Model Theory

no. 2 vol. 4 2025

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Model Theory (ISSN 2832-904X electronic, 2832-9058 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

MT peer review and production are managed by EditFlow® from MSP.

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https://msp.org/

Model Theory
Vol. 4, No. 2, 2025
https://doi.org/10.2140/mt.2025.4.101

Definable compactness in o-minimal structures

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We characterize the notion of definable compactness for topological spaces definable in o-minimal structures, answering questions posed by Peterzil and Steinhorn (*J. London Math. Soc.* (2) **59**:3 (1999), 769–786) and Johnson (*J. Symb. Log.* **83**:4 (2018), 1477–1500). Specifically, we prove the equivalence of various definitions of definable compactness in the literature, including those in terms of definable curves, definable types and definable downward directed families of closed sets.

1. Introduction

In the study of first-order topological theories various definable notions of topological compactness have been helpful tools in tame settings by isolating classes of topological objects with desirable properties. The first of such notions was introduced in *o-minimal* theories for definable manifold spaces in [Peterzil and Steinhorn 1999], and corresponds to the property that every definable curve converges (here curve-compactness). This property was crucial in formulating Pillay's conjecture [2004, Conjecture 1.1] about o-minimal definably compact groups and their relationship with compact Lie groups. The research that led to the solution of this conjecture provided a deeper understanding of the relationship between neostability and tame topology. In particular results in o-minimal forking were used to reach another reasonable notion of o-minimal definable compactness [Peterzil and Pillay 2007]: for every definable family of closed sets with the finite intersection property there exists a finite set that intersects each set in the family (here transversal-compactness). On the other hand, Thomas [2012] and Walsberg [2015] generalized and applied curve-compactness to study topologies arising from o-minimal definable norms and metrics respectively. In collaboration with the author [Andújar Guerrero et al. 2021], they also explored a third notion of definable compactness within o-minimality: every downward directed definable family of nonempty closed sets has nonempty intersection (here filter-compactness). This definition has been independently studied by Johnson [2018] in the context of o-minimal quotient spaces, and in a general model-theoretic setting by Fornasiero [2015]. The o-minimal exploration of definable compactness (through the various notions mentioned above)

MSC2020: primary 03C64; secondary 54A05, 54D30.
Keywords: o-minimality, types, definable compactness, definable topological spaces.

has yielded in particular that, in many cases, definably compact spaces are definably homeomorphic to a set with the canonical o-minimal "Euclidean" topology (see Chapter 7 in [Andújar Guerrero 2021a]). Hrushovski and Loeser [2016] explored the tame topology of valued fields, including the introduction of yet another notion of definable compactness: every definable type has a limit (here type-compactness), where a limit is a point in every closed set in the type. Recently, some of these notions have also been approached in the p-adic setting in [Andújar Guerrero and Johnson 2024], and in the local o-minimal setting in [Fujita 2024].

In the present paper we prove the equivalence of all the above notions of definable compactness in the setting of Hausdorff definable topological spaces (Definition 5.1) in o-minimal structures. We also show that, if we drop the Hausdorffness assumption, curve-compactness is strictly weaker than all the other properties. Our main result is the following (see Sections 2.2 and 5.1 for definitions).

Theorem A. *Fix an o-minimal structure $\mathcal{M} = (M, <, \ldots)$. Let (X, τ) be a definable topological space in $\mathcal{M}$. The following are equivalent*:

(1) *Every downward directed definable family of nonempty τ-closed sets has nonempty intersection* (*filter-compactness*).

(2) *Every definable type $p \in S_X(M)$ has a limit*; *i.e.*, *there is a point in the intersection of every τ-closed set in p* (*type-compactness*).

(3) *Every definable family of τ-closed sets that extends to a definable type in $S_X(M)$ has nonempty intersection.*

(4) *Every consistent definable family of τ-closed sets admits a finite transversal*; *i.e.*, *there exists a finite set that intersects every set in the family* (*transversal-compactness*).

(5) *Every definable family $\mathcal{C}$ of τ-closed sets with the (m, n)-property, where $m \geq n > \dim \bigcup \mathcal{C}$, has a finite transversal.*

(6) *Every definable family $\mathcal{C}$ of τ-closed sets with the (m, n)-property, where $m \geq n$ and n is greater than the* VC*-codensity of $\mathcal{C}$, has a finite transversal.*

Moreover all the above imply and, if τ is Hausdorff or $\mathcal{M}$ has definable choice, are equivalent to:

(7) *Every definable curve in X is τ-completable* (*curve-compactness*).

Theorem A and Remark 5.17 provide a positive answer to [Johnson 2018, Question 4.14], which asks whether curve-compactness and filter-compactness are equivalent for o-minimal definable manifold spaces.

In light of Theorem A we may present the following definition.

Definition 1.1. Let (X, τ) be a definable topological space in an o-minimal structure. We say that (X, τ) is *definably compact* if it satisfies any (all) of the conditions (1)–(6) in Theorem A.

We also prove that, if $\mathcal{M}$ is an o-minimal expansion of the real line $(\mathbb{R}, <)$, then every definable topological space in $\mathcal{M}$ is definably compact if and only if it is compact in the classical sense (Corollary 5.19), which provides a positive answer (Remark 5.20) to part of [Peterzil and Steinhorn 1999, Question 2.5]. Furthermore, we show that definable compactness is definable uniformly in families (Proposition 5.22). Additionally, throughout the paper we comment on other reasonable notions of definable compactness, including definitions in terms of externally definable sets (Remark 5.9), chains (paragraph above Lemma 5.14) and nets (Remark 5.18).

We prove Theorem A using o-minimal combinatorial and geometrical facts which are either known to hold in more general settings or can be conjectured to do so. These facts include known characterizations of o-minimal nonforking formulas, the Alon–Kleitman–Matoušek (p, q)-theorem for VC classes, and two geometrical facts of independent interest about o-minimal types (Propositions 4.1 and 4.3), the first of which can be understood as a strong form of distal cell decomposition. Hence this paper can be seen as a road map to characterizing definable compactness in various NIP settings.

The structure of the paper is as follows. In Section 2 we include preliminaries. In Section 3 we gather the necessary literature results on Vapnik–Chervonenkis theory and on forking, extracting some easy corollaries. In Section 4 we prove our two main results about o-minimal types. In Section 5 we introduce our topological framework and prove Theorem A through a series of propositions, as well our other results on definable compactness.

This paper has been largely extracted from [Andújar Guerrero 2021b], which includes independent proofs within o-minimality of Fact 3.6 and of a version of Corollary 3.7 where $m \geq n > \mathrm{vc}^*(\mathcal{S})$ is substituted by $m \geq n > \dim \bigcup \mathcal{S}$, thus avoiding largely the use of forking or VC literature.

2. Preliminaries

2.1. ***Conventions.*** We fix a language $\mathcal{L} = \{<, \dots\}$ and a first-order $\mathcal{L}$-structure $\mathcal{M} = (M, <, \dots)$ expanding a dense linear order without endpoints. For a set of parameters A we denote by $\mathcal{L}(A)$ the expansion of $\mathcal{L}$ by symbols for elements in A. Throughout unless otherwise specified "definable" means "$\mathcal{L}(M)$-definable in $\mathcal{M}$". All variables and parameters $x, a, u \dots$ are n-tuples for some $n < \omega$. We denote the length of a variable or parameter x by $|x|$. We denote ordered pairs of tuples by $\langle x, y \rangle$. We use n, m, k and l to denote natural numbers.

Unless stated otherwise, any formula we consider is in $\mathcal{L}(M)$. For any formula $\varphi(x)$ and set $A \subseteq M^{|x|}$, let $\varphi(A) = \{a \in A : \mathcal{M} \models \varphi(a)\}$. For simplicity we write $\varphi(M)$ to mean $\varphi(M^{|x|})$. A (uniformly) definable family of sets is a family of the form $\{\varphi(M, b) : b \in \psi(M)\}$ for some formulas $\varphi(x, y)$ and $\psi(y)$, where we may always assume that $\varphi(x, y) \in \mathcal{L}$ (i.e., a formula without parameters). For any two formulas $\varphi(x)$ and $\psi(x)$, we write $\varphi(x) \vdash \psi(x)$ to mean $\mathcal{M} \models \forall x(\varphi(x) \to \psi(x))$. For sets of formulas $p(x)$ and $q(x)$ on free variables x, we write $p(x) \vdash q(x)$ to mean that, for every formula $\varphi(x) \in q(x)$, there is a finite subset $p'(x) \subseteq p(x)$ such that $\wedge p'(x) \vdash \varphi(x)$.

For a given n, let π denote the projection $M^{n+1} \to M^n$ onto the first n coordinates, where n will often be omitted and clear from context. For a family $\mathcal{S}$ of subsets of M^{n+1} let $\pi(\mathcal{S}) = \{\pi(S) : S \in \mathcal{S}\}$.

Recall that $\mathcal{M}$ is o-minimal if every definable subset of M is a finite union of points and intervals with endpoints in $M \cup \{-\infty, +\infty\}$. For background in o-minimality we direct the reader to [van den Dries 1998]. We will use, in particular, the existence of uniform cell decompositions found in Chapter 3, Proposition 3.5 of that work. We use the following notation related to o-minimal cells: given two partial functions $f, g : M^n \to M \cup \{-\infty, +\infty\}$, with domains $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ respectively, let $(f, g) = \{\langle x, t\rangle : x \in \mathrm{dom}(f) \cap \mathrm{dom}(g),\ f(x) < t < g(x)\}$ (we relax thus the classical notation throughout by allowing that f and g have different domains). Whenever $\mathcal{M}$ is o-minimal, we refer jointly to the order topology on M and induced product topology on M^n as the *Euclidean topology*. Given a definable set $X \subseteq M^n$, we denote its closure in the Euclidean topology by $\mathrm{cl}(X)$, and its frontier by $\partial(X) = \mathrm{cl}(X) \setminus X$. We also denote the o-minimal dimension of X by $\dim X$.

2.2. ***Intersecting families of sets and refinements.*** We say that a family of sets $\mathcal{S}$ is *n-consistent* if every subfamily of cardinality at most n has nonempty intersection. A family is *consistent* if it is n-consistent for every n. We say that $\mathcal{S}$ is *n-inconsistent* if every subfamily of cardinality n has empty intersection.

A family of sets $\mathcal{S}$ has the *(p, q)-property*, for cardinals $p \geq q > 0$, if the sets in $\mathcal{S}$ are nonempty and, for every p distinct sets in $\mathcal{S}$, there exist q among them with nonempty intersection. Note that $\mathcal{S}$ does not have the (p, q)-property if and only if it either contains the empty set or there exists a subfamily of $\mathcal{S}$ of size p that is q-inconsistent.

A family of sets $\mathcal{S}$ is *downward directed* if, for every $F_0, F_1 \in \mathcal{S}$, there exists $F_2 \in \mathcal{S}$ such that $F_2 \subseteq F_0 \cap F_1$. Equivalently if for every finite $\mathcal{F} \subseteq \mathcal{S}$ there exists $F \in \mathcal{S}$ with $F \subseteq \bigcap \mathcal{F}$.

Given a family of sets $\mathcal{S}$ and a set X let $X \cap \mathcal{S} = \mathcal{S} \cap X = \{S \cap X : S \in \mathcal{S}\}$. Observe that, if $\mathcal{S}$ is downward directed, then, for every set X, it holds that $X \cap \mathcal{S}$ is downward directed too.

Given two families of sets $\mathcal{S}$ and $\mathcal{F}$, we say that $\mathcal{F}$ *is a refinement of* $\mathcal{S}$, or that $\mathcal{F}$ *refines* $\mathcal{S}$ if, for every $S \in \mathcal{S}$, there exists $F \in \mathcal{F}$ with $F \subseteq S$. Observe that, if $\mathcal{F}$ is a downward directed refinement of $\mathcal{S}$, then, for every finite subfamily $\mathcal{G} \subseteq \mathcal{S}$, there exists some $F \in \mathcal{F}$ with $F \subseteq \bigcap \mathcal{G}$.

Given a family of sets $\mathcal{S}$ and a set X we say that X is a *transversal of* $\mathcal{S}$ if it intersects every set in $\mathcal{S}$ (i.e., $\varnothing \notin X \cap \mathcal{S}$). In this paper we are interested in the property that a definable family of sets has a finite transversal, as a weakening of the property of having nonempty intersection (i.e., having a transversal of size 1).

The following lemma will be used throughout the paper. We leave the easy proof to the reader.

Lemma 2.1. *Let $\mathcal{S}$ be a downward directed family of sets and $\mathcal{X}$ be a finite covering of a set X. If $S \cap X \neq \varnothing$ for every $S \in \mathcal{S}$, then there exists some $Y \in \mathcal{X}$ such that $S \cap Y \neq \varnothing$ for every $S \in \mathcal{S}$. In particular, if $\mathcal{S}$ has a finite transversal, then $\bigcap \mathcal{S} \neq \varnothing$.*

2.3. ***Type preliminaries.*** All the types that we consider are consistent and, unless otherwise specified, complete over M. We denote the set of these types by $S(M)$. We denote by $S_n(M)$ the set of n-types in $S(M)$. We resort often and without warning to the common model-theoretic convention of identifying types with the family of sets defined by formulas in it. For a definable set $X \subseteq M^n$, we denote by $S_X(M)$ the family of all types $p \in S_n(M)$ with $X \in p$ (namely types that concentrate on X). We will investigate partial types which are downward directed,[1] and the refinement relation between partial types.

Recall that a type $p(x) \in S(M)$ is definable if, for every formula $\varphi(x, y) \in \mathcal{L}$, there is another formula $\psi(y) \in \mathcal{L}(M)$ such that $\psi(M) = \{b \in M^{|y|} : \psi(x, b) \in p(x)\}$. It is definable over $A \subseteq M$ if these formulas $\psi(y)$ can be chosen in $\mathcal{L}(A)$. Note that, if a type is definable, then its projection $\pi(p)$ is definable too.

Given a formula $\varphi(x)$ let $\varphi^1(x) = \varphi(x)$ and $\varphi^0(x) = \neg\varphi(x)$. Given a type $p(x)$ and a formula $\varphi(x, y)$, recall that the restriction of $p(x)$ to $\varphi(x, y)$ is the subtype $p|_\varphi(x) = \{\varphi^i(x, b) \in p(x) : i \in \{0, 1\},\ b \in M^{|y|}\}$. We denote by $p|^1_\varphi(x)$ the restriction of $p(x)$ to "positive" instances of $\varphi(x, y)$, i.e., $p|^1_\varphi(x) = \{\varphi(x, b) \in p(x) : b \in M^{|y|}\}$.

3. O-minimal VC theory and forking

3.1. VC ***theory.*** The following is an ad hoc presentation of the notion of VC-codensity and related results, with applications in Sections 3.2 and 5. For a more standard treatment of Vapnik–Chervonenkis (VC) theory in a model-theoretic context see [Aschenbrenner et al. 2016].

[1] In the literature this property among types is also denoted by 1-*compressible*.

A pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a family of subsets of X, is called a *set system*. For a subfamily $\mathcal{F} \subseteq \mathcal{S}$, let $\mathrm{BA}(\mathcal{F})$ denote the collection of *Boolean atoms* of $\mathcal{F}$, by which we mean the family of all maximal nonempty intersections of sets in $\mathcal{F} \cup \{X \setminus S : S \in \mathcal{F}\}$. The *dual shatter function* of $\mathcal{S}$ is the function $\pi^*_{\mathcal{S}} : \omega \to \omega$ given by

$$\pi^*_{\mathcal{S}}(n) = \max_{\mathcal{F} \subseteq \mathcal{S},\, |\mathcal{F}| = n} |\mathrm{BA}(\mathcal{F})|.$$

The VC-*codensity of* $\mathcal{S}$, denoted by $\mathrm{vc}^*(\mathcal{S})$, is the infimum over all real numbers $r \geq 0$ such that $\pi^*_{\mathcal{S}}(n) = O(n^r)$ (that is, $\pi^*_{\mathcal{S}}(n)/n^r$ is bounded at infinity). Observe that $\mathrm{vc}^*(\mathcal{S})$ is independent of the ambient set X, and so throughout we omit it from our terminology. A theory T is *NIP* (*not the independence property*) if every definable family of sets in every model of T has finite VC-codensity. Every o-minimal theory is NIP [van den Dries 1998, Chapter 5].

For convenience we state the Alon–Kleitman–Matoušek (p, q)-theorem in terms of VC-codensity. For a finer statement see [Matoušek 2004, Theorem 4].

Fact 3.1 (Alon–Kleitman–Matoušek (p, q)-theorem [Matoušek 2004]). Let $p \geq q > 0$ be natural numbers and let $\mathcal{S}$ be a set system such that $\mathrm{vc}^*(\mathcal{S}) < q$. Then there is $n < \omega$ such that, for every finite subfamily $\mathcal{F} \subseteq \mathcal{S}$, if $\mathcal{F}$ has the (p, q)-property, then it has a transversal of size at most n.

The following easy corollary will be used in the proof of Corollary 3.7.

Corollary 3.2. *Let $p \geq q > 0$ be natural numbers and let $\mathcal{S}$ be a set system such that $\mathrm{vc}^*(\mathcal{S}) < q$. If $\mathcal{S}$ has the (p, q)-property, then, for every $0 < q' < \omega$, there exists some natural number $p' = p'(q') \geq q'$ such that $\mathcal{S}$ has the (p', q')-property. In particular, $\mathcal{S}$ has the (ω, q')-property for every $0 < q' < \omega$.*

Proof. Let $\mathcal{S}$ be as in the corollary, satisfying the (p, q)-property. Let n be as described by Fact 3.1. For any given $q' > 0$, let $p' = n(q' - 1) + 1$. Consider an arbitrary subfamily $\mathcal{F}$ of $\mathcal{S}$ of size p'. By Fact 3.1, $\mathcal{F}$ has a transversal A of size at most n. By the definition of p', there must exist some $a \in A$ such that $|F \in \mathcal{F} : a \in F| \geq q'$. It follows that $\mathcal{S}$ has the (p', q')-property. □

The following fact is a reformulation of the main result for weakly o-minimal structures (a class which contains o-minimal structures) in [Aschenbrenner et al. 2016] by Aschenbrenner, Dolich, Haskell, Macpherson and Starchenko. It was previously proved for o-minimal structures by Wilkie (unpublished) and Johnson and Laskowski [2010], and for o-minimal expansions of the field of reals by Karpinski and Macintyre [2000].

Fact 3.3 [Aschenbrenner et al. 2016, Theorem 6.1]. Let $\mathcal{M}$ be an o-minimal structure and let $\mathcal{S}$ be a definable family of subsets of M^n. Then $\mathrm{vc}^*(\mathcal{S}) \leq n$.

We will apply Fact 3.3 in subsequent sections through the slight improvement given by the next corollary.

Corollary 3.4. *Let $\mathcal{M}$ be an o-minimal structure and let $\mathcal{S}$ be a definable family of sets with $n = \dim \bigcup \mathcal{S}$. Then $\mathrm{vc}^*(\mathcal{S}) \leq n$.*

The proof of Corollary 3.4 follows immediately from the following lemma and o-minimal cell decomposition, the latter implying that, if X is a definable set in an o-minimal structure $\mathcal{M}$ with $\dim X \leq n$, then X admits a finite partition into definable subsets, each of which is in definable bijection with a subset of M^n.

Lemma 3.5. *Let $\mathcal{S}$ be a set system and let $X_1, \ldots, X_m$ be sets such that $\bigcup \mathcal{S} \subseteq \bigcup_{i \leq m} X_i$. Then*

$$\mathrm{vc}^*(\mathcal{S}) = \max_{1 \leq i \leq m} \mathrm{vc}^*(X_i \cap \mathcal{S}).$$

Proof. First note that, for every $i \leq m$ and finite subfamily $\mathcal{F} \subseteq \mathcal{S}$, we have $\mathrm{BA}(X_i \cap \mathcal{F}) \leq \mathrm{BA}(\mathcal{F}) + 1$, meaning that $\pi^*_{X_i \cap \mathcal{S}}(n) \leq \pi^*_{\mathcal{S}}(n) + 1$ for every n, and consequently $\mathrm{vc}^*(X_i \cap \mathcal{S}) \leq \mathrm{vc}^*(\mathcal{S})$.

For the opposite inequality, let $\mathcal{F}$ be a finite subfamily of $\mathcal{S}$. Observe that

$$\mathrm{BA}(\mathcal{F}) \leq \mathrm{BA}(X_1 \cap \mathcal{F}) + \cdots + \mathrm{BA}(X_m \cap \mathcal{F}).$$

Consequently

$$\pi^*_{\mathcal{S}}(n) \leq \pi^*_{X_1 \cap \mathcal{S}}(n) + \cdots + \pi^*_{X_m \cap \mathcal{S}}(n)$$

for every n. It follows that, for any real number $r \geq 0$, if $\pi^*_{X_i \cap \mathcal{S}}(n) = O(n^r)$ for all $i \leq m$, then $\pi^*_{\mathcal{S}}(n) = O(n^r)$. Hence there must exist some $i \leq m$ such that $\mathrm{vc}^*(\mathcal{S}) \leq \mathrm{vc}^*(X_i \cap \mathcal{S})$. □

Since throughout this paper p and q are employed as terminology for types, in subsequent sections we address the (p, q)-property in terms of m and n, e.g., the (m, n)-property.

3.2. ***Forking, dividing and definable types.*** In this section we recall some facts about nonforking formulas in o-minimal theories, and derive some consequences which we will need in Section 5. This is the subject of ongoing research among NIP theories [Simon 2015]. Throughout we fix an $|M|^+$-saturated elementary extension $\mathcal{U} = (U, \ldots)$ of $\mathcal{M}$.

Recall that a formula $\varphi(x, b) \in \mathcal{L}(U)$ is said to *n-divide over $A \subseteq U$*, for some $n \geq 1$, if there exists a sequence of elements $(b_i)_{i<\omega}$ in $U^{|b|}$, with $\mathrm{tp}(b_i/A) = \mathrm{tp}(b/A)$ for every i, such that $\{\varphi(x, b_i) : i < \omega\}$ is n-inconsistent. Equivalently, $\varphi(x, b)$ is said to n-divide over A if the family $\{\varphi(U, b') : \mathrm{tp}(b'/A) = \mathrm{tp}(b/A)\}$ does not have the (ω, n)-property. A formula $\varphi(x, b)$ *divides* if it n-divides for some n. Conversely, a formula $\varphi(x, b)$ does not divide over A if and only if the family $\{\varphi(U, b') : \mathrm{tp}(b'/A) = \mathrm{tp}(b/A)\}$ has the (ω, n)-property for every n. Hence, not dividing is an intersection property.

A formula *forks over A* if it implies a finite disjunction of formulas that divide each over A. In NTP_2 theories (a class which includes NIP and simple theories) forking and dividing over a model are equivalent notions [Chernikov and Kaplan 2012, Theorem 1.1].

The next equivalence was proved first for o-minimal expansions of ordered fields[2] by Dolich [2004] (where he considered forking over small sets and not just models) and for unpackable VC-minimal theories, a class which includes o-minimal theories, by Cotter and Starchenko [2012]. The best generalization up to date is due to Simon and Starchenko [2014], and applies to a large class of dp-minimal theories (for details and precise definitions of unpackable VC-minimal and dp-minimal theory see [Cotter and Starchenko 2012; Simon and Starchenko 2014] respectively). We state the result for o-minimal theories.

Fact 3.6. Let T be an o-minimal $\mathcal{L}$-theory with monster model $\mathcal{U}$. Let $\mathcal{M} \models T$ and $\varphi(x, b) \in \mathcal{L}(U)$. The following are equivalent:

(i) $\varphi(x, b)$ does not fork (equivalently, by [Chernikov and Kaplan 2012], does not divide) over M.

(ii) $\varphi(x, b)$ extends to an M-definable type in $S_{|x|}(U)$.

In Section 5.2 we will apply Fact 3.6 in the form of the following corollary.

Corollary 3.7. *Let $\mathcal{M}$ be an o-minimal structure and $\mathcal{S}$ be a definable family of nonempty subsets of M^k. If there exist natural numbers $m \geq n > \mathrm{vc}^*(\mathcal{S})$ such that $\mathcal{S}$ has the (m, n)-property, then there exists a finite covering $\{\mathcal{S}_1, \ldots, \mathcal{S}_l\}$ of $\mathcal{S}$ by definable subfamilies such that, for every $i \leq l$, the family $\mathcal{S}_i$ extends to a definable type in $S_k(M)$.*

Proof. Let $\varphi(x, y) \in \mathcal{L}$ and $\psi(y) \in \mathcal{L}(M)$ be formulas such that $\mathcal{S} = \{\varphi(M, b) : b \in \psi(M)\}$. If $\mathcal{S}$ does not admit a covering as described in the corollary then, by model-theoretic compactness, there exists some $\boldsymbol{b} \in \psi(U)$ such that $\varphi(U, \boldsymbol{b})$ does not extend to an M-definable type in $S_{|x|}(U)$. On the other hand, by Corollary 3.2, the family $\mathcal{S}$ has the (ω, n)-property for every $n > 0$, and consequently the formula $\varphi(x, \boldsymbol{b})$ does not divide over M. So, by Fact 3.6, $\varphi(x, \boldsymbol{b})$ extends to an M-definable type in $S_{|x|}(U)$, contradiction. □

Remark 3.8. By [Andújar Guerrero 2021b, Theorem 3.21; Cotter and Starchenko 2012, Corollary 5.6], Fact 3.6 still holds if we substitute M with any (small) set $A \subseteq U$. It follows that, in Corollary 3.7, if $\mathcal{S}$ is A-definable for some $A \subseteq M$, then the finite covering $\{\mathcal{S}_1, \ldots, \mathcal{S}_l\}$ can be chosen so that each $\mathcal{S}_i$ extends to an A-definable type in $S_k(M)$.

[2]Dolich specifically works with "nice" o-minimal theories, a certain class of theories which includes o-minimal expansions of ordered fields.

Remark 3.9. There is a close relation between (p, q)-theorems and so-called fractional Helly theorems (see [Matoušek 2004]), both of which branched from the classical Helly theorem. In its infinite version, this classical theorem states that every family of closed and bounded convex subsets of $\mathbb{R}^n$ that is $(n+1)$-consistent has nonempty intersection. Aschenbrenner and Fischer [2011, Theorem B] proved a definable version of Helly's theorem (i.e., for definable families of closed and bounded convex sets) in definably complete expansions of real closed fields.

Our Theorem A and the arguments in Section 3.2 of [Aschenbrenner and Fischer 2011] allow an obvious generalization of the o-minimal part of Aschenbrenner and Fischer's definable Helly theorem, by asking that the sets be definably compact and closed in some (any) definable topology, instead of closed and bounded in the Euclidean sense. Perhaps more interestingly, by using Corollary 3.7 to adapt the second proof of Theorem B in [loc. cit.] (the one right below Theorem 3.7), one may show that, in an o-minimal expansion $\mathcal{M}$ of an ordered field, every definable family of convex subsets of M^n that is $(n+1)$-consistent extends to a definable type in $S_n(M)$.

4. O-minimal types

Throughout this section we assume that our structure $\mathcal{M}$ is o-minimal. Our aim is to investigate the relationship between definable types and definable downward directed families of sets, in order to apply the results in Section 5. Our two main results, Propositions 4.1 and 4.3, are of independent interest.

Proposition 4.1 below can be seen as a strong nonparameter form of distal cell decomposition within o-minimality (see Theorem 21(2) in [Chernikov and Simon 2015]). It implies that every definable family of sets that extends to a definable type admits a refinement given by a definable downward directed family.

Proposition 4.1. *Let $p(x) \in S(M)$ be a type and $\varphi(x, y)$ be a formula. There exists another formula $\psi(x, z)$ such that $p|^1_\psi(x)$ is downward directed and*

$$p|^1_\psi \vdash p|_\varphi.$$

In particular, for every finite subtype $q \subseteq p|_\varphi$, there exists $c \in M^{|z|}$ such that $\psi(x, c) \in p(x)$ and $\psi(x, c) \vdash q(x)$.

To prove the above proposition we will use the following easy lemma, whose proof we leave to the reader.

Lemma 4.2. *Let $p(x)$ be a type and $q_1(x), \ldots, q_k(x)$ be finitely many partial subtypes of $p(x)$. Suppose that, for every $i \leq k$, there exists a formula $\psi_i(x, z_i)$ such that $p|^1_{\psi_i}$ is downward directed and $p|^1_{\psi_i} \vdash q_i$. Then the conjunction*

$$\psi(x, z_1, \ldots, z_k) = \text{“}\bigwedge_{i \leq k} \psi_i(x, z_i)\text{”}$$

satisfies that $p|^1_\psi$ *is downward directed and*

$$p|^1_\psi \vdash \bigcup_{i \leq k} q_i.$$

We now present the proof of the proposition.

Proof of Proposition 4.1. We proceed by induction on $|x|$. We may assume throughout that $p(x)$ is not realized, since otherwise it suffices to have $\psi(x, z)$ be the formula $x = z$, where $|x| = |z|$.

Case $|x| = 1$: By o-minimality it suffices to have $\psi(x, z_1, z_2)$, with $|z_1| = |z_2| = 1$, be one of the three formulas

$$(z_1 < x) \wedge (x < z_2), \quad z_1 < x, \quad x < z_1.$$

Case $|x| > 1$: Throughout let $x = (u, t)$, where $|t| = 1$. Recall that $\pi(p) \in S_{|u|}(M)$ denotes the projection of the type p to the first $|u|$ coordinates; i.e., $\pi(p)(u)$ is the family of all formulas $\lambda(u)$ such that $\lambda(u) \wedge (t = t)$ is in $p(x)$.

Suppose that there exists a definable partial function $f : M^{|x|-1} \to M$ whose graph is contained in p. By extending f if necessary to a constant function outside its domain we may assume that the domain of f is in fact $M^{|x|-1}$. We may apply the induction hypothesis to the type $\pi(p)$ and formula

$$\varphi_f(u, y) = \text{“}\exists t((t = f(u)) \wedge \varphi(u, t, y))\text{”},$$

and obtain a formula $\psi_f(u, z_f)$ as described in the proposition. This allows us to construct our desired formula ψ as follows:

$$\psi(x, z_f) = \psi(u, t, z_f) = \text{“}(t = f(u)) \wedge \psi_f(u, z_f)\text{”}.$$

We show that $\psi(x, z_f)$ has the desired properties. Observe that, since the graph of f is contained in p, for every $b \in M^{|y|}$ and $i \in \{0, 1\}$, the formula $\varphi^i(x, b)$ belongs in p if and only if $\varphi^i_f(u, b)$ belongs in $\pi(p)$. An analogous statement holds for $\psi(x, z_f)$ and $\psi_f(u, z_f)$. In particular, we may define

$$C = \{c \in M^{|z_f|} : \psi_f(u, c) \in \pi(p)\} = \{c \in M^{|z_f|} : \psi(x, c) \in p\}.$$

Since, by the induction hypothesis, the family of formulas $\{\psi_f(u, c) : c \in C\}$ is downward directed, the same clearly holds for $p|^1_\psi = \{\psi(x, c) : c \in C\}$. Moreover, for any formula of the form $\varphi^i(x, b)$ in $p(x)$, where $b \in M^{|y|}$ and $i \in \{0, 1\}$, there exists $c \in C$ such that $\psi_f(u, c) \vdash \varphi^i_f(u, b)$, and so $\psi(x, c) \in p$ and $\psi(x, c) \vdash \varphi^i(x, b)$. Hence $p|^1_\psi \vdash p|_\varphi$.

Hence onwards we assume that there does not exist a definable partial function $f : M^{|x|-1} \to M$ whose graph is contained in p.

In the next paragraphs we reduce the remainder of the proof to the case where, for every $b \in M^{|y|}$, if the formula $\varphi(x, b)$ is in p, then it defines a set of the form $(f_b, +\infty)$ for some partial function $M^{|x|-1} \to M \cup \{-\infty\}$.

By o-minimal uniform cell decomposition [van den Dries 1998, Chapter 3, Proposition 3.5], there exist finitely many formulas $\sigma_1(x, y), \ldots, \sigma_k(x, y)$ such that, for every $b \in M^{|y|}$, the family $\{\sigma_1(M, b), \ldots, \sigma_k(M, b)\}$ is an o-minimal cell decomposition of $M^{|x|}$ compatible with $\varphi(M, b)$. Observe that

$$\bigcup_{i \leq k} p|^1_{\sigma_i} \vdash p|_\varphi.$$

By Lemma 4.2, it suffices to pass to an arbitrary $i \leq k$ and prove the proposition for $p|^1_{\sigma_i}$ in place of $p|_\varphi$. Hence onwards let us assume that, for every $b \in M^{|y|}$, the formula $\varphi(x, b)$ defines a cell and, moreover, if $\varphi(x, b) \in p$, then, by the assumption on p, this cell is of the form (f_b, g_b) for f_b and g_b partial functions $M^{|x|-1} \to M \cup \{-\infty, +\infty\}$ with the same domain and with $f_b < g_b$. Additionally, to prove the proposition it suffices to find $\psi(x, z)$ such that $p|^1_\psi$ is downward directed and $p|^1_\psi(x) \vdash p|^1_\varphi(x)$.

Recall the notation $x = (u, t)$, with $|t| = 1$. Let $B = \{b \in M^{|y|} : \varphi(x, b) \in p\}$. Let $\varphi_0(x, y) (= \varphi_0(u, t, y))$ denote the formula $\exists s (s \leq t) \wedge \varphi(u, s, y)$, and similarly let $\varphi_1(x, y)$ be the formula $\exists s (s \geq t) \wedge \varphi(u, s, y)$. That is, for every $b \in B$, the formulas $\varphi_0(x, b)$ and $\varphi_1(x, b)$ define the sets $(f_b, +\infty)$ and $(-\infty, g_b)$ respectively. In particular, when $b \in B$, the formula $\varphi(x, b)$ is equivalent to the conjunction $\varphi_0(x, b) \wedge \varphi_1(x, b)$. So $p|^1_{\varphi_0} \cup p|^1_{\varphi_1} \vdash p|^1_\varphi$. By Lemma 4.2, to prove the proposition it suffices to find formulas $\psi_0(x, z_0)$ and $\psi_1(x, z_1)$ such that, for every $j \in \{0, 1\}$, the restriction $p|^1_{\psi_j}$ is downward directed and $p|^1_{\psi_j} \vdash p|^1_{\varphi_j}$. We prove this for $j = 0$, since the remaining case is analogous. For simplicity of notation we also assume that φ is equivalent to φ_0.

Consider the formula

$$\theta(u, y, y') = \text{“}\exists s\, \varphi(u, s, y) \wedge \exists t\, \varphi(u, t, y') \wedge \forall t (\varphi(u, t, y') \to \varphi(u, t, y))\text{”}.$$

For every $b, b' \in M^{|y|}$ note that it holds that

$$\theta(u, b, b') \wedge \varphi(x, b') \vdash \varphi(x, b). \tag{1}$$

In particular, if b and b' are in B, then $\theta(u, b, b')$ defines the set of all u such that $f_b(u) \leq f_{b'}(u)$.

Recall the notation $\pi(p)$ for the projection of p to the first $|u| = |x| - 1$ coordinates. By the induction hypothesis on the formula $\theta(u, y, y')$ and the type $\pi(p)$, there exists a formula $\xi(u, z_\xi)$ such that $\pi(p)|^1_\xi$ is downward directed and $\pi(p)|^1_\xi \vdash \pi(p)|_\theta$.

Finally, let $z = (z_\xi, y)$ and

$$\psi(x, z) = \psi(u, t, z_\xi, y) = \text{“}\xi(u, z_\xi) \wedge \varphi(x, y)\text{”}.$$

Clearly by construction $p|^1_\psi \vdash p|^1_\varphi$. We show that $p|^1_\psi$ is downward directed.

Let $D = \{d \in M^{|z_\xi|} : \xi(u, d) \in \pi(p)\}$. Note that $\xi(u, d) \wedge \varphi(x, b)$ belongs in p if and only if $b \in B$ and $d \in D$. Let us fix $b, b' \in B$ and $d, d' \in D$. Recall that $\varphi(M, b) = (f_b, +\infty)$ and $\varphi(M, b') = (f_{b'}, +\infty)$. Consider the formula $\zeta(u, b, b') =$ "$\exists s \varphi(u, s, b) \wedge \exists t \varphi(u, t, b')$", which defines the intersection of the domains of f_b and $f_{b'}$. Clearly $\zeta(u, b, b') \in \pi(p)$. Observe that the sets $\theta(M, b, b')$ and $\theta(M, b', b)$ cover $\zeta(M, b, b')$, and so at least one of them belongs in $\pi(p)$. Without loss of generality we assume that $\theta(u, b, b') \in \pi(p)$.

Let $d'' \in D$ be such that $\xi(u, d'') \vdash \theta(u, b, b')$. By (1) we have that

$$\xi(u, d'') \wedge \varphi(x, b') \vdash \varphi(x, b).$$

By downward directedness let $d''' \in D$ be such that

$$\xi(u, d''') \vdash \xi(u, d) \wedge \xi(u, d') \wedge \xi(u, d'').$$

We conclude that

$$\xi(u, d''') \wedge \varphi(x, b') \vdash \xi(u, d) \wedge \varphi(x, b) \wedge \xi(u, d') \wedge \varphi(x, b'),$$

or equivalently

$$\psi(x, d''', b') \vdash \psi(x, d, b) \wedge \psi(x, d', b').$$

So $p|^1_\psi$ is downward directed. □

It seems likely that Proposition 4.1 is also true in weakly o-minimal structures. As far as the author knows, it is open among distal dp-minimal structures.

The following proposition shows that every definable downward directed family of nonempty sets extends to a definable type $p(x) \in S(M)$, and furthermore that $p(x)$ can be chosen so that, for some formula $\psi(x, z)$, the restriction $p|^1_\psi(x)$ is a basis (in the sense of filter basis) of cells for $p(x)$. We present a shorter proof than the one in [Andújar Guerrero 2021b, Lemma 2.7], applying ideas communicated to the author by Will Johnson.

Proposition 4.3. *Let $\varphi(x, y)$ be a formula and $B \subseteq M^{|y|}$ be such that the family $\{\varphi(x, b) : b \in B\}$ is consistent and downward directed. Then there exists a type $p(x) \in S(M)$ with $\{\varphi(x, b) : b \in B\} \subseteq p(x)$, and a formula $\psi(x, z)$ such that $p|^1_\psi$ defines a family of cells, is downward directed, and $p|^1_\psi \vdash p$. Furthermore, if B is definable, then $p(x)$ can be chosen definable too.*

In particular, for every definable downward directed family of nonempty sets $\mathcal{S}$, there exists a definable downward directed family of cells $\mathcal{F}$ which refines $\mathcal{S}$ and furthermore $\mathcal{F}$ generates a definable type in $S(M)$.

Proof. We devote most of the proof to showing the existence of $p(x)$ and $\psi(x, z)$ as described in the proposition except for the condition that $p|^1_\psi$ defines a family of cells. In the next two paragraphs we describe how, once we have these, by passing

if necessary to a formula in a cell decomposition of $\psi(x, z)$ we may assume that $p|^1_\psi$ defines a family of cells, completing the proof.

Applying uniform cell decomposition [van den Dries 1998, Chapter 3, Proposition 3.5] to the formula $\psi(x, z)$, let $\sigma_i(x, z)$, for $i \leq k$, denote formulas such that, for every $c \in M^{|z|}$, the sets $\sigma_i(M, c)$, for $i \leq k$, are a cell partition of $\psi(M, c)$. We claim that there exists some $i \leq k$ such that the family $p|^1_{\sigma_i}$ is downward directed and $p|^1_{\sigma_i} \vdash p|^1_\psi$ (hence $p|^1_{\sigma_i} \vdash p$). To see this let $p|^1_\psi = \{\psi(x, c) : c \in C\}$ and, for every $i \leq k$, let $p|^1_{\sigma_i} = \{\sigma_i(x, c) : c \in C(i)\}$. We show that there exists $i \leq k$ such that, for every $c \in C$, there exists some $c' \in C(i)$ with $\sigma_i(x, c') \vdash \psi(x, c)$ (i.e., $p|^1_{\sigma_i}$ refines $p|^1_\psi$); hence $p|^1_{\sigma_i} \vdash p|^1_\psi$ and, using the facts that $p|^1_\psi \vdash p$ and $p|^1_\psi$ is downward directed, it is also easy to derive that $p|^1_{\sigma_i}$ is downward directed.

Towards a contradiction suppose that, for every $i \leq k$, there exists some $c_i \in C$ such that $\sigma_i(x, c) \nvdash \psi(x, c_i)$ for every $c \in C(i)$. By downward directedness of $p|^1_\psi$, let $c_{k+1} \in C$ be such that $\psi(x, c_{k+1}) \vdash \bigwedge_{i \leq k} \psi(x, c_i)$. It follows that $\sigma_i(x, c) \nvdash \psi(x, c_{k+1})$ for every $i \leq k$ and $c \in C(i)$. However this contradicts the facts that $\psi(x, c_{k+1}) \in p(x)$ and $\vdash \psi(x, c_{k+1}) \leftrightarrow \bigvee_{i \leq k} \sigma_i(x, c_{k+1})$, which imply that there exists some $i \leq k$ with $\sigma_i(x, c_{k+1}) \in p(x)$ (i.e., $c_{k+1} \in C(i)$) and $\sigma_i(x, c_{k+1}) \vdash \psi(x, c_{k+1})$.

We now begin the proof of the existence of a type $p(x) \in S(M)$ extending $\{\varphi(x, b) : b \in B\}$ and a formula $\psi(x, z)$ satisfying that $p|^1_\psi$ is downward directed and $p|^1_\psi \vdash p$ (i.e., $p|^1_\psi$ is a basis for p). We prove the case where B is definable. In the general case the same proof applies by considering throughout, instead of definable families of sets, subfamilies of fibers of definable sets in general. To make the presentation more succinct, we work explicitly with set notation rather than formulas.

We introduce some useful terminology. For a definable family of nonempty sets $\mathcal{F}$, let $d(\mathcal{F})$ denote the smallest $n \geq 0$ such that, for every set $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ with $G \subseteq F$ and $\dim(G) = n$. Let $c(\mathcal{F})$ denote the smallest $m \geq 1$ such that, for every set $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ with $G \subseteq F$ such that G has exactly m definably connected components.

Let $\mathcal{S} = \{\varphi(M, b) : b \in B\}$. Recall that a family of sets $\mathcal{F}$ is a refinement of $\mathcal{S}$ if, for every $S \in \mathcal{S}$, there exists $F \in \mathcal{F}$ with $F \subseteq S$. Let $\mathcal{DR}$ denote the collection of all definable downward directed refinements of $\mathcal{S}$ which do not contain the empty set. Throughout we fix $n = \min\{d(\mathcal{F}) : \mathcal{F} \in \mathcal{DR}\}$ and $m = \min\{c(\mathcal{F}) : \mathcal{F} \in \mathcal{DR},\ d(\mathcal{F}) = n\}$. We also fix $\mathcal{F} \in \mathcal{DR}$ with $d(\mathcal{F}) = n$ and $c(\mathcal{F}) = m$. We show that $\mathcal{F}$ generates a (clearly definable) type in $S_{|x|}(M)$.

Towards a contradiction we assume that $\mathcal{F}$ does not generate a type in $S_{|x|}(M)$, meaning that there exists a definable set $X \subseteq M^{|x|}$ satisfying that, for every $F \in \mathcal{F}$, $F \cap X \neq \varnothing$ and $F \setminus X \neq \varnothing$. Let us fix some $F_0 \in \mathcal{F}$ with $\dim F_0 = n$.

Consider the boundary of $F_0 \cap X$ in F_0, i.e., the set

$$Z = F_0 \cap (\partial(F_0 \cap X) \cup \partial(F_0 \setminus X)).$$

Since $\dim F_0 = n$, by o-minimality we have that $\dim Z < n$. It follows that $\mathcal{F} \cap Z$ is a downward directed refinement of $\mathcal{S}$ composed of sets of dimension lower than n. By the definition of n, there must exist a set $F_1 \in \mathcal{F}$ with $F_1 \cap Z = \varnothing$. Now let $\mathcal{G} = \{F \cap X : F \in \mathcal{F},\ F \subseteq F_0 \cap F_1\}$. By the downward directedness of $\mathcal{F}$ and the definition of X, the definable family $\mathcal{G}$ is a downward directed refinement of $\mathcal{S}$ that does not contain the empty set (i.e., $\mathcal{G} \in \mathcal{DR}$). By the definition of n it follows that $d(\mathcal{G}) = n$. We show that $c(\mathcal{G}) < m$, contradicting the definition of m.

We show that, for every $F \in \mathcal{F}$ with $F \subseteq F_0 \cap F_1$, the intersection $F \cap X \in \mathcal{G}$ has strictly fewer definably connected components than F. In particular, this implies that, for every set $F \in \mathcal{F}$ with $F \subseteq F_0 \cap F_1$, if $G \in \mathcal{F}$ is a subset of F with exactly m definably connected components, then $G \cap X \in \mathcal{G}$ has less than m definably connected components, and so $c(\mathcal{G}) < m$ as desired.

Let Y denote the interior of $F_0 \cap X$ in F_0, i.e., $Y = F_0 \setminus \mathrm{cl}(F_0 \setminus X)$. Let $C \subseteq F_0$ be a definably connected set. If $C \cap Z = \varnothing$, then by the definition of Z clearly C must be a subset of either Y or $F_0 \setminus (Y \cup Z)$. Since $Y \subseteq F_0 \cap X \subseteq Y \cup Z$, we know C must be a subset of either $F_0 \cap X$ or $F_0 \setminus X$. Now let us fix a set $F \in \mathcal{F}$ with $F \subseteq F_0 \cap F_1$. Since $F_1 \cap Z = \varnothing$, we have that $F \cap Z = \varnothing$, and so every definably connected component C of F is a subset of either $F_0 \cap X$ or $F_0 \setminus X$. Finally recall that, by the definition of X, the sets $F \cap X$ and $F \setminus X$ are both nonempty. Consequently we conclude that the set $F \cap X$ (as well as $F \setminus X$) has a positive number of definably connected components that is fewer than the number of definably connected components of F. □

In Proposition 4.3, whenever B is definable, one may wonder if $p(x)$ can always be chosen definable over the same parameters as B. This was proved to be false in general by Johnson in [Andújar Guerrero 2021b, Appendix B]. Nevertheless, by [loc. cit., Proposition 2.17] it does hold that every definable downward directed family $\{\varphi(x, b) : b \in B\}$ extends to a type in $S_{|x|}(M)$ definable over the same parameters as B. This can also be proved using Corollary 3.7 and Remark 3.8. For a similar result see [Hrushovski and Loeser 2016, Lemma 4.2.18].

Remark 4.4. Observe that Propositions 4.1 and 4.3 together yield a strong density result for types $p(x)$ satisfying that there is a formula $\varphi(x, y)$ such that $p|_\varphi$ is downward directed and $p|_\varphi \vdash p$, namely types which have a basis (in the sense of filter basis) given by their restriction to a single formula. This is discussed in [Andújar Guerrero 2021b, Remarks 2.13 and 2.22]. In any o-minimal structure every 1-type is of this kind (it is either realized or has a basis of open intervals). On the other hand, it was shown in [Andújar Guerrero et al. 2021, Corollary 32]

that, in an o-minimal expansion of an ordered group, every definable type of this kind contains at least one set of dimension at most 2 (and of dimension at most 1 in o-minimal expansions of ordered fields). Using the Marker–Steinhorn theorem [1994, Theorem 2.1] one derives that, in any o-minimal expansion of the group of reals, there are n-types that do not have a basis given by their restriction to a single formula for every $n > 2$ ($n > 1$ in o-minimal expansions of the field of reals).

5. O-minimal definable compactness

5.1. ***Topological preliminaries.*** We introduce definable (explicitly in the sense of [Flum and Ziegler 1980]) topological spaces and various related definitions.

Definition 5.1. A *definable topological space* (X, τ), with $X \subseteq M^n$, is a topological space such that there exists a definable family of subsets of X which is a basis for τ.

Any definable set in an o-minimal structure with its induced Euclidean topology is a definable topological space. For other examples within o-minimality, see the definable manifold spaces studied in [Pillay 1988] and [van den Dries 1998, Chapter 10], the definable Euclidean quotient spaces of the latter work and [Johnson 2018], the definable normed spaces of [Thomas 2012], and the definable metric spaces of [Walsberg 2015]. See moreover the author's doctoral dissertation [Andújar Guerrero 2021a] for an exhaustive exploration of o-minimal definable topological spaces. For a foundational treatment of definable tame topology generalizing o-minimality, see [Pillay 1987]. For an exploration of dp-minimal tame topology, see the more recent work [Simon and Walsberg 2019], and related independent work [Dolich and Goodrick 2022].

Onwards we contextualize topological notions related to a given topological space (X, τ) by adding the prefix τ, e.g., τ-open, τ-closure etc. We recall some standard definitions.

Definition 5.2. Let (X, τ) be a definable topological space. A *definable curve in* X is a definable map $\gamma : (a, b) \to X$ for some $-\infty \leq a < b \leq +\infty$. We say that it *$\tau$-converges* to $x \in X$ (i.e., x is a *τ-limit of* γ) as $t \to a$ if, for every τ-neighborhood A of x, there exists $t_A \in (a, b)$ such that $\gamma(s) \in A$ whenever $s \in (a, t_A)$. The notion of τ-convergence as $t \to b$ is defined analogously. We denote by $\lim^{\tau}_{t \to a} \gamma(t)$ (respectively $\lim^{\tau}_{t \to b} \gamma(t)$) the set of τ-limit points of x as $t \to a$ (respectively $t \to b$).

We say that γ is *τ-completable* if it τ-converges as $t \to a$ and as $t \to b$.

Given a definable topological space (X, τ) and a set $Y \subseteq X$ we denote the τ-closure of Y by $\mathrm{cl}_\tau(Y)$. It is easy to check that a τ-limit of a definable curve $\gamma : (a, b) \to Y \subseteq X$ is always contained in $\mathrm{cl}_\tau(Y)$. Furthermore, if τ is Hausdorff, then the sets $\lim^{\tau}_{t \to a} \gamma(t)$ and $\lim^{\tau}_{t \to b} \gamma(t)$ are always either empty or a singleton, and in the latter case we abuse terminology by identifying them with their single point.

We will use these facts in Section 5.2 without explanation. To erase ambiguity, at times we also use side convergence notation $t \to a^+$ and $t \to b^-$ (e.g., $\lim^{\tau}_{t\to a^+} \gamma(t)$), with the standard meaning.

The following definition is borrowed from [Hrushovski and Loeser 2016].

Definition 5.3. Let (X, τ) be a definable topological space and p be a (possibly partial) type with $X \in p$. We say that $x \in X$ is a *τ-limit*[3] *of p* if x is contained in the τ-closure of every subset of X in p. If $p(x) \in S_X(M)$, then this is equivalent to saying that x is contained in every τ-closed set in p.

We now present various definitions extracted from the literature (for references see Section 1) which seek to capture the notion of definable compactness. We mostly maintain consistency with [Andújar Guerrero and Johnson 2024] in the names. (In particular we avoid using the adjective "definable" in our terminology to enable an easier read.) A more general approach to definable compactness, including more definitions than the ones in this paper, can be found in [Fornasiero 2015].

Definition 5.4. Let (X, τ) be a definable topological space. Then (X, τ) is

(1) *curve-compact* if every definable curve in X is τ-completable,

(2) *filter-compact* if every downward directed definable family of nonempty τ-closed subsets of X has nonempty intersection,

(3) *type-compact* if every definable type $p(x) \in S_X(M)$ has a τ-limit in X,

(4) *transversal-compact* if every consistent definable family of τ-closed subsets of X has a finite transversal.

The equivalence between curve-compactness and filter-compactness was proved for definable topological spaces in o-minimal expansions of ordered fields in [Andújar Guerrero et al. 2021, Corollary 44]. In this paper we present a deeper characterization in the general o-minimal setting.

5.2. ***Characterizing definable compactness.*** In this section we prove our results on definable compactness for definable topological spaces in o-minimal structures. Throughout we assume that our underlying structure $\mathcal{M}$ is o-minimal.

We devote most of the section to proving the characterization of definable compactness given by Theorem A, which we divide into three propositions. Proposition 5.5 provides the equivalence (2)⇔(3) in the theorem. In Proposition 5.7 we prove, using results from previous sections, the equivalence between (1), (3), (4), (5) and (6). Finally, in Proposition 5.12 we prove the implication (1) ⇒ (7), and the reverse implication when τ is Hausdorff or $\mathcal{M}$ has definable choice. We follow

[3]Fornasiero [2015], as well as Thomas, Walsberg and the author [Andújar Guerrero et al. 2021], use the word "specialization" (borrowed from real algebraic geometry) to refer to limits of types. Here we use instead the terminology from [Hrushovski and Loeser 2016, Chapter 4].

it with an example (Example 5.16) showing that the implication (1) $\Rightarrow$ (7) is strict in general. Throughout we also discuss other notions of definable compactness, and end the section with two additional results: definable compactness is equivalent to classical compactness in o-minimal expansions of $(\mathbb{R}, <)$ (Corollary 5.19), and definable compactness is definable in families (Proposition 5.22).

The equivalence (1) $\Leftrightarrow$ (2) in Proposition 5.5 below corresponds to the equivalence (2) $\Leftrightarrow$ (3) in Theorem A. Note that the proof of this equivalence does not use o-minimality. Hence this characterization of type-compactness holds in any model-theoretic structure. Furthermore, the equivalence of type-compactness with classical compactness always holds whenever the underlying structure $\mathcal{M}$ satisfies that every type in $S(M)$ is definable, as we point out in Remark 5.6 below.

Proposition 5.5. *Let (X, τ) be a definable topological space. The following are equivalent*:

(1) *(X, τ) is type-compact.*

(2) *Every definable family of τ-closed sets that extends to a definable type in $S_X(M)$ has nonempty intersection.*

If $\mathcal{M}$ expands $(\mathbb{R}, <)$, then (1) and (2) are also equivalent to

(3) *(X, τ) is compact.*

Proof. To prove (1) $\Rightarrow$ (2), suppose that (X, τ) is type-compact and let $\mathcal{C}$ be a definable family of τ-closed sets that extends to a definable type $p \in S_X(M)$. Let $x \in X$ be a τ-limit of p. Then clearly $x \in \bigcap \mathcal{C}$.

The key element to the rest of the proof is the fact that every closed set in a topological space is an intersection of basic closed sets.

To prove (2) $\Rightarrow$ (1), let $p \in S_X(M)$ be a definable type. Let $\mathcal{B}$ denote a definable basis (of opens) for the topology τ. Now let $\mathcal{C}$ denote the definable family of basic τ-closed sets in p, i.e., the family of sets C in p of the form $X \setminus B$ for some $B \in \mathcal{B}$. If (2) holds, then there exists some $x \in X$ with $x \in \bigcap \mathcal{C}$. In this case it follows that x is a τ-limit of p.

Finally, suppose that $\mathcal{M}$ expands $(\mathbb{R}, <)$. Clearly, if (X, τ) is compact, then it is type-compact. Conversely, suppose that (X, τ) is type-compact and let $\mathcal{C}$ be a consistent family of τ-closed sets. The intersection $\bigcap \mathcal{C}$ can be rewritten as an intersection of basic closed sets. In particular, we may assume that $\mathcal{C}$ contains only definable sets. Now, by the Marker–Steinhorn theorem [1994, Theorem 2.1], every type over $M = \mathbb{R}$ is definable. Consequently $\mathcal{C}$ extends to a definable type $p \in S_X(M)$. Let x be a τ-limit of p. Then $x \in \bigcap \mathcal{C}$. So (X, τ) is compact. $\square$

Remark 5.6. Note that the equivalence between type-compactness and classical topological compactness shown in Proposition 5.5 holds in every structure satisfying

that all types are definable. For example it remains true in the field of p-adic numbers $(\mathbb{Q}_p, +, \cdot)$, as observed in [Andújar Guerrero and Johnson 2024, Theorem 8.15].

More specifically, if $\varphi(x, y)$ defines a basis $\mathcal{B}$ for the topology τ, i.e., $\mathcal{B} = \{\varphi(M, b) : b \in M^{|y|}\}$, then to have the equivalence between type-compactness and classical compactness it suffices to have that every maximal consistent subfamily of $\{X \setminus B : B \in \mathcal{B}\}$ is definable, which occurs in particular whenever every φ-type (i.e., restrictions of types in $S_{|x|}(M)$ to $\varphi(x, y)$) is definable. Observe that the latter always holds whenever $\varphi(x, y)$ is stable although, as already noted in the proof of [Pillay 1987, Proposition 1.2], every infinite T_1 topological space that has a basis defined by a stable formula must be discrete, and consequently not compact.

Proposition 5.7 below corresponds to the equivalence between (1), (3), (4), (5) and (6) in Theorem A. Its proof relies on Proposition 4.1 and Corollaries 3.4 and 3.7.

Proposition 5.7. *Let (X, τ) be a definable topological space. The following are equivalent*:

(1) *(X, τ) is filter-compact.*

(2) *Every definable family of τ-closed sets that extends to a definable type in $S_X(M)$ has nonempty intersection.*

(3) *Every definable family $\mathcal{C}$ of τ-closed sets with the (m, n)-property, where $m \geq n > \mathrm{vc}^*(\mathcal{S})$, has a finite transversal.*

(4) *Every definable family $\mathcal{C}$ of τ-closed sets with the (m, n)-property, where $m \geq n > \dim \bigcup \mathcal{C}$, has a finite transversal.*

(5) *(X, τ) is transversal-compact.*

Proof. Note that, if a downward directed family of sets has a finite transversal, then, by Lemma 2.1, it has nonempty intersection. Hence (3), (4) and (5) each imply (1). We prove (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Observe that implication (3) $\Rightarrow$ (4) follows from Corollary 3.4, and implication (4) $\Rightarrow$ (5) is trivial, completing the proof.

Proof of (1) $\Rightarrow$ (2). Suppose that (X, τ) is filter-compact and let $\mathcal{C} = \{\varphi(M, b) : b \in B\}$ be a definable family of τ-closed sets that extends to a definable type $p(x) \in S_X(M)$. Let $\psi(x, z)$ be as given by Proposition 4.1 for $\varphi(x, y)$ and $p(x)$. Let $\mathcal{F} = \{\psi(X, c) : \psi(x, c) \in p(x),\ c \in M^{|z|}\}$. By Proposition 4.1, $\mathcal{F}$ is a definable downward directed family of subsets of X which refines $\mathcal{C}$. Let $\mathcal{D} = \{\mathrm{cl}_\tau(F) : F \in \mathcal{F}\}$. Clearly $\mathcal{D}$ is a definable downward directed family of τ-closed sets, so by filter-compactness there exists $a \in \bigcap \mathcal{D}$. Moreover observe that, since the sets in $\mathcal{C}$ are closed, $\mathcal{D}$ is still a refinement of $\mathcal{C}$, implying that $\bigcap \mathcal{D} \subseteq \bigcap \mathcal{C}$, and so $a \in \bigcap \mathcal{C}$.

Proof of (2) $\Rightarrow$ (3). Let $\mathcal{C}$ be a definable family of τ-closed subsets of X with the (m, n)-property, where $m \geq n > \mathrm{vc}^*(\mathcal{C})$. By Corollary 3.7 there exists a finite covering $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ of $\mathcal{C}$ by definable subfamilies, each of which extends to a

definable type in $S_X(M)$. If property (2) holds, then, for each $i \leq k$, there exists some $a_i \in \bigcap \mathcal{C}_i$ in X. The family $\{a_1, \ldots, a_k\}$ is clearly a transversal of $\mathcal{C}$. □

Remark 5.8. We remark that, although omitted from the proof above, the implication (2) ⇒ (1) in Proposition 5.7 (i.e., by Proposition 5.5 the implication *type-compactness* ⇒ *filter-compactness*) can be shown to follow easily from Proposition 4.3. In fact we claim that this implication, as well as (2) ⇒ (5), hold in a more general dp-minimal setting by [Simon and Starchenko 2014, Theorem 5] (see the discussion above Fact 3.6). Additionally, the equivalence (3) ⇔ (5) holds in all in NIP structures by recent work [Kaplan 2024, Corollary 4.9].

Remark 5.9. It was shown in [Peterzil and Steinhorn 1999, Theorem 2.1] that a definable set with the o-minimal Euclidean topology is curve-compact if and only if it is closed and bounded. In [Peterzil and Pillay 2007, Theorem 2.1] the authors extracted from [Dolich 2004] the following. Suppose that our o-minimal structure $\mathcal{M}$ has definable choice (e.g., expands an ordered group). Let $\mathcal{U} = (U, \ldots)$ be a monster model and $\varphi(x, b)$ be a formula in $\mathcal{L}(U)$ such that $\varphi(U, b)$ is closed and bounded (in the Euclidean topology in $U^{|x|}$). If the family $\{\varphi(U, b') : \mathrm{tp}(b'/M) = \mathrm{tp}(b/M)\}$ is consistent, then $\varphi(U, b)$ has a point in $M^{|x|}$. Using a straightforward model-theoretic compactness argument they derive from this that every closed and bounded Euclidean space is transversal-compact [Peterzil and Pillay 2007, Corollary 2.2(i)].

Let (X, τ) be a definable topological space (in $\mathcal{M}$), whose definition in $\mathcal{U}$ we denote by $(X(\mathcal{U}), \tau(\mathcal{U}))$. The property that every formula $\varphi(x, b) \in \mathcal{L}(U)$, satisfying that $\varphi(U, b)$ is $\tau(\mathcal{U})$-closed and the family $\{\varphi(U, b') : \mathrm{tp}(b'/M) = \mathrm{tp}(b/M)\}$ is consistent, satisfies that $\varphi(M, b) \neq \varnothing$, is labeled Dolich's property in [Fornasiero 2015]. As mentioned in the previous paragraph, this property implies transversal-compactness (without any assumption on $\mathcal{M}$), and furthermore one may show, using [Kaplan 2024, Theorem 1.5], that the converse implication (transversal-compactness ⇒ Dolich's property) holds in all NIP structures.

Theorem A completes the characterization of closed and bounded definable sets with the Euclidean topology. Furthermore, it generalizes the aforementioned result of [Peterzil and Pillay 2007] in three ways. First, we drop the assumption of having definable choice in $\mathcal{M}$. Second, we weaken the consistency assumption to having an appropriate (n, m)-property (in their work they actually observe that it suffices to have k-consistency for some k in terms of $|x|$ and $|b|$). Third, we establish, by means of the equivalence with transversal-compactness mentioned in the paragraph above, the relationship between Dolich's property and the other compactness notions in the full generality of any o-minimal definable topological space.

We now prove the connection within o-minimality between filter-compactness and curve-compactness stated in Theorem A, that is, that filter-compactness implies

curve-compactness, and that both notions are equivalent when the topology is Hausdorff or when the underlying o-minimal structure has definable choice. This is Proposition 5.12. We follow the proposition with an example of a non-Hausdorff topological space definable in the dense linear order without endpoints $(M, <)$ that is curve-compact but not filter-compact.

The next lemma allows us to apply definable choice in certain instances even when the underlying structure $\mathcal{M}$ may not have the property.

Lemma 5.10 (definable choice in compact Hausdorff spaces). *Let C be a definable nonempty τ-closed set in a curve-compact Hausdorff definable topological space (X, τ). Let $A \subseteq M$ be such that τ and C are A-definable. Then there exists a point $x \in C \cap \operatorname{dcl}(A)$, where $\operatorname{dcl}(A)$ denotes the set of finite tuples of elements in the definable closure of A.*

Consequently, for every A-definable family $\{\varphi(M, b) : b \in B\}$ of nonempty subsets of X there exists an A-definable choice function $h : B \to X$ such that $h(b) \in \operatorname{cl}_\tau(\varphi(M, b))$ for every $b \in B$.

Proof. We prove the first paragraph of the lemma. The uniform result is derived in the usual way by the use of first-order logic compactness.

For this proof we adopt the convention of the one-point Euclidean space $M^0 = \{\mathbf{0}\}$. In particular, any projection $M^k \to M^0$ is simply the constant function $\mathbf{0}$, and any relation $E \subseteq M^0 \times M^k$ is definable if and only if its projection to M^k is.

Let C, (X, τ) and A be as in the lemma, with $X \subseteq M^m$. Let $n \leq m$ be such that there exists an A-definable function $f : D \subseteq M^n \to C$ for D a nonempty set. If n can be chosen to be zero, then the lemma follows. We prove that this is the case by backwards induction on n.

Note that n can always be chosen equal to m, by letting f be the identity on C. Consider a positive $n \leq m$. For every $x \in M^{n-1}$, let D_x denote the fiber $\{t \in M : \langle x, t\rangle \in D\}$. For each $x \in \pi(D)$, let $s_x = \sup D_x$, and consider the A-definable set $F = \{x \in \pi(D) : s_x \in D_x\}$.

If $F \neq \varnothing$, then let g be the map $x \mapsto f(s_x) : F \to C$. If $F = \varnothing$, then let g be the map $x \mapsto \lim^\tau_{t \to s_x^-} f(x, t) : \pi(D) \to C$ which, by curve-compactness and Hausdorffness, is well-defined. In both cases g is an A-definable nonempty partial function $M^{n-1} \to C$. □

Remark 5.11. Let $A \subseteq M$ and $\mathcal{C}$ be an A-definable family of nonempty τ-closed sets in a curve-compact Hausdorff A-definable topological space (X, τ). Lemma 5.10 implies that, if $\mathcal{C}$ has a finite transversal, then it also has one of the same size in $\operatorname{dcl}(A)$. To prove this it suffices to note that, for every $k \geq 1$, the set of k-tuples of points corresponding to a transversal of $\mathcal{C}$ is A-definable and closed in the product topology, which can easily be shown to be A-definable and curve-compact.

It follows that, whenever $\mathcal{M}$ has definable choice or τ is Hausdorff, the finite transversals in Theorem A (statements (4), (5) and (6)) can always be assumed to be definable over the same parameters as the family of closed sets $\mathcal{C}$ and topology τ.

Proposition 5.12. *Let (X, τ) be a definable topological space. If (X, τ) is filter-compact, then it is curve-compact.*

Suppose that either τ is Hausdorff or $\mathcal{M}$ has definable choice. Then (X, τ) is filter-compact if and only if it is curve-compact.

We prove the left-to-right direction through a short lemma.

Lemma 5.13. *Let (X, τ) be a filter-compact definable topological space. Then (X, τ) is curve-compact.*

Proof. Let $\gamma : (a, b) \to X$ be a definable curve in X. Consider the definable family of τ-closed nested sets $\mathcal{C}_\gamma = \{\mathrm{cl}_\tau\, \gamma[(a, t)] : a < t < b\}$. By filter-compactness, there exists $x \in \bigcap \mathcal{C}_\gamma$. By o-minimality, observe that γ satisfies that it τ-converges to x as $t \to a$. Similarly one shows that γ also τ-converges as $t \to b$. □

We now prove a simpler case of the left-to-right implication in Proposition 5.12. We do so by implicitly introducing a weakening of filter-compactness corresponding to the property that every definable family of nonempty closed sets that is nested has nonempty intersection (say chain-compactness). We show that, when $\mathcal{M}$ has definable choice or the underlying topology is Hausdorff, curve-compactness implies chain-compactness (the reverse implication always holds within o-minimality by the proof of Lemma 5.13). On the other hand, Example 5.16 describes a (non-Hausdorff) definable topological space in $(M, <)$ that is curve-compact but not chain-compact. It is unclear whether chain-compactness is equivalent to definable compactness (Definition 1.1) in the general setting of o-minimal definable topological spaces.

Lemma 5.14. *Let (X, τ) be a definable topological space. Suppose that either τ is Hausdorff or $\mathcal{M}$ has definable choice. Let $\mathcal{C}$ be a nested definable family of nonempty τ-closed subsets of X. If (X, τ) is curve-compact, then $\bigcap \mathcal{C} \neq \varnothing$.*

Proof. Let (X, τ) and $\mathcal{C} = \{\varphi(M, b) : b \in B\}$, with $B \subseteq M^n$, be as in the lemma. We assume that (X, τ) is curve-compact and show that $\bigcap \mathcal{C} \neq \varnothing$. We proceed by induction on n.

Case $n = 1$: Consider the definable total preorder $\preceq$ in B given by $b \preceq c$ if and only if $\varphi(M, b) \subseteq \varphi(M, c)$. If B has a minimum $\boldsymbol{b}$ with respect to $\preceq$, then $\varphi(M, \boldsymbol{b}) \subseteq \varphi(M, c)$ for every $c \in B$, and the result follows. We suppose that $(B, \preceq)$ does not have a minimum and consider the definable nested family of (necessarily infinite) sets $\{(-\infty, b)_\preceq : b \in B\}$, where $(-\infty, b)_\preceq = \{c \in B : c \preceq b\}$ for every $b \in B$. Now let $a = \sup\{\inf(-\infty, b)_\preceq : b \in B\}$, where the infimum and supremum are taken in $M \cup \{-\infty, +\infty\}$ with respect to the order $<$ in M. We show that one of the

families $\{(a,t) : t > a\}$ or $\{(t,a) : t < a\}$ is a refinement of $\{(-\infty, b)_{\preceq} : b \in B\}$. We prove the case where $a \in M$, since the case where $a \in \{-\infty, +\infty\}$ is similar but more straightforward.

Towards a contradiction suppose that $\{(-\infty, b)_{\preceq} : b \in B\}$ does not have a refinement as described. Then, by o-minimality, there exist some $b_1 \in B$ and $t_1 > a$ such that $(-\infty, b_1)_{\preceq} \cap (a, t_1) = \varnothing$, and similarly there exist $b_2 \in B$ and $t_2 < a$ such that $(-\infty, b_2)_{\preceq} \cap (t_2, a) = \varnothing$. Additionally, by the definition of a, there exists $b_3 \in B$ such that $\inf(-\infty, b_3)_{\preceq} > t_2$. Finally let $b_4 \in B$ be such that $a \notin (-\infty, b_4)_{\preceq}$. Now let $i \leq 4$ be such that b_i is the minimum with respect to $\preceq$ in $\{b_1, b_2, b_3, b_4\}$. Then observe that $\inf(-\infty, b_i)_{\preceq} \geq t_1$, contradicting the definition of a.

Hence onwards we assume that the family of intervals $\{(a,t) : t > a\}$ is a refinement of $\{(-\infty, b)_{\preceq} : b \in B\}$; the case where the refinement is given by the family $\{(t,a) : t < a\}$ is analogous. This means that, for every $b \in B$, there exists an element $t(b) > a$ such that $\varphi(M, c) \subset \varphi(M, b)$ for every $c \in (a, t(b))$.

Since either $\mathcal{M}$ has definable choice or τ is Hausdorff, there exists by Lemma 5.10 a definable function $f : B \to \bigcup \mathcal{C}$ satisfying that $f(b) \in \varphi(M, b)$ for every $b \in B$. By the above paragraph it follows that, for every $b \in B$, if $c \in (a, t(b))$, then $f(c) \in \varphi(M, b)$. Let $d > a$ be such that $(a, d) \subseteq B$ and γ be the restriction of f to (a, d). We derive that, for every $C \in \mathcal{C}$, it holds that $\lim^{\tau}_{t \to a} \gamma(t) \subseteq C$. Since (X, τ) is curve-compact, we conclude that $\bigcap \mathcal{C} \neq \varnothing$.

Case $n > 1$: For every $x \in \pi(B)$, let $\mathcal{C}_x$ denote the family $\{\varphi(M, x, t) : t \in B_x\}$, where $B_x = \{t \in M : \langle x, t \rangle \in B\}$, and set $C(x) := \bigcap \mathcal{C}_x$. By the case $n = 1$ the definable family of τ-closed sets $\mathcal{D} = \{C(x) : x \in \pi(B)\}$ does not contain the empty set. Clearly $\bigcap \mathcal{D} = \bigcap \mathcal{C}$. We observe that the family $\mathcal{D}$ is nested and the result follows from the induction hypothesis.

Given $x, y \in \pi(B)$, if, for every $C \in \mathcal{C}_x$, there is $C' \in \mathcal{C}_y$ with $C' \subseteq C$, then $\bigcap \mathcal{C}_y \subseteq \bigcap \mathcal{C}_x$. Otherwise, there is $C \in \mathcal{C}_x$ such that, for every $C' \in \mathcal{C}_y$, it holds that $C \subseteq C'$, in which case $\bigcap \mathcal{C}_x \subseteq C \subseteq \bigcap \mathcal{C}_y$. □

We may now prove Proposition 5.12.

Proof of Proposition 5.12. By Lemma 5.13 we must only prove the right-to-left implication in the second paragraph. Let (X, τ), with $X \subseteq M^m$, be a curve-compact definable topological space, where either τ is Hausdorff or otherwise $\mathcal{M}$ has definable choice. Let $\mathcal{C}$ be a definable downward directed family of nonempty subsets of X, not necessarily τ-closed. We show that $\bigcap \{\mathrm{cl}_\tau(C) : C \in \mathcal{C}\} \neq \varnothing$.

We proceed by induction on $n = \min\{\dim C : C \in \mathcal{C}\}$. Applying Proposition 4.3, after passing to a refinement of $\mathcal{C}$ if necessary, we may assume that $\mathcal{C}$ is a downward directed family of cells that generates a type in $S_m(M)$. If $n = 0$, then there exists a finite set in $\mathcal{C}$ and so (see Lemma 2.1) $\bigcap \mathcal{C} \neq \varnothing$. Hence onwards we assume that $n > 0$. We begin by proving the case $n = m$. Hence suppose that every $C \in \mathcal{C}$ is

an open cell $C = (f_C, g_C)$ for functions $f_C, g_C : \pi(C) \to M \cup \{-\infty, +\infty\}$ with $f_C < g_C$. Onwards recall the notation fixed in the last paragraph of Section 2.1. For every $C \in \mathcal{C}$, consider the definable set $D(C) = \bigcap\{\mathrm{cl}_\tau(f_C, g_{C'}) : C' \in \mathcal{C}\}$.

Claim 5.15. *For every $C \in \mathcal{C}$ it holds that $D(C) \neq \varnothing$.*

Proof. Let us fix $C = (f, g)$ and, for every $x \in \pi(C)$, let c_x denote a point in $\lim^\tau_{t \to f(x)^+} \langle x, t\rangle$. If τ is Hausdorff then, for every $x \in \pi(C)$, there is a unique choice for c_x; otherwise we use definable choice to pick c_x definably in x. The definable set $C^0 = \{c_x : x \in \pi(C)\}$ has dimension at most $\dim(\pi(C)) = \dim(C) - 1 = n - 1$. For every $C' = (f', g') \in \mathcal{C}$, since $C \cap C' \neq \varnothing$, the definable set $\{x \in \pi(C) \cap \pi(C') : f(x) < g'(x)\} \supseteq \pi(C \cap C')$ is nonempty, and so $C^0 \cap \mathrm{cl}_\tau(f, g') \neq \varnothing$.

Note that, because $\mathcal{C}$ is downward directed, the definable family of nonempty sets $\{C^0 \cap \mathrm{cl}_\tau(f, g_{C'}) : C' \in \mathcal{C}\}$ is downward directed. Since $\dim C^0 \leq n-1$, by inductive hypothesis there is a point that belongs in the τ-closure of $C^0 \cap \mathrm{cl}_\tau(f, g_{C'})$ — in particular in $\mathrm{cl}_\tau(f, g_{C'})$ — for all $C' \in \mathcal{C}$. Hence $D(C) \neq \varnothing$. □

Note that, for every $C \in \mathcal{C}$, it holds that $D(C) \subseteq \mathrm{cl}_\tau(C)$. We now show that the definable family of nonempty (by Claim 5.15) τ-closed sets $\{D(C) : C \in \mathcal{C}\}$ is nested. Then, by Lemma 5.14, $\bigcap\{D(C) : C \in \mathcal{C}\} \neq \varnothing$, and thus $\bigcap\{\mathrm{cl}_\tau(C) : C \in \mathcal{C}\} \neq \varnothing$.

Let us fix $C_1 = (f_1, g_1)$ and $C_2 = (f_2, g_2)$ in $\mathcal{C}$. We may partition $B = \pi(C_1) \cap \pi(C_2)$ into the definable sets

$$B_1 = \{x \in B : f_1(x) \leq f_2(x)\} \quad \text{and} \quad B_2 = \{x \in B : f_1(x) > f_2(x)\}.$$

Since $\mathcal{C}$ is a basis (i.e., a downward directed generating family) for a type in $S_m(M)$, there exists some $i \in \{1, 2\}$ and $C \in \mathcal{C}$ such that $\pi(C) \subseteq B_i$. Without loss of generality suppose that $i = 1$, and let us fix $C_3 \in \mathcal{C}$ with $\pi(C_3) \subseteq B_1$. For an arbitrary set $C = (f, g) \in \mathcal{C}$, let $C' = (f', g') \in \mathcal{C}$ be such that $C' \subseteq C \cap C_3$. Then, clearly $(f_2, g') \subseteq (f_1, g') \subseteq (f_1, g)$. It follows that $D(C_2) \subseteq D(C_1)$. This completes the proof of the case $n = m > 0$.

Finally, we describe how the proof in the case $0 < n < m$ can be obtained by adapting the arguments above. Fix $\widehat{C} \in \mathcal{C}$ with $\dim \widehat{C} = n < m$ and a projection $\hat{\pi} : M^m \to M^n$ such that $\hat{\pi}|_{\widehat{C}} : \widehat{C} \to \hat{\pi}(\widehat{C})$ is a bijection. By passing to a refining subfamily of $\mathcal{C}$ if necessary, we may assume that every set in $\mathcal{C}$ is contained in $\widehat{C}$. By the definition of n it follows that the definable downward directed family $\hat{\pi}(\mathcal{C}) = \{\hat{\pi}(C) : C \in \mathcal{C}\}$ contains only open cells in M^n. Note moreover that this family is a basis for a type in $S_n(M)$.

Set $h := (\hat{\pi}|_{\widehat{C}})^{-1}$ and, for every $C \in \mathcal{C}$, let $\hat{\pi}(C) = (\hat{f}_C, \hat{g}_C)$. Then the proof in the case $n = m$ can be applied by letting $D(C)$ be the intersection $\bigcap\{\mathrm{cl}_\tau\, h[(\hat{f}_C, \hat{g}_{C'})] : C' \in \mathcal{C}\}$, and letting C^0 be a set given by points c_x in $\lim^\tau_{t \to \hat{f}(x)^+} h(x, t)$ chosen definably in $x \in \pi(\hat{\pi}(C))$. □

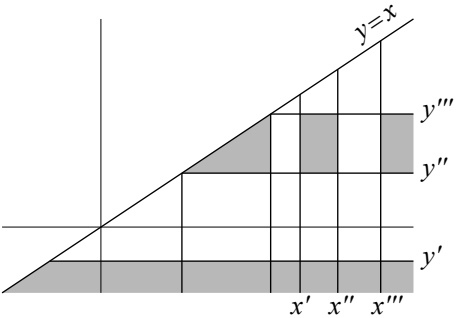

Figure 1. Depicting (in gray) the set $A(x', x'', x''', y', y'', y''')$.

The following is an example of a non-Hausdorff definable topological space in the unbounded dense linear order $(M, <)$ that is curve-compact but not definably compact. In particular, the space admits a definable nested family of nonempty closed sets with empty intersection (see the comments above Lemma 5.14).

Example 5.16. Let $X = \{\langle x, y\rangle \in M^2 : y < x\}$. Consider the family $\mathcal{B}$ of subsets of X of the form

$$\begin{aligned} &A(x', x'', x''', y', y'', y''') \\ &\quad = \{\langle x, y\rangle : y < y', y < x\} \\ &\qquad\qquad \cup \{\langle x, y\rangle : y'' < y < y''' \wedge (y < x < y''' \vee x' < x < x'' \vee x''' < x)\} \end{aligned}$$

definable uniformly in $y' < y'' < y''' < x' < x'' < x'''$.

Given any $A_0 = A(x_0', x_0'', x_0''', y_0', y_0'', y_0''')$ and $A_1 = A(x_1', x_1'', x_1''', y_1', y_1'', y_1''')$ in $\mathcal{B}$, and any $\langle x, y\rangle \in A_1 \cap A_2$, since every set in $\mathcal{B}$ is open in the Euclidean topology, we may find $y'' < y < y''' < x' < x < x''$ such that the box $(x', x'') \times (y'', y''')$ is a subset of $A_1 \cap A_2$. Let $y' < \min\{y'', y_0', y_1'\}$ and $x''' > \max\{x'', x_0''', x_1'''\}$. Then $\langle x, y\rangle \in A(x', x'', x''', y', y'', y''') \subseteq A_1 \cap A_2$. Hence the family $\mathcal{B}$ is a definable basis for a topology $\tilde{\tau}$ on X.

This topology is T_1, i.e., every singleton is $\tilde{\tau}$-closed. For every $y \in M$,

$$\lim^{\tilde{\tau}}_{t \to y^+} \langle t, y\rangle = \lim^{\tilde{\tau}}_{t \to +\infty} \langle t, y\rangle = (M \times \{y\}) \cap X,$$

and, for every $x \in M$,

$$\lim^{\tilde{\tau}}_{t \to x^-} \langle x, t\rangle = (M \times \{x\}) \cap X \quad \text{and} \quad \lim^{\tilde{\tau}}_{t \to -\infty} \langle x, t\rangle = X.$$

In particular, $\tilde{\tau}$ is not Hausdorff.

Now suppose that $\mathcal{M} = (M, <)$. By quantifier elimination we know that in this structure any definable partial map $M \to M$ is piecewise either constant or the identity. Let $\gamma = (\gamma_0, \gamma_1) : (a, b) \to X$ be an injective definable curve in X, where γ_0 and γ_1 are the projections to the first and second coordinates respectively. Let

$I = (a, c) \subseteq (a, b)$ be an interval where γ_0 and γ_1 are either constant or the identity. Since the graph of the identity is disjoint from X and γ is injective it must be that γ_i is constant and γ_{1-i} is the identity on I for some $i \in \{0, 1\}$.

Suppose that $\gamma_1|_I$ is constant with value y. Then, by the observations made above about the topology $\tilde{\tau}$, the curve γ satisfies that it $\tilde{\tau}$-converges as $t \to a$ to either $\langle a, y\rangle$ (if $y < a$) or $(M \times \{y\}) \cap X$ (if $a = y$). On the other hand, if $\gamma_0|_I$ has a constant value x, then γ $\tilde{\tau}$-converges as $t \to a$ to either $\langle x, a\rangle$ (if $a > -\infty$) or the whole space X (if $a = -\infty$). Analyzing the limit as $t \to b$ similarly allows us to conclude that γ is $\tilde{\tau}$-completable. Hence the space $(X, \tilde{\tau})$ is curve-compact.

Meanwhile, the definable nested family of $\tilde{\tau}$-closed sets $\{X \cap (M \times [b, +\infty)) : b \in M\}$ has empty intersection. In particular, $(X, \tilde{\tau})$ is not definably compact.

Remark 5.17. Question 4.14 of [Johnson 2018] asks whether curve-compactness and filter-compactness are equivalent for o-minimal definable manifold spaces [van den Dries 1998, Chatper 10, Section 1]. While these spaces are not necessarily Hausdorff observe that, because they admit a covering by finitely many Hausdorff open subspaces, every definable curve in them converges to only finitely many points. It follows that the proofs of Lemmas 5.10 and 5.14 and Proposition 5.12 adapt to these spaces without any assumption that they are Hausdorff or that $\mathcal{M}$ has definable choice. Hence we can answer Johnson's question in the affirmative. In fact, by Theorem A, every definable manifold space is curve-compact if and only if it is definably compact.

In the next remark we relate definable compactness to definable nets.

Remark 5.18. In [Andújar Guerrero et al. 2021, Section 6], Thomas, Walsberg and the author introduce the notion of *definable net* $\gamma : (B, \preceq) \to (X, \tau)$ to mean a definable map from a definable directed set $(B, \preceq)$ into a definable topological space (X, τ). Recall that a (*Kelley*) *subnet* of γ is a net $\gamma' : (B', \preceq') \to (X, \tau)$ such that $\gamma' = \gamma \circ \mu$, where $\mu : B' \to B$ satisfies that, for every $b \in B$, there exists $c \in B'$ satisfying that $\mu(d) \succeq b$ for every $d \succeq c$. We say that such a net γ' is definable if $(B', \preceq')$ and μ are definable.

Classically, a topological space is compact if and only if every net in it has a converging subnet. Following the classical proof of this result one may show that, in any model-theoretic structure (regardless of the axiom of o-minimality), filter-compactness implies that every definable net has a definable converging subnet (say net-compactness). The reverse implication follows whenever the structure has definable choice. In the o-minimal case one may easily show that net-compactness implies curve-compactness and so, by Theorem A, net-compactness implies definable compactness whenever the topology is Hausdorff or $\mathcal{M}$ has definable choice. See [Andújar Guerrero et al. 2021, Corollary 44] for a detailed proof of the equivalence between net-compactness, curve-compactness and filter-compactness in o-minimal

expansions of ordered groups. Furthermore, one may show that Example 5.16 above is not net-compact. The author is unaware of whether net-compactness implies definable compactness always within o-minimality.

In general topology there are further characterizations of compactness, such as the property that every net has a cluster point, which are not addressed in this paper.

In light of Definition 1.1, the following corollary is a direct consequence of Proposition 5.5.

Corollary 5.19. *Suppose that $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, <)$ and let (X, τ) be a definable topological space. Then (X, τ) is definably compact if and only if it is compact.*

Remark 5.20. Part of [Peterzil and Steinhorn 1999, Question 2.5] asks whether curve-compactness is equivalent to classical compactness among definable manifold spaces in o-minimal expansions of $(\mathbb{R}, <)$. Theorem A, Remark 5.17 and Corollary 5.19 above provide a positive answer to this question. The other part of [Peterzil and Steinhorn 1999, Question 2.5] asks whether curve-compactness for o-minimal definable manifold spaces is maintained after passing to an o-minimal expansion. Recall that Example 5.16 describes a (non-Hausdorff) definable topological space $(X, \tilde{\tau})$ in an arbitrary dense linear order $(M, <)$ which is curve-compact but not definably compact. The linear order $(M, <)$ could be chosen to have an o-minimal expansion $\mathcal{N}$ with definable choice, in which case, by Theorem A, in $\mathcal{N}$ the space $(X, \tilde{\tau})$ would lose the property of curve-compactness. Hence the above question has a negative answer when directed at all (non-Hausdorff) definable topological spaces. On the other hand, by Corollary 5.19, definable compactness is always maintained after passing to an expansion in o-minimal structures over $(\mathbb{R}, <)$. It remains open however whether the same is true in arbitrary o-minimal structures.

Notice that all the characterizations of definable compactness in Theorem A are upfront expressible with infinitely many sentences (possibly with parameters) in the language of $\mathcal{M}$, i.e., you have to check all relevant definable families of closed sets or all definable curves. Recall that, by [Peterzil and Steinhorn 1999, Theorem 2.1] (or alternatively [Johnson 2018, Proposition 3.10]) and Theorem A, a definable set with the Euclidean topology is definably compact if and only if it is closed and bounded. Being closed and bounded is expressible by a single formula (in the same parameters used to define the set). Furthermore, given a definable family of sets with the Euclidean topology, the subfamily of those that are closed and bounded is definable.

We generalize this last observation to all definably compact spaces definable in o-minimal structures, showing along the way that definable compactness is expressible with a single formula (in the same parameters that define the topological space). We prove this for type-compactness and apply Theorem A. We use the following fact regarding strict prodefinability of the space of definable types from

[Kovacsics et al. 2021], which can also be proved using Propositions 4.1 and 4.3, together with the Marker–Steinhorn theorem [1994, Theorem 2.1]. Recall that throughout this section our structure $\mathcal{M}$ is o-minimal.

Fact 5.21 [Kovacsics et al. 2021, Theorems 2.4.8 and 3.2.1]. For every formula $\varphi(x, y)$, there exists a formula $\psi(y, z)$ and a definable (over $\varnothing$) set $Z \subseteq M^{|z|}$ with the following properties. For every $d \in Z$, there exists a definable type $p(x) \in S_{|x|}(M)$ such that

$$\{b \in M^{|y|} : \varphi(x, b) \in p(x)\} = \psi(M, d), \tag{2}$$

and, vice versa, for every definable type $p \in S_{|x|}(M)$ there exists some $d \in Z$ satisfying (2).

Proposition 5.22. *Let $\{(X_c, \tau_c) : c \in C\}$, be a definable family of topological spaces, i.e., there exists a definable set $B \subseteq M^{n+m}$ and some formula $\sigma(x, u, v)$, with $|u| = n$ and $|v| = m$, such that C is the projection of B to the last m coordinates and, for every $c \in C$, the family $\{\sigma(M, b, c) : b \in B_c\}$, where $B_c = \{b \in M^n : \langle b, c\rangle \in B\}$, is a basis for τ_c. Then the subfamily of all definably compact spaces is definable, i.e., there exists a definable set $D \subseteq C$ such that (X_c, τ_c) is definably compact if and only if $c \in D$.*

In particular, given any definable family of subsets of a definable topological space, the subfamily of those that are definably compact is definable.

Proof. Let $\varphi(x, u, v)$ be a formula such that, for every $c \in C \subseteq M^{|v|}$, the family $\{\varphi(M, b, c) : b \in B_c\}$ is a basis of closed sets for τ_c. Recall that, as observed in the proof of Proposition 5.5, a type has a limit if and only if the intersection of all *basic* closed sets in it is nonempty.

Let $\psi(u, v, z)$ and $Z \subseteq M^{|z|}$ be as given by Fact 5.21 with $y = (u, v)$. It follows that, for any fixed $c \in C$, the space (X_c, τ_c) is type-compact if and only if either X_c is empty or the following holds:

$$\forall z \in Z\, (\exists x\, \forall u\, (\psi(u, c, z) \wedge u \in B_c \to \varphi(x, u, c))).$$

Since the set Z is definable, this completes the proof. □

In the case where $\mathcal{M}$ is an o-minimal expansion of an ordered field, the completeness of the theory of tame pairs [van den Dries and Lewenberg 1995] can be used to circumvent some of the proofs in this paper. In particular Thomas, Walsberg and the author [Andújar Guerrero et al. 2021, Corollary 47] use it to prove the equivalence between curve-compactness and type-compactness among definable topological spaces, drawing inspiration from Walsberg's previous proof [2015, Proposition 6.6] of the equivalence between curve-compactness and sequential compactness among definable metric spaces in the case where $\mathcal{M}$ expands the field of reals.

Acknowledgements

The author thanks Sergei Starchenko for the idea of a characterization of definable compactness in terms of VC theory and finite transversals, which motivated much of the work in this paper. He also thanks Margaret Thomas, Pantelis Eleftheriou, Matthias Aschenbrenner, and the anonymous referee for extensive feedback on various versions of the paper. In particular, he thanks Professor Aschenbrenner for suggesting the terminology "n-consistent" and "n-inconsistent". The author also thanks Antongiulio Fornasiero [2015] for sharing his unpublished notes on definable compactness. Finally, the author is deeply indebted to Will Johnson for some very helpful conversations on the contents of Section 4, which resulted in the current proof to Proposition 4.3, and in Johnson's authoring of [Andújar Guerrero 2021b, Appendix A], a counterexample to a parameter version of Proposition 4.3.

During the research into the contents of this paper the author was supported by the Institut Henri Poincaré (during the 2018 Program on Model Theory, Combinatorics and Valued Fields), the Canada Natural Sciences and Engineering Research Council (NSERC) Discovery Grant RGPIN-06555-2018, the Graduate School and Department of Mathematics at Purdue University, the Fields Institute for Research in Mathematical Sciences (during the 2021 Thematic Program on Trends in Pure and Applied Model Theory), and the Engineering and Physical Sciences Research Council (EPSRC) Grant EP/V003291/1.

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Received 29 Feb 2024. Revised 10 Jan 2025.

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Model Theory
Vol. 4, No. 2, 2025
https://doi.org/10.2140/mt.2025.4.131

The classification of dp-minimal integral domains

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We classify dp-minimal integral domains, building off the existing classification of dp-minimal fields and dp-minimal valuation rings. We show that if R is a dp-minimal integral domain, then R is a field or a valuation ring or arises from the following construction: there is a dp-minimal valuation overring $\mathcal{O} \supseteq R$, a proper ideal I in $\mathcal{O}$, and a finite subring $R_0 \subseteq \mathcal{O}/I$ such that R is the preimage of R_0 in $\mathcal{O}$.

1. Introduction

A dp-minimal domain is an integral domain whose first order theory is of dp-rank 1; see Section 2 for the definition. In [d'Elbée and Halevi 2021], the first two authors studied under which algebraic conditions dp-minimal domains are in fact valuation rings. After noting that any such ring R must be local, the answer was that R is a valuation ring if and only if its residue field is infinite or its residue field is finite and its maximal ideal is principal. The third author gave a complete classification of dp-minimal fields and valuation rings in [Johnson 2023a] (see Section 2.2 for the statements of these results), and a classification of dp-minimal Noetherian domains in [Johnson 2023b].

In this paper we continue the arguments of [d'Elbée and Halevi 2021; Johnson 2023a; 2023b], leading to a classification of dp-minimal integral domains.

Theorem. *Let R be an integral domain. Then R is dp-minimal if and only if one of the following holds*:

(1) *R is a dp-minimal field.*

(2) *R is a dp-minimal valuation ring.*

(3) *There is a valuation subring $\mathcal{O}$ of $K = \mathrm{Frac}(R)$, a proper ideal $I \lhd \mathcal{O}$, and a finite subring R_0 of $\mathcal{O}/I$ such that R is the preimage of R_0 under the quotient map $\mathcal{O} \to \mathcal{O}/I$, and the valuation ring $(\mathcal{O}, +, \cdot)$ is dp-minimal.*

Moreover, in (3) *the ring $\mathcal{O}$, ideal I, and subring R_0 can be chosen to be definable in R.*

MSC2020: 03C60, 13G05.
Keywords: dp-minimality, integral domains, valued fields.

See Facts 2.2 and 2.3 for the classification of dp-minimal fields and valuation rings.

The "if" direction of this theorem — rings satisfying (3) are dp-minimal — is the relatively easy Proposition 3.2. The "only if" direction — any dp-minimal domain arises from one of the three constructions — is the much harder Theorem 6.19.

Here are two typical examples of the construction in (3):

Example 1.1. Let $K = \mathbb{Q}_3(i)$, where $i = \sqrt{-1}$. Let $\mathcal{O}$ be the natural valuation ring on K, namely $\mathcal{O} = \mathbb{Z}_3[i]$. Then $\mathcal{O}/27\mathcal{O}$ is a finite ring extending $\mathbb{Z}/27\mathbb{Z}$. Taking $R_0 = \mathbb{Z}/27\mathbb{Z}$, we get a dp-minimal integral domain $R \subseteq \mathcal{O}$, consisting of those $x \in \mathcal{O}$ which are congruent mod 27 to an element of $\{0, 1, 2, \ldots, 26\}$. One can describe R more directly as $R = \mathbb{Z} + 27\mathcal{O}$. Note that R is not a valuation ring.

Example 1.2. Let K be the field of Hahn series $\mathbb{F}_3^{\mathrm{alg}}((t^{\mathbb{Q}}))$, a model of $\mathrm{ACVF}_{3,3}$. Let $\mathcal{O}$ be the natural valuation ring on K, and let I be the (nondefinable) ideal $I = \{x \in \mathcal{O} : v(x) > \pi\}$. There is a finite subring $R_0 \subseteq \mathcal{O}/I$ of size 3^7 given by

$$R_0 = \{a + bt + ct^2 + dt^3 + I : a \in \mathbb{F}_3,\ b, c, d \in \mathbb{F}_9\}.$$

This yields a dp-minimal integral domain $R \subseteq \mathcal{O}$. Explicitly, R is the set of $x \in \mathcal{O}$ such that there are $a \in \mathbb{F}_3$ and $b, c, d \in \mathbb{F}_9$ with $v(x - (a + bt + ct^2 + dt^3)) > \pi$. Again, R is not a valuation ring.

During the proof of our main theorem, we prove (or observe) the following results which might be interesting in their own right:

Proposition (Proposition 4.9). *If $\mathcal{O}$ is a $\vee$-definable valuation subring of some sufficiently saturated field, then $\mathcal{O}$ is externally definable.*

Proposition (Proposition 6.20). *Let $(K, +, \cdot, A)$ be a dp-minimal expansion of a field K by an infinite, proper additive subgroup $(A, +) \subseteq (K, +)$. Then there is a nontrivial definable valuation ring on K.*

Observation (Fact 4.18). If R is an NIP commutative ring, then $R^{00} = R^0$.

We also make some preliminary observations about more general dp-minimal commutative rings (not domains), showing that each is the product of a finite ring and a dp-minimal henselian local ring, and in dp-minimal local rings, the prime ideals are linearly ordered (Proposition 7.1).

1.1. ***Sketch of the proof.*** In order to prove our main theorem, there are two things to check. First, we must show that the construction in part (3) of the theorem really does give dp-minimal integral domains. The proof is relatively easy, using Shelah expansions, and is given in Section 3.

The much more difficult direction is to show that if R is a dp-minimal integral domain other than a field or valuation ring, then R arises via the construction

in part (3). The proof of this fact occupies Sections 4–6. We call such rings R *exceptional dp-minimal domains*. The fraction field $K = \mathrm{Frac}(R)$ turns out to be a dp-minimal field. Using the classification of dp-minimal fields, we divide into three cases: K is either *ACVF-like*, *pCF-like*, or *RCVF-like* (see Section 5).

Corollary 5.6 in [d'Elbée and Halevi 2021] essentially says that R must be comparable with any definable valuation ring on K (see Fact 4.3). Using this, we can quickly rule out the RCVF-like case, at least when R is exceptional (Proposition 6.2).

Proposition 3.14 in [d'Elbée and Halevi 2021] gives a dichotomy for the behavior of $\sqrt{\mathfrak{m}^{00}}$, the radical of the 00-connected component of the maximal ideal $\mathfrak{m}$ of R: it is either $\mathfrak{m}$ itself or the unique second-largest prime ideal in R (see Proposition 4.14 and Lemma 4.16). We show that these two cases correspond exactly to the ACVF-like case and the pCF-like case, respectively (Proposition 6.3). Combining this with [d'Elbée and Halevi 2021, Corollary 5.6] again, we finish the pCF-like case in Proposition 6.7.

The ACVF-like case is the hardest to deal with. Assuming that $\mathrm{Frac}(R)$ is ACVF-like, say that R is a *good domain* if $\mathfrak{m}^{00}$ is definable, and a *rogue domain* otherwise (Definition 6.8). Good domains are almost trivial to deal with (see the proof of Theorem 6.19); all the work goes into showing that *rogue domains do not exist* (Proposition 6.18). In Section 6.3, we analyze rogue domains, proving enough facts about their structure to get a contradiction. The proof is technical, but involves showing that the set $\mathcal{O} = \mathfrak{m}^{00} : \mathfrak{m}^{00} = \{x \in K : x\mathfrak{m}^{00} \subseteq \mathfrak{m}^{00}\}$ is the coarsest externally definable valuation ring on K, and studying the interactions between R and $\mathcal{O}$.

The proofs in Section 6 depend on a number of technical tools, which we collect in Section 4:

- In Section 4.1 we review some basic facts about NIP and dp-minimal domains from [d'Elbée and Halevi 2021; Johnson 2022].
- In Section 4.2 we show certain prime ideals in R are 00-connected.
- In Section 4.3 we show that $\vee$-definable valuation rings are externally definable, and use this to show that R^{00} is externally definable when R is a dp-minimal domain.
- In Section 4.4 we strengthen and improve the $\sqrt{\mathfrak{m}^{00}}$ dichotomy from [d'Elbée and Halevi 2021, Proposition 3.14], leveraging the earlier results on 00-connectedness and external definability.
- In Section 4.5 we use tools from topological ring theory to observe that $R^{00} = R^0$ and analyze the structure of R/R^{00} in certain cases using the classification of locally compact fields.
- In Section 4.6 we show that any dp-minimal domain R with an infinite definable additive proper subgroup also admits an infinite definable proper subring

(Proposition 4.22). We subsequently strengthen this to get an externally definable nontrivial valuation ring (Proposition 6.6) and ultimately a definable nontrivial valuation ring (Proposition 6.20).

2. Preliminaries

2.1. ***Notation.*** We employ fairly standard model theoretic notation and conventions; see [Tent and Ziegler 2012; Simon 2015].

We briefly review the definition of dp-rank and dp-minimality. If $\Sigma(x)$ is a partial type and κ is a cardinal, an *ict-pattern* of depth κ in $\Sigma(x)$ is an array of formulas $(\varphi_\alpha(x, b_{\alpha,i}) : \alpha < \kappa,\ i < \omega)$ over an elementary extension, such that for every function $\eta : \kappa \to \omega$, the type

$$\Sigma(x) \cup \{\varphi_\alpha(x, b_{\alpha,i}) : i = \eta(\alpha)\} \cup \{\neg\varphi_\alpha(x, b_{\alpha,i}) : i \neq \eta(\alpha)\}$$

is consistent. The *dp-rank of* $\Sigma(x)$, written dp-rk$(\Sigma(x))$, is the supremum of cardinals κ such that there is an ict-pattern of depth κ in $\Sigma(x)$. (This definition coincides with the alternative definition of dp-rank using indiscernible sequences; see [Kaplan et al. 2013, Lemma 2.5, Proposition 2.6].) The dp-rank of a definable set D, written dp-rk(D), is the dp-rank of the formula defining D. A one-sorted structure M is *dp-minimal* if dp-rk$(M) = 1$. In particular, dp-rk$(M) > 0$, so M must be infinite, with our definition of "dp-minimal". A one-sorted theory is dp-minimal if its models are dp-minimal.

By a ring we mean a (possibly trivial) commutative ring with identity. An ideal in R can be the improper ideal R. A "maximal ideal" means a maximal proper ideal. A local ring is a ring with a unique maximal ideal, possibly a field. Valuations can be trivial, so valuation rings can be fields.

If $M = (R, +, \cdot, \dots)$ is an expansion of an integral domain, then the structures M and $(\mathrm{Frac}(R), R, +, \cdot, \dots)$ are obviously interdefinable in M^{eq}, so for example saturation and definability pass from one to the other. We mostly do not distinguish between the two structures.

We assume the reader is familiar with valued fields. Given a valued field (K, v) we usually denote its value group by Γ and its residue field by k. Any cut in the value group Γ is of the form $\{x \in \Gamma : x > \gamma\}$ or $\{x \in \Gamma : x \geq \gamma\}$ for some $\gamma \in \Gamma' \succ \Gamma$. Thus any ideal of the valuation ring is of the form $\{x \in K : v(x) \square \gamma\}$, where $\square \in \{>, \geq\}$. An ideal I is principal if and only if $I = \{x \in K : v(x) \geq \gamma\}$ for some $\gamma \in \Gamma$.

We do not assume that we are working in a highly saturated monster model, unless stated otherwise.

2.2. ***Dp-minimal valuation rings and fields.*** We recall the classification of dp-minimal valuation rings and fields given in [Johnson 2023a]. First recall the following.

Fact 2.1 [Jahnke et al. 2017, Proposition 5.1]. *Let* $(\Gamma, \leq, +)$ *be a nontrivial ordered abelian group. Then* Γ *is dp-minimal if and only if* $|\Gamma/n\Gamma|$ *is finite for all* $n > 0$.

For example, $\mathbb{Z}$ and $\mathbb{R}$ are dp-minimal as ordered abelian groups.

Fact 2.2. *An infinite field* $(K, +, \cdot)$ *is dp-minimal if and only if there is a henselian defectless valuation ring* $\mathcal{O} \subseteq K$ *with maximal ideal* $\mathfrak{m}$ *such that*:

(1) *The value group* $\Gamma := K^\times/\mathcal{O}^\times$ *is trivial or dp-minimal as an ordered abelian group.*

(2) *The residue field* $k := \mathcal{O}/\mathfrak{m}$ *is algebraically closed, real closed, or* p*-adically closed for some prime* p.

(3) *If the residue field* k *is algebraically closed of characteristic* $p > 0$, *then the interval* $[-v(p), v(p)] \subseteq \Gamma$ *is* p*-divisible, where* $v(p) = +\infty$ *when* $\mathrm{char}(K) = p$.

In (2), we mean "p-adically closed" in the broad sense: K is p-adically closed if $K \equiv K'$ for some finite extension $K'/\mathbb{Q}_p$. Fact 2.2 is proved in [Johnson 2023a, Theorem 7.3], but only under the extra assumption that K is sufficiently saturated(!). For completeness, we have included a proof in the unsaturated case in the Appendix.

Next, dp-minimal valuation rings[1] are classified as follows:

Fact 2.3 [Johnson 2023a, Theorems 1.5 and 1.6]. *Let* $\mathcal{O}$ *be an infinite valuation ring with fraction field* K, *maximal ideal* $\mathfrak{m}$, *residue field* $k = \mathcal{O}/\mathfrak{m}$, *and value group* $\Gamma = K^\times/\mathcal{O}^\times$. *Then* $(\mathcal{O}, +, \cdot)$ *is dp-minimal if and only if the following conditions hold*:

(1) $(k, +, \cdot)$ *and* $(\Gamma, +, \leq)$ *are dp-minimal or finite.*

(2) $\mathcal{O}$ *is henselian and defectless.*

(3) *One of the following cases holds*:

- k *is finite of characteristic* $p > 0$, K *has characteristic* 0, *and the interval* $[-v(p), v(p)] \subseteq \Gamma$ *is finite.*
- k *is infinite of characteristic* $p > 0$, *and the interval* $[-v(p), v(p)] \subseteq \Gamma$ *is* p*-divisible.*
- k *has characteristic* 0.

For example, $\mathbb{Z}_p$ and $\mathbb{Q}_p[\![t]\!]$ are dp-minimal valuation rings, but $\mathbb{F}_p[\![t]\!]$ is not.

[1]The citation is to the classification of dp-minimal *valued fields* $(K, \mathcal{O})$ rather than dp-minimal *valuation rings* $\mathcal{O}$, but the distinction does not matter because the two structures $\mathcal{O}$ and $(K, \mathcal{O})$ have the same dp-rank by Fact 4.2 below. Therefore, a valuation ring $\mathcal{O}$ is dp-minimal if and only if the corresponding valued field $(K, \mathcal{O})$ is dp-minimal.

3. Constructing dp-minimal integral domains

Recall that if M is a structure, the *Shelah expansion* M^{Sh} is the expansion of M by all externally definable sets. If M is NIP, then the definable sets in M^{Sh} are precisely the externally definable sets in M, and M^{Sh} is also NIP [Shelah 2009; Simon 2015, Section 3.3].

The following fact is well known [Onshuus and Usvyatsov 2011, Observation 3.8], but we had trouble understanding the proof, so we include a more detailed proof for completeness:

Fact 3.1. *M^{Sh} has the same dp-rank as M, and in fact, if $D \subseteq M^n$ is definable, then D has the same dp-rank whether considered in M or in M^{Sh}.*

Proof. Let L and L^{Sh} be the languages of M and M^{Sh}. Let $\mathbb{M}^*$ be a monster model elementary extension of M^{Sh}, and let $\mathbb{M}$ be its reduct to L. (Note that $\mathbb{M}^*$ is probably not the Shelah expansion of $\mathbb{M}$.) Let $\psi(x, c)$ be the $L(M)$-formula defining D in M and M^{Sh}. Suppose $\text{dp-rk}(\psi(x, c)) \geq \kappa$ in $\mathbb{M}^*$. We must show $\text{dp-rk}(\psi(x, c)) \geq \kappa$ in $\mathbb{M}$. By definition of dp-rank, there are L^{Sh}-formulas $\varphi_\alpha(x, y)$, elements a_η in $\mathbb{M}$ for $\eta : \kappa \to \omega$, and elements $b_{\alpha,i}$ in $\mathbb{M}$ for $\alpha < \kappa$ and $i < \omega$, such that

$$\mathbb{M} \models \psi(a_\eta, c) \text{ for each } \eta,$$
$$\mathbb{M}^* \models \varphi_\alpha(a_\eta, b_{\alpha,i}) \iff \eta(\alpha) = i \text{ for each } \eta, \alpha, i.$$

Each L^{Sh}-formula $\varphi_\alpha(x, y)$ is equivalent *on the small set M* to an L-formula $\theta_\alpha(x, y, e_\alpha)$, with e_α in $\mathbb{M}$. Moving the e_α by an automorphism in $\mathrm{Aut}(\mathbb{M}^*/M)$, we may assume that $\mathrm{tp}^{L^{\mathrm{Sh}}}(\bar{a}\bar{b}/M\bar{e})$ is finitely satisfiable in M. For any η, α, i, the L^{Sh}-formula

$$\neg(\theta_\alpha(x, y, e_\alpha) \leftrightarrow \varphi_\alpha(x, y))$$

is not satisfiable in M, so $(a_\eta, b_{\alpha,i})$ cannot satisfy it, and so

$$\mathbb{M}^* \models \theta_\alpha(a_\eta, b_{\alpha,i}, e_\alpha) \leftrightarrow \varphi_\alpha(a_\eta, b_{\alpha,i}).$$

Then

$$\mathbb{M} \models \theta_\alpha(a_\eta, b_{\alpha,i}, e_\alpha) \iff \eta(\alpha) = i,$$

and we have an ict-pattern of depth κ in D, in the original language L, showing that $\text{dp-rk}(D) \geq \kappa$ in M. □

In light of Fact 3.1, the following is nearly trivial:

Proposition 3.2. *Let $\mathcal{O}$ be a dp-minimal valuation ring, let $I \subseteq \mathcal{O}$ be a proper ideal (not necessarily definable), let R_0 be a finite subring of $\mathcal{O}/I$, and let $R \subseteq \mathcal{O}$ be the preimage of R_0 under the quotient map $\mathcal{O} \to \mathcal{O}/I$. Then R is a dp-minimal integral domain.*

Proof. Let K be the fraction field of $\mathcal{O}$, let Γ be the value group, and let $v : K \to \Gamma$ be the valuation. As an ideal in a valuation ring, I must have the form $\{x \in K : v(x) > \xi\}$ for some ξ in an elementary extension of Γ. The set $\{\gamma \in \Gamma : \gamma > \xi\}$ is externally definable, so it is definable in the Shelah expansion of $\mathcal{O}$. Then $\mathcal{O}/I$, R_0, and R are all interpretable in the Shelah expansion. It follows that R is externally definable in $\mathcal{O}$. Then $(\mathcal{O}, +, \cdot, R)$ is a reduct of the Shelah expansion $\mathcal{O}^{\mathrm{Sh}}$, so $(\mathcal{O}, +, \cdot, R)$ is dp-minimal, implying that $(R, +, \cdot)$ is dp-minimal (as $R \subseteq \mathcal{O}$). □

Remark 3.3. For which dp-minimal valuation rings $\mathcal{O}$ can we find an ideal I and a finite subring $R_0 \subseteq \mathcal{O}/I$? To begin with, $\mathcal{O}$ must have positive residue characteristic, or else $\mathcal{O}$ and $\mathcal{O}/I$ are $\mathbb{Q}$-algebras, which have no finite subrings. If the residue characteristic of $\mathcal{O}$ is $p > 0$, then there are two possibilities for $\mathcal{O}$, by Fact 2.3(3):

(1) The residue field of $\mathcal{O}$ is finite. Then $\mathcal{O}$ has mixed characteristic and is finitely ramified. The quotient $\mathcal{O}/I$ can only have positive characteristic if the ideal I contains $p^n\mathcal{O}$ for some n. Then R is the pullback of some subring of $\mathcal{O}/p^n\mathcal{O}$. Conversely, every subring of $\mathcal{O}/p^n\mathcal{O}$ gives a possibility for R, because $\mathcal{O}/p^n\mathcal{O}$ is finite. Example 1.1 is a typical example.

(2) The residue field of $\mathcal{O}$ can be infinite, such as a model of ACF_p. Example 1.2 is a typical example. When $\mathcal{O}$ has equicharacteristic p, there is at least one finite subring of $\mathcal{O}/I$ for any proper ideal I, namely $\mathbb{F}_p \subseteq \mathcal{O}/I$. When $\mathcal{O}$ has mixed characteristic, the ideal I must contain $p^n\mathcal{O}$ for some n, but there are many such ideals I because the convex hull of $\mathbb{Z} \cdot v(p)$ is p-divisible by Fact 2.3(3).

Over the course of the paper, we will see that these two cases behave very differently. For example, note that R has finite index in $\mathcal{O}$ in the first case, but infinite index in $\mathcal{O}$ in the second case (because $\mathcal{O}/I$ is infinite). In the first case, R is definable in the structure $(\mathcal{O}, +, \cdot)$, but in the second case this can fail (take I nondefinable). Later, we will see that R/R^{00} is infinite in the first case, but finite in the second case (Proposition 6.3).

4. Tools

We review some known and new results on NIP and dp-minimal rings and domains. Unless specified otherwise, any ring below is not assumed to be pure, i.e., there might be some more structure.

4.1. ***Basic facts about NIP and dp-minimal domains.*** Every NIP ring has finitely many maximal ideals [d'Elbée and Halevi 2021, Proposition 2.1] and if it is a dp-minimal domain then its prime ideals are linearly ordered by inclusion and so it is either a field or a local ring [d'Elbée and Halevi 2021, Corollaries 2.3 and 2.5].[2]

[2]Recall that in NIP theories dp-minimality and inp-minimality coincide.

Fact 4.1. *Any prime ideal* $\mathfrak{p}$ *in a ring* R *is externally definable.*

When R is NIP, this is [Johnson 2022, Proposition 2.13], but NIP turns out to be unnecessary.

Proof. Let $\Sigma(x)$ be the partial type $\{a \mid x : a \in R \setminus \mathfrak{p}\} \cup \{b \nmid x : b \in \mathfrak{p}\}$. Then $\Sigma(x)$ is finitely satisfiable — any finite subtype

$$\{a_1 \mid x,\ a_2 \mid x, \ldots,\ a_n \mid x,\ b_1 \nmid x,\ b_2 \nmid x, \ldots,\ b_m \nmid x\} \subseteq \Sigma(x)$$

is realized by the product $a_1 a_2 \cdots a_n$. If $R' \succ R$ is an elementary extension containing an element c realizing $\Sigma(x)$, then the formula $x \nmid c$ defines a set whose trace on R is $\mathfrak{p}$. □

As a consequence, if $(R, +, \cdot, \ldots)$ is an NIP ring and $\mathfrak{p}$ is any prime ideal, then the expansion $(R, +, \cdot, \ldots, \mathfrak{p})$ is NIP of the same dp-rank (Fact 3.1). The bi-interpretable structure $(R_{\mathfrak{p}}, R, +, \cdot, \ldots)$ is also NIP, and in fact has the same dp-rank as R by the following:

Fact 4.2. *If* R *is a definable ring and* $\mathfrak{p}$ *is a definable prime ideal in some structure, then the definable ring* $R_{\mathfrak{p}}$ *has the same dp-rank as* R*. In particular, if* R *is an integral domain then* $\mathrm{Frac}(R)$ *has the same dp-rank as* R*.*

This is essentially [d'Elbée and Halevi 2021, Proposition 2.8(2)] (purity of the structure R is inessential), or [Johnson 2020, Lemma 10.25] for the case of $\mathrm{Frac}(R)$.

Fact 4.3. *Let* $M = (K, R, \mathcal{O}, +, \cdot, \ldots)$ *be an expansion of a field together with a predicate for a subring and a valuation ring, with* $\mathrm{Frac}(R) = \mathrm{Frac}(\mathcal{O}) = K$*. If* M *is dp-minimal then either* $R \subseteq \mathcal{O}$ *or* $\mathcal{O} \subseteq R$*.*

This is essentially [d'Elbée and Halevi 2021, Corollary 5.6]. Again, purity of the structure R is inessential.

4.2. 00-*connectedness of prime ideals.* For any type-definable group G in a sufficiently saturated NIP structure we let G^{00} be its 00-connected component [Simon 2015, Section 8.1.3].

Remark 4.4. Suppose G is an NIP group, possibly with additional structure. The following facts are well known:

- If G is sufficiently saturated, then G^{00} is definable if and only if G^{00} has finite index in G.
- If G is sufficiently saturated and $\widehat{G} \succ G$ is sufficiently saturated, then G^{00} is definable if and only if $(\widehat{G})^{00}$ is definable.
- We can talk about "G^{00} is definable" without assuming saturation, by going to a sufficiently saturated elementary extension. The choice of the sufficiently saturated elementary extension does not matter by the previous point.

- If $G \equiv H$, then G^{00} is definable if and only if H^{00} is definable.
- If G^{00} is definable and H is a reduct of G then H^{00} is definable. Note that the converse fails, though — it can happen that G^{00} is not definable, but becomes definable in a reduct. For example, if G is the circle group in RCF, then G^{00} is not definable, but it becomes definable in the pure group reduct.

More generally, analogous results hold when G is a definable group in an NIP structure M.

From now on, whenever G is some NIP expansion of a group, by "G^{00} is definable" we mean that $(\widehat{G})^{00}$ is definable for some/every sufficiently saturated elementary extension $\widehat{G} \succ G$.

Fact 4.5 [Johnson 2021, proof of Lemma 2.6]. (1) *Let R be a sufficiently saturated NIP integral domain and $I \trianglelefteq R$ a type-definable ideal. Then I^{00} is also an ideal of R.*

(2) *Let R be a sufficiently saturated NIP local domain with maximal ideal $\mathfrak{m}$. If $R/\mathfrak{m}$ is infinite then $\mathfrak{m} = \mathfrak{m}^{00}$ and more generally, $I = I^{00}$ for any definable ideal I.*

We prove a similar result for $\vee$-definable valuation rings.

Proposition 4.6. *Let K be a sufficiently saturated NIP field and $\mathcal{O}$ some $\vee$-definable valuation subring with maximal ideal $\mathfrak{m}$. Then $\mathfrak{m}$ is type-definable and unless $\mathfrak{m}$ is principal and $\mathcal{O}/\mathfrak{m}$ is finite, $\mathfrak{m}^{00} = \mathfrak{m}$.*

Proof. For nonzero $x \in K$, we have $x \in \mathfrak{m}$ if and only if $x^{-1} \notin \mathcal{O}$, so $\mathfrak{m}$ is type-definable and $\mathfrak{m}^{00}$ is an ideal in $\mathcal{O}$. Let v be the valuation on $K = \mathrm{Frac}(\mathcal{O})$ associated to $\mathcal{O}$.

Now, we have two cases:

Case 1: There is a minimal positive element in the value group of $\mathcal{O}$. Then $\mathfrak{m}$ is a principal ideal $t\mathcal{O}$, where t is a local uniformizer. The fact that $\mathfrak{m} = t\mathcal{O}$ means that $\mathfrak{m}$ is $\vee$-definable. It follows that $\mathcal{O}$ and $\mathfrak{m}$ are definable. As $\mathcal{O}/\mathfrak{m}$ is infinite in this situation, $\mathfrak{m} = \mathfrak{m}^{00}$ by Fact 4.5(2).

Case 2: There is no minimal positive element in the value group of $\mathcal{O}$. Suppose for the sake of contradiction that $\mathfrak{m} \neq \mathfrak{m}^{00}$. Take an element $a \in \mathfrak{m} \setminus \mathfrak{m}^{00}$, so $v(a) > 0$. If $x \in \mathfrak{m}^{00}$, then $x\mathcal{O} \subseteq \mathfrak{m}^{00}$ so $a \notin x\mathcal{O}$, implying that $x \in a\mathcal{O}$. This shows that $\mathfrak{m}^{00} \subseteq a\mathcal{O}$.

Because there is no minimal positive element in the value group, we can find a_1 with $0 < v(a_1) < v(a)$. Then find a_2 with $0 < v(a_2) < v(a_1)$. Continuing in this way, we can find a sequence

$$a = a_0, a_1, a_2, a_3, \ldots$$

with $v(a_0) > v(a_1) > v(a_2) > \cdots > 0$. Equivalently, there is a sequence $a_0, a_1, a_2, \ldots$ with

- $a_0 = a$,
- $a_i/a_j \in \mathfrak{m}$ for $i < j$,
- $a_i \in \mathfrak{m}$ for all i.

Since $\mathfrak{m}$ is type-definable, these conditions are type-definable, and we can use compactness to get a long sequence $\{a_i\}_{i<\kappa}$ satisfying the conditions above. We get a very long ascending chain of subgroups between $\mathfrak{m}^{00}$ and $\mathfrak{m}$:

$$\mathfrak{m}^{00} \subseteq a_0\mathcal{O} \subsetneq a_1\mathcal{O} \subsetneq a_2\mathcal{O} \subsetneq \cdots \subseteq \mathfrak{m}.$$

But the total number of abstract groups between $\mathfrak{m}^{00}$ and $\mathfrak{m}$ is $2^{|\mathfrak{m}/\mathfrak{m}^{00}|}$, so by taking $\kappa > 2^{|\mathfrak{m}/\mathfrak{m}^{00}|}$ we get a contradiction. □

Corollary 4.7. *Let R be a sufficiently saturated dp-minimal domain. For any nonmaximal type-definable prime ideal $\mathfrak{p}$, $\mathfrak{p}R_\mathfrak{p} = \mathfrak{p}$, $R_\mathfrak{p}$ is a henselian valuation ring with maximal ideal $\mathfrak{p}$ and it satisfies $\mathfrak{p}^{00} = \mathfrak{p}$.*

Proof. The fact that $R_\mathfrak{p}$ is a henselian valuation ring with maximal ideal $\mathfrak{p}$ is [d'Elbée and Halevi 2021, Proposition 3.8, Theorem 3.9]. To show connectedness we use Proposition 4.6: $R_\mathfrak{p}$ is $\vee$-definable since $a \in R_\mathfrak{p}$ if and only if $a^{-1} \notin \mathfrak{p}$, so we only need to show that $R_\mathfrak{p}/\mathfrak{p}$ is not finite. Otherwise, $R/\mathfrak{p}$ injects into the field $R_\mathfrak{p}/\mathfrak{p}$, so the former is a finite integral domain, i.e., a field, contradicting nonmaximality of $\mathfrak{p}$. □

If R is dp-minimal and $a \in R$ is nonzero, then there is a unique maximal prime ideal P of R with the property $a \notin P$ [d'Elbée and Halevi 2021, Lemma 3.3]. We denote this prime ideal P_a.

Corollary 4.8. *If R is a sufficiently saturated dp-minimal domain and $a \in R$ is nonzero and not a unit, then P_a is a nonmaximal type-definable prime ideal, and $P_a^{00} = P_a$.*

Proof. Since a is not a unit, it is in the maximal ideal, so P_a must be nonmaximal. By [d'Elbée and Halevi 2021, Proposition 3.8 and Theorem 3.9], R_{P_a} is a valuation ring with maximal ideal P_a. By [d'Elbée and Halevi 2021, Remark 3.4], R_{P_a} is the localization $S^{-1}R$ with $S = \{a^n : n \in \mathbb{N}\}$. Then $R_{P_a} = \bigcup_{n=0}^{\infty} a^{-n}R$, so R_{P_a} is $\vee$-definable. The maximal ideal of a $\vee$-definable valuation ring is type-definable, so P_a is type-definable. The 00-connectedness of P_a then follows by Corollary 4.7. □

4.3. *External definability of $\vee$-definable valuation rings and R^{00}.*

Proposition 4.9. *Let $\mathcal{O}$ be a $\vee$-definable valuation subring of some sufficiently saturated field K. Then $\mathcal{O}$ is externally definable. If K is NIP, then every ideal of $\mathcal{O}$ is externally definable.*

Proof. As observed in the proof of Proposition 4.6, the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ is type-definable. By compactness, there is some definable set D with $\mathfrak{m} \subseteq D \subseteq \mathcal{O}$.

Let $\Sigma(x)$ be the set of formulas

$$\Sigma(x) = \{x \in aD : a \notin \mathcal{O}\} \cup \{x \notin bD : b \in \mathcal{O}\}.$$

First suppose that $\Sigma(x)$ is finitely satisfiable. Then there is an element c realizing $\Sigma(x)$ in an elementary extension N of the original model. Then

$$a \notin \mathcal{O} \implies c \in aD(N) \implies a \in cD(N)^{-1},$$
$$b \in \mathcal{O} \implies c \notin bD(N) \implies b \notin cD(N)^{-1}.$$

Then $\mathcal{O}$ is externally definable as the complement of $M \cap cD(N)^{-1}$.

So we may assume $\Sigma(x)$ is not finitely satisfiable. Then there are $a_1, \ldots, a_n \notin \mathcal{O}$ and $b_1, \ldots, b_m \in \mathcal{O}$ such that

$$\bigcap_{i=1}^{n} a_i D \subseteq \bigcup_{i=1}^{m} b_i D.$$

Note that $b_i D \subseteq \mathcal{O}$ for each i, because $b_i \in \mathcal{O}$ and $D \subseteq \mathcal{O}$. Conversely, for v the valuation associated to $\mathcal{O}$, if $x \in \mathcal{O}$ then $v(x) \geq 0$ and $v(a_i) < 0$ (for each i), so $v(x/a_i) > 0$ and $x/a_i \in \mathfrak{m} \subseteq D$, implying that $x \in a_i D$. So $\mathcal{O} \subseteq a_i D$ for each i. Then

$$\mathcal{O} \subseteq \bigcap_{i=1}^{n} a_i D \subseteq \bigcup_{i=1}^{m} b_i D \subseteq \mathcal{O}.$$

Then equality holds, so $\mathcal{O}$ equals the definable set $\bigcap_{i=1}^{n} a_i D$.

The final statement follows since expanding the structure by the externally definable set $\mathcal{O}$ does not change the set of externally definable sets, assuming NIP. Ideals of definable valuation rings are externally definable. □

Lemma 4.10. *Let R be a sufficiently saturated dp-minimal domain with fraction field K. Define*

$$\mathcal{O} = R^{00} : R^{00} := \{x \in K : xR^{00} \subseteq R^{00}\} \subseteq K.$$

Then $\mathcal{O}$ is a valuation ring. Further, if $a \in K$, then $a \in \mathcal{O}$ if and only if $aR/(aR \cap R)$ is finite. In particular, $\mathcal{O} = R^{00} : R^{00}$ is a $\vee$-definable and externally definable valuation ring.

Proof. The proof that $R^{00} : R^{00}$ is a valuation ring is similar to the proof of [d'Elbée and Halevi 2021, Proposition 3.14(1)].

Claim 4.11. *$R^{00} : R^{00}$ is a valuation ring.*

Proof. It is routine to show that $R^{00} : R^{00}$ is a ring, so it is sufficient to prove that it is a valuation ring, i.e., for all $a \in K$, $aR^{00} \subseteq R^{00}$ or $a^{-1}R^{00} \subseteq R^{00}$. Let $a \in K$. By

[Chernikov et al. 2015, Proposition 4.5], $aR^{00}/(R^{00} \cap aR^{00})$ or $R^{00}/(R^{00} \cap aR^{00})$ is small. Neither R^{00} nor aR^{00} has any type-definable subgroups of small index, so either $aR^{00} = R^{00} \cap aR^{00}$ or $R^{00} = R^{00} \cap aR^{00}$. Equivalently, $aR^{00} \subseteq R^{00}$ or $R^{00} \subseteq aR^{00}$, i.e., $a^{-1}R^{00} \subseteq R^{00}$. □

If $a \in \mathcal{O}$ then by definition $aR^{00} \subseteq R^{00}$. It follows that $aR^{00} \subseteq aR \cap R \subseteq aR$, so $aR \cap R$ has bounded, hence finite index in aR. For the other direction, for $a \in K$, if $aR \cap R$ has finite index in aR then $(aR)^{00} \subseteq aR \cap R \subseteq aR$. In particular, $(aR)^{00} \subseteq R$. However, $(aR)^{00} = aR^{00}$ so $aR^{00} \subseteq R$ and $aR^{00} = (aR^{00})^{00} \subseteq R^{00}$. It follows that $\mathcal{O}$ is defined by the disjunction of conditions of the form "$aR/(aR \cap R)$ is finite". By Proposition 4.9, $\mathcal{O}$ is externally definable. □

Corollary 4.12. *Let R be a sufficiently saturated dp-minimal domain with fraction field $K = \mathrm{Frac}(R)$ and let $\mathcal{O} = R^{00} : R^{00}$. Then R^{00} is an externally definable ideal of $\mathcal{O}$, and the ring R/R^{00} is either finite or dp-minimal.*

If, furthermore, R^{00} is a nonmaximal prime ideal of R, then R^{00} is the maximal ideal of $\mathcal{O}$ and $\mathcal{O}$ is equal to the localization of R at R^{00}.

Proof. If R is a field then $R = R^{00}$ and we have nothing to show; so assume otherwise. Then R^{00} is an ideal of the valuation ring $\mathcal{O}$, so by Lemma 4.10 and Proposition 4.9, it is also externally definable.

Thus, adding R^{00} as a predicate preserves dp-minimality (Fact 3.1), so R/R^{00} is either finite or dp-minimal as well.

If R^{00} is a nonmaximal prime ideal $\mathfrak{p}$ of R then by Corollary 4.7 the localization $R_{\mathfrak{p}}$ is a valuation ring with maximal ideal $\mathfrak{p}$. Note that $\mathcal{O}' = \mathfrak{m}' : \mathfrak{m}'$ for any valuation ring $(\mathcal{O}', \mathfrak{m}')$, and so

$$\mathcal{O} := R_{00} : R_{00} = \mathfrak{p} : \mathfrak{p} = R_{\mathfrak{p}}.$$

Thus $\mathcal{O}$ is the localization $R_{\mathfrak{p}}$ and R^{00} is the maximal ideal of $\mathcal{O}$. □

4.4. ***The $\sqrt{\mathfrak{m}^{00}}$ dichotomy.*** We observe that the proof of [d'Elbée and Halevi 2021, Proposition 3.14] had a gap[3], which we correct now.

Fact 4.13. *Let G be an abelian dp-minimal group in a structure M.*

(1) *If M is sufficiently saturated and X, Y are type-definable subgroups of G, then $X/(X \cap Y)$ or $Y/(X \cap Y)$ is small (relative to the degree of saturation).*

(2) *If X, Y are definable subgroups of G, then $X/(X \cap Y)$ or $Y/(X \cap Y)$ is finite.*

(3) *If X, Y are externally definable subgroups of G, then $X/(X \cap Y)$ or $Y/(X \cap Y)$ is finite.*

[3] In the proof of [d'Elbée and Halevi 2021, Proposition 3.14], we misunderstood the notation "$< \infty$" in [Kaplan and Shelah 2013, Proposition 3.12; Chernikov et al. 2015, Proposition 4.5(2)] to mean "finite", when it in fact meant "small".

Part (1) is [Chernikov et al. 2015, Proposition 4.5(2)]. It easily implies (2), which then implies (3) by passing to the Shelah expansion.

Proposition 4.14 [d'Elbée and Halevi 2021, Proposition 3.14]. *Let R be a sufficiently saturated dp-minimal domain with maximal ideal $\mathfrak{m}$ and fraction field K. Then*

(1) $\mathfrak{m}^{00} : \mathfrak{m}^{00}$ *is a valuation overring of* R;

(2) $\sqrt{\mathfrak{m}^{00}} = \{a \in \mathfrak{m} \mid R/aR \text{ is infinite}\}$;

(3) *for every* $a \in \mathfrak{m}$ *with* R/aR *finite,* $\sqrt{\mathfrak{m}^{00}} = P_a$.

In particular, exactly one of the following holds:

- R/aR *is infinite for all* $a \in \mathfrak{m}$ *and in this case* $\sqrt{\mathfrak{m}^{00}} = \mathfrak{m}$.
- *There is* $a \in \mathfrak{m}$ *with* R/aR *finite, and for any such* a, $\sqrt{\mathfrak{m}^{00}} = P_a$.

Proof. Before beginning the proof, we make a general observation about local rings: if $I \subseteq J \subsetneq R$ are proper ideals in a local ring R, and J/I is finite, then $J \subseteq \sqrt{I}$. Indeed, for any $b \in J$, the pigeonhole principle gives $b^n \equiv b^m \pmod{I}$ for some $n < m < \omega$. Then $b^n(1 - b^{m-n}) \in I$, implying that $b^n \in I$, because $1 - b^{m-n}$ is a unit. In particular, if $\mathfrak{p} \subsetneq \mathfrak{q}$ are prime ideals, then $\mathfrak{q}/\mathfrak{p}$ is infinite, or else $\mathfrak{q} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$. (Compare with [d'Elbée and Halevi 2021, Remark 3.7].)

(1) It is routine to show that $\mathfrak{m}^{00} : \mathfrak{m}^{00}$ is a ring, so it is sufficient to prove that it is a valuation ring, i.e., for all $a \in K$, $a\mathfrak{m}^{00} \subseteq \mathfrak{m}^{00}$ or $a^{-1}\mathfrak{m}^{00} \subseteq \mathfrak{m}^{00}$. This follows immediately by the same argument as in Claim 4.11.

(2) If $R/\mathfrak{m}$ is infinite then $\mathfrak{m} = \mathfrak{m}^{00}$ by Fact 4.5 and the equality is obvious. We assume that $R/\mathfrak{m}$ is finite and show that $a \in \mathfrak{m} \setminus \sqrt{\mathfrak{m}^{00}}$ if and only if R/aR is finite. When $R/\mathfrak{m}$ is finite, note that $\mathfrak{m}^{00} = R^{00}$, and so $\mathfrak{m}^{00}$ is externally definable by Corollary 4.12.

Let $a \in \mathfrak{m} \setminus \sqrt{\mathfrak{m}^{00}}$. By Fact 4.13(3) we have that $aR/(\mathfrak{m}^{00} \cap aR)$ is finite or $\mathfrak{m}^{00}/(\mathfrak{m}^{00} \cap aR)$ is finite. If it was the former, then by the remarks about local rings at the start of the proof,

$$a \in aR \subseteq \sqrt{\mathfrak{m}^{00} \cap aR} \subseteq \sqrt{\mathfrak{m}^{00}},$$

a contradiction. Consequently, $\mathfrak{m}^{00}/(\mathfrak{m}^{00} \cap aR)$ is finite. By the definition of $\mathfrak{m}^{00}$, we have $\mathfrak{m}^{00} \subseteq aR$ and so $|\mathfrak{m}/aR|$ is finite, which implies that $|R/aR| = |R/\mathfrak{m}||\mathfrak{m}/aR|$ is finite as well.

Conversely, if R/aR is finite, then

$$\mathfrak{m}^{00} = R^{00} = (aR)^{00} = aR^{00} = a\mathfrak{m}^{00}.$$

Suppose for the sake of contradiction that $a \in \sqrt{\mathfrak{m}^{00}}$. Take n minimal such that $a^n \in \mathfrak{m}^{00}$. Then $a^n \in \mathfrak{m}^{00} = a\mathfrak{m}^{00}$, implying $a^{n-1} \in \mathfrak{m}^{00}$, which contradicts the choice of n.

(3) Assume that R/aR is finite. By (2), $a \notin \sqrt{\mathfrak{m}^{00}}$ so $a \notin \mathfrak{p}$ for some prime ideal $\mathfrak{p} \supseteq \mathfrak{m}^{00}$. By choice of P_a, we have $\sqrt{\mathfrak{m}^{00}} \subseteq \mathfrak{p} \subseteq P_a$. For the other inclusion, note that $P_a \subsetneq \mathfrak{m}$, and so $\mathfrak{m}/P_a$ is infinite by the remarks on local rings at the start of the proof. If $b \in P_a$, then $bR \subseteq P_a \subsetneq \mathfrak{m} \subseteq R$, so R/bR is infinite, and $b \in \sqrt{\mathfrak{m}^{00}}$ by (2). This shows the reverse inclusion $P_a \subseteq \sqrt{\mathfrak{m}^{00}}$. □

Corollary 4.15. *Let R be a sufficiently saturated dp-minimal domain with maximal ideal $\mathfrak{m}$ and fraction field K. Then $\mathfrak{m} = \mathfrak{m}^{00}$ if and only if $\mathfrak{m}^{00}$ is prime and R/aR is infinite for all $a \in \mathfrak{m}$.*

Proof. Assume that $\mathfrak{m} = \mathfrak{m}^{00}$; so it is prime. Assume further that R/aR is finite for some $a \in \mathfrak{m}$. By Proposition 4.14, $\mathfrak{m} = P_a$ so the latter is a maximal ideal which gives that a is a unit, contradicting the choice of a. So R/aR is infinite for any $a \in \mathfrak{m}$.

If $\mathfrak{m}^{00}$ is prime and R/aR is infinite for all $a \in \mathfrak{m}$ we conclude that $\mathfrak{m}^{00} = \mathfrak{m}$ by Proposition 4.14. □

Corollary 4.8 yields a strengthening of Proposition 4.14.

Lemma 4.16. *Let R be a sufficiently saturated dp-minimal domain with maximal ideal $\mathfrak{m}$.*

(1) *If R/aR is infinite for every $a \in \mathfrak{m}$, then $\sqrt{\mathfrak{m}^{00}} = \mathfrak{m}$.*

(2) *If $a \in \mathfrak{m}$ and R/aR is finite, then $\mathfrak{m}^{00} = P_a$, and $\mathfrak{m}^{00}$ is the largest nonmaximal prime ideal.*

Proof. Proposition 4.14 proves (1), and shows that in the setting of (2), we have $P_a = \sqrt{\mathfrak{m}^{00}}$. But $P_a = P_a^{00}$ by Corollary 4.8, and so

$$P_a = P_a^{00} \subseteq \mathfrak{m}^{00} \subseteq \sqrt{\mathfrak{m}^{00}} = P_a.$$

Equality must hold, and $\mathfrak{m}^{00} = P_a$. In particular, $\mathfrak{m}^{00}$ is a nonmaximal prime ideal. If $\mathfrak{m}^{00}$ fails to be the largest nonmaximal prime ideal, then there is a prime ideal $\mathfrak{p}$ with $\mathfrak{m}^{00} \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$. Take $b \in \mathfrak{m} \setminus \mathfrak{p}$. Enlarging $\mathfrak{p}$, we may assume $\mathfrak{p}$ is the largest prime ideal not containing b, i.e., $\mathfrak{p} = P_b$. Then

$$P_b = P_b^{00} \subseteq \mathfrak{m}^{00} \subsetneq P_b,$$

by Corollary 4.8 and the fact that $P_b \subseteq \mathfrak{m}$. This is absurd, so $\mathfrak{m}^{00}$ is truly the second-largest prime ideal. □

4.5. ***The topological ring R/R^{00}.*** Let R be an NIP ring.

The "logic topology" on R/R^{00} is the topology where a set $X \subseteq R/R^{00}$ is closed if its preimage $\widetilde{X} \subseteq R$ is type-definable. The logic topology makes R/R^{00} into a compact Hausdorff topological ring; see [Simon 2015, Section 8.1].

In contrast to compact Hausdorff topological groups, the following holds:

Fact 4.17 [Ribes and Zalesskii 2010, Proposition 5.1]. *A topological ring is compact Hausdorff if and only if it is profinite.*

As a direct consequence:

Fact 4.18. *If R is an NIP ring then R/R^{00} is profinite and so $R^{00} = R^0$.*

The following should be well known, but we include a proof for completeness.

Fact 4.19. *Let A be an infinite compact Hausdorff integral domain such that every nontrivial principal ideal has finite index. Then $F = \mathrm{Frac}(A)$ is either isomorphic to a finite extension of $\mathbb{Q}_p$ or to $\mathbb{F}_q((t))$.*

Proof. We first make some observations.

(1) If $a \in A \setminus \{0\}$, then the principal ideal aA is closed because it is the image of the compact set A under the continuous function $f(x) = ax$. Since aA has finite index, aA is a clopen ideal.

(2) Every nonzero ideal I contains a principal ideal, and therefore is clopen and has finite index.

(3) Conversely, the zero ideal is not open, since A is infinite.

(4) The set of units $A^\times$ is closed, being the projection of the compact set

$$\{(x, y) \in A^2 : xy = 1\}$$

onto the first coordinate. If A is a field, this contradicts the previous point, so A is not a field.

(5) By [Ribes and Zalesskii 2010, Proposition 5.1.2], the open ideals form a neighborhood basis of 0. By the points above, the nonzero ideals of A form a neighborhood basis of 0, and the family

$$\{I + a : I \trianglelefteq A \text{ a nonzero ideal}, \ a \in A\}$$

is a basis for the topology.

(6) Every commutative profinite ring is a cartesian product of profinite local rings [Ribes and Zalesskii 2010, Exercise 5.1.3(2)] so as A is an integral domain it must be a local profinite ring.

Let $F = \mathrm{Frac}(A)$. By [Prestel and Ziegler 1978, Example 1.2], the family

$$\{I + a : I \trianglelefteq A \text{ a nonzero ideal}, \ a \in F\}$$

is a basis for a nondiscrete Hausdorff ring topology on F. Moreover, this ring topology is a field topology because A is local; see the proof of [Prestel and Ziegler 1978, Theorem 2.2(b)]. One can easily see that A is clopen in this topology. By the observations above, the induced topology on A is the original compact Hausdorff

topology on A. Then F is a locally compact field, because it is covered by the compact open translates $A + b$.

As F is a locally compact field, it is isomorphic to $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or $\mathbb{F}_q((t))$ [Milne 2020, Remark 7.49]. Since $\mathbb{R}$ and $\mathbb{C}$ have no compact subrings it must be one of the latter two. □

Corollary 4.20. *Let R be a sufficiently saturated NIP integral domain with R^{00} a prime ideal satisfying $R^{00} = aR^{00}$ for any $a \in R \setminus R^{00}$. Then* $\mathrm{Frac}(R/R^{00})$ *is either finite or a finite extension of $\mathbb{Q}_p$.*

Proof. Since R^{00} is a prime ideal, R/R^{00} is a compact Hausdorff integral domain; assume it is infinite. Note that every nonzero principal ideal in R/R^{00} has finite index. Indeed if $a \in R \setminus R^{00}$ then $aR \supseteq (aR)^{00} = aR^{00} = R^{00}$, so aR has finite index in R. By Fact 4.19, $\mathrm{Frac}(R/R^{00})$ is either a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$. Since R^{00} is a prime ideal it is externally definable, by Fact 4.1. Then R/R^{00} and hence $\mathrm{Frac}(R/R^{00})$ is NIP. Since $\mathbb{F}_q((t))$ has IP [Kaplan et al. 2011, Theorem 4.3], $\mathrm{Frac}(R/R^{00})$ must instead be a finite extension of $\mathbb{Q}_p$. □

Corollary 4.21. *Let R be a sufficiently saturated dp-minimal integral domain with maximal ideal $\mathfrak{m}$. If $R/\mathfrak{m}$ is finite and R^{00} is a prime ideal then* $\mathrm{Frac}(R/R^{00})$ *is either finite or a finite extension of $\mathbb{Q}_p$.*

Proof. If R/R^{00} is finite then it is a finite integral domain, and hence a finite field. Assume that R/R^{00} is infinite.

In particular, $\mathfrak{p} = R^{00}$ is a nonmaximal prime ideal so it is the maximal ideal of the valuation overring $R_{\mathfrak{p}}$ of R (Corollary 4.7). This implies that $aR^{00} = R^{00}$ for any $a \in R_{\mathfrak{p}} \setminus R^{00}$, and hence in particular for all $a \in R \setminus R^{00}$. The result now follows from Corollary 4.20. □

4.6. ***Subrings from additive subgroups.*** In this section, we prove the existence of an infinite definable proper subring in dp-minimal structures $(K, +, \cdot, A)$, where $(K, +, \cdot)$ is a field and A is an infinite additive proper subgroup, without any saturation assumption. This will be used later.

Proposition 4.22. *Let $M = (K, +, \cdot, A)$ be a dp-minimal expansion of a field with A a proper infinite additive subgroup. Then $R = A : A = \{x \in K : xA \subseteq A\}$ is an infinite definable proper subring of K and* $\mathrm{Frac}(R) = K$.

Proof. There is no harm in assuming sufficient saturation of M. Also, we will later pass to the Shelah expansion; this too causes no harm.

The definable set A is neither finite nor cofinite, so $(K, +, \cdot, A)$ is not strongly minimal.

By [Johnson 2018, Theorem 1.3], M has a Hausdorff nondiscrete definable V-topology called the *canonical topology*. The canonical topology is characterized

by the fact that

$$\{D - D : D \subseteq K \text{ definable and infinite}\}$$

is a neighborhood basis of 0, where $D - D = \{x - y : x, y \in D\}$. Taking $D = A$, we see that $A = A - A$ is a neighborhood of 0.

By saturation, there exists an externally definable valuation inducing this topology [Halevi et al. 2020, Proposition 3.5]. So after passing to the Shelah expansion, we may assume the existence of a definable valuation v inducing this topology. Let Γ be its value group.

Claim 4.23. *A is a bounded set with respect to the topology induced by v.*

Proof. Note that $(K, +)^0$ is an ideal in K of small index, so $(K, +)^0 = (K, +)$ and so K has no definable proper subgroups of finite index. Therefore K/A is infinite. Let $(c_i)_{i<\omega}$ be different representatives of distinct classes.

Assume towards a contradiction that A is not bounded, so for any $\gamma \in \Gamma$ there exists an element $a \in A$ with $v(a) < \gamma$. By saturation, and the fact that A is infinite, we can find a sequence of elements $(a_s)_{s<\omega}$ of A satisfying $v(a_s) < v(a_t) < v(c_i)$ for any $i < \omega$ and any $t < s < \omega$. Now consider the formulas $(x \in A + c_i)_{i<\omega}$ and $(v(x) = v(a_s))_{s<\omega}$; they contradict dp-minimality. □

The definable set R is certainly a definable subring of K, because A is an additive subgroup. Since A is a bounded neighborhood of 0, there is an open neighborhood $V \ni 0$ such that $V \cdot A \subseteq A$. Then R contains the infinite set V. The ring R is a proper subring because if we take $a \in A \setminus \{0\}$ and $b \notin A$ then $b/a \notin R$. Finally, $\mathrm{Frac}(R) = K$, since $(K, \mathrm{Frac}(R))$ is dp-minimal so $[K : \mathrm{Frac}(R)] = 1$. □

5. A trichotomy for dp-minimal fields

Definition 5.1. Let K be an infinite field.

(1) K is *ACVF-like* if there is a henselian valuation v on K with algebraically closed residue field.

(2) K is *pCF-like* if there is a henselian valuation v on K with finite residue field of characteristic p.

(3) K is *RCVF-like* if there is a henselian valuation v on K with real closed residue field.

We allow v to be trivial.

For example, $\mathbb{F}_p^{\mathrm{alg}}$ and $\mathbb{C}((t))$ are ACVF-like, $\mathbb{R}$ and $\mathbb{R}((t))$ are RCVF-like, and $\mathbb{Q}_3(i)$ and $\mathbb{Q}_3((t))$ are 3CF-like.

If there is a henselian valuation w on K such that Kw is p-adically closed, then we can compose w with the canonical valuation on Kw, getting a henselian valuation v on K with Kv finite of characteristic p. Thus K is pCF-like.

Remark 5.2. If K is a dp-minimal field, then K is ACVF-like, pCF-like, or RCVF-like. In fact, if $\mathcal{O}_\infty$ is the intersection of all 0-definable valuation rings on K, then $\mathcal{O}_\infty$ is henselian and its residue field is algebraically closed, real closed, or finite. See [Johnson 2023a, Theorem 4.8] or Fact 2.2.

In fact, the three cases are mutually exclusive:

Proposition 5.3. *An infinite field K satisfies at most one of the following*:

(1) *K is ACVF-like.*

(2) *K is pCF-like.*

(3) *K is RCVF-like.*

In the second case, p is uniquely determined.

Proof. Otherwise, there are two henselian valuation rings $\mathcal{O}_1, \mathcal{O}_2$ whose residue fields k_1, k_2 are of a different nature. There are two cases:

(1) The two valuations are comparable, say, $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Then $\mathcal{O}_2$ is a coarsening of $\mathcal{O}_1$, so k_1 is the residue field of some henselian valuation w on k_2. If k_2 is finite, then w is trivial, so $k_1 = k_2$ and k_1 is finite of the same characteristic as k_1. If k_2 is algebraically closed or real closed, then (k_2, w) is a model of ACVF or RCVF, respectively, and so k_1 is algebraically closed or real closed. In each case, k_1 has the same nature as k_2.

(2) The two valuations are incomparable. Let $\mathcal{O}_3$ be the join $\mathcal{O}_1 \cdot \mathcal{O}_2$ and let k_3 be its residue field. Then $\mathcal{O}_1$ and $\mathcal{O}_2$ induce independent henselian valuations w_1, w_2 on k_3, and the residue fields of w_1 and w_2 are k_1 and k_2, respectively. Because k_3 has two independent nontrivial henselian valuations, it must be separably closed [Engler and Prestel 2005, Theorem 4.4.1], and then the two residue fields k_1 and k_2 must be algebraically closed [Engler and Prestel 2005, Theorem 3.2.11]. Then k_1 and k_2 again have the same nature. □

Moreover, everything is determined by the complete theory of $(K, +, \cdot)$:

Lemma 5.4. *Let K_1, K_2 be two dp-minimal fields. If $K_1 \equiv K_2$, then K_1 is ACVF-like (RCVF-like, pCF-like) if and only if K_2 is ACVF-like (RCVF-like, pCF-like).*

Proof. Let $\mathcal{O}_i$ be a henselian valuation ring on K_i whose residue field is algebraically closed, real closed, or finite. By Robinson joint consistency, there is a structure $(K_3, \mathcal{O}_1', \mathcal{O}_2')$ with elementary embeddings $(K_i, \mathcal{O}_i) \hookrightarrow (K_3, \mathcal{O}_i')$ for $i = 1, 2$. In particular, $\mathcal{O}_i'$ is a henselian valuation ring on K_3 whose residue field has the same nature as the residue field of $\mathcal{O}_i$. By Proposition 5.3, the residue fields of $\mathcal{O}_1'$ and $\mathcal{O}_2'$ must have the same nature. Therefore, the residue fields of $\mathcal{O}_1$ and $\mathcal{O}_2$ must have the same nature. □

Remark 5.5. Let K be a sufficiently saturated dp-minimal field, possibly with extra structure. Suppose that K is pCF-like. Let $\mathcal{O}_{\text{can}}$ be the intersection of all definable valuation rings on K. By [Johnson 2023a, Theorem 4.7], $\mathcal{O}_{\text{can}}$ is henselian, definable, and has finite residue field, because K is pCF-like rather than ACVF-like or RCVF-like.[4] We claim that $\mathcal{O}_{\text{can}}^{00}$ is the second largest prime ideal in $\mathcal{O}_{\text{can}}$, and that $\mathcal{O}_{\text{can}}/\mathcal{O}_{\text{can}}^{00}$ is a ring of characteristic 0.

The valuation ring $\mathcal{O}_{\text{can}}$ is dp-minimal with finite residue field, so the interval $[-v(p), v(p)]$ in the value group of $\mathcal{O}_{\text{can}}$ is finite by Fact 2.3. Then the value group is discretely ordered and so the maximal ideal $\mathfrak{m}_{\text{can}}$ of $\mathcal{O}_{\text{can}}$ is a principal ideal. Since $\mathcal{O}_{\text{can}}/\mathfrak{m}_{\text{can}}$ is finite, $\mathcal{O}_{\text{can}}^{00} = \mathfrak{m}_{\text{can}}^{00}$ is the second largest prime ideal in $\mathcal{O}_{\text{can}}$ by Lemma 4.16.

Because $\mathcal{O}_{\text{can}}^{00}$ is a nonmaximal prime ideal in $\mathcal{O}_{\text{can}}$, the quotient $\mathcal{O}_{\text{can}}/\mathcal{O}_{\text{can}}^{00}$ is an integral domain but not a field. In particular, it is infinite. By Corollary 4.21, the fraction field of $\mathcal{O}_{\text{can}}/\mathcal{O}_{\text{can}}^{00}$ must be a finite extension of $\mathbb{Q}_p$, and must therefore have characteristic 0.

6. Proof of the main theorem

6.1. ***Exceptional dp-minimal domains.*** An integral domain which is neither a field nor a valuation ring is called *exceptional*. Our goal is to understand exceptional dp-minimal domains. By [d'Elbée and Halevi 2021, Lemma 3.6(1)], any exceptional dp-minimal domain R is a local domain with finite residue field $R/\mathfrak{m}$. In particular, for such domains, $R^{00} = \mathfrak{m}^{00}$. Note also that R is infinite.

Remark 6.1. Let R be an exceptional dp-minimal domain (possibly with extra structure). If $\mathcal{O}$ is an externally definable valuation ring on $\text{Frac}(R)$, then $R \subseteq \mathcal{O}$. First, we may assume that $\mathcal{O}$ is definable by passing to R^{Sh}. Then Fact 4.3 shows that $R \subseteq \mathcal{O}$ or $\mathcal{O} \subseteq R$. The latter cannot happen as R is not a valuation ring.

Recall that every infinite dp-minimal field is either ACVF-like, RCVF-like or pCF-like, and only one of these properties holds (Remark 5.2 and Proposition 5.3).

Proposition 6.2. *If R is an exceptional dp-minimal domain, then* $\text{Frac}(R)$ *is ACVF-like or pCF-like.*

Proof. We may assume that R is sufficiently saturated. Its fraction field $K = \text{Frac}(R)$ is dp-minimal by Fact 4.2. Suppose for the sake of contradiction that K is RCVF-like. Then K admits a henselian valuation with real closed residue field. By Ax–Kochen–Ershov, K is elementarily equivalent to the field of Hahn series $L = \mathbb{R}((t^{\Gamma}))$ for some dp-minimal ordered abelian group Γ. This field L is orderable. By [Jahnke et al. 2017, proof of Theorem 6.2], any field-order on L is definable. As a result there is a definable field-ordering $\leq$ on K as well. Then $(K, R, \leq)$

[4] The notation reflects the fact that $\mathcal{O}_{\text{can}}$ is the canonical henselian valuation ring on K.

is dp-minimal. Let $\mathcal{O}$ be the convex hull of $\mathbb{Z}$ in $(K, \leq)$. It is easily seen to be a valuation subring of K and since it is convex, it is externally definable. By Remark 6.1, $R \subseteq \mathcal{O}$. By saturation, $R \subseteq [-n, n]$ for some integer n, which is absurd. □

The next proposition shows that the two cases of Lemma 4.16 correspond to the case where $\mathrm{Frac}(R)$ is ACVF-like and pCF-like.

Proposition 6.3. *Let R be an exceptional dp-minimal domain with maximal ideal $\mathfrak{m}$.*

(1) *If R/aR is infinite for every $a \in \mathfrak{m}$, then $\mathrm{Frac}(R)$ is ACVF-like.*

(2) *If R/aR is finite for some $a \in \mathfrak{m}$, then $\mathrm{Frac}(R)$ is pCF-like.*

Proof. Since $\mathrm{Frac}(R)$ is either ACVF-like or pCF-like by Proposition 6.2, it suffices to show that $\mathrm{Frac}(R)$ is pCF-like if and only if R/aR is finite for some $a \in \mathfrak{m}$. There is no harm in assuming that R is sufficiently saturated.

First suppose that $\mathrm{Frac}(R)$ is pCF-like. Let $\mathcal{O}_{\mathrm{can}}$ be the canonical henselian valuation ring, as in Remark 5.5. Then $\mathcal{O}_{\mathrm{can}}$ is a definable henselian valuation ring with finite residue field, the second largest prime ideal of $\mathcal{O}_{\mathrm{can}}$ is $\mathcal{O}_{\mathrm{can}}^{00}$, and the quotient ring $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$ has characteristic zero.

By Remark 6.1, as R is exceptional we have $R \subseteq \mathcal{O}_{\mathrm{can}}$ and hence $R^{00} \subseteq \mathcal{O}_{\mathrm{can}}^{00} \cap R$. Note that $\mathcal{O}_{\mathrm{can}}^{00} \cap R$ is a prime ideal of R. The quotient $R/(R \cap \mathcal{O}_{\mathrm{can}}^{00})$ injects into $\mathcal{O}_{\mathrm{can}}/\mathcal{O}_{\mathrm{can}}^{00}$, so it has characteristic zero. In contrast, $R/\mathfrak{m}$ is finite. Therefore $R \cap \mathcal{O}_{\mathrm{can}}^{00} \neq \mathfrak{m}$. Since $\mathfrak{m}^{00} = R^{00}$ is contained in the nonmaximal prime ideal $R \cap \mathcal{O}_{\mathrm{can}}^{00}$, we have $\sqrt{\mathfrak{m}^{00}} \subsetneq \mathfrak{m}$. By Lemma 4.16(1), R/aR is finite for some $a \in \mathfrak{m}$.

Conversely, suppose that R/aR is finite for some $a \in \mathfrak{m}$. By Lemma 4.16(2), $R^{00} = \mathfrak{m}^{00}$ is a nonmaximal prime ideal of R. In particular, R/R^{00} is infinite. Then $k = \mathrm{Frac}(R/R^{00})$ is a finite extension of $\mathbb{Q}_p$ by Corollary 4.21. Let $\mathcal{O}_p$ be the standard definable valuation ring of k.

Let $K = \mathrm{Frac}(R)$ and let $\mathcal{O} = R^{00} : R^{00}$; it is an externally definable valuation overring with maximal ideal R^{00} (Corollary 4.12), so $\mathcal{O}/R^{00}$ is its residue field. Now using the quotient map, the structure $(\mathcal{O}/R^{00}, R/R^{00})$ is dp-minimal, and hence so is $(\mathcal{O}/R^{00}, \mathrm{Frac}(R/R^{00}))$. As R/R^{00} is infinite, $\mathrm{Frac}(R/R^{00})$ is an infinite subfield of $\mathcal{O}/R^{00}$. Hence they must be equal by dp-minimality. Composing the valuation $\mathcal{O}_p$ on k with $\mathcal{O}$, we are supplied with an externally definable valuation on K with finite residue field; so $K = \mathrm{Frac}(R)$ is pCF-like. □

Combining with Lemma 4.16, we get the following dichotomy for exceptional dp-minimal domains:

Corollary 6.4. *Let R be an exceptional dp-minimal domain with maximal ideal $\mathfrak{m}$. Then exactly one of the following holds*:

(1) $\mathrm{Frac}(R)$ *is ACVF-like, R/aR is infinite for every $a \in \mathfrak{m}$, and $\sqrt{\mathfrak{m}^{00}} = \mathfrak{m}$.*

(2) $\mathrm{Frac}(R)$ *is pCF-like,* R/aR *is finite for some* $a \in \mathfrak{m}$, $\mathfrak{m}^{00}$ *is the second-largest prime ideal in* R, *and* $\mathfrak{m}^{00} = P_a$ *for any* $a \in \mathfrak{m}$ *such that* R/aR *is finite.*

Lemma 6.5. *Let* R_0 *be an exceptional dp-minimal integral domain with fraction field* $K_0 = \mathrm{Frac}(R_0)$. *If* K_0 *is ACVF-like, then there is an externally definable nontrivial valuation ring in the structure* $(K_0, R_0, +, \cdot)$.

Proof. Let $R \succ R_0$ be a sufficiently saturated elementary extension and $K = \mathrm{Frac}(R)$. Then R is exceptional and K is ACVF-like. Let $\mathcal{O} = R^{00} : R^{00}$ and $\mathcal{O}_0 = \mathcal{O} \cap K$ (see Lemma 4.10). Then $\mathcal{O}_0$ is an externally definable valuation ring on K_0. We only need to show that $\mathcal{O}_0$ is nontrivial.

Let $\mathfrak{m}_0$ and $\mathfrak{m}$ be the maximal ideals of R_0 and R. Take nonzero $a \in \mathfrak{m}_0 \subseteq \mathfrak{m}$. By Corollary 6.4, $\mathfrak{m} = \sqrt{\mathfrak{m}^{00}}$, and so $a^n \in \mathfrak{m}^{00} = R^{00}$ for some n. On the other hand, $1 \notin \mathfrak{m}^{00}$. Since $\mathfrak{m}^{00}$ is an ideal in $\mathcal{O}$, we must have $v(a^n) \neq v(1)$, where v is the valuation on K induced by $\mathcal{O}$. Then the restriction of v to K_0 is nontrivial, and $\mathcal{O}_0$ is nontrivial. □

By [Halevi et al. 2020, Proposition 3.5], every sufficiently saturated field with a definable V-topology admits a nontrivial externally definable valuation ring. The following gives the same conclusion by assuming the existence of a definable infinite proper additive subgroup, rather than assuming saturation.

Proposition 6.6. *Let* $(K, +, \cdot, A)$ *be a dp-minimal expansion of a field* K *by an infinite, proper additive subgroup* $(A, +) \subsetneq (K, +)$. *Then there is a nontrivial externally definable valuation ring on* K.

Proof. By Proposition 4.22, $R = A : A$ is an infinite definable proper subring of K and $\mathrm{Frac}(A) = K$. If it is a valuation ring, there is nothing to show so we assume it is exceptional. If $\mathrm{Frac}(R)$ is pCF-like, then it admits a definable valuation (the canonical valuation). So we assume that $\mathrm{Frac}(R)$ is ACVF-like. The result now follows from Lemma 6.5. □

After completing the classification, we will see that "externally definable" can be changed to "definable" (Proposition 6.20).

6.2. ***The pCF-like case.*** Let R be a dp-minimal domain with maximal ideal $\mathfrak{m}$, not necessarily pure, and $K = \mathrm{Frac}(R)$. In the following, we frequently use the fact that if R is exceptional then $R \subseteq \mathcal{O}_0$ for any externally definable valuation $\mathcal{O}_0$ of $K = \mathrm{Frac}(R)$ (Remark 6.1).

Proposition 6.7. *Let* R *be an exceptional dp-minimal domain such that* $K = \mathrm{Frac}(R)$ *is pCF-like. Let* $\mathcal{O}_{\mathrm{can}}$ *be the canonical henselian valuation on* K *as in Remark 5.5.*

(1) R *has finite index in* $\mathcal{O}_{\mathrm{can}}$.

(2) *In fact, there is some* n *such that* $\mathcal{O}_{\mathrm{can}} \supseteq R \supseteq p^n \mathcal{O}_{\mathrm{can}}$, *and each inclusion has finite index.*

In particular, R arises via the construction in Proposition 3.2 in a definable manner: $\mathcal{O}_{\text{can}}$ is a definable valuation subring of K and R is the preimage of $R/p^n\mathcal{O}_{\text{can}}$ under the definable map $\mathcal{O}_{\text{can}} \to \mathcal{O}_{\text{can}}/p^n\mathcal{O}_{\text{can}}$.

Proof. Replacing R with an elementary extension, we may assume R is sufficiently saturated for both points.

(1) By Remark 6.1, $R \subseteq \mathcal{O}_{\text{can}}$. As both rings are definable, it suffices to show $\mathcal{O}_{\text{can}}/R$ is finite, or equivalently, small. Equivalently, we must show that $R^{00} = \mathcal{O}_{\text{can}}^{00}$. Suppose not. Then $R^{00} \subsetneq \mathcal{O}_{\text{can}}^{00} \subseteq \mathfrak{m}_{\text{can}}$, where $\mathfrak{m}_{\text{can}}$ is the maximal ideal of $\mathcal{O}_{\text{can}}$.

Let $\mathfrak{p} = R^{00} = \mathfrak{m}^{00}$. By Corollary 6.4, $\mathfrak{p}$ is the second-largest prime ideal of R. By [d'Elbée and Halevi 2021, Proposition 3.8 and Theorem 3.9], the localization $R_{\mathfrak{p}}$ is a valuation ring with maximal ideal $\mathfrak{p}$. The fact that $\mathfrak{p} \subseteq \mathfrak{m}_{\text{can}}$ implies the reverse inclusion on valuation rings: $\mathcal{O}_{\text{can}} \subseteq R_{\mathfrak{p}}$. Then $\mathfrak{p}$ is closed under multiplication by elements of $\mathcal{O}_{\text{can}}$, so $\mathfrak{p}$ is an ideal in $\mathcal{O}_{\text{can}}$.

Let v be the valuation on K associated to $\mathcal{O}_{\text{can}}$. Both $\mathfrak{p}$ and $\mathcal{O}_{\text{can}}^{00}$ are ideals in $\mathcal{O}_{\text{can}}$, so they are defined by cuts in the value group. The fact that $\mathfrak{p} \subsetneq \mathcal{O}_{\text{can}}^{00}$ implies that there is some closed ball $B_{\geq\gamma}(0)$ (with respect to the valuation induced by $\mathcal{O}_{\text{can}}$) separating the two:

$$\mathfrak{p} \subseteq B_{\geq\gamma}(0) \subseteq \mathcal{O}_{\text{can}}^{00}.$$

By Fact 4.18, the compact Hausdorff ring R/R^{00} is profinite, which means that R^{00} is a filtered intersection of definable ideals $I \lhd R$ with finite index. The ball $B_{\geq\gamma}(0)$ is definable, so by saturation there is some finite-index definable ideal I such that $I \subseteq B_{\geq\gamma}(0) \subseteq \mathcal{O}_{\text{can}}^{00}$. Then there is a ring homomorphism

$$R/I \to \mathcal{O}_{\text{can}}/\mathcal{O}_{\text{can}}^{00}.$$

However, $\mathcal{O}_{\text{can}}/\mathcal{O}_{\text{can}}^{00}$ has characteristic 0 (Remark 5.5) and R/I is finite, so this is absurd.

(2) If m is the index of R in $\mathcal{O}_{\text{can}}$, then m annihilates the group $(\mathcal{O}_{\text{can}}/R, +)$, meaning that $R \supseteq m\mathcal{O}_{\text{can}}$. Write m as $m_0 p^n$, where m_0 is prime to p. Then m_0 is invertible in $\mathcal{O}_{\text{can}}$, so

$$\mathcal{O}_{\text{can}} \supseteq R \supseteq m\mathcal{O}_{\text{can}} = p^n\mathcal{O}_{\text{can}}.$$

Finally, $\mathcal{O}_{\text{can}}/p^n\mathcal{O}_{\text{can}}$ has finite index because $\mathcal{O}_{\text{can}}$ has finite residue field and $v(p)$ is a finite multiple of the minimum positive element in the valuation group (Fact 2.3). □

6.3. ***Rogue domains.*** Let R be a dp-minimal domain with maximal ideal $\mathfrak{m}$, not necessarily pure, and $K = \text{Frac}(R)$. Whenever R is sufficiently saturated we let $\mathcal{O}$ be the externally definable valuation ring $R^{00} : R^{00}$, $\mathfrak{m}_0$ its maximal ideal, v the corresponding valuation, Γ its value group, and $\boldsymbol{k}_0$ its residue field.

Definition 6.8. A *rogue domain* is an exceptional dp-minimal domain $(R, +, \cdot, \dots)$, possibly with extra structure, such that $\mathrm{Frac}(R)$ is ACVF-like and R^{00} is *not* definable.

We will eventually show that rogue domains do not exist.

Remark 6.9. Remark 4.4 has the following consequences.

(1) If R is a rogue domain and $R \equiv S$, then S is a rogue domain. In particular, elementary extensions of rogue domains are rogue.

(2) Any dp-minimal expansion of a rogue domain is a rogue domain. In particular, Shelah expansions of rogue domains are rogue.

(3) If R is a sufficiently saturated rogue domain, then R/R^{00} is infinite.

Lemma 6.10. *Let R be a sufficiently saturated rogue domain with fraction field K, and let $\mathcal{O} = R^{00} : R^{00}$. Then $\mathcal{O} \subsetneq \mathcal{O}'$ for any definable valuation ring $\mathcal{O}'$ on K.*

Proof. Otherwise, $\mathcal{O} \supseteq \mathcal{O}'$, because any two externally definable valuation rings are comparable (essentially by Fact 4.3). The definable valuation ring $\mathcal{O}'$ is henselian. If its residue field is finite, then K is pCF-like, contradicting the fact that K is ACVF-like. Thus $\mathcal{O}'$ has infinite residue field. By saturation, $\mathcal{O}'$ has residue field of large cardinality, as does the coarsening $\mathcal{O}$. In particular, the cardinality of the residue field $\boldsymbol{k}_0$ of $\mathcal{O}$ is much larger than that of R/R^{00}.

Let $S \subseteq R$ be a set of coset representatives of R^{00} in R, chosen so that $0 \in S$. Let v be the valuation associated with $\mathcal{O}$ and let res be the residue map. Let $\mathrm{rv} : K^\times \to K^\times/(1 + \mathfrak{m}_0)$ be the quotient map, where $\mathfrak{m}_0$ is the maximal ideal of $\mathcal{O}$.

Let

$$B = \{\mathrm{res}(x/y) : x, y \in S \setminus \{0\} \text{ and } v(x) = v(y)\} \subseteq \boldsymbol{k}_0.$$

Because $|R/R^{00}|$ is much smaller than $|\boldsymbol{k}_0|$, we have $B \subsetneq \boldsymbol{k}_0$. Take $c \in \mathcal{O}$ such that $\mathrm{res}(c) \in \boldsymbol{k}_0^\times \setminus B$. Note that $c \in \mathcal{O}^\times$ since $\mathrm{res}(c) \neq 0$. The fact that $c \in \mathcal{O}^\times$ implies that $cR^{00} = R^{00}$, by definition of $\mathcal{O} = R^{00} : R^{00}$. Then

$$R^{00} = c(R^{00}) = (cR)^{00} \subseteq cR,$$

so $R^{00} \subseteq cR \cap R$. We will arrive at a contradiction by showing that $R^{00} = cR \cap R$, and hence R^{00} is definable.

Otherwise, take $a_1 \in cR \cap R \setminus R^{00}$. Then $b_1 := a_1/c \in R$. If $b_1 \in R^{00}$, then $a_1 = cb_1 \in cR^{00} = R^{00}$, a contradiction. So $b_1 \notin R^{00}$.

By choice of S, there are some $a_2, b_2 \in S$ such that

$$a_2 \equiv a_1 \pmod{R^{00}}, \qquad b_2 \equiv b_1 \pmod{R^{00}}.$$

If $a_2 = 0$, then $a_1 \in R^{00}$, a contradiction. So $a_2 \in S \setminus \{0\}$. Similarly, $b_2 \in S \setminus \{0\}$ because $b_1 \notin R^{00}$.

Note that $a_1 - a_2 \in R^{00}$ but $a_1 \notin R^{00}$. As R^{00} is an ideal in $\mathcal{O}$ (so given by a cut), we have $v(a_1 - a_2) > v(a_1)$ and so $\mathrm{rv}(a_1) = \mathrm{rv}(a_2)$. Similarly, $v(b_1 - b_2) > v(b_1)$ and so also $\mathrm{rv}(b_1) = \mathrm{rv}(b_2)$. Consequently, since $c \in \mathcal{O}^\times$,

$$\mathrm{rv}(c) = \mathrm{rv}(a_1/b_1) = \mathrm{rv}(a_1)/\mathrm{rv}(b_1) = \mathrm{rv}(a_2)/\mathrm{rv}(b_2) = \mathrm{rv}(a_2/b_2).$$

In particular, $v(a_2/b_2) = v(c) = 0$, so $v(a_2) = v(b_2)$. By definition of B,

$$\mathrm{res}(c) = \mathrm{rv}(c) = \mathrm{rv}(a_2/b_2) = \mathrm{res}(a_2/b_2) \in B,$$

contradicting the choice of c. □

Corollary 6.11. *Let R be a rogue domain with fraction field K and let $\mathcal{O}'$ be a definable valuation ring on K. If $a \in K$ and $aR/(aR \cap R)$ is finite, then $a \in \mathcal{O}'$.*

Proof. We may replace R with a sufficiently saturated elementary extension. By Lemma 4.10, $a \in \mathcal{O} = R^{00} : R^{00}$ and so we conclude by Lemma 6.10. □

Lemma 6.12. *Let R be a sufficiently saturated rogue domain with fraction field K, and let $\mathcal{O} = R^{00} : R^{00}$. Then $\mathcal{O}$ is the finest externally definable valuation ring on K.*

Proof. The ring $\mathcal{O}$ is externally definable by Corollary 4.12. Fix an externally definable valuation ring $\mathcal{O}' \subseteq K$; it suffices to show that $\mathcal{O} \subseteq \mathcal{O}'$. By Lemma 4.10, we must show that if $a \in K$ and $aR/(aR \cap R)$ is finite, then $a \in \mathcal{O}'$. This holds by Corollary 6.11 applied to the Shelah expansion R^{Sh} of R, in which $\mathcal{O}'$ becomes a definable valuation ring (but saturation is lost). The Shelah expansion is still a rogue domain (Remark 6.9). □

Corollary 6.13. *Let R be a sufficiently saturated rogue domain with fraction field $K = \mathrm{Frac}(R)$, let $\mathcal{O}$ be the externally definable valuation ring $\mathcal{O} = R^{00} : R^{00} \subseteq K$, and let $\boldsymbol{k}_0$ be the residue field of $\mathcal{O}$.*

(1) *There are no externally definable nontrivial valuation rings on $\boldsymbol{k}_0$.*

(2) *If $A \subseteq \boldsymbol{k}_0$ is an externally definable additive subgroup, then A is finite or $A = \boldsymbol{k}_0$.*

Proof. The first point is a direct consequence of Lemma 6.12: if there were a nontrivial externally definable valuation on $\boldsymbol{k}_0$ then we could compose it with $\mathcal{O}$, getting a new externally definable valuation $\mathcal{O}' \subsetneq \mathcal{O}$, contradicting the lemma. The second statement is now a direct consequence of Proposition 6.6, noting that saturation was not assumed there. □

To a first approximation, the following lemma says that rogue domains cannot arise from Proposition 3.2.

Lemma 6.14. *Let R be a rogue domain. Let $\mathcal{O}'$ be an externally definable valuation ring on $K = \mathrm{Frac}(R)$ and let I be an ideal in $\mathcal{O}'$. If $I \subseteq R$, then the group R/I is infinite.*

Proof. Note that I is externally definable. Passing to the Shelah expansion, we may assume that $\mathcal{O}'$ and I are definable. Passing to an elementary extension, we may assume the structure is sufficiently saturated. Both changes are acceptable by Remark 6.9. Because K is ACVF-like, the residue field of $\mathcal{O}'$ must be infinite. Then every definable ideal is 00-connected by Fact 4.5(2), so $I = I^{00}$. If R/I is finite, then $R^{00} = I^{00} = I$, and R^{00} is definable, contradicting the fact that R is a rogue domain. □

Suppose the rogue domain R is sufficiently saturated. As R^{00} is an ideal in $\mathcal{O}$, it is defined by a cut Ξ in the value group:

$$R^{00} = \{x \in K : v(x) > \Xi\}.$$

The cut Ξ cannot be the cut $+\infty$ or $-\infty$, since R^{00} is neither 0 nor K.

Lemma 6.15. (1) *The value group Γ is densely ordered (not necessarily divisible).*

(2) *Ξ is not the cut γ^+ infinitesimally above some $\gamma \in \Gamma$. Equivalently, the set $\{\gamma \in \Gamma : \gamma < \Xi\}$ has no maximum.*

(3) *There is an increasing sequence $\gamma_0 < \gamma_1 < \gamma_2 < \cdots$ in Γ such that the following things hold*:

$$R/\{x \in R : v(x) \geq \gamma_i\} \text{ is finite for each } i,$$
$$\lim_{i\to\infty} \left|R/\{x \in R : v(x) \geq \gamma_i\}\right| = \infty.$$

(4) *If $\gamma_0 < \gamma_1 < \gamma_2 < \cdots$ is a sequence as in the previous point, then Ξ is the limit of the sequence, in the sense that*

$$\{x \in \Gamma : x > \Xi\} = \bigcap_{i=0}^{\infty} \{x \in \Gamma : x > \gamma_i\}.$$

Proof. Each closed ball $B_{\geq\gamma}(0) = \{x \in K : v(x) \geq \gamma\}$ is $\vee$-definable, because it is a scaled copy of the $\vee$-definable ring $\mathcal{O}$. Similarly, each open ball $B_{>\gamma}(0)$ is type-definable, being a scaled copy of the type-definable maximal ideal $\mathfrak{m} \lhd \mathcal{O}$.

If Γ is not densely ordered, then it is discretely ordered, and every closed ball is an open ball (of a different radius). In particular, $\mathcal{O}$ is type-definable, and therefore definable, contradicting Lemma 6.10, which says that $\mathcal{O}$ is strictly finer than any definable valuation ring. Therefore Γ is densely ordered, proving (1).

Claim 6.16. *If $\gamma > \Xi$, then $R/(R \cap B_{\geq\gamma}(0))$ is infinite, while if $\gamma < \Xi$, then $R/(R \cap B_{\geq\gamma}(0))$ is finite and $R \cap B_{\geq\gamma}(0)$ is definable.*

Proof. If $\gamma > \Xi$, then $B_{\geq\gamma}(0) \subseteq R^{00}$ because R^{00} is the "ball of radius Ξ", so to speak. Therefore $R \cap B_{\geq\gamma}(0) \subseteq R \cap R^{00} = R^{00}$, and

$$[R : R \cap B_{\geq\gamma}(0)] \geq [R : R^{00}].$$

Since R^{00} is not definable, $[R : R^{00}]$ is not finite.

Conversely, if $\gamma < \Xi$, then $B_{\geq\gamma}(0) \supseteq R^{00}$. The intersection $R \cap B_{\geq\gamma}(0)$ is $\vee$-definable and

$$[R : R \cap B_{\geq\gamma}(0)] \leq [R : R^{00}],$$

so $R \cap B_{\geq\gamma}(0)$ has small index in R. In general, $\vee$-definable subgroups of small index have finite index and are definable, by a compactness argument (analogous to the fact that open subgroups of a compact Hausdorff group are clopen of finite index). □

Now we can prove (2). Suppose for the sake of contradiction that $\Xi = \gamma^+$ for some $\gamma \in \Gamma$, so that

$$R^{00} = \{x \in K : x > \gamma\} = B_{>\gamma}(0).$$

Note that

$$R \supseteq R \cap B_{\geq\gamma}(0) \supseteq R \cap B_{>\gamma}(0) = B_{>\gamma}(0) = R^{00}.$$

The element γ is less than the cut γ^+, so $R/(R \cap B_{\geq\gamma}(0))$ is finite by the claim. On the other hand, $R/B_{>\gamma}(0) = R/R^{00}$ is infinite because R is a rogue domain. Therefore, $(R \cap B_{\geq\gamma}(0))/B_{>\gamma}(0)$ is infinite. It is an externally definable subgroup of $B_{\geq\gamma}(0)/B_{>\gamma}(0)$. Since the latter is (externally) definably isomorphic to the residue field of $\mathcal{O}$, by Corollary 6.13, we must have

$$\begin{aligned}(R \cap B_{\geq\gamma}(0))/B_{>\gamma}(0) &= B_{\geq\gamma}(0)/B_{>\gamma}(0),\\ R \cap B_{\geq\gamma}(0) &= B_{\geq\gamma}(0),\\ R &\supseteq B_{\geq\gamma}(0).\end{aligned}$$

As $B_{\geq\gamma}(0) \supseteq R^{00}$, $R/B_{\geq\gamma}(0)$ has finite index and $B_{\geq\gamma}(0)$ is an ideal in an externally definable valuation ring, contradicting Lemma 6.14. This proves (2). In particular, the set $\{\gamma \in \Gamma : \gamma < \Xi\}$ has no maximum.

For $\gamma \in \Gamma$, let

$$f(\gamma) = [R : R \cap B_{\geq\gamma}(0)].$$

Clearly, f is weakly increasing:

$$\gamma \leq \gamma' \implies R \cap B_{\geq\gamma}(0) \supseteq R \cap B_{\geq\gamma'}(0) \implies f(\gamma) \leq f(\gamma').$$

Moreover, $f(\gamma)$ is finite if and only if $\gamma < \Xi$, by Claim 6.16. To prove (3) and (4), it suffices to show that $f(\gamma)$ is unbounded as γ approaches Ξ from below.

Suppose not. Then there is some $\gamma_0 < \Xi$ and some n such that

$$\gamma_0 < \gamma < \Xi \implies f(\gamma) = n.$$

It follows that $R \cap B_{\geq\gamma}(0)$ must be some fixed subgroup G of R of index n, independent of $\gamma < \Xi$. Moreover, G is definable by Claim 6.16.

On the other hand, the fact that there is no maximum element below Ξ means that

$$\begin{aligned} R^{00} &= \{x \in K : v(x) > \Xi\} \\ &= \{x \in K : v(x) \geq \gamma \text{ for every } \gamma \in \Gamma \text{ with } \gamma < \Xi\} \\ &= \bigcap_{\gamma < \Xi} B_{\geq \gamma}(0) \end{aligned}$$

and so

$$R^{00} = R \cap R^{00} = \bigcap_{\gamma < \Xi} (R \cap B_{\geq \gamma}(0)) = G.$$

Thus R^{00} is definable, contradicting the fact that R is rogue. □

Lemma 6.17. *Let R be a rogue domain. Let $\mathcal{O}'$ be an externally definable valuation ring and let v' be the corresponding valuation. There cannot exist an ascending sequence $\gamma'_0 < \gamma'_1 < \gamma'_2 < \cdots$ in the value group of v' such that*

$$R/\{x \in R : v'(x) \geq \gamma'_i\} \text{ is finite for each } i,$$
$$\lim_{i \to \infty} \left| R/\{x \in R : v'(x) \geq \gamma'_i\} \right| = \infty.$$

Proof. Replacing R with its Shelah expansion, we reduce to the case where $\mathcal{O}'$ is definable. Replacing R with an elementary extension, we may assume that R is sufficiently saturated. Both changes preserve the fact that R is rogue, by Remark 6.9. Furthermore, the existence of the ascending sequence is also preserved in the elementary extension.

Now let $\mathcal{O}$ be the usual valuation ring $R^{00} : R^{00}$, and let $v : K \to \Gamma$ be its corresponding valuation. By Lemma 6.10, $\mathcal{O} \subsetneq \mathcal{O}'$, meaning that v' is a strict coarsening of v. In other words, v' is the composition

$$K \xrightarrow{v} \Gamma \to \Gamma/\Delta$$

for some nontrivial convex subgroup $\Delta \subseteq \Gamma$.

Let γ_i be an element in Γ lifting $\gamma'_i \in \Gamma/\Delta$. Note that

$$\{x \in R : v'(x) \geq \gamma'_{i+1}\} \subseteq \{x \in R : v(x) \geq \gamma_i\} \subseteq \{x \in R : v'(x) \geq \gamma'_i\}$$

and so

$$R/\{x \in R : v(x) \geq \gamma_i\} \text{ is finite for each } i,$$
$$\lim_{i \to \infty} \left| R/\{x \in R : v(x) \geq \gamma_i\} \right| = \infty.$$

By Lemma 6.15, the cut Ξ defining R^{00} is the limit of the γ_i:

$$x \in R^{00} \iff v(x) > \Xi \iff (v(x) > \gamma_i \text{ for every } i) \iff (v(x) \geq \gamma_i \text{ for every } i).$$

For fixed i, we have

$$v'(x) > \gamma'_i \implies v(x) > \gamma_i \implies v(x) \geq \gamma_i \implies v'(x) \geq \gamma'_i.$$

Therefore, the following four conditions are sequentially weaker:

(1) $v'(x) > \gamma_i'$ for every i.

(2) $v(x) > \gamma_i$ for every i.

(3) $v(x) \geq \gamma_i$ for every i.

(4) $v'(x) \geq \gamma_i'$ for every i.

But the first and fourth are equivalent because the sequence $\gamma_0', \gamma_1', \ldots$ is strictly increasing. Thus all four conditions are equivalent to each other, and also to the condition $x \in R^{00}$. In summary,

$$x \in R^{00} \iff (v'(x) > \gamma_i' \text{ for all } i).$$

It follows that $\mathcal{O}' \cdot R^{00} \subseteq R^{00}$, or equivalently that $\mathcal{O}' \subseteq R^{00} : R^{00} = \mathcal{O}$, contradicting Lemma 6.10. □

Of course we can apply Lemma 6.17 to $\mathcal{O}' = \mathcal{O}$ itself, and the conclusion directly contradicts parts (3) and (4) of Lemma 6.15. Therefore, *rogue domains do not exist*, and we have proven the following:

Proposition 6.18. *Let R be a sufficiently saturated exceptional dp-minimal domain such that* Frac(R) *is ACVF-like. Then R^{00} is definable and has finite index in R. As a consequence, the valuation ring $\mathcal{O} = R^{00} : R^{00}$ is definable as well.*

With Propositions 6.2 and 6.7, we can now prove the main theorem of the paper.

Theorem 6.19. *Let R be a dp-minimal integral domain. If R is not a valuation ring, then there exists a valuation subring $\mathcal{O}$ of $K =$* Frac(R)*, a proper ideal $I \trianglelefteq \mathcal{O}$, and a finite subring of R_0 of $\mathcal{O}/I$ such that R is the preimage of R_0 under the quotient map $\mathcal{O} \to \mathcal{O}/I$. The data $(\mathcal{O}, I, R_0)$ can be chosen to be definable in R.*

Proof. Let R be a dp-minimal integral domain which is not a valuation ring, i.e., it is exceptional. By Proposition 6.2, Frac(R) is either ACVF-like or pCF-like. If it is the latter, we conclude by Proposition 6.7, so assume that it is ACVF-like. There is no harm in assuming that R is sufficiently saturated. By Proposition 6.18, $I = R^{00}$ is a definable ideal in the dp-minimal definable valuation ring $\mathcal{O} = R^{00} : R^{00}$, and R^{00} has finite index in R. The desired conclusion follows easily: take

$$R_0 = R/R^{00} \subseteq \mathcal{O}/R^{00}.$$ □

Incidentally, we can now strengthen Proposition 6.6:

Proposition 6.20. *Let $(K, +, \cdot, A)$ be a dp-minimal expansion of a field K by an infinite, proper additive subgroup $(A, +) \subseteq (K, +)$. Then there is a nontrivial definable valuation ring on K.*

Proof. There is no harm in assuming sufficient saturation. By Proposition 4.22, the ring $R = A : A$ is an infinite definable proper subring of K with $\mathrm{Frac}(R) = K$. If R is a valuation ring, we are done. Otherwise, R is exceptional. If K is pCF-like, then the canonical valuation is a nontrivial definable valuation. If K is ACVF-like, Proposition 6.18 gives a nontrivial valuation ring $\mathcal{O} = R^{00} : R^{00}$. □

7. Remarks on dp-minimal commutative rings

Having classified dp-minimal integral domains, it is natural to ask what can be said about more general dp-minimal commutative rings. As a first step, we prove the following:

Proposition 7.1. (1) *Every dp-minimal* (*commutative*) *ring has the form* $R \times S$, *where* R *is a dp-minimal henselian local ring and* S *is a finite ring.*

(2) *If* R *is a dp-minimal local ring, then the prime ideals of* R *are linearly ordered. Consequently, there is a unique minimal prime ideal.*

(3) *If* R *is a dp-minimal local ring, then every prime ideal containing the zero divisors is comparable to any principal ideal.*

Proof. (1) If R is a dp-finite commutative ring, then R decomposes as a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a henselian local ring [Johnson 2023b, Theorem 1.3]. The decomposition is definable, and so

$$\text{dp-rk}(R) = \sum_{i=1}^{n} \text{dp-rk}(R_i).$$

In the case where R is dp-minimal, it follows that one of the R_i is a dp-minimal henselian local ring and the rest of the factors are finite.

(2) Otherwise, there are two nonmaximal prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ which are incomparable. Then $(\mathfrak{p} + \mathfrak{q})/\mathfrak{p}$ is a nonzero ideal in the integral domain $R/\mathfrak{p}$. Since $\mathfrak{p}$ is nonmaximal, $R/\mathfrak{p}$ is not a field, so $R/\mathfrak{p}$ is infinite and every nonzero ideal is infinite. Thus $(\mathfrak{p} + \mathfrak{q})/\mathfrak{p}$ is infinite. Similarly, $(\mathfrak{p} + \mathfrak{q})/\mathfrak{q}$ is infinite. Equivalently, $\mathfrak{p}/(\mathfrak{p} \cap \mathfrak{q})$ and $\mathfrak{q}/(\mathfrak{p} \cap \mathfrak{q})$ are infinite. As $\mathfrak{p}$, $\mathfrak{q}$ are externally definable, this contradicts dp-minimality (Fact 4.13).

(3) Let $\mathfrak{p}$ be a prime ideal of R. Recall that there is a ring homomorphism $\varphi : R \to R_{\mathfrak{p}}$ with

$$\ker \varphi = \{c \in R \mid \exists s \notin \mathfrak{p}\ cs = 0\}.$$

We denote by R^{φ}, $\mathfrak{p}^{\varphi}$, a^{φ}, etc. the images in $R_{\mathfrak{p}}$. Note that $\mathfrak{p}^{\varphi}$ is a prime ideal of R^{φ} and $\mathfrak{m}^{\varphi}$ is the maximal ideal of R^{φ}.

Claim 7.2. $\mathfrak{p}^{\varphi} = \mathfrak{p}^{\varphi} R_{\mathfrak{p}}$.

Proof. It is easy to check that $R^\varphi \cap \mathfrak{p}^\varphi R_\mathfrak{p} = \mathfrak{p}^\varphi$, so it is enough to prove that $R^\varphi = R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$. We may assume that $\mathfrak{p}^\varphi$ is nonmaximal in R^φ. By Facts 4.1 and 4.2, $(R_\mathfrak{p}, R^\varphi, \mathfrak{p}^\varphi, \ldots)$ is dp-minimal, and hence by Fact 4.13, one of $R^\varphi/\mathfrak{p}^\varphi$ or $\mathfrak{p}^\varphi R_\mathfrak{p}/\mathfrak{p}^\varphi$ has finite index as an abelian group. As $\mathfrak{p}^\varphi$ is nonmaximal, $R/\mathfrak{p}^\varphi$ is infinite, so $\mathfrak{p}^\varphi R_\mathfrak{p}/\mathfrak{p}^\varphi$ is finite. In particular, extending a set of representatives of $\mathfrak{p}^\varphi R_\mathfrak{p}/\mathfrak{p}^\varphi$ by the element 1 yields that $R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$ is a finitely generated R^φ-module. Let $\mathfrak{m}^\varphi$ be the maximal ideal of R^φ. Then one easily checks that $R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$. Since $R^\varphi/\mathfrak{m}^\varphi$ surjects onto $(R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})/(\mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})$, $1+\mathfrak{m}^\varphi+\mathfrak{p}^\varphi R_\mathfrak{p}$ generates $(R^\varphi+\mathfrak{p}^\varphi R_\mathfrak{p})/(\mathfrak{m}^\varphi+\mathfrak{p}^\varphi R_\mathfrak{p})$ as an R^φ-module. Restricting the module action from R^φ to the Jacobson ideal $\mathfrak{m}^\varphi$ of R^φ, we have $\mathfrak{m}^\varphi(R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}) = \mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$. By Nakayama's lemma, the generator $1+\mathfrak{m}^\varphi+\mathfrak{p}^\varphi R_\mathfrak{p}$ of the R^φ-module $(R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})/(\mathfrak{m}^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p})$ lifts to a generator of $R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$ as an R^φ-module, so $R^\varphi = R^\varphi + \mathfrak{p}^\varphi R_\mathfrak{p}$. □

To conclude, assume that $\mathfrak{p}$ contains all the zero divisors of R. Then the map $\varphi : R \to R_\mathfrak{p}$ is injective. Let $a \in R$. If $a \in \mathfrak{p}$ then we are done, otherwise, $a \notin \mathfrak{p}$ and hence $(1/a^\varphi)\mathfrak{p}^\varphi \subseteq \mathfrak{p}^\varphi R_\mathfrak{p} = \mathfrak{p}^\varphi$, so that if $b \in \mathfrak{p}$ there exists $c \in \mathfrak{p}$ such that $b^\varphi = a^\varphi c^\varphi$. As φ is injective, $b = ac$, so we conclude that $\mathfrak{p} \subseteq aR$. □

The comparability of prime ideals fails for rings of dp-rank 2: consider the fiber product $\mathcal{O} \times_k \mathcal{O}$, where $(K, \mathcal{O}) \models \mathrm{ACVF}$ and k is the residue field.

If R is a henselian local dp-minimal ring and $\mathfrak{p}$ is the unique minimal prime ideal, then $\mathfrak{p}$ is the nilradical $\sqrt{0}$, so every element of $\mathfrak{p}$ is nilpotent. The quotient $R/\mathfrak{p}$ is a dp-minimal integral domain, whose structure we understand by the main theorems of this paper.

Appendix: proof of Fact 2.2

Recall Fact 2.2:

Fact 2.2. *An infinite field $(K, +, \cdot)$ is dp-minimal if and only if there is a henselian defectless valuation ring $\mathcal{O} \subseteq K$ with maximal ideal $\mathfrak{m}$ such that*:

(1) *The value group $\Gamma := K^\times/\mathcal{O}^\times$ is dp-minimal as an ordered abelian group (possibly trivial).*

(2) *The residue field $k := \mathcal{O}/\mathfrak{m}$ is algebraically closed, real closed, or p-adically closed for some prime p.*

(3) *If the residue field k is algebraically closed of characteristic $p > 0$, then the interval $[-v(p), v(p)] \subseteq \Gamma$ is p-divisible, where $v(p) = +\infty$ when $\mathrm{char}(K) = p$.*

Proof. The "if" direction holds by the classification of dp-minimal valuation rings (Fact 2.3 above, originally from [Johnson 2023a, Theorems 1.5 and 1.6]), together with the well-known fact that ACF, RCF, and pCF are dp-minimal. It remains to prove the "only if" direction. Suppose K is dp-minimal.

Claim A.1. *There is a henselian valuation ring $\mathcal{O}$ on K whose residue field is p-adically closed, real closed, or algebraically closed.*

Proof. Let $\mathcal{O}_\infty$ be the intersection of all 0-definable valuation rings on K. By [Johnson 2023a, Theorem 1.2], $\mathcal{O}_\infty$ is a henselian valuation ring whose residue field k_∞ is finite, real closed, or algebraically closed. In the latter two cases, we can simply take $\mathcal{O} = \mathcal{O}_\infty$. Suppose we are in the first case: k_∞ is finite. Since $\mathcal{O}_\infty$ is henselian, the expansion $(K, \mathcal{O}_\infty)$ is dp-minimal by [Halevi and Hasson 2019, Proposition 5.14]. By the classification of dp-minimal valuation rings (Fact 2.3), $\mathcal{O}_\infty$ is finitely ramified. Let $v_\infty : K^\times \to \Gamma_\infty$ be the valuation induced by $\mathcal{O}_\infty$. Let Δ be the convex hull of $\mathbb{Z} \cdot v_\infty(p)$ in Γ_∞. Note that $\Delta \cong \mathbb{Z}$ by finite ramification. Coarsening by the convex subgroup Δ, we get a factorization of the place $K \to k_\infty$ into a composition of two places

$$K \xrightarrow{\Gamma_\infty/\Delta} k \xrightarrow{\Delta} k_\infty,$$

where the labels on the arrows show the value groups. The fact that $K \to k_\infty$ is henselian implies that $K \to k$ and $k \to k_\infty$ are henselian. The fact that $v_\infty(p) \in \Delta$ implies that $K \to k$ is equicharacteristic 0 and $k \to k_\infty$ is mixed characteristic. Then $k \to k_\infty$ is a finitely ramified henselian valuation with value group $\Delta \cong \mathbb{Z}$ and finite residue field, so k is p-adically closed. Take $\mathcal{O}$ to be the valuation ring associated to $K \to k$. □

If $\mathcal{O}$ is the henselian valuation ring from the claim, then $\mathcal{O}$ satisfies condition (2) of Fact 2.2. Moreover, $(K, \mathcal{O})$ is a dp-minimal valued field by [Halevi and Hasson 2019, Proposition 5.14]. By the classification of dp-minimal valued fields (Fact 2.3), the valuation is defectless and satisfies conditions (1) and (3) of Fact 2.2. □

Acknowledgments

D'Elbée is fully supported by the UKRI Horizon Europe Guarantee Scheme, grant no. EP/Y027833/1. Halevi was supported by ISF grant no. 290/19. Johnson was supported by the National Natural Science Foundation of China (grant no. 12101131) and the Ministry of Education of China (grant no. 22JJD110002).

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Received 28 Jun 2024. Revised 5 Dec 2024.

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Model Theory
Vol. 4, No. 2, 2025
https://doi.org/10.2140/mt.2025.4.163

Soluble Lie rings of finite Morley rank

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As a corollary to a stronger linearisation result (Theorem A), we prove the finite Morley rank version of the Lie–Kolchin–Malcev theorem on Lie algebras (Corollary A2). We also classify Lie ring actions on modules of characteristic $\neq 2, 3$ and Morley rank 2 (Theorem B).

1. Introduction

The present paper, on Lie rings, should not be read as a by-product of the theory of groups of finite Morley rank, for the following reasons (more details in [Deloro and Tindzogho Ntsiri 2024]).

• The topics are disjoint. We do not believe in a Lie–Chevalley "exponential" correspondence [Deloro and Tindzogho Ntsiri 2024, Question 14] at this level of generality (and we would not recommend telling experts in modular Lie algebras that they just do group theory in disguise).

• Lie rings have more simple configurations. Should the Cherlin–Zilber conjecture on groups and our driving question [Deloro and Tindzogho Ntsiri 2024, "log CZ", p. 247] of similar inspiration both be true, then the latter would handle *more* objects than the former. Indeed, positive characteristic collapses of infinite-dimensional, geometric algebras known as "Lie–Cartan" would emerge, and have no counterparts in algebraic group theory.

• The setting remains open-ended [Deloro and Tindzogho Ntsiri 2024, Question 13]. It is not clear yet whether finiteness of the Morley rank will prove the right context. In the case of groups, it is already quite hard. But for the moment, things go smoothly with Lie rings, so finite-dimensionality is a decent candidate for the future. Our exploratory work stays in finite Morley rank mainly to accelerate the development of the theory; if successful, generalisations will follow spontaneously.

We thus contend that despite group-theoretic experience, model-theoretic algebra of Lie rings is, if not new, young. Zilber [1982] had understood that the characteristic 0 case reduces to Lie algebra. Nesin [1989a] saw Lie rings as part of a general

MSC2020: primary 03C60, 17B50; secondary 20F11.
Keywords: Lie rings, Morley rank, model-theoretic algebra.

study of nonassociative rings. Then [Rosengarten 1991] studied with success Lie rings of Morley rank 1, 2, 3. Our [Deloro and Tindzogho Ntsiri 2024] proved that in characteristic $\neq 2, 3$, there are no simple Lie rings of Morley rank 4.

In the present paper we solve two basic problems, which should become pervasive tools in model-theoretic algebra of Lie rings.

(1) The analogue of Lie's theorem that the commutator algebra of a complex, finite-dimensional, soluble Lie algebra is nilpotent. This first step towards [Deloro and Tindzogho Ntsiri 2024, Question 10] is taken in Section 2.

(2) Classification of 2-dimensional modules for abstract Lie rings of finite Morley rank. This solution to [Deloro and Tindzogho Ntsiri 2024, Question 8] is given in Section 3.

1.1. ***Present results*** **(*for quick reference*).** There is terminology and notation in Section 1.2.

Theorem A (Section 2.1). *Let $\mathfrak{g}$ be a Lie ring of finite Morley rank (not necessarily connected). Let V be a definable, irreducible, faithful $\mathfrak{g}$-module. Suppose that* $\operatorname{char} V > \dim V$ *and that $\mathfrak{g}$ has an infinite abelian ideal $\mathfrak{a} \trianglelefteq \mathfrak{g}$. Then $\mathfrak{a} \sqsubseteq Z(\mathfrak{g})$ and the configuration is definably linear: there is an infinite field $\mathbb{K}$ such that $V \in \mathbb{K}$-**Vect**, $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V : \mathbb{K}\text{-}\mathbf{Vect})$, and $\mathfrak{a} \hookrightarrow \mathbb{K}\operatorname{Id}_V$, all definably.*

Corollary A1 (Section 2.2). *Let $\mathfrak{r}$ be a connected, soluble Lie ring of finite Morley rank. Let V be a definable, irreducible, faithful $\mathfrak{r}$-module. Suppose* $\operatorname{char} V > \dim V$. *Then $\mathfrak{r}$ is abelian and the configuration is definably linear: there is an infinite field $\mathbb{K}$ such that $V \simeq \mathbb{K}_+$ and $\mathfrak{r} \hookrightarrow \mathbb{K}\operatorname{Id}_V$, all definably.*

Corollary A2 (Section 2.2). *Let $\mathfrak{r}$ be a connected, soluble Lie ring of finite Morley rank. Suppose $(\mathfrak{r}; +)$ has prime exponent $> \dim \mathfrak{r}$.*

(i) *There is a largest definable, connected, nilpotent ideal $F^\circ(\mathfrak{r}) \trianglelefteq \mathfrak{r}$.*

(ii) *$\mathfrak{r}'$ is nilpotent, viz., $\mathfrak{r}' \sqsubseteq F^\circ(\mathfrak{r})$.*

(iii) *If $\mathfrak{u} \sqsubseteq \mathfrak{r}$ is a definable, connected subring consisting of $\mathfrak{r}$-nilpotent elements (viz., $(\forall a \in \mathfrak{u})(\exists n \in \mathbb{N})(\operatorname{ad}_a^n(\mathfrak{r}) = 0)$), then $\mathfrak{u} \sqsubseteq F^\circ(\mathfrak{r})$.*

Theorem B (Section 3). *Let $\mathfrak{g}$ be a connected Lie ring of finite Morley rank acting definably and faithfully on an irreducible module V of characteristic $\neq 2, 3$. Suppose* $\dim V = 2$. *Then there is a definable field $\mathbb{K}$ such that, all definably,*

- *either $V \simeq \mathbb{K}_+$ and $\mathfrak{g} \hookrightarrow \mathbb{K}\operatorname{Id}_V$;*
- *or $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{K})$ in its natural action on $V \simeq \mathbb{K}^2$;*
- *or $\mathfrak{g} \simeq \mathfrak{gl}_2(\mathbb{K})$ in its natural action on $V \simeq \mathbb{K}^2$.*

1.2. *Terminology and notation.*

- Inside Lie rings, we use $\leq$ for subgroups, $\sqsubseteq$ for subrings, and $\trianglelefteq$ for ideals.
- For V an abelian group, $(\mathrm{End}(V); +, \cdot)$ is an associative ring. When there is a notion of definability around, we let $\mathrm{DefEnd}(V)$ be the subring of *definable* endomorphisms; $\mathrm{DefEnd}(V)$ itself need not be definable.
- Both $\mathrm{End}(V)$ and $\mathrm{DefEnd}(V)$ bear a Lie ring structure by letting

$$[\![f, g]\!] = f \circ g - g \circ f.$$

- A Lie ring $\mathfrak{g}$ *acts* on V if there is a morphism

$$\rho : (\mathfrak{g}; +, [\cdot, \cdot]) \to (\mathrm{End}(V); +, [\![\cdot, \cdot]\!]).$$

When there is a notion of definability around, the action is *definable* if $(\mathfrak{g}; +, [\cdot, \cdot])$, $(V; +)$, and the map $\mathfrak{g} \times V \to V$ taking (g, v) to $\rho(g)(v)$ are definable. We omit ρ from notation.

- A Lie ring acting definably on V generates an *invariant* associative subring of $\mathrm{DefEnd}(V)$. Here, *invariance* is a model-theoretic notion generalising type-definability. For our purpose, $\bigvee$-definability is a decent proxy.
- In case V is definable and connected we call V a $\mathfrak{g}$-*module*; we avoid the phrase otherwise. In particular, "V is an irreducible $\mathfrak{g}$-module" means that V has no definable, connected, nontrivial, proper, $\mathfrak{g}$-invariant subgroup.
- Let V be a definable, irreducible $\mathfrak{g}$-module in a theory of finite Morley rank. Then either
  - $(V; +)$ has prime exponent $p > 0$, called the *characteristic* of V;
  - or $(V; +)$ is divisible, in which case we say V has *characteristic* 0.

This follows from Macintyre's theorem on abelian groups of ordinal Morley rank. (By rigidity phenomena, there can be no nontrivial definable module structure on a divisible torsion group; also, by [Deloro and Tindzogho Ntsiri 2024, Theorem 0], we have no remaining interest in characteristic 0.)

- $\mathbb{K}$-**Vect** is the category of $\mathbb{K}$-vector spaces.
- The general linearisation result [Deloro 2024] has a consequence on modules of Morley rank 1 [Deloro and Tindzogho Ntsiri 2024, Corollary 1]: they linearise without solubility.
- Morley rank is denoted by dim. It has been noted that a weaker assumption than finiteness of the Morley rank might suffice for model-theoretic algebra on Lie rings [Deloro and Tindzogho Ntsiri 2024, Question 13]. It would be interesting (though

possibly very challenging) to see what happens if one allows definable field derivations, as opposed to the finite Morley rank context [Deloro and Tindzogho Ntsiri 2024, Lemma G].

2. Linearisation, and Lie's theorem

Our ad hoc [Deloro and Tindzogho Ntsiri 2024, Corollary 2] was a disappointingly special case of a desirable result, viz., "Lie's theorem" for abstract Lie rings of finite Morley rank. Its group analogue, sometimes called the "Lie–Kolchin–Malcev theorem", states that the commutator subgroup of a connected, soluble group of finite Morley rank is nilpotent. This was first proved for groups of finite Morley rank in [Nesin 1989b]. We obtain the Lie ring version (Corollary A2 below) as a consequence of Theorem A, a linearisation result similar to [Poizat 2001, Theorem 3.9]. Poizat had extracted the latter from [Nesin 1989b].

Though new, Corollary A2 is not unexpected. It is "presumed" without a proof as [Nesin 1989a, Theorem 18]. However Nesin did not make the necessary assumptions on the characteristic (a personal conversation confirmed he did not know about low-characteristic counterexamples). Also, in our context, lack of conjugacy prevents us from retrieving a global linear structure; this sharply contrasts with [Nesin 1989b; Poizat 2001, Theorem 3.9]. Thus Theorem A is not as straightforward as it looks.

2.1. *Linearisation theorem.*

Theorem A. *Let $\mathfrak{g}$ be a Lie ring of finite Morley rank (not necessarily connected). Let V be a definable, irreducible, faithful $\mathfrak{g}$-module. Suppose that* $\operatorname{char} V > \dim V$ *and that $\mathfrak{g}$ has an infinite abelian ideal $\mathfrak{a} \trianglelefteq \mathfrak{g}$. Then $\mathfrak{a} \sqsubseteq Z(\mathfrak{g})$ and the configuration is definably linear: there is an infinite field $\mathbb{K}$ such that $V \in \mathbb{K}$-**Vect**, $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V : \mathbb{K}\text{-}\mathbf{Vect})$, and $\mathfrak{a} \hookrightarrow \mathbb{K}\,\mathrm{Id}_V$, all definably.*

Proof. Let $\hat{\mathfrak{a}} = Z(C_{\mathfrak{g}}(\mathfrak{a})) \sqsupseteq \mathfrak{a}$. Then $\hat{\mathfrak{a}} \trianglelefteq \mathfrak{g}$ is an infinite abelian ideal, and is definable. So we may assume that $\mathfrak{a}$ is definable. Both $\mathfrak{g}$ and $\mathfrak{a}$ generate invariant, unbounded, associative subrings of $\operatorname{DefEnd}(V)$. By [Deloro 2024, Corollary 1], it is enough to prove $\mathfrak{a} \sqsubseteq Z(\mathfrak{g})$. So suppose not. We seek a contradiction.

Notation. Let $W \leq V$ be $\mathfrak{a}$-irreducible.

Step 1. (i) *No nonzero ideal of $\mathfrak{g}$ may centralise W.*

(ii) *There is $h \in \mathfrak{g}$ such that $[h, \mathfrak{a}]$ does not centralise W.*

(iii) *W has no nontrivial, finite $\mathfrak{a}$-invariant subgroups.*

Proof. (i) Let $0 < \mathfrak{b} \trianglelefteq \mathfrak{g}$ be any nonzero ideal. Then $C_V^\circ(\mathfrak{b})$ is a $\mathfrak{g}$-submodule of V. By faithfulness, it is proper, so by $\mathfrak{g}$-irreducibility, it is trivial. Hence $\mathfrak{b}$ may not centralise W.

(ii) Let $\mathfrak{b} = [\mathfrak{g}, \mathfrak{a}] \trianglelefteq \mathfrak{g}$. We supposed $\mathfrak{a} \not\subseteq Z(\mathfrak{g})$, so $\mathfrak{b} \neq 0$. By (i), $\mathfrak{b}$ acts nontrivially on W. So there is $h \in \mathfrak{g}$ such that $[h, \mathfrak{a}]$ does not centralise W.

(iii) Let $\mathfrak{b} = \mathfrak{a}^\circ \trianglelefteq \mathfrak{g}$. Since $\mathfrak{a}$ is infinite, $\mathfrak{b} \neq 0$. By (i), $\mathfrak{b}$ acts nontrivially on W, so $\bar{\mathfrak{a}} = \mathfrak{a}/C_{\mathfrak{a}}(W)$ is infinite. Let $\mathbb{K} = C_{\mathrm{DefEnd}(W)}(\bar{\mathfrak{a}})$. By abelianity, $\bar{\mathfrak{a}} \leq \mathbb{K}$. Linearising [Deloro 2024], $\mathbb{K}$ is an infinite definable field. So $\bar{\mathfrak{a}}$ acts by scalars, and hence freely, on W. As it is infinite, there can be no finite nontrivial $\mathfrak{a}$-invariant subgroup of W. ◇

An element h like in (ii) is now fixed. For $a \in \mathfrak{a}$, let $a^{[i]} = \mathrm{ad}_h^i(a)$. In particular, $a' = [h, a]$. In this notation, for $i \geq 0$ one has in $\mathrm{DefEnd}(V)$

$$h^i a = \sum_{j=0}^{i} \binom{i}{j} a^{[j]} h^{i-j}.$$

Notation. Set $S_0 = W$ and for $i \geq 1$, let $S_i = \sum_{j=0}^{i-1} h^j W$.

The sum giving S_i thus has i terms. (For the computations below, this notation is slightly preferable.)

Step 2. (iv) *Each S_i is an $\mathfrak{a}$-module.*

(v) *If $w \in W \setminus \{0\}$ and $i \geq 0$ are such that $h^i w \in S_i$, then $S_{i+1} = S_i$.*

(vi) *There is $k \leq \dim V$ such that $\sum_{n \geq 0} h^n W = S_k = \bigoplus_{i=0}^{k-1} h^i W > S_{k-1}$.*

Proof. (iv) Definability and connectedness are obvious; $\mathfrak{a}$-invariance is by induction. Indeed, let $i \geq 0$ be such that S_i is $\mathfrak{a}$-invariant. Let $a \in \mathfrak{a}$ and $w \in W$. One has

$$ah^i \cdot w = h^i a \cdot w - \sum_{j=1}^{i} \binom{i}{j} a^{[j]} h^{i-j} \cdot w \in S_{i+1},$$

which proves $\mathfrak{a}$-invariance of S_{i+1}.

(v) Let $W' = \{w \in W : h^i w \in S_i\} \neq 0$. For $w \in W'$ and $a \in \mathfrak{a}$ one has

$$h^i a \cdot w = \sum_{j=0}^{i} \binom{i}{j} a^{[j]} h^{i-j} \cdot w.$$

Each S_{i-j} is an $\mathfrak{a}$-module by (iv), so $h^i a \cdot w \in S_i$. Hence W' is a nonzero, definable, $\mathfrak{a}$-invariant subgroup of W. By (iii), it is infinite; so $\mathfrak{a}$-irreducibility of W implies $(W')^\circ = W$. Hence $h^i W \leq S_i$.

(vi) If the sum defining S_i is direct, then $i \leq \dim V$. So there is k maximal such that the sum giving S_k is direct. Then $k \leq \dim V$. Moreover, $S_k \cap h^k W \neq 0$, so there is $w \in W \setminus \{0\}$ with $h^k w \in S_k$. Then $S_{k+1} = S_k$ by (v), and $S_\infty = S_k$. ◇

Notation. • With k as in (vi), let $X = S_k/S_{k-1}$.

• Let $\mathbb{K} = C_{\mathrm{DefEnd}(X)}(\mathfrak{a})$.

Step 3. (vii) *Taking w to $(h^{k-1}\cdot w \bmod S_{k-1})$ defines an isomorphism $W \simeq X$ of $\mathfrak{a}$-modules.*

(viii) *$\mathbb{K}$ is an infinite, definable field.*

(ix) *h induces a natural morphism $\eta \in \mathrm{DefEnd}(X)$; for $a \in \mathfrak{a}$, one has $\eta a = a\eta + ka'$.*

Proof. (vii) Consider the additive morphism $W \to X$, $w \mapsto (h^{k-1}\cdot w \bmod S_{k-1})$. Since $S_k = h^{k-1}W + S_{k-1}$, it is onto. Its $\mathfrak{a}$-covariance results from

$$h^{k-1}a = \sum_{i=0}^{k-1} \binom{k-1}{i} a^{[i]} h^{k-1-i}.$$

If $w \in W \setminus \{0\}$ is such that $h^{k-1}w \in S_{k-1}$, then (v) implies $S_{k-1} = S_k$, against the definition of k. So the morphism is also injective.

(viii) By (vii), $W \simeq X$ as $\mathfrak{a}$-modules. By construction, X is therefore $\mathfrak{a}$-irreducible. The image $\mathfrak{a}/C_{\mathfrak{a}}(X)$ is infinite by (i). Linearising the abelian action, $\mathbb{K}$ is an infinite, definable field.

(ix) Let $x \in X$, say $x = (h^{k-1}w \bmod S_{k-1})$ with $w \in W$. Let $\eta(x) = (h^k w \bmod S_{k-1})$. This is well-defined. Indeed, if $h^{k-1}w \in S_{k-1}$, then (v) implies $w = 0$, so $h^k w = 0$ and $(h^k \bmod S_{k-1}) = 0$.

Keep the same notation, and let $a \in \mathfrak{a}$. Then $h^{k-1}aw \equiv ah^{k-1}w\ [S_{k-1}]$, so $h^{k-1}aw$ represents $a\cdot x$ and $\eta(a\cdot x) = (h^k aw \bmod S_{k-1})$. Now,

$$h^k aw \equiv ah^k w + ka'h^{k-1}w\ [S_{k-1}],$$

so $\eta(a\cdot x) = (ah^k w + ka'h^{k-1}w \bmod S_{k-1}) = a\cdot\eta(x) + ka'\cdot x$. This proves $\eta a = a\eta + ka'$. ◇

Step 4. (x) *For $\lambda \in \mathbb{K}$, let $\delta(\lambda) = [\![\eta, \lambda]\!]$. Then δ is a definable derivation of $\mathbb{K}$.*

(xi) *Contradiction.*

Proof. (x) The $[\![\cdot,\cdot]\!]$-bracket is computed in $\mathrm{DefEnd}(X)$ and therefore makes sense. It yields $\mathrm{ad}_\eta(\lambda)$, which results in a derivation on $\mathrm{DefEnd}(X)$. One must however check $\delta(\lambda) \in \mathbb{K}$, viz., that $\delta(\lambda)$ commutes with the action of $\mathfrak{a}$.

By (ix), one has, for $\lambda \in \mathbb{K} = C_{\mathrm{DefEnd}(X)}(\mathfrak{a})$,

$$\begin{aligned}(\eta\lambda - \lambda\eta)a &= (\eta a)\lambda - \lambda(\eta a)\\ &= a\eta\lambda + ka'\lambda - \lambda a\eta - \lambda ka'\\ &= a(\eta\lambda - \lambda\eta).\end{aligned}$$

Therefore $\delta(\lambda) \in \mathbb{K}$.

(xi) An infinite field of finite Morley rank has no nontrivial definable derivations by [Deloro and Tindzogho Ntsiri 2024, Lemma G]. So for $\lambda \in \mathbb{K}$ one has $\delta(\lambda) = 0$. In particular, for (the image in $\mathbb{K}$ of) $a \in \mathfrak{a}$, one has $[\![\eta, a]\!] = 0$. By (ix), ka' acts

trivially on X. By (vi) and by assumption, $k \le \dim V < \operatorname{char} V$. So a' acts trivially on X, but $X \simeq W$ [$\mathfrak{a}$-**Mod**] by (vii). So $[h, \mathfrak{a}]$ acts trivially on W, against (ii). $\diamond$

This contradiction completes the proof: $\mathfrak{a} \sqsubseteq Z(\mathfrak{g})$, so we may linearise. $\square$

2.2. *Corollaries.*

Corollary A1. *Let $\mathfrak{r}$ be a connected, soluble Lie ring of finite Morley rank. Let V be a definable, irreducible, faithful $\mathfrak{r}$-module. Suppose* $\operatorname{char} V > \dim V$. *Then $\mathfrak{r}$ is abelian and the configuration is definably linear: there is an infinite field $\mathbb{K}$ such that $V \simeq \mathbb{K}_+$ and $\mathfrak{r} \hookrightarrow \mathbb{K}\,\mathrm{Id}_V$, all definably.*

Proof. Let $\mathfrak{a}$ be the last nontrivial commutator subring. By the theorem, the configuration is linear, say $V \simeq \mathbb{K}_+^n$ and $\mathfrak{r} \hookrightarrow \mathfrak{gl}_n(\mathbb{K})$. By Lie's theorem, $\mathfrak{r}$ (or the Lie algebra it generates, viz., its $\mathbb{K}$-linear span) being soluble has a common eigenvector in V. Irreducibility forces $n = 1$; so $\mathfrak{r}$ is abelian. $\square$

In the following, one must assume existence of a characteristic. In practice, all our soluble Lie rings will be definable subrings of simple Lie rings of finite Morley rank, so this requirement will be met.

Corollary A2. *Let $\mathfrak{r}$ be a connected, soluble Lie ring of finite Morley rank. Suppose $(\mathfrak{r}; +)$ has prime exponent* $> \dim \mathfrak{r}$.

(i) *There is a largest definable, connected, nilpotent ideal $F^\circ(\mathfrak{r}) \trianglelefteq \mathfrak{r}$.*

(ii) *$\mathfrak{r}'$ is nilpotent, viz., $\mathfrak{r}' \sqsubseteq F^\circ(\mathfrak{r})$.*

(iii) *If $\mathfrak{u} \sqsubseteq \mathfrak{r}$ is a definable, connected subring consisting of $\mathfrak{r}$-nilpotent elements (viz., $(\forall a \in \mathfrak{u})(\exists n \in \mathbb{N})(\mathrm{ad}_a^n(\mathfrak{r}) = 0)$), then $\mathfrak{u} \sqsubseteq F^\circ(\mathfrak{r})$.*

Proof. This does follow the group case, as planned by Nesin, mostly induction on $\dim \mathfrak{r}$. Good references are [Borovik and Nesin 1994, Corollary 9.9 and Theorem 9.21]. $\square$

Remarks. • As is well-known (and which makes the claim in [Nesin 1989a, Theorem 18] look hasty), there are algebraic counterexamples to nilpotence of $\mathfrak{r}'$ if $(\mathfrak{r}; +)$ has exponent $\le \dim \mathfrak{r}$.

• We introduce only a connected version of the Fitting ideal, so (i) is obvious. One could wonder what happens with the nonconnected version [Borovik and Nesin 1994, §7.2].

• There remains of course to develop a theory of Cartan subrings [Deloro and Tindzogho Ntsiri 2024, Question 10]. Only then should one attack the analogue of the Cherlin–Zilber conjecture. Fortunately Theorem B is too small to require advanced tools.

3. Modules of Morley rank 2

The following is an analogue of [Deloro 2009, Theorem A]. Its proof may look easy; it however relies on [Deloro and Tindzogho Ntsiri 2024], which has no analogue in the group case.

Theorem B. *Let $\mathfrak{g}$ be a connected Lie ring of finite Morley rank acting definably and faithfully on an irreducible module V of characteristic $\neq 2, 3$. Suppose* $\dim V = 2$. *Then there is a definable field $\mathbb{K}$ such that, all definably*:

- *either $V \simeq \mathbb{K}_+$ and $\mathfrak{g} \hookrightarrow \mathbb{K}\,\mathrm{Id}_V$;*
- *or $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{K})$ in its natural action on $V \simeq \mathbb{K}^2$;*
- *or $\mathfrak{g} \simeq \mathfrak{gl}_2(\mathbb{K})$ in its natural action on $V \simeq \mathbb{K}^2$.*

Proof. Let $\mathfrak{g}$ be a connected Lie ring of finite Morley rank. Let V be a definable, faithful, irreducible $\mathfrak{g}$-module of characteristic $\neq 2, 3$ with $\dim V = 2$.

Step 1 (reductions). (i) *We may suppose that $\mathfrak{g}$ is nonsoluble.*

(ii) *We may suppose that for every $v \in V \setminus \{0\}$, the centraliser $C^\circ_\mathfrak{g}(v)$ is proper.*

(iii) *We may suppose that for every $v \in V \setminus \{0\}$, the centraliser $C^\circ_\mathfrak{g}(v)$ is soluble.*

(iv) *Let $\mathfrak{h} \sqsubseteq \mathfrak{g}$ be a definable, connected subring. Suppose there is $v \in V \setminus \{0\}$ such that $\mathfrak{h} \cap C_\mathfrak{g}(v) \neq 0$. Then $\mathfrak{h}$ is soluble if and only if V is $\mathfrak{h}$-reducible.*

Proof. (i) If $\mathfrak{g}$ is soluble, then Corollary A1 gives the desired description.

(ii) Let $R(V) = \{v \in V : \mathfrak{g} \cdot v = 0\}$, a finite subgroup by $\mathfrak{g}$-irreducibility. Let $\overline{V} = V/R(V)$ and $\pi : V \twoheadrightarrow \overline{V}$ be the natural projection. Let $R_2(V) = \pi^{-1}(R(\overline{V}))$. Then $R_2(V)$ is finite and $\mathfrak{g}$-invariant, so by connectedness $\mathfrak{g} \cdot R_2(V) = 0$ and $R_2(V) \leq R(V)$. Hence $R(\overline{V}) = 0$.

Suppose the problem is solved for modules with $R(V) = 0$. Then there is a definable field $\mathbb{K}$ such that $(\mathfrak{g}, \overline{V})$ is either $(\mathfrak{gl}_2(\mathbb{K}), \mathbb{K}^2)$ or $(\mathfrak{sl}_2(\mathbb{K}), \mathbb{K}^2)$, both in the natural action. Let $v \in V \setminus R(V)$, with image $\bar{v} = \pi(v) \in \overline{V}$. In the natural action, $\mathfrak{g} \cdot \bar{v} = \overline{V}$, so by connectedness $\mathfrak{g} \cdot v = V$. Let $w \in R(V)$. Then there is $a \in \mathfrak{g}$ such that $w = a \cdot v$. So $a \in C_\mathfrak{g}(\bar{v})$. But $C_\mathfrak{g}(v) \sqsubseteq C_\mathfrak{g}(\bar{v})$ and $C_\mathfrak{g}(\bar{v})$ takes v to $R(V)$, so $C^\circ_\mathfrak{g}(\bar{v}) \sqsubseteq C^\circ_\mathfrak{g}(v)$. In the natural action, centralisers of vectors of $\overline{V}$ in $\mathfrak{g}$ are connected. Hence,

$$a \in C_\mathfrak{g}(\bar{v}) = C^\circ_\mathfrak{g}(\bar{v}) = C^\circ_\mathfrak{g}(v) \sqsubseteq C_\mathfrak{g}(v),$$

meaning $w = a \cdot v = 0$, as wanted. We may therefore assume $R(V) = 0$.

(iii) We use induction on $\dim \mathfrak{g}$. Let $v \in V \setminus \{0\}$ and $\mathfrak{c} = C^\circ_\mathfrak{g}(v)$. By (ii), $\mathfrak{c}$ is proper. If nonsoluble, then V is $\mathfrak{c}$-irreducible, since otherwise one may linearise in dimension 1 [Deloro and Tindzogho Ntsiri 2024, Corollary 1] twice and find $\mathfrak{c}'' = 0$

(more details in the proof of (iv)). By induction $\mathfrak{c}$ is either $\mathfrak{sl}_2(\mathbb{K})$ or $\mathfrak{gl}_2(\mathbb{K})$ in their natural action, a contradiction to $\mathfrak{c} = C^\circ_{\mathfrak{g}}(v)$.

(iv) Suppose $\mathfrak{h}$ is soluble. If V is $\mathfrak{h}$-irreducible, then by Corollary A1, $\mathfrak{h}$ acts freely on V, against its meeting a centraliser. Suppose V is $\mathfrak{h}$-reducible, say $0 < W < V$ is an $\mathfrak{h}$-series. Linearising in dimension 1 twice, $\mathfrak{h}'$ acts trivially on both W and V/W, so $\mathfrak{h}'' = 0$. Hence $\mathfrak{h}$ is soluble. ◇

Step 2 (real-life cases). (v) *If* $\dim \mathfrak{g} = 3$, *then we are done.*

(vi) *If* $\dim \mathfrak{g} = 4$, *then we are done.*

Proof. (v) Suppose $\dim \mathfrak{g} = 3$. It is nonsoluble by (i), so by [Rosengarten 1991] or [Deloro and Tindzogho Ntsiri 2024], $\mathfrak{g}$ has a finite centre and $\mathfrak{g}/Z(\mathfrak{g}) \simeq \mathfrak{sl}_2(\mathbb{K})$ for some definable field $\mathbb{K}$.

We prove $Z(\mathfrak{g}) = 0$ and $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{K})$. Let $\bar{\mathfrak{g}} = \mathfrak{g}/Z(\mathfrak{g})$. Let $\bar{h} \in \bar{\mathfrak{g}}$ induce a weight decomposition $\bar{\mathfrak{g}} = E_{-2}(\bar{h}) \oplus E_0(\bar{h}) \oplus E_2(\bar{h})$. Here $E_k(\bar{h}) = \ker^\circ(\mathrm{ad}_{\bar{h}} - k\,\mathrm{Id}) \leq \bar{\mathfrak{g}}$ (see [Deloro and Tindzogho Ntsiri 2024, §2.1]). Let $h \in \mathfrak{g}$ lie above $\bar{h}$. We turn to eigenspaces for h in $\mathfrak{g}$, simply denoted by E_k. Lifting eigenspaces [Deloro and Tindzogho Ntsiri 2024, Lemma D] and using connectedness, $\mathfrak{g} = E_{-2} \oplus E_0 \oplus E_2$. If $z \in Z(\mathfrak{g})$, then z decomposes as $z = z_{-2} + z_0 + z_2$; applying h shows $z_{-2} = z_2 = 0$. Hence $Z(\mathfrak{g}) \sqsubseteq E_0$. Fix $e \in E_2 \setminus \{0\}$. Then e is not central, so by connectedness $[E_{-2}, e] = E_0$. In particular, there is $f \in E_{-2}$ such that $[f, e] = z$. But then $[\bar{f}, \bar{e}] = 0$, forcing one to be trivial in $\bar{\mathfrak{g}}$. Since $Z(\mathfrak{g}) \sqsubseteq E_0$, one of e or f is trivial in $\mathfrak{g}$, and so is z. Thus $Z(\mathfrak{g}) = 0$ and $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{K})$.

Let $\mathfrak{b} \sqsubset \mathfrak{sl}_2(\mathbb{K})$ be a Borel subring. If V is $\mathfrak{b}$-irreducible, then by Corollary A1, $\mathfrak{b}$ is abelian, a contradiction. So there is a $\mathfrak{b}$-series $0 < W < V$. By Corollary A1 again, $\mathfrak{b}'$ acts trivially on W and V/W. In particular, $\mathfrak{b}'$ acts quadratically, viz., for $x \in \mathfrak{b}'$ one has $x^2 \cdot V = 0$.

Finally, V has no nontrivial, finite $\mathfrak{g}$-invariant subgroup by (ii). This is enough to follow the proof of [Deloro 2013, Variation 11], which definably provides a $\mathbb{K}$-linear structure on V turning it into the natural representation.

(vi) Suppose $\dim \mathfrak{g} = 4$. By [Deloro and Tindzogho Ntsiri 2024], $\mathfrak{g}$ cannot be simple; so there is an infinite, proper, definable, connected $I \lhd \mathfrak{g}$.

If $\dim I = 1$, then I is abelian and we linearise using Theorem A: there is a definable field $\mathbb{K}$ such that $V \in \mathbb{K}$-**Vect** and $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V : \mathbb{K}\text{-}\mathbf{Vect})$. Clearly $\dim \mathbb{K} = 1$ and the $\mathbb{K}$-linear dimension of V is 2; we are done.

If $\dim I = 2$, then by [Rosengarten 1991, Theorem 4.2.1] I, $\mathfrak{g}/I$, and $\mathfrak{g}$ are soluble, against (iv).

Therefore $\dim I = 3$, and by (v), $I \simeq \mathfrak{sl}_2(\mathbb{K})$ in its natural action on $V \simeq \mathbb{K}^2$.

Let W be a $\mathbb{K}$-vector line in $V \simeq \mathbb{K}^2$. Fix $w \in W \setminus \{0\}$. Let $\mathfrak{c} = C^\circ_{\mathfrak{g}}(w)$; then $\mathfrak{u} = C^\circ_I(w)$ and $\mathfrak{b} = N^\circ_I(\mathfrak{u})$. Observing the natural representation, $\mathfrak{u}$ is a nilpotent

$\mathbb{K}$-subalgebra of I and $\mathfrak{b}$ the Borel $\mathbb{K}$-subalgebra of I containing $\mathfrak{u}$. Moreover, $\mathfrak{u} = C_I^\circ(W)$ and $W = C_V^\circ(\mathfrak{u})$. So $\mathfrak{c}$ normalises $\mathfrak{u}$ and W.

If $\dim \mathfrak{c} = 3$, then $\dim \mathfrak{u} = \dim(\mathfrak{c} \cap I)^\circ \geq 2$, a contradiction. Hence $\dim \mathfrak{c} = 2$. Let $\mathfrak{d} = \mathfrak{b} + \mathfrak{c}$. Since $\mathfrak{c} \not\sqsubseteq I$, one has $\dim \mathfrak{d} \geq 3$. Since $\mathfrak{d}$ normalises W, one has $\dim \mathfrak{d} = 3$ by irreducibility of $\mathfrak{g}$.

Now start from three distinct $\mathbb{K}$-vector lines W_1, W_2, W_3 in V. Fix $w_i \in W_i \setminus \{0\}$ in each and define $\mathfrak{c}_i, \mathfrak{u}_i, \mathfrak{b}_i, \mathfrak{d}_i$ accordingly. For $i \neq j$, let $\mathfrak{d}_{i,j} = (\mathfrak{d}_i \cap \mathfrak{d}_j)^\circ$, which normalises W_i and W_j. By Lie's theorem, $\mathfrak{d}'_{i,j} = 0$, so $\mathfrak{d}_{i,j}$ is abelian. Since $\dim(\mathfrak{d}_{i,j} \cap I) \leq 1$, one has $\dim \mathfrak{d}_{i,j} = 2$, and $(\mathfrak{d}_{i,j} \cap I)^\circ$ is a Cartan $\mathbb{K}$-subalgebra of I. In particular, $\mathfrak{d}_{i,j}$ does not normalise W_k, implying $\mathfrak{d}_{i,j} \not\sqsubseteq \mathfrak{d}_{i,k}$.

Finally let $\mathfrak{a} = (\mathfrak{d}_1 \cap \mathfrak{d}_2 \cap \mathfrak{d}_3)^\circ$, so $\dim \mathfrak{a} = 1$. Let $\mathfrak{h} = C_{\mathfrak{g}}^\circ(\mathfrak{a})$. Then $\mathfrak{h}$ contains $\mathfrak{d}_{i,j}$ for each pair. So $\mathfrak{h} \cap I$ is a subring of I containing two distinct Cartan subalgebras of I; hence $I \leq \mathfrak{h}$. By abelianity of $\mathfrak{a}$ [Rosengarten 1991, Theorem 4.1.1], also $\mathfrak{a} \leq \mathfrak{h}$. Thus $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, and we are back to the first case above. ◇

Definition. *A definable, connected subring $\mathfrak{u} \sqsubseteq \mathfrak{g}$ is **V-nilpotent** if for all $a \in \mathfrak{u}$, there is an integer n such that $a^n = 0$* [End(V)].

In this terminology, ad-nilpotence becomes $\mathfrak{g}$-nilpotence, where $\mathfrak{g}$ is the adjoint module. In general, if $\mathfrak{g}$ is faithful on V and $\mathfrak{u} \sqsubseteq \mathfrak{g}$ is V-nilpotent with lengths bounded by n, then $\mathfrak{u}$ is $\mathfrak{g}$-nilpotent with lengths bounded by $2n - 1$. (This relies on the formula $\mathrm{ad}_a^k(x) = \sum_{i=0}^k \binom{k}{i} a^{k-i} x a^i$ [End(V)].) We shall use $\mathfrak{g}$-nilpotence of V-nilpotent subrings in (xi) below.

The following manages to elude an $N_\circ^\circ$-theory, which would still be interesting to have, with the proviso that Witt's algebras are simple and $N_\circ^\circ$; see [Deloro and Tindzogho Ntsiri 2024, Question 11].

Step 3 (nilpotent analysis). (vii) *If $v \in V \setminus \{0\}$, then $C_{\mathfrak{g}}^\circ(v)$ contains a nontrivial, V-nilpotent subring.*

Let $\mathfrak{u}$ be a maximal V-nilpotent subring.

(viii) *There exists a unique nontrivial, proper $\mathfrak{u}$-submodule $W_{\mathfrak{u}} < V$. One has $\mathfrak{u} = C_{\mathfrak{g}}^\circ(W_{\mathfrak{u}}, V/W_{\mathfrak{u}})$ and $W_{\mathfrak{u}} = C_V^\circ(\mathfrak{u})$.*

(ix) *$N_{\mathfrak{g}}^\circ(\mathfrak{u}) = N_{\mathfrak{g}}^\circ(W_{\mathfrak{u}})$ is a Borel subring of $\mathfrak{g}$.*

Proof. (vii) Let $\mathfrak{c} = C_{\mathfrak{g}}^\circ(v)$. By (iii), $\mathfrak{c}$ is soluble, so by (iv), it is reducible. Let $0 < W < V$ be a $\mathfrak{c}$-series, so $\dim W = 1$. Let $\mathfrak{c}_1 = C_{\mathfrak{c}}(W)$ and $\mathfrak{c}_2 = C_{\mathfrak{c}}(V/W)$. (We do not use connected components here.) By definition, $\mathfrak{u} = (\mathfrak{c}_1 \cap \mathfrak{c}_2)^\circ$ is V-nilpotent.

We claim $\mathfrak{c} = \mathfrak{c}_1 \cup \mathfrak{c}_2$. Otherwise, linearising in dimension 1, any $a \in \mathfrak{c} \setminus (\mathfrak{c}_1 \cup \mathfrak{c}_2)$ acts like a nonzero scalar (hence freely) on both W and V/W. The latter forces $v \in W$ and the former $v = 0$, a contradiction. So $\mathfrak{c} = \mathfrak{c}_1$ or $\mathfrak{c} = \mathfrak{c}_2$.

Suppose $\mathfrak{u} = 0$. Then $\mathfrak{c}_2$ acts with a finite kernel on W, so $\dim \mathfrak{c}_2 \leq \dim W = 1$. Likewise $\dim \mathfrak{c}_1 \leq \dim(V/W) = 1$. Since $\mathfrak{c}$ equals $\mathfrak{c}_1$ or $\mathfrak{c}_2$, we have $\dim \mathfrak{c} \leq 1$

and $\dim \mathfrak{g} \leq 3$. By (v), $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{K})$ in the natural action. But there, $\mathfrak{c}$ *is* a nontrivial V-nilpotent subring, a contradiction.

(viii) Any V-nilpotent ring is easily seen to be V-nilpotent of class ≤ 2, hence abelian. If V is $\mathfrak{u}$-irreducible, then $\mathfrak{u}$ acts freely on V, against V-nilpotence. So there is a $\mathfrak{u}$-series $0 < W < V$. The action of $\mathfrak{u}$ on W is by (possibly trivial) scalars; V-nilpotence forces triviality, so $W \leq C_V^\circ(\mathfrak{u})$ and equality holds. In particular, $\mathfrak{u}$ may not normalise another nontrivial, proper submodule of V, which proves uniqueness. Always by V-nilpotence, $\mathfrak{u}$ centralises V/W. Thus $\mathfrak{u} \sqsubseteq C_\mathfrak{g}^\circ(W, V/W)$. Since $C_\mathfrak{g}^\circ(W, V/W)$ is V-nilpotent, maximality of $\mathfrak{u}$ forces equality.

(ix) Clearly $N_\mathfrak{g}^\circ(\mathfrak{u}) \sqsubseteq N_\mathfrak{g}^\circ(W_\mathfrak{u})$. Moreover, $N_\mathfrak{g}^\circ(W_\mathfrak{u})$ is soluble by reducibility. (For full rigour: apply (iv). One may observe that $N_\mathfrak{g}^\circ(W_\mathfrak{u})$ meets $C_\mathfrak{g}^\circ(w)$ at least in $\mathfrak{u} \neq 0$ for any $w \in W \setminus \{0\}$, or that this direction of (iv) did not require meeting a centraliser.) So there is a Borel subring $\mathfrak{b} \sqsubset \mathfrak{g}$ containing $N_\mathfrak{g}^\circ(W_\mathfrak{u})$. Now $\mathfrak{u} \sqsubseteq N_\mathfrak{g}^\circ(\mathfrak{u}) \sqsubseteq \mathfrak{b}$, so $\mathfrak{b}$ meets a point centraliser. By (iv), $\mathfrak{b}$ is reducible; there is a $\mathfrak{b}$-series $0 < W < V$. Then $\mathfrak{u} \sqsubseteq \mathfrak{b}$ acts on W, so $W = W_\mathfrak{u}$ by (viii). Thus $\mathfrak{b}$ normalises $W_\mathfrak{u}$. By Lie's theorem in the form of Corollary A1, $\mathfrak{b}'$ acts trivially on $W_\mathfrak{u}$ and $V/W_\mathfrak{u}$, so by (viii) one has $\mathfrak{b}' \sqsubseteq C_\mathfrak{g}^\circ(W_\mathfrak{u}, V/W_\mathfrak{u}) = \mathfrak{u}$. Therefore $\mathfrak{u}$ is an ideal of $\mathfrak{b}$, viz., $\mathfrak{b} \sqsubseteq N_\mathfrak{g}^\circ(\mathfrak{u})$. $\diamond$

Step 4 (rigid, then disjoint, centralisers). (x) *If* $w \in W_\mathfrak{u} \setminus \{0\}$, *then* $C_\mathfrak{g}^\circ(w) = C_\mathfrak{g}^\circ(W_\mathfrak{u})$.

(xi) *There are two maximal distinct V-nilpotent subrings.*

(xii) *We are done.*

Proof. (x) Let $w \in W_\mathfrak{u} \setminus \{0\}$ and $\mathfrak{c} = C_\mathfrak{g}^\circ(w)$ (not the same as in (vii)). By (iii), $\mathfrak{c}$ is soluble, and hence reducible by (iv). Now $\mathfrak{u} \sqsubseteq \mathfrak{c}$, so every $\mathfrak{c}$-module is $\mathfrak{u}$-invariant. By (viii) the only possible nontrivial, proper $\mathfrak{c}$-submodule of V is $W_\mathfrak{u}$. If $\mathfrak{c}$ acts on $W_\mathfrak{u}$ nontrivially, then it acts freely, a contradiction. So $\mathfrak{c}$ acts trivially on $W_\mathfrak{u}$, and $\mathfrak{c} \sqsubseteq C_\mathfrak{g}^\circ(W_\mathfrak{u}) \sqsubseteq C_\mathfrak{g}^\circ(w) = \mathfrak{c}$.

(xi) Suppose not. Let $\mathfrak{u}$ be the unique such; let $W = C_V^\circ(\mathfrak{u})$ and $\mathfrak{b} = N_\mathfrak{g}^\circ(\mathfrak{u})$. However intuitively pathological, this configuration takes a little time to dispose of. We use (viii) and (ix) implicitly.

Let $v_2 \in V \setminus C_V(\mathfrak{u})$ and $\mathfrak{c}_2 = C_\mathfrak{g}^\circ(v_2)$, so that $\operatorname{codim} \mathfrak{c}_2 \leq 2$. By (vii), $\mathfrak{c}_2$ contains a nontrivial, V-nilpotent subring $\mathfrak{a}_2 \neq 0$. By assumption, $\mathfrak{a}_2 \sqsubseteq \mathfrak{u}$. Now $\mathfrak{c}_2$ being soluble by (iii) must normalise a 1-dimensional submodule by (iv), say W'. Then $\mathfrak{a}_2$ centralises W'; if $W' \neq W$, then $\mathfrak{a}_2 \neq 0$ centralises $W + W' = V$, a contradiction. Hence $W' = W$ and $\mathfrak{c}_2 \leq N_\mathfrak{g}^\circ(W) = \mathfrak{b}$.

Since $v_2 \notin C_V(\mathfrak{u})$, we have $v_2 \notin W$. So the action of $\mathfrak{c}_2$ on V/W is not free; since $\dim(V/W) = 1$, the action is therefore trivial, viz., $\mathfrak{c}_2 \leq C_\mathfrak{g}^\circ(V/W)$. Now $v_2 \notin C_V(\mathfrak{u})$ implies $\mathfrak{u} \not\sqsubseteq \mathfrak{c}_2$, and therefore $\mathfrak{c}_2 \subsetneq \mathfrak{b} \subsetneq \mathfrak{g}$. Then $\dim \mathfrak{c}_2 = \dim \mathfrak{g} - 2$ and $\dim \mathfrak{b} = \dim \mathfrak{g} - 1$. It follows $\mathfrak{b} = \mathfrak{u} + \mathfrak{c}_2 \leq C_\mathfrak{g}^\circ(V/W) < \mathfrak{g}$ so $\mathfrak{b} = C_\mathfrak{g}^\circ(V/W)$.

Now recall that $\mathfrak{u}$ is $\mathfrak{g}$-nilpotent, viz., ad-nilpotent as a subring of $\mathfrak{g}$. Since $\dim(\mathfrak{g}/\mathfrak{b}) = 1$, the ring $\mathfrak{u}$ acts trivially on $\mathfrak{g}/\mathfrak{b}$. Let $u \in \mathfrak{u} \setminus \{0\}$. Then we have $[u, \mathfrak{g}] \leq \mathfrak{b} \leq C_{\mathfrak{g}}(V/W)$. Let $x \in \mathfrak{g}$, and $v \in V$. Then

$$xu \cdot v = [x, u] \cdot v + u \cdot xv \in [u, \mathfrak{g}] \cdot v + u \cdot V \leq C_{\mathfrak{g}}(V/W) \cdot v + W \leq W.$$

Since $0 < uV \leq W$ implies $uV = W$, this yields $xW \leq W$. So any $x \in \mathfrak{g}$ normalises W, against irreducibility.

(xii) By (xi), let $\mathfrak{u}_1 \neq \mathfrak{u}_2$ be two distinct maximal V-nilpotent subrings. Let $W_i = C_V^\circ(\mathfrak{u}_i)$ be given by (viii); notice $W_1 \neq W_2$. Fix $w_i \in W_i \setminus \{0\}$ and let $\mathfrak{c}_i = C_{\mathfrak{g}}^\circ(w_i)$. By (x), $\mathfrak{c}_1 \cap \mathfrak{c}_2$ centralises $W_1 + W_2 = V$, so $\mathfrak{c}_1 \cap \mathfrak{c}_2 = 0$. Since $\operatorname{codim} \mathfrak{c}_i = \dim(\mathfrak{g} \cdot v_i) \leq 2$, this gives

$$2 \dim(\mathfrak{g} - 2) \leq \dim \mathfrak{c}_1 + \dim \mathfrak{c}_2 \leq \mathfrak{g},$$

whence $\dim \mathfrak{g} \leq 4$. By (v) and (vi), we are done. ◇

This completes the proof of Theorem B. □

Remark (on (v)). Presumably the following should hold by Chevalley's basis theorem. Let $\mathfrak{g}$ be a connected Lie ring of finite Morley rank. Suppose $\mathfrak{g}$ is perfect and there is an algebraically closed field $\mathbb{K}$ such that $\mathfrak{g}/Z(\mathfrak{g})$ is a simple, finite-dimensional $\mathbb{K}$-Lie algebra *of Chevalley type* $(A_n, \ldots, G_2)$. If the characteristic is large enough, then $Z(\mathfrak{g}) = 0$. This would be an analogue to [Altınel and Cherlin 1999]. We do not know what happens with algebras "of Lie–Cartan type".

It would be interesting to translate [Deloro 2009, Theorem B] for Lie rings. This would certainly use [Deloro 2013, Variations 11 and 14]. The thorough discussion of pathological "partially quadratic" $\mathfrak{sl}_2(\mathbb{K})$-modules in characteristic 3 [Deloro 2013, §4.3] is not expected to have an impact here.

Acknowledgements

After [Deloro and Tindzogho Ntsiri 2024], which was carried in Luminy and in Franceville thanks to CIRM, we were given another opportunity to meet (this time in Paris) by CIRM and the EMS. Many thanks to both institutions; extra, personal thanks to Olivia Barbarroux, Pascal Hubert, and Carolin Pfaff.

We also greatly benefited from Ulla Karhumäki's energetic thoughts. The first author completed his share of the work in the Mathematics Village of Şirince, Turkey; best regards to Ali Nesin.

Last, we heartfully thank the copy editor for a superb job.

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Received 8 Oct 2024. Revised 23 Mar 2025.

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msp

Model Theory
Vol. 4, No. 2, 2025
https://doi.org/10.2140/mt.2025.4.177

Neostability transfers in derivation-like theories

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Motivated by structural properties of differential field extensions, we introduce the notion of a theory T being derivation-like with respect to another model complete theory T_0. We prove that when T admits a model companion T_+, several model-theoretic properties transfer from T_0 to T_+. These properties include completeness, quantifier elimination, stability, simplicity, and NSOP_1. We also observe that, aside from the theory of differential fields, examples of derivation-like theories are plentiful.

1. Introduction

Extending the argument of simplicity of the theory ACFA, Chatzidakis and Pillay [1998] studied the abstract condition of adding an automorphism to a first-order theory T_0 and proved that if such an expanded theory has a model companion T_0A, then T_0A is simple whenever T_0 is stable (this stable-to-simple transfer result has been further generalised in [Blossier and Martin-Pizarro 2019]). In this paper we propose an abstract analogue of this where instead of adding an automorphism, we expand T_0 to a theory T that satisfies certain conditions which resemble structural properties of derivations.

Recall that given a difference field (K, σ), the automorphism σ extends (not necessarily uniquely) to the separable closure K^{sep}. In the case of a differential field (K, δ) much more is true: the derivation extends *uniquely* to any separably algebraic extension. This is a crucial difference between the theories of difference fields and differential fields; for instance, it is one of the reasons why DCF_0 has quantifier elimination while ACFA does not. Another structural property of differential fields (or even differential rings) is that given two differential fields (K, δ_1) and (L, δ_2) with a common differential subfield (E, δ), the tensor product $K \otimes_E L$ has a *unique* derivation extending those on K and L (note that this property also holds for difference fields).

We extract the above two properties of differential field extensions to an abstract setup and define (in Definition 3.1) the notion of a theory T being derivation-like

Omar León Sánchez was supported by EPSRC grant EP/V03619X/1.
MSC2020: 03C45, 03C60, 12H05.
Keywords: model theory, neostability, differential fields.

with respect to a complete and model complete theory T_0 equipped with an invariant ternary relation $\downarrow^0$. The motivating example, of course, is that the theory DF_0 of differential fields in characteristic 0 is derivation-like with respect to ACF_0 equipped with the algebraic disjointness relation $\downarrow^{\mathrm{alg}}$. In Section 4, we provide several other instances of derivation-like theories; in particular, we note that the recently developed theory DCCM of compact complex manifolds with meromorphic vector fields, introduced in [Moosa 2024], is derivation-like.

In Section 3, under the assumption that T has a model companion T_+ (and some assumptions on $\downarrow^0$), we prove that several model-theoretic properties transfer from T_0 to T_+. In particular, completeness and quantifier elimination transfer, the model-theoretic dcl and acl have a natural description, and the neostability properties of stability, simplicity, NSOP_1 (under additional conditions), and rosiness transfer from T_0 to T_+.

While most neostability properties have local combinatorial descriptions, for each of the four mentioned above there is a theorem of a form similar to the Kim–Pillay theorem — one indicating a semantic way to characterise the given property in terms of the existence of an independence relation satisfying certain conditions. These appear in [Kim 2014; Kim and Pillay 1997; Conant et al. 2025; Adler 2009], respectively, and will be stated in full in Theorem 2.2 below.

By inducing (from $\downarrow^0$) a natural independence relation

$$A \downarrow^+_C B \quad \Longleftrightarrow \quad \mathrm{acl}(AC) \downarrow^0_{\mathrm{acl}(C)} \mathrm{acl}(BC)$$

on the model companion T_+ and using Theorem 2.2, in Section 3 we are able to prove that stability and simplicity transfer from T_0 to T_+.

Theorem 1.1. *Suppose $\downarrow^0$ is nonforking independence in T_0 and that for every model $M \models T_+$ we have that $\mathrm{dcl}_0(M) \models T_0$. Suppose also that T is derivation-like with respect to $(T_0, \downarrow^0)$.*

(1) *If T_0 is stable, then T_+ is stable and $\downarrow^+$ is nonforking independence.*

(2) *If T_0 is simple, then T_+ is simple and $\downarrow^+$ is nonforking independence.*

We also prove the following transfer result for NSOP_1.

Theorem 1.2. *Suppose $\downarrow^0$ is Kim-independence in T_0, that T_0 has an independence relation $\downarrow^1$ such that $\downarrow^0_M \Longrightarrow \downarrow^1_M$ for every $M \models T_+$, and that $T_0 \subseteq T_+$. Suppose also that T is derivation-like with respect to $(T, \downarrow^1)$.*

If T_0 is NSOP_1, then T_+ is NSOP_1 and $\downarrow^+$ is Kim-independence.

In particular, when T_0 is the theory of a very slim field, the relation $\downarrow^{\mathrm{alg}}$ is a natural choice for $\downarrow^1$. Finally, assuming that both T_0 and T_+ eliminate imaginaries, we obtain that rosiness also transfers.

Theorem 1.3. *Suppose that T_0 eliminates imaginaries and is rosy with strict independence relation $\downarrow^0$. Suppose also that T is derivation-like with respect to $(T_0, \downarrow^0)$ and that T_+ eliminates imaginaries. Then $\downarrow^+$ is a strict independence relation on T_+, and hence T_+ is rosy.*

Our method of proof relies on a detailed study of how the individual properties of independence relations constituting Definition 2.1 transfer from T_0 to T_+. This is explicitly done in Theorems 3.12, 3.14, and 3.15.

As consequences of these theorems, we obtain the following familiar results on the stability and simplicity of theories of fields with operators.

Corollary 1.4. (1) $\mathrm{SCF}_{p,e}$ *(for $e < \infty$) is stable, and forking independence coincides with algebraic disjointness in the language with constant symbols for a fixed p-basis and λ-functions.*

(2) DCCM *is stable, and forking independence coincides with that in* CCM.

(3) *The model companion of a bounded PAC differential field of characteristic* 0 *is simple, and forking independence coincides with algebraic disjointness.*

(4) CODF *is rosy.*

And we obtain novel results.

Corollary 1.5. (1) $\mathrm{SDCF}^{\lambda}_{p,\infty}$*, the theory of separably differentially closed fields of characteristic p and infinite differential degree of imperfection, is stable, and forking independence coincides with algebraic disjointness and p-disjointness in the language with the λ-functions.*

(2) *The model companion of an ω-free PAC differential field of characteristic* 0 *is* NSOP_1.

Stability of $\mathrm{SDCF}^{\lambda}_{p,\infty}$ was established in [Ino and León Sánchez 2023] by counting types; here we give a characterisation of forking.

Conventions. We assume that all our theories are closed under deductions.

2. Preliminaries

As discussed in the introduction, our definitions and proofs rely on the notion of abstract independence relations and the fact that stability, simplicity, NSOP_1, and rosiness all have a theorem of the form of the Kim–Pillay theorem — one characterising the given property in terms of the existence of a ternary relation satisfying certain properties. In this section, we collect the necessary material and use it freely.

We follow the treatment of independence relations in [Adler 2009]. We fix a complete first-order theory T_0.

Definition 2.1. A relation $\downarrow^0$ on triples of small subsets of a monster model $\mathcal{U}_0$ of T_0 is called an independence relation if it is invariant under automorphisms and satisfies the following eight properties:

(1) *normality*: $A \downarrow^0_C B \Longrightarrow A \downarrow^0_C BC$;

(2) *monotonicity*: $A \downarrow^0_C BD \Longrightarrow A \downarrow^0_C B$;

(3) *base monotonicity*: $A \downarrow^0_C BD \Longrightarrow A \downarrow^0_{CD} B$;

(4) *transitivity*: $A \downarrow^0_C B$ and $A \downarrow^0_B D \Longrightarrow A \downarrow^0_C D$ for $C \subseteq B \subseteq D$;

(5) *symmetry*: $A \downarrow^0_C B \Longrightarrow B \downarrow^0_C A$;

(6) *full existence*: for any A, B, C there is $A' \equiv_C A$ with $A' \downarrow^0_C B$;

(7) *finite character*: if $A_0 \downarrow^0_C B$ for all finite $A_0 \subseteq A$ then $A \downarrow^0_C B$;

(8) *local character*: for any A there is a cardinal $\kappa = \kappa(A)$ such that for any B there is $C \subseteq B$ with $|C| < \kappa$ such that $A \downarrow^0_C B$.

There are other properties that are generally of interest:

- *existence*: for any A and C we have $A \downarrow^0_C C$;
- *extension*: if $A \downarrow^0_C B$ then for any D there is $A' \equiv_{BC} A$ with $A' \downarrow^0_C BD$;
- *antireflexivity*: if $a \downarrow^0_C a$ then $a \in \operatorname{acl}(C)$ (an independence relation is called strict if it satisfies antireflexivity);
- *chain local character*: for a finite tuple a and a regular cardinal $\kappa > |T_0|$, for every continuous chain of models $(M_i)_{i<\kappa}$ with $|M_i| < \kappa$ there is $j < \kappa$ such that $a \downarrow^0_{M_j} \bigcup_{i<\kappa} M_i$;
- *independence theorem over M*: if $A_1 \downarrow^0_M A_2$, $a_1 \downarrow^0_M A_1$, $a_2 \downarrow^0_M A_2$, and $a_1 \equiv_M a_2$, then there is $a \models \operatorname{tp}(a_1/MA_1) \cup \operatorname{tp}(a_2/MA_2)$ with $a \downarrow^0_M A_1A_2$;
- *stationarity over M*: if $M \subseteq A$, $a \downarrow^0_M A$, $b \downarrow^0_M A$, and $a \equiv_M b$, then $a \equiv_A b$.

We say that *monotonicity*, *symmetry*, *finite character*, *existence*, or *extension* holds over models if the property holds when C is a small model of T_0, and we say that $\downarrow^0_M$ satisfies the property if it holds when $C = M$. The *independence theorem* or *stationarity* holds over models if the property holds when M is a small model of T_0. We say that *transitivity* holds over models if the property holds when C and B are small models of T_0.

We collect the various theorems of the form of the Kim–Pillay theorem.

Theorem 2.2. (i) [Kim 2014] *The theory T_0 is stable if and only if it admits an independence relation $\downarrow^0$ (that is, one satisfying* (1)–(8) *above) which satisfies stationarity over models. In this case $\downarrow^0$ coincides with forking independence.*

(ii) [Kim and Pillay 1997] *The theory T_0 is simple if and only if it admits an independence relation $\downarrow^0$ which satisfies the independence theorem over models. In this case $\downarrow^0$ coincides with forking independence.*

(iii) [Conant et al. 2025] *The theory T_0 is* NSOP_1 *if and only if it admits an invariant ternary relation $\downarrow^0$ with chain local character and which over models satisfies monotonicity, transitivity, symmetry, finite character, existence, extension, and the independence theorem. In this case $\downarrow^0$ coincides with Kim-independence over models.*

(iv) [Adler 2009] *Suppose T_0 eliminates imaginaries. Then T_0 is rosy if and only if it admits a strict independence relation.*

Finally, we define some notation that we use freely throughout the next section.

Definition 2.3. Let $\mathcal{L}$ be a language, let M and N be two $\mathcal{L}$-structures, and let $X \subseteq M$ be some subset.

(1) $M \leq_{\mathcal{L}} N$ means that M is an $\mathcal{L}$-substructure of N.

(2) $\langle X \rangle_{\mathcal{L}}$ is the $\mathcal{L}$-structure generated by X (inside M).

(3) $\mathrm{diag}^M_{\mathcal{L}}(X)$ is the quantifier-free $\mathcal{L}$-diagram of X inside M; that is, the set of all quantifier free $\mathcal{L}(C)$-sentences true in M, where $C = \{c_x : x \in X\}$ is a set of new constant symbols and M is expanded to an $\mathcal{L}(C)$-structure by interpreting c_x as x.

3. Main results

We fix the following data:

- $\mathcal{L}_0 \subseteq \mathcal{L}$ are two (first-order) languages, possibly multisorted.
- T_0 is a complete and model complete $\mathcal{L}_0$-theory equipped with an automorphism invariant ternary relation $\downarrow^0$, we denote by $\mathcal{U}_0$ a monster model of T_0, and, unless otherwise stated, acl_0 refers to model-theoretic algebraic closure taken with respect to the language $\mathcal{L}_0$ in $\mathcal{U}_0$.
- T is an $\mathcal{L}$-theory such that $T_0^{\forall} \subseteq T$.

Definition 3.1. We say that T is derivation-like with respect to $(T_0, \downarrow^0)$ if whenever $A, B, C \models T^{\forall}$ (with C a common $\mathcal{L}$-substructure of A and B) are such that $A, B \leq_{\mathcal{L}_0} \mathcal{U}_0$, $\mathrm{acl}_0(C) \cap A = \mathrm{acl}_0(C) \cap B = C$, and $A \downarrow^0_C B$, we have that

(i) there exists $M \models T$ such that $M \leq_{\mathcal{L}_0} \mathcal{U}_0$ and $A, B \leq_{\mathcal{L}} M$, and

(ii) for any M as in (i) and any $\mathcal{L}_0$-structure D such that

$$\langle A, B \rangle_{\mathcal{L}_0} \leq_{\mathcal{L}_0} D \leq_{\mathcal{L}_0} \mathrm{acl}_0(A, B) \cap M,$$

we have that $D \leq_{\mathcal{L}} M$ and, moreover, this $\mathcal{L}$-structure on D is the unique one expanding its $\mathcal{L}_0$-structure, making it a model of $T^{\forall}$, and extending the $\mathcal{L}$-structures of A and B.

Note that part (i) of the definition is, in some sense, a strong form of independent amalgamation.

Remark 3.2. (1) Suppose T is derivation-like with respect to $(T_0, \downarrow^0)$ and $M \models T$ with $M \leq_{\mathcal{L}_0} \mathcal{U}_0$. We note that if $A \leq_{\mathcal{L}} M$ is such that

$$A \downarrow^0_A A,$$

then $\mathrm{acl}_0(A) \cap M \leq_{\mathcal{L}} M$. Indeed, taking B and C equal to A, part (ii) of the definition yields

$$\mathrm{acl}_0(A) \cap M = \mathrm{acl}_0(A, B) \cap M \leq_{\mathcal{L}} M.$$

More generally, for any $\mathcal{L}_0$-structure D such that $A \leq_{\mathcal{L}_0} D \leq_{\mathcal{L}_0} \mathrm{acl}_0(A) \cap M$, we have that $D \leq_{\mathcal{L}} M$ and this $\mathcal{L}$-structure on D is the unique one expanding its $\mathcal{L}_0$-structure, making it a model of $T^{\forall}$, and extending the $\mathcal{L}$-structure of A.

(2) We say that T is *almost* derivation-like with respect to $(T_0, \downarrow^0)$ if in condition (ii) of Definition 3.1 we restrict only to $A = B = C$. Some of the preliminary results of this section also hold for almost derivation-like theories.

(3) When $T_0 \subseteq T$, we may weaken condition (ii) by restricting to only those cases where D is dcl_0-closed: the results of this section continue to hold.

Example 3.3. We highlight the distinction made above in Remark 3.2(3) between $T_0 \subseteq T$ and $T_0 \nsubseteq T$.

Let $T_0 = \mathrm{ACF}_p$ and $T = \mathrm{SCF}_{p,e}$ with $e > 0$. Then $T_0^{\forall} \subseteq T$ but $T_0 \nsubseteq T$ since fields of positive degree of imperfection are not algebraically closed.

If $T_0 = \mathrm{ACF}_0$ and $T = \mathrm{DCF}_0$, then $T_0 \subseteq T$ since every differentially closed field is necessarily algebraically closed.

The following assumptions will be in place throughout the rest of this section.

Assumption 3.4. (i) From now on T is a derivation-like theory with respect to $(T_0, \downarrow^0)$.

(ii) We assume that T has a model companion T_+ and that $T \subseteq T_+$. We fix a monster model $\mathcal{U}_+$ of a completion of T_+. Since $T_0^{\forall} \subseteq T_+$, without loss of generality we may assume that $\mathcal{U}_+ \leq_{\mathcal{L}_0} \mathcal{U}_0$. The notation acl_+ refers to model-theoretic algebraic closure taken in $\mathcal{U}_+$ (with respect to the language $\mathcal{L}$).

(iii) If $T_0 \nsubseteq T_+$, we further assume that T_0 has quantifier elimination.

Let $\mathcal{L}_0^*$ be some language expanding $\mathcal{L}_0$, and set $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}_0^*$. Let T_0^* be an expansion by definitions of T_0 to the language $\mathcal{L}_0^*$ (for instance, the Morleyisation

of T_0). Also, expand T and T_+ to T^* and T^*_+, respectively, to the language $\mathcal{L}^*$ using the same definitions as for T^*_0.

Remark 3.5. The following can be readily checked:

(1) $(T^*_0)^\forall \subseteq T^*$.

(2) $\mathop{\smile\!\!\!\!\!\!|}^0$ is naturally an invariant ternary relation on $\mathcal{U}_0$ as a model of T^*_0.

(3) T^* is derivation-like with respect to $(T^*_0, \mathop{\smile\!\!\!\!\!\!|}^0)$.

(4) T^*_+ is the model companion of T^* and $T^* \subseteq T^*_+$.

(5) $\mathcal{U}_0$ and $\mathcal{U}_+$ remain monster models of T^*_0 and T^*_+, respectively, and $\mathcal{U}_+ \preceq_{\mathcal{L}^*_0} \mathcal{U}_0$.

For (1), in particular to obtain $\mathcal{U}_+ \preceq_{\mathcal{L}^*_0} \mathcal{U}_0$, we need to ensure that any new symbols of $\mathcal{L}^*_0$ have compatible interpretations in $M \preceq_{\mathcal{L}_0} N$ for $M \models T$ and $N \models T_0$; that is, that for any $\mathcal{L}_0$-formula ϕ defining a new symbol we have

$$\phi(M) = \phi(N) \cap M.$$

If $T_0 \subseteq T$, then $M \preceq N$ as T_0 is model complete, and this is automatic. If $T_0 \not\subseteq T$, this can be ensured by assuming T_0 has quantifier elimination in the language $\mathcal{L}_0$. This is the basis for Assumption 3.4(iii).

Lemma 3.6. *Assume $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies full existence. Let $A \preceq_{\mathcal{L}} \mathcal{U}_+$. If $T^*_0 \cup \operatorname{diag}^{\mathcal{U}_0}_{\mathcal{L}^*_0}(A)$ is complete, then $T^*_+ \cup \operatorname{diag}^{\mathcal{U}_+}_{\mathcal{L}^*}(A)$ is complete.*

Proof. Let $K \models T^*_+ \cup \operatorname{diag}^{\mathcal{U}_+}_{\mathcal{L}^*}(A)$. We show that $K \equiv_A \mathcal{U}_+$ as $\mathcal{L}^*$-structures. First note that $K \models (T^*_0)^\forall$, and hence it $\mathcal{L}^*_0$-embeds in some $K' \models T^*_0$. Now by the completeness of $T^*_0 \cup \operatorname{diag}^{\mathcal{U}_0}_{\mathcal{L}^*_0}(A)$, K' $\mathcal{L}^*_0$-embeds inside $\mathcal{U}_0$ over A. Let L be an $\mathcal{L}^*$-elementary substructure of $\mathcal{U}_+$ containing A. Use full existence to find a copy of L with $L' \mathop{\smile\!\!\!\!\!\!|}^0_A K$ and $\operatorname{tp}^{\mathcal{U}_0}_{\mathcal{L}^*_0}(L'/A) = \operatorname{tp}^{\mathcal{U}_0}_{\mathcal{L}^*_0}(L/A)$. This last fact means that L induces an isomorphic $\mathcal{L}^*$-structure on L'. The partial $\mathcal{L}^*_0$-elementary map $A \to A$ from K to L' extends to a partial $\mathcal{L}^*_0$-elementary map $\operatorname{acl}^{\mathcal{U}_0}_{\mathcal{L}^*_0}(A) \cap K \to \operatorname{acl}^{\mathcal{U}_0}_{\mathcal{L}^*_0}(A) \cap L'$. By Remark 3.2(1), this map must be an $\mathcal{L}^*$-isomorphism (note that full existence yields $A \mathop{\smile\!\!\!\!\!\!|}^0_A A$). So we may assume that A is relatively $\operatorname{acl}_{\mathcal{L}^*}$-closed in K and L'.

Since T^* is derivation-like, there is some $M \models T^*$ such that $M \preceq_{\mathcal{L}^*_0} \mathcal{U}_0$ and $K, L' \preceq_{\mathcal{L}^*} M$. Since T^*_+ is the model companion of T^*, there is some $N \models T^*_+$ extending M as an $\mathcal{L}^*$-structure. Now, T^*_+ is model complete, so $L' \preceq N \succeq K$ as $\mathcal{L}^*$-structures. Finally $K \equiv_A N \equiv_A L' \equiv_A L \equiv_A \mathcal{U}_+$. □

We collect some immediate corollaries.

Corollary 3.7. *Assume $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies full existence. Suppose T^*_0 is the model companion of some inductive $\mathcal{L}^*_0$-theory S.*

(1) *$T_+ \cup S$ is the model companion of $T \cup S$.*

(2) *If T_0^* is the model completion of S, then $T_+ \cup S$ is the model completion of $T \cup S$.*

(3) *If T_0^* has quantifier elimination, then T_+^* has quantifier elimination.*

Proof. This is precisely the analogous result of Theorem 7.2 of [Tressl 2005]. The argument is given in more detail in Theorem 3.9 of [Mohamed 2023]. □

As a result, we now *assume* in all cases that T_0 has quantifier elimination — if $T_0 \subseteq T$, then we Morleyise and use Remark 3.5; if $T_0 \not\subseteq T$, we assumed in Assumption 3.4(iii) that T_0 has quantifier elimination in $\mathcal{L}_0$. Hence by Corollary 3.7, T_+ also has quantifier elimination.

Lemma 3.8. *Assume $\downarrow^0$ satisfies monotonicity, symmetry, full existence, and antireflexivity. Then for any $A \subset \mathcal{U}_+$, we have*

$$\mathrm{acl}_+(A) = \mathrm{acl}_0(\langle A \rangle_{\mathcal{L}}) \cap \mathcal{U}_+.$$

Proof. By full existence, we have that $A \downarrow^0_A A$, and so by Remark 3.2(1) we have that

$$F := \mathrm{acl}_0(\langle A \rangle_{\mathcal{L}}) \cap \mathcal{U}_+$$

is an $\mathcal{L}$-substructure of $\mathcal{U}_+$. As T_0 has quantifier elimination, we get $F \subseteq \mathrm{acl}_+(A)$.

For the other containment, consider $a \in \mathrm{acl}_+(A)$. Let K be an elementary $\mathcal{L}$-substructure of $\mathcal{U}_+$ containing all (finitely many) realisations of $\mathrm{tp}_+(a/F)$. By full existence, there is an $\mathcal{L}_0$-substructure L of $\mathcal{U}_0$ with $L \downarrow^0_F K$ and $\mathrm{tp}_0(L/F) = \mathrm{tp}_0(K/F)$. The latter induces an $\mathcal{L}$-structure on L, via some $\sigma \in \mathrm{Aut}_{\mathcal{L}_0}(\mathcal{U}_0/F)$ with $L = \sigma(K)$, making L a model of T_+ and an $\mathcal{L}$-extension of F. Since T is derivation-like, there is $M \models T$ with $M \leq_{\mathcal{L}_0} \mathcal{U}_0$ such that K and L are $\mathcal{L}$-substructures of M. Let $N \models T_+$ be an $\mathcal{L}$-extension of M. Since T_+ is model complete and $K \models T_+$ is a common substructure of N and $\mathcal{U}_+$, there is an elementary $\mathcal{L}$-embedding $\phi : N \to \mathcal{U}_+$ over K. Let $L' = \phi(L)$. We first note that

$$\mathrm{tp}_+^{\mathcal{U}_+}(a/F) = \mathrm{tp}_+^K(a/F) = \mathrm{tp}_+^L(\sigma(a)/F) = \mathrm{tp}_+^N(\sigma(a)/F) = \mathrm{tp}_+^{\mathcal{U}_+}(\phi(\sigma(a))/F),$$

and so $\phi(\sigma(a)) \in K$ (as K contains all realisations of $\mathrm{tp}_+^{\mathcal{U}_+}(a/F)$).

We now claim that $\mathrm{tp}_0(L/K) = \mathrm{tp}_0(L'/K)$. First note that

$$\mathrm{qftp}_0^{\mathcal{U}_0}(L/K) = \mathrm{qftp}_0^M(L/K) = \mathrm{qftp}_0^N(L/K) = \mathrm{qftp}_0^{\mathcal{U}_+}(L'/K) = \mathrm{qftp}_0^{\mathcal{U}_0}(L'/K).$$

Since T_0 has quantifier elimination, it follows that $\mathrm{tp}_0(L/K) = \mathrm{tp}_0(L'/K)$.

Now, by invariance, we have $L' \downarrow^0_F K$. Then monotonicity and symmetry imply that $\phi(\sigma(a)) \downarrow^0_F \phi(\sigma(a))$. By antireflexivity, $\phi(\sigma(a)) \in \mathrm{acl}_0(F) \cap \mathcal{U}_+ = F$. Since $\phi \circ \sigma$ fixes F pointwise, we get that $a \in F$, as desired. □

We now observe that in derivation-like theories with $T_0 \subseteq T_+$ we have a natural description of dcl.

Lemma 3.9. *Assume $\downarrow^0$ satisfies monotonicity, symmetry, full existence, and antireflexivity. In addition, assume that $T_0 \subseteq T_+$. Then for any $A \subset \mathcal{U}_+$, we have*

$$\mathrm{dcl}_+(A) = \mathrm{dcl}_0(\langle A \rangle_{\mathcal{L}}).$$

Proof. As $T_0 \subseteq T_+$, we have

$$\mathrm{dcl}_0(\langle A \rangle_{\mathcal{L}}) \subseteq \mathrm{dcl}_+(A).$$

For the other containment, let $a \in \mathrm{dcl}_+(A)$ and let σ be an $\mathcal{L}_0$-automorphism of $\mathcal{U}_0$ fixing $\langle A \rangle_{\mathcal{L}}$ pointwise. We aim to show that $\sigma(a) = a$. Note that the assumption $T_0 \subseteq T_+$ implies $\mathcal{U}_+ \preceq_{\mathcal{L}_0} \mathcal{U}_0$; this together with Lemma 3.8 yields

$$\sigma(\mathrm{dcl}_+(A)) \leq_{\mathcal{L}_0} \sigma(\mathrm{acl}_0(\langle A \rangle_{\mathcal{L}})) = \mathrm{acl}_0(\langle A \rangle_{\mathcal{L}}) \leq_{\mathcal{L}_0} \mathcal{U}_+.$$

Because T is derivation-like, by Remark 3.2(1) we have that $\sigma(\mathrm{dcl}_+(A)) \leq_{\mathcal{L}} \mathcal{U}_+$ and this is the unique $\mathcal{L}$-structure expanding its $\mathcal{L}_0$-structure, making it a model of $T^{\forall}$, and extending $\langle A \rangle_{\mathcal{L}}$. It follows from this and the fact that T_+ has quantifier elimination (by Corollary 3.7(3)) that σ restricted to $\mathrm{dcl}_+(A)$ is a partial $\mathcal{L}$-elementary map of $\mathcal{U}_+$. Thus, we may extend this restriction to an automorphism ρ of $\mathcal{U}_+$ (which fixes A). But then, as $a \in \mathrm{dcl}_+(A)$, we have that $a = \rho(a) = \sigma(a)$. □

Remark 3.10. We note that in the proofs of Lemmas 3.6, 3.8, and 3.9, condition (ii) of derivation-like (Definition 3.1) was only used when $A = B = C$. Namely, we applied Remark 3.2(1), and so all results stated so far hold when T is almost derivation-like with respect to $(T_0, \downarrow^0)$ in the sense of Remark 3.2(2).

Definition 3.11. Define the following relation on triples of small subsets of $\mathcal{U}_+$:

$$A \downarrow^+_C B \quad \Longleftrightarrow \quad \mathrm{acl}_+(AC) \downarrow^0_{\mathrm{acl}_+(C)} \mathrm{acl}_+(BC).$$

The following provides a detailed description of how independence properties of $\downarrow^0$ transfer to $\downarrow^+$.

Theorem 3.12. (1) *The relation $\downarrow^+$ is invariant and normal.*

(2) *If $\downarrow^0$ satisfies any of monotonicity, symmetry, finite character, then so does $\downarrow^+$.*

(3) *If $\downarrow^0$ is transitive and monotone, then $\downarrow^+$ is transitive.*

(4) *If $\downarrow^0$ satisfies base monotonicity, finite character, and local character, then $\downarrow^+$ has local character.*

(5) *If $\downarrow^0$ satisfies normality, monotonicity, base monotonicity, transitivity, symmetry, and full existence, then $\downarrow^+$ satisfies base monotonicity.*

(6) *If $\downarrow^0$ satisfies monotonicity, symmetry, and full existence, then $\downarrow^+$ satisfies full existence.*

(7) *If* $\mathop{\smile\!\!\!\!\!\!|}^{0}$ *satisfies monotonicity and extension and* T_+ *has quantifier elimination, then* $\mathop{\smile\!\!\!\!\!\!|}^{+}$ *satisfies extension.*

In addition, if $T_0 \subseteq T_+$*, then* (2), (3), *and* (7) *also hold for the corresponding properties stated over models.*

Proof. Invariance. Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{+}_{C} B$ and $\mathrm{tp}_+(ABC) = \mathrm{tp}_+(A'B'C')$. Then

$$\mathrm{tp}_+(\mathrm{acl}_+(ABC)) = \mathrm{tp}_+(\mathrm{acl}_+(A'B'C')),$$

and similar arguments to the proofs above (using quantifier elimination for T_0) show that

$$\mathrm{tp}_0(\mathrm{acl}_+(ABC)) = \mathrm{tp}_0(\mathrm{acl}_+(A'B'C')).$$

Invariance for $\mathop{\smile\!\!\!\!\!\!|}^{0}$ then means that $A' \mathop{\smile\!\!\!\!\!\!|}^{+}_{C'} B'$.

Normality. This is by definition — it does not require normality of $\mathop{\smile\!\!\!\!\!\!|}^{0}$.

Monotonicity. Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{+}_{C} B$ and $D \subseteq B$. Then $\mathrm{acl}_+(AC) \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(BC)$. Also $\mathrm{acl}_+(DC) \subseteq \mathrm{acl}_+(BC)$, so by monotonicity for $\mathop{\smile\!\!\!\!\!\!|}^{0}$, we have that

$$\mathrm{acl}_+(AC) \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(DC);$$

that is, $A \mathop{\smile\!\!\!\!\!\!|}^{+}_{C} D$.

Transitivity. This follows from transitivity and monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^{0}$.

Symmetry. This follows from symmetry of $\mathop{\smile\!\!\!\!\!\!|}^{0}$.

Finite character. This follows from finite character of $\mathop{\smile\!\!\!\!\!\!|}^{0}$ and the fact that acl_+ is finitary.

Local character. Precisely the same proof as in Theorem 2.1 of [Blossier and Martin-Pizarro 2019] applies here.

Base monotonicity. Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{+}_{C} B$ and $C \subseteq D \subseteq B$. We may also assume that $A \supseteq C$ by normality. Then $\mathrm{acl}_+(A) \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(B)$. By monotonicity, we have $\mathrm{acl}_+(A) \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(D)$. Since T is derivation-like,

$$\mathrm{acl}_0(\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)) \cap \mathcal{U}_+ \preceq_{\mathcal{L}} \mathcal{U}_+.$$

So $\langle AD\rangle_{\mathcal{L}} \subseteq \mathrm{acl}_0(\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)) \cap \mathcal{U}_+$, and so by Lemma 3.8 we have

$$\mathrm{acl}_+(AD) = \mathrm{acl}_0(\langle AD\rangle_{\mathcal{L}}) \cap \mathcal{U}_+ \subseteq \mathrm{acl}_0(\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)) \cap \mathcal{U}_+.$$

By base monotonicity and normality for $\mathop{\smile\!\!\!\!\!\!|}^{0}$, we get

$$\mathrm{acl}_+(A)\,\mathrm{acl}_+(D) \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathrm{acl}_+(D)} \mathrm{acl}_+(B).$$

By full existence, we get

$$\mathrm{acl}_0(\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)) \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)} \mathrm{acl}_+(B),$$

and by symmetry, transitivity, and monotonicity, $\mathrm{acl}_+(AD) \downarrow^0_{\mathrm{acl}_+(D)} \mathrm{acl}_+(B)$; that is, $A \downarrow^+_D B$.

Full existence. Suppose a, A, B are given inside $\mathcal{U}_+$. We need to find $a' \in \mathcal{U}_+$ such that $\mathrm{tp}_+(a'/A) = \mathrm{tp}_+(a/A)$ and $a' \downarrow^+_A B$. Let K be some small $\mathcal{L}$-elementary substructure of $\mathcal{U}_+$ containing a, A, B. Write $C = \mathrm{acl}_+(A)$. Use full existence for $\downarrow^0$ to find $L \leq_{\mathcal{L}_0} \mathcal{U}_0$ with $L \downarrow^0_C K$ and $\mathrm{tp}_0(L/C) = \mathrm{tp}_0(K/C)$. Let $\sigma \in \mathrm{Aut}(\mathcal{U}_0/C)$ be the $\mathcal{L}_0$-automorphism taking K to L. This automorphism then induces an $\mathcal{L}$-structure on L. Since T is derivation-like, there is some $M \models T$ such that $K, L \leq_{\mathcal{L}} M \leq_{\mathcal{L}_0} \mathcal{U}_0$. Since T_+ is the model companion of T, there is some $N \models T_+$ extending M. Let $\phi : N \to \mathcal{U}_+$ be the $\mathcal{L}$-elementary embedding of N inside $\mathcal{U}_+$ that fixes K. Then

$$\mathrm{tp}_+^{\mathcal{U}_+}(a/C) = \mathrm{tp}_+^K(a/C) = \mathrm{tp}_+^L(\sigma(a)/C) = \mathrm{tp}_+^N(\sigma(a)/C) = \mathrm{tp}_+^{\mathcal{U}_+}(\phi\sigma(a)/C).$$

We also have the following chain of equalities of quantifier-free $\mathcal{L}_0$-types:

$$\mathrm{qftp}_0^{\mathcal{U}_0}(L/K) = \mathrm{qftp}_0^M(L/K) = \mathrm{qftp}_0^N(L/K) = \mathrm{qftp}_0^{\mathcal{U}_+}(\phi(L)/K) = \mathrm{qftp}_0^{\mathcal{U}_0}(\phi(L)/K).$$

As T_0 has quantifier elimination, this yields $\mathrm{tp}_0(L/K) = \mathrm{tp}_0(\phi(L)/K)$. Invariance then gives $\phi(L) \downarrow^0_C K$, and monotonicity gives $\mathrm{acl}_+(C, \phi\sigma(a)) \downarrow^0_C \mathrm{acl}_+(AB)$; that is, $\phi\sigma(a) \downarrow^+_A B$.

Extension. Suppose $A \downarrow^+_C B$ and $D \supseteq B$ is given. We need to find $A' \equiv_{BC} A$ with $A' \downarrow^+_C D$. As usual we may assume $C \subseteq A, B$ and that these parameter sets are acl_+-closed. Let K be a small $\mathcal{L}$-elementary substructure of $\mathcal{U}_+$ containing all of these sets. Use extension for $\downarrow^0$ to find $A' \downarrow^0_C K$ with $\mathrm{tp}_0(A'/B) = \mathrm{tp}_0(A/B)$. This $\mathcal{L}_0$-isomorphism induces an $\mathcal{L}$-isomorphic structure on A'. By the derivation-like axiom, there is some $M \models T$ such that $M \leq_{\mathcal{L}_0} \mathcal{U}_0$ with $A', K \leq_{\mathcal{L}} M$. Since T_+ is the model companion of T, extend M to some $N \models T_+$, and let $\phi : N \to \mathcal{U}_+$ be an $\mathcal{L}$-elementary embedding of N in $\mathcal{U}_+$ which fixes K. Then

$$\mathrm{qftp}_+^{\mathcal{U}_+}(A/B) = \mathrm{qftp}_+^K(A/B) = \mathrm{qftp}_+^M(A'/B) = \mathrm{qftp}_+^N(A'/B) = \mathrm{qftp}_+^{\mathcal{U}_+}(\phi(A')/B).$$

By quantifier elimination for T_+, $\mathrm{tp}_+(A/B) = \mathrm{tp}_+(\phi(A')/B)$.

As usual,

$$\begin{aligned}\mathrm{qftp}_0^{\mathcal{U}_0}(A'/K) &= \mathrm{qftp}_0^M(A'/K) = \mathrm{qftp}_0^N(A'/K) \\ &= \mathrm{qftp}_0^{\mathcal{U}_+}(\phi(A')/K) = \mathrm{qftp}_0^{\mathcal{U}_0}(\phi(A')/K),\end{aligned}$$

and by quantifier elimination for T_0 and invariance and monotonicity for $\downarrow^0$, we get $\phi(A') \downarrow^0_C D$.

For the final clause of the statement, note that the arguments provided in (2), (3), and (7) hold when working over models since $T_0 \subseteq T_+$. As an example, we will

give the proof that if $\downarrow^0$ satisfies monotonicity over models, then $\downarrow^+$ satisfies monotonicity over models.

Monotonicity over models. Suppose $A \downarrow^+_C B$ and $D \subseteq B$, where $C \models T_+$. Then $\mathrm{acl}_+(AC) \downarrow^0_C \mathrm{acl}_+(BC)$ — since C is acl_+-closed — and $\mathrm{acl}_+(DC) \subseteq \mathrm{acl}_+(BC)$. Now since $T_0 \subseteq T_+$, C is also a model of T_0, and so by monotonicity over models for $\downarrow^0$, we have that

$$\mathrm{acl}_+(AC) \downarrow^0_C \mathrm{acl}_+(DC);$$

that is, $A \downarrow^+_C D$. □

Using the above theorem, we observe that rosiness transfers.

Corollary 3.13. *Assume both T_0 and T_+ admit elimination of imaginaries. If T_0 is rosy, then so is T_+.*

Proof. By Theorem 2.2(iv), it suffices to show that T_+ admits a strict independence relation. Taking $\downarrow^0$ to be any strict independence relation (which exists by rosiness of T_0), Theorem 3.12 yields that $\downarrow^+$ is an independence relation; thus, it suffices to show that $\downarrow^+$ satisfies antireflexivity. Suppose $a \downarrow^+_C a$. Then, by symmetry and monotonicity of $\downarrow^0$, we get $a \downarrow^0_{\mathrm{acl}_+(C)} a$; and so, by antireflexivity of $\downarrow^0$, we obtain

$$a \in \mathrm{acl}_0(\mathrm{acl}_+(C)) \cap \mathcal{U}_+ = \mathrm{acl}_+(C),$$

where the last equality uses Lemma 3.8. □

We now address the transfer of the independence theorem.

Theorem 3.14. *Let $M \subset \mathcal{U}_+$ with $M = \mathrm{acl}_+(M)$, and suppose $T_0 \subseteq T$. Assume the following*:

(1) *T_0 has, in addition to $\downarrow^0$, an independence relation $\downarrow^1$ such that $\downarrow^0_M \Longrightarrow \downarrow^1_M$.*

(2) *T is derivation-like with respect to $(T_0, \downarrow^1)$.*

(3) *And $\downarrow^0_M$ satisfies monotonicity, symmetry, and extension.*

If $\downarrow^0$ satisfies the independence theorem over M, then so does $\downarrow^+$.

Proof. Let $A_1 \downarrow^+_M A_2$, $a_1 \downarrow^+_M A_1$, $a_2 \downarrow^+_M A_2$, and $\mathrm{tp}_+(a_1/M) = \mathrm{tp}_+(a_2/M)$. We will show that there is $a \downarrow^+_M A_1A_2$ realising $\mathrm{tp}_+(a_1/MA_1) \cup \mathrm{tp}_+(a_2/MA_2)$. Let $N \models T_+$ be some $\mathcal{L}$-elementary substructure of $\mathcal{U}_+$ containing all of the above subsets.

Note that, by Theorem 3.12, $\downarrow^+_M$ satisfies symmetry and extension (the fact that T is derivation-like with respect to the independence relation $\downarrow^1$ yields that T_+ has quantifier elimination by Corollary 3.7). Thus, we may assume that A_1, A_2, a_1, and a_2 are all acl_+-closed and contain M (note that $T_0 \subseteq T$ implies that they are also acl_0-closed).

Claim 1. *There is some $a \in \mathcal{U}_0$ with $a \downarrow^0_M N$ and $a \models \mathrm{tp}_0(a_1/A_1) \cup \mathrm{tp}_0(a_2/A_2)$.*

Proof of claim. Note first that by definition of $\downarrow^+$, we have that $A_1 \downarrow^0_M A_2$, $a_1 \downarrow^0_M A_1$, $a_2 \downarrow^0_M A_2$, and $\mathrm{tp}_0(a_1/M) = \mathrm{tp}_0(a_2/M)$. By the independence theorem for $\downarrow^0$, there is $a \in \mathcal{U}_0$ with $a \downarrow^0_M A_1A_2$ and $a \models \mathrm{tp}_0(a_1/A_1) \cup \mathrm{tp}_0(a_2/A_2)$. Now by extension for $\downarrow^0_M$, we may assume that $a \downarrow^0_M N$. □

Claim 2. *Inside $\mathcal{U}_0$, there are $\mathcal{L}_0$-isomorphic copies N_1 and N_2 of N, both containing a, with $N_1 \downarrow^1_a N_2$ and $N \downarrow^1_{A_1A_2} N_1N_2$.*

Proof of claim. Note first that, by assumption and the fact that $a \downarrow^0_M N$, we have that $a \downarrow^1_M N$. Now for $i = 1, 2$, let N_i' be the copy of N coming from the $\mathcal{L}_0$-automorphism $A_ia_i \mapsto A_ia$. By full existence for $\downarrow^1$, let $N_i \equiv^0_{A_ia} N_i'$ with $N_1 \downarrow^1_{A_1a} N$ and $N_2 \downarrow^1_{A_2a} NN_1$. Then $N \downarrow^1_{A_1} N_1$ and $N \downarrow^1_{A_2} N_2$ by transitivity. From $a \downarrow^1_M N$ we get $a \downarrow^1_{A_1} A_2$, and so $A_1a \downarrow^1_{A_1} A_2$. Along with $A_1 \downarrow^1_M A_2$, transitivity gives $A_1a \downarrow^1_M A_2$, so that $A_1a \downarrow^1_a A_2$ by base monotonicity. This implies $A_1 \downarrow^1_a A_2$ and $N_1 \downarrow^1_a A_2$. This last part implies $N_1 \downarrow^1_a A_2a$ and along with $N_2 \downarrow^1_{A_2a} NN_1$ implies $N_1 \downarrow^1_a N_2$. Also, $N \downarrow^1_{A_1A_2} N_1$ by base monotonicity since $A_1A_2 \subseteq N$. From $NN_1 \downarrow^1_{A_2a} N_2$, we get $N \downarrow^1_{A_2aN_1} N_2$, and hence $N \downarrow^1_{A_2N_1} N_2$ since $a \in N_1$. Combining this with $N \downarrow^1_{A_1A_2} N_1$ gives $N \downarrow^1_{A_1A_2} N_1N_2$. □

Claim 3. *There is some model of T which is an $\mathcal{L}$-extension of N, N_1, and N_2.*

Proof of claim. Define $\mathcal{L}$-structures on N_1 and N_2 such that (N_i, A_i, a) is $\mathcal{L}$-isomorphic to (N, A_i, a_i). So $N_i \models T_+$ for $i = 1, 2$. Note that since a_i is an $\mathcal{L}$-substructure of N, a is also an $\mathcal{L}$-substructure of N_i. Now $N_1 \downarrow^1_a N_2$, and a is relatively acl_0-closed in N_1 and N_2; since T is derivation-like with respect to $(T_0, \downarrow^1)$, there is some $P \models T$ such that $P \leq_{\mathcal{L}_0} \mathcal{U}_0$ and $N_1, N_2 \leq_{\mathcal{L}} P$. Since $T_0 \subseteq T$ and using part (ii) of the definition of derivation-like (Definition 3.1), we have that $\mathrm{acl}_0(N_1N_2)$ is an $\mathcal{L}$-substructure of P. By the uniqueness clause of part (ii) of derivation-like and the fact that $A_1 \downarrow^1_M A_2$, we have that $\mathrm{acl}_0(A_1A_2)$ is equipped with an $\mathcal{L}$-structure that makes it simultaneously an $\mathcal{L}$-substructure of N and an $\mathcal{L}$-substructure of $\mathrm{acl}_0(N_1N_2)$. Now $N \downarrow^1_{A_1A_2} N_1N_2$, and so $N \downarrow^1_{\mathrm{acl}_0(A_1A_2)} \mathrm{acl}_0(N_1N_2)$ by invariance, base monotonicity, monotonicity, transitivity, and full existence. Now by part (i) of the derivation-like axiom we may find some $S \models T$ with $S \leq_{\mathcal{L}_0} \mathcal{U}_0$ and $N, \mathrm{acl}_0(N_1N_2) \leq_{\mathcal{L}} S$. So S is an $\mathcal{L}$-extension of N, N_1, and N_2, as desired. □

Now S extends to some $S' \models T_+$. Let $j : S' \to \mathcal{U}_+$ be an $\mathcal{L}$-elementary embedding of S' in $\mathcal{U}_+$ that fixes N. Then

$$\mathrm{tp}_+^{\mathcal{U}_+}(a_1/A_1) = \mathrm{tp}_+^{N}(a_1/A_1) = \mathrm{tp}_+^{N_1}(a/A_1) = \mathrm{tp}_+^{S'}(a/A_1) = \mathrm{tp}_+^{\mathcal{U}_+}(j(a)/A_1),$$

and similarly we have $j(a) \equiv^+_{A_2} a_2$.

As T_0 has quantifier elimination, we have $\mathrm{tp}_0(a/N) = \mathrm{tp}_0(j(a)/N)$. Now, by construction of a, we have $a \mathop{\smile\!\!\!\!\!\!|}^0_M N$, and by monotonicity and invariance, we get $j(a) \mathop{\smile\!\!\!\!\!\!|}^0_M \mathrm{acl}_+(A_1A_2)$, and so $j(a) \mathop{\smile\!\!\!\!\!\!|}^+_M A_1A_2$. □

The following addresses transfer of stationarity.

Theorem 3.15. *Suppose $M \subset \mathcal{U}_+$ with $M = \mathrm{acl}_+(M)$ and $\mathrm{dcl}_0(M) = \mathrm{acl}_0(M)$. Suppose also that $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies base monotonicity, extension, and full existence. If $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies stationarity over $\mathrm{dcl}_0(M)$, then $\mathop{\smile\!\!\!\!\!\!|}^+$ satisfies stationarity over M.*

Proof. Note that, by full existence and Corollary 3.7, T_+ has quantifier elimination. Now suppose $M \subset N \subset \mathcal{U}_+$, $a, b \in \mathcal{U}_+$ with $\mathrm{tp}_+(a/M) = \mathrm{tp}_+(b/M)$, $a \mathop{\smile\!\!\!\!\!\!|}^+_M N$, and $b \mathop{\smile\!\!\!\!\!\!|}^+_M N$. Since $\mathop{\smile\!\!\!\!\!\!|}^+$ satisfies extension (by Theorem 3.12(7)), we may assume that N is a model of T_+. Let $K_a = \mathrm{acl}_+(Ma)$ and $K_b = \mathrm{acl}_+(Mb)$. By definition of $\mathop{\smile\!\!\!\!\!\!|}^+$, we have that $K_a \mathop{\smile\!\!\!\!\!\!|}^0_M N$ and $K_b \mathop{\smile\!\!\!\!\!\!|}^0_M N$. By extension for $\mathop{\smile\!\!\!\!\!\!|}^0$, the same independence holds after replacing N for some N_0 containing $N \cup \mathrm{dcl}_0(M)$. Hence, by base monotonicity, $K_a \mathop{\smile\!\!\!\!\!\!|}^0_{\mathrm{dcl}_0(M)} N_0$ and $K_b \mathop{\smile\!\!\!\!\!\!|}^0_{\mathrm{dcl}_0(M)} N_0$. Note that $\mathrm{tp}_0(K_a/\mathrm{dcl}_0(M)) = \mathrm{tp}_0(K_b/\mathrm{dcl}_0(M))$. Then by stationarity for $\mathop{\smile\!\!\!\!\!\!|}^0$ over $\mathrm{dcl}_0(M)$,

$$\mathrm{tp}_0(K_a/N) = \mathrm{tp}_0(K_b/N).$$

This implies that there is an $\mathcal{L}_0$-isomorphism $\langle K_aN\rangle_{\mathcal{L}_0} \to \langle K_bN\rangle_{\mathcal{L}_0}$ taking $a \mapsto b$ and fixing N.

Note that M is an $\mathcal{L}$-substructure of K_a, K_b, and N. Moreover, by Remark 3.2(1) and full existence, $\mathrm{acl}_0(M) \cap \mathcal{U}_+ = \mathrm{acl}_+(M) = M$.

By the derivation-like axiom, $\langle K_aN\rangle_{\mathcal{L}_0}$ and $\langle K_bN\rangle_{\mathcal{L}_0}$ are $\mathcal{L}$-substructures of $\mathcal{U}_+$. By its uniqueness clause, this $\mathcal{L}_0$-isomorphism must be an $\mathcal{L}$-isomorphism. So $\mathrm{qftp}_+(a/N) = \mathrm{qftp}_+(b/N)$. By quantifier elimination for T_+, we have

$$\mathrm{tp}_+(a/N) = \mathrm{tp}_+(b/N).$$ □

Corollary 3.16. *Suppose $\mathop{\smile\!\!\!\!\!\!|}^0$ is nonforking independence.*

(1) *Assume that $\mathrm{dcl}_0(M) \models T_0$ whenever $M \models T_+$. If T_0 is stable, then T_+ is stable and $\mathop{\smile\!\!\!\!\!\!|}^+$ is nonforking independence.*

(2) *Assume $T_0 \subseteq T$. If T_0 is simple, then T_+ is simple and $\mathop{\smile\!\!\!\!\!\!|}^+$ is nonforking independence.*

Proof. Part (1) follows from Theorems 3.12 and 3.15, while part (2) follows from Theorems 3.12 and 3.14 (note that in the latter we take $\mathop{\smile\!\!\!\!\!\!|}^1 = \mathop{\smile\!\!\!\!\!\!|}^0$). □

Example 3.17. In Corollary 3.16 we required that $\mathrm{dcl}_0(M) \models T_0$ for every $M \models T_+$. If $T_0 \subseteq T_+$, then this holds automatically since T_0 is model complete. Otherwise, Example 3.3 provides a case where this requirement holds. Suppose $T_0 = \mathrm{ACF}_p$ and $T = \mathrm{SCF}_{p,e}$. Then for any set A, $\mathrm{dcl}_0(A)$ coincides with the perfect closure of the field generated by A. So, if $M \models \mathrm{SCF}_{p,e}$, then $\mathrm{dcl}_0(M) \models \mathrm{ACF}_p$.

We now aim to prove a similar result on the transfer of NSOP_1. We will need to restrict to the case of fields to apply Theorem 3.14 with a particular choice of $\mathop{\smile\!\!\!\!\!\!|}^1$.

Assume T_0 is an $\mathcal{L}_0$-theory of fields. We say that a relation $\mathop{\smile\!\!\!\!\!\!|}^1$ on T_0 *implies* $\mathcal{L}_0$-*compositums* if for all $K, L \leq_{\mathcal{L}_0} \mathcal{U}_0$ satisfying $K \mathop{\smile\!\!\!\!\!\!|}^1_E L$, for some $E = \mathrm{acl}_0(E) \cap K = \mathrm{acl}_0(E) \cap L$, the compositum $K \cdot L$ is an $\mathcal{L}_0$-substructure of $\mathcal{U}_0$. Following [Junker and Koenigsmann 2010], we say that T_0 is *very* $\mathcal{L}_0$-*slim* if for every $F \leq_{\mathcal{L}_0} \mathcal{U}_0$ we have that

$$\mathrm{acl}_0(F) = F^{\mathrm{alg}} \cap \mathcal{U}_0.$$

Define the relation $\mathop{\smile\!\!\!\!\!\!|}^1$ on small subsets of $\mathcal{U}_0$ by

$$A \mathop{\smile\!\!\!\!\!\!|}^1_C B \quad \Longleftrightarrow \quad \langle AC \rangle_{\mathcal{L}_0} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_{\langle C \rangle_{\mathcal{L}_0}} \langle BC \rangle_{\mathcal{L}_0}, \tag{1}$$

where $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$ denotes algebraic independence in fields.

Fact 3.18. *Assume $\mathop{\smile\!\!\!\!\!\!|}^1$ implies $\mathcal{L}_0$-compositums. The relation $\mathop{\smile\!\!\!\!\!\!|}^1$ as defined above is an independence relation if and only if T_0 is very $\mathcal{L}_0$-slim.*

The proof is an adaptation of Theorem 2.1 of [Junker and Koenigsmann 2010]. Some details are provided in Lemma 4.4.7 of [Mohamed 2024].

Corollary 3.19. *Assume that T_0 is very $\mathcal{L}_0$-slim, that $\mathop{\smile\!\!\!\!\!\!|}^1$ implies $\mathcal{L}_0$-compositums, that T is derivation-like with respect to $(T_0, \mathop{\smile\!\!\!\!\!\!|}^1)$, and that $T_0 \subseteq T$. If T_0 is NSOP_1 and $\mathop{\smile\!\!\!\!\!\!|}^0$ is Kim-independence, then T_+ is NSOP_1 and $\mathop{\smile\!\!\!\!\!\!|}^+$ is Kim-independence.*

Proof. By [Ramsey 2018, Proposition 3.9.26], if two subfields are Kim-independent over a submodel, then they are algebraically independent. So $\mathop{\smile\!\!\!\!\!\!|}^0_M \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^1_M$ holds, and $\mathop{\smile\!\!\!\!\!\!|}^+$ satisfies the independence theorem over models by Theorem 3.14 (noting that $\mathop{\smile\!\!\!\!\!\!|}^1$ is an independence relation by Fact 3.18). Existence over models and chain local character each transfer from $\mathop{\smile\!\!\!\!\!\!|}^0$ to $\mathop{\smile\!\!\!\!\!\!|}^+$ since every model of T_+ is also a model of T_0. The remaining conditions of Theorem 2.2(iii) hold by Theorem 3.12 (note that (7) of that theorem, the transfer of existence, does apply as T_+ has quantifier elimination; this is by Corollary 3.7 and the fact that $\mathop{\smile\!\!\!\!\!\!|}^1$ satisfies full existence). □

4. Examples

In this section we observe that there are plenty of examples of theories that are derivation-like, and hence to which the results of the previous section apply (when the model companion exists).

4.1. ***Separably closed fields and Hasse–Schmidt fields.*** Fix a prime $p > 0$ and a finite nonnegative integer e. Let $\mathcal{L}_0$ be the language of fields, $T_0 = \mathrm{ACF}_p$, and $\mathop{\smile\!\!\!\!\!\!|}^0$ forking independence (which coincides with algebraic disjointness $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$). Let $\mathcal{L}_{b,\lambda}$ be the language of fields expanded by constants $b = (b_1, \ldots, b_e)$ and unary function symbols $(\lambda_i)_{i \in p^e}$. Let $T_{b,\lambda}$ be the theory of fields of characteristic p

together with sentences specifying that b is a p-basis and that the λ_i are interpreted as the λ-functions with respect to b (in some fixed order of the p-monomials).

Lemma 4.1. *The theory $T_{b,\lambda}$ is derivation-like with respect to* $(\mathrm{ACF}_p, \downarrow^{\mathrm{alg}})$.

Proof. With $\mathcal{U}_0$ a monster model of ACF_p, let $A, B, C \models T^{\forall}_{b,\lambda}$ be as in the definition of derivation-like. Since $C \leq_{\mathcal{L}_{b,\lambda}} A$, we have that A/C is a separable field extension. This, together with the fact that $C = C^{\mathrm{alg}} \cap A$, implies that the field extension A/C is regular (i.e., separable and relatively algebraically closed). This, together with $A \downarrow^{\mathrm{alg}}_C B$, implies that A and B are linearly disjoint over C. Linear disjointness implies that b is p-independent in the compositum $A \cdot B$ (and hence a p-basis). It follows that $A \cdot B \models T_{b,\lambda}$ and $A, B \leq_{\mathcal{L}_{b,\lambda}} A \cdot B \leq_{\mathcal{L}_0} \mathcal{U}_0$. This shows condition (i) of Definition 3.1. Condition (ii) follows from the fact that p-bases are preserved when passing to separably algebraic extensions. □

The model companion of $T_{b,\lambda}$ is $\mathrm{SCF}_{p,e}$. In this case for any $M \models \mathrm{SCF}_{p,e}$ we have that $\mathrm{dcl}^{\mathrm{ACF}_p}(M) \models \mathrm{ACF}_p$ (since perfect closures of separably closed fields remain separably closed). Thus, our Corollary 3.16(1) applies and recovers the well known fact that $\mathrm{SCF}_{p,e}$ is stable and (in the language $\mathcal{L}_{b,\lambda}$) forking independence coincides with algebraic disjointness.

In a similar fashion we can also recover the context of iterative Hasse–Schmidt derivations from [Ziegler 2003]. Let $\mathcal{L}_\partial$ be the language of fields expanded by unary function symbols

$$((\partial_{1,j})_{j=1}^{\infty}, \ldots, (\partial_{e,j})_{j=1}^{\infty}).$$

Let T_∂ be the theory of fields of characteristic p expanded by sentences specifying that $(\partial_{i,j})_{j=1}^{\infty}$ is an iterative Hasse–Schmidt derivation and that, for different i, they pairwise commute.

Lemma 4.2. *The theory T_∂ is derivation-like with respect to* $(\mathrm{ACF}_p, \downarrow^{\mathrm{alg}})$.

Proof. Let $A, B, C \models T^{\forall}_{\partial}$ be as the definition of derivation-like. By Lemmas 2.3 and 2.4 from [Ziegler 2003], after possibly passing to a purely inseparable extension of the separable closure of C, we may assume that C is strict and separably closed. Strictness implies that A/C is a separable extension. Thus, since C is separably closed, A/C is a regular field extension; this implies that A and B are linearly disjoint over C. It follows that $A \cdot B$ is isomorphic to the quotient field of $A \otimes_C B$. The Hasse–Schmidt derivations extend uniquely to $A \otimes_C B$ and this yields an $\mathcal{L}_\partial$-structure on $A \cdot B$ making it a model of T_∂ (see for instance Lemma 2.5 of [Ziegler 2003]). This yields condition (i) of derivation-like. Since Hasse–Schmidt fields have a smallest strict extension (see [Ziegler 2003, Lemma 2.4]) and separably algebraic extensions are étale, condition (ii) of derivation-like follows. □

The model companion of T_∂ is $\mathrm{SCH}_{p,e}$ (using the notation from [Ziegler 2003]). Recall that the latter is the theory of fields equipped with strict and iterative Hasse–Schmidt derivations that pairwise commute and whose underlying field is a model of $\mathrm{SCF}_{p,e}$. As in the SCF case above, for any $M \models \mathrm{SCH}_{p,e}$ we have that $\mathrm{dcl}^{\mathrm{ACF}_p}(M) \models \mathrm{ACF}_p$. Thus, Corollary 3.16(1) applies and recovers the fact that $\mathrm{SCH}_{p,e}$ is stable and in the language $\mathcal{L}_\partial$ forking independence coincides with algebraic disjointness.

4.2. ***$\mathcal{D}$-fields in characteristic 0.*** Let $\mathcal{L}_0$ be an expansion of the field language and T_0 a complete and model complete $\mathcal{L}_0$-theory of fields of characteristic 0. As before, we denote by $\mathop{\smile\!\!\!\!\!\!|}^0$ an invariant ternary relation on a monster model $\mathcal{U}_0 \models T_0$. Recall that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$ denotes the algebraic disjointness relation.

We say that $\mathop{\smile\!\!\!\!\!\!|}^0$ *implies algebraic disjointness* if

$$K \mathop{\smile\!\!\!\!\!\!|}^0_E L \quad \Longrightarrow \quad K \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_E L$$

for K, L $\mathcal{L}_0$-substructures of $\mathcal{U}_0$ and E a common $\mathcal{L}_0$-substructure of K and L. Recall from the previous section that $\mathop{\smile\!\!\!\!\!\!|}^0$ *implies $\mathcal{L}_0$-compositums* if for all $K, L \leq_{\mathcal{L}_0} \mathcal{U}_0$ satisfying $K \mathop{\smile\!\!\!\!\!\!|}^0_E L$, for some $E = \mathrm{acl}_0(E) \cap K = \mathrm{acl}_0(E) \cap L$, the compositum $K \cdot L$ is an $\mathcal{L}_0$-substructure of $\mathcal{U}_0$. Recall also that T_0 is *very $\mathcal{L}_0$-slim* if for every $F \leq_{\mathcal{L}_0} \mathcal{U}_0$ we have that

$$\mathrm{acl}_0(F) = F^{\mathrm{alg}} \cap \mathcal{U}_0.$$

Following the general framework of [Moosa and Scanlon 2014], let $\mathcal{D}$ be a finite-dimensional algebra over a field k of characteristic 0 equipped with a k-basis $\epsilon_0 = 1, \epsilon_1, \ldots, \epsilon_d$ such that $\mathcal{D}$ is a local ring with residue field k. A $\mathcal{D}$-field K is a field which is also a k-algebra equipped with a sequence of operators $(\partial_i : K \to K)_{i=1}^d$ such that the map $K \to K \otimes_k \mathcal{D}$ defined by

$$a \mapsto a \otimes \epsilon_0 + \partial_1(a) \otimes \epsilon_1 + \cdots + \partial_d(a) \otimes \epsilon_d$$

is a k-algebra homomorphism. Let $\mathcal{L}_\mathcal{D}$ be the language $\mathcal{L}_0$ expanded by the language of k-algebras and the unary function symbols $\{\partial_1, \ldots, \partial_d\}$. Let $T_\mathcal{D}$ be $\mathcal{L}_\mathcal{D}$-theory consisting of T_0 together with the theory of $\mathcal{D}$-fields. In addition, let $T_{\mathcal{D}^*}$ be $T_\mathcal{D}$ expanded by sentences specifying that the ∂_i pairwise commute.

Remark 4.3. Let $\mathcal{D} = \mathbb{Q}[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^2$. In this case, the theory $T_\mathcal{D}$ is the theory of differential fields of characteristic 0 with d many (not necessarily commuting) derivations whose underlying field is a model of T_0. The theory $T_{\mathcal{D}^*}$ is similar but requires the derivations to pairwise commute.

Lemma 4.4. *Suppose $\mathop{\smile\!\!\!\!\!\!|}^0$ implies algebraic disjointness and $\mathcal{L}_0$-compositums. Also assume T_0 is very $\mathcal{L}_0$-slim. Then the theories $T_\mathcal{D}$ and $T_{\mathcal{D}^*}$ are derivation-like with respect to $(T_0, \mathop{\smile\!\!\!\!\!\!|}^0)$.*

Proof. First we prove $T_{\mathcal{D}}$ is derivation-like. Let $K, L, E \models T_{\mathcal{D}}^{\forall}$ be as in the definition of derivation-like. Since $E = \mathrm{acl}_0(E) \cap K$ and T_0 is very $\mathcal{L}_0$-slim, K/E is a regular field extension. Since $\mathop{\smile\!\!\!\!\!\!|}^{0}$ implies algebraic disjointness, it follows that K and L are linearly disjoint over E. Then $K \cdot L$ is isomorphic to the quotient field of $K \otimes_E L$. Since $\mathop{\smile\!\!\!\!\!\!|}^{0}$ implies $\mathcal{L}_0$-compositums and $\mathcal{D}$-structures extend uniquely to $K \otimes_E L$ (see [Beyarslan et al. 2019, Proposition 2.20]), this yields an $\mathcal{L}_{\mathcal{D}}$-structure on $K \cdot L$. As we are in characteristic 0, this $\mathcal{D}$-structure extends to all of $\mathcal{U}_0$ (recall from [Beyarslan et al. 2019, Lemma 2.7(3)] that $\mathcal{D}$-structures always extend to smooth extensions) which yields condition (i) of derivation-like. Since algebraic extensions are étale (in characteristic 0), condition (ii) follows (recall from [Beyarslan et al. 2019, Lemma 2.7(2)] that $\mathcal{D}$-structures extend uniquely to étale extensions).

For $T_{\mathcal{D}^*}$, the same argument works by simply noting that uniqueness of the $\mathcal{D}$-structure on $K \otimes_E L$ forces the ∂_i to commute. And similarly when passing to algebraic extensions (as they are étale in characteristic 0). To extend the $\mathcal{D}$-structure from $K \cdot L$ to $\mathcal{U}_0$, first extend to a transcendence basis in a trivial way to force commutativity of the ∂_i and after this the unique extension to $\mathcal{U}_0$ necessarily commutes. This sort of argument is spelled out in Example 4.4.11 of [Mohamed 2024]. □

For the remainder of this section we set $\mathop{\smile\!\!\!\!\!\!|}^{0}$ to be the relation we defined in (1) at the end of Section 3:

$$A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B \quad \Longleftrightarrow \quad \langle AC \rangle_{\mathcal{L}_0} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_{\langle C \rangle_{\mathcal{L}_0}} \langle BC \rangle_{\mathcal{L}_0}. \tag{2}$$

Note that this particular relation implies algebraic disjointness. We recall Fact 3.18 along with an additional fact.

Fact 4.5. (1) *Assume $\mathop{\smile\!\!\!\!\!\!|}^{0}$ implies $\mathcal{L}_0$-compositums. The relation $\mathop{\smile\!\!\!\!\!\!|}^{0}$ as defined in (2) above is an independence relation if and only if T_0 is very $\mathcal{L}_0$-slim.*

(2) *Suppose $\mathcal{L}_0 = \mathcal{L}_{\mathrm{fields}}(C)$, where C is a set of constant symbols. If models of T_0 are large fields, then T_0 is very $\mathcal{L}_0$-slim.*

We note that (2) is an adaptation of Theorem 5.4 of [Junker and Koenigsmann 2010] and appears in Lemma 4.4.10 of [Mohamed 2024].

Corollary 4.6. *Suppose models of T_0 are large fields, $\mathcal{L}_0 = \mathcal{L}_{\mathrm{fields}}(C)$ for C a set of constant symbols, and $\mathop{\smile\!\!\!\!\!\!|}^{0}$ is given as in (2). Assume $T_{\mathcal{D}}$ and $T_{\mathcal{D}^*}$ have model companions $T_{\mathcal{D}}^{+}$ and $T_{\mathcal{D}^*}^{+}$, respectively.*

(i) *If T_0 is simple, then so are $T_{\mathcal{D}}^{+}$ and $T_{\mathcal{D}^*}^{+}$.*

(ii) *If T_0 is* NSOP_1, *then so are $T_{\mathcal{D}}^{+}$ and $T_{\mathcal{D}^*}^{+}$.*

(iii) *Assume T_0, $T_{\mathcal{D}}^{+}$, and $T_{\mathcal{D}^*}^{+}$ all eliminate imaginaries. If T_0 is rosy, then so are $T_{\mathcal{D}}^{+}$ and $T_{\mathcal{D}^*}^{+}$.*

Proof. Part (i) follows immediately by Corollary 3.16(2), (ii) by Corollary 3.19, and (iii) by Corollary 3.13. □

Remark 4.7. Under the hypothesis of Corollary 4.6 (and recall that we are in characteristic 0), an immediate application is that

(i) if T_0 is the theory of a bounded PAC field, then $T_{\mathcal{D}}^+$ and $T_{\mathcal{D}^*}^+$ are simple,

(ii) the theory CODF (closed ordered differential fields in one derivation) is rosy.

Indeed, for (i) recall that bounded PAC fields are simple, while for (ii), recall that the theory RCF is rosy and eliminates imaginaries; also, CODF eliminates imaginaries by [Cubides Kovacsics and Point 2023].

We also note that under the hypothesis of Corollary 4.6, the model companions $T_{\mathcal{D}}^+$ and $T_{\mathcal{D}^*}^+$ in fact exist. Existence of $T_{\mathcal{D}}^+$ is one of the main results of [Mohamed 2023] and, moreover, the simplicity claimed in Remark 4.7(i) already appears there. Existence of $T_{\mathcal{D}^*}^+$ will appear in a forthcoming paper.[1] In the case when $\mathcal{D} = \mathbb{Q}[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^2$ (i.e., in the context of differential fields of characteristic 0, see Remark 4.3) and T_0 is the theory of a bounded PAC field, the existence of $T_{\mathcal{D}^*}^+$ is an instance of the main result of [Tressl 2005] and its simplicity already appears in [Hoffmann and León Sánchez 2022].

A natural question is whether Corollary 3.19 applies when T_0 is the theory of an ω-free PAC field of characteristic 0. The authors do not know if such a theory can be made model complete in a language $\mathcal{L}_{\text{fields}}(C)$ for some set of constant symbols C, which is required for Corollary 4.6 and to argue as above that $T_{\mathcal{D}}^+$ and $T_{\mathcal{D}^*}^+$ exist. However, in the case $\mathcal{D} = \mathbb{Q}[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^2$ — the context of differential fields of characteristic 0 (see Remark 4.3) — we may use results of [Fornasiero and Terzo 2024] on algebraically bounded structures with generic derivations. Any PAC field of characteristic 0 is very $\mathcal{L}_{\text{fields}}$-slim by Corollary 4.5 of [Chatzidakis and Pillay 1998]. In Section 30.2 of [Fried and Jarden 1986], the authors expand by definitions an ω-free PAC field to a language $\mathcal{L}_{\text{fields}}(R_n : n > 0)$ and show that, in this language, T_0 is model complete. Here the symbols R_n are predicates that ensure that extensions are regular field extensions. This theory is then very $\mathcal{L}_{\text{fields}}(R_n : n > 0)$-slim, and hence algebraically bounded. By Theorems 4.5 and 5.12 of [Fornasiero and Terzo 2024], $T_{\mathcal{D}}^+$ and $T_{\mathcal{D}^*}^+$ exist. By Corollary 3.19 of this paper, both are NSOP_1.

4.3. ***Differential fields in positive characteristic.*** In this section we apply our results in the context of separably differentially closed fields [Ino and León Sánchez 2023]. Fix a prime $p > 0$. Let $\mathcal{L}_\lambda$ be the language of fields expanded by the (countably many) λ-functions, namely, by functions $(\lambda_{n,i} : n \in \omega, i \in p^n)$, where

[1] As part of joint work of the first author with Jan Dobrowolski.

$\lambda_{n,i}$ is $(n+1)$-ary. We denote by $\mathrm{SCF}^{\lambda}_{p,\infty}$ the theory of separably closed fields of infinite (algebraic) degree of imperfection expanded by sentences specifying that the $\lambda_{n,i}$ are to be interpreted as the λ-functions; that is, for $(\bar{a}, b)$, if $\bar{a}$ is p-dependent or $(\bar{a}, b)$ is p-independent then $\lambda_{n,i}(\bar{a}, b) = 0$; otherwise,

$$b = \sum_{i \in p^n} (\lambda_{n,i}(\bar{a}, b))^p \, m_i(\bar{a}),$$

where the $m_i(\bar{a})$ denote the p-monomials (with some fixed order). We denote forking independence in $\mathrm{SCF}^{\lambda}_{p,\infty}$ by $\mathop{\smile\!\!\!\!\!\!|}^{0}$, and $\mathcal{U}_0$ is a monster model. Recall from [Srour 1986] that for $\mathcal{L}_\lambda$-substructures K, L, E of $\mathcal{U}_0$ we have

$$K \mathop{\smile\!\!\!\!\!\!|}^{0}_{E} L \quad \Longleftrightarrow \quad K \text{ and } L \text{ are algebraically disjoint and } p\text{-disjoint over } E.$$

Let $\mathcal{L}_{\lambda,\delta}$ be the expansion of $\mathcal{L}_\lambda$ by a (single) unary function symbol δ and let $\mathrm{DF}^{\lambda}_{p}$ be the theory of differential fields of characteristic p with sentences specifying that the $\lambda_{n,i}$ are the λ-functions.

Lemma 4.8. *The theory* $\mathrm{DF}^{\lambda}_{p}$ *is derivation-like with respect to* $(\mathrm{SCF}^{\lambda}_{p,\infty}, \mathop{\smile\!\!\!\!\!\!|}^{0})$.

Proof. Let $K, L, E \models (\mathrm{DF}^{\lambda}_{p})^{\forall}$ be as in the definition of derivation-like. Since $K \mathop{\smile\!\!\!\!\!\!|}^{0}_{E} L$, K and L are p-disjoint over E, and so $K \cdot L$ is an $\mathcal{L}_\lambda$-substructure of $\mathcal{U}_0$.

On the other hand, since $E \leq_{\mathcal{L}_{\lambda,\delta}} K$, we have that K/E is a separable field extension. This, together with the fact that $E = E^{\mathrm{alg}} \cap K$, implies that K/E is a regular field extension. This, together with $K \mathop{\smile\!\!\!\!\!\!|}^{0}_{E} L$, implies that K and L are linearly disjoint over E. Linear disjointness implies $K \cdot L$ is isomorphic to the quotient field of $K \otimes_E L$. This yields a derivation on $K \cdot L$ making it a model of $\mathrm{DF}^{\lambda}_{p}$. This yields condition (i) of derivation-like. Since separably algebraic extensions are étale, condition (ii) follows. □

For $\epsilon \in \mathbb{N} \cup \{\infty\}$, recall that a differential field (K, δ) of characteristic p is said to have differential degree of imperfection ϵ if

$$[C_K : K^p] = p^\epsilon.$$

Here C_K denotes the field of δ-constants of (K, δ). When $\epsilon = \infty$ the above equality should be understood as the degree $[C_K : K^p]$ being infinite. See [Ino and León Sánchez 2023] for further details.

In [Ino and León Sánchez 2023] it was shown that $\mathrm{DF}^{\lambda}_{p}$ has a model companion; namely, the theory $\mathrm{SDCF}^{\lambda}_{p,\infty}$ of separably differentially closed fields of characteristic p of infinite differential degree of imperfection expanded by the λ-functions. We note that in [Ino and León Sánchez 2023] the authors work in the language of the so-called differential λ-functions, denoted by $\ell_{n,i}$, but the result on the existence of the model companion also holds working with the algebraic λ-functions (the argument is spelled out in [Mohamed 2024, Fact 4.4.16]). We note

that $\mathrm{SCF}^{\lambda}_{p,\infty} \subseteq \mathrm{SDCF}^{\lambda}_{p,\infty}$. Thus, Corollary 3.16(1) applies and recovers the fact that $\mathrm{SDCF}^{\lambda}_{p,\infty}$ is stable; furthermore, it shows that, in the language $\mathcal{L}_{\lambda,\delta}$, forking independence coincides with algebraic disjointness and p-disjointness (which is not explicitly stated in [Ino and León Sánchez 2023]).

We conclude this section by noting that, unfortunately, our results do not seem to apply to the theory DCF_p of differentially closed fields of characteristic $p > 0$, studied in [Wood 1973]. Recall that DCF_p is the model companion of DF_p, the theory of differential fields (of characteristic p) in the language of differential fields $\mathcal{L}_\delta$. In this language the theory DCF_p does not eliminate quantifiers, but Wood [1973] showed that it suffices to add the *p-th root on constants* function ℓ_0; namely, the unique function satisfying

$$\begin{cases} (\ell_0(x))^p = x & \text{when } \delta(x) = 0, \\ \ell_0(x) = 0 & \text{otherwise.} \end{cases} \tag{3}$$

The theory of differentially perfect fields (i.e., those (K, δ) such that $C_K = K^p$) is denoted by $\mathrm{DPF}^{\ell_0}_p$ and can be axiomatised by expanding DF_p by a sentence specifying (3) above. It follows that $\mathrm{DCF}^{\ell_0}_p$ is the model completion of $\mathrm{DPF}^{\ell_0}_p$. One could ask whether $\mathrm{DPF}^{\ell_0}_p$ is derivation-like with respect to $\mathrm{SCF}^{\lambda}_{p,\infty}$ (note that the underlying field of a model of DCF_p is a model of $\mathrm{SCF}_{p,\infty}$). We now prove this is not the case.

Lemma 4.9. *The theory* $\mathrm{DPF}^{\ell_0}_p$ ***is not*** *derivation-like with respect to* $(\mathrm{SCF}^{\lambda}_{p,\infty}, \mathop{\smile\!\!\!\!\!\!|}^{0})$.

Proof. Consider the function field $K = \mathbb{F}_p(t)$ with standard derivation $\delta = d/dt$. Note that $(K, \delta) \models \mathrm{DPF}_p$ since $C_K = \mathbb{F}_p$. Inside the model $\mathcal{U}_0$ of $\mathrm{SCF}^{\lambda}_{p,\infty}$, find s such that $t \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathbb{F}_p} s$ and $\mathrm{tp}^{\mathrm{SCF}^{\lambda}_{p,\infty}}(t/\mathbb{F}_p) = \mathrm{tp}^{\mathrm{SCF}^{\lambda}_{p,\infty}}(s/\mathbb{F}_p)$. Equip $L = \mathbb{F}_p(s)$ with the derivation $\delta = d/ds$. We argue that there cannot be an M as in condition (i) of derivation-like. It there were, M would be a model of $\mathrm{DPF}^{\ell_0}_p$. In other words, $C_M = M^p$. Since $K \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathbb{F}_p} L$, we obtain that K and L are p-disjoint over $\mathbb{F}_p$, and so $K \cdot L = \mathbb{F}_p(t, s)$ is an $\mathcal{L}_\lambda$-substructure of M. Hence, the extension $M/\mathbb{F}_p(t, s)$ is separable, which implies that $\mathbb{F}_p(t, s)$ is differentially perfect. But, since $\delta(t-s) = 0$, this would imply that $t - s$ has a p-th root in $\mathbb{F}_p(t, s)$, which is impossible (as t and s are algebraically independent). □

One could further ask whether $\mathrm{DPF}^{\ell_0}_p$ is derivation-like with respect to ACF_p. Again, this is not the case.

Lemma 4.10. *The theory* $\mathrm{DPF}^{\ell_0}_p$ ***is not*** *derivation-like with respect to* $(\mathrm{ACF}_p, \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}})$.

Proof. Consider the function field $K = \mathbb{F}_p(t)$ equipped with the standard derivation $\delta = d/dt$. Let x be a differential indeterminate over K. Let $s := t + x^p$. Then, the derivation on $M := K\langle x \rangle = K(x, \delta x, \dots)$ restricts to the standard derivation $\delta = d/ds$ on $L := \mathbb{F}_p(s)$. Note that both K and L are differentially

perfect and $K \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_{\mathbb{F}_p} L$. However, the algebraic closure of the compositum $K \cdot L$ contains x but it does not contain $\delta(x)$; namely, it is not a differential subfield. In other words, condition (ii) of derivation-like does not hold. □

Remark 4.11. (i) While DCF_p is a stable theory, the two proofs above show that forking independence does not have an obvious algebraic description.[2] Indeed, the proof of Lemma 4.9 shows that $\mathop{\smile\!\!\!\!\!\!|}^{0} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{SCF}_{p,\infty}}$ does not satisfy full existence in DCF_p; while the proof of Lemma 4.10 shows that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$ does not satisfy base monotonicity in DCF_p. Currently the authors are not aware of an algebraic description of forking independence in this theory.

(ii) We leave it as an exercise to check that $\mathrm{DPF}_p^{\ell_0}$ is almost derivation-like with respect to $(\mathrm{ACF}_p, \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}})$ in the sense of Remark 3.10, and hence Lemmas 3.6 and 3.8 apply to the theory $\mathrm{DCF}_p^{\ell_0}$.

4.4. ***CCMs with meromorphic vector fields.*** Our final examples demonstrates that our results apply beyond theories of fields. In this section we observe that the theory, recently formulated in [Moosa 2024], of compact complex manifolds equipped with a "differential" structure fits into our setup.

Recall that the theory CCM (compact complex manifolds) is the theory of the multisorted structure consisting of all compact complex manifolds (or rather all reduced and irreducible compact complex-analytic spaces) by naming as basic relations all closed complex-analytic subsets of finite cartesian products of sorts. See [Moosa 2005] for further details on this theory. However, Moosa [2024] works in the seemingly more general setup of "compactifiable" (rather than compact) complex-analytic spaces. Namely, he works in a definable expansion of CCM where there is a sort for each irreducible meromorphic variety. We denote by $\mathcal{L}_0$ the language of this expansion and continue to denote the theory of the expanded structure by CCM. The advantage of this expansion is that now sorts are closed under taking tangent bundles. We refer the reader to [Moosa 2024, §2] for further details and explanations.

In the language $\mathcal{L}_\nabla = \mathcal{L}_0 \cup \{\nabla_S : S \text{ is a sort of } \mathcal{L}_0\}$, where each ∇_S is a function symbol from sort S to TS, Moosa considers the universal $\mathcal{L}_\nabla$-theory $\mathrm{CCM}^\forall_\nabla$ obtained by adding to $\mathrm{CCM}^\forall$ axioms specifying that $\nabla_S : S \to TS$ is a section to $\pi : TS \to S$ together with a compatibility condition of ∇ with definable meromorphic maps between sorts (see [Moosa 2024, Definition 3.3]). It turns out that, somewhat unintentionally, Moosa has proven that $\mathrm{CCM}^\forall_\nabla$ is derivation-like.

Lemma 4.12. *The theory* $\mathrm{CCM}^\forall_\nabla$ *is derivation-like with respect to* $(\mathrm{CCM}, \mathop{\smile\!\!\!\!\!\!|}^{0})$ *(here* $\mathop{\smile\!\!\!\!\!\!|}^{0}$ *denotes forking independence).*

[2]These examples grew out of discussions with Amador Martin-Pizarro.

Proof. Moosa [2024, Lemma 6.2] proved a form of independent amalgamation that readily yields condition (i) of derivation-like. In addition, in [Moosa 2024, Lemma 6.1], he proves the uniqueness of differential CCM-structures of dcl-closed sets inside $\mathrm{acl}^{\mathrm{CCM}}$-closures, yielding condition (ii) of derivation-like (or rather the weakening observed in Remark 3.2(3)). □

In [Moosa 2024, Theorem 5.5], Moosa proves that the theory $\mathrm{CCM}^{\forall}_{\nabla}$ admits a model companion, which he denotes by DCCM. Our results then yield some of the model-theoretic properties of DCCM deduced in §6 and §7 of [Moosa 2024]; e.g., completeness, quantifier elimination, description acl and dcl, and stability.

4.5. ***Theories with an automorphism.*** For our final example, we exhibit some $\mathcal{L}_0$-theories T_0 where the $\mathcal{L}_0(\sigma)$-theory $T = T_0 \cup \{\text{“}\sigma \text{ is an } \mathcal{L}_0\text{-automorphism”}\}$ is derivation-like with respect to T_0. We would like to thank the referee of this paper for suggesting this example.

Let T_0 be a complete and model-complete $\mathcal{L}_0$-theory with $\mathop{\smile\!\!\!\!\!\!|}^{0}$ denoting nonforking independence, and suppose that dcl_0 and acl_0 coincide in T_0. This, together with Remark 3.2(3), immediately yields condition (ii) of derivation-like. Indeed, suppose we are given $\mathcal{L}$-substructures A and B of $M \models T$. Let $D = \mathrm{dcl}_0(D)$ be an $\mathcal{L}_0$-substructure of M such that

$$\langle A, B\rangle_0 \leq_{\mathcal{L}_0} D \leq_{\mathcal{L}_0} \mathrm{acl}_0(A, B) \cap M.$$

Then $D = \mathrm{dcl}_0(A, B)$. Now $d \in D$ is definable over A and B by some $\mathcal{L}_0$-formula $\phi(x, a, b)$. Then $\phi(x, \sigma(a), \sigma(b))$ defines some element e. Since D is dcl_0-closed, $e \in D$. We have that $\sigma|_M(d) = e$, and hence $D \leq_{\mathcal{L}} M$, and that any $\mathcal{L}_0$-automorphism of D must send $d \mapsto e$, and hence this is the unique $\mathcal{L}_0$-automorphism on D making it a model of $T^{\forall}$ and extending the ones on A and B.

Example 4.13. (1) Let $T_0 = \mathrm{Infset}$ be the theory of an infinite set. T_0 is stable (see Section 5.7 in [Tent and Ziegler 2012]), and $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B \iff A \cap B \subseteq C$. Now if $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$, then $\sigma_A \cup \sigma_B$ is an automorphism of AB; this yields condition (i) of derivation-like. So, the theory of infinite sets with an automorphism is derivation-like with respect to the theory of an infinite set, and hence Infset A is stable (note that Infset A exists as Infset has the DMP [Kikyo and Pillay 2000, Fact 1.5]).

(2) Let K be a field, $\mathcal{L}_0$ be the language of K-vector spaces, and T_0 the theory of infinite K-vector spaces. Then T_0 is stable (see Section 5.7 in [Tent and Ziegler 2012]), and $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B \iff \dim(A/C) = \dim(A/BC)$. Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$. Let β_C be a basis of C, and extend β_C to bases β_A and β_B of A and B, respectively. Now $\beta = \beta_A \cup \beta_B$ is a basis of $\mathrm{span}(AB)$. Defining $\sigma|_\beta = \sigma|_{\beta_A} \cup \sigma|_{\beta_B}$ and extending K-linearly gives the desired amalgam for condition (i) of derivation-like. Hence, the theory of infinite K-vector spaces with an automorphism is derivation-like with

respect to the theory of infinite K-vector spaces, and hence $\mathrm{Vect}_K\ A$ is stable (note that $\mathrm{Vect}_K\ A$ exists as Vect_K has the DMP [Kikyo and Pillay 2000, Fact 1.5]).

Acknowledgements

The authors would like to thank Jan Dobrowolski and Amador Martin-Pizarro for several useful discussions that led to improvements of this paper, and the anonymous referee of this paper for their very helpful comments that improved the clarity of our exposition and suggestion of a new class of examples.

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Received 17 Sep 2024. Revised 27 Mar 2025.

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Model Theory

no. 2 vol. 4 2025