfrak{p}^\varphi$ or $\mathfrak{p}^\varphi R_\mathfrak{p}/\mathfrak{p}^\varphi$ has finite index as an abelian group. As $\mathfrak{p}^\varphi$ is nonmaximal, $R/\mathfrak{p}^\varphi$ is infinite, so $\mathfrak{p}^\varphi R_\mathfrak{p}/\mathfrak{p}^\varphi$ is finite. In particular, extending a set of representatives of $\mathfrak{p}^\varphi R_\mathfrak{p}/\mathfrak{p}^\varphi$ by the element 1 yields that $R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$ is a finitely generated R^φ-module. Let $\mathfrak{m}^\varphi$ be the maximal ideal of R^φ. Then one easily checks that $R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$. Since $R^\varphi/\mathfrak{m}^\varphi$ surjects onto $(R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})/(\mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})$, $1 + \mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$ generates $(R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})/(\mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})$ as an R^φ-module. Restricting the module action from R^φ to the Jacobson ideal $\mathfrak{m}^\varphi$ of R^φ, we have $\mathfrak{m}^\varphi(R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}) = \mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$. By Nakayama's lemma, the generator $1 + \mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$ of the R^φ-module $(R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})/(\mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})$ lifts to a generator of $R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$ as an R^φ-module, so $R^\varphi = R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$. □

To conclude, assume that $\mathfrak{p}$ contains all the zero divisors of R. Then the map $\varphi : R \to R_\mathfrak{p}$ is injective. Let $a \in R$. If $a \in \mathfrak{p}$ then we are done, otherwise, $a \notin \mathfrak{p}$ and hence $(1/a^\varphi)\mathfrak{p}^\varphi \subseteq \mathfrak{p}^\varphi R_\mathfrak{p} = \mathfrak{p}^\varphi$, so that if $b \in \mathfrak{p}$ there exists $c \in \mathfrak{p}$ such that $b^\varphi = a^\varphi c^\varphi$. As φ is injective, $b = ac$, so we conclude that $\mathfrak{p} \subseteq aR$. □

The comparability of prime ideals fails for rings of dp-rank 2: consider the fiber product $\mathcal{O} \times_k \mathcal{O}$, where $(K, \mathcal{O}) \models \mathrm{ACVF}$ and k is the residue field.

If R is a henselian local dp-minimal ring and $\mathfrak{p}$ is the unique minimal prime ideal, then $\mathfrak{p}$ is the nilradical $\sqrt{0}$, so every element of $\mathfrak{p}$ is nilpotent. The quotient $R/\mathfrak{p}$ is a dp-minimal integral domain, whose structure we understand by the main theorems of this paper.

Appendix: proof of Fact 2.2

Recall Fact 2.2:

Fact 2.2. *An infinite field $(K, +, \cdot)$ is dp-minimal if and only if there is a henselian defectless valuation ring $\mathcal{O} \subseteq K$ with maximal ideal $\mathfrak{m}$ such that*:

(1) *The value group $\Gamma := K^\times/\mathcal{O}^\times$ is dp-minimal as an ordered abelian group (possibly trivial).*

(2) *The residue field $k := \mathcal{O}/\mathfrak{m}$ is algebraically closed, real closed, or p-adically closed for some prime p.*

(3) *If the residue field k is algebraically closed of characteristic $p > 0$, then the interval $[-v(p), v(p)] \subseteq \Gamma$ is p-divisible, where $v(p) = +\infty$ when* $\mathrm{char}(K) = p$.

Proof. The "if" direction holds by the classification of dp-minimal valuation rings (Fact 2.3 above, originally from [Johnson 2023a, Theorems 1.5 and 1.6]), together with the well-known fact that ACF, RCF, and pCF are dp-minimal. It remains to prove the "only if" direction. Suppose K is dp-minimal.

Claim A.1. *There is a henselian valuation ring $\mathcal{O}$ on K whose residue field is p-adically closed, real closed, or algebraically closed.*

Proof. Let $\mathcal{O}_\infty$ be the intersection of all 0-definable valuation rings on K. By [Johnson 2023a, Theorem 1.2], $\mathcal{O}_\infty$ is a henselian valuation ring whose residue field k_∞ is finite, real closed, or algebraically closed. In the latter two cases, we can simply take $\mathcal{O} = \mathcal{O}_\infty$. Suppose we are in the first case: k_∞ is finite. Since $\mathcal{O}_\infty$ is henselian, the expansion $(K, \mathcal{O}_\infty)$ is dp-minimal by [Halevi and Hasson 2019, Proposition 5.14]. By the classification of dp-minimal valuation rings (Fact 2.3), $\mathcal{O}_\infty$ is finitely ramified. Let $v_\infty : K^\times \to \Gamma_\infty$ be the valuation induced by $\mathcal{O}_\infty$. Let Δ be the convex hull of $\mathbb{Z} \cdot v_\infty(p)$ in Γ_∞. Note that $\Delta \cong \mathbb{Z}$ by finite ramification. Coarsening by the convex subgroup Δ, we get a factorization of the place $K \to k_\infty$ into a composition of two places

$$K \xrightarrow{\Gamma_\infty/\Delta} k \xrightarrow{\Delta} k_\infty,$$

where the labels on the arrows show the value groups. The fact that $K \to k_\infty$ is henselian implies that $K \to k$ and $k \to k_\infty$ are henselian. The fact that $v_\infty(p) \in \Delta$ implies that $K \to k$ is equicharacteristic 0 and $k \to k_\infty$ is mixed characteristic. Then $k \to k_\infty$ is a finitely ramified henselian valuation with value group $\Delta \cong \mathbb{Z}$ and finite residue field, so k is p-adically closed. Take $\mathcal{O}$ to be the valuation ring associated to $K \to k$. □

If $\mathcal{O}$ is the henselian valuation ring from the claim, then $\mathcal{O}$ satisfies condition (2) of Fact 2.2. Moreover, $(K, \mathcal{O})$ is a dp-minimal valued field by [Halevi and Hasson 2019, Proposition 5.14]. By the classification of dp-minimal valued fields (Fact 2.3), the valuation is defectless and satisfies conditions (1) and (3) of Fact 2.2. □

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