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Yatir Halevi, Itay Kaplan and Saharon Shelah

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Suppose that G is a graph of cardinality μ^+ with chromatic number $\chi(G) \geq \mu^+$. One possible reason that this could happen is if G contains a clique of size μ^+. We prove that this is indeed the case when the edge relation is stable. When G is a random graph (which is simple but not stable), this is not true. But still if in general the complete theory of G is simple, G must contain finite cliques of unbounded sizes.

1. Introduction

The chromatic number $\chi(G)$ of a graph G is the minimal cardinal κ for which there exists a vertex coloring with κ colors such that connected vertices get different colors.

Research around graphs having an uncountable chromatic number has a long history; see, e.g., [Komjáth 2011, Section 3]. This topic is set-theoretic in nature, with many results being independent of the axioms of set theory (ZFC). In [Halevi et al. 2022; 2024] we studied a specific conjecture (Taylor's strong conjecture) in the context of *stable* graphs (in ZFC): a graph whose first-order theory is stable (a model-theoretic notion of tameness; see Section 2 for all the definitions). It turns out that a very close relative of this conjecture holds for stable graphs (although it does not hold in general).

More specifically, we showed that if a stable graph has chromatic number $> \beth_2(\aleph_0)$ then this implies the presence of all the finite subgraphs of a shift graph $\mathrm{Sh}_n(\omega)$ for some $0 < n < \omega$, where for a cardinal κ, the shift graph $\mathrm{Sh}_n(\kappa)$ is the graph whose vertices are increasing n-tuples of ordinals in κ, and two such tuples s, t are connected if for every $1 \leq i \leq n-1$, $s(i) = t(i-1)$ or vice-versa (see Example 2.4). In turn, this implies that the chromatic numbers of elementary extensions of said graph are unbounded. An important example for this paper is the

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case $n = 1$: $\mathrm{Sh}_1(\kappa)$ is the complete graph on κ. For more on this result, see [Halevi et al. 2022; 2024].

In this paper, we take the first step towards identifying the n in the previous paragraph, by considering not only the chromatic number of the graph, but its *cardinality* as well, as we now explain.

Bounds for the chromatic number of the shift graph were computed by Erdős and Hajnal [1966]: assuming the generalized continuum hypothesis (GCH),

$$\chi(\mathrm{Sh}_{n+1}(\kappa^{+n})) = \kappa$$

for all $n < \omega$; see Fact 2.6. In particular (and trivially), $\chi(\mathrm{Sh}_1(\kappa)) = \kappa$. Thus, it makes sense to ask the following question:

Question 1.1. Suppose that G is a stable graph and for simplicity also assume GCH. Assume that for every cardinal κ, there is some $G' \equiv G$ (i.e., $\mathrm{Th}(G) = \mathrm{Th}(G')$) of cardinality $|G'| \leq \kappa^{+n}$ satisfying $\chi(G') \geq \kappa^+$. Then is it true that for some $m \leq n$, G contains all finite subgraphs of $\mathrm{Sh}_m(\omega)$?

Remark 1.2. Proposition 3.12 and Remark 3.13 explain why we restrict ourselves to successor cardinals.

In this paper we deal with the case $n = 1$, and we manage to give a satisfying solution to this case (and more) assuming that the theory of G is simple or that the edge relation is stable (both weaker assumptions than stability of the theory). The following sums up the main results of this paper:

Main Theorem 1.3 (Propositions 3.2 and 3.11). *Let $G = (V, E)$ be a graph and $T = \mathrm{Th}(G, E)$ be its first-order theory. Assume that $|G| = \mu^+$ and $\chi(G) \geq \mu^+$ for some infinite cardinal μ.*

(1) *Assuming T is a simple theory then G contains cliques of any finite size.*

(2) *Assuming the edge relation E is stable then G contains an infinite clique of cardinality μ^+.*

Note that the conclusion of item (1) (together with Löwenheim–Skolem and compactness) implies the existence of $G' \succ G$ of cardinality μ^+ that contains an infinite clique of cardinality μ^+. The conclusion of item (2) is stronger: we can find such a clique already in G itself.

Organization of the paper. In Section 2 we go over the relevant basic definitions in model theory and graph theory. Section 3 contains the proof of the main results. Section 4 reframes the main results in terms of the function hinted at in Question 1.1. Finally, in the Appendix we analyze an example of Hajnal and Komjáth that refutes Taylor's strong conjecture and shows that the theory of this graph is not stable (in fact, we show more: that it is not simple and has IP).

2. Preliminaries

We use small latin letters a, b, c for tuples and capital letters A, B, C for sets. We also employ the standard model-theoretic abuse of notation and write $a \in A$ even for tuples when the length of the tuple is immaterial or understood from context.

Stability and simplicity. We use fairly standard model-theoretic terminology and notation; see, for example, [Tent and Ziegler 2012]. We gather some of the needed notions. For stability, the reader may also consult with [Shelah 1978].

We denote by $\mathrm{tp}(a/A)$ the complete type of a over A. A structure M is κ-*saturated*, for a cardinal κ, if any type p over A with $|A| < \kappa$ is realized in M. The structure M is *saturated* if it is $|M|$-saturated.

The *monster model* of a complete theory T, denoted here by $\mathbb{U}$, is a large saturated model containing all sets and models (as elementary substructures) we encounter.[1] All subsets and models are *small*, i.e., of cardinality $< |\mathbb{U}|$.

Given a first-order theory T, a formula $\varphi(x, y)$ is *stable* if we cannot find elements $\langle a_i, b_j \in \mathbb{U} : i, j < \omega \rangle$ such that $\mathbb{U} \models \varphi(a_i, b_j)$ if and only if $i < j$.

For any formula $\varphi(x, y)$ we set $\varphi(y, x)^{\mathrm{opp}} = \varphi(x, y)$; it is the same formula but with the roles of the variables replaced. The following is folklore.

Fact 2.1. *Work in a complete first-order theory T with infinite models. Let $\varphi(x, y)$ be a stable formula. There is a formula $\psi(y, z)$ such that any φ-type p over a model M is definable by an instance of ψ over M, i.e., for any such φ-type p there is an element $c \in M$ such that $\varphi(x, b) \in p$ if and only if $M \models \psi(b, c)$. Moreover, $\psi(y, z)$ can be chosen such that for any $c \in M$, $\psi(y, c)$ is equivalent to a boolean combination of instances of φ^{opp}.*

Proof. By [Shelah 1978, Theorem II.2.12], there is some $\psi(y, z)$ such that for any φ-type p over any set A ($|A| \geq 2$), there is some $c_p \in A$ such that $\psi(y, c_p)$ defines p. By [Pillay 1996, Lemma 2.2(i)], if $p \in S_\varphi(M)$, where M is a model, then p is definable by a boolean combination of instances of φ^{opp}. But then, as M is a model, $\psi(y, c_p)$ is equivalent to such a boolean combination. □

A theory T is *stable* if all formulas are stable.

Next we define simplicity. We give an equivalent definition, using the notion of dividing for types; see [Tent and Ziegler 2012, Proposition 7.2.5]. Given a first-order theory T with a monster model $\mathbb{U}$, a formula $\varphi(x, b)$ with $b \in \mathbb{U}$ *divides* over A if there is a sequence of realizations $\langle b_i \in \mathbb{U} : i < \omega \rangle$ of $\mathrm{tp}(b/A)$ such

[1] There are set-theoretic issues in assuming that such a model exists, but these are overcome by standard techniques from set theory that ensure the generalized continuum hypothesis from some point on while fixing a fragment of the universe; see [Halevi and Kaplan 2023]. The reader can just accept this or alternatively assume that $\mathbb{U}$ is merely κ-saturated and κ-strongly homogeneous for large enough κ.

that $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent for some $k < \omega$ (every subset of size k is inconsistent). A complete type p over B divides over A if it contains some formula which divides over A.

The theory T is *simple* if for every complete type p over B there is some $A \subseteq B$ with $|A| \leq |T|$ such that p does not divide over A. Every stable theory is simple.

The main tool we use from simplicity theory is forking calculus. Nonforking independence is a 3-place relation on sets (or tuples) denoted by $\perp$. We do not go over all the properties that nonforking independence enjoys in simple theories; see [Tent and Ziegler 2012, Chapter 7] for more information.

Graph theory. Here we gather some facts on graphs and the chromatic number of graphs (see also [Halevi et al. 2022]).

By a *graph* we mean a pair $G = (V, E)$, where $E \subseteq V^2$ is symmetric and irreflexive. A *graph homomorphism* between $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a map $f : V_1 \to V_2$ such that $f(e) \in E_2$ for every $e \in E_1$. If f is injective we say that f embeds G_1 into G_2 a subgraph. If in addition we require that $f(e) \in E_2$ if and only if $e \in E_1$, we say that f embeds G_1 into G_2 as an induced subgraph.

Definition 2.2. Let $G = (V, E)$ be a graph.

(1) For a cardinal κ, a *vertex coloring* (or just coloring) of cardinality κ is a function $c : V \to \kappa$ such that $x \mathrel{E} y$ implies $c(x) \neq c(y)$ for all $x, y \in V$.

(2) The *chromatic number* $\chi(G)$ is the minimal cardinality of a vertex coloring of G.

The following is easy and well known.

Fact 2.3. *Let $G = (V, E)$ be a graph. If $V = \bigcup_{i \in I} V_i$ then*

$$\chi(G) \leq \sum_{i \in I} \chi(V_i, E \restriction V_i).$$

Proof. Let $c_i : V_i \to \mu_i$ be a coloring of $(V_i, E \restriction V_i)$. Define a coloring

$$c : V \to \bigcup \{\mu_i \times \{i\} : i \in I\}$$

by choosing for $v \in V$ an $i_v \in I$ such that $v \in V_{i_v}$ and setting $c(v) = (c_{i_v}(v), i_v)$. □

Example 2.4. For any finite number $r \geq 1$ and any linearly ordered set $(A, <)$, let $\mathrm{Sh}_r(A)$ (*the shift graph on* A) be the following graph: its set of vertices is the set of strictly increasing r-tuples $\langle s_0 < \cdots < s_{r-1} \rangle$ from A, and we put an edge between s and t if for every $1 \leq i \leq r-1$, $s_i = t_{i-1}$, or vice-versa. It is an easy exercise to show that $\mathrm{Sh}_r(A)$ is a connected graph. If $r = 1$ this gives K_A, the complete graph on A.

Example 2.5 (symmetric shift graph). Let $r \geq 1$ be any natural number and A any set. The *symmetric shift graph* $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is defined similarly as the shift graph but with set of vertices the set of distinct r-tuples. Note that $\mathrm{Sh}_r(A)$ is an induced subgraph of $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ (and that for $r = 1$ they are both the complete graph on A).

Since for any infinite set A, $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is definable in $(A, =)$, it is stable. Moreover, for any two infinite sets A and B, $\mathrm{Sh}_r^{\mathrm{sym}}(A) \equiv \mathrm{Sh}_r^{\mathrm{sym}}(B)$. Since every infinite set A is saturated, $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is saturated.

Fact 2.6 [Halevi et al. 2022, Fact 2.6; Erdős and Hajnal 1968, proof of Theorem 2]. *Let $1 \leq r < \omega$ be a natural number and μ be an infinite cardinal. Then*

$$\chi(\mathrm{Sh}_r^{\mathrm{sym}}(\beth_{r-1}(\mu))) \leq \mu,$$
$$\chi(\mathrm{Sh}_r(\beth_{r-1}(\mu)^+)) \geq \mu^+.$$

Remark 2.7. Note that it follows that $\chi(\mathrm{Sh}_r(\beth_{r-1}(\mu))) \leq \mu$.

Lemma 2.8. *If A is infinite then $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is a saturated model of $\mathrm{Th}(\mathrm{Sh}_r^{\mathrm{sym}}(\omega))$ of cardinality $|A|$.*

Proof. This follows easily from the fact that $(A, =)$ is saturated and that $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is definable in $(A, =)$. □

We prove two easy results on the theory of the shift graphs.

Lemma 2.9. *Every model of $T = \mathrm{Th}(\mathrm{Sh}_r(\omega))$ (of cardinality λ) can be embedded as an induced subgraph of $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ for some infinite set A (of cardinality λ).*

Proof. Since $\mathrm{Sh}_r(\omega)$ is an induced subgraph of $\mathrm{Sh}_r^{\mathrm{sym}}(\omega)$, the former satisfies the universal theory of $\mathrm{Sh}_r^{\mathrm{sym}}(\omega)$. Thus every model M of T can be embedded as an induced subgraph of a model of $\mathrm{Th}(\mathrm{Sh}_r^{\mathrm{sym}}(\omega))$. By Lemma 2.8, and the universality of saturated models, M can be embedded as an induced subgraph of $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ for some infinite set A of cardinality $|M|$. □

Lemma 2.10. *For any cardinal μ, the following holds for the graph $(\mathrm{Sh}_2(\mu), E)$:*

(1) *Its complete theory is not stable.*

(2) *Its graph relation is stable.*

(3) *It is triangle-free.*

Proof. (1) For any $1 < n < \omega$, let X_n be the definable set

$$\{x \in \mathrm{Sh}_2(\mu) : (x \mathrel{E} (0, n)) \wedge \neg(x \mathrel{E} (0, 1))\}.$$

It is easily seen that $X_n = \{(n, k) : k > n\}$. For any $0 < m < \omega$, let

$$Y_m = \{x \in \mathrm{Sh}_2(\mu) : (x \mathrel{E} (m, m+1)) \wedge \neg(x \mathrel{E} (m-1, m+1))\};$$

it is easily seen that $Y_m = \{(k, m) : k < m\}$.

For any pair of natural numbers (n, m) with $n > 1$ and $n < m$, let $\psi_{n,m}(x, y)$ be the formula (with parameters n, m)

$$\exists z(x \in X_n \wedge y \in Y_m \wedge (z\, E\, x) \wedge (z\, E\, y)).$$

For any l with $l < m - n$, let $a_l = (n, n+l) \in X_n$, and for any $0 < k < m - n$ let $b_k = (n+k, m) \in Y_m$. It is easily checked that $\psi_{n,m}(a_l, b_k)$ if and only if $l < k$. Since all the $\psi_{n,m}$ are uniformly definable with parameters, by compactness we get that the theory of $\mathrm{Sh}_2(\mu)$ is not stable.

(2) One can either see directly that the graph relation is stable, or note that since $\mathrm{Sh}_2(\mu)$ is an induced subgraph of the stable graph $\mathrm{Sh}_2^{\mathrm{sym}}(\mu)$, its edge relation must also be stable.

(3) Suppose $(a, b) E (c, d) E (e, f) E (a, b)$. Note also $(c, d) E (a, b) E (e, f) E (c, d)$. Hence, without loss of generality $b = c$. It follows easily that $e = d$. So $a < b < d < f$. Now either $a = f$ or $b = e = d$, so either way we get a contradiction. □

Remark 2.11. (1) Since $\mathrm{Sh}_2(\mu)$ is definable in $(\mu, <)$, it has NIP (not the independence property (IP); see, e.g., [Simon 2015]).

(2) One can also note that $\mathrm{Sh}_4^{\mathrm{sym}}(\mu)$ is triangle-free.

Every graph is a (not necessarily induced) subgraph of a stable graph (e.g., a large enough complete graph). On the other hand, every shift graph $\mathrm{Sh}_n(\mu)$ is an induced subgraph of a stable graph (the symmetric shift graph). This raises the following:

Question 2.12. Is every graph with a stable edge relation an induced subgraph of a stable graph?

3. Infinite cliques

In this section we prove the main results of the paper.

Simple graphs. We start with the following technical result.

Lemma 3.1. *Let M be some structure, in a language L, and assume that $T = \mathrm{Th}(M)$ is simple. Let M' be some expansion of M to a language $L' \supseteq L$. Let $<$ be a linear order on the universe of M and $\alpha, \beta \in M$ elements satisfying that*

(1) $(\mathrm{dcl}_{L'}(\alpha), <)$ *and* $(\mathrm{dcl}_{L'}(\beta), <)$ *are well-orders, and*

(2) *for any* $\gamma \in \mathrm{dcl}_{L'}(\alpha) \cup \mathrm{dcl}_{L'}(\beta)$, $\gamma \downarrow_{\mathrm{dcl}_{L'}(\gamma) \cap \{\varepsilon \in M : \varepsilon < \gamma\}} \{\varepsilon \in M : \varepsilon < \gamma\}$.

Then

$$\mathrm{dcl}_{L'}(\alpha) \underset{\mathrm{dcl}_{L'}(\alpha) \cap \mathrm{dcl}_{L'}(\beta)}{\downarrow} \mathrm{dcl}_{L'}(\beta).$$

Proof. Let $\alpha, \beta \in M$ be elements as in the statement, and let $\boldsymbol{a}^\alpha = \mathrm{dcl}_{L'}(\alpha)$ and $\boldsymbol{a}^\beta = \mathrm{dcl}_{L'}(\beta)$. Set $\Omega = \boldsymbol{a}^\alpha \cup \boldsymbol{a}^\beta$ and for any $\gamma \in \Omega$ let $\Omega_{<\gamma} = \{\varepsilon \in \Omega : \varepsilon < \gamma\}$. Since $(\Omega, <)$ is a finite union of well-ordered sets, it is also well-ordered.

Claim 3.1.1. *For $\gamma \in \Omega$, if*

$$\boldsymbol{a}^\alpha \cap \Omega_{<\gamma} \underset{\boldsymbol{a}^\alpha \cap \boldsymbol{a}^\beta \cap \Omega_{<\gamma}}{\perp} \boldsymbol{a}^\beta \cap \Omega_{<\gamma}$$

then

$$\boldsymbol{a}^\alpha \cap \Omega_{\leq\gamma} \underset{\boldsymbol{a}^\alpha \cap \boldsymbol{a}^\beta \cap \Omega_{\leq\gamma}}{\perp} \boldsymbol{a}^\beta \cap \Omega_{\leq\gamma}.$$

Proof. By symmetry, we deal with the case $\gamma \in \boldsymbol{a}^\alpha$. We first prove that

$$\boldsymbol{a}^\alpha \cap \Omega_{\leq\gamma} \underset{\boldsymbol{a}^\alpha \cap \boldsymbol{a}^\beta \cap \Omega_{<\gamma}}{\perp} \boldsymbol{a}^\beta \cap \Omega_{<\gamma}. \tag{3-1}$$

By hypothesis (2), $\gamma \perp_{\mathrm{dcl}_{L'}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}} \{\varepsilon : \varepsilon < \gamma\}$. Since

$$\mathrm{dcl}_{L'}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\} \subseteq \boldsymbol{a}^\alpha \cap \Omega_{<\gamma} \subseteq \{\varepsilon : \varepsilon < \gamma\},$$

we get that $\gamma \perp_{\boldsymbol{a}^\alpha \cap \Omega_{<\gamma}} \{\varepsilon : \varepsilon < \gamma\}$, so $\gamma \perp_{\boldsymbol{a}^\alpha \cap \Omega_{<\gamma}} \boldsymbol{a}^\beta \cap \Omega_{<\gamma}$ and

$$\boldsymbol{a}^\alpha \cap \Omega_{\leq\gamma} \underset{\boldsymbol{a}^\alpha \cap \Omega_{<\gamma}}{\perp} \boldsymbol{a}^\beta \cap \Omega_{<\gamma}.$$

By assumption and transitivity, we conclude (3-1).

If $\gamma \notin \boldsymbol{a}^\beta$ then we are done; so assume that $\gamma \in \boldsymbol{a}^\beta$ as well. In this case, by properties of forking we get that $\boldsymbol{a}^\alpha \cap \Omega_{\leq\gamma} \perp_{\boldsymbol{a}^\alpha \cap \boldsymbol{a}^\beta \cap \Omega_{\leq\gamma}} \boldsymbol{a}^\beta \cap \Omega_{\leq\gamma}$, which is what we wanted to prove. □

The proof now follows by induction: the successor step is Claim 3.1.1 and the limit case follows since forking is witnessed by a formula. □

We move to the main result of this section on simplicity.

Proposition 3.2. *Let T be a complete simple theory of graphs in the language of graphs $L = \{E\}$. Let μ be an infinite cardinal and $G = (V, E) \vDash T$ with $|G| \leq 2^\mu$. If $\chi(G) \geq \mu^+$ then there exists $G \equiv G'$ with $|G'| = \mu^+$ that contains a clique of cardinality μ^+.*

Proof. By compactness and Löwenheim–Skolem, it is sufficient to show that G contains arbitrary large finite cliques. Furthermore, by passing to an elementary extension, we may assume that $|G| = 2^\mu$. Additionally, after renaming elements, we may assume that $V = 2^\mu$. Assume that $\chi(G) \geq \mu^+$.

As T is simple, for every nonzero $\alpha \in V$, the type $\mathrm{tp}(\alpha/\{\beta : \beta < \alpha\})$ does not fork over some nonempty countable subset $A_\alpha \subseteq \{\beta : \beta < \alpha\}$. Enumerate A_α as $\langle c_{\alpha,n} : n < \omega\rangle$, possibly with repetitions. Let F_n be the function mapping a nonzero $\alpha \in V$ to $c_{\alpha,n}$ (and $F_n(0) = 0$).

Let $L' \supseteq L \cup \{F_n : n < \omega\} \cup \{<\}$ be a language containing Skolem functions and G' an expansion of G to L' with Skolem functions such that the F^n are interpreted as above. Let $T' = \mathrm{Th}(G')$. As usual $\mathrm{Sk}(-)$ denotes the Skolem hull in L', i.e., $\mathrm{Sk}(C)$ is the structure generated by C in G'. Unless specified otherwise, whatever is done below is done in L.

By forking base monotonicity and the choice of the functions F_n, for any $\alpha \in V$, $\mathrm{tp}(\alpha/\{\beta : \beta < \alpha\})$ does not fork over $\mathrm{Sk}(\{\alpha\}) \cap \{\beta : \beta < \alpha\}$.

Let Δ be the collection of all of formulas in one variable x over $\varnothing$ in L' and let $\Delta = \bigcup_{l<\omega} \Delta_l$ be an increasing union of finite subsets. Let $\langle t_i(x) : i < \omega\rangle$ be some enumeration of all the terms in L', with $t_0(x) = x$. Thus

$$\mathrm{Sk}(\{\alpha\}) = \{t_i(\alpha) : i < \omega\}.$$

Enumerate the 2^μ functions from μ to $\{0, 1\}$ by $\langle \eta_\alpha \rangle_{\alpha<2^\mu}$, without repetitions. For any finite subset $u \subseteq \mu$ and $n < \omega$ we define a relation $R_{u,n}$ on

$$\mathrm{Dom}(R_{u,n}) := \big\{\alpha \in V = 2^\mu : \eta_{t_i(\alpha)} \restriction u \neq \eta_{t_j(\alpha)} \restriction u \quad \text{for all } i < j < n \text{ such that } t_i(\alpha) \neq t_j(\alpha)\big\}.$$

Let $\alpha\ R_{u,n}\ \beta$ (for $\alpha, \beta \in \mathrm{Dom}(R_{u,n})$) if

(1) for all $j < n$, $\eta_{t_j(\alpha)} \restriction u = \eta_{t_j(\beta)} \restriction u$ and

(2) $\mathrm{tp}_{\Delta_n}(\alpha) = \mathrm{tp}_{\Delta_n}(\beta)$.

Note that $R_{u,n}$ is an equivalence relation on $\mathrm{Dom}(R_{u,n})$.

Claim 3.2.1. *There exists $\alpha \in V$ such that for every finite subset $u \subseteq \mu$ and $n < \omega$, if $\alpha \in \mathrm{Dom}(R_{u,n})$ then*

$$\exists \beta \in [\alpha]_{R_{u,n}} (\alpha\ E\ \beta).$$

Proof. Note that each $R_{u,n}$ has only finitely many classes on $\mathrm{Dom}(R_{u,n})$. Assume towards a contradiction that for every $\alpha \in V$ we can find some $u_\alpha \subseteq \mu$ and $n_\alpha < \omega$ that satisfy the negation of the statement. Now map α to $(u_\alpha, n_\alpha, [\alpha]_{R_{u_\alpha,n_\alpha}})$. This is easily a legal coloring of V by μ colors, contradicting $\chi(G) \geq \mu^+$. □

Let $\alpha \in V$ be as supplied by Claim 3.2.1. For any $n < \omega$ let

$$u_n = \big\{\min\{\varepsilon < \mu : \eta_{t_i(\alpha)}(\varepsilon) \neq \eta_{t_j(\alpha)}(\varepsilon)\} : i < j < n \text{ such that } t_i(\alpha) \neq t_j(\alpha)\big\};$$

it is a finite subset of μ. Easily, $\alpha \in \mathrm{Dom}(R_{u_n,n})$ for all $n < \omega$. For $n < \omega$ let β_n be the element given by Claim 3.2.1 for (u_n, n) (and α).

Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω and let $\widetilde{G} = (G')^\omega/\mathcal{U}$ be the corresponding ultrapower in the language L'. We let $\widetilde{G} = (\widetilde{V}, E)$. Set $\widetilde{\alpha} = [\alpha]_{\mathcal{U}}$, $\widetilde{\beta} = [\beta_n]_{\mathcal{U}}$; so $\widetilde{\alpha}\, E\, \widetilde{\beta}$. We make some observations. By definition of $\widetilde{\alpha}$, $\mathrm{tp}_{L'}(\widetilde{\alpha}) = \mathrm{tp}_{L'}(\alpha)$ and by definition of the relations $R_{u_n,n}$, this type is also equal to $\mathrm{tp}_{L'}(\widetilde{\beta})$.

Set $\boldsymbol{a}^{\widetilde{\alpha}} = \langle t_i(\widetilde{\alpha}) : i < \omega\rangle$ (and likewise for α), and $\boldsymbol{a}^{\widetilde{\beta}} = \langle t_i(\widetilde{\beta}) : i < \omega\rangle$.

Claim 3.2.2. *For all* $i, j < \omega$, *if* $t_i(\widetilde{\alpha}) = t_j(\widetilde{\beta})$ *then* $t_i(\widetilde{\alpha}) = t_i(\widetilde{\beta})$.

Proof. First note that for all $i < j < \omega$, if $t_i(\widetilde{\alpha}) = t_j(\widetilde{\alpha})$ then the same holds for $\widetilde{\beta}$ and vice versa.

Assume the claim is not true and that $i < j$. If $t_i(\widetilde{\alpha}) = t_j(\widetilde{\alpha})$ then by the first paragraph we are done. So assume not. We can find some n large enough for which $t_i(\beta_n) \neq t_j(\beta_n)$, $t_i(\alpha) = t_j(\beta_n)$ and $i < j < n$.

As $\beta_n \, R_{u_n,n} \, \alpha$, $\eta_{t_i(\beta_n)} \restriction u_n = \eta_{t_i(\alpha)} \restriction u_n$ so by assumption $\eta_{t_i(\beta_n)} \restriction u_n = \eta_{t_j(\beta_n)} \restriction u_n$, contradicting the definition of $\mathrm{Dom}(R_{u_n,n})$. □

As $(\boldsymbol{a}^\alpha, <)$ is a well-order (as a substructure of 2^μ) so are $(\boldsymbol{a}^{\widetilde{\alpha}}, <)$ and $(\boldsymbol{a}^{\widetilde{\beta}}, <)$. With the aim of applying Lemma 3.1 with $M = \widetilde{G}$, we prove the following.

Claim 3.2.3. *For any* $\gamma \in \boldsymbol{a}^{\widetilde{\alpha}} \cup \boldsymbol{a}^{\widetilde{\beta}}$, $\mathrm{tp}_L(\gamma/\{\varepsilon \in \widetilde{G} : \varepsilon < \gamma\})$ *does not fork over* $\mathrm{Sk}(\gamma) \cap \{\varepsilon \in \widetilde{G} : \varepsilon < \gamma\}$.

Proof. Assume that $\gamma \in \boldsymbol{a}^{\widetilde{\alpha}}$. The proof only uses the fact that $\mathrm{tp}_{L'}(\widetilde{\alpha}) = \mathrm{tp}_{L'}(\alpha)$. Hence, the same proof also works for $\gamma \in \boldsymbol{a}^{\widetilde{\beta}}$. Recall that $\mathrm{tp}_{L'}(\widetilde{\alpha}) = \mathrm{tp}_{L'}(\widetilde{\beta})$. Let $t(x)$ be a term (in L') for which $\gamma = t(\widetilde{\alpha})$. We get that for $\gamma' := t(\alpha)$, $\mathrm{tp}_{L'}(\gamma) = \mathrm{tp}_{L'}(\gamma')$.

If $\mathrm{tp}_L(\gamma/\{\varepsilon : \varepsilon < \gamma\})$ forks over $\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}$ then by symmetry of forking there is a tuple c of elements from $\{\varepsilon : \varepsilon < \gamma\}$ such that

$$\mathrm{tp}_L(c/\{\gamma\} \cup (\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}))$$

forks over $\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}$. Let $\varphi(y, z)$ be a formula over $\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}$, satisfied by (c, γ), such that $\varphi(y, \gamma)$ forks over $\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}$. Since $\mathrm{tp}_{L'}(\gamma) = \mathrm{tp}_{L'}(\gamma')$, $\varphi(y, \gamma')$ forks over $\mathrm{Sk}(\{\gamma'\}) \cap \{\varepsilon : \varepsilon < \gamma'\}$.

On the other hand, we know that $\exists y < z \, \varphi(y, z)$ is in $\mathrm{tp}_{L'}(\gamma/\, \mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\})$ so $\exists y < z \, \varphi(y, z)$ is in $\mathrm{tp}_{L'}(\gamma'/\, \mathrm{Sk}(\{\gamma'\}) \cap \{\varepsilon : \varepsilon < \gamma'\})$. Consequently, there exists a tuple c' of elements in $\{\varepsilon \in V : \varepsilon < \gamma'\}$ for which $\varphi(c', \gamma')$ holds, contradicting the fact that $\mathrm{tp}_L(\gamma'/\{\varepsilon : \varepsilon < \gamma'\})$ does not fork over $\mathrm{Sk}(\{\gamma'\}) \cap \{\varepsilon : \varepsilon < \gamma'\}$ (by symmetry of nonforking). □

Recall that $\mathrm{tp}_{L'}(\widetilde{\alpha}) = \mathrm{tp}_{L'}(\widetilde{\beta})$. Also, $\boldsymbol{a}^{\widetilde{\alpha}}$ and $\boldsymbol{a}^{\widetilde{\beta}}$ enumerate elementary substructures of $\widetilde{G} \restriction L$. Setting $\boldsymbol{a}^0 = \langle t_i(\widetilde{\alpha}) : t_i(\widetilde{\alpha}) = t_i(\widetilde{\beta}), i < \omega \rangle$ by Claim 3.2.2, we have $\mathrm{tp}_L(\boldsymbol{a}^{\widetilde{\alpha}}/\boldsymbol{a}^0) = \mathrm{tp}_L(\boldsymbol{a}^{\widetilde{\beta}}/\boldsymbol{a}^0)$. Note that since $\boldsymbol{a}^0$ is closed under the chosen Skolem functions, it enumerates an elementary substructure as well.

By Claim 3.2.3 and Lemma 3.1 (which we are allowed to use since $(\boldsymbol{a}^{\widetilde{\alpha}}, <)$ and $(\boldsymbol{a}^{\widetilde{\beta}}, <)$ are well-orders), $\boldsymbol{a}^{\widetilde{\alpha}} \perp_{\boldsymbol{a}^0} \boldsymbol{a}^{\widetilde{\beta}}$. So by the independence theorem for simple theories [Tent and Ziegler 2012, Lemma 7.4.8], we may find an indiscernible sequence starting with $\boldsymbol{a}^{\widetilde{\alpha}}$ and $\boldsymbol{a}^{\widetilde{\beta}}$ (in some elementary extension). Thus, as $\widetilde{\alpha} \; E \; \widetilde{\beta}$ we can find an infinite clique in that elementary extension. □

Given Proposition 3.2, it is a natural question to ask whether we can necessarily find an infinite clique already in G itself. The following proposition shows that you cannot hope for an uncountable clique in general.

Proposition 3.3. *Let μ be an infinite cardinal. Any graph of cardinality μ without an uncountable clique can be embedded in a random graph of cardinality μ with no uncountable clique.*

Remark 3.4. Every random graph has a simple theory and contains a countable infinite clique, so countable cliques cannot be avoided.

Proof of Proposition 3.3. Let μ be an infinite cardinal and let $G_0 = (V_0, E_0)$ be the given graph. Without loss of generality, $V_0 = \{2\alpha : \alpha < \mu\}$.

Let $\langle u_\gamma \rangle_{\gamma<\mu}$ enumerate all finite subsets of μ such that each finite subset occurs μ times. In particular, for any $\gamma < \alpha < \mu$ there is some $\alpha < \gamma' < \mu$ for which $u_\gamma = u_{\gamma'}$.

Let $V = \mu$. We define a new graph $G = (V, E)$ extending G_0 such that for $\alpha < \beta < \mu$, if $\beta = 2\gamma + 1$ for some $\gamma < \mu$ and $\alpha \in u_\gamma$ we let $\{\alpha, \beta\}$ be an edge.

We claim that G has the desired properties.

We note that G is a random graph. Indeed, let $X = \{\alpha_1, \dots, \alpha_n\}$, $Y = \{\beta_1, \dots, \beta_n\}$ be two disjoint sets of vertices. Let $\gamma < \mu$ be larger than the α_i and β_j and satisfying that $u_\gamma = \{\alpha_1, \dots, \alpha_n\}$. Then $2\gamma + 1$ is connected to each of the α_i and to none of the β_j.

Finally, assume that G contains a clique C of cardinality $\aleph_1$. By assumption, $C \cap V_0$ must be at most countable, so there must be a clique of cardinality $\aleph_1$ consisting of odd ordinals in μ. Let U be the first ω of those and let $2\gamma + 1 \in C$ be an ordinal larger than any of the ordinals in U. But $2\gamma + 1$ can only be connected to finitely many vertices which are smaller, a contradiction. □

Corollary 3.5. *For any infinite cardinal μ there exists a random graph of cardinality and chromatic number μ with no uncountable clique.*

Proof. Apply Proposition 3.3 to any graph G_0 of cardinality μ with no infinite clique and $\chi(G_0) = \mu$ (for example the triangle-free graph from [Erdős and Rado 1960]).

Let G be the graph supplied by the proposition. Since G_0 embeds into G and $|G| = \mu$ it follows that $\chi(G) = \mu$. □

Question 3.6. Is there a theory of simple graphs such that for every cardinal μ we can find a graph of cardinality and chromatic number μ with no infinite cliques at all?

Question 3.7. Does an analog of Proposition 3.2 hold for other model-theoretic tame graphs, such as NSOP_1 and NIP?

Graphs with stable edge relation. Before getting into the main result we prove a technical lemma which may be interesting on its own.

Lemma 3.8. *Let T be a first-order theory, μ an infinite cardinal and $\varphi(x, y)$ a stable formula. Let $M \vDash T$ with $|M| = \mu^+$ and assume that M is an increasing continuous union of elementary substructures $\langle M_\alpha \rangle_{\alpha<\mu^+}$ each of cardinality at most μ.*

Let $\psi(y, z)$ be a uniform definition of φ-types (as in Fact 2.1). For any $a \in M$ and $\alpha < \mu^+$, let $c_{a,\alpha} \in M_\alpha$ be such that $\psi(y, c_{a,\alpha})$ defines $\operatorname{tp}_\varphi(a/M_\alpha)$. Then there exists a club $\mathcal{C} \subseteq \mu^+$ of limit ordinals satisfying that for any $\delta \in \mathcal{C}$ and $a \in M \setminus M_\delta$:

For any β such that $\delta < \beta < \mu^+$ there is $b \in M \setminus M_\beta$ for which $\operatorname{tp}_\varphi(b/M_\beta)$ is definable by $\psi(y, c_{a,\delta})$. $(\dagger)_{a,\delta}$

Proof. Let F be the set of all limit ordinals $\delta < \mu^+$ such that for any $a \in M \setminus M_\delta$, $(\dagger)_{a,\delta}$ holds. It is enough to show that F contains a club $\mathcal{C}$.

Suppose F does not contain a club. Then $\mu^+ \setminus F$ is stationary. Let $\bar{F}$ be the set of all limit ordinals in $\mu^+ \setminus F$; this set is also stationary as an intersection of a club with a stationary set. By definition, for any limit ordinal $\delta \in \bar{F}$ there is some $a_\delta \in M \setminus M_\delta$ and $\beta_\delta > \delta$ such that for any $b \in M \setminus M_{\beta_\delta}$, $\operatorname{tp}_\varphi(b/M_{\beta_\delta})$ is not defined by $\psi(y, c_{a_\delta,\delta})$.

For any limit ordinal $\delta \in \bar{F}$, let $f(\delta)$ be the minimal ordinal ε for which $c_{a_\delta,\delta} \in M_\varepsilon$. Note that as $c_{a_\delta,\delta}$ is a finite tuple and δ is a limit ordinal, necessarily $f(\delta) < \delta$. By Fodor's lemma [Jech 2003, Theorem 8.7] there exists a stationary subset $S \subseteq \bar{F}$ and $\varepsilon < \mu^+$ for which $f(\delta) = \varepsilon$ for any $\delta \in S$.

By definition, $c_{a_\delta,\delta} \in M_\varepsilon$ for any $\delta \in S$. As $|M_\varepsilon| \le \mu$, by the pigeonhole principle there is an unbounded subset $S' \subseteq S$ (i.e., of cardinality μ^+) for which $c := c_{a_{\delta_1},\delta_1} = c_{a_{\delta_2},\delta_2}$ for any $\delta_1, \delta_2 \in S'$.

Now, pick any $\delta \in S'$. By our assumption there is some $\beta_\delta > \delta$ such that for any $b \in M \setminus M_{\beta_\delta}$, $\operatorname{tp}_\varphi(b/M_{\beta_\delta})$ is not defined by $\psi(y, c_{a_\delta,\delta}) = \psi(y, c)$.

Let $\beta' > \beta_\delta$ be an element in S' (it is unbounded) and let $a_{\beta'}$ be the corresponding element. So $a_{\beta'} \notin M_{\beta'}$ and in particular $a_{\beta'} \notin M_{\beta_\delta}$, and thus $\operatorname{tp}_\varphi(a_{\beta'}/M_{\beta_\delta})$ is not defined by $\psi(y, c_{a_\delta,\delta}) = \psi(y, c)$. On the other hand, by choice of S', $\operatorname{tp}_\varphi(a_{\beta'}/M_{\beta'})$ and thus also $\operatorname{tp}_\varphi(a_{\beta'}/M_{\beta_\delta})$ is defined by $\psi(y, c)$, a contradiction. □

We phrase forking symmetry for a stable formula in a form useful to us.

Fact 3.9 (Harrington). *Let T be a first-order theory with monster model $\mathbb{U}$. Let $\varphi(x, y)$ be a stable symmetric[2] formula and $\psi(y, z)$ a formula uniformly defining φ-types. For any two small models N_1 and N_2 and elements $a, b \in \mathbb{U}$, if $\psi(y, c_a)$ defines $\operatorname{tp}_\varphi(a/N_1)$, $\psi(y, c_b)$ defines $\operatorname{tp}_\varphi(b/N_2)$ and $c_a \in N_2$, $c_b \in N_1$ then*

$$\psi(a, c_b) \iff \psi(b, c_a).$$

[2] A formula $\varphi(x, y)$ is *symmetric* if x and y have the same sort and $\varphi(x, y) \equiv \varphi(y, x)$.

Proof. Let $p(x) = \mathrm{tp}_\varphi(a/N_1)$ and $q(y) = \mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/N_2)$. Note that as φ is symmetric, $\psi(x, c_b)$ defines $q(y)$.

Let $\widetilde{p} \supseteq p$ be the global φ-type that $\psi(y, c_a)$ defines and $\widetilde{q} \supseteq q$ be the global φ-type that $\psi(x, c_b)$ defines. By [Tent and Ziegler 2012, Lemma 8.3.4] and using Fact 2.1, $\widetilde{p} \vdash \psi(x, c_b)$ if and only if $\widetilde{q} \vdash \psi(y, c_a)$; but as $c_a \in N_2$ and $c_b \in N_1$, we conclude. □

The following is due to [Engelking and Karłowicz 1965]; see also [Rinot 2012] for a streamlined presentation.

Fact 3.10. *For cardinals $\kappa \leq \lambda \leq \mu \leq 2^\lambda$ the following are equivalent*:

(1) $\lambda^{<\kappa} = \lambda$.

(2) *There exists a collection of functions $\langle f_i : \mu \to \lambda \rangle_{i<\lambda}$ such that for every $X \in [\mu]^{<\kappa}$ and every function $f : X \to \lambda$, there exists some $i < \lambda$ with $f \subseteq f_i$.*

We now prove the main result of this section.

Proposition 3.11. *Let $L = \{E\}$ be the language of graphs and T be an L-theory specifying that E is a symmetric and irreflexive stable relation. For any infinite cardinal μ and $G \vDash T$ with $|G| = \mu^+$, if $\chi(G) \geq \mu^+$ then G contains an infinite clique of cardinality μ^+.*

Proof. For ease of notation, write $\varphi(x, y) = E(x, y)$ and let $\psi(y, z)$ be a uniform definition for φ-types (as in Fact 2.1). Assume that $\chi(G) \geq \mu^+$.

By Lemma 3.8, there exists an increasing continuous family of elementary substructures $\langle G_\alpha \prec G : 0 < \alpha < \mu^+ \rangle$, with $|G_\alpha| = \mu$ and $\bigcup_{0<\alpha<\mu^+} G_\alpha = G$, satisfying:

$$\text{For any } 0 < \delta < \mu^+ \text{ and } a \in G \setminus G_\delta \text{ there exists } c_{a,\delta} \in G_\delta \text{ such that } \psi(y, c_{a,\delta}) \text{ defines } \mathrm{tp}_\varphi(a/G_\delta) \text{ satisfying that for any } \beta \text{ with } \delta < \beta < \mu^+ \text{ there is } b \notin G_\beta \text{ for which } \mathrm{tp}_\varphi(b/G_\beta) \text{ is defined by } \psi(y, c_{a,\delta}). \tag{†}$$

Indeed, Lemma 3.8 supplies a club $\mathcal{C} \subseteq \mu^+$ for which the above holds. Since the order type of $\mathcal{C}$ is μ^+, by restricting to the models indexed by ordinals in $\mathcal{C}$ we get the desired sequence of elementary substructures.

Set $G_0 = \varnothing$. For any $a \in G$ let $\alpha_0^a < \mu^+$ be minimal such that $a \in G_{\alpha_0+1}$. Let $c_0^a = c_{a,\alpha_0^a} \in G_{\alpha_0^a}$; in particular, $\psi(y, c_0^a)$ defines $\mathrm{tp}_\varphi(a/G_{\alpha_0^a})$. Let $\alpha_1^a < \alpha_0^a$ be such that $c_0^a \in G_{\alpha_1^a+1} \setminus G_{\alpha_1^a}$. Let $c_1^a = c_{a,\alpha_1^a}$. Likewise we continue and find a sequence $\langle (c_{k-1}^a, \alpha_k^a) : 1 \leq k \leq n_a \rangle$ satisfying

- $\alpha_k^a < \alpha_{k-1}^a$,
- $c_{k-1}^a = c_{a,\alpha_{k-1}^a}$ — and in particular $\psi(y, c_{k-1}^a)$ defines $\mathrm{tp}_\varphi(a/G_{\alpha_{k-1}^a})$,
- $c_{k-1}^a \in G_{\alpha_k^a+1} \setminus G_{\alpha_k^a}$,
- $\alpha_{n_a}^a = 0$.

Note that $\alpha_0^a = 0$ if and only if $n_a = 0$ and then there are no c's.

We now define a coloring. By Engelking–Karłowicz (Fact 3.10, applied to $\omega \le \mu \le \mu^+ \le 2^\mu$), there exists a family of functions $\{g_\beta : \mu^+ \to \mu \mid \beta < \mu\}$ satisfying that for any finite subset $X \subseteq \mu^+$ and every function $f : X \to \mu$ there is some function g_β, $\beta < \mu$, with $f \subseteq g_\beta$.

For any such $a \in G$ define a function $f_a : \{\alpha_0^a, \dots, \alpha_{n_a}^a\} \to \omega$ by setting $f_a(\alpha_i^a) = i$. Let $\beta^a < \mu$ be minimal such that $f_a \subseteq g_{\beta^a}$.

For any $\alpha < \mu^+$, let $s_\alpha : G_\alpha \to \mu$ be a bijection. For simplicity, we also denote by s_α the induced bijection between G_α^n and μ^n. Let $Y = \aleph_0 \times \mu^2 \times \mu^{<\omega}$; note that $|Y| = \mu$.

Let $\phi : G \to Y$ be a function mapping $a \in G$ to

$$(n_a, \beta^a, s_{\alpha_0^a+1}(a), (s_{\alpha_i^a+1}(c_{i-1}^a))_{1 \le i \le n_a}).$$

Since ϕ cannot be a legal coloring there exist distinct elements $a, b \in G$ satisfying that $\phi(a) = \phi(b)$ and $\varphi(a, b)$ holds, i.e., $a \mathrel{E} b$.

Without loss of generality, assume that $\alpha_0^a \ge \alpha_0^b$. Note that necessarily, $\alpha_0^a > \alpha_0^b$, since otherwise, as $s_{\alpha_0^a+1}(a) = s_{\alpha_0^b+1}(b)$ we conclude that $a = b$, a contradiction. In particular, $n_a > 0$ and so also $n_b = n_a > 0$.

Also, if $\alpha_i^a = \alpha_j^b$ for some i, j then $i = j$, since by assumption $\beta^a = \beta^b$.

Let $n > 0$ be minimal such that $\alpha_n^a = \alpha_n^b$ (such n exists since this equality holds for $n = n_a > 0$). Consider the (ordered) set $A' = \{\alpha_i^a, \alpha_i^b : i < n\}$. Note that the elements in A' are distinct. Now let $A = A' \cup \{\alpha_n^a = \alpha_n^b\}$.

We call an element $\alpha_i^a \in A$ (with $i > 0$) an *a-pivot* if there exists an element $\alpha_j^b \in A$ with $\alpha_{i-1}^a > \alpha_j^b > \alpha_i^a$ (and likewise a *b-pivot* for $\alpha_i^b \in A$ with $i > 0$). Note that $\alpha_n^a = \alpha_n^b$ is either an a-pivot or a b-pivot.

We prove the following by (downward) induction.

Claim 3.11.1. *For every a-pivot $\alpha_i^a \in A$ with $i > 0$, $\psi(b, c_{i-1}^a)$ holds, and for every b-pivot $\alpha_i^b \in A$ with $i > 0$, $\psi(a, c_{i-1}^b)$ holds.*

Proof. Let $\alpha_{t+1}^a \in A$ be an a-pivot (possibly $t + 1 = n$). Let $v < n$ be minimal with $\alpha_t^a > \alpha_v^b > \alpha_{t+1}^a$; α_v^b is either a b-pivot and $v = s + 1$ for some s or $v = 0$. Assume, first, that α_{s+1}^b is a b-pivot, so $\psi(a, c_s^b)$ holds by the induction hypothesis. As $c_s^b \in G_{\alpha_{s+1}^b+1} \subseteq G_{\alpha_t^a}$ and $c_t^a \in G_{\alpha_{t+1}^a+1} \subseteq G_{\alpha_{s+1}^b} \subseteq G_{\alpha_s^b}$, we can apply Fact 3.9 with $\operatorname{tp}_\varphi(a/G_{\alpha_t^a})$ and $\operatorname{tp}_\varphi(b/G_{\alpha_s^b})$ and conclude that $\psi(b, c_t^a)$ holds.

Now assume that $v = 0$. Since $b \in G_{\alpha_0^b+1} \subseteq G_{\alpha_t^a}$ and $\varphi(a, b)$ holds, then as $\psi(y, c_t^a)$ defines $\operatorname{tp}_\varphi(a/G_{\alpha_t^a})$ we get that $\psi(b, c_t^a)$ holds, as needed.

The case where $\alpha_{t+1}^b \in A$ is a b-pivot is proved similarly. $\square$

Note that $c_{n-1}^a \in G_{\alpha_n^a+1} = G_{\alpha_n^b+1} \ni c_{n-1}^b$. As $\phi(a) = \phi(b)$, we necessarily have $c := c_{n-1}^a = c_{n-1}^b$. If α_n^a is an a-pivot we have that $\psi(a, c)$ holds and if it is a b-pivot then $\psi(b, c)$ holds. Assume the former holds (the proof where the latter holds is identical).

We inductively construct a sequence $(d_\alpha)_{\alpha<\mu^+}$ in V such that

- $\psi(d_\alpha, c)$ holds for all $\alpha < \mu^+$,
- $\varphi(d_\alpha, d_\beta)$ holds for all $\alpha \neq \beta < \mu^+$.

Suppose we constructed $(d_\alpha)_{\alpha<\gamma}$. Let δ be such that $\alpha^a_{n-1} < \delta < \mu^+$ and $d_\alpha \in G_\delta$ for any $\alpha < \gamma$. By (†) there exists $d_\gamma \in G \setminus G_\delta$ such that $\mathrm{tp}_\varphi(d_\gamma/G_\delta)$ is definable by $\psi(y, c)$.

Since $\psi(d_\alpha, c)$ holds for any $\alpha < \gamma$ it follows that $\varphi(d_\gamma, d_\alpha)$ holds (and also $\varphi(d_\alpha, d_\gamma)$ by symmetry). Additionally, $\psi(y, c)$ defines both $\mathrm{tp}_\varphi(a/G_{\alpha^a_{n-1}})$ and $\mathrm{tp}_\varphi(d_\gamma/G_{\alpha^a_{n-1}})$; hence $a \equiv^\varphi_{G_{\alpha^a_{n-1}}} d_\gamma$. As $c \in G_{\alpha^a_{n-1}}$ and $\psi(y, c)$ is equivalent to a boolean combination of instances of φ^{opp}-formulas over $G_{\alpha^a_{n-1}}$, it follows by symmetry of φ that $a \equiv^{\varphi^{\mathrm{opp}}}_{G_{\alpha^a_{n-1}}} d_\gamma$, so $\psi(d_\gamma, c)$ holds as well. □

Given Proposition 3.11 (and Proposition 3.2), it is natural to ask under which conditions on the cardinality of G does the proposition hold. The following example shows that it fails for strong limit cardinals.

Proposition 3.12. *Let λ be a strong limit cardinal.*[3] *There exists a nonstable graph with stable edge relation G of cardinality λ, with $\chi(G) \geq \lambda$, for which we cannot embed arbitrarily large finite cliques. In fact, it is triangle-free.*

Proof. For any $\mu < \lambda$, let $G_\mu = \mathrm{Sh}_2(\beth_1(\mu)^+)$. By Lemma 2.10, it is not stable but has a stable edge relation, and it is triangle-free. By Fact 2.6, $\chi(G_\mu) \geq \mu^+$.

Let $G = \bigoplus_{\mu<\lambda} G_\mu$ be the direct sum of all of these graphs. Thus $\chi(G) \geq \lambda$ and $|G| = \lambda$. The graph G is not stable but its edge relation is stable. On the other hand, since each of the G_μ is triangle-free, we cannot embed arbitrarily large finite cliques into G. □

Remark 3.13. By Remark 2.11(2), we may replace $\mathrm{Sh}_2(\beth_1(\mu)^+)$ by $\mathrm{Sh}^{\mathrm{sym}}_4(\beth_2(\mu)^+)$ and arrive at a stable graph with the prescribed properties.

4. The chromatic spectrum

We rephrase the results of the previous section in a different manner.

For T a theory of graphs and μ an infinite cardinal, let

$$\mathrm{ch}_{\mathrm{T}}(\mu) = \min\{|G| : G \vDash T,\, \chi(G) \geq \mu\}.$$

We employ the convention that $\min \varnothing = \infty$. Note that if $\mathrm{ch}_{\mathrm{T}}(\mu) = \infty$ then $\mathrm{ch}_{\mathrm{T}}(\lambda) = \infty$ for all $\lambda \geq \mu$.

[3]That means that $2^\mu < \lambda$ for any $\mu < \lambda$. Any such cardinal is a limit cardinal. An example is $\beth_\omega(\aleph_0)$.

Remark 4.1. (1) Since for any graph G, $\chi(G) \leq |G|$, we have $\mathrm{ch_T}(\mu) \geq \mu$.

(2) Let $T = \mathrm{Th}(\mathrm{Sh}_k(\omega))$. By Fact 2.6, $\mathrm{ch_T}(\lambda^+) \leq \beth_{k-1}(\lambda)^+$. On the other hand, if towards a contradiction we assume that $\mathrm{ch_T}(\lambda^+) < \beth_{k-1}(\lambda)^+$ then we can find $M \vDash T$ with $\chi(M) \geq \lambda^+$ and $|M| \leq \beth_{k-1}(\lambda)$. By Löwenheim–Skolem we can assume that $|M| = \beth_{k-1}(\lambda)$. By Lemma 2.9, we may embed M into $\mathrm{Sh}_n^{\mathrm{sym}}(\beth_{k-1}(\lambda))$, so $\chi(M) \leq \lambda$ by Fact 2.6, a contradiction. We conclude that $\mathrm{ch_T}(\lambda^+) = \beth_{k-1}(\lambda)^+$.

We can now rephrase the main result of [Halevi et al. 2024] using this function:

Proposition 4.2. *The following are equivalent for a stable theory of graphs T*:

(1) *There are some $G \vDash T$ and a natural number k such that G contains* $\mathrm{Sh}_k(n)$ *for all n.*

(2) $\mathrm{ch_T}(\beth_2(\aleph_0)^+) < \infty$.

(3) *For any cardinal μ*, $\mathrm{ch_T}(\mu) < \infty$.

(4) *For any cardinal μ*, $\mathrm{ch_T}(\mu) < \beth_\omega(\mu)$.

Proof. $\neg(1) \implies \neg(2)$. By the main theorem of [Halevi et al. 2024, Corollary 6.2], $\chi(G) \leq \beth_2(\aleph_0)$ for all $G \vDash T$, i.e., $\mathrm{ch_T}(\beth_2(\aleph_0)^+) = \infty$.

$(4) \implies (3) \implies (2)$. These are easy.

$(1) \implies (4)$. Let μ be an infinite cardinal. By compactness and (1) we can embed $\mathrm{Sh}_k(\beth_{k-1}(\mu)^+)$ in a large enough model of T. So by Löwenheim–Skolem we can find a model G of cardinality $\beth_{k-1}(\mu)^+$ with $\chi(G) \geq \chi(\mathrm{Sh}_k(\beth_{k-1}(\mu)^+)) \geq \mu^+ \geq \mu$ (using Fact 2.6). Consequently, $\mathrm{ch_T}(\mu) \leq \beth_{k-1}(\mu)^+ < \beth_\omega(\mu)$. □

Next, we phrase the results of the previous section for simple graphs and graphs with stable edge relation using $\mathrm{ch_T}$.

Proposition 4.3. *Let T be a theory of graphs and assume that either T is simple or the edge relation is stable. The following are equivalent*:

(1) *T proves the existence of arbitrarily large finite cliques.*

(2) *For any infinite cardinal μ*, $\mathrm{ch_T}(\mu) = \mu$.

(3) *For any infinite cardinal μ*, $\mathrm{ch_T}(\mu^+) = \mu^+$.

(4) *There exists an infinite cardinal μ for which* $\mathrm{ch_T}(\mu^+) = \mu^+$.

If T is simple then they are also equivalent to:

(5) *There exists an infinite cardinal μ with* $\mathrm{ch_T}(\mu^+) \leq 2^\mu$.

(6) *For any infinite cardinal μ*, $\mathrm{ch_T}(\mu^+) \leq 2^\mu$.

Proof. $(1) \implies (2)$. By compactness and Löwenheim–Skolem, there is a model G of cardinality μ which has an infinite clique of cardinality μ. Thus $\chi(G) \geq \mu$, so $\mathrm{ch_T}(\mu) = \mu$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) These are easy.

(4) $\Longrightarrow$ (1). Assume that (4) holds and let μ be an infinite cardinal for which $\mathrm{ch}_{\mathrm{T}}(\mu^+) = \mu^+$. Thus there exists a model G with $\chi(G) \geq \mu^+$ and $|G| = \mu^+$. By Proposition 3.2 for the simple case and Proposition 3.11 for the stable edge relation case, G contains arbitrarily large finite cliques.

Assume that T is simple.

(4) $\Longrightarrow$ (5) $\Longrightarrow$ (1). The first implication is easy and the second uses Proposition 3.2 as above.

(3) $\Longrightarrow$ (6) $\Longrightarrow$ (5). These are easy. □

Is there an analog to Proposition 4.3 for general shift graphs? Here is a reasonable suggestion:

Conjecture 4.4. Suppose that T is stable. If $\mathrm{ch}_{\mathrm{T}}(\mu^+) \leq \beth_{n-1}(\mu)^+$ for all cardinals μ, then for some $m \leq n$, there is an embedding of $\mathrm{Sh}_m(\omega)$ in any ω-saturated model of T.

Note this is exactly Question 1.1 without assuming GCH.

Appendix: An example by Hajnal and Komjáth

In this section we present an example due to Hajnal and Komjáth [1984, Theorem 4]. This is an example of a graph of size continuum whose chromatic number is $\aleph_1$ which does not contain all finite subgraphs of any shift graph $\mathrm{Sh}_n(\omega)$. They gave it as an example refuting Taylor's strong conjecture (which does hold outright for ω-stable graphs with a close relative of it holding for stable theories in general by [Halevi et al. 2022; 2024]). The main goal here is to prove that this example has the independence property (IP) (thus is not stable) and furthermore that its theory is not simple.

Definition A.1. A graph $G = (V, E)$ is called *special* if there exists a partial order $\prec$ on V satisfying that

(1) if $x \mathrel{E} y$ then either $x \prec y$ or $y \succ x$ and

(2) there is no circuit $C = \langle x_0, \dots, x_{n-1}\rangle$, $n \geq 3$, of the form

$$x_0 \prec x_1 \prec \cdots \prec x_{m-1} \prec x_m \succ x_{m+1} \succ \cdots \succ x_{n-1} \succ x_0.$$

Proposition A.2 [Hajnal and Komjáth 1984, Theorem 4]. *Let $G = (V, E)$ be a special graph as witnessed by $\prec$. Then for all $n \geq 1$, G does not contain all the finite subgraphs of* $\mathrm{Sh}_n(\omega)$.

Proof. Assume towards a contradiction that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \geq 1$. If $n = 1$ then it must contain a triangle, obviously contradicting Definition A.1(2). So we assume that $n \geq 2$. Coloring pairs of $<$-increasing tuples, by Ramsey there is some integer r such that if $f : \mathrm{Sh}_n(r) \to G$ is an embedding, there is $A \subseteq r$, $|A| = 2n+1$ such that either $f(a_0, \dots, a_{n-1}) \prec f(a_1, \dots, a_n)$ for all strictly increasing $n+1$-tuples $(a_0, \dots, a_n)$ from A or $f(a_0, \dots, a_{n-1}) \succ f(a_1, \dots, a_n)$ for all strictly increasing $n+1$-tuples $(a_0, \dots, a_n)$ from A.

Assume the former occurs and that for simplicity $A = \{0, \dots, 2n\}$. Then

$$(0, \dots, n-1) \prec (1, \dots, n-1, n+1) \prec \dots \prec (n-1, n+1, \dots, 2n-1) \prec (n+1, \dots, 2n)$$

and

$$(0, \dots, n-1) \prec (1, \dots, n-1, n) \prec \dots \prec (n, n+1, \dots, 2n-1) \prec (n+1, \dots, 2n),$$

which is a contradiction. □

Proposition A.3 [Hajnal and Komjáth 1984, Theorem 4]. *There exists a graph $G = (V, E)$ with $|V| = 2^{\aleph_0}$ satisfying the following properties*:

(1) *G is special and in particular for every $n \geq 1$ it does not contain all finite subgraphs of* $\mathrm{Sh}_n(\omega)$.

(2) $\chi(G) = \aleph_1$.

(3) *G has IP and in particular is not stable* (*in fact*, *the edge relation has IP*).

(4) *G is not simple.*

Proof. Let $\{T_\alpha : \alpha < \aleph_1\}$ be a collection of disjoint sets with $|T_\alpha| = 2^{\aleph_0}$ for each $\alpha < \aleph_1$ and set $V = \bigcup_{\alpha < \aleph_1} T_\alpha$. We define an edge relation on V turning it to a graph satisfying our desired properties.

To define the edge relation we define, for $x \in T_\alpha$ and $\alpha < \aleph_1$,

$$G(x) = \{y \in T_{<\alpha} : x \mathrel{E} y\},$$

where $T_{<\alpha} = \bigcup_{\beta < \alpha} T_\beta$, by induction on α.

If $\alpha = \beta + 1$ we let $G(x) = \varnothing$ for every $x \in T_\alpha$, so assume that α is a limit ordinal and that $G(x)$ has already been defined for $x \in T_{<\alpha}$.

For $\gamma < \alpha$ and $y \in T_{<\alpha}$, we say that *y is γ-covered* if there exists $\alpha_0 < \dots < \alpha_m$ with $\alpha_0 \leq \gamma$, and $x_i \in T_{\alpha_i}$ with $x_m = y$ such that $x_0 \mathrel{E} x_1 \mathrel{E} \cdots \mathrel{E} x_m$. Note that any $y \in T_\gamma$ is γ-covered, as witnessed by the trivial path.

Let $\mathcal{W}_\alpha$ be the collection of all subsets $W \subseteq T_{<\alpha}$ satisfying that

- $W = \{x_n : n < \omega\}$ is countable,
- $x_n \in T_{\alpha_n}$ and $\alpha_n < \alpha_m$ whenever $n < m < \omega$,
- $\sup\{\alpha_n : n < \omega\} = \alpha$,
- no x_n is α_{n-1}-covered for $0 < n < \omega$.

Obviously, $|\mathcal{W}_\alpha| \le 2^{\aleph_0}$; choose some enumeration $\mathcal{W}_\alpha = \{W_\gamma : \gamma < 2^{\aleph_0}\}$ and $T_\alpha = \{t_\gamma : \gamma < 2^{\aleph_0}\}$ and set $G(t_\gamma) = W_\gamma$. That is, for $y \in T_{<\alpha}$ and $x = t_\gamma \in T_\alpha$, $x\ E\ y$ if and only if $y \in W_\gamma$. Let $G = (V, E)$.

We show that G satisfies the properties listed in the statement.

We first show (1). Let $C = \langle x_0, \dots, x_{n-1}\rangle \subseteq V$, $n \ge 3$, be a circuit in G with $x_i \in T_{\alpha_i}$ and

$$\alpha_0 < \alpha_1 < \cdots < \alpha_{m-1} < \alpha_m > \alpha_{m+1} > \cdots > \alpha_{n-1} > \alpha_0$$

for some $0 < m < n-1$. If $\alpha_{m-1} = \alpha_{m+1}$, then as the elements of C are distinct, x_m would be connected to two vertices in $T_{\alpha_{m-1}} = T_{\alpha_{m+1}}$, contradicting our construction. So assume without loss of generality that $\alpha_{m-1} < \alpha_{m+1}$. Thus $x_{m-1}, x_{m+1} \in G(x_m)$ and x_{m+1} is α_{m-1}-covered as witnessed by $\alpha_0 < \alpha_{n-1} < \cdots < \alpha_{m+1}$ and $\alpha_{m-1} \ge \alpha_0$. On the other hand, since $\alpha_{m-1} < \alpha_{m+1}$, we get a contradiction.

We show (2). Towards a contradiction, assume there exists a legal coloring $c : G \to \aleph_0$. We say that a color $n < \omega$ is *small* if there is a $\gamma_n < \aleph_1$ such that every point $x \in V$ with $c(x) = n$ is γ_n-covered; otherwise, call n *large*. For any small $n < \omega$ choose γ_n minimal satisfying the above. Put $\gamma = \sup\{\gamma_n : n \text{ small}\} < \aleph_1$. We note that there exist large n; indeed, take any $x \in T_{\gamma+1}$ and let $n < \omega$ be with $c(x) = n$. Since x is not connected to any $y \in T_{<\gamma+1}$ it cannot be γ-covered.

If $n < \aleph_0$ is large then for every $\alpha < \aleph_1$ there exists $x \in V$ with $c(x) = n$ which is not α-covered.

Let $\langle m_i < \aleph_0 : i < \omega\rangle$ be a sequence (possibly with repetitions), containing all large colors. By definition of m_0 being large, there exist α_0 and $x_0 \in T_{\alpha_0}$ with $c(x_0) = m_0$ which is not γ-covered (so necessarily $\gamma < \alpha_0$). We continue inductively and for any $n < \omega$ we find $x_n \in T_{\alpha_n}$ with $c(x_n) = m_n$ which is not α_{n-1}-covered (so necessarily $\alpha_{n-1} < \alpha_n$).

Suppose $\alpha = \sup\{\alpha_n : n < \omega\}$, which is necessarily a limit ordinal, and let $W = \{x_n : n < \omega\} \subseteq T_{<\alpha}$. By definition, there exists an element $x \in T_\alpha$ with $G(x) = W$. In particular $c(x) \ne m_n$ for all $n < \omega$, so $c(x) = k$ is small, i.e., it is γ_k-covered. But by the definition above, any $y \in G(x) = W$ is not γ-covered, so it cannot be that x is $\gamma_k \le \gamma$-covered, a contradiction.

To show that $\chi(G) = \aleph_1$, note that $c : V \to \aleph_1$ defined by $c(x) = \alpha_x$ for $x \in T_{\alpha_x}$ is a legal coloring.

To show (3), choose for each $n < \omega$ some $x_n \in T_n$. Then, for all unbounded subsets $W \subseteq \omega$ there exists a unique $x \in T_\omega$ with $G(x) = \{x_n : n \in W\}$, giving IP.

Item (4) is a direct consequence of Proposition 3.2 (with $\mu = \aleph_0$). □

Question A.4. Can one find such a counterexample which is stable? Simple?

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References

[Engelking and Karłowicz 1965] R. Engelking and M. Karłowicz, "Some theorems of set theory and their topological consequences", *Fund. Math.* **57** (1965), 275–285. MR

[Erdős and Hajnal 1966] P. Erdős and A. Hajnal, "On chromatic number of graphs and set-systems", *Acta Math. Acad. Sci. Hungar.* **17** (1966), 61–99. MR

[Erdős and Hajnal 1968] P. Erdős and A. Hajnal, "On chromatic number of infinite graphs", pp. 83–98 in *Theory of graphs* (Tihany, Hungary, 1966), edited by P. Erdős and G. Katona, Academic Press, New York, 1968. MR

[Erdős and Rado 1960] P. Erdős and R. Rado, "A construction of graphs without triangles having preassigned order and chromatic number", *J. London Math. Soc.* **35** (1960), 445–448. MR

[Hajnal and Komjáth 1984] A. Hajnal and P. Komjáth, "What must and what need not be contained in a graph of uncountable chromatic number?", *Combinatorica* **4**:1 (1984), 47–52. MR

[Halevi and Kaplan 2023] Y. Halevi and I. Kaplan, "Saturated models for the working model theorist", *Bull. Symb. Log.* **29**:2 (2023), 163–169. MR

[Halevi et al. 2022] Y. Halevi, I. Kaplan, and S. Shelah, "Infinite stable graphs with large chromatic number", *Trans. Amer. Math. Soc.* **375**:3 (2022), 1767–1799. MR

[Halevi et al. 2024] Y. Halevi, I. Kaplan, and S. Shelah, "Infinite stable graphs with large chromatic number, II", *J. Eur. Math. Soc.* (*JEMS*) **26**:12 (2024), 4585–4614. MR

[Jech 2003] T. Jech, *Set theory*, 3rd ed., Springer, 2003. MR

[Komjáth 2011] P. Komjáth, "The chromatic number of infinite graphs — a survey", *Discrete Math.* **311**:15 (2011), 1448–1450. MR

[Pillay 1996] A. Pillay, *Geometric stability theory*, Oxford Logic Guides **32**, Oxford University Press, 1996. MR

[Rinot 2012] A. Rinot, "The Engelking–Karłowicz theorem, and a useful corollary", blog post, 2012, available at https://blog.assafrinot.com/?p=2054.

[Shelah 1978] S. Shelah, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics **92**, North-Holland Publishing Co., Amsterdam, 1978. MR

[Simon 2015] P. Simon, *A guide to NIP theories*, Lecture Notes in Logic **44**, Association for Symbolic Logic, Chicago, 2015. MR

[Tent and Ziegler 2012] K. Tent and M. Ziegler, *A course in model theory*, Lecture Notes in Logic **40**, Association for Symbolic Logic, La Jolla, CA, 2012. MR

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YATIR HALEVI:

ybenarih@campus.haifa.ac.il
Department of Mathematics, University of Haifa, Haifa, Israel

ITAY KAPLAN:

kaplan@math.huji.ac.il
Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel

SAHARON SHELAH:

shelah@math.huji.ac.il
Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel

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