Model Theory

no. 3 vol. 4 2025

$M \not\models T$

msp

Model Theory

msp.org/mt

See inside back cover or msp.org/mt for submission instructions.

Model Theory (ISSN 2832-904X electronic, 2832-9058 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004 is published continuously online.

MT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

Model Theory
Vol. 4, No. 3, 2025
https://doi.org/10.2140/mt.2025.4.203

Classification properties for some ternary structures

Alberto Miguel-Gómez

We provide a model-theoretic classification of the countable homogeneous $\boldsymbol{H}_4$-free 3-hypertournament studied by Cherlin, Hubička, Konečný, and Nešetřil. Our main result is that the theory of this structure is SOP_3, TP_2, and NSOP_4. We offer two proofs of this fact: one is a direct proof, and the other employs part of the abstract machinery recently developed by Mutchnik.

1. Introduction

One of the major current programmes in model theory concerns the extension of methods from stability and simplicity theory to more general classification properties. An example of this program was the development of a structure theory for NSOP_1 theories, mostly carried out in [Kaplan and Ramsey 2020]. The key idea of this development was to generalise the notion of *dividing*, central to the setting of simple theories, to that of *Kim-dividing*, intended to capture the idea of dividing at a generic scale. Kaplan and Ramsey were able to essentially complete the study of NSOP_1 theories using this notion and the derived notion of *Kim-independence*, given by non-Kim-forking over a model.

A natural question is whether we can keep extending these techniques further down the NSOP_n hierarchy introduced by Shelah [1996]. Mutchnik [2022b] has shown that the classes of NSOP_2 and NSOP_1 coincide, and it remains open whether there are any NSOP_3 SOP_1 theories. In contrast, there have been many known examples of natural NSOP_4 SOP_3 theories. The first such examples, already appearing in [Shelah 1996], were of a combinatorial nature.

MSC2020: 03C45.
Keywords: NSOP_4, homogeneous hypertournaments, Conant-independence.

A contribution to the study of NSOP_4 theories was Conant's study [2017] of free amalgamation theories, which includes many previously known examples such as the generic digraphs of Henson [1972]. In a different direction, the Hrushovski constructions of Evans and Wong [2009] provided, in some cases, new examples of strictly NSOP_4 structures. More recently, new algebraic examples of strictly NSOP_4 structures have been found, including the generic c-nilpotent Lie algebras over $\mathbb{F}_p$ for $c > 2$ and p prime of d'Elbée, Müller, Ramsey and Siniora [d'Elbée et al. 2025] and the curve-excluding fields of [Johnson and Ye 2025]. A common feature of all these examples is the presence of an invariant independence relation defined over models satisfying full existence, symmetry, and stationarity.

A more recent contribution towards a systematic theory of independence in the context of NSOP_4 theories is due to Mutchnik. The notion of *Conant-dividing* (also appearing in the literature as *strong Kim-dividing* in [Kaplan et al. 2019] and, later, as *universally Kim-dividing* in [Kruckman and Ramsey 2024]) is studied in depth in [Mutchnik 2022a]. It aims to capture the notion of dividing at a maximally generic scale. A connection is achieved between the related notion of Conant-independence and NSOP_4 by showing that the symmetry of the former implies the latter. Furthermore, in the same paper, Mutchnik generalises the structure theory for NSOP_1 theories relative to a choice of an invariant independence relation satisfying full existence and stationarity over models and gives criteria for identifying Conant-independence as one of these relative notions of Kim-independence.

In this document, we present the first example of a strictly NSOP_4 structure with no known invariant independent relations satisfying full existence and stationarity, namely, the $\boldsymbol{H}_4$*-free* 3*-hypertournament*. More precisely, we prove the following.

Theorem 1.0.1. *The theory of the countable homogeneous* $\boldsymbol{H}_4$*-free* 3*-hypertournament is* SOP_3, TP_2, *and* NSOP_4.

This result contrasts with the situation for the other three known homogeneous 3-hypertournaments, all of which are NTP_2. We also relate our structure to higher-arity versions of stability and NIP; namely, those studied by Terry and Wolf [2021] and by Abd Aldaim, Conant, and Terry [Abd Aldaim et al. 2025] in terms of NFOP_n, and those introduced by Shelah [2014] and further developed by Chernikov, Hempel, Palacín, Takeuchi and others [Chernikov et al. 2019; Hempel 2016; Chernikov and Hempel 2019; 2021] in terms of NIP_n. In particular, we show that the homogeneous $\boldsymbol{H}_4$-free 3-hypertournament is IP_2 and NFOP_3.

There is a long history of interactions between model theory and the combinatorial study of tournaments, i.e., directed graphs (V, E) such that, for all distinct $a, b \in V$, exactly one of $E(a, b)$ and $E(b, a)$ holds. As a result of this investigation, Lachlan [1984] famously classified the countable homogeneous tournaments into three structures up to isomorphism: the generic tournament, the countable dense linear order without endpoints, and the homogeneous local order.

Around the same time, a combinatorial generalisation of tournaments to higher arities was introduced by Assous [1986], which was soon slightly modified into the notion of an *n-hypertournament*. In an attempt to extend Lachlan's classification of homogeneous tournaments, Cherlin studied the homogeneous 4-*constrained* 3-hypertournaments [Cherlin 2022, Appendix B, §B1.4.1], showing that there exist only four up to isomorphism. It remains open whether there exist any other countable homogeneous 3-hypertournaments beyond these four cases. Here, we focus on one of these four structures, namely, the $\boldsymbol{H}_4$-free 3-hypertournament. We introduce its definition in Section 3 and show some of its basic model-theoretic properties, which include weak elimination of imaginaries and the existence of global types Lascar-invariant over A for A a nonempty set.

Cherlin's four homogeneous 3-hypertournaments have recently appeared in connection with some fundamental questions in structural Ramsey theory. One of the major open questions in this area was posed by Bodirsky, Pinsker and Tsankov [Bodirsky et al. 2011]: Does every homogeneous structure in a finite relational language have a homogeneous expansion by finitely many relations which is Ramsey? In [Cherlin et al. 2021], Cherlin, Hubička, Konečný, and Nešetřil set out to find Ramsey expansions of each of the four homogeneous 4-constrained 3-hypertournaments in finite relational languages and were able to find them for all except one of the four: the $\boldsymbol{H}_4$-free 3-hypertournament. Thus, this example serves as motivation for extending the traditional techniques of structural Ramsey theory.

In Section 4, we offer the first proof of Theorem 1.0.1. Nonetheless, the main interest of this example for us lies in the second proof of NSOP_4 that we offer in Section 6. As preliminary work towards this proof, in Section 5, we adapt the notion of strong Lascar independence from the master's thesis of Tartarotti [2023] to the context of the Kim-dividing order introduced by Mutchnik [2022a]. Tartarotti defines strong Lascar independence in terms of minimal extensions of types with respect to the fundamental order and uses this towards a proof of Lascar's reconstruction theorem. Using analogous ideas, we obtain an independence relation we can use to characterise, abstractly, the notions of relative Kim's lemma and strong witnessing property, which play an important role in [Mutchnik 2022a].

In contrast to most known NSOP_4 examples, including all of those we have mentioned before, there cannot exist any independence relation over models of the theory of the $\boldsymbol{H}_4$-free 3-hypertournament satisfying full existence, symmetry, and stationarity. In our second proof of NSOP_4 (Section 6), we show the following.

Theorem 1.0.2. *In the $\boldsymbol{H}_4$-free 3-hypertournament, there is a nonstationary independence relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{hti}}$ satisfying full existence and the relative Kim's lemma. In particular, Conant-independence coincides with $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$.*

Although our proof uses some of the concepts and results that Mutchnik [2022a] introduces, the specific methods applied to examples in that paper use the existence of an independence relation satisfying monotonicity, full existence, and stationarity over models, which we have not been able to find in the $\boldsymbol{H}_4$-free 3-hypertournament. We conjecture that, in fact, there is no such independence relation defined over models of this theory. However, regardless of the outcome of this conjecture, the present example remains a theoretical novelty in the context of NSOP_4 as the first application of Mutchnik's concepts with a nonstationary independence relation.

2. Conventions and preliminaries

Since we need to keep track of elements and tuples in many of the proofs concerning 3-hypertournaments, we use a to denote an element, the bar notation $\bar{a}$ to denote a tuple, and A to denote a set. For an n-tuple $\bar{a}$, we write $\bar{a} = (a_0, \dots, a_{n-1})$. As usual, we denote by AB the union of sets $A \cup B$.

For an ordinal α, we denote by $\alpha^{<\omega}$ the tree of finite sequences of elements of α, and we denote its partial order by $\trianglelefteq$. For $\eta \in \alpha^{\omega}$ and $i \in \omega$, we write $\eta|_i$ for the restriction of η to the first i entries. For $\eta, \nu \in \alpha^{<\omega}$, we write $\eta^\frown\nu$ to denote the concatenation of η and ν as sequences.

2.1. ***Fraïssé's theorem.*** We quickly review some of the main concepts and results from the theory of homogeneous structures that we employ throughout the present work. More details can be found in, e.g., [Cameron 1990, Sections 2.6–2.8].

Definition 2.1.1. Let $\mathcal{L}$ be a relational language.

(i) We say an $\mathcal{L}$-structure M is *homogeneous* if, for all $A, B \subset M$ finite substructures, any isomorphism $f : A \to B$ extends to an automorphism $g : M \to M$.

(ii) The *age* of M is the class of all $\mathcal{L}$-structures isomorphic to finite substructures of M.

(iii) We say a class $\mathcal{C}$ of finite $\mathcal{L}$-structures has the *amalgamation property*, or AP, if, for all $A, B_1, B_2 \in \mathcal{C}$ and embeddings $f_i : A \to B_i$ for $i = 1, 2$, there exist some $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ for $i = 1, 2$ making the following diagram commute:

$$\begin{tikzcd} & C & \\ B_1 \arrow[ur, dashed, "g_1"] & & B_2 \arrow[ul, dashed, "g_2"'] \\ & A \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

We say that $\mathcal{C}$ has the *strong amalgamation property*, or SAP, if the above holds and, in addition, whenever there are $b_i \in B_i$, $i = 1, 2$, such that $g_1(b_1) = g_2(b_2)$, there is some $a \in A$ such that $b_1 = f_1(a)$ and $b_2 = f_2(a)$.

To prove that a class has AP, it is enough to show it for $|B_i \setminus f_i(A)| = 1,\ i = 1, 2$.

Let us note in passing that many model-theorists call a structure as in (i) above *ultrahomogeneous* instead. However, in what follows, we adopt the convention from [Cherlin et al. 2021] and use "homogeneous" as above.

Definition 2.1.2. Let $\mathcal{L}$ be a relational language. We say a class $\mathcal{C}$ of finite $\mathcal{L}$-structures is a (resp. *strong*) *amalgamation class* if it is closed under substructures and isomorphisms, has countably many isomorphism classes, and has AP (resp. SAP).

Fact 2.1.3 (Fraïssé's theorem). Let $\mathcal{L}$ be a relational language and $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. Then $\mathcal{C}$ is an amalgamation class if and only if there is a countable homogeneous $\mathcal{L}$-structure M such that $\mathcal{C}$ is the age of M. This M is unique up to isomorphism. (We call M the *Fraïssé limit* of $\mathcal{C}$.)

Fact 2.1.4 [Cameron 1990, 2.22]. Let M be a countable homogeneous structure, and let $T = \mathrm{Th}(M)$. Then T is ω-categorical and has quantifier elimination.

Fact 2.1.5 [Cameron 1990, 2.15]. Let M be a countable homogeneous structure in a relational language. Then the age of M is a strong amalgamation class if and only if $\mathrm{acl}(A) = A$ for all finite $A \subset M$.

2.2. *Generalised stability theory.* From now on, let T be a complete theory and $\mathbb{M} \models T$ be a monster model, i.e., a sufficiently saturated and strongly homogeneous model. As usual, we assume that all elements, tuples, and sets are small and embed into $\mathbb{M}$. Types defined over $\mathbb{M}$ are called *global*. Let us recall the relevant definitions of the classification properties we use.

Definition 2.2.1. Let T be a complete theory.

(i) We say T has the *tree property* (or is TP) if there are a formula $\varphi(\bar{x}, \bar{y})$, $k \in \omega$, and a tree $(\bar{a}_\eta)_{\eta \in \omega^{<\omega}}$ such that

- for all $\eta \in \omega^\omega$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta|_i}) : i \in \omega\}$ is consistent modulo T,
- for all $\eta \in \omega^{<\omega}$, any k-subset of $\{\varphi(\bar{x}, \bar{a}_{\eta^\frown\langle i\rangle}) : i \in \omega\}$ is inconsistent modulo T.

Otherwise, we say T is *simple*.

(ii) We say T has the *tree property of the second kind* (or is TP_2) if there are a formula $\varphi(\bar{x}, \bar{y})$, $k \in \omega$, and an array $(\bar{a}_{i,j})_{i,j \in \omega}$ such that

- for all $f : \omega \to \omega$, the set $\{\varphi(\bar{x}, \bar{a}_{i,f(i)}) : i \in \omega\}$ is consistent modulo T,
- for all $i \in \omega$, any k-subset of $\{\varphi(\bar{x}, \bar{a}_{i,j}) : j \in \omega\}$ is inconsistent modulo T.

Otherwise, we say T is NTP_2.

(iii) We say T has the 1-*strong order property* (or is SOP_1) if there are a formula $\varphi(\bar{x}, \bar{y})$ and a tree $(\bar{a}_\eta)_{\eta \in 2^{<\omega}}$ such that

- for all $\eta \in 2^\omega$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta|_i}) : i \in \omega\}$ is consistent modulo T,
- for all $\eta, \nu \in 2^{<\omega}$ with $\nu^\frown\langle 0\rangle \trianglelefteq \eta$, the set $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{\nu^\frown\langle 1\rangle})\}$ is inconsistent modulo T.

Otherwise, we say T is NSOP_1.

(iv) Let $3 \le n \in \omega$. We say T has the n-*strong order property* (or is SOP_n) if there are a formula $\varphi(\bar{x}, \bar{y})$ with $|\bar{x}| = |\bar{y}|$ and a sequence $(\bar{a}_i)_{i\in\omega}$ such that

- $\models \varphi(\bar{a}_i, \bar{a}_j)$ for all $i < j$,
- the set $\{\varphi(\bar{x}_0, \bar{x}_1), \varphi(\bar{x}_1, \bar{x}_2), \dots, \varphi(\bar{x}_{n-2}, \bar{x}_{n-1}), \varphi(\bar{x}_{n-1}, \bar{x}_0)\}$ is inconsistent modulo T.

Otherwise, we say T is NSOP_n.

Another recent programme in model theory consists in extending the traditional binary classification-theoretic properties to higher arities. Two of the most fruitful notions in this direction are the following, introduced by Shelah [2014] and by Terry and Wolf [2021] (extended by Abd Aldaim, Conant, and Terry in [Abd Aldaim et al. 2025]), respectively.

Definition 2.2.2. Let T be a complete theory.

(i) Let $0 < n \in \omega$. We say T has the n-*independence property* (or is IP_n) if there are sequences $(\bar{a}_{0,i})_{i\in\omega}, \dots, (\bar{a}_{n-1,i})_{i<\omega}$ and $(\bar{b}_I)_{I\subseteq\omega^n}$ and a formula $\varphi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{y})$ such that

$$\models \varphi(\bar{a}_{0,i_0}, \dots, \bar{a}_{n-1,i_{n-1}}, \bar{b}_I) \iff (i_0, \dots, i_{n-1}) \in I$$

for all $i_0, \dots, i_{n-1} \in \omega$ and $I \subseteq \omega^n$. Otherwise, we say T is NIP_n. When $n = 1$, we just say T is NIP.

(ii) Let $2 \le n \in \omega$. We say T has the n-*functional order property* (or is FOP_n) if there are sequences $(\bar{a}_f)_{f:\omega^{n-1}\to\omega}$ and $(\bar{b}_{0,i})_{i\in\omega}, \dots, (\bar{b}_{n-1,i})_{i\in\omega}$ and a formula $\varphi(\bar{x}_0, \dots, \bar{x}_n)$ such that

$$\models \varphi(\bar{a}_f, \bar{b}_{1,i_1}, \dots, \bar{b}_{n,i_n}) \iff i_n \le f(i_1, \dots, i_{n-1})$$

for all $i_1, \dots, i_n \in \omega$ and $f : \omega^{n-1} \to \omega$. Otherwise, we say T is NFOP_n.

The following is a diagram of the (known) implications between the above notions. This combines many results from the literature; a good compilation of most of them containing all the relevant references is [Conant 2012] (the relation between NIP_n and NFOP_n is more recent and appears in [Abd Aldaim et al. 2025]):

$$\begin{array}{ccccccccccc}
\mathrm{NFOP}_2 & \Longrightarrow & \mathrm{NIP}_2 & \Longrightarrow & \mathrm{NFOP}_3 & \Longrightarrow & \mathrm{NIP}_3 & \Longrightarrow & \cdots & & \\
\Uparrow & & & & & & & & & & \\
\mathrm{NIP} & \Longrightarrow & \mathrm{NTP}_2 & & & & & & & & \\
\Uparrow & & \Uparrow & & & & & & & & \\
\text{stable} & \Longrightarrow & \text{simple} & \Longrightarrow & \mathrm{NSOP}_1 & \Longrightarrow & \mathrm{NSOP}_3 & \Longrightarrow & \mathrm{NSOP}_4 & \Longrightarrow & \cdots
\end{array}$$

It is open whether there exists an NSOP_n theory for any $n \geq 3$ which is NTP_2 but not simple.

2.3. ***A primer on independence relations.*** Formal definitions of the notion of an "independence relation" are commonplace in the literature; see, e.g., [Adler 2005]. However, these definitions fall short of capturing more recent developments within model theory, such as the notion of Kim-independence in NSOP_1 theories or that of Conant-independence that we study below. For this reason, we follow [d'Elbée 2023] and freely use the term *independence relation* to denote an $\mathrm{Aut}(\mathbb{M})$-invariant ternary relation $\mathop{\smile\kern-0.9em\mid}$ on subsets of our fixed monster model; i.e., $A \mathop{\smile\kern-0.9em\mid}_C B$ if and only if $\sigma(A) \mathop{\smile\kern-0.9em\mid}_{\sigma(C)} \sigma(B)$ for all $\sigma \in \mathrm{Aut}(\mathbb{M})$. (This use of the term is already prefigured in [Chernikov and Kaplan 2012], although the authors of that paper refer to this as a "pre-independence relation".)

There are several properties that an independence relation may satisfy, which already appear, in some form, in [Shelah 1978]. Their explicit versions in this abstract setting can be traced back to [Baldwin 1988]:

- *left monotonicity*: For all $A, B, C \subset \mathbb{M}$, if $A \mathop{\smile\kern-0.9em\mid}_C B$ and $A' \subseteq A$, then $A' \mathop{\smile\kern-0.9em\mid}_C B$.
- *right monotonicity*: For all $A, B, C \subseteq \mathbb{M}$, if $A \mathop{\smile\kern-0.9em\mid}_C B$ and $B' \subseteq B$, then $A \mathop{\smile\kern-0.9em\mid}_C B'$.
- *right transitivity*: For all $A \subset \mathbb{M}$ and $D \subseteq C \subseteq B \subset \mathbb{M}$, if $A \mathop{\smile\kern-0.9em\mid}_C B$ and $A \mathop{\smile\kern-0.9em\mid}_D C$, then $A \mathop{\smile\kern-0.9em\mid}_D B$.
- *existence*: For all tuples $\bar{a} \subset \mathbb{M}$ and sets $C \subset \mathbb{M}$, we have $\bar{a} \mathop{\smile\kern-0.9em\mid}_C C$.
- *full existence*: For all tuples $\bar{a} \subset \mathbb{M}$ and sets $B, C \subseteq \mathbb{M}$, there is $\bar{a}' \equiv_C \bar{a}$ such that $\bar{a}' \mathop{\smile\kern-0.9em\mid}_C B$.
- *left extension*: For all $A, A', B, C \subset \mathbb{M}$, if $A \mathop{\smile\kern-0.9em\mid}_C B$ and $A \subseteq A'$, there is some $B' \equiv_{AC} B$ such that $A' \mathop{\smile\kern-0.9em\mid}_C B'$.
- *right extension*: For all $A, B, B', C \subseteq \mathbb{M}$, if $A \mathop{\smile\kern-0.9em\mid}_C B$ and $B \subseteq B'$, there is some $A' \equiv_{BC} A$ such that $A' \mathop{\smile\kern-0.9em\mid}_C B'$.
- *stationarity over models*: For $A, A', B \subset \mathbb{M}$ and $M \prec \mathbb{M}$, if $A \mathop{\smile\kern-0.9em\mid}_M B$, $A' \mathop{\smile\kern-0.9em\mid}_M B$, and $A \equiv_M A'$, then $A \equiv_{MB} A'$.

We often say that $\mathop{\smile\!\!\!\!\!\!\!\mid}$ satisfies *monotonicity* if it satisfies both left and right monotonicity. We call $\mathop{\smile\!\!\!\!\!\!\!\mid}$ an independence relation *over models* if its base can only be a model, i.e., if $A \mathop{\smile\!\!\!\!\!\!\!\mid}_C B$ is defined only if C is a model of T.

Example 2.3.1. As is standard in the literature, for tuples $\bar{a}$, $\bar{b}$ and sets C, let us define $\bar{a} \mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{i}}_C \bar{b}$ if and only if $\mathrm{tp}(\bar{a}/C\bar{b})$ has a global extension Lascar-invariant over C. Restricting $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{i}}$ to be defined only over models, the definition is equivalent to the existence of a global M-invariant extension of $\mathrm{tp}(\bar{a}/M\bar{b})$. In any theory T, $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{i}}$ satisfies invariance and monotonicity, among other properties not discussed here.

Notation. Given two independence relations $\mathop{\smile\!\!\!\!\!\!\!\mid}^1$ and $\mathop{\smile\!\!\!\!\!\!\!\mid}^2$, we write $\mathop{\smile\!\!\!\!\!\!\!\mid}^1 \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^2$ if, whenever $A \mathop{\smile\!\!\!\!\!\!\!\mid}^1_C B$, we have $A \mathop{\smile\!\!\!\!\!\!\!\mid}^2_C B$.

We also recall some operations on abstract independence relations which will be useful for the later sections. The first one comes, in its explicit form, from the thesis of Adler [2005, Definition 1.16].

Definition 2.3.2. Given an independence relation $\mathop{\smile\!\!\!\!\!\!\!\mid}$, we define $\mathop{\smile\!\!\!\!\!\!\!\mid}^*$ by

$$A \mathop{\smile\!\!\!\!\!\!\!\mid}^*_C B \iff \text{for all } B' \supseteq B, \text{ there exists } A' \equiv_{BC} A \text{ such that } A' \mathop{\smile\!\!\!\!\!\!\!\mid}_C B'.$$

It is immediate from the definition that

- $\mathop{\smile\!\!\!\!\!\!\!\mid}^* \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}$,
- if $\mathop{\smile\!\!\!\!\!\!\!\mid}^0 \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}$ and $\mathop{\smile\!\!\!\!\!\!\!\mid}^0$ satisfies right extension, then $\mathop{\smile\!\!\!\!\!\!\!\mid}^0 \Rightarrow \mathop{\smile\!\!\!\!\!\!\!\mid}^*$.

Fact 2.3.3 [Adler 2005, Lemma 1.17]. If $\mathop{\smile\!\!\!\!\!\!\!\mid}$ satisfies right monotonicity, then $\mathop{\smile\!\!\!\!\!\!\!\mid}^*$ satisfies right monotonicity and right extension. If, in addition, $\mathop{\smile\!\!\!\!\!\!\!\mid}$ satisfies left monotonicity, then so does $\mathop{\smile\!\!\!\!\!\!\!\mid}^*$.

Example 2.3.4. The classical example of this operation is that the nonforking independence relation, denoted by $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{f}}$, is the result of applying the above operation to the nondividing independence relation, denoted by $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{d}}$. Explicitly: $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{f}} = (\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{d}})^*$.

The second operation has been explicitly defined by d'Elbée [2023, Definition 3.2.9].

Definition 2.3.5. Given an independence relation $\mathop{\smile\!\!\!\!\!\!\!\mid}$, we define $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{opp}}$ by

$$A \mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{opp}}_C B \iff B \mathop{\smile\!\!\!\!\!\!\!\mid}_C A.$$

Remark 2.3.6. It is clear from the definition that, if $\mathop{\smile\!\!\!\!\!\!\!\mid}$ satisfies left monotonicity/left extension (resp. right monotonicity/right extension), then $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{opp}}$ satisfies right monotonicity/right extension (resp. left monotonicity/left extension).

Example 2.3.7. The coheir independence relation, denoted by $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{u}}$, is the result of applying the opp operation to the heir independence relation, denoted by $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{h}}$. Explicitly: $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{u}} = (\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{h}})^{\mathrm{opp}}$.

2.4. *Relative Kim-independence.* Given an independence relation $\mathop{\smile\!\!\!\!\!\!|}$, we can relativise the notions appearing in the theory of NSOP$_1$ using the methods in [Mutchnik 2022a] which originated in [Adler 2005]. We start with a notion from the theory of abstract independence relations [Chernikov and Kaplan 2012, Definition 2.26].

Definition 2.4.1. Let T be a complete theory and $\mathop{\smile\!\!\!\!\!\!|}$ be an independence relation over subsets of $\mathbb{M} \models T$. We say a global type q is $\mathop{\smile\!\!\!\!\!\!|}$*-free* over M if, for all $B \supset M$ and $\bar{a} \models q|_B$, we have $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M B$.

Example 2.4.2. A global type is $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$-free over M if and only if it is M-invariant.

Definition 2.4.3. Let T be a complete theory, $M \models T$, and $\bar{a}$, $\bar{b}$ be tuples.

(i) We say that $(\bar{b}_i)_{i\in\omega}$ is an $\mathop{\smile\!\!\!\!\!\!|}$*-Morley sequence* over M if, for all $i \in \omega$, we have $\bar{b}_i \equiv_M \bar{b}_0$ and $\bar{b}_i \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}_{<i}$. For any global type q which is $\mathop{\smile\!\!\!\!\!\!|}$-free over M, we say $(\bar{b}_i)_{i\in\omega}$ is a *Morley sequence in q over M* if $\bar{b}_i \models q|_{M\bar{b}_{<i}}$ for all $i < \omega$. Every Morley sequence in a global type $\mathop{\smile\!\!\!\!\!\!|}$-free over M is $\mathop{\smile\!\!\!\!\!\!|}$-Morley over M.

(ii) We say a formula $\varphi(\bar{x}, \bar{b})$ $\mathop{\smile\!\!\!\!\!\!|}$*-Kim-divides* over M if there is some global extension $q \supset \mathrm{tp}(\bar{b}/M)$ $\mathop{\smile\!\!\!\!\!\!|}$-free over M and a Morley sequence $(\bar{b}_i)_{i\in\omega}$ in q over M with $\bar{b}_0 = \bar{b}$ such that the set $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$ is inconsistent.

(iii) We say a formula $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\!|}$*-Kim-forks* over M if there exist formulas $\psi_i(x, c_i)$ for $i < n$ such that $\varphi(x, b) \vdash \bigvee_{i<n} \psi_i(x, c_i)$ and each $\psi_i(x, c_i)$ $\mathop{\smile\!\!\!\!\!\!|}$-Kim-divides over M.

(iv) We say $\bar{a}$ is $\mathop{\smile\!\!\!\!\!\!|}$*-Kim-independent* from $\bar{b}$ over M if $\mathrm{tp}(\bar{a}/M\bar{b})$ does not contain any formula that $\mathop{\smile\!\!\!\!\!\!|}$-Kim-forks over M.

Example 2.4.4. We call Morley sequences in global M-invariant types *M-invariant Morley sequences*. We refer to $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$-Kim-independence simply as *Kim-independence*.

Recently, Mutchnik [2022a] has generalised the fundamental order introduced by Poizat [1979] to Kim-dividing, extending the work of Ben Yaacov and Chernikov [2014], who already generalised it to dividing. Our presentation follows that of the fundamental order given by Pillay [1983].

Definition 2.4.5. Let T be a complete theory, $M \models T$, and $r \in S(M)$.

(i) Let p be a global M-invariant extension of r. The *Kim-dividing class of p*, denoted by $\mathrm{cl}_K(p)$, is defined to be the set of formulas $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_M$ such that $\varphi(\bar{x}, \bar{a})$ is consistent for some (equivalently any) realisation $\bar{a} \models p|_M$ and there is $(\bar{a}_i)_{i\in\omega}$ with $\bar{a}_i \models p|_{M\bar{a}_{<i}}$ for all $i \in \omega$ and $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ inconsistent. We may equivalently ask that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ be inconsistent for all $(\bar{a}_i)_{i\in\omega}$ as above.

(ii) Given global M-invariant extensions p and q of r, we write $p \leq_K q$ if $\mathrm{cl}_K(p) \subseteq \mathrm{cl}_K(q)$. We say p is *least in the Kim-dividing order*, or *$\leq_K$-least*, if $p \leq_K q$ for all global M-invariant extensions q of r.

Mutchnik relativises Kim's lemma to a choice of independence relation.

Definition 2.4.6. We say an independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ defined over models satisfies the *relative Kim's lemma* if, for all global types q and models $M \models T$, if q is $\mathop{\smile\!\!\!\!\!\!\mid}$-free over M, then q is a $\leq_K$-least extension of $q|_M$.

Remark 2.4.7. Note that, by the definition of the Kim-dividing order, if $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies the relative Kim's lemma, then every global type that is $\mathop{\smile\!\!\!\!\!\!\mid}$-free over M must be M-invariant.

Example 2.4.8. Another way of restating the version of Kim's lemma for NSOP_1 theories by Kaplan and Ramsey [2020, Theorem 3.16] is the following (hence the terminology): a theory T is NSOP_1 if and only if $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$ satisfies the relative Kim's lemma.

Remark 2.4.9. In the special case where $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies stationarity over models, some of the previous definitions may be simplified. This is due to the following easy observations:

- If $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies full existence and stationarity over models, then every type $p \in S(M)$ has a unique global extension that is $\mathop{\smile\!\!\!\!\!\!\mid}$-free over M. Such an extension is additionally M-invariant.
- If $\mathop{\smile\!\!\!\!\!\!\mid}$ satisfies stationarity over models, then every $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence over M is M-indiscernible.

In the context of relative Kim-independence from [Mutchnik 2022a], the relevant independence relation satisfies stationarity over models by assumption, and hence the above observations apply. We want to use this machinery in the context of a nonstationary independence relation; see Section 6. The observations in this remark serve as a point of comparison between the two approaches and will not be used later.

3. Basic properties

3.1. ***Definition of the H_4-free 3-hypertournament.*** The definitions and results included in this subsection can be found, explicitly or implicitly, in [Cherlin 2022, Appendix B, §B1.4.1] and [Cherlin et al. 2021]. Let $\mathcal{L} = \{R\}$, where R is a ternary relation symbol.

Definition 3.1.1. Let T be the $\mathcal{L}$-theory axiomatised by the following formulas:

(i) $\forall x_1 x_2 x_3 \left(R(x_1, x_2, x_3) \to \bigwedge_{i \neq j} x_i \neq x_j\right)$.

(ii) $\forall x_1 x_2 x_3 \,(R(x_1, x_2, x_3) \leftrightarrow R(x_2, x_3, x_1) \leftrightarrow R(x_3, x_1, x_2))$.

(iii) $\forall x_1 x_2 x_3 \left(\bigwedge_{i \neq j} x_i \neq x_j \to (\neg R(x_1, x_2, x_3) \leftrightarrow R(x_3, x_2, x_1))\right)$.

We call any model $M \models T$ a 3-*hypertournament*.

The general classification of all countable homogeneous 3-hypertournaments is still open. As a partial answer to this problem, Cherlin's result classifies those homogeneous 3-hypertournaments that arise from looking at the possible structures that can occur on any subset of four points contained in them.

Remark 3.1.2. Let M be a 3-hypertournament. Given a linear order $\leq$ on M, define a 3-uniform hypergraph $\widehat{M}$ on M in $\mathcal{L}$ by declaring, for $a \leq b \leq c$, $\{a, b, c\} \in R^{\widehat{M}}$ if and only if $(a, b, c) \in R^M$. Formally, a linear order on M determines an isomorphism of categories between the age of M with embeddings to the category of 3-uniform hypergraphs on finite subsets of M with embeddings.

There are three 4-point substructures of a 3-hypertournament up to isomorphism:

- $\boldsymbol{C}_4$: there is an order $\leq$ on C_4 (the underlying set of four points) such that $\widehat{\boldsymbol{C}}_4$ is a complete 3-uniform hypergraph. In different orders, $\widehat{\boldsymbol{C}}_4$ might have no hyperedges at all, or exactly two hyperedges which intersect at two vertices that are adjacent relative to the ordering on the four points.
- $\boldsymbol{O}_4$: for any linear order $\leq$ on O_4, $\widehat{\boldsymbol{O}}_4$ has an odd number of hyperedges.
- $\boldsymbol{H}_4$: for any linear order $\leq$ on H_4, $\widehat{\boldsymbol{H}}_4$ has exactly two hyperedges intersecting in vertices $a < b$ such that there is a unique $c \in H_4$ with $a < c < b$.

For any $\boldsymbol{A} \in \{\boldsymbol{C}_4, \boldsymbol{O}_4, \boldsymbol{H}_4\}$, given a 3-hypertournament M and a substructure $B \subseteq M$, we say B is an $\boldsymbol{A}$-*structure* if it is isomorphic to $\boldsymbol{A}$, and we call M $\boldsymbol{A}$-*free* if it contains no $\boldsymbol{A}$-structures.

Definition 3.1.3. Let $\mathcal{C}$ be a class of finite 3-hypertournaments. We say that $\mathcal{C}$ is 4-*constrained* if there is $S \subseteq \{\boldsymbol{C}_4, \boldsymbol{O}_4, \boldsymbol{H}_4\}$ such that a finite 3-hypertournament M belongs to the class $\mathcal{C}$ if and only if any substructure $B \subseteq M$ with $|B| = 4$ is an $\boldsymbol{A}$-structure for some $\boldsymbol{A} \in S$.

As alluded to in [Cherlin et al. 2021], Cherlin earlier proved that there are exactly four 4-constrained classes of 3-hypertournaments that are (strong) amalgamation classes in. Here, we focus on just one of these classes, namely, that of finite 3-hypertournaments omitting $\boldsymbol{H}_4$.

Remark 3.1.4. Since we focus on $\boldsymbol{H}_4$ from now on, let us note that we may also define it as, up to isomorphism, a structure on four points $\{a, b, c, d\}$ such that

$$\models R(a, b, c) \wedge R(a, c, d) \wedge R(a, d, b) \wedge R(b, d, c).$$

We often use without mention the following easy observation: if $a, b, c, d \in M$ are elements of a 3-hypertournament such that $M \models R(a, b, c) \wedge R(a, b, d)$, then $\{a, b, c, d\}$ is $\boldsymbol{H}_4$-free.

We include the proof of strong amalgamation for the sake of completeness.

Lemma 3.1.5. *The class of finite $\boldsymbol{H}_4$-free 3-hypertournaments is a strong amalgamation class.*

Proof. We may assume that our amalgamation problem is of the form $Ab_1 \leftarrow A \rightarrow Ab_2$ for $b_1, b_2 \notin A$. Let $\leq$ be a linear order on Ab_1b_2 such that $b_1 < A < b_2$. This generates an amalgamation problem in the category of finite 3-uniform hypergraphs, so we can freely amalgamate to obtain a solution $\widehat{C}$. Applying the inverse functor, we obtain a finite 3-hypertournament C which is in our class. Indeed, by construction, the only potential $\boldsymbol{H}_4$-structure must be of the form $\{a_1, a_2, b_1, b_2\}$ with $a_1, a_2 \in A$. Because $\widehat{C}$ is obtained by free amalgamation, no hyperedges involving both b_1 and b_2 appear in $\widehat{C}$. So it follows that $C \models R(b_1, b_2, a_1) \wedge R(b_1, b_2, a_2)$, which implies that $\{a_1, a_2, b_1, b_2\}$ is $\boldsymbol{H}_4$-free. $\square$

By Fact 2.1.3, the Fraïssé limit of the class in Lemma 3.1.5 exists, and we denote its theory by $T_{\boldsymbol{H}_4\text{-free}}$. By Facts 2.1.4 and 2.1.5, $T_{\boldsymbol{H}_4\text{-free}}$ is ω-categorical and has quantifier elimination and trivial algebraic closure. Often, we refer to the countable model of $T_{\boldsymbol{H}_4\text{-free}}$ as *the* $\boldsymbol{H}_4$-free 3-hypertournament.

The three remaining countable homogeneous 3-hypertournaments are determined by the constrained classes they come from. These are

- the *generic* 3-*hypertournament*, which is the Fraïssé limit of the 4-constrained class determined by $S = \{\boldsymbol{C}_4, \boldsymbol{O}_4, \boldsymbol{H}_4\}$. Using, e.g., disjoint 3-amalgamation, one can show that the generic 3-hypertournament is supersimple with trivial forking; see [Kruckman 2019, Theorem 3.14].
- the *even* 3-*hypertournament*, which is the Fraïssé limit of the 4-constrained class determined by $S = \{\boldsymbol{C}_4, \boldsymbol{H}_4\}$. As before, one can show that the even 3-hypertournament is supersimple with trivial forking.
- the *cyclic* 3-*hypertournament*, which is the Fraïssé limit of the 4-constrained class determined by $S = \{\boldsymbol{C}_4\}$. One can show that the cyclic 3-hypertournament is bi-interpretable with $(\mathbb{Q}, \mathrm{cyc})$, and hence it is distal and dp-minimal.

The goal of this paper is to provide similar classification results for the $\boldsymbol{H}_4$-free 3-hypertournament.

3.2. ***Existence of invariant extensions.*** In this section, we prove that $T_{\boldsymbol{H}_4\text{-free}}$ is locally $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{i}}$-extensible but not $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{i}}$-extensible, properties which we define below. During the course of the proof, some other basic model-theoretic properties of $T_{\boldsymbol{H}_4\text{-free}}$ are established, such as weak elimination of imaginaries. From now on, let $\mathbb{M} \models T_{\boldsymbol{H}_4\text{-free}}$ be a monster model.

Definition 3.2.1 (cf. [Chernikov and Kaplan 2012]). Let T be a complete theory.

(i) We say T is $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{i}}$-*extensible* if $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{i}}$ satisfies existence; i.e., every type $p \in S(A)$ has a global extension q which is Lascar-invariant over A.

(ii) We say T is *locally* $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$*-extensible* if it is $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$-extensible after adding finitely many parameters.

Remark 3.2.2. For (local) $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$-extensibility, since $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ satisfies right transitivity and monotonicity (see Example 2.3.1), it suffices to check the condition from the definition for 1-types.

Lemma 3.2.3. $T_{\boldsymbol{H}_4\text{-free}}$ *has weak elimination of imaginaries*; i.e., *for every imaginary e, there is some real $\bar{c}$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{c})$ and $\bar{c} \in \mathrm{acl}(e)$.*

Proof. Let $M \models T_{\boldsymbol{H}_4\text{-free}}$ be countable. By [Poizat 2000, Lemma 16.17], it suffices to show that

$$H := \langle \mathrm{Aut}(M/A), \mathrm{Aut}(M/B)\rangle = \mathrm{Aut}(M/A \cap B) \tag{3-1}$$

for finite algebraically closed $A, B \subset M$. Since M has trivial algebraic closure, by [Li 2018, Proposition 4.3], it is enough to prove (3-1) for $A = Ca$, $B = Cb$ for some finite set $C \subseteq M$ and distinct elements $a, b \in M$. We may assume $a, b \notin C$.

Claim 1. If $b' \equiv_C b$, then there is $a' \models \mathrm{tp}(a/Cb)$ such that $b' \equiv_{Ca'} b$.

Proof of Claim 1. Write $C = \{c_1, \dots, c_n\}$ and $p(x, b, \bar{c}) := \mathrm{tp}(a/b\bar{c})$, and let $b' \models \mathrm{tp}(b/\bar{c})$. First note that, if $b' = b$, then we can pick $a' = a$. Hence, assuming that $b' \neq b$, we need to show that $p(x, b, \bar{c}) \cup \{R(x, b, c_i) \leftrightarrow R(x, b', c_i) : i \in [n]\}$ is consistent. We split the proof of this claim into two cases. In both, our strategy is to build a finite structure that satisfies the formulas in the set.

<u>Case 1</u>: $b' = a$. Let $\Sigma(x, b, a)$ be the extension of $p(x, b, \bar{c})$ given by adding the following formulas:

(i) $R(x, a, c_i) \leftrightarrow R(a, b, c_i)$ for all $i \in [n]$.

(ii) $R(x, a, b)$.

We claim that Σ is consistent modulo $T_{\boldsymbol{H}_4\text{-free}}$. Since Σ extends p, it suffices to show that 4-point sets containing x and a are $\boldsymbol{H}_4$-free. But this also follows immediately from the fact that Σ extends p, using Remark 3.1.4.

<u>Case 2</u>: $b' \neq a$. This time, let $\Sigma(x, b, b', a)$ be the extension of $p(x, b, \bar{c})$ given by adding the following formulas:

(i) $R(x, b', c_i) \leftrightarrow R(a, b, c_i)$ for all $i \in [n]$.

(ii) $R(x, a, b) \wedge R(x, a, b')$.

(iii) $R(x, a, c_i)$ for all $i \in [n]$.

(iv) $R(x, b, b') \leftrightarrow R(a, b, b')$.

As in Case 1, we show that Σ is consistent modulo $T_{\boldsymbol{H}_4\text{-free}}$ by showing that 4-point sets containing x and at least one of a or b' are $\boldsymbol{H}_4$-free. By (iii), Σ implies that any set of the form $\{x, a, c_i, c_j\}$ is $\boldsymbol{H}_4$-free, and by (ii) and (iii), so is $\{x, a, b, c_i\}$.

Moreover, by (i) and since $b \equiv_C b'$, Σ implies that any set of the form $\{x, b', c_i, c_j\}$ or $\{x, b, b', c_i\}$ is $\boldsymbol{H}_4$-free. Finally, by (ii), Σ implies that $\{x, a, b, b'\}$ is also $\boldsymbol{H}_4$-free, and (ii) and (iii) combined imply that so is $\{x, a, b', c_i\}$.

Therefore, in either case, Σ defines a type over $a, b, b', \bar{c}$, so we can choose $a' \models \Sigma$ in M. By (i) in both cases, this a' works. □

We now show (3-1). Note that $\leq$ is clear. For $\geq$, let $g \in \mathrm{Aut}(M/C)$, and let $b' := g(b)$. Then, by Claim 1, there is $a' \equiv_B a$ such that $b \equiv_{Ca'} b'$.

Claim 2. $\mathrm{Aut}(M/Ca') \leq H$.

Proof of Claim 2. Since $a' \equiv_B a$, there is some $h \in \mathrm{Aut}(M/B)$ sending a' to a. Hence, for any $k \in \mathrm{Aut}(M/Ca')$, we have that $hkh^{-1} \in \mathrm{Aut}(M/A) \leq H$, and as $h \in \mathrm{Aut}(M/B) \leq H$, it follows that $k \in H$. □

Thus, since $b \equiv_{Ca'} b'$, there is some $k \in \mathrm{Aut}(M/Ca')$ such that $k(b) = b'$, that is, $k(b) = g(b)$. Therefore, $k^{-1}g \in H$, and so, since $k \in H$ by Claim 2, $g \in H$. □

Remark 3.2.4. Claim 1 above is a strong property for which we need to restrict ourselves to both a and b being singletons. See Remark 4.0.4 for a counterexample to the version of the claim obtained by replacing b with a pair of elements.

Corollary 3.2.5. *For all $\bar{a}, \bar{b}$ and A, $\bar{a} \equiv_{\mathrm{acl}^{\mathrm{eq}}(A)} \bar{b}$ if and only if $\bar{a} \equiv_{\mathrm{acl}(A)} \bar{b}$.*

Proof. This follows directly from [Casanovas and Farré 2004, Proposition 3.4]. □

Proposition 3.2.6. *$T_{\boldsymbol{H}_4\text{-free}}$ is not $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$-extensible.*

Proof. Note that, for any two pairs of distinct points ab and cd, we have $ab \equiv cd$ in $T_{\boldsymbol{H}_4\text{-free}}$. Since $T_{\boldsymbol{H}_4\text{-free}}$ has trivial algebraic closure, we get $ab \equiv_{\mathrm{acl}(\varnothing)} cd$, and so, by Corollary 3.2.5, it follows that $ab \equiv_{\mathrm{acl}^{\mathrm{eq}}(\varnothing)} cd$. Finally, since $T_{\boldsymbol{H}_4\text{-free}}$ is ω-categorical, it follows that $ab \equiv^{\mathrm{Ls}} cd$.

Now assume, for contradiction, that there is some global 1-type p which is Lascar-invariant over $\varnothing$. Then, for any $a \neq b$, we have $ab \equiv^{\mathrm{Ls}} ba$, and thus $R(x, a, b) \leftrightarrow R(x, b, a) \in p(x)$. Hence, p is inconsistent modulo $T_{\boldsymbol{H}_4\text{-free}}$, a contradiction. □

Remark 3.2.7. It follows that there cannot exist any canonical independence relation on the finite subsets of the countable homogeneous $\boldsymbol{H}_4$-free 3-hypertournament in the sense of Kaplan and Simon [2019], since they all satisfy stationarity over $\varnothing$, and thus give rise to global invariant types. This contrasts with the situation for the countable homogeneous tournaments; see [Kaplan and Simon 2019, Example 4.4].

Proposition 3.2.8. *$T_{\boldsymbol{H}_4\text{-free}}$ is locally $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$-extensible.*

Proof. Let $p(\bar{x}) \in S(A)$ with $A \neq \varnothing$. Fix $\bar{a}^* \in A$. Let $q(\bar{x})$ be the global expansion of $p(\bar{x})$ obtained by adding the following formulas:

(i) $R(x_i, a, b)$ for all $a \in A$, $b \notin A$, and i.

(ii) $R(x_i, b_1, b_2) \leftrightarrow R(a^*_i, b_1, b_2)$ for all $b_1, b_2 \notin A$ and i.

We claim that this is consistent. Indeed, since Σ extends p, any potential $\boldsymbol{H}_4$-structure must contain elements outside of A, and by (i), q implies that any set of the form $\{x_i, a, a', b\}$, $\{x_i, a, b, b'\}$, or $\{x_i, x_j, a, b\}$ with $a, a' \in A$ and $b, b' \notin A$ is $\boldsymbol{H}_4$-free. Also, by (ii), q implies that 4-point sets of the form $\{x_i, b, b', b''\}$, $\{x_i, x_j, b, b'\}$, or $\{x_i, x_j, x_k, b\}$ with $b, b', b'' \notin A$ are $\boldsymbol{H}_4$-free. Therefore, q is a global type.

Furthermore, suppose that $b_1 b_2 \equiv_A b'_1 b'_2$. Since $\bar{a}^* \in A$, this means that

$$\models R(a_i^*, b_1, b_2) \leftrightarrow R(a_i^*, b'_1, b'_2)$$

for all i, and thus, by (ii), $R(x_i, b_1, b_2) \leftrightarrow R(x_i, b'_1, b'_2) \in q$. Similarly,

$$\models R(a_i^*, a_j^*, b_1) \leftrightarrow R(a_i^*, a_j^*, b'_1)$$

for all i, j, which implies by (ii) that $R(x_i, x_j, b_1) \leftrightarrow R(x_i, x_j, b'_1) \in q$. Therefore, q is a global A-invariant extension of p, and so, by the argument from Proposition 3.2.6, q is Lascar-invariant over A. □

Corollary 3.2.9. *In $T_{\boldsymbol{H}_4\text{-free}}$, $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ satisfies left extension over nonempty bases.*

Proof. This is an immediate application of Lemma 3.23 in [Chernikov and Kaplan 2012]. □

4. First proof of NSOP$_4$

In this section, we prove the main classification results for $T_{\boldsymbol{H}_4\text{-free}}$ using concrete constructions instead of the more abstract machinery that will be used in the following sections. In what follows, we work in a monster model $\mathbb{M} \models T_{\boldsymbol{H}_4\text{-free}}$ and, as usual, assume that all sets and tuples are in $\mathbb{M}$ and all models elementarily embed in $\mathbb{M}$.

Proposition 4.0.1. *$T_{\boldsymbol{H}_4\text{-free}}$ is* NFOP$_3$ *and* IP$_2$.

Proof. NFOP$_3$ follows directly from [Abd Aldaim et al. 2025, Proposition 3.23].

We claim that the formula $R(x_1, x_2; y)$ has IP$_2$. Fix a total order $<^*$ on $\mathcal{P}(\omega^2)$. Let $\Sigma(x_{0,i}, x_{1,j}, y_I)_{i,j\in\omega, I\subseteq\omega^2}$ be the following set of formulas:

(i) $R(x_{0,i}, x_{1,j}, y_I) \Leftrightarrow (i, j) \in I$.

(ii) $R(x_{n,i}, x_{n,j}, x_{n,k}) \Leftrightarrow i < j < k$ for any $n \in \{0, 1\}$.

(iii) $R(x_{n,i}, x_{n,j}, x_{n',l}) \Leftrightarrow i < j$ for any l and $\{n, n'\} = \{0, 1\}$.

(iv) $R(x_{n,i}, x_{n,j}, y_I) \Leftrightarrow i < j$ for any $I \subseteq \omega^2$ and $n \in \{0, 1\}$.

(v) $R(x_{n,i}, y_I, y_J) \Leftrightarrow I <^* J$ for any i and $n \in \{0, 1\}$.

(vi) $R(y_I, y_J, y_K) \Leftrightarrow I <^* J <^* K$.

We claim that this is a complete type. The only thing that needs to be checked is that it does not generate any $\boldsymbol{H}_4$-structures. By (i) and (ii), Σ implies that any 4-point set containing only elements from $\bar{x}_0$ and $\bar{x}_1$ is $\boldsymbol{H}_4$-free, and, by (vi), it also implies that those containing only y_I's are $\boldsymbol{H}_4$-free. By (iv), $\Sigma \vdash R(x_{n,i}, x_{n,j}, y_I) \leftrightarrow R(x_{n,i}, x_{n,j}, y_J)$ for all I, J, so that $\{x_{n,i}, x_{n,j}, y_I, y_J\}$ is $\boldsymbol{H}_4$-free and, by (v), $\Sigma \vdash R(x_{n,i}, y_I, y_J) \leftrightarrow R(x_{n',j}, y_I, y_J)$ for $\{n, n'\} = \{0, 1\}$, so that $\{x_{n,i}, x_{n',j}, y_I, y_J\}$ is $\boldsymbol{H}_4$-free. Moreover, also by (iv), Σ implies that $\{x_{n,i}, x_{n,j}, x_{n',l}, y_I\}$ is $\boldsymbol{H}_4$-free. Finally, since $<^*$ is a total order, by (v) Σ implies that $\{x_{n,i}, y_I, y_J, y_K\}$ is $\boldsymbol{H}_4$-free.

Since these are all possible $\boldsymbol{H}_4$-structures, it follows that Σ is consistent, and thus we can choose a realisation $(a_{0,i}, a_{1,j}, b_I)_{i,j\in\omega, I\subseteq\omega^2} \models \Sigma$. By (i), this realisation witnesses IP_2 for the formula $R(x_1, x_2, y)$. □

Proposition 4.0.2. *$T_{\boldsymbol{H}_4\text{-free}}$ is* SOP_3.

Proof. Fix some elements a, b from the monster model. We prove that the formula $\varphi(x, y) := R(x, y, a) \wedge R(x, b, y)$ witnesses SOP_3. First, assume $\varphi(x_0, x_1) \wedge \varphi(x_1, x_2) \wedge \varphi(x_2, x_0)$ is consistent. Let (c_0, c_1, c_2) be a realisation. Then the substructure on $\{c_0, c_1, c_2, a\}$ implies that $\models R(c_0, c_1, c_2)$, since otherwise we obtain an $\boldsymbol{H}_4$-structure. Similarly, the substructure on $\{c_0, c_1, c_2, b\}$ implies that $\models R(c_0, c_2, c_1)$, so we get a contradiction. Hence $\varphi(x_0, x_1) \wedge \varphi(x_1, x_2) \wedge \varphi(x_2, x_0)$ is inconsistent.

Now let $\Sigma(x_i)_{i\in\omega}$ be the set containing the following formulas over ab:

(i) $R(x_i, x_j, a) \wedge R(x_i, b, x_j)$ for all $i < j$.

(ii) $R(x_i, x_j, x_k) \Leftrightarrow i < j < k$.

(iii) $R(x_i, b, a)$ for all i.

By (ii), Σ implies that any 4-point set containing only x_i's is $\boldsymbol{H}_4$-free and, by (i), so is any 4-point set of the form $\{x_i, x_j, x_k, a\}$ or $\{x_i, x_j, x_k, b\}$. Thus, any potential $\boldsymbol{H}_4$-structure must be on a set of the form $\{x_i, x_j, a, b\}$. But Σ implies that this is $\boldsymbol{H}_4$-free by (iii).

Therefore, $\Sigma(x_i)_{i\in\omega}$ is consistent, and so we can choose a realisation $(c_i)_{i\in\omega} \models \Sigma$. By stipulation, $\models \varphi(c_i, c_j)$ for all $i < j$. □

Proposition 4.0.3. *$T_{\boldsymbol{H}_4\text{-free}}$ is* TP_2.

Proof of Proposition 4.0.3 (version 1). For convenience, let us expand $\mathcal{L}$ by adding two constant symbols e and f. Let $\bar{b} = (c, d)$ be a pair such that

$$\models R(c, d, e) \wedge R(d, e, f) \wedge R(c, e, f) \wedge R(c, f, d).$$

Now let $\varphi(x, \bar{b})$ be the following $\mathcal{L}(e, f)$-formula:

$$R(x, c, d) \wedge R(x, d, e) \wedge R(x, e, f) \wedge R(x, c, f) \wedge R(x, e, c) \wedge R(x, f, d).$$

It is easy to see from the definition that, for any $a \models \varphi(x, \bar{b})$, the set $\{a, c, d, e, f\}$ is $\boldsymbol{H}_4$-free, and therefore $\varphi(x, \bar{b})$ isolates a complete type in $\mathcal{L}(e, f)$.

Now let $\Sigma(\bar{v}_i)_{i\in\omega}$ with $\bar{v}_i := (x_i, y_i)$ be the following set of formulas:

(i) $\bar{v}_i \equiv_{ef} \bar{b}$ for all i.

(ii) $R(e, y_j, x_i) \wedge R(f, x_i, y_j) \Leftrightarrow i < j$.

(iii) $R(e, x_i, x_j) \wedge R(f, x_i, x_j) \wedge R(e, y_i, y_j) \wedge R(f, y_i, y_j) \Leftrightarrow i < j$.

(iv) $R(x_i, x_j, x_k) \wedge R(y_i, y_j, y_k) \Leftrightarrow i < j < k$.

(v) $R(x_i, x_j, y_n) \Leftrightarrow i < j$ and $R(x_m, y_k, y_l) \Leftrightarrow k < l$ for all m and n.

We claim that $\Sigma(\bar{v}_i)_{i\in\omega}$ is consistent. Indeed, by (i), it suffices to consider those 4-sets containing elements from different $\bar{v}_i$'s. It is straightforward to see using (iv) and (v) that 4-sets containing only x_i's and y_j's are $\boldsymbol{H}_4$-free. Similarly, by (ii) and (iii), 4-sets containing only x_i's, y_j's and exactly one of e or f are also $\boldsymbol{H}_4$-free, and similarly for those 4-sets containing both e and f and elements from exactly one of $\bar{x}$ or $\bar{y}$. So we just need to check that sets of the form $\{e, f, x_i, y_j\}$ are $\boldsymbol{H}_4$-free. But these are $\boldsymbol{H}_4$-free by (i). Note that a realisation will give an ef-indiscernible sequence.

Hence $\Sigma(\bar{v}_i)_{i\in\omega}$ defines a type in $\mathcal{L}(e, f)$, so we can choose a realisation $\bar{b}_i := (c_i, d_i)_{i\in\omega} \models \Sigma$. Then, by (ii), the substructure on $\{x, d_j, e, c_i\}$ implies that $\varphi(x, \bar{b}_i) \wedge \varphi(x, \bar{b}_j) \vdash R(x, d_j, c_i)$ for $i < j$, since otherwise we obtain an $\boldsymbol{H}_4$-structure. Similarly, the substructure on $\{x, c_i, f, d_j\}$ implies that $\varphi(x, \bar{b}_i) \wedge \varphi(x, \bar{b}_j) \vdash R(x, c_i, d_j)$ for $i < j$. Therefore, the set $\{\varphi(x, \bar{b}_i) : i \in \omega\}$ is 2-inconsistent.

To make this into an instance of TP_2, we just take copies of this indiscernible sequence. Formally, we choose an array $(\bar{b}_{i,j})_{i,j\in\omega} = (c_{i,j}, d_{i,j})_{i,j\in\omega}$ such that:

(i′) For all $i \in \omega$, $(\bar{b}_{i,j})_{j\in\omega} \equiv_{ef} (\bar{b}_j)_{j\in\omega}$.

(ii′) For any $i \neq j$, l and m, we have

$$\models R(z, c_{j,m}, d_{j,m}) \wedge R(z, c_{j,m}, e) \wedge R(z, c_{j,m}, f) \wedge R(z, d_{j,m}, e) \wedge R(z, d_{j,m}, f)$$

for any $z \in \bar{b}_{i,l}$.

(iii′) For any $i < j < k$, any l, m, n, and $z_\ell \in \bar{b}_{\ell,u}$ for $\ell \in \{i, j, k\}$ and $u \in \{l, m, n\}$, we have $\models R(z_i, z_j, z_k)$.

It is then easy to see that (i′) implies that $\{\varphi(x, \bar{b}_{i,j}) : j \in \omega\}$ is 2-inconsistent for each $i \in \omega$ by the argument from the previous paragraph, and that (ii′) and (iii′) jointly imply that $\{\varphi(x, \bar{b}_{i,f(i)}) : i \in \omega\}$ is consistent for any $f : \omega \to \omega$. □

Remark 4.0.4. Let us note, using the choice of elements from the previous proof, that the inconsistency of $\varphi(x, \bar{b}_0) \wedge \varphi(x, \bar{b}_1)$ implies that, for any $a \models \varphi(x, \bar{b}_0)$, we cannot find any $a' \equiv_{ef\bar{b}_0} a$ such that $\bar{b}_1 \equiv_{efa'} \bar{b}_0$. This shows that we cannot extend Claim 1 from the proof of Lemma 3.2.3 to tuples of finite length greater than 1.

Proposition 4.0.5. *$T_{\boldsymbol{H}_4\text{-free}}$ is* NSOP$_4$.

Proof of Proposition 4.0.5 (*version 1*). It suffices to show, for any indiscernible sequence $(\bar{b}_i)_{i<\omega}$ with common intersection $\bar{c}$ (possibly empty) and letting $p(\bar{x}, \bar{y}) := \mathrm{tp}(\bar{b}_0\bar{b}_1)$, that $p(\bar{x}_0, \bar{x}_1) \wedge p(\bar{x}_1, \bar{x}_2) \wedge p(\bar{x}_2, \bar{x}_3) \wedge p(\bar{x}_3, \bar{x}_0)$ is consistent. By indiscernibility, we can equivalently take $(\bar{b}_i)_{i<\omega}$ to be $\bar{c}$-indiscernible and consist only of those elements outside of $\bar{c}$ from each tuple.

To that end, let $(\bar{b}_i)_{i<\omega}$ be a $\bar{c}$-indiscernible sequence as above, and let $p(\bar{x}, \bar{y}) := \mathrm{tp}(\bar{b}_0\bar{b}_1/\bar{c})$. Let $n := |\bar{b}_0|$. Notice that p is of the following form:

$$p(\bar{x}, \bar{y}) = \mathrm{tp}(\bar{x}/\bar{c}) \wedge \mathrm{tp}(\bar{y}/\bar{c}) \wedge \mathrm{tp}(\bar{x}\bar{y}) \wedge \bigwedge_{j,k<n} R(c_i, x_j, y_k)^{\varepsilon(i,j,k)},$$

where $\varepsilon(i, j, k) \in \{0, 1\}$. (Note that j and k can be equal.)

Define $\Sigma(\bar{x})$ with $|\bar{x}| = n$ as the set containing the following formulas:

(i) $\bar{b}_0\bar{b}_1 \equiv_{\bar{c}} \bar{b}_1\bar{x}$.

(ii) $R(c_i, b_{0,j}, x_k)$ for all i and $j, k < n$ (not necessarily distinct).

(iii) $R(b_{0,i}, x_j, b_{1,k})$ for all $i, j, k < n$.

We claim that this is $\boldsymbol{H}_4$-free. Indeed, by (i), any potential $\boldsymbol{H}_4$-structures involving $\bar{b}_0$, $\bar{b}_1$ and $\bar{x}$ arise from the last big conjunction in $p(\bar{b}_0, \bar{b}_1)$ and $p(\bar{b}_1, \bar{x})$. It also follows from (i) and (iii) that there cannot be an $\boldsymbol{H}_4$-structure which does not contain any element from $\bar{c}$. Thus, it suffices to consider possible $\boldsymbol{H}_4$-structures on sets of the form $\{c_i, b_{0,j}, b_{1,k}, x_l\}$ for some fixed i, j, k and l.

- If $\varepsilon(i, j, k) = \varepsilon(i, k, l) = 1$, then $\models R(c_i, b_{0,j}, b_{1,k})$ and $\Sigma(\bar{x}) \vdash R(c_i, b_{0,j}, x_l)$ by (ii). Therefore, $\Sigma(\bar{x})$ implies that $\{c_i, b_{0,j}, b_{1,k}, x_l\}$ is $\boldsymbol{H}_4$-free.
- If $\varepsilon(i, j, k) = \varepsilon(i, k, l) = 0$, then $\models R(b_{0,j}, c_i, b_{1,k})$ and $\Sigma(\bar{x}) \vdash R(b_{0,j}, x_l, b_{1,k})$ by (iii). Hence, once again, $\Sigma(\bar{x})$ implies that $\{c_i, b_{0,j}, b_{1,k}, x_l\}$ is $\boldsymbol{H}_4$-free.
- If $\varepsilon(i, j, k) = 1$ but $\varepsilon(i, k, l) = 0$, then we have $\models R(c_i, b_{0,j}, b_{1,k})$ and $\Sigma(\bar{x}) \vdash R(c_i, x_l, b_{1,k})$ by (i), and so $\Sigma(\bar{x})$ implies that $\{c_i, b_{0,j}, b_{1,k}, x_l\}$ is $\boldsymbol{H}_4$-free. A similar argument works if $\varepsilon(i, j, k) = 0$ but $\varepsilon(i, k, l) = 1$.

Therefore $\Sigma(\bar{x})$ defines a partial type over $\bar{b}_0\bar{b}_1\bar{c}$ (note that we do not specify $\mathrm{tp}(\bar{b}_0\bar{x})$ since it is not needed for our purposes), and so we choose a realisation $\bar{b}_2^* \models \Sigma$.

Now define $\Gamma(\bar{y})$ with $|\bar{y}| = n$ as the set containing the following formulas:

(i′) $\bar{y}\bar{b}_0 \equiv_{\bar{c}} \bar{b}_2^*\bar{y} \equiv_{\bar{c}} \bar{b}_0\bar{b}_1$.

(ii′) $R(b_{0,i}, b_{2,j}^*, y_k)$ for all $i, j, k < n$.

Once again, by the two stipulations, it suffices to show that no set of the form $\{c_i, b_{2,j}^*, y_k, b_{0,l}\}$ gives rise to an $\boldsymbol{H}_4$-structure for some fixed i, j, k and l.

- If $\varepsilon(i, j, k) = \varepsilon(i, k, l) = 1$, then $\Gamma(\bar{y}) \vdash R(c_i, b_{2,j}^*, y_k) \wedge R(b_{0,l}, b_{2,j}^*, y_k)$, and thus $\Gamma(\bar{y})$ implies $\{c_i, b_{2,j}^*, y_k, b_{0,l}\}$ is $\boldsymbol{H}_4$-free.

- If $\varepsilon(i, j, k) = \varepsilon(i, k, l) = 0$, then we have $\models R(c_i, b_{0,l}, b^*_{2,j})$ and $\Gamma(\bar{y}) \vdash R(c_i, b_{0,l}, y_k)$. Hence, once again, $\Gamma(\bar{y})$ implies that $\{c_i, b^*_{2,j}, y_k, b_{0,l}\}$ is $\boldsymbol{H}_4$-free.
- If $\varepsilon(i, j, k) = 1$ but $\varepsilon(i, k, l) = 0$, then $\Gamma(\bar{y}) \vdash R(c_i, b^*_{2,j}, y_k) \wedge R(c_i, b_{0,l}, y_k)$, and so $\Gamma(\bar{y})$ implies that $\{c_i, b^*_{2,j}, y_k, b_{0,l}\}$ is $\boldsymbol{H}_4$-free. A similar argument works if $\varepsilon(i, j, k) = 0$ but $\varepsilon(i, k, l) = 1$.

Hence, again, $\Gamma(\bar{y})$ defines a type over $\bar{b}_0\bar{b}^*_2\bar{c}$, and so by ω-saturation we can choose a realisation $\bar{b}^*_3 \models \Gamma$. Then $\models p(\bar{b}_0, \bar{b}_1) \wedge p(\bar{b}_1, \bar{b}^*_2) \wedge p(\bar{b}^*_2, \bar{b}^*_3) \wedge p(\bar{b}^*_3, \bar{b}_0)$. □

This completes our first proof of Theorem 1.0.1.

Remark 4.0.6. Surprisingly, we have not been able to produce a direct proof that the $\boldsymbol{H}_4$-free 3-hypertournament is NSOP, despite the fact that NSOP$_4$ is a stronger property.

5. $\leq_K$-independence

In order to apply Mutchnik's framework [2022a] to offer a different proof of the above classification results for the $\boldsymbol{H}_4$-free 3-hypertournament, we need to slightly generalise them. In their current form, the results from that paper apply whenever we have an independence relation satisfying, among other properties, stationarity over models. However, as will become clear in the next section, we want to apply them in the context of a nonstationary independence relation. To achieve this, in this section, we introduce an independence relation having a close relationship with *strong Lascar independence* as defined in [Tartarotti 2023].

5.1. *Defining $\leq_K$-independence.* In this section, we work in a general theory T with monster model $\mathbb{M}$.

Definition 5.1.1. For $M \models T$, we define $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\leq_K}_M \bar{b}$ if $\mathrm{tp}(\bar{a}/M\bar{b})$ extends to a $\leq_K$-least extension of $\mathrm{tp}(\bar{a}/M)$.

We introduce some auxiliary notation that will be useful for the following proofs. Let $\sigma \in \mathrm{Aut}(\mathbb{M})$. For a formula $\varphi(\bar{x}) \in \mathcal{L}(M)$, let us write $\sigma(\varphi(\bar{x})) \in \mathcal{L}(\sigma(M))$ for the formula where the parameters are shifted by σ. Similarly, for a global type q, we write $\sigma(q) := \{\sigma(\varphi(\bar{x})) : \varphi(\bar{x}) \in q\}$.

Lemma 5.1.2. *Let q be a global M-invariant type, and let $\sigma \in \mathrm{Aut}(\mathbb{M})$. Then $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$ if and only if $\sigma(\varphi(\bar{x}, \bar{y})) \in \mathrm{cl}_K(\sigma(q))$.*

Proof. It suffices to show one of the implications. First note that, as q is M-invariant, $\sigma(q)$ is $\sigma(M)$-invariant. Now suppose $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$. Let $(\bar{b}_i)_{i\in\omega}$ be a Morley sequence in $\sigma(q)$ over $\sigma(M)$. Then, for any i, since $\bar{b}_i \models \sigma(q)|_{\sigma(M)\bar{b}_{<i}}$ and σ is an automorphism, it follows that $\sigma^{-1}(\bar{b}_i) \models q|_{M\sigma^{-1}(\bar{b}_{<i})}$. Therefore, $(\sigma^{-1}(\bar{b}_i))_{i\in\omega}$ is

a Morley sequence in q over M, and thus, by assumption, $\{\varphi(\bar{x}, \sigma^{-1}(\bar{b}_i)) : i \in \omega\}$ is inconsistent. Hence $\{\sigma(\varphi(\bar{x}, \bar{b}_i)) : i \in \omega\}$ is inconsistent, as required. $\square$

Corollary 5.1.3. *$\mathop{\smile\!\!\!\!\!\mid}^{\leq_K}$ is an independence relation (i.e., it is* $\mathrm{Aut}(\mathbb{M})$*-invariant).*

Proof. By Lemma 5.1.2, we have that q is a $\leq_K$-least extension of $q|_M$ if and only if $\sigma(q)$ is a $\leq_K$-least extension of $\sigma(q)|_{\sigma(M)}$ for any $\sigma \in \mathrm{Aut}(\mathbb{M})$. $\square$

Remark 5.1.4. It is immediate from the definition that $\leq_K$-independence satisfies right monotonicity and right extension. In particular, it has existence if and only if it has full existence. M-$\mathop{\smile\!\!\!\!\!\mid}^{\leq_K}$-free types are precisely the global types which are $\leq_K$-least over their restrictions to M.

Lemma 5.1.5. *If $\mathop{\smile\!\!\!\!\!\mid}^{i}$ satisfies left extension over models, then $\mathop{\smile\!\!\!\!\!\mid}^{\leq_K}$ satisfies left monotonicity.*

Proof. Given a type $p(\bar{x}, \bar{y}) \in S(M)$ and a global M-invariant extension $q(\bar{x}, \bar{y})$, let us write $\tilde{p}(\bar{y})$ and $\tilde{q}(\bar{y})$ for their respective restrictions to complete types in the variables $\bar{y}$. It suffices to show that, if $q(\bar{x}, \bar{y})$ is a $\leq_K$-least extension of $p(\bar{x}, \bar{y}) \in S(M)$, then $\tilde{q}(\bar{y})$ is a $\leq_K$-least extension of $\tilde{p}(\bar{y})$.

Suppose $\varphi(\bar{z}, \bar{y}) \in \mathrm{cl}_K(\tilde{q}(\bar{y}))$. Then $\varphi(\bar{z}, \bar{y}) \in \mathrm{cl}_K(q(\bar{x}, \bar{y}))$, and so, since $q(\bar{x}, \bar{y})$ is a $\leq_K$-least extension of $p(\bar{x}, \bar{y})$, it follows that $\varphi(\bar{z}, \bar{y}) \in \mathrm{cl}_K(r(\bar{x}, \bar{y}))$ for all global M-invariant extensions $r(\bar{x}, \bar{y})$ of $p(\bar{x}, \bar{y})$. Now, for any global M-invariant extension $\tilde{r}(\bar{y})$ of $\tilde{p}(\bar{y})$, we can find, by left extension for $\mathop{\smile\!\!\!\!\!\mid}^{i}$, a global M-invariant completion of $p(\bar{x}, \bar{y}) \cup \tilde{r}(\bar{y})$, and thus it follows that $\varphi(\bar{z}, \bar{y}) \in \mathrm{cl}_K(\tilde{r}(\bar{y}))$. So $\tilde{q}(\bar{y})$ is a $\leq_K$-least extension of $\tilde{p}(\bar{y})$. $\square$

Beyond satisfying the relative Kim's lemma, $\leq_K$-independence also provides a criterion for this property.

Proposition 5.1.6. *Suppose $\mathop{\smile\!\!\!\!\!\mid}$ is an independence relation satisfying full existence. Then $\mathop{\smile\!\!\!\!\!\mid}$ satisfies the relative Kim's lemma if and only if $\mathop{\smile\!\!\!\!\!\mid} \Rightarrow \mathop{\smile\!\!\!\!\!\mid}^{\leq_K}$.*

Proof. ($\Rightarrow$) Suppose $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_M \bar{b}$. By full existence, $\mathrm{tp}(\bar{a}/M\bar{b})$ extends to a global M-$\mathop{\smile\!\!\!\!\!\mid}$-free type q which, by the relative Kim's lemma, is $\leq_K$-least. So $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\leq_K}_M \bar{b}$.

($\Leftarrow$) Suppose q is a global type that is $\mathop{\smile\!\!\!\!\!\mid}$-free over $M \models T$. Since $\mathop{\smile\!\!\!\!\!\mid} \Rightarrow \mathop{\smile\!\!\!\!\!\mid}^{\leq_K}$, it follows that q is also $\mathop{\smile\!\!\!\!\!\mid}^{\leq_K}$-free over M, and thus, by Remark 5.1.4, q is a $\leq_K$-least extension of $q|_M$. $\square$

5.2. ***A criterion for the strong witnessing property.*** The following property was introduced by Mutchnik [2022a, Definition 3.23].

Definition 5.2.1. Let T be a complete theory.

(i) We say a global type q is a *strong witnessing extension* of $q|_M$ for $M \models T$ if, for all $\bar{a} \models q|_{M\bar{b}}$ and all $\bar{c}$, there is some $\bar{c}' \equiv_{M\bar{a}} \bar{c}$ such that $\mathrm{tp}(\bar{a}\bar{c}'/M\bar{b})$ extends to a $\leq_K$-least extension of $\mathrm{tp}(\bar{a}\bar{c}/M)$.

(ii) We say T has the *strong witnessing property* if, for all small $M \models T$, every $p \in S(M)$ has a global strong witnessing extension.

There are no known NSOP_4 counterexamples to the above property. The main application is given in [Mutchnik 2022a, Theorem 3.25], where it is shown that any theory with the strong witnessing property is either simple or TP_2.

We introduce the following operation: given an independence relation $\mathop{\smile\!\!\!\!\!\mid}$, define

$$A \mathop{\smile\!\!\!\!\!\mid}_C^{\mathrm{le}} B \quad\Longleftrightarrow\quad \text{for all } D \supseteq A, \text{ there exists } B' \equiv_{AC} B \text{ with } D \mathop{\smile\!\!\!\!\!\mid}_C B'.$$

Some versions of this operation have previously appeared in the literature, although not at this level of generality; see, e.g., [Dobrowolski and Kamsma 2022]. It is immediate from this definition that:

- $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{le}} \Rightarrow \mathop{\smile\!\!\!\!\!\mid}$.
- If $\mathop{\smile\!\!\!\!\!\mid}^0 \Rightarrow \mathop{\smile\!\!\!\!\!\mid}$ and $\mathop{\smile\!\!\!\!\!\mid}^0$ satisfies left extension, then $\mathop{\smile\!\!\!\!\!\mid}^0 \Rightarrow \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{le}}$.
- $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{le}} = ((\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{opp}})^*)^{\mathrm{opp}}$.

In particular, combining the above properties with Fact 2.3.3 and Remark 2.3.6, it follows that, whenever $\mathop{\smile\!\!\!\!\!\mid}$ satisfies left monotonicity, $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{le}}$ satisfies left monotonicity and left extension, and if in addition $\mathop{\smile\!\!\!\!\!\mid}$ satisfies right monotonicity, so does $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{le}}$.

Lemma 5.2.2. *Let $M \models T$, and let q be a global type. The following are equivalent*:

(i) *q is a strong witnessing extension of $q|_M$.*

(ii) *q is $(\mathop{\smile\!\!\!\!\!\mid}^{\leq_K})^{\mathrm{le}}$-free over M.*

Proof. (i) $\Rightarrow$ (ii) Suppose q is a strong witnessing extension of $q|_M$. Let $\bar{a} \models q|_{M\bar{b}}$. Take some $\bar{c}$. Then, by strong witnessing, there is $\bar{c}' \equiv_{M\bar{a}} \bar{c}$ such that $\bar{a}\bar{c}' \mathop{\smile\!\!\!\!\!\mid}_M^{\leq_K} \bar{b}$, and thus, by invariance, there is $\bar{b}' \equiv_{M\bar{a}} \bar{b}$ such that $\bar{a}\bar{c} \mathop{\smile\!\!\!\!\!\mid}_M^{\leq_K} \bar{b}'$. Therefore, we have $\bar{a}\ (\mathop{\smile\!\!\!\!\!\mid}_M^{\leq_K})^{\mathrm{le}}\ \bar{b}$, as required.

(ii) $\Rightarrow$ (i) Suppose q is $(\mathop{\smile\!\!\!\!\!\mid}^{\leq_K})^{\mathrm{le}}$-free over M. Take $\bar{a} \models q|_{M\bar{b}}$ and some $\bar{c}$. Then, by assumption and invariance, there is $\bar{c}' \equiv_{M\bar{a}} \bar{c}$ such that $\bar{a}\bar{c}' \mathop{\smile\!\!\!\!\!\mid}_M^{\leq_K} \bar{b}$. Therefore, q is a strong witnessing extension of $q|_M$, as required. □

Theorem 5.2.3. *If there is an independence relation $\mathop{\smile\!\!\!\!\!\mid}$ over models of T satisfying full existence*, *left extension*, *and the relative Kim's lemma*, *then T satisfies the strong witnessing property.*

Moreover, *if*, *in addition*, *$\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{i}}$ satisfies left extension over models in T*, *then the converse holds.*

Proof. Suppose that such an independence relation $\mathop{\smile\!\!\!\!\!\mid}$ exists. By Proposition 5.1.6, we have $\mathop{\smile\!\!\!\!\!\mid} \Rightarrow \mathop{\smile\!\!\!\!\!\mid}^{\leq_K}$, and, since the relation $\mathop{\smile\!\!\!\!\!\mid}$ satisfies left extension, we conclude that $\mathop{\smile\!\!\!\!\!\mid} \Rightarrow (\mathop{\smile\!\!\!\!\!\mid}^{\leq_K})^{\mathrm{le}}$. But now, since $\mathop{\smile\!\!\!\!\!\mid}$ satisfies full existence, every type $p \in S(M)$ has a global M-$\mathop{\smile\!\!\!\!\!\mid}$-free extension, which by the above is also $(\mathop{\smile\!\!\!\!\!\mid}^{\leq_K})^{\mathrm{le}}$-free over M. Thus, by Lemma 5.2.2, T satisfies the strong witnessing property.

For the "moreover" part, suppose that T satisfies the strong witnessing property. In particular, by Lemma 5.2.2, this entails that every type $p \in S(M)$ has a global M-$(\mathop{\smile\!\!\!\!\!\!\mid}^{\leq_K})^{\mathrm{le}}$-free extension, which implies that $(\mathop{\smile\!\!\!\!\!\!\mid}^{\leq_K})^{\mathrm{le}}$ satisfies full existence. In particular, by Proposition 5.1.6, $(\mathop{\smile\!\!\!\!\!\!\mid}^{\leq_K})^{\mathrm{le}}$ satisfies the relative Kim's lemma. Finally, as $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$ satisfies left extension by assumption, $\mathop{\smile\!\!\!\!\!\!\mid}^{\leq_K}$ satisfies left monotonicity by Lemma 5.1.5, and thus $(\mathop{\smile\!\!\!\!\!\!\mid}^{\leq_K})^{\mathrm{le}}$ satisfies left extension. □

Corollary 5.2.4. *Suppose there is an independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ over models satisfying full existence, left extension, and the relative Kim's lemma. Then T is either simple or* TP_2.

Proof. This is just a reformulation of [Mutchnik 2022a, Theorem 3.25]. □

6. Second proof of NSOP_4

6.1. ***Triviality of Conant-independence.*** Our second proof of Theorem 1.0.1 goes via properties of independence relations, using several results from [Mutchnik 2022a]. First, we recall the key result we employ in this section.

Definition 6.1.1. Let T be a complete theory and $M \models T$.

(i) We say $\varphi(\bar{x}, \bar{b})$ *Conant-divides over M* if, for all M-invariant Morley sequences $(\bar{b}_i)_{i\in\omega}$ with $\bar{b}_0 = \bar{b}$, the set $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$ is inconsistent.

(ii) We say $\varphi(x, b)$ *Conant-forks over M* if there exist formulas $\psi_i(x, c_i)$ for $i < n$ such that $\varphi(x, b) \vdash \bigvee_{i<n} \psi_i(x, c_i)$ and each $\psi_i(x, c_i)$ Conant-divides over M.

(iii) We say $\bar{a}$ is *Conant-independent* from $\bar{b}$ over M if $\mathrm{tp}(\bar{a}/M\bar{b})$ does not contain a formula that Conant-forks over M.

Remark 6.1.2. In the theory $T_{\boldsymbol{H}_4\text{-free}}$, since $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$ satisfies left extension over models by Corollary 3.2.9, we have that $\varphi(\bar{x}, \bar{b})$ Conant-forks over M if and only if it Conant-divides over M. The proof of this fact is exactly the same as that of [Mutchnik 2022b, Proposition 5.2] after replacing all instances of "coheir Morley sequence over M" by "M-invariant Morley sequence," since left extension is the main property of coheir extensions used there. (Note that Conant-dividing has a different definition in [Mutchnik 2022b] than the one we are using in this document.)

Fact 6.1.3 [Mutchnik 2022a, Theorem 6.2]. If Conant-independence is symmetric, then T is NSOP_4.

Now work in $\mathbb{M} \models T_{\boldsymbol{H}_4\text{-free}}$. Let us define an independence relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ht}}$ over subsets of $\mathbb{M}$ as follows: $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ht}}_C B$ if and only if $A \cap B \subseteq C$ and $\models R(a, c, b)$ for all $a \in A \setminus C$, $c \in C$, and $b \in B \setminus C$.

Lemma 6.1.4. $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ht}}$ *is an independence relation satisfying monotonicity, full existence, and left extension.*

Proof. Invariance and monotonicity are immediate. Full existence and left extension follow similar proofs to those of consistency we have done before. □

Remark 6.1.5. $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$ does not satisfy stationarity (over arbitrary subsets) nor symmetry. In fact, if $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}_C B$ and $\varnothing \neq C \subset A, B$, then, for all $a \in A \setminus C,\ c \in C$, and $b \in B \setminus C$, we have $\models R(a, c, b)$, and thus $\not\models R(b, c, a)$. Hence $B \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}_C A$.

So $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$ is neither a stationary weak independence relation in the sense of [Li 2019] nor a free amalgamation relation in the sense of [Conant 2017]. It is also different from the independence relations studied in [Mutchnik 2022a].

Remark 6.1.6. We can characterise $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$-Morley sequences over some $M \models T_{\boldsymbol{H}_4\text{-free}}$ starting at some $\bar{b}_0$ disjoint from M as precisely those sequences $(\bar{b}_i)_{i\in\omega}$ such that $\models R(m, b_{i,k}, b_{j,l})$ for all $i < j$ and any indices k, l and elements $m \in M$.

Proposition 6.1.7. *For (possibly infinite) tuples $\bar{a}$ and $\bar{b}$ and a model $M \models T_{\boldsymbol{H}_4\text{-free}}$, $\mathrm{tp}(\bar{a}/M\bar{b})$ does not $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$-Kim-divide over M if and only if $\bar{a} \cap \bar{b} \subseteq M$.*

Proof. It is enough to prove the right-to-left direction. Suppose $\bar{a} \cap \bar{b} \subseteq M$. Note that we may assume that $\bar{a}$ and $\bar{b}$ are both finite tuples. Also note that, if either $\bar{a} \subseteq \bar{b}$ or $\bar{b} \subseteq \bar{a}$ hold, then there is nothing to do. So, without loss of generality, suppose that $\bar{a} \cap \bar{b} = \varnothing$ (in particular, they are disjoint from M). Let $(\bar{b}_i)_{i\in\omega}$ be an $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$-Morley sequence over M starting with $\bar{b}$. Let $\varphi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{a}/M\bar{b})$, which, since $T_{\boldsymbol{H}_4\text{-free}}$ is ω-categorical, we may assume is a formula isolating a complete type over $M'\bar{b}$ for some finite $M' \subset M$. We claim that $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$ is consistent.

We show that

$$\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\} \cup \{R(x_k, b_{i,l}, b_{j,l'}) : i < j \in \omega\}$$

is consistent. Note that, by the condition on elements from different indexed tuples from the $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$-Morley sequence and the fact that φ isolates a complete type, all 4-substructures containing only variables and elements from the $\bar{b}_i$'s are $\boldsymbol{H}_4$-free. Moreover, since φ isolates a type, we have, for any $i, j \in \omega$, that $R(x_k, m, b_{i,l}) \in \varphi(\bar{x}, \bar{b}_i)$ if and only if $R(x_k, m, b_{j,l}) \in \varphi(\bar{x}, \bar{b}_j)$, and thus the set $\{x_k, m, b_{i,l}, b_{j,l}\}$ is $\boldsymbol{H}_4$-free. Finally, since $(\bar{b}_i)_{i\in\omega}$ is an $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$-Morley sequence over M, we have $\models R(m, b_{i,l}, b_{j,l'})$ whenever $i < j$, and so it follows that $\{x, m, b_{i,l}, b_{j,l'}\}$ is $\boldsymbol{H}_4$-free. These are all the possible $\boldsymbol{H}_4$-substructures.

Therefore, $\mathrm{tp}(\bar{a}/M\bar{b})$ does not $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$-Kim-divide over M, as required. □

Lemma 6.1.8. *Every type $p \in S(M)$ has a global M-invariant extension that is $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}$-free over M.*

Proof. It suffices to show that p has a global M-invariant extension q such that $R(x, m, a) \in q$ for all $m \in M$ and $a \notin M$. Let us write $p(\bar{x}) = \mathrm{tp}(\bar{b}/M)$ for some realisation $\bar{b}$ in the monster. We may assume, without loss of generality, that $\bar{b}$ is disjoint from M. Fix some tuple $\bar{m}$ from M of the same length as $\bar{b}$. Then it is enough to show that the extension of $\mathrm{tp}(\bar{b}/M)$ obtained by adding the formulas

(i) $R(x, m, a)$ for all $m \in M$ and $a \notin M$,

(ii) $\varphi(\bar{x}, \bar{a})$ whenever $\bar{a} \cap M = \varnothing$ and $\models \varphi(\bar{m}, \bar{a})$

is consistent. (Note that M-invariance is clear.)

As usual, it suffices to check all possible $\boldsymbol{H}_4$-substructures. Note that, by (ii), no 4-point set containing only variables and elements outside of M can be $\boldsymbol{H}_4$. So the only possibilities are $\{x_i, x_j, m, a\}$, $\{x_i, m, m', a\}$ and $\{x_i, m, a, a'\}$ for $m, m' \in M$ and $a, a' \notin M$. But all these are $\boldsymbol{H}_4$-free by (i). $\square$

Proof of Proposition 4.0.5 (*version* 2). We claim that $\bar{a}$ is Conant-independent from $\bar{b}$ over M if and only if $\bar{a} \cap \bar{b} \subseteq M$. The left-to-right direction is always true. For the converse, suppose that $\bar{a} \cap \bar{b} \subseteq M$. By Proposition 6.1.7, $\operatorname{tp}(\bar{a}/M\bar{b})$ does not $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}$-Kim-divide over M. Moreover, by Lemma 6.1.8, there exists an M-invariant $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}$-Morley sequence $(\bar{b}_i)_{i\in\omega}$ over M with $\bar{b}_0 = b$. So, for all $\varphi(\bar{x}, \bar{b}) \in \operatorname{tp}(\bar{a}/M\bar{b})$, $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$ is consistent, and thus $\varphi(\bar{x}, \bar{b})$ does not Conant-divide over M. Therefore, by Remark 6.1.2, $\bar{a}$ is Conant-independent from $\bar{b}$ over M.

In particular, it follows that Conant-independence is symmetric. Thus, by Fact 6.1.3, $T_{\boldsymbol{H}_4\text{-free}}$ is NSOP$_4$. $\square$

6.2. *Relative Kim's lemma.* We focus on another one of the notions introduced by Mutchnik, namely, the relative Kim's lemma. In view of Remark 2.4.7, we immediately notice that $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}$ cannot satisfy the relative Kim's lemma, as global M-$\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}$-free types might fail to be M-invariant.

To fix this, let us define $\bar{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{hti}}_M B$ if and only if $\bar{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{i}}_M B$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}_M B$.

Lemma 6.2.1. $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{hti}}$ *is an independence relation satisfying monotonicity*, *full existence*, *and left extension.*

Proof. (Monotonicity) This holds, as it is satisfied by both $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}$ and $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{i}}$.

(Full existence) Let $\bar{a}$ be a tuple, $M \models T$, and B a set. Let $p := \operatorname{tp}(\bar{a}/M)$. By Lemma 6.1.8, there is a global M-invariant extension q of p which is $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}$-free over M. Let $\bar{a}' \models q|_{MB}$, so that $\bar{a}' \equiv_M \bar{a}$. Then $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{i}}_M B$ by M-invariance of q, and $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}_M B$ since q is $\mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ht}}$-free over M. Hence $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{hti}}_M B$.

(Left extension) This proof is analogous to that of Lemma 6.1.8: Suppose that $\bar{a}$, $M \models T$, and B are given such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{hti}}_M B$, and $\bar{a}'$ is a tuple extending $\bar{a}$. We may write $\bar{a}'$ as the concatenation of $\bar{a}$ with some other tuple $\bar{c}$. Let $\bar{b}$ be a (possibly infinite) tuple enumerating B. Fix a tuple $\bar{m}$ in M of the same length as $\bar{c}$. We now want to show the consistency of the extension of $\operatorname{tp}(\bar{b}/M\bar{a})$ by the following set of formulas:

(i) $R(c_i, m, x_j)$ for all indices i, j of the appropriate length and $m \in M$.

(ii) $\varphi(\bar{c}, \bar{x}) \leftrightarrow \varphi(\bar{m}, \bar{x})$ for any $\mathcal{L}$-formula φ.

As usual, it suffices to show that this does not generate any $\boldsymbol{H}_4$-structures. In this case, the proof is analogous to that of Lemma 6.1.8, so we omit it. Pick a realisation $\bar{b}'$ of this type. By (i), we have $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ht}}_M \bar{b}'$, and, by (ii), $\mathrm{tp}(\bar{a}'/M\bar{b}')$ is M-invariant. Therefore, letting B' be the set enumerated by $\bar{b}'$, it follows that $B' \equiv_{M\bar{a}} B$ and $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{hti}}_M B'$. $\square$

We can now also see that the Morley sequences appearing in Lemma 6.1.8 are precisely $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{hti}}$-Morley sequences. So, from Proposition 6.1.7, we immediately obtain the following.

Corollary 6.2.2. *For (possibly infinite) tuples $\bar{a}$ and $\bar{b}$ and a model $M \models T_{\boldsymbol{H}_4\text{-free}}$, $\mathrm{tp}(\bar{a}/M\bar{b})$ does not $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{hti}}$-Kim-divide over M if and only if $\bar{a} \cap \bar{b} \subseteq M$.*

Our goal now is to prove that $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{hti}}$-free global types are $\leq_K$-least. At this point, ω-categoricity is particularly useful.

Lemma 6.2.3. *Let T be ω-categorical, $M \models T$, and q, r be global M-invariant types with $q|_M = r|_M$. Assume that, for all formulas $\varphi(\bar{x}, \bar{y})$ isolating a complete type over some finite subset of M, if $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$, then $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(r)$. Then $\mathrm{cl}_K(q) \subseteq \mathrm{cl}_K(r)$.*

Proof. Let $\psi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$. Since T is ω-categorical, there are finitely many types over a finite set of parameters and every such type is principal, so we can write $\psi(\bar{x}, \bar{y}) := \bigvee_{i<l} \varphi_i(\bar{x}, \bar{y})$, where each φ_i isolates a complete type over the set of parameters in ψ. Since $\varphi_i(\bar{x}, \bar{y}) \vdash \psi(\bar{x}, \bar{y})$ for each i, it follows that $\varphi_i(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$, and thus, by assumption, $\varphi_i(\bar{x}, \bar{y}) \in \mathrm{cl}_K(r)$ for all $i < l$. Thus, for all M-invariant Morley sequences $(\bar{b}_j)_{j\in\omega}$ in r, the set $\{\varphi_i(\bar{x}, \bar{b}_j) : j \in \omega\}$ is inconsistent for all $i < l$.

Assume, for contradiction, that $\psi(\bar{x}, \bar{y}) \notin \mathrm{cl}_K(r)$. So there exists some M-invariant Morley sequence $(\bar{c}_i)_{i\in\omega}$ in r such that $\{\psi(\bar{x}, \bar{c}_i) : i \in \omega\}$ is consistent. Let $\bar{a}$ be a realisation. Then, by the pigeonhole principle, we can find an infinite subsequence $(\bar{c}_{i_j})_{j\in\omega}$ and some k such that $\bar{a} \models \{\varphi_k(\bar{x}, \bar{c}_{i_j}) : j \in \omega\}$. But note that $(\bar{c}_{i_j})_{j\in\omega}$ is also an M-invariant Morley sequence in r, a contradiction. $\square$

This result tells us that it suffices to look at types over finite parameter sets to determine $\leq_K$-minimality.

Remark 6.2.4. Note that, for a model $M \models T_{\boldsymbol{H}_4\text{-free}}$ and a tuple $\bar{b}$, if $\varphi(\bar{x}, \bar{b}) \vdash \bar{x} \cap \bar{b} \not\subseteq M$, then $\varphi(\bar{x}, \bar{b})$ Conant-divides over M. In other words, for every global M-invariant extension q of $\mathrm{tp}(\bar{b}/M)$, we have $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$.

Lemma 6.2.5. *Let q be a global type $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{hti}}$-free over $M \models T_{\boldsymbol{H}_4\text{-free}}$. For all $\bar{b} \models q|_M$ and all formulas $\varphi(\bar{x}, \bar{b})$ isolating a complete type over $M'\bar{b}$ with $M' \subset M$ finite, if $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$, then $\varphi(\bar{x}, \bar{b}) \vdash \bar{x} \cap \bar{b} \not\subseteq M$.*

Proof. Suppose that $\varphi(\bar{x}, \bar{b}) \vdash \bar{x} \cap \bar{b} \subseteq M$. Since q is a global M-$\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}^{\mathrm{hti}}$-free type, by Corollary 6.2.2, $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$ is consistent for any $(\bar{b}_i)_{i\in\omega}$ such that $\bar{b}_i \models q|_{M\bar{b}_{<i}}$ for all $i \in \omega$. This means that $\varphi(\bar{x}, \bar{y}) \notin \mathrm{cl}_K(q)$. □

Proposition 6.2.6. $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}^{\mathrm{hti}}$ *satisfies the relative Kim's lemma in* $T_{\boldsymbol{H}_4\text{-free}}$.

Proof. Let $p \in S(M)$, and let q be a global extension of p $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}^{\mathrm{hti}}$-free over M. We want to show that q is a $\leq_K$-least extension of p. By Lemma 6.2.3, it is enough to show that, if $\varphi(\bar{x}, \bar{y})$ isolates a complete type over a finite set, then $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$ implies $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(r)$ for all global M-invariant extensions r of p. But notice that, by Lemma 6.2.5, $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(q)$ implies that $\varphi(\bar{x}, \bar{b}) \vdash \bar{x} \cap \bar{b} \not\subseteq M$ for any $\bar{b} \models q|_M$. Hence, by Remark 6.2.4, it follows that $\varphi(\bar{x}, \bar{y}) \in \mathrm{cl}_K(r)$. □

Proof of Proposition 4.0.3 (*version* 2). By Proposition 4.0.2, $T_{\boldsymbol{H}_4\text{-free}}$ is SOP_3, and so, in particular, it is not simple. Since $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}^{\mathrm{hti}}$ satisfies full existence and left extension by Lemma 6.2.1 and the relative Kim's lemma by Proposition 6.2.6, it follows by Corollary 5.2.4 that $T_{\boldsymbol{H}_4\text{-free}}$ is TP_2. □

This completes our second proof of Theorem 1.0.1.

Remark 6.2.7. Note that, by [Conant 2017, Lemma 4.3], there does not exist an independence relation $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}$ over subsets of $\mathbb{M} \models T_{\boldsymbol{H}_4\text{-free}}$ satisfying invariance, full existence, symmetry, and stationarity over models, because, for any pair $ab \in \mathbb{M}$ of distinct elements and $M \models T_{\boldsymbol{H}_4\text{-free}}$, if $\models R(m, a, b)$ for some $m \in M$, then $\models \neg R(m, b, a)$, and so $ab \not\equiv_M ba$. This means that $T_{\boldsymbol{H}_4\text{-free}}$ is not a free amalgamation theory, so we cannot apply Conant's methods to show NSOP_4. The same argument also shows that we cannot use the criterion from [d'Elbée et al. 2025, Theorem 5.10] either.

Remark 6.2.8. Mutchnik [2022a] uses his Theorem 3.25 (compare Corollary 5.2.4) in the context of a theory with an independence relation $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}$ that satisfies monotonicity, full existence, and stationarity over models. It is shown there that, if $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}$ satisfies a generalisation of freedom, known as generalised freedom, then satisfying the relative Kim's lemma and the symmetry of $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}$-Kim-independence are equivalent, which means that we do not need to check them separately as we did here. However, it is unclear whether one can find such an independence relation in $T_{\boldsymbol{H}_4\text{-free}}$.

Question 6.2.9. Is there an independence relation $\mathop{\smile\hskip-0.9em\raise0.7ex\hbox{$|$}}$ in $T_{\boldsymbol{H}_4\text{-free}}$ satisfying monotonicity, full existence, and stationarity over models?

Acknowledgments

The present work was completed during my PhD at Imperial College London supported by an EPSRC scholarship. I am deeply grateful to my supervisors, David Evans and Charlotte Kestner, for their guidance, suggestions, and constant support.

I would also like to thank Nicholas Ramsey, Scott Mutchnik, and Paolo Marimon for the enlightening discussions on the results of this paper, and Christian d'Elbée for the encouragement to write this article. Finally, I thank the anonymous referee for a very careful reading of this paper and many useful suggestions to improve its quality.

References

[Abd Aldaim et al. 2025] A. Abd Aldaim, G. Conant, and C. Terry, "Higher arity stability and the functional order property", *Selecta Math.* (*N.S.*) **31**:3 (2025), art. id. 59. MR

[Adler 2005] H. Adler, "Explanation of Independence", preprint, 2005. arXiv math/0511616

[Assous 1986] R. Assous, "Enchaînabilité et seuil de monomorphie des tournois n-aires", *Discrete Math.* **62**:2 (1986), 119–125. MR

[Baldwin 1988] J. T. Baldwin, *Fundamentals of stability theory*, Springer, 1988. MR

[Ben Yaacov and Chernikov 2014] I. Ben Yaacov and A. Chernikov, "An independence theorem for NTP_2 theories", *J. Symb. Log.* **79**:1 (2014), 135–153. MR

[Bodirsky et al. 2011] M. Bodirsky, M. Pinsker, and T. Tsankov, "Decidability of definability", pp. 321–328 in 26*th annual IEEE symposium on logic in computer science — LICS* 2011, IEEE Computer Soc., Los Alamitos, CA, 2011. MR

[Cameron 1990] P. J. Cameron, *Oligomorphic permutation groups*, London Mathematical Society Lecture Note Series **152**, Cambridge University Press, 1990. MR

[Casanovas and Farré 2004] E. Casanovas and R. Farré, "Weak forms of elimination of imaginaries", *MLQ Math. Log. Q.* **50**:2 (2004), 126–140. MR

[Cherlin 2022] G. Cherlin, *Homogeneous ordered graphs, metrically homogeneous graphs, and beyond, II: 3-multi-graphs and 2-multi-tournaments*, Lecture Notes in Logic **54**, Cambridge University Press, 2022. MR

[Cherlin et al. 2021] G. Cherlin, J. Hubička, M. Konečný, and J. Nešetřil, "Ramsey expansions of 3-hypertournaments", pp. 696–701 in *Extended abstracts EuroComb* 2021, edited by J. Nešetřil et al., Trends in Mathematics: Research Perspectives CRM Barcelona **14**, Springer, 2021. MR

[Chernikov and Hempel 2019] A. Chernikov and N. Hempel, "Mekler's construction and generalized stability", *Israel J. Math.* **230**:2 (2019), 745–769. MR

[Chernikov and Hempel 2021] A. Chernikov and N. Hempel, "On n-dependent groups and fields II", *Forum Math. Sigma* **9** (2021), art. id. e38. MR

[Chernikov and Kaplan 2012] A. Chernikov and I. Kaplan, "Forking and dividing in NTP_2 theories", *J. Symbolic Logic* **77**:1 (2012), 1–20. MR

[Chernikov et al. 2019] A. Chernikov, D. Palacin, and K. Takeuchi, "On n-dependence", *Notre Dame J. Form. Log.* **60**:2 (2019), 195–214. MR

[Conant 2012] G. Conant, "Dividing lines in unstable theories", expository article, 2012, see https://homepages.math.uic.edu/~gconant//Math/Properties_in_Unstable_Theories.pdf.

[Conant 2017] G. Conant, "An axiomatic approach to free amalgamation", *J. Symb. Log.* **82**:2 (2017), 648–671. MR

[d'Elbée 2023] C. d'Elbée, "Axiomatic theory of independence relations in model theory", preprint, 2023. arXiv 2308.07064

[d'Elbée et al. 2025] C. d'Elbée, I. Müller, N. Ramsey, and D. Siniora, "Model-theoretic properties of nilpotent groups and Lie algebras", *J. Algebra* **662** (2025), 640–701. MR

[Dobrowolski and Kamsma 2022] J. Dobrowolski and M. Kamsma, "Kim-independence in positive logic", *Model Theory* **1** (2022), 55–113. MR

[Evans and Wong 2009] D. M. Evans and M. W. H. Wong, "Some remarks on generic structures", *J. Symbolic Logic* **74**:4 (2009), 1143–1154. MR

[Hempel 2016] N. Hempel, "On n-dependent groups and fields", *MLQ Math. Log. Q.* **62**:3 (2016), 215–224. MR

[Henson 1972] C. W. Henson, "Countable homogeneous relational structures and $\aleph_0$-categorical theories", *J. Symbolic Logic* **37** (1972), 494–500. MR

[Johnson and Ye 2025] W. Johnson and J. Ye, "Curve-excluding fields", *J. Eur. Math. Soc.* (online publication March 2025).

[Kaplan and Ramsey 2020] I. Kaplan and N. Ramsey, "On Kim-independence", *J. Eur. Math. Soc.* (*JEMS*) **22**:5 (2020), 1423–1474. MR

[Kaplan and Simon 2019] I. Kaplan and P. Simon, "Automorphism groups of finite topological rank", *Trans. Amer. Math. Soc.* **372**:3 (2019), 2011–2043. MR

[Kaplan et al. 2019] I. Kaplan, N. Ramsey, and S. Shelah, "Local character of Kim-independence", *Proc. Amer. Math. Soc.* **147**:4 (2019), 1719–1732. MR

[Kruckman 2019] A. Kruckman, "Disjoint n-amalgamation and pseudofinite countably categorical theories", *Notre Dame J. Form. Log.* **60**:1 (2019), 139–160. MR

[Kruckman and Ramsey 2024] A. Kruckman and N. Ramsey, "A new Kim's lemma", *Model Theory* **3**:3 (2024), 825–860. MR

[Lachlan 1984] A. H. Lachlan, "Countable homogeneous tournaments", *Trans. Amer. Math. Soc.* **284**:2 (1984), 431–461. MR

[Lascar and Poizat 1979] D. Lascar and B. Poizat, "An introduction to forking", *J. Symbolic Logic* **44**:3 (1979), 330–350. MR

[Li 2018] Y. Li, "Simplicity of the automorphism groups of some binary homogeneous structures determined by triangle constraints", preprint, 2018. arXiv 1806.01671

[Li 2019] Y. Li, "Automorphism groups of homogeneous structures with stationary weak independence relations", preprint, 2019. arXiv 1911.08540

[Mutchnik 2022a] S. Mutchnik, "Conant-independence and generalized free amalgamation", preprint, 2022. arXiv 2210.07527

[Mutchnik 2022b] S. Mutchnik, "On NSOP_2 theories", preprint, 2022. arXiv 2206.08512

[Pillay 1983] A. Pillay, *An introduction to stability theory*, Oxford Logic Guides **8**, Oxford University Press, 1983. MR

[Poizat 2000] B. Poizat, *A course in model theory*, Springer, 2000. MR

[Shelah 1978] S. Shelah, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics **92**, North-Holland Publishing Co., 1978. MR

[Shelah 1996] S. Shelah, "Toward classifying unstable theories", *Ann. Pure Appl. Logic* **80**:3 (1996), 229–255. MR

[Shelah 2014] S. Shelah, "Strongly dependent theories", *Israel J. Math.* **204**:1 (2014), 1–83. MR

[Tartarotti 2023] M. Tartarotti, *The Lascar group and reconstruction problems in model theory*, master's thesis, University of Oxford, 2023, see https://mira-tartarotti.github.io/tartarotti_msc_thesis.pdf.

[Terry and Wolf 2021] C. Terry and J. Wolf, "Higher-order generalizations of stability and arithmetic regularity", preprint, 2021. arXiv 2111.01739

Received 17 May 2024. Revised 30 Jul 2025.

ALBERTO MIGUEL-GÓMEZ:

a.miguel-gomez22@imperial.ac.uk
Department of Mathematics, Imperial College London, London, United Kingdom

Model Theory
Vol. 4, No. 3, 2025
https://doi.org/10.2140/mt.2025.4.231

Infinite cliques in simple and stable graphs

Yatir Halevi, Itay Kaplan and Saharon Shelah

Suppose that G is a graph of cardinality μ^+ with chromatic number $\chi(G) \geq \mu^+$. One possible reason that this could happen is if G contains a clique of size μ^+. We prove that this is indeed the case when the edge relation is stable. When G is a random graph (which is simple but not stable), this is not true. But still if in general the complete theory of G is simple, G must contain finite cliques of unbounded sizes.

1. Introduction

The chromatic number $\chi(G)$ of a graph G is the minimal cardinal κ for which there exists a vertex coloring with κ colors such that connected vertices get different colors.

Research around graphs having an uncountable chromatic number has a long history; see, e.g., [Komjáth 2011, Section 3]. This topic is set-theoretic in nature, with many results being independent of the axioms of set theory (ZFC). In [Halevi et al. 2022; 2024] we studied a specific conjecture (Taylor's strong conjecture) in the context of *stable* graphs (in ZFC): a graph whose first-order theory is stable (a model-theoretic notion of tameness; see Section 2 for all the definitions). It turns out that a very close relative of this conjecture holds for stable graphs (although it does not hold in general).

More specifically, we showed that if a stable graph has chromatic number $> \beth_2(\aleph_0)$ then this implies the presence of all the finite subgraphs of a shift graph $\mathrm{Sh}_n(\omega)$ for some $0 < n < \omega$, where for a cardinal κ, the shift graph $\mathrm{Sh}_n(\kappa)$ is the graph whose vertices are increasing n-tuples of ordinals in κ, and two such tuples s, t are connected if for every $1 \leq i \leq n-1$, $s(i) = t(i-1)$ or vice-versa (see Example 2.4). In turn, this implies that the chromatic numbers of elementary extensions of said graph are unbounded. An important example for this paper is the

Halevi would like to thank the Israel Science Foundation for its support of this research (grant no. 555/21 and 290/19). Kaplan would like to thank the Israel Science Foundation for their support of this research (grant no. 804/22). Shelah would like to thank the Israel Science Foundation (grants no. 1838/19 and 2320/23). This is paper no. 1256 in Shelah's publication list.
MSC2020: 03C45, 05C15, 03C50.
Keywords: chromatic number, stable graphs, simple graphs, infinite cliques, Taylor's conjecture.

case $n = 1$: $\mathrm{Sh}_1(\kappa)$ is the complete graph on κ. For more on this result, see [Halevi et al. 2022; 2024].

In this paper, we take the first step towards identifying the n in the previous paragraph, by considering not only the chromatic number of the graph, but its *cardinality* as well, as we now explain.

Bounds for the chromatic number of the shift graph were computed by Erdős and Hajnal [1966]: assuming the generalized continuum hypothesis (GCH),

$$\chi(\mathrm{Sh}_{n+1}(\kappa^{+n})) = \kappa$$

for all $n < \omega$; see Fact 2.6. In particular (and trivially), $\chi(\mathrm{Sh}_1(\kappa)) = \kappa$. Thus, it makes sense to ask the following question:

Question 1.1. Suppose that G is a stable graph and for simplicity also assume GCH. Assume that for every cardinal κ, there is some $G' \equiv G$ (i.e., $\mathrm{Th}(G) = \mathrm{Th}(G')$) of cardinality $|G'| \leq \kappa^{+n}$ satisfying $\chi(G') \geq \kappa^+$. Then is it true that for some $m \leq n$, G contains all finite subgraphs of $\mathrm{Sh}_m(\omega)$?

Remark 1.2. Proposition 3.12 and Remark 3.13 explain why we restrict ourselves to successor cardinals.

In this paper we deal with the case $n = 1$, and we manage to give a satisfying solution to this case (and more) assuming that the theory of G is simple or that the edge relation is stable (both weaker assumptions than stability of the theory). The following sums up the main results of this paper:

Main Theorem 1.3 (Propositions 3.2 and 3.11). *Let $G = (V, E)$ be a graph and $T = \mathrm{Th}(G, E)$ be its first-order theory. Assume that $|G| = \mu^+$ and $\chi(G) \geq \mu^+$ for some infinite cardinal μ.*

(1) *Assuming T is a simple theory then G contains cliques of any finite size.*

(2) *Assuming the edge relation E is stable then G contains an infinite clique of cardinality μ^+.*

Note that the conclusion of item (1) (together with Löwenheim–Skolem and compactness) implies the existence of $G' \succ G$ of cardinality μ^+ that contains an infinite clique of cardinality μ^+. The conclusion of item (2) is stronger: we can find such a clique already in G itself.

Organization of the paper. In Section 2 we go over the relevant basic definitions in model theory and graph theory. Section 3 contains the proof of the main results. Section 4 reframes the main results in terms of the function hinted at in Question 1.1. Finally, in the Appendix we analyze an example of Hajnal and Komjáth that refutes Taylor's strong conjecture and shows that the theory of this graph is not stable (in fact, we show more: that it is not simple and has IP).

2. Preliminaries

We use small latin letters a, b, c for tuples and capital letters A, B, C for sets. We also employ the standard model-theoretic abuse of notation and write $a \in A$ even for tuples when the length of the tuple is immaterial or understood from context.

Stability and simplicity. We use fairly standard model-theoretic terminology and notation; see, for example, [Tent and Ziegler 2012]. We gather some of the needed notions. For stability, the reader may also consult with [Shelah 1978].

We denote by $\mathrm{tp}(a/A)$ the complete type of a over A. A structure M is *κ-saturated*, for a cardinal κ, if any type p over A with $|A| < \kappa$ is realized in M. The structure M is *saturated* if it is $|M|$-saturated.

The *monster model* of a complete theory T, denoted here by $\mathbb{U}$, is a large saturated model containing all sets and models (as elementary substructures) we encounter.[1] All subsets and models are *small*, i.e., of cardinality $< |\mathbb{U}|$.

Given a first-order theory T, a formula $\varphi(x, y)$ is *stable* if we cannot find elements $\langle a_i, b_j \in \mathbb{U} : i, j < \omega\rangle$ such that $\mathbb{U} \vDash \varphi(a_i, b_j)$ if and only if $i < j$.

For any formula $\varphi(x, y)$ we set $\varphi(y, x)^{\mathrm{opp}} = \varphi(x, y)$; it is the same formula but with the roles of the variables replaced. The following is folklore.

Fact 2.1. *Work in a complete first-order theory T with infinite models. Let $\varphi(x, y)$ be a stable formula. There is a formula $\psi(y, z)$ such that any φ-type p over a model M is definable by an instance of ψ over M, i.e., for any such φ-type p there is an element $c \in M$ such that $\varphi(x, b) \in p$ if and only if $M \vDash \psi(b, c)$. Moreover, $\psi(y, z)$ can be chosen such that for any $c \in M$, $\psi(y, c)$ is equivalent to a boolean combination of instances of φ^{opp}.*

Proof. By [Shelah 1978, Theorem II.2.12], there is some $\psi(y, z)$ such that for any φ-type p over any set A ($|A| \geq 2$), there is some $c_p \in A$ such that $\psi(y, c_p)$ defines p. By [Pillay 1996, Lemma 2.2(i)], if $p \in S_\varphi(M)$, where M is a model, then p is definable by a boolean combination of instances of φ^{opp}. But then, as M is a model, $\psi(y, c_p)$ is equivalent to such a boolean combination. $\square$

A theory T is *stable* if all formulas are stable.

Next we define simplicity. We give an equivalent definition, using the notion of dividing for types; see [Tent and Ziegler 2012, Proposition 7.2.5]. Given a first-order theory T with a monster model $\mathbb{U}$, a formula $\varphi(x, b)$ with $b \in \mathbb{U}$ *divides* over A if there is a sequence of realizations $\langle b_i \in \mathbb{U} : i < \omega\rangle$ of $\mathrm{tp}(b/A)$ such

[1] There are set-theoretic issues in assuming that such a model exists, but these are overcome by standard techniques from set theory that ensure the generalized continuum hypothesis from some point on while fixing a fragment of the universe; see [Halevi and Kaplan 2023]. The reader can just accept this or alternatively assume that $\mathbb{U}$ is merely κ-saturated and κ-strongly homogeneous for large enough κ.

that $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent for some $k < \omega$ (every subset of size k is inconsistent). A complete type p over B divides over A if it contains some formula which divides over A.

The theory T is *simple* if for every complete type p over B there is some $A \subseteq B$ with $|A| \le |T|$ such that p does not divide over A. Every stable theory is simple.

The main tool we use from simplicity theory is forking calculus. Nonforking independence is a 3-place relation on sets (or tuples) denoted by $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}$. We do not go over all the properties that nonforking independence enjoys in simple theories; see [Tent and Ziegler 2012, Chapter 7] for more information.

Graph theory. Here we gather some facts on graphs and the chromatic number of graphs (see also [Halevi et al. 2022]).

By a *graph* we mean a pair $G = (V, E)$, where $E \subseteq V^2$ is symmetric and irreflexive. A *graph homomorphism* between $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a map $f : V_1 \to V_2$ such that $f(e) \in E_2$ for every $e \in E_1$. If f is injective we say that f embeds G_1 into G_2 a subgraph. If in addition we require that $f(e) \in E_2$ if and only if $e \in E_1$, we say that f embeds G_1 into G_2 as an induced subgraph.

Definition 2.2. Let $G = (V, E)$ be a graph.

(1) For a cardinal κ, a *vertex coloring* (or just coloring) of cardinality κ is a function $c : V \to \kappa$ such that $x \mathrel{E} y$ implies $c(x) \neq c(y)$ for all $x, y \in V$.

(2) The *chromatic number* $\chi(G)$ is the minimal cardinality of a vertex coloring of G.

The following is easy and well known.

Fact 2.3. *Let $G = (V, E)$ be a graph. If $V = \bigcup_{i \in I} V_i$ then*

$$\chi(G) \le \sum_{i \in I} \chi(V_i, E \restriction V_i).$$

Proof. Let $c_i : V_i \to \mu_i$ be a coloring of $(V_i, E \restriction V_i)$. Define a coloring

$$c : V \to \bigcup \{\mu_i \times \{i\} : i \in I\}$$

by choosing for $v \in V$ an $i_v \in I$ such that $v \in V_{i_v}$ and setting $c(v) = (c_{i_v}(v), i_v)$. $\square$

Example 2.4. For any finite number $r \ge 1$ and any linearly ordered set $(A, <)$, let $\mathrm{Sh}_r(A)$ (*the shift graph on A*) be the following graph: its set of vertices is the set of strictly increasing r-tuples $\langle s_0 < \cdots < s_{r-1} \rangle$ from A, and we put an edge between s and t if for every $1 \le i \le r-1$, $s_i = t_{i-1}$, or vice-versa. It is an easy exercise to show that $\mathrm{Sh}_r(A)$ is a connected graph. If $r = 1$ this gives K_A, the complete graph on A.

Example 2.5 (symmetric shift graph). Let $r \geq 1$ be any natural number and A any set. The *symmetric shift graph* $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is defined similarly as the shift graph but with set of vertices the set of distinct r-tuples. Note that $\mathrm{Sh}_r(A)$ is an induced subgraph of $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ (and that for $r = 1$ they are both the complete graph on A).

Since for any infinite set A, $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is definable in $(A, =)$, it is stable. Moreover, for any two infinite sets A and B, $\mathrm{Sh}_r^{\mathrm{sym}}(A) \equiv \mathrm{Sh}_r^{\mathrm{sym}}(B)$. Since every infinite set A is saturated, $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is saturated.

Fact 2.6 [Halevi et al. 2022, Fact 2.6; Erdős and Hajnal 1968, proof of Theorem 2]. *Let $1 \leq r < \omega$ be a natural number and μ be an infinite cardinal. Then*

$$\chi(\mathrm{Sh}_r^{\mathrm{sym}}(\beth_{r-1}(\mu))) \leq \mu,$$
$$\chi(\mathrm{Sh}_r(\beth_{r-1}(\mu)^+)) \geq \mu^+.$$

Remark 2.7. Note that it follows that $\chi(\mathrm{Sh}_r(\beth_{r-1}(\mu))) \leq \mu$.

Lemma 2.8. *If A is infinite then $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is a saturated model of $\mathrm{Th}(\mathrm{Sh}_r^{\mathrm{sym}}(\omega))$ of cardinality $|A|$.*

Proof. This follows easily from the fact that $(A, =)$ is saturated and that $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ is definable in $(A, =)$. □

We prove two easy results on the theory of the shift graphs.

Lemma 2.9. *Every model of $T = \mathrm{Th}(\mathrm{Sh}_r(\omega))$ (of cardinality λ) can be embedded as an induced subgraph of $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ for some infinite set A (of cardinality λ).*

Proof. Since $\mathrm{Sh}_r(\omega)$ is an induced subgraph of $\mathrm{Sh}_r^{\mathrm{sym}}(\omega)$, the former satisfies the universal theory of $\mathrm{Sh}_r^{\mathrm{sym}}(\omega)$. Thus every model M of T can be embedded as an induced subgraph of a model of $\mathrm{Th}(\mathrm{Sh}_r^{\mathrm{sym}}(\omega))$. By Lemma 2.8, and the universality of saturated models, M can be embedded as an induced subgraph of $\mathrm{Sh}_r^{\mathrm{sym}}(A)$ for some infinite set A of cardinality $|M|$. □

Lemma 2.10. *For any cardinal μ, the following holds for the graph $(\mathrm{Sh}_2(\mu), E)$:*

(1) *Its complete theory is not stable.*

(2) *Its graph relation is stable.*

(3) *It is triangle-free.*

Proof. (1) For any $1 < n < \omega$, let X_n be the definable set

$$\{x \in \mathrm{Sh}_2(\mu) : (x\ E\ (0, n)) \wedge \neg(x\ E\ (0, 1))\}.$$

It is easily seen that $X_n = \{(n, k) : k > n\}$. For any $0 < m < \omega$, let

$$Y_m = \{x \in \mathrm{Sh}_2(\mu) : (x\ E\ (m, m+1)) \wedge \neg(x\ E\ (m-1, m+1))\};$$

it is easily seen that $Y_m = \{(k, m) : k < m\}$.

For any pair of natural numbers (n, m) with $n > 1$ and $n < m$, let $\psi_{n,m}(x, y)$ be the formula (with parameters n, m)

$$\exists z(x \in X_n \wedge y \in Y_m \wedge (z\, E\, x) \wedge (z\, E\, y)).$$

For any l with $l < m - n$, let $a_l = (n, n + l) \in X_n$, and for any $0 < k < m - n$ let $b_k = (n + k, m) \in Y_m$. It is easily checked that $\psi_{n,m}(a_l, b_k)$ if and only if $l < k$. Since all the $\psi_{n,m}$ are uniformly definable with parameters, by compactness we get that the theory of $\mathrm{Sh}_2(\mu)$ is not stable.

(2) One can either see directly that the graph relation is stable, or note that since $\mathrm{Sh}_2(\mu)$ is an induced subgraph of the stable graph $\mathrm{Sh}_2^{\mathrm{sym}}(\mu)$, its edge relation must also be stable.

(3) Suppose $(a, b)E(c, d)E(e, f)E(a, b)$. Note also $(c, d)E(a, b)E(e, f)E(c, d)$. Hence, without loss of generality $b = c$. It follows easily that $e = d$. So $a < b < d < f$. Now either $a = f$ or $b = e = d$, so either way we get a contradiction. □

Remark 2.11. (1) Since $\mathrm{Sh}_2(\mu)$ is definable in $(\mu, <)$, it has NIP (not the independence property (IP); see, e.g., [Simon 2015]).

(2) One can also note that $\mathrm{Sh}_4^{\mathrm{sym}}(\mu)$ is triangle-free.

Every graph is a (not necessarily induced) subgraph of a stable graph (e.g., a large enough complete graph). On the other hand, every shift graph $\mathrm{Sh}_n(\mu)$ is an induced subgraph of a stable graph (the symmetric shift graph). This raises the following:

Question 2.12. Is every graph with a stable edge relation an induced subgraph of a stable graph?

3. Infinite cliques

In this section we prove the main results of the paper.

Simple graphs. We start with the following technical result.

Lemma 3.1. *Let M be some structure, in a language L, and assume that $T = \mathrm{Th}(M)$ is simple. Let M' be some expansion of M to a language $L' \supseteq L$. Let $<$ be a linear order on the universe of M and $\alpha, \beta \in M$ elements satisfying that*

(1) $(\mathrm{dcl}_{L'}(\alpha), <)$ *and* $(\mathrm{dcl}_{L'}(\beta), <)$ *are well-orders, and*

(2) *for any* $\gamma \in \mathrm{dcl}_{L'}(\alpha) \cup \mathrm{dcl}_{L'}(\beta)$, $\gamma \downarrow_{\mathrm{dcl}_{L'}(\gamma) \cap \{\varepsilon \in M : \varepsilon < \gamma\}} \{\varepsilon \in M : \varepsilon < \gamma\}$.

Then

$$\mathrm{dcl}_{L'}(\alpha) \mathop{\downarrow}_{\mathrm{dcl}_{L'}(\alpha) \cap \mathrm{dcl}_{L'}(\beta)} \mathrm{dcl}_{L'}(\beta).$$

Proof. Let $\alpha, \beta \in M$ be elements as in the statement, and let $\boldsymbol{a}^\alpha = \mathrm{dcl}_{L'}(\alpha)$ and $\boldsymbol{a}^\beta = \mathrm{dcl}_{L'}(\beta)$. Set $\Omega = \boldsymbol{a}^\alpha \cup \boldsymbol{a}^\beta$ and for any $\gamma \in \Omega$ let $\Omega_{<\gamma} = \{\varepsilon \in \Omega : \varepsilon < \gamma\}$. Since $(\Omega, <)$ is a finite union of well-ordered sets, it is also well-ordered.

Claim 3.1.1. *For $\gamma \in \Omega$, if*

$$\boldsymbol{a}^\alpha \cap \Omega_{<\gamma} \mathop{\downarrow}_{\boldsymbol{a}^\alpha \cap \boldsymbol{a}^\beta \cap \Omega_{<\gamma}} \boldsymbol{a}^\beta \cap \Omega_{<\gamma}$$

then

$$\boldsymbol{a}^\alpha \cap \Omega_{\leq\gamma} \mathop{\downarrow}_{\boldsymbol{a}^\alpha \cap \boldsymbol{a}^\beta \cap \Omega_{\leq\gamma}} \boldsymbol{a}^\beta \cap \Omega_{\leq\gamma}.$$

Proof. By symmetry, we deal with the case $\gamma \in \boldsymbol{a}^\alpha$. We first prove that

$$\boldsymbol{a}^\alpha \cap \Omega_{\leq\gamma} \mathop{\downarrow}_{\boldsymbol{a}^\alpha \cap \boldsymbol{a}^\beta \cap \Omega_{<\gamma}} \boldsymbol{a}^\beta \cap \Omega_{<\gamma}. \tag{3-1}$$

By hypothesis (2), $\gamma \downarrow_{\mathrm{dcl}_{L'}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}} \{\varepsilon : \varepsilon < \gamma\}$. Since

$$\mathrm{dcl}_{L'}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\} \subseteq \boldsymbol{a}^\alpha \cap \Omega_{<\gamma} \subseteq \{\varepsilon : \varepsilon < \gamma\},$$

we get that $\gamma \downarrow_{\boldsymbol{a}^\alpha \cap \Omega_{<\gamma}} \{\varepsilon : \varepsilon < \gamma\}$, so $\gamma \downarrow_{\boldsymbol{a}^\alpha \cap \Omega_{<\gamma}} \boldsymbol{a}^\beta \cap \Omega_{<\gamma}$ and

$$\boldsymbol{a}^\alpha \cap \Omega_{\leq\gamma} \mathop{\downarrow}_{\boldsymbol{a}^\alpha \cap \Omega_{<\gamma}} \boldsymbol{a}^\beta \cap \Omega_{<\gamma}.$$

By assumption and transitivity, we conclude (3-1).

If $\gamma \notin \boldsymbol{a}^\beta$ then we are done; so assume that $\gamma \in \boldsymbol{a}^\beta$ as well. In this case, by properties of forking we get that $\boldsymbol{a}^\alpha \cap \Omega_{\leq\gamma} \downarrow_{\boldsymbol{a}^\alpha \cap \boldsymbol{a}^\beta \cap \Omega_{\leq\gamma}} \boldsymbol{a}^\beta \cap \Omega_{\leq\gamma}$, which is what we wanted to prove. □

The proof now follows by induction: the successor step is Claim 3.1.1 and the limit case follows since forking is witnessed by a formula. □

We move to the main result of this section on simplicity.

Proposition 3.2. *Let T be a complete simple theory of graphs in the language of graphs $L = \{E\}$. Let μ be an infinite cardinal and $G = (V, E) \models T$ with $|G| \leq 2^\mu$. If $\chi(G) \geq \mu^+$ then there exists $G \equiv G'$ with $|G'| = \mu^+$ that contains a clique of cardinality μ^+.*

Proof. By compactness and Löwenheim–Skolem, it is sufficient to show that G contains arbitrary large finite cliques. Furthermore, by passing to an elementary extension, we may assume that $|G| = 2^\mu$. Additionally, after renaming elements, we may assume that $V = 2^\mu$. Assume that $\chi(G) \geq \mu^+$.

As T is simple, for every nonzero $\alpha \in V$, the type $\mathrm{tp}(\alpha/\{\beta : \beta < \alpha\})$ does not fork over some nonempty countable subset $A_\alpha \subseteq \{\beta : \beta < \alpha\}$. Enumerate A_α as $\langle c_{\alpha,n} : n < \omega\rangle$, possibly with repetitions. Let F_n be the function mapping a nonzero $\alpha \in V$ to $c_{\alpha,n}$ (and $F_n(0) = 0$).

Let $L' \supseteq L \cup \{F_n : n < \omega\} \cup \{<\}$ be a language containing Skolem functions and G' an expansion of G to L' with Skolem functions such that the F^n are interpreted as above. Let $T' = \mathrm{Th}(G')$. As usual $\mathrm{Sk}(-)$ denotes the Skolem hull in L', i.e., $\mathrm{Sk}(C)$ is the structure generated by C in G'. Unless specified otherwise, whatever is done below is done in L.

By forking base monotonicity and the choice of the functions F_n, for any $\alpha \in V$, $\mathrm{tp}(\alpha/\{\beta : \beta < \alpha\})$ does not fork over $\mathrm{Sk}(\{\alpha\}) \cap \{\beta : \beta < \alpha\}$.

Let Δ be the collection of all of formulas in one variable x over $\varnothing$ in L' and let $\Delta = \bigcup_{l<\omega} \Delta_l$ be an increasing union of finite subsets. Let $\langle t_i(x) : i < \omega\rangle$ be some enumeration of all the terms in L', with $t_0(x) = x$. Thus

$$\mathrm{Sk}(\{\alpha\}) = \{t_i(\alpha) : i < \omega\}.$$

Enumerate the 2^μ functions from μ to $\{0, 1\}$ by $\langle \eta_\alpha\rangle_{\alpha<2^\mu}$, without repetitions. For any finite subset $u \subseteq \mu$ and $n < \omega$ we define a relation $R_{u,n}$ on

$$\mathrm{Dom}(R_{u,n}) := \big\{\alpha \in V = 2^\mu : \eta_{t_i(\alpha)} \restriction u \neq \eta_{t_j(\alpha)} \restriction u \text{ for all } i < j < n \text{ such that } t_i(\alpha) \neq t_j(\alpha)\big\}.$$

Let $\alpha\ R_{u,n}\ \beta$ (for $\alpha, \beta \in \mathrm{Dom}(R_{u,n})$) if

(1) for all $j < n$, $\eta_{t_j(\alpha)} \restriction u = \eta_{t_j(\beta)} \restriction u$ and

(2) $\mathrm{tp}_{\Delta_n}(\alpha) = \mathrm{tp}_{\Delta_n}(\beta)$.

Note that $R_{u,n}$ is an equivalence relation on $\mathrm{Dom}(R_{u,n})$.

Claim 3.2.1. *There exists $\alpha \in V$ such that for every finite subset $u \subseteq \mu$ and $n < \omega$, if $\alpha \in \mathrm{Dom}(R_{u,n})$ then*

$$\exists \beta \in [\alpha]_{R_{u,n}} (\alpha\ E\ \beta).$$

Proof. Note that each $R_{u,n}$ has only finitely many classes on $\mathrm{Dom}(R_{u,n})$. Assume towards a contradiction that for every $\alpha \in V$ we can find some $u_\alpha \subseteq \mu$ and $n_\alpha < \omega$ that satisfy the negation of the statement. Now map α to $(u_\alpha, n_\alpha, [\alpha]_{R_{u_\alpha,n_\alpha}})$. This is easily a legal coloring of V by μ colors, contradicting $\chi(G) \geq \mu^+$. $\square$

Let $\alpha \in V$ be as supplied by Claim 3.2.1. For any $n < \omega$ let

$$u_n = \big\{\min\{\varepsilon < \mu : \eta_{t_i(\alpha)}(\varepsilon) \neq \eta_{t_j(\alpha)}(\varepsilon)\} : i < j < n \text{ such that } t_i(\alpha) \neq t_j(\alpha)\big\};$$

it is a finite subset of μ. Easily, $\alpha \in \mathrm{Dom}(R_{u_n,n})$ for all $n < \omega$. For $n < \omega$ let β_n be the element given by Claim 3.2.1 for (u_n, n) (and α).

Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω and let $\widetilde{G} = (G')^\omega/\mathcal{U}$ be the corresponding ultrapower in the language L'. We let $\widetilde{G} = (\widetilde{V}, E)$. Set $\widetilde{\alpha} = [\alpha]_{\mathcal{U}}$, $\widetilde{\beta} = [\beta_n]_{\mathcal{U}}$; so $\widetilde{\alpha}\, E\, \widetilde{\beta}$. We make some observations. By definition of $\widetilde{\alpha}$, $\mathrm{tp}_{L'}(\widetilde{\alpha}) = \mathrm{tp}_{L'}(\alpha)$ and by definition of the relations $R_{u_n,n}$, this type is also equal to $\mathrm{tp}_{L'}(\widetilde{\beta})$.

Set $\boldsymbol{a}^{\widetilde{\alpha}} = \langle t_i(\widetilde{\alpha}) : i < \omega\rangle$ (and likewise for α), and $\boldsymbol{a}^{\widetilde{\beta}} = \langle t_i(\widetilde{\beta}) : i < \omega\rangle$.

Claim 3.2.2. *For all $i, j < \omega$, if $t_i(\widetilde{\alpha}) = t_j(\widetilde{\beta})$ then $t_i(\widetilde{\alpha}) = t_i(\widetilde{\beta})$.*

Proof. First note that for all $i < j < \omega$, if $t_i(\widetilde{\alpha}) = t_j(\widetilde{\alpha})$ then the same holds for $\widetilde{\beta}$ and vice versa.

Assume the claim is not true and that $i < j$. If $t_i(\widetilde{\alpha}) = t_j(\widetilde{\alpha})$ then by the first paragraph we are done. So assume not. We can find some n large enough for which $t_i(\beta_n) \neq t_j(\beta_n)$, $t_i(\alpha) = t_j(\beta_n)$ and $i < j < n$.

As $\beta_n \, R_{u_n,n} \, \alpha$, $\eta_{t_i(\beta_n)} \restriction u_n = \eta_{t_i(\alpha)} \restriction u_n$ so by assumption $\eta_{t_i(\beta_n)} \restriction u_n = \eta_{t_j(\beta_n)} \restriction u_n$, contradicting the definition of $\mathrm{Dom}(R_{u_n,n})$. □

As $(\boldsymbol{a}^\alpha, <)$ is a well-order (as a substructure of 2^μ) so are $(\boldsymbol{a}^{\widetilde{\alpha}}, <)$ and $(\boldsymbol{a}^{\widetilde{\beta}}, <)$. With the aim of applying Lemma 3.1 with $M = \widetilde{G}$, we prove the following.

Claim 3.2.3. *For any $\gamma \in \boldsymbol{a}^{\widetilde{\alpha}} \cup \boldsymbol{a}^{\widetilde{\beta}}$, $\mathrm{tp}_L(\gamma / \{\varepsilon \in \widetilde{G} : \varepsilon < \gamma\})$ does not fork over $\mathrm{Sk}(\gamma) \cap \{\varepsilon \in \widetilde{G} : \varepsilon < \gamma\}$.*

Proof. Assume that $\gamma \in \boldsymbol{a}^{\widetilde{\alpha}}$. The proof only uses the fact that $\mathrm{tp}_{L'}(\widetilde{\alpha}) = \mathrm{tp}_{L'}(\alpha)$. Hence, the same proof also works for $\gamma \in \boldsymbol{a}^{\widetilde{\beta}}$. Recall that $\mathrm{tp}_{L'}(\widetilde{\alpha}) = \mathrm{tp}_{L'}(\widetilde{\beta})$. Let $t(x)$ be a term (in L') for which $\gamma = t(\widetilde{\alpha})$. We get that for $\gamma' := t(\alpha)$, $\mathrm{tp}_{L'}(\gamma) = \mathrm{tp}_{L'}(\gamma')$.

If $\mathrm{tp}_L(\gamma / \{\varepsilon : \varepsilon < \gamma\})$ forks over $\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}$ then by symmetry of forking there is a tuple c of elements from $\{\varepsilon : \varepsilon < \gamma\}$ such that

$$\mathrm{tp}_L(c / \{\gamma\} \cup (\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}))$$

forks over $\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}$. Let $\varphi(y, z)$ be a formula over $\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}$, satisfied by (c, γ), such that $\varphi(y, \gamma)$ forks over $\mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\}$. Since $\mathrm{tp}_{L'}(\gamma) = \mathrm{tp}_{L'}(\gamma')$, $\varphi(y, \gamma')$ forks over $\mathrm{Sk}(\{\gamma'\}) \cap \{\varepsilon : \varepsilon < \gamma'\}$.

On the other hand, we know that $\exists y < z \, \varphi(y, z)$ is in $\mathrm{tp}_{L'}(\gamma / \mathrm{Sk}(\gamma) \cap \{\varepsilon : \varepsilon < \gamma\})$ so $\exists y < z \, \varphi(y, z)$ is in $\mathrm{tp}_{L'}(\gamma' / \mathrm{Sk}(\{\gamma'\}) \cap \{\varepsilon : \varepsilon < \gamma'\})$. Consequently, there exists a tuple c' of elements in $\{\varepsilon \in V : \varepsilon < \gamma'\}$ for which $\varphi(c', \gamma')$ holds, contradicting the fact that $\mathrm{tp}_L(\gamma' / \{\varepsilon : \varepsilon < \gamma'\})$ does not fork over $\mathrm{Sk}(\{\gamma'\}) \cap \{\varepsilon : \varepsilon < \gamma'\}$ (by symmetry of nonforking). □

Recall that $\mathrm{tp}_{L'}(\widetilde{\alpha}) = \mathrm{tp}_{L'}(\widetilde{\beta})$. Also, $\boldsymbol{a}^{\widetilde{\alpha}}$ and $\boldsymbol{a}^{\widetilde{\beta}}$ enumerate elementary substructures of $\widetilde{G} \restriction L$. Setting $\boldsymbol{a}^0 = \langle t_i(\widetilde{\alpha}) : t_i(\widetilde{\alpha}) = t_i(\widetilde{\beta}), i < \omega \rangle$ by Claim 3.2.2, we have $\mathrm{tp}_L(\boldsymbol{a}^{\widetilde{\alpha}} / \boldsymbol{a}^0) = \mathrm{tp}_L(\boldsymbol{a}^{\widetilde{\beta}} / \boldsymbol{a}^0)$. Note that since $\boldsymbol{a}^0$ is closed under the chosen Skolem functions, it enumerates an elementary substructure as well.

By Claim 3.2.3 and Lemma 3.1 (which we are allowed to use since $(\boldsymbol{a}^{\widetilde{\alpha}}, <)$ and $(\boldsymbol{a}^{\widetilde{\beta}}, <)$ are well-orders), $\boldsymbol{a}^{\widetilde{\alpha}} \downarrow_{\boldsymbol{a}^0} \boldsymbol{a}^{\widetilde{\beta}}$. So by the independence theorem for simple theories [Tent and Ziegler 2012, Lemma 7.4.8], we may find an indiscernible sequence starting with $\boldsymbol{a}^{\widetilde{\alpha}}$ and $\boldsymbol{a}^{\widetilde{\beta}}$ (in some elementary extension). Thus, as $\widetilde{\alpha} \, E \, \widetilde{\beta}$ we can find an infinite clique in that elementary extension. □

Given Proposition 3.2, it is a natural question to ask whether we can necessarily find an infinite clique already in G itself. The following proposition shows that you cannot hope for an uncountable clique in general.

Proposition 3.3. *Let μ be an infinite cardinal. Any graph of cardinality μ without an uncountable clique can be embedded in a random graph of cardinality μ with no uncountable clique.*

Remark 3.4. Every random graph has a simple theory and contains a countable infinite clique, so countable cliques cannot be avoided.

Proof of Proposition 3.3. Let μ be an infinite cardinal and let $G_0 = (V_0, E_0)$ be the given graph. Without loss of generality, $V_0 = \{2\alpha : \alpha < \mu\}$.

Let $\langle u_\gamma \rangle_{\gamma<\mu}$ enumerate all finite subsets of μ such that each finite subset occurs μ times. In particular, for any $\gamma < \alpha < \mu$ there is some $\alpha < \gamma' < \mu$ for which $u_\gamma = u_{\gamma'}$.

Let $V = \mu$. We define a new graph $G = (V, E)$ extending G_0 such that for $\alpha < \beta < \mu$, if $\beta = 2\gamma + 1$ for some $\gamma < \mu$ and $\alpha \in u_\gamma$ we let $\{\alpha, \beta\}$ be an edge.

We claim that G has the desired properties.

We note that G is a random graph. Indeed, let $X = \{\alpha_1, \dots, \alpha_n\}$, $Y = \{\beta_1, \dots, \beta_n\}$ be two disjoint sets of vertices. Let $\gamma < \mu$ be larger than the α_i and β_j and satisfying that $u_\gamma = \{\alpha_1, \dots, \alpha_n\}$. Then $2\gamma + 1$ is connected to each of the α_i and to none of the β_j.

Finally, assume that G contains a clique C of cardinality $\aleph_1$. By assumption, $C \cap V_0$ must be at most countable, so there must be a clique of cardinality $\aleph_1$ consisting of odd ordinals in μ. Let U be the first ω of those and let $2\gamma + 1 \in C$ be an ordinal larger than any of the ordinals in U. But $2\gamma + 1$ can only be connected to finitely many vertices which are smaller, a contradiction. □

Corollary 3.5. *For any infinite cardinal μ there exists a random graph of cardinality and chromatic number μ with no uncountable clique.*

Proof. Apply Proposition 3.3 to any graph G_0 of cardinality μ with no infinite clique and $\chi(G_0) = \mu$ (for example the triangle-free graph from [Erdős and Rado 1960]).

Let G be the graph supplied by the proposition. Since G_0 embeds into G and $|G| = \mu$ it follows that $\chi(G) = \mu$. □

Question 3.6. Is there a theory of simple graphs such that for every cardinal μ we can find a graph of cardinality and chromatic number μ with no infinite cliques at all?

Question 3.7. Does an analog of Proposition 3.2 hold for other model-theoretic tame graphs, such as NSOP_1 and NIP?

Graphs with stable edge relation. Before getting into the main result we prove a technical lemma which may be interesting on its own.

Lemma 3.8. *Let T be a first-order theory, μ an infinite cardinal and $\varphi(x, y)$ a stable formula. Let $M \vDash T$ with $|M| = \mu^+$ and assume that M is an increasing continuous union of elementary substructures $\langle M_\alpha \rangle_{\alpha<\mu^+}$ each of cardinality at most μ.*

Let $\psi(y, z)$ be a uniform definition of φ-types (as in Fact 2.1). For any $a \in M$ and $\alpha < \mu^+$, let $c_{a,\alpha} \in M_\alpha$ be such that $\psi(y, c_{a,\alpha})$ defines $\mathrm{tp}_\varphi(a/M_\alpha)$. Then there exists a club $\mathcal{C} \subseteq \mu^+$ of limit ordinals satisfying that for any $\delta \in \mathcal{C}$ and $a \in M \setminus M_\delta$:

$$\text{For any } \beta \text{ such that } \delta < \beta < \mu^+ \text{ there is } b \in M \setminus M_\beta \text{ for which } \mathrm{tp}_\varphi(b/M_\beta) \text{ is definable by } \psi(y, c_{a,\delta}). \qquad (\dagger)_{a,\delta}$$

Proof. Let F be the set of all limit ordinals $\delta < \mu^+$ such that for any $a \in M \setminus M_\delta$, $(\dagger)_{a,\delta}$ holds. It is enough to show that F contains a club $\mathcal{C}$.

Suppose F does not contain a club. Then $\mu^+ \setminus F$ is stationary. Let $\bar{F}$ be the set of all limit ordinals in $\mu^+ \setminus F$; this set is also stationary as an intersection of a club with a stationary set. By definition, for any limit ordinal $\delta \in \bar{F}$ there is some $a_\delta \in M \setminus M_\delta$ and $\beta_\delta > \delta$ such that for any $b \in M \setminus M_{\beta_\delta}$, $\mathrm{tp}_\varphi(b/M_{\beta_\delta})$ is not defined by $\psi(y, c_{a_\delta,\delta})$.

For any limit ordinal $\delta \in \bar{F}$, let $f(\delta)$ be the minimal ordinal ε for which $c_{a_\delta,\delta} \in M_\varepsilon$. Note that as $c_{a_\delta,\delta}$ is a finite tuple and δ is a limit ordinal, necessarily $f(\delta) < \delta$. By Fodor's lemma [Jech 2003, Theorem 8.7] there exists a stationary subset $S \subseteq \bar{F}$ and $\varepsilon < \mu^+$ for which $f(\delta) = \varepsilon$ for any $\delta \in S$.

By definition, $c_{a_\delta,\delta} \in M_\varepsilon$ for any $\delta \in S$. As $|M_\varepsilon| \le \mu$, by the pigeonhole principle there is an unbounded subset $S' \subseteq S$ (i.e., of cardinality μ^+) for which $c := c_{a_{\delta_1},\delta_1} = c_{a_{\delta_2},\delta_2}$ for any $\delta_1, \delta_2 \in S'$.

Now, pick any $\delta \in S'$. By our assumption there is some $\beta_\delta > \delta$ such that for any $b \in M \setminus M_{\beta_\delta}$, $\mathrm{tp}_\varphi(b/M_{\beta_\delta})$ is not defined by $\psi(y, c_{a_\delta,\delta}) = \psi(y, c)$.

Let $\beta' > \beta_\delta$ be an element in S' (it is unbounded) and let $a_{\beta'}$ be the corresponding element. So $a_{\beta'} \notin M_{\beta'}$ and in particular $a_{\beta'} \notin M_{\beta_\delta}$, and thus $\mathrm{tp}_\varphi(a_{\beta'}/M_{\beta_\delta})$ is not defined by $\psi(y, c_{a_\delta,\delta}) = \psi(y, c)$. On the other hand, by choice of S', $\mathrm{tp}_\varphi(a_{\beta'}/M_{\beta'})$ and thus also $\mathrm{tp}_\varphi(a_{\beta'}/M_{\beta_\delta})$ is defined by $\psi(y, c)$, a contradiction. □

We phrase forking symmetry for a stable formula in a form useful to us.

Fact 3.9 (Harrington). *Let T be a first-order theory with monster model $\mathbb{U}$. Let $\varphi(x, y)$ be a stable symmetric[2] formula and $\psi(y, z)$ a formula uniformly defining φ-types. For any two small models N_1 and N_2 and elements $a, b \in \mathbb{U}$, if $\psi(y, c_a)$ defines $\mathrm{tp}_\varphi(a/N_1)$, $\psi(y, c_b)$ defines $\mathrm{tp}_\varphi(b/N_2)$ and $c_a \in N_2$, $c_b \in N_1$ then*

$$\psi(a, c_b) \iff \psi(b, c_a).$$

[2]A formula $\varphi(x, y)$ is *symmetric* if x and y have the same sort and $\varphi(x, y) \equiv \varphi(y, x)$.

Proof. Let $p(x) = \mathrm{tp}_\varphi(a/N_1)$ and $q(y) = \mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/N_2)$. Note that as φ is symmetric, $\psi(x, c_b)$ defines $q(y)$.

Let $\widetilde{p} \supseteq p$ be the global φ-type that $\psi(y, c_a)$ defines and $\widetilde{q} \supseteq q$ be the global φ-type that $\psi(x, c_b)$ defines. By [Tent and Ziegler 2012, Lemma 8.3.4] and using Fact 2.1, $\widetilde{p} \vdash \psi(x, c_b)$ if and only if $\widetilde{q} \vdash \psi(y, c_a)$; but as $c_a \in N_2$ and $c_b \in N_1$, we conclude. $\square$

The following is due to [Engelking and Karłowicz 1965]; see also [Rinot 2012] for a streamlined presentation.

Fact 3.10. *For cardinals $\kappa \leq \lambda \leq \mu \leq 2^\lambda$ the following are equivalent*:

(1) $\lambda^{<\kappa} = \lambda$.

(2) *There exists a collection of functions $\langle f_i : \mu \to \lambda \rangle_{i<\lambda}$ such that for every $X \in [\mu]^{<\kappa}$ and every function $f : X \to \lambda$, there exists some $i < \lambda$ with $f \subseteq f_i$.*

We now prove the main result of this section.

Proposition 3.11. *Let $L = \{E\}$ be the language of graphs and T be an L-theory specifying that E is a symmetric and irreflexive stable relation. For any infinite cardinal μ and $G \vDash T$ with $|G| = \mu^+$, if $\chi(G) \geq \mu^+$ then G contains an infinite clique of cardinality μ^+.*

Proof. For ease of notation, write $\varphi(x, y) = E(x, y)$ and let $\psi(y, z)$ be a uniform definition for φ-types (as in Fact 2.1). Assume that $\chi(G) \geq \mu^+$.

By Lemma 3.8, there exists an increasing continuous family of elementary substructures $\langle G_\alpha \prec G : 0 < \alpha < \mu^+ \rangle$, with $|G_\alpha| = \mu$ and $\bigcup_{0<\alpha<\mu^+} G_\alpha = G$, satisfying:

$$\text{For any } 0 < \delta < \mu^+ \text{ and } a \in G \setminus G_\delta \text{ there exists } c_{a,\delta} \in G_\delta \text{ such that } \psi(y, c_{a,\delta}) \text{ defines } \mathrm{tp}_\varphi(a/G_\delta) \text{ satisfying that for any } \beta \text{ with } \delta < \beta < \mu^+ \text{ there is } b \notin G_\beta \text{ for which } \mathrm{tp}_\varphi(b/G_\beta) \text{ is defined by } \psi(y, c_{a,\delta}). \tag{†}$$

Indeed, Lemma 3.8 supplies a club $\mathcal{C} \subseteq \mu^+$ for which the above holds. Since the order type of $\mathcal{C}$ is μ^+, by restricting to the models indexed by ordinals in $\mathcal{C}$ we get the desired sequence of elementary substructures.

Set $G_0 = \varnothing$. For any $a \in G$ let $\alpha_0^a < \mu^+$ be minimal such that $a \in G_{\alpha_0+1}$. Let $c_0^a = c_{a,\alpha_0^a} \in G_{\alpha_0^a}$; in particular, $\psi(y, c_0^a)$ defines $\mathrm{tp}_\varphi(a/G_{\alpha_0^a})$. Let $\alpha_1^a < \alpha_0^a$ be such that $c_0^a \in G_{\alpha_1^a+1} \setminus G_{\alpha_1^a}$. Let $c_1^a = c_{a,\alpha_1^a}$. Likewise we continue and find a sequence $\langle (c_{k-1}^a, \alpha_k^a) : 1 \leq k \leq n_a \rangle$ satisfying

- $\alpha_k^a < \alpha_{k-1}^a$,
- $c_{k-1}^a = c_{a,\alpha_{k-1}^a}$ — and in particular $\psi(y, c_{k-1}^a)$ defines $\mathrm{tp}_\varphi(a/G_{\alpha_{k-1}^a})$,
- $c_{k-1}^a \in G_{\alpha_k^a+1} \setminus G_{\alpha_k^a}$,
- $\alpha_{n_a}^a = 0$.

Note that $\alpha_0^a = 0$ if and only if $n_a = 0$ and then there are no c's.

We now define a coloring. By Engelking–Karłowicz (Fact 3.10, applied to $\omega \le \mu \le \mu^+ \le 2^\mu$), there exists a family of functions $\{g_\beta : \mu^+ \to \mu \mid \beta < \mu\}$ satisfying that for any finite subset $X \subseteq \mu^+$ and every function $f : X \to \mu$ there is some function g_β, $\beta < \mu$, with $f \subseteq g_\beta$.

For any such $a \in G$ define a function $f_a : \{\alpha_0^a, \ldots, \alpha_{n_a}^a\} \to \omega$ by setting $f_a(\alpha_i^a) = i$. Let $\beta^a < \mu$ be minimal such that $f_a \subseteq g_{\beta^a}$.

For any $\alpha < \mu^+$, let $s_\alpha : G_\alpha \to \mu$ be a bijection. For simplicity, we also denote by s_α the induced bijection between G_α^n and μ^n. Let $Y = \aleph_0 \times \mu^2 \times \mu^{<\omega}$; note that $|Y| = \mu$.

Let $\phi : G \to Y$ be a function mapping $a \in G$ to

$$(n_a, \beta^a, s_{\alpha_0^a+1}(a), (s_{\alpha_i^a+1}(c_{i-1}^a))_{1 \le i \le n_a}).$$

Since ϕ cannot be a legal coloring there exist distinct elements $a, b \in G$ satisfying that $\phi(a) = \phi(b)$ and $\varphi(a, b)$ holds, i.e., $a \mathrel{E} b$.

Without loss of generality, assume that $\alpha_0^a \ge \alpha_0^b$. Note that necessarily, $\alpha_0^a > \alpha_0^b$, since otherwise, as $s_{\alpha_0^a+1}(a) = s_{\alpha_0^b+1}(b)$ we conclude that $a = b$, a contradiction. In particular, $n_a > 0$ and so also $n_b = n_a > 0$.

Also, if $\alpha_i^a = \alpha_j^b$ for some i, j then $i = j$, since by assumption $\beta^a = \beta^b$.

Let $n > 0$ be minimal such that $\alpha_n^a = \alpha_n^b$ (such n exists since this equality holds for $n = n_a > 0$). Consider the (ordered) set $A' = \{\alpha_i^a, \alpha_i^b : i < n\}$. Note that the elements in A' are distinct. Now let $A = A' \cup \{\alpha_n^a = \alpha_n^b\}$.

We call an element $\alpha_i^a \in A$ (with $i > 0$) an *a-pivot* if there exists an element $\alpha_j^b \in A$ with $\alpha_{i-1}^a > \alpha_j^b > \alpha_i^a$ (and likewise a *b-pivot* for $\alpha_i^b \in A$ with $i > 0$). Note that $\alpha_n^a = \alpha_n^b$ is either an a-pivot or a b-pivot.

We prove the following by (downward) induction.

Claim 3.11.1. *For every a-pivot $\alpha_i^a \in A$ with $i > 0$, $\psi(b, c_{i-1}^a)$ holds, and for every b-pivot $\alpha_i^b \in A$ with $i > 0$, $\psi(a, c_{i-1}^b)$ holds.*

Proof. Let $\alpha_{t+1}^a \in A$ be an a-pivot (possibly $t + 1 = n$). Let $v < n$ be minimal with $\alpha_t^a > \alpha_v^b > \alpha_{t+1}^a$; α_v^b is either a b-pivot and $v = s + 1$ for some s or $v = 0$. Assume, first, that α_{s+1}^b is a b-pivot, so $\psi(a, c_s^b)$ holds by the induction hypothesis. As $c_s^b \in G_{\alpha_{s+1}^b+1} \subseteq G_{\alpha_t^a}$ and $c_t^a \in G_{\alpha_{t+1}^a+1} \subseteq G_{\alpha_{s+1}^b} \subseteq G_{\alpha_s^b}$, we can apply Fact 3.9 with $\mathrm{tp}_\varphi(a/G_{\alpha_t^a})$ and $\mathrm{tp}_\varphi(b/G_{\alpha_s^b})$ and conclude that $\psi(b, c_t^a)$ holds.

Now assume that $v = 0$. Since $b \in G_{\alpha_0^b+1} \subseteq G_{\alpha_t^a}$ and $\varphi(a, b)$ holds, then as $\psi(y, c_t^a)$ defines $\mathrm{tp}_\varphi(a/G_{\alpha_t^a})$ we get that $\psi(b, c_t^a)$ holds, as needed.

The case where $\alpha_{t+1}^b \in A$ is a b-pivot is proved similarly. □

Note that $c_{n-1}^a \in G_{\alpha_n^a+1} = G_{\alpha_n^b+1} \ni c_{n-1}^b$. As $\phi(a) = \phi(b)$, we necessarily have $c := c_{n-1}^a = c_{n-1}^b$. If α_n^a is an a-pivot we have that $\psi(a, c)$ holds and if it is a b-pivot then $\psi(b, c)$ holds. Assume the former holds (the proof where the latter holds is identical).

We inductively construct a sequence $(d_\alpha)_{\alpha<\mu^+}$ in V such that

- $\psi(d_\alpha, c)$ holds for all $\alpha < \mu^+$,
- $\varphi(d_\alpha, d_\beta)$ holds for all $\alpha \neq \beta < \mu^+$.

Suppose we constructed $(d_\alpha)_{\alpha<\gamma}$. Let δ be such that $\alpha^a_{n-1} < \delta < \mu^+$ and $d_\alpha \in G_\delta$ for any $\alpha < \gamma$. By (†) there exists $d_\gamma \in G \setminus G_\delta$ such that $\mathrm{tp}_\varphi(d_\gamma/G_\delta)$ is definable by $\psi(y, c)$.

Since $\psi(d_\alpha, c)$ holds for any $\alpha < \gamma$ it follows that $\varphi(d_\gamma, d_\alpha)$ holds (and also $\varphi(d_\alpha, d_\gamma)$ by symmetry). Additionally, $\psi(y, c)$ defines both $\mathrm{tp}_\varphi(a/G_{\alpha^a_{n-1}})$ and $\mathrm{tp}_\varphi(d_\gamma/G_{\alpha^a_{n-1}})$; hence $a \equiv^\varphi_{G_{\alpha^a_{n-1}}} d_\gamma$. As $c \in G_{\alpha^a_{n-1}}$ and $\psi(y, c)$ is equivalent to a boolean combination of instances of φ^{opp}-formulas over $G_{\alpha^a_{n-1}}$, it follows by symmetry of φ that $a \equiv^{\varphi^{\mathrm{opp}}}_{G_{\alpha^a_{n-1}}} d_\gamma$, so $\psi(d_\gamma, c)$ holds as well. □

Given Proposition 3.11 (and Proposition 3.2), it is natural to ask under which conditions on the cardinality of G does the proposition hold. The following example shows that it fails for strong limit cardinals.

Proposition 3.12. *Let λ be a strong limit cardinal.*[3] *There exists a nonstable graph with stable edge relation G of cardinality λ, with $\chi(G) \geq \lambda$, for which we cannot embed arbitrarily large finite cliques. In fact, it is triangle-free.*

Proof. For any $\mu < \lambda$, let $G_\mu = \mathrm{Sh}_2(\beth_1(\mu)^+)$. By Lemma 2.10, it is not stable but has a stable edge relation, and it is triangle-free. By Fact 2.6, $\chi(G_\mu) \geq \mu^+$.

Let $G = \bigoplus_{\mu<\lambda} G_\mu$ be the direct sum of all of these graphs. Thus $\chi(G) \geq \lambda$ and $|G| = \lambda$. The graph G is not stable but its edge relation is stable. On the other hand, since each of the G_μ is triangle-free, we cannot embed arbitrarily large finite cliques into G. □

Remark 3.13. By Remark 2.11(2), we may replace $\mathrm{Sh}_2(\beth_1(\mu)^+)$ by $\mathrm{Sh}^{\mathrm{sym}}_4(\beth_2(\mu)^+)$ and arrive at a stable graph with the prescribed properties.

4. The chromatic spectrum

We rephrase the results of the previous section in a different manner.

For T a theory of graphs and μ an infinite cardinal, let

$$\mathrm{ch}_{\mathrm{T}}(\mu) = \min\{|G| : G \vDash T, \chi(G) \geq \mu\}.$$

We employ the convention that $\min \varnothing = \infty$. Note that if $\mathrm{ch}_{\mathrm{T}}(\mu) = \infty$ then $\mathrm{ch}_{\mathrm{T}}(\lambda) = \infty$ for all $\lambda \geq \mu$.

[3]That means that $2^\mu < \lambda$ for any $\mu < \lambda$. Any such cardinal is a limit cardinal. An example is $\beth_\omega(\aleph_0)$.

Remark 4.1. (1) Since for any graph G, $\chi(G) \le |G|$, we have $\mathrm{ch_T}(\mu) \ge \mu$.

(2) Let $T = \mathrm{Th}(\mathrm{Sh}_k(\omega))$. By Fact 2.6, $\mathrm{ch_T}(\lambda^+) \le \beth_{k-1}(\lambda)^+$. On the other hand, if towards a contradiction we assume that $\mathrm{ch_T}(\lambda^+) < \beth_{k-1}(\lambda)^+$ then we can find $M \vDash T$ with $\chi(M) \ge \lambda^+$ and $|M| \le \beth_{k-1}(\lambda)$. By Löwenheim–Skolem we can assume that $|M| = \beth_{k-1}(\lambda)$. By Lemma 2.9, we may embed M into $\mathrm{Sh}_n^{\mathrm{sym}}(\beth_{k-1}(\lambda))$, so $\chi(M) \le \lambda$ by Fact 2.6, a contradiction. We conclude that $\mathrm{ch_T}(\lambda^+) = \beth_{k-1}(\lambda)^+$.

We can now rephrase the main result of [Halevi et al. 2024] using this function:

Proposition 4.2. *The following are equivalent for a stable theory of graphs T*:

(1) *There are some $G \vDash T$ and a natural number k such that G contains* $\mathrm{Sh}_k(n)$ *for all n.*

(2) $\mathrm{ch_T}(\beth_2(\aleph_0)^+) < \infty$.

(3) *For any cardinal μ*, $\mathrm{ch_T}(\mu) < \infty$.

(4) *For any cardinal μ*, $\mathrm{ch_T}(\mu) < \beth_\omega(\mu)$.

Proof. $\neg(1) \implies \neg(2)$. By the main theorem of [Halevi et al. 2024, Corollary 6.2], $\chi(G) \le \beth_2(\aleph_0)$ for all $G \vDash T$, i.e., $\mathrm{ch_T}(\beth_2(\aleph_0)^+) = \infty$.

$(4) \implies (3) \implies (2)$. These are easy.

$(1) \implies (4)$. Let μ be an infinite cardinal. By compactness and (1) we can embed $\mathrm{Sh}_k(\beth_{k-1}(\mu)^+)$ in a large enough model of T. So by Löwenheim–Skolem we can find a model G of cardinality $\beth_{k-1}(\mu)^+$ with $\chi(G) \ge \chi(\mathrm{Sh}_k(\beth_{k-1}(\mu)^+)) \ge \mu^+ \ge \mu$ (using Fact 2.6). Consequently, $\mathrm{ch_T}(\mu) \le \beth_{k-1}(\mu)^+ < \beth_\omega(\mu)$. □

Next, we phrase the results of the previous section for simple graphs and graphs with stable edge relation using $\mathrm{ch_T}$.

Proposition 4.3. *Let T be a theory of graphs and assume that either T is simple or the edge relation is stable. The following are equivalent*:

(1) *T proves the existence of arbitrarily large finite cliques.*

(2) *For any infinite cardinal μ*, $\mathrm{ch_T}(\mu) = \mu$.

(3) *For any infinite cardinal μ*, $\mathrm{ch_T}(\mu^+) = \mu^+$.

(4) *There exists an infinite cardinal μ for which* $\mathrm{ch_T}(\mu^+) = \mu^+$.

If T is simple then they are also equivalent to:

(5) *There exists an infinite cardinal μ with* $\mathrm{ch_T}(\mu^+) \le 2^\mu$.

(6) *For any infinite cardinal μ*, $\mathrm{ch_T}(\mu^+) \le 2^\mu$.

Proof. $(1) \implies (2)$. By compactness and Löwenheim–Skolem, there is a model G of cardinality μ which has an infinite clique of cardinality μ. Thus $\chi(G) \ge \mu$, so $\mathrm{ch_T}(\mu) = \mu$.

(2) $\implies$ (3) $\implies$ (4) These are easy.

(4) $\implies$ (1). Assume that (4) holds and let μ be an infinite cardinal for which $\mathrm{ch_T}(\mu^+) = \mu^+$. Thus there exists a model G with $\chi(G) \geq \mu^+$ and $|G| = \mu^+$. By Proposition 3.2 for the simple case and Proposition 3.11 for the stable edge relation case, G contains arbitrarily large finite cliques.

Assume that T is simple.

(4) $\implies$ (5) $\implies$ (1). The first implication is easy and the second uses Proposition 3.2 as above.

(3) $\implies$ (6) $\implies$ (5). These are easy. □

Is there an analog to Proposition 4.3 for general shift graphs? Here is a reasonable suggestion:

Conjecture 4.4. Suppose that T is stable. If $\mathrm{ch_T}(\mu^+) \leq \beth_{n-1}(\mu)^+$ for all cardinals μ, then for some $m \leq n$, there is an embedding of $\mathrm{Sh}_m(\omega)$ in any ω-saturated model of T.

Note this is exactly Question 1.1 without assuming GCH.

Appendix: An example by Hajnal and Komjáth

In this section we present an example due to Hajnal and Komjáth [1984, Theorem 4]. This is an example of a graph of size continuum whose chromatic number is $\aleph_1$ which does not contain all finite subgraphs of any shift graph $\mathrm{Sh}_n(\omega)$. They gave it as an example refuting Taylor's strong conjecture (which does hold outright for ω-stable graphs with a close relative of it holding for stable theories in general by [Halevi et al. 2022; 2024]). The main goal here is to prove that this example has the independence property (IP) (thus is not stable) and furthermore that its theory is not simple.

Definition A.1. A graph $G = (V, E)$ is called *special* if there exists a partial order $\prec$ on V satisfying that

(1) if $x \mathrel{E} y$ then either $x \prec y$ or $y \succ x$ and

(2) there is no circuit $C = \langle x_0, \ldots, x_{n-1}\rangle$, $n \geq 3$, of the form

$$x_0 \prec x_1 \prec \cdots \prec x_{m-1} \prec x_m \succ x_{m+1} \succ \cdots \succ x_{n-1} \succ x_0.$$

Proposition A.2 [Hajnal and Komjáth 1984, Theorem 4]. *Let $G = (V, E)$ be a special graph as witnessed by $\prec$. Then for all $n \geq 1$, G does not contain all the finite subgraphs of* $\mathrm{Sh}_n(\omega)$.

Proof. Assume towards a contradiction that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \geq 1$. If $n = 1$ then it must contain a triangle, obviously contradicting Definition A.1(2). So we assume that $n \geq 2$. Coloring pairs of $<$-increasing tuples, by Ramsey there is some integer r such that if $f : \mathrm{Sh}_n(r) \to G$ is an embedding, there is $A \subseteq r$, $|A| = 2n+1$ such that either $f(a_0, \ldots, a_{n-1}) \prec f(a_1, \ldots, a_n)$ for all strictly increasing $n+1$-tuples $(a_0, \ldots, a_n)$ from A or $f(a_0, \ldots, a_{n-1}) \succ f(a_1, \ldots, a_n)$ for all strictly increasing $n+1$-tuples $(a_0, \ldots, a_n)$ from A.

Assume the former occurs and that for simplicity $A = \{0, \ldots, 2n\}$. Then

$$(0, \ldots, n-1) \prec (1, \ldots, n-1, n+1) \\ \prec \cdots \prec (n-1, n+1, \ldots, 2n-1) \prec (n+1, \ldots, 2n)$$

and

$$(0, \ldots, n-1) \prec (1, \ldots, n-1, n) \prec \cdots \prec (n, n+1, \ldots, 2n-1) \prec (n+1, \ldots, 2n),$$

which is a contradiction. □

Proposition A.3 [Hajnal and Komjáth 1984, Theorem 4]. *There exists a graph $G = (V, E)$ with $|V| = 2^{\aleph_0}$ satisfying the following properties*:

(1) *G is special and in particular for every $n \geq 1$ it does not contain all finite subgraphs of* $\mathrm{Sh}_n(\omega)$.

(2) $\chi(G) = \aleph_1$.

(3) *G has IP and in particular is not stable* (*in fact*, *the edge relation has IP*).

(4) *G is not simple.*

Proof. Let $\{T_\alpha : \alpha < \aleph_1\}$ be a collection of disjoint sets with $|T_\alpha| = 2^{\aleph_0}$ for each $\alpha < \aleph_1$ and set $V = \bigcup_{\alpha<\aleph_1} T_\alpha$. We define an edge relation on V turning it to a graph satisfying our desired properties.

To define the edge relation we define, for $x \in T_\alpha$ and $\alpha < \aleph_1$,

$$G(x) = \{y \in T_{<\alpha} : x\, E\, y\},$$

where $T_{<\alpha} = \bigcup_{\beta<\alpha} T_\beta$, by induction on α.

If $\alpha = \beta+1$ we let $G(x) = \varnothing$ for every $x \in T_\alpha$, so assume that α is a limit ordinal and that $G(x)$ has already been defined for $x \in T_{<\alpha}$.

For $\gamma < \alpha$ and $y \in T_{<\alpha}$, we say that y *is γ-covered* if there exists $\alpha_0 < \cdots < \alpha_m$ with $\alpha_0 \leq \gamma$, and $x_i \in T_{\alpha_i}$ with $x_m = y$ such that $x_0\, E\, x_1\, E \cdots E\, x_m$. Note that any $y \in T_\gamma$ is γ-covered, as witnessed by the trivial path.

Let $\mathcal{W}_\alpha$ be the collection of all subsets $W \subseteq T_{<\alpha}$ satisfying that

- $W = \{x_n : n < \omega\}$ is countable,
- $x_n \in T_{\alpha_n}$ and $\alpha_n < \alpha_m$ whenever $n < m < \omega$,
- $\sup\{\alpha_n : n < \omega\} = \alpha$,
- no x_n is α_{n-1}-covered for $0 < n < \omega$.

Obviously, $|\mathcal{W}_\alpha| \leq 2^{\aleph_0}$; choose some enumeration $\mathcal{W}_\alpha = \{W_\gamma : \gamma < 2^{\aleph_0}\}$ and $T_\alpha = \{t_\gamma : \gamma < 2^{\aleph_0}\}$ and set $G(t_\gamma) = W_\gamma$. That is, for $y \in T_{<\alpha}$ and $x = t_\gamma \in T_\alpha$, $x \mathrel{E} y$ if and only if $y \in W_\gamma$. Let $G = (V, E)$.

We show that G satisfies the properties listed in the statement.

We first show (1). Let $C = \langle x_0, \dots, x_{n-1}\rangle \subseteq V$, $n \geq 3$, be a circuit in G with $x_i \in T_{\alpha_i}$ and

$$\alpha_0 < \alpha_1 < \cdots < \alpha_{m-1} < \alpha_m > \alpha_{m+1} > \cdots > \alpha_{n-1} > \alpha_0$$

for some $0 < m < n-1$. If $\alpha_{m-1} = \alpha_{m+1}$, then as the elements of C are distinct, x_m would be connected to two vertices in $T_{\alpha_{m-1}} = T_{\alpha_{m+1}}$, contradicting our construction. So assume without loss of generality that $\alpha_{m-1} < \alpha_{m+1}$. Thus $x_{m-1}, x_{m+1} \in G(x_m)$ and x_{m+1} is α_{m-1}-covered as witnessed by $\alpha_0 < \alpha_{n-1} < \cdots < \alpha_{m+1}$ and $\alpha_{m-1} \geq \alpha_0$. On the other hand, since $\alpha_{m-1} < \alpha_{m+1}$, we get a contradiction.

We show (2). Towards a contradiction, assume there exists a legal coloring $c : G \to \aleph_0$. We say that a color $n < \omega$ is *small* if there is a $\gamma_n < \aleph_1$ such that every point $x \in V$ with $c(x) = n$ is γ_n-covered; otherwise, call n *large*. For any small $n < \omega$ choose γ_n minimal satisfying the above. Put $\gamma = \sup\{\gamma_n : n \text{ small}\} < \aleph_1$. We note that there exist large n; indeed, take any $x \in T_{\gamma+1}$ and let $n < \omega$ be with $c(x) = n$. Since x is not connected to any $y \in T_{<\gamma+1}$ it cannot be γ-covered.

If $n < \aleph_0$ is large then for every $\alpha < \aleph_1$ there exists $x \in V$ with $c(x) = n$ which is not α-covered.

Let $\langle m_i < \aleph_0 : i < \omega\rangle$ be a sequence (possibly with repetitions), containing all large colors. By definition of m_0 being large, there exist α_0 and $x_0 \in T_{\alpha_0}$ with $c(x_0) = m_0$ which is not γ-covered (so necessarily $\gamma < \alpha_0$). We continue inductively and for any $n < \omega$ we find $x_n \in T_{\alpha_n}$ with $c(x_n) = m_n$ which is not α_{n-1}-covered (so necessarily $\alpha_{n-1} < \alpha_n$).

Suppose $\alpha = \sup\{\alpha_n : n < \omega\}$, which is necessarily a limit ordinal, and let $W = \{x_n : n < \omega\} \subseteq T_{<\alpha}$. By definition, there exists an element $x \in T_\alpha$ with $G(x) = W$. In particular $c(x) \neq m_n$ for all $n < \omega$, so $c(x) = k$ is small, i.e., it is γ_k-covered. But by the definition above, any $y \in G(x) = W$ is not γ-covered, so it cannot be that x is $\gamma_k \leq \gamma$-covered, a contradiction.

To show that $\chi(G) = \aleph_1$, note that $c : V \to \aleph_1$ defined by $c(x) = \alpha_x$ for $x \in T_{\alpha_x}$ is a legal coloring.

To show (3), choose for each $n < \omega$ some $x_n \in T_n$. Then, for all unbounded subsets $W \subseteq \omega$ there exists a unique $x \in T_\omega$ with $G(x) = \{x_n : n \in W\}$, giving IP.

Item (4) is a direct consequence of Proposition 3.2 (with $\mu = \aleph_0$). $\square$

Question A.4. Can one find such a counterexample which is stable? Simple?

Acknowledgments

We thank the anonymous referee for the careful reading and useful comments.

References

[Engelking and Karłowicz 1965] R. Engelking and M. Karłowicz, "Some theorems of set theory and their topological consequences", *Fund. Math.* **57** (1965), 275–285. MR

[Erdős and Hajnal 1966] P. Erdős and A. Hajnal, "On chromatic number of graphs and set-systems", *Acta Math. Acad. Sci. Hungar.* **17** (1966), 61–99. MR

[Erdős and Hajnal 1968] P. Erdős and A. Hajnal, "On chromatic number of infinite graphs", pp. 83–98 in *Theory of graphs* (Tihany, Hungary, 1966), edited by P. Erdős and G. Katona, Academic Press, New York, 1968. MR

[Erdős and Rado 1960] P. Erdős and R. Rado, "A construction of graphs without triangles having preassigned order and chromatic number", *J. London Math. Soc.* **35** (1960), 445–448. MR

[Hajnal and Komjáth 1984] A. Hajnal and P. Komjáth, "What must and what need not be contained in a graph of uncountable chromatic number?", *Combinatorica* **4**:1 (1984), 47–52. MR

[Halevi and Kaplan 2023] Y. Halevi and I. Kaplan, "Saturated models for the working model theorist", *Bull. Symb. Log.* **29**:2 (2023), 163–169. MR

[Halevi et al. 2022] Y. Halevi, I. Kaplan, and S. Shelah, "Infinite stable graphs with large chromatic number", *Trans. Amer. Math. Soc.* **375**:3 (2022), 1767–1799. MR

[Halevi et al. 2024] Y. Halevi, I. Kaplan, and S. Shelah, "Infinite stable graphs with large chromatic number, II", *J. Eur. Math. Soc.* (*JEMS*) **26**:12 (2024), 4585–4614. MR

[Jech 2003] T. Jech, *Set theory*, 3rd ed., Springer, 2003. MR

[Komjáth 2011] P. Komjáth, "The chromatic number of infinite graphs — a survey", *Discrete Math.* **311**:15 (2011), 1448–1450. MR

[Pillay 1996] A. Pillay, *Geometric stability theory*, Oxford Logic Guides **32**, Oxford University Press, 1996. MR

[Rinot 2012] A. Rinot, "The Engelking–Karłowicz theorem, and a useful corollary", blog post, 2012, available at https://blog.assafrinot.com/?p=2054.

[Shelah 1978] S. Shelah, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics **92**, North-Holland Publishing Co., Amsterdam, 1978. MR

[Simon 2015] P. Simon, *A guide to NIP theories*, Lecture Notes in Logic **44**, Association for Symbolic Logic, Chicago, 2015. MR

[Tent and Ziegler 2012] K. Tent and M. Ziegler, *A course in model theory*, Lecture Notes in Logic **40**, Association for Symbolic Logic, La Jolla, CA, 2012. MR

Received 19 Aug 2024. Revised 23 Jul 2025.

YATIR HALEVI:

ybenarih@campus.haifa.ac.il
Department of Mathematics, University of Haifa, Haifa, Israel

ITAY KAPLAN:

kaplan@math.huji.ac.il
Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel

SAHARON SHELAH:

shelah@math.huji.ac.il
Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel

Model Theory
Vol. 4, No. 3, 2025
https://doi.org/10.2140/mt.2025.4.251

Forking and invariant types in regular ordered abelian groups

Akash Hossain

We give a characterization of forking in regular ordered abelian groups. In particular, we prove that $C \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{f}}_A B$ if and only if $c \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{f}}_A B$ for each singleton $c \in \operatorname{dcl}(AC)$ in these structures.

Introduction

The nonforking global extensions of some unary type in the theory of divisible ordered abelian groups (DOAG) are very easy to describe: they correspond to the invariant cuts, and there are at most two of them for any base parameter set. Now, if we look at a finite tuple $\vec{c} = (c_1, \ldots, c_n)$ in DOAG, a necessary condition for $\operatorname{tp}(\vec{c}/AB)$ to be nonforking over A ($c_1, \ldots, c_n \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{f}}_A B$) is to have $d \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{f}}_A B$ for every $\mathbb{Q}$-linear combination d of the c_i. As the type in DOAG of a tuple is characterized by the cuts of the $\mathbb{Q}$-linear combinations of its components, one can wonder whether this condition is sufficient. We show in this paper that the answer is yes:

Theorem 1. *Let $M \models$ DOAG, let A, B be parameter subsets of M, and $\vec{c} = (c_1, \ldots, c_n) \in M^n$. Let $\kappa = \max(|AB|, 2^{\aleph_0})^+$, and suppose M is κ-saturated and strongly-κ-homogeneous. Then $\vec{c} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathbf{f}}_A B$ if and only if every closed bounded interval of the $\mathbb{Q}$-span of B that has a point in that of $A\vec{c}$ already has a point in that of A.*

Moreover, by using quantifier elimination in the Presburger language, we can extend our results to find a characterization of forking in the whole class of regular ordered abelian groups (ROAG, the theory of the ordered subgroups of $\mathbb{R}$, the ordered groups for which every interval having at least n elements admits an n-divisible element). These are our main theorems:

The author was partially supported by GeoMod AAPG2019 (ANR-DFG), Geometric and Combinatorial Configurations in Model Theory. The author would like to thank the referee for their valuable comments and advice.

MSC2020: primary 03C64; secondary 06F20.

Keywords: ordered abelian groups, divisible ordered abelian groups, regular ordered abelian groups, forking, dividing, model theory.

Theorem 2. *Let $M \models$ ROAG, let A, B be parameter subsets of M, and $\vec{c} = (c_1, \dots, c_n) \in M^n$. Let $\kappa = \max(|AB|, 2^{\aleph_0})^+$, and suppose M is κ-saturated and strongly-κ-homogeneous. If M is discrete, interpret 1 as its least positive element, else interpret 1 as 0 (this is the standard interpretation of 1 in the Presburger language). Let C be the subgroup of M generated by c, and A' and B' the relative divisible closures in M of the respective subgroups of M generated by $A \cup \{1\}$ and $AB \cup \{1\}$, respectively. Then $\vec{c} \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{f}}_A B$ if and only if $\vec{c} \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{d}}_A B$, if and only if $\mathrm{tp}(\vec{c}/AB)$ admits a global $\mathrm{Aut}(M^{\mathrm{eq}}/A' \cup \mathrm{acl}^{\mathrm{eq}}(\varnothing))$-invariant extension in $S(M^{\mathrm{eq}})$, if and only if the following conditions hold:*

(1) *Every closed bounded interval of B' that has a point in C already has a point in the divisible closure of A'.*

(2) *For all prime l, if $[M : lM]$ is infinite, then for all $N > 0$, we have*

$$(C + l^N M) \cap (B' + l^N M) = A' + l^N M.$$

Theorem 3. *With the same assumptions as in Theorem 2, $\mathrm{tp}(\vec{c}/AB)$ admits an $\mathrm{Aut}(M/A)$-invariant extension inside $S(M)$ if and only if $\vec{c} \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{f}}_A B$ and, additionally, for every prime l for which $[M : lM]$ is finite, we have $C \subseteq \bigcap_{N>0} A' + l^N M$.*

This description of forking is the most simple and natural way to assemble together the (well-known) descriptions of forking in the theory of torsion-free abelian groups, and in DLO. Our results clearly imply that forking in ROAG is a phenomenon that happens in dimension one: we have $C \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{f}}_A B$ if and only if, for each singleton $c \in \mathrm{acl}(AC)$ (which coincides here with $\mathrm{dcl}(AC)$), we have $c \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{f}}_A B$. Note that this particular property of unary forking is well-known in stable 1-based theories. This raises the question of how this property relates to local modularity in, say, o-minimal theories. It also suggests that one might define over unstable theories a notion (satisfied by ROAG) which would extend that of 1-basedness in the stable world. Note that we cannot hope to have this property in infinite (expansions of) fields, as it does not even hold in ACF.

Our main results rely heavily on our description of forking in DOAG, which really is the core of our paper. This theory is an enrichment of DLO and the theory of nontrivial $\mathbb{Q}$-vector spaces, and it is a reduct of RCF, three very common theories in which we know exactly what is forking. However, the current literature lacks a satisfying description of forking in DOAG, and this paper fills that gap. Simon [2011, Proposition 2.5] gave a characterization of forking in dimension one (essentially the one we describe in the first condition of Theorem 2) for dp-minimal ordered structures, which include dp-minimal ordered groups. One can also use Theorem 13.7 from [Haskell et al. 2008] to establish our characterization of forking in DOAG in the particular case where the base parameter set is an Archimedean-complete model.

Dolich [2004] gave a characterization of forking in o-minimal expansions of real-closed fields (see [Starchenko 2008] for a survey). This very nice characterization naturally links forking with our intuition on o-minimal geometry. The independence notion introduced by Dolich is shown to be stronger or equal to forking in any o-minimal theory, but, for technical reasons, one needs to be in an expansion of RCF to prove equality. In these extensions, [Dolich 2004, Section 8] shows that their independence coincides with the abstract model-theoretic notion of "non-1-dividing in semisimple theories". We do not know whether nonforking coincides with the independence of Dolich in DOAG, however we point out in Section 1.5 that forking does coincide with 1-dividing.

Our work is very close in spirit to [Mennuni 2022; Hils and Mennuni 2024] on the domination monoid. They focus on a different notion, in a larger class of theories. However, while they describe the space of invariant types (over arbitrary small parameter sets) up to equidomination, we essentially study the space of A-invariant types (with A fixed) up to A-interdefinability. One new difficulty that comes with having to deal with A-invariant types instead of arbitrary invariant types is that we have to deal with *Archimedean extensions*, the analogue of immediate extensions for the Archimedean/convex valuation (see the end of page 58 of [Gravett 1956] for a formal definition).

This paper aims to be the first step towards a systematic characterization of forking in every complexity class of arbitrary ordered abelian groups. Interesting future work could be to get a nice characterization of forking for dp-minimal ordered groups, as quantifier elimination is not very complicated in this class, and the case in dimension one is already covered by the work of Simon.

Let us give a brief outline of the proof of our description of forking in DOAG. Let A, B, $\vec{c}$ be as in Theorem 2, so that the conditions of the list hold. First of all, we show that $\vec{c}$ is A-interdefinable with a tuple $\vec{d}$ which is under "normal form". After that, we split $\vec{d}$ into fibers of ad hoc group valuations with respect to which being in normal form ensures that $\vec{d}$ is *separated* (see Definition 1.23). For each subtuple $\vec{d}'$ of this partition of $\vec{d}$, we build a global A-invariant extension of $\mathrm{tp}(\vec{d}'/AB)$. Finally, we show that we can "glue" all those extensions into a global A-invariant extension of the full type $\mathrm{tp}(\vec{d}/AB)$, and we are done. We will try to be as explicit as possible in our manipulations, and we will show that our ad hoc valuations interact in a meaningful way with the model-theoretic notions of tensor product and weak orthogonality, which play an important role in [Mennuni 2022; Hils and Mennuni 2024].

In the first section of this paper, we write basic definitions and give a more detailed and formalized outline of the proof of our result in DOAG.

The way we build our global invariant extension is typically a bottom-up approach. However, the concepts have to be introduced in a particular order to be coherent, and it will be more convenient for us to present the steps of this process in a top-down order.

In the second section, we define our ad hoc valuations, and we show, given global A-invariant extensions of the $\mathrm{tp}(\vec{d}'/AB)$, how to glue them into a global A-invariant extension of $\mathrm{tp}(\vec{d}/AB)$.

In the third section, we show how to build a global A-invariant extension of each of the $\mathrm{tp}(\vec{d}'/AB)$.

In the fourth section, we show how to build the normal form $\vec{d}$ from $\vec{c}$. The transformation of a tuple into a tuple under normal form is carried out in a very procedural way which goes through many steps, where we apply successively A-translations and maps from $\mathrm{GL}^n(\mathbb{Q})$ to $\vec{c}$. Once $\vec{d}$ is built, our characterization of forking for DOAG will be established.

We extend our results in DOAG to ROAG in the remaining sections. We use quantifier elimination to describe the types, their global extensions and their conjugates. We end the paper by proving our main result, Theorem 7.1, where we give our characterization of forking for ROAG.

Let us also describe the outline of the proof. In ROAG, we can see a complete type as a union of countably many partial types (Φ-types) which are "independent" of each other (see Lemma 5.7). For each such partial type, we give algebraic characterizations (essentially the conditions of Theorem 2) of when there exists a partial global extension which does not fork over A, and when this extension is A-invariant. Our results in DOAG allow us to deal with partial types that involve equality and order. The results about the other partial types may already follow from literature on the model theory of torsion-free abelian groups, but we still prove them explicitly for completeness.

1. Forking in DOAG

1.1. *Cut-independence.*

Notation. In the theory of some total order $<$, we adopt the standard notation to work with an arbitrary interval (be it closed, open, half-open, possibly with infinite bounds), by writing its lower bound on the left, and its upper bound on the right. For instance, if we consider, say, an interval $]a, b[$, then it is implicit that $a \leqslant b$.

Assumptions 1.1. Let $(M, <, \ldots)$ be the expansion of an infinite totally ordered first-order structure. Let $C \supseteq A \subseteq B$ be parameter subsets of M. Suppose M is $|AB|^+$-saturated and strongly-$|AB|^+$-homogeneous.

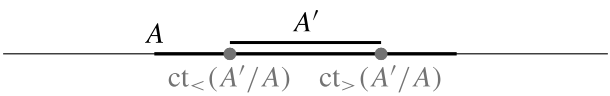
In this paper, we will manipulate various linear orders (not only ordered abelian groups, but also the chains of their convex subgroups, and other ad hoc orders). In these linear orders, the cuts will be seen as type-definable sets. The formal definition is as follows:

Definition 1.2. Let $c \in M$. The *cut* of c over A is the A-type-definable set $\mathrm{ct}(c/A)$, defined as the intersection of every interval containing c with bounds in $A \cup \{\pm\infty\}$ (a singleton is a closed interval).

Let P be some partition of M into convex sets. Then there is a natural linear order on P, the only linear order that makes the projection $M \to P$ an increasing map.

The set of cuts over A is clearly a partition of M into convex sets, and therefore the above paragraph applies.

Let us also write $x > A$ when $\forall a \in A,\ x > a$. If $\varnothing \neq A' \subseteq A$, define $\mathrm{ct}_>(A'/A)$ as the cut over A that corresponds to the elements that are $> A'$, and strictly smaller than any element of A that is $> A'$. Define $\mathrm{ct}_<(A'/A)$ similarly. More generally, if X is some (type/$\vee$)-definable subset of M, we write $\mathrm{ct}_>(X/A) = \mathrm{ct}_>(X(A)/A)$, and similarly for $\mathrm{ct}_<(X/A)$.



Definition 1.3. Let us define the following independence notions, which are ternary relations defined on the small subsets of M:

- $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{d}}$ is the standard nondividing independence (see [Tent and Ziegler 2012, Definition 7.1.2]).
- $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{f}}$ is the standard nonforking independence (see [Tent and Ziegler 2012, Definition 7.1.7]).
- Let $c_1, \dots, c_n \in M$. Define $c_1 \dots c_n \mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{inv}}_A B$ when $\mathrm{tp}(c_1 \dots c_n/AB)$ admits a global (in $S(M)$) $\mathrm{Aut}(M/A)$-invariant extension. Define $C \mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{inv}}_A B$ when $\vec{c} \mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{inv}}_A B$ for each finite tuple $\vec{c}$ of C.
- $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{bo}}$ (for "bounded orbit") is a weaker notion than $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{inv}}$, where we require the orbit under $\mathrm{Aut}(M/A)$ of the global extension to have a bounded cardinal instead of being a single point.

Fact 1.4. *The inclusions* $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{inv}} \subseteq \mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{bo}} \subseteq \mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{f}} \subseteq \mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{d}}$ *always hold in any first-order structure.*

Remark 1.5. Note that in DOAG, if A is not included in $\{0\}$, then $\mathrm{dcl}(A)$ is a model. In an o-minimal theory (in fact in any NIP theory), $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{f}}$, $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{d}}$ and $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{inv}}$ all coincide over models, and one can easily check by hand that it is also the case in DOAG for $A \subseteq \{0\}$. More precisely, the fact that $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{f}} = \mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{d}}$ follows from [Chernikov and Kaplan 2012, Theorem 1.1], while $\mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{f}} = \mathop{\smile\hskip-0.9em ^| \ }^{\mathbf{inv}}$ over models trivially follows from the fact that nonforking coincides with the independence notion given by Lascar-invariance (see for instance [Hrushovski and Pillay 2011, Proposition 2.1]).

As a result, the abstract equality $\mathop{\smile\hspace{-0.9em}\mid}^{\mathbf{f}} = \mathop{\smile\hspace{-0.9em}\mid}^{\mathbf{inv}}$ is already well-known in DOAG, though this equality does not help in any way to relate forking with more concrete geometric phenomena.

We show in Theorem 7.1 that $\mathop{\smile\hspace{-0.9em}\mid}^{\mathbf{f}}$, $\mathop{\smile\hspace{-0.9em}\mid}^{\mathbf{d}}$ and $\mathop{\smile\hspace{-0.9em}\mid}^{\mathbf{bo}}$ all coincide in regular groups.

Now, we are looking at these abstract model-theoretic notions in ordered abelian groups, which are structures that come with much more concrete geometric notions (the atomic definable sets are the solution sets of $\mathbb{Q}$-linear equations and inequations with parameters). The theory DOAG is even o-minimal. Thus we expect to find a purely geometric description of our abstract independence relations. The most basic example of a dependence behavior is the following:

Example 1.6. Suppose $M \models \mathrm{DLO}$, and identify some elementary substructure of M with $\mathbb{Q}$. Let $A = \varnothing$, $B = \{0, 2\}$, $C = \{1\}$. Then $\mathrm{acl}(AC) \cap \mathrm{acl}(AB) = \mathrm{acl}(A) = A$, but $C \not\mathop{\smile\hspace{-0.9em}\mid}^{\mathbf{d}}_{A} B$. Indeed, if we define $I_n = [4n, 4n+2]$, then the $(I_n)_{n<\omega}$ are pairwise-disjoint and A-conjugates, thus they divide over A, and $1 \in I_0$, which is B-definable.

In this example, the definable set that forks is an interval which is disjoint from some conjugates. We can see with the same reasoning that such intervals always divide:

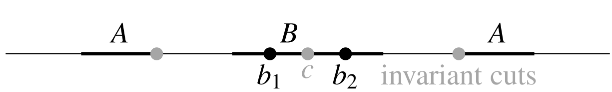
Lemma 1.7. *In the expansion of some total order, let I be an interval with nonempty interior such that its bounds are A-conjugates. Then I divides over A.*

Corollary 1.8. *Suppose that the theory of M is o-minimal, and let $I = [b_1, b_2]$ be an interval which is closed, bounded and disjoint from* $\mathrm{dcl}(A)$ (*in particular, b_1 and b_2 are A-conjugates*). *Then I divides over A.*

Note that we only consider bounded intervals: the bounds are not in $\{\pm\infty\}$, for they must be A-conjugates. Those considerations allow us to characterize forking in dimension one:

Proposition 1.9. *Suppose the theory of M is o-minimal. Then, for every singleton $c \in M$, we have $c \mathop{\smile\hspace{-0.9em}\mid}^{\mathbf{inv}}_{A} B$ if and only if every closed interval with bounds in* $\mathrm{dcl}(AB)$ *containing c has a point in* $\mathrm{dcl}(A)$.

Proof. The left-to-right implication is a consequence of Corollary 1.8 and Fact 1.4. For the other direction, assume the condition on the right holds. Then the elements of $\mathrm{dcl}(AB)$ having the same cut as c must all be either strictly smaller or strictly larger than c. Then, either $\mathrm{ct}_{<}(\mathrm{ct}(c/\mathrm{dcl}(A))/M)$ or $\mathrm{ct}_{>}(\mathrm{ct}(c/\mathrm{dcl}(A))/M)$ corresponds to a global (and complete by o-minimality) unary type that extends $\mathrm{tp}(c/AB)$. As $\mathrm{ct}(c/\mathrm{dcl}(A))$ is A-invariant, so are both those types, and we get the proposition.



□

Definition 1.10. Let R be $\mathbb{Z}$ or $\mathbb{Q}$. Suppose M is an expansion of some R-module, and let f be some n-ary $\varnothing$-definable function in this structure. We write $f \in \mathrm{LC}^n(R)$ when there exist $\lambda_1, \dots, \lambda_n \in R$ such that f coincides with the function $x_1, \dots, x_n \mapsto \sum_i \lambda_i x_i$. We write $\mathrm{LC}(R)$ when n is implicit.

Assumptions 1.11. Work in DOAG, in the language of ordered $\mathbb{Q}$-vector spaces. In this language, every substructure is a definably closed $\mathbb{Q}$-vector-subspace. The independence notions we are looking at are insensitive to definable closure (i.e., we have $C \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $\mathrm{dcl}(AC) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{dcl}(A)} \mathrm{dcl}(AB)$), so we fix $C \geqslant A \leqslant B$ $\mathbb{Q}$-vector subspaces of M.

It is well-known that DOAG is complete, has quantifier elimination and is o-minimal in this language. Moreover, for $\vec{c} = (c_1, \dots, c_n), \vec{d} = (d_1, \dots, d_n) \in M^n$, we have

$$\vec{c} \equiv_A \vec{d} \iff \forall f \in \mathrm{LC}^n(\mathbb{Q}),\ \mathrm{ct}(f(\vec{c})/A) = \mathrm{ct}(f(\vec{d})/A).$$

By Proposition 1.9, one can see that $C \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{inv}}_A B$ if and only if for all $c_1, \dots, c_n \in C$, there exists a global extension p of $\mathrm{tp}(c_1 \dots c_n/B)$ so that for all $f \in \mathrm{LC}^n(\mathbb{Q})$ and all closed intervals I with bounds in M not disjoint from A, we have $p(\vec{x}) \models f(\vec{x}) \notin I$.

The above description of $\mathop{\smile\!\!\!\!\!\!|}^{\mathbf{inv}}$ is still not geometric, because it is existential in $p \in S(M)$. There is a geometric description of a weaker version of this, where we swap the first two quantifiers:

Definition 1.12. Define $C \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{cut}}_A B$ if the following equivalent conditions hold:

- $\forall \vec{c} \in C^n,\ \forall f \in \mathrm{LC}^n(\mathbb{Q}),\ (f(\vec{c})) \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{inv}}_A B$ (the quantifier over p has been swapped with the one over f).
- $\forall c \in C,\ c \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{inv}}_A B$.
- Every closed interval with bounds in B that has a point in C already has a point in A.

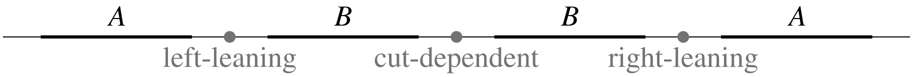
For $c_1, \dots, c_n \in M$, we define $c_1 \dots c_n \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{cut}}_A B$ when $(A + \mathbb{Q}c_1 + \dots + \mathbb{Q}c_n) \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{cut}}_A B$.

Remark 1.13. If c is a singleton from $C \setminus A$, then we have $c \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{cut}}_A B$ if and only if at least one of the following conditions holds:

- Every bounded B-definable closed interval included in $\mathrm{ct}(c/A)$ (viewed as an A-type-definable set) has all its points smaller than c.
- Every bounded B-definable closed interval included in $\mathrm{ct}(c/A)$ has all its points larger than c.

Definition 1.14. If the first condition of Remark 1.13 holds, then we say that c *leans right* with respect to A and B. If the second condition holds, then c *leans left*.

The two conditions hold at the same time if and only if $\mathrm{ct}(c/\operatorname{dcl}(A))$ has no point in $\operatorname{dcl}(AB)$.



It is easy to see that c leans right or left with respect to A and B if and only if there exists $A' \subseteq A$ so that $\mathrm{ct}(c/B) = \mathrm{ct}_>(A'/B)$ or $\mathrm{ct}_<(A'/B)$, respectively.

This definition also makes sense (and will be used) in more abstract linear orders that do not come from ordered groups.

The last item of Definition 1.12 is a purely geometric description which depends only on A, B, C. This description is simple and uses very little algebra: we have to replace A (resp. B, C) by the $\mathbb{Q}$-vector subspace generated by A (resp. AB, AC) as done in the notation, then we take every point from C, we completely ignore the algebraic relations between those points, and we check for each point a condition that only depends on the linear order.

Lemma 1.15. *We have* $\mathop{\smile\hspace{-0.9em}|}^{\mathbf{d}} \subseteq \mathop{\smile\hspace{-0.9em}|}^{\mathbf{cut}}$ *in* DOAG.

Proof. See Corollary 1.8. □

It actually turns out that this nice, but weak independence notion $\mathop{\smile\hspace{-0.9em}|}^{\mathbf{cut}}$ is no weaker than $\mathop{\smile\hspace{-0.9em}|}^{\mathbf{inv}}$ in DOAG. This is the fundamental result of this paper:

Theorem 1.16. *In* DOAG, *we have* $\mathop{\smile\hspace{-0.9em}|}^{\mathbf{inv}} = \mathop{\smile\hspace{-0.9em}|}^{\mathbf{cut}}$.

By Fact 1.4 and Lemma 1.15, we just need to prove that $\mathop{\smile\hspace{-0.9em}|}^{\mathbf{cut}} \subseteq \mathop{\smile\hspace{-0.9em}|}^{\mathbf{inv}}$.

Assumptions 1.17. Recall from Assumptions 1.11 that $B \geqslant A \leqslant C$ are $\mathbb{Q}$-vector-subspaces of M. We assume $C \mathop{\smile\hspace{-0.9em}|}^{\mathbf{cut}}_A B$. Let $\vec{c} = (c_1, \ldots, c_n) \in C$.

Our goal is to build a global A-invariant extension of $\mathrm{tp}(\vec{c}/B)$. This will be achieved in Sections 2–4, and we deal with ROAG and examples in the remaining sections.

1.2. ***Outline of the proof.*** Although $\mathop{\smile\hspace{-0.9em}|}^{\mathbf{cut}}$ and $\mathop{\smile\hspace{-0.9em}|}^{\mathbf{inv}}$ are very combinatorial notions, we have to set up a rather technical algebraic machinery to prove that they coincide. Let us explain in this subsection what we do conceptually. The reader has to keep in mind the outline of the proof given in the introduction; here we give a more detailed outline.

Assumptions 1.18. On top of Assumptions 1.17, we assume that M is $|B|^+$-saturated and strongly $|B|^+$-homogeneous.

We wish to show $c_1 \ldots c_n \mathop{\smile\hspace{-0.9em}|}^{\mathbf{inv}}_A B$. For this, as announced in the introduction, we partition our tuple $\vec{c}$ into smaller subtuples that we call "blocks", and we build a global $\mathrm{Aut}(M/A)$-invariant extension of the type of each block. We get a family of global $\mathrm{Aut}(M/A)$-invariant types that we call (in Definition 1.22) a "block extension", and what is left for us is to "glue" those types into a global $\mathrm{Aut}(M/A)$-invariant extension of the full type of $\vec{c}$. The partition of our tuple will be that given by fibers of an ad hoc valuation val^3_B on M (more precisely, a map which factors through a valuation on M/B; see Definition 1.20). The way we glue the block extension is via a tensor product of each of its types (the order of the factors has to be chosen well). We give more formal definitions and statements around valuations, orthogonality and tensor products in Sections 1.3 and 1.4.

There are three key steps in the proof:

(I) We replace $\vec{c}$ by a well-chosen A-interdefinable tuple $\vec{d}$, the "normal form". The following *normal form lemma* specifies the properties that we expect of $\vec{d}$, as well as guarantees its existence:

(NF3) If $\vec{c} \mathop{\smile\!\!\!\!|}^{\mathbf{cut}}_A B$, then $\vec{c}$ is A-interdefinable with some tuple $\vec{d}$ which is val^3_B-separated (as defined in Definition 1.23).

We prove the normal form lemma in Section 4 (Theorem 4.21).

(II) We find a block extension of the partition of $\vec{d}$ given by val^3_B, as guaranteed by the *block extension lemma*:

(BE3) If $\vec{d}$ is a val^3_B-separated tuple, and $\vec{d} \mathop{\smile\!\!\!\!|}^{\mathbf{cut}}_A B$, then the type over B of each fiber of $\vec{d}$ by val^3_B admits a global $\mathrm{Aut}(M/A)$-invariant extension (the family of these global types is a val^3_B-*block extension*, as defined in Definition 1.22). Moreover, this block extension is *strong*, i.e., the union of those global types is consistent with $\mathrm{tp}(\vec{d}/B)$.

We prove the block extension lemma in Section 3 (Corollary 3.24).

(III) We glue the block extension into a global invariant extension of the full type of $\vec{d}$, as guaranteed by the *gluing lemma*:

(GL3) Given $\vec{d}$ a val^3_B-separated tuple, and $(p_i)_i$ a strong val^3_B-block extension of $\vec{d}$, some tensor product of the $(p_i)_i$ is a global $\mathrm{Aut}(M/A)$-invariant extension of $\mathrm{tp}(\vec{d}/B)$.

We prove the gluing lemma in Section 2 (Corollary 2.46).

Theorem 1.16 immediately follows from these lemmas:

Proof. The inclusion $\mathop{\smile\!\!\!\!|}^{\mathbf{inv}} \subseteq \mathop{\smile\!\!\!\!|}^{\mathbf{cut}}$ follows from Fact 1.4 and Lemma 1.15.

For the other direction, assume $\vec{c} \mathop{\smile\!\!\!\!|}^{\mathbf{cut}}_A B$. Let $\vec{d}$ be given by (NF3). By definition, $\mathop{\smile\!\!\!\!|}^{\mathbf{cut}}$ is insensitive to definable closure, so we have $\vec{d} \mathop{\smile\!\!\!\!|}^{\mathbf{cut}}_A B$. By (BE3) and (GL3), we have $\vec{d} \mathop{\smile\!\!\!\!|}^{\mathbf{inv}}_A B$. We conclude that $\vec{c} \mathop{\smile\!\!\!\!|}^{\mathbf{inv}}_A B$, as $\mathop{\smile\!\!\!\!|}^{\mathbf{inv}}$ is insensitive to definable closure. $\square$

As the definition of val^3_B and the proof of (GL3) involve a sequence of several techniques of different nature, our approach is to build two intermediate coarser valuations val^1_B, val^2_B (val^3_B refines val^2_B, which refines val^1_B) for which the proofs of the intermediate gluing lemmas

(GL1) Given $\vec{d}$ a val^1_B-separated tuple, and $(p_i)_i$ a val^1_B-block extension of $\vec{d}$, $(p_i)_i$ is weakly orthogonal.

(GL2) Given $\vec{d}$ a val^2_B-separated tuple, the types over B of each fiber of $\vec{d}$ by val^2_B are weakly orthogonal. In particular, any val^2_B-block extension of $\vec{d}$ is strong.

as well as their formal definitions, are more atomic.

Note that a normal form for val^3_B is automatically a normal form for the two coarser valuations: finding a normal form is easier for coarser valuations. On the other hand, finding a block extension is harder, as a (strong) block extension for val^1_B or val^2_B automatically induces a (strong) block extension of val^3_B.

The coarser valuations also turn out to play an important role in the description of the Stone space of A-invariant types, which we do not perform in this paper (see Remark 3.3.31 from the author's PhD thesis [Hossain 2024]).

1.3. *Valuations.* The content of this subsection is well-known in the literature, and most of the proofs are omitted.

Assumptions 1.19. In this subsection, let G be an abelian group.

Definition 1.20. Let H be a subgroup of G, and Γ a linearly ordered set with a least element $-\infty$. A map $\mathrm{val}: G \to \Gamma$ is an *H-valuation* if it satisfies the following axioms for all $x, y \in G$:

- $\mathrm{val}(x-y) \leqslant \max(\mathrm{val}(x), \mathrm{val}(y))$.
- $\mathrm{val}(x) = -\infty$ if and only if $x \in H$.

A *valuation* over G is a $\{0\}$-valuation.

If val' is another H-valuation of G, we say that val' *refines* val if there exists an increasing map $f : \mathrm{val}'(G) \to \mathrm{val}(G)$ such that $\mathrm{val} = \mathrm{val}' \circ f$.

Equivalently, an H-valuation can be seen as a valuation over G/H.

Remark 1.21. Let us remark for the readers unfamiliar with valuations that if $\mathrm{val}(x) < \mathrm{val}(y)$, then it follows from the axioms that $\mathrm{val}(x+y) = \mathrm{val}(y)$.

The way we partition our tuple is with fibers of some ad hoc valuations:

Definition 1.22. Let val be a B-valuation over M. Given our tuple $\vec{c}$, the val-*blocks* of $\vec{c}$ are defined as the maximal subtuples of $\vec{c}$ of elements of equal value. They form a partition of $\vec{c}$.

By abuse of notation, if $\vec{c_i} = (\dots, c_{ij}, \dots)$ is a val-block of $\vec{c}$, we define $\mathrm{val}(\vec{c_i}) = \mathrm{val}(c_{ij})$, which does not depend on the choice of j.

A *weak* val-*block extension* of $\vec{c}$ is a family $(p_i)_i$ of global $\mathrm{Aut}(M/A)$-invariant extensions of the types over B of each val-block of $\vec{c}$. Such a block extension is *strong* when $\bigcup_i p_i$ is consistent with $\mathrm{tp}(\vec{c}/B)$.

In this paper, whenever we use the terminology of block extensions, it is always for global $\mathrm{Aut}(M/A)$-invariant extensions of types over B; the parameter sets A, B do not change.

In particular, if val′ refines val, then the val′-blocks of some tuple form a finer partition than its val-blocks.

Definition 1.23. Let val be an A-valuation over M. We say that the tuple $\vec{c}$ is val-*separated* when the following conditions hold:

- $\forall (\lambda_i)_i \in \mathbb{Q},\ \mathrm{val}\big(\sum_i \lambda_i c_i\big) = \max_i(\mathrm{val}(\lambda_i c_i))$.
- $\forall i,\ c_i \notin A$.

The notion of separatedness is an algebraic way to state that the val-blocks of a tuple are independent from each other. One can note that $\vec{c}$ is val-separated if and only if each of its val-blocks is. Moreover, a finite tuple that is separated with respect to some A-valuation must be a lift of some $\mathbb{Q}$-free tuple of M/A, i.e., a tuple in M which maps via the canonical surjection to a $\mathbb{Q}$-free tuple in M/A. In particular, such a tuple is $\mathbb{Q}$-free.

Remark 1.24. The tuple $\vec{c}$ is clearly A-interdefinable with the lift of a $\mathbb{Q}$-free tuple from C/A. Moreover, if $\vec{d}$ is a tuple that is A-interdefinable with $\vec{c}$, then one can easily show that we have $\vec{c} \mathop{\perp}_A B$ if and only if $\vec{d} \mathop{\perp}_A B$ for $\perp \in \{\perp^{\mathbf{cut}}, \perp^{\mathbf{inv}}\}$.

The next lemma gives us a canonical way to build an H-valuation from a preorder over G satisfying certain conditions:

Lemma 1.25. *Let P be a preorder over G. Let $\sim$ be the associated equivalence relation over G, π the quotient map, and $<$ the associated order on $G/\sim$. Suppose that $<$ is linear, and that for all $x, y \in G$, if $\pi(x) < \pi(y)$ then $\pi(x+y) = \pi(y)$ (the same implication as in Remark 1.21). Then the following conditions hold:*

- *$G/\sim$ has a least element, which is $\pi(0)$.*
- *The fiber $\pi^{-1}(\pi(0))$ is a subgroup of G.*
- *π is a $\pi^{-1}(\pi(0))$-valuation.*

For example, if P is the divisibility relation on an integral local ring R ($P(x, y)$ when y divides x), then P is a preorder satisfying the hypothesis of Lemma 1.25, i.e., $\pi(x) < \pi(y)$ implies $\pi(x+y) = \pi(y)$. In that case, $R/\sim$ is in natural correspondence with the poset of principal ideals of R, and $<$ is linear if and only if R is a valuation ring. Then, the group valuation given by the proposition is the natural ring valuation on R.

Assumptions 1.26. Suppose now G is an ordered abelian group.

Proposition 1.27. *Let $x, y \in G$. The following conditions are equivalent*:

(1) $x \in \bigcup_{n<\omega} [-n|y|, n|y|]$.

(2) *For every convex subgroup H of G, if $y \in H$, then $x \in H$.*

(3) *The convex subgroup of G generated by x is included in the one generated by y.*

Definition 1.28. These equivalent conditions define a relation on G which is clearly a preorder satisfying the necessary conditions of Lemma 1.25, with $\pi^{-1}(\pi(0)) = \{0\}$. This gives a valuation over G, which we call Δ, the *Archimedean valuation*. For $x \in G$, $\Delta^{-1}(\Delta(x))$ is called the *Archimedean class* of x. By abuse of notation, we may identify the Archimedean value of an element with its Archimedean class.

The idea behind the statement $\Delta(x) < \Delta(y)$ is that x is "infinitesimal" compared to y, or that y is "infinitely greater" than x.

Equivalently,

$$\begin{aligned}\Delta(x) < \Delta(y) &\iff x \in \bigcap_{n>0} \left]-\frac{1}{n}|y|, \frac{1}{n}|y|\right[\\ &\iff |y| \in \bigcap_{n>0} \left]n|x|, +\infty\right[.\end{aligned}$$

Let us draw a picture (not scaled correctly) which intuitively shows how the Archimedean classes look like. Suppose G has exactly four Archimedean classes $\Delta(0) < \delta_1 < \delta_2 < \delta_3$. Then G looks like Figure 1.

There is no standard terminology in the modern literature when it comes to the Archimedean valuation and related notions. Some use the keyword "convex" to refer to all those notions, others use the very obscure names "i-extension" or "i-completeness". We draw our inspiration from older sources: we choose to use the terminology of [Gravett 1956], which, on top of being a good introductory paper, sets up notation which we find more intuitive and easier to work with.

Remark 1.29. A convex subgroup $H \leqslant G$ is equal to the union of the convex subgroups of G generated by each $x \in H$. So H is entirely determined by its direct image $\Delta(H)$, which is an initial segment of $\Delta(G)$. We can note that the set $\mathcal{C}(G)$ of convex subgroups of G is totally ordered by inclusion, and that $H \mapsto \Delta(H)$ is an order isomorphism between $\mathcal{C}(G)$ and the set of nonempty initial segments of $\Delta(G)$ ordered by inclusion.

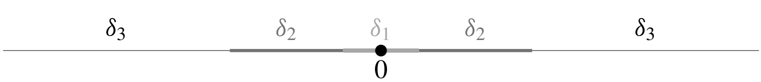


Figure 1. A group with four Archimedean classes.

We can also note that the cosets of some convex subgroup $H \leqslant G$ are all convex, because the translations are strictly increasing. So the quotient of G by some convex subgroup H is naturally endowed with a total order, as defined in Definition 1.2.

Definition 1.30. Let H be a subgroup of G, and $\delta \in \Delta(G)$. We define

$$H_{<\delta} = \{x \in H \mid \Delta(x) < \delta\}.$$

We introduce a similar notation for the conditions $\leqslant \delta$, $= \delta$, ... More generally, if $P \subseteq \Delta(G)$, we define $H_{\in P} = \{x \in H \mid \Delta(x) \in P\}$.

1.4. *Orthogonality and tensor product.* We work with Assumptions 1.18, i.e., M is a monster model of DOAG, $C \geqslant A \leqslant B$ are $\mathbb{Q}$-vector subspaces of M, $C \mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathbf{cut}}_A B$, and $\vec{c} = (c_1, \ldots, c_n)$ is a tuple from C.

Definition 1.31. Let $p_1, \ldots, p_n \in S(A)$. We say that the tuple of types $(p_i)_i$ is *weakly orthogonal* if $\bigcup_i p_i(\vec{x}_i)$ is a complete type over A in the union of the pairwise-disjoint tuples of variables $\vec{x}_1, \ldots, \vec{x}_n$.

There is a link in [Mennuni 2022, Proposition 4.5] between orthogonality and convex subgroups. We state a generalization:

Lemma 1.32. *Let G be some $\operatorname{Aut}(M/A)$-invariant $\mathbb{Q}$-vector subspace of M, and take $c, d \in M$. If $c \in (G + A) \not\ni d$, then $\operatorname{tp}(c/A)$ and $\operatorname{tp}(d/A)$ are weakly orthogonal.*

This holds in particular when G is some $\operatorname{Aut}(M/A)$-invariant convex subgroup.

Proof. Suppose towards a contradiction that they are not. Then $\operatorname{tp}(d/A)$ is not complete in $S(Ac)$, i.e., $\operatorname{ct}(d/A)$ has a point in $A + \mathbb{Q} \cdot c$, and this point is in $A + G$. By strong homogeneity, there exists an automorphism $\sigma \in \operatorname{Aut}(M/A)$ such that $\sigma(d) \in A + G$. As G is $\operatorname{Aut}(M/A)$-invariant, we have $d \in A + G$, a contradiction. □

Example 1.33. Suppose A is the lexicographical product

$$\mathbb{Q} \times_{\text{lex}} \mathbb{Q} \times_{\text{lex}} \mathbb{Q} \leqslant \mathbb{R} \times_{\text{lex}} \mathbb{R} \times_{\text{lex}} \mathbb{R}.$$

Let $c_1 = \big(0, 0, \sqrt{2}\big)$, $c_2 = \big(\sqrt{2}, 0, 0\big)$, $c_3 \in M$ such that

$$c_3 \in \bigcap_{N>0} \Big](0, N, 0), \Big(\frac{1}{N}, 0, 0\Big)\Big[.$$

Let G be the convex subgroup $\bigcap_{N>0} \big](-1/N, 0, 0), (1/N, 0, 0)\big[$. Then we can show by Lemma 1.32 that $\operatorname{tp}(c_2/A)$ is weakly orthogonal to both $\operatorname{tp}(c_1/A)$ and $\operatorname{tp}(c_3/A)$ using G.

Actually, we need not choose G to show that the type of c_1 is weakly orthogonal to that of c_2; we may choose instead, for instance, the convex subgroup $\bigcap_{N>0} \big](0, -1/N, 0), (0, 1/N, 0)\big[$. In fact, this works for any $\operatorname{Aut}(M/A)$-invariant convex subgroup of G which properly extends $\bigcap_{N>0} \big](0, 0, -1/N), (0, 0, 1/N)\big[$.

The same cannot be said for c_3: G seems to be the only valid witness for orthogonality. Indeed, c_2 belongs to $H=\bigcup_{N<\omega}\big[(-N,0,0),(N,0,0)\big]$, which is the least $\mathrm{Aut}(M/A)$-invariant convex subgroup properly extending G, whereas $c_3\notin A+H'$, with $H'=\bigcup_{N<\omega}\big[(0,-N,0),(0,N,0)\big]$ the largest $\mathrm{Aut}(M/A)$-invariant proper convex subgroup of G (note that H' witnesses orthogonality between c_3 and c_1).

We conclude that the type of c_2 is weakly orthogonal to the two other types, but for very different reasons. As it is harder to prove weak orthogonality between c_2 and c_3, c_2 is, in some sense that will be made formal in the next section, more related to c_3 than it is to c_1.

This example suggests that there are "several layers" of weak orthogonality in DOAG. We understand them by defining several ways to partition our tuple (with val^1_B, val^2_B, val^3_B) that refine each other. The finer the partition, the easier it becomes to find a block extension, the harder it is to "glue" the elements of the block extension. Actually, the reason why c_2 is more related to c_3 in the above example is exactly because $\mathrm{val}^1_A(c_1)\neq\mathrm{val}^1_A(c_2)=\mathrm{val}^1_A(c_3)$, and the reason why c_2 and c_3 are still weakly orthogonal is because $\mathrm{val}^2_A(c_2)\neq\mathrm{val}^2_A(c_3)$.

In model theory, the standard way to "glue" global invariant types is via the tensor product:

Definition 1.34. Let p, q be $\mathrm{Aut}(M/A)$-invariant global types. We define the *tensor product* of p by q to be the $\mathrm{Aut}(M/A)$-invariant complete global type

$$p(\vec{x})\otimes q(\vec{y})=\big\{\varphi(\vec{x},\vec{y},\vec{m})\mid \vec{m}\in M^n,\ \exists\vec{a}\models q_{|A\vec{m}},\ p(\vec{x})\models\varphi(\vec{x},\vec{a},\vec{m})\big\}.$$

The tensor product is associative, but not commutative in general. However, two types that are weakly orthogonal must necessarily commute. Conversely:

Fact 1.35. *In* DOAG (*in fact in any distal theory*; *see* [Simon 2013, Proposition 2.17]), *two global* $\mathrm{Aut}(M/A)$*-invariant types that commute must be weakly orthogonal.*

Lemma 1.36. *Let F_1, F_2 be closed subspaces of $S(A)$. Suppose p_1 and p_2 are weakly orthogonal for all $p_i\in F_i$. Then the set $F_1\times F_2$, seen as a topological subspace of $S(A)$, is exactly the topological direct product $F_1\times_{\mathrm{Top}}F_2$, via the homeomorphism*

$$h:p_1(\vec{x}_1)\cup p_2(\vec{x}_2)\mapsto(p_1(\vec{x}_1),p_2(\vec{x}_2)).$$

Proof. The map h is clearly a continuous bijection between compact separated spaces, and thus it is a homeomorphism. □

1.5. ***Relation to Dolich-independence.*** We make some short comments on how our result on DOAG relates to the work of Dolich [2004], their Section 8 in particular.

Dolich defines in their paper a geometric independence notion in terms of "halfway-definable cells", which we call $\mathop{\smile\!\!\!\!\!\!\perp}^{\text{Dolich}}$. It is shown that it coincides with $\mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{f}}$ in any o-minimal expansion of RCF. To motivate their results, Dolich also gives five axioms for independence notions (allegedly from unpublished notes of Shelah), four of which are always satisfied by nonforking in any theory, and the fifth, called "chain condition", is a weakening of the independence theorem. They also define an independence notion called "non-1-dividing", with a combinatorial definition which strengthens that of nonforking; let us write it $\mathop{\smile\!\!\!\!\!\!\perp}^{1\text{-}\mathbf{d}}$. They claim that, in case some independence relation satisfies the five axioms in a given theory, non-1-dividing is the weakest relation satisfying those axioms, and they show that $\mathop{\smile\!\!\!\!\!\!\perp}^{\text{Dolich}}$ satisfies the five axioms in any o-minimal theory. In particular, we have $\mathop{\smile\!\!\!\!\!\!\perp}^{\text{Dolich}} \subseteq \mathop{\smile\!\!\!\!\!\!\perp}^{1\text{-}\mathbf{d}} \subseteq \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{f}}$ in any o-minimal theory, and those inclusions are equalities in expansions of RCF.

Note that, if we generalize the independence notion $C \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{cut}}_A B$ to any o-minimal theory by the definition "*any closed bounded interval with bounds in* $\operatorname{dcl}(AB)$ *having a point in* $\operatorname{dcl}(AC)$ *already has a point in* $\operatorname{dcl}(A)$", then it follows from Corollary 1.8 that $\mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{d}} \subseteq \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{cut}}$, and hence $\mathop{\smile\!\!\!\!\!\!\perp}^{\text{Dolich}} \subseteq \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{cut}}$. This inclusion is strict in general, for there is an example in RCF where $C \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{cut}}_A B$ holds and $C \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{alg}}_A B$ fails, which implies $C \not\mathop{\smile\!\!\!\!\!\!\perp}^{\text{Dolich}}_A B$. So the main difference between the geometric independence notion introduced by Dolich and ours is that ours is easily shown to be weaker than nonforking in the general case, and the difficulties come when we prove that it is actually as strong as nonforking in DOAG, while $\mathop{\smile\!\!\!\!\!\!\perp}^{\text{Dolich}}$ is clearly stronger than nonforking, and it is difficult to prove the other direction in RCF.

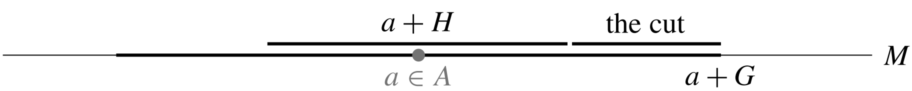
While we do not know whether $\mathop{\smile\!\!\!\!\!\!\perp}^{\text{Dolich}} = \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{f}}$ in DOAG (it would be a strengthening of our result), we remark that $\mathop{\smile\!\!\!\!\!\!\perp}^{1\text{-}\mathbf{d}} = \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{f}}$ in any theory (DOAG in particular) where $\mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{f}} = \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{inv}}$. Indeed, one may easily show that $\mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{inv}}$ satisfies the chain condition, using the fact that it satisfies extension. If $\mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{inv}} = \mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{f}}$, then it follows that $\mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{f}}$ satisfies the five axioms. By maximality of $\mathop{\smile\!\!\!\!\!\!\perp}^{1\text{-}\mathbf{d}}$, we have $\mathop{\smile\!\!\!\!\!\!\perp}^{\mathbf{f}} \subseteq \mathop{\smile\!\!\!\!\!\!\perp}^{1\text{-}\mathbf{d}}$, and the other inclusion always holds.

2. How to glue types via ad hoc valuations

This section has a dual purpose. On one hand, we want to establish the gluing properties (GL3), (GL2), (GL1). On the other, we need to set up the machinery necessary to prove those properties, including the definition of the valuations val^i_A for $i \in \{1, 2, 3\}$. This machinery will also be used in Sections 3 and 4. Our approach is to go back and forth between those two subjects (the gluing properties and the machinery), but before we do that and dive into the formal details, we introduce to the reader the geometric ideas that are at play, while keeping it somewhat informal.

A cut over some $\mathbb{Q}$-vector subspace A can be written as exactly one of the two following forms:

- A coset of some A-type-definable convex subgroup G. Examples of such cuts include singletons of A (with G trivial), irrational numbers (with $A = \mathbb{Q}$ and G the group of infinitesimals), power series of infinite support with $A = \mathbb{R}(t)$ and $G = \bigcap_n]-t^n, t^n[$ (the order is that for which t is positive and infinitesimal), or the singletons c_1, c_2 from Example 1.33. Those points can always be written as the limit of some sort of sequence of elements of A, either a pseudo-Cauchy sequence (as in Kaplansky theory) with respect to the Archimedean valuation, or some sequence whose behavior mimics that of a standard Cauchy sequence of real numbers.
- An A-translate of one of the two connected components of $G \setminus H$, with G some nontrivial A-type-definable convex subgroup, and H the greatest proper A-$\vee$-definable convex subgroup of G.



Examples of such cuts include (A-translates of) elements which are infinitesimal (G the group of infinitesimals, H trivial) or infinite (H the convex closure of A, $G =]-\infty, +\infty[$) with respect to A, or intermediate elements such as c_3 in Example 1.33. Extensions of A generated by those elements always have a new (not in $\Delta(A)$) Archimedean class.

We call the second case *ramified*, because it is the analogue for the Archimedean valuation of ramified points in valued fields. The first case is the analogue of residual and immediate points, and we call it *Archimedean*. The essential data of the cut of some singleton c, what really matters for the study of independence, is whether the cut is Archimedean or ramified, and which A-type-definable group G is involved (we call this group $G(c/A)$). This classification of 1-types is already interesting on its own, but it turns out that it can very naturally be extended in two crucial directions:

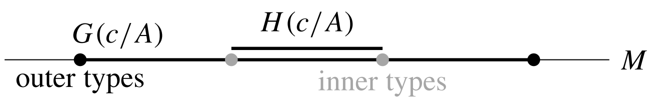
- Another description of independence in dimension one: if M is a monster model extending A, what are the type-definable groups G corresponding to global $\mathrm{Aut}(M/A)$-invariant extensions of some unary type of a singleton c over A?
- A description of types in any dimension, in terms of the description in dimension one.

Let us discuss the first direction. Let c be such that $c \mathop{\perp\!\!\!\!\perp}^{\mathbf{cut}}_A M$. First of all, by some saturation arguments, global $\mathrm{Aut}(M/A)$-invariant cuts are always ramified with respect to M. Secondly, if we define $H(c/A)$ (even in the Archimedean case) as the greatest proper A-$\vee$-definable convex subgroup of $G(c/A)$, then the interval $]H(c/A), G(c/A)[$ (in the chain of convex subgroups) can be seen as some approximation of the *radius of the cut* of c over A. The intuition is that in order to have independence, $]H(c/M), G(c/M)[$ has to be a finer approximation, i.e., we must have

$$H(c/A) \leqslant H(c/M) < G(c/M) \leqslant G(c/A)$$

and any other setting will imply forking dependence. Moreover, if all those inclusions were strict, then some phenomenon similar to Corollary 1.8 would occur, and lead to forking dependence. In conclusion, c must be ramified with respect to M, and satisfy one of the two following conditions:

- $H(c/M) = H(c/A)$, and $G(c/M)$ is the least M-type-definable convex subgroup which strictly extends $H(c/A)$. We call this type *inner*.
- $G(c/M) = G(c/A)$, and $H(c/M)$ is the greatest M-$\vee$-definable proper convex subgroup of $G(c/A)$. We call this type *outer*.



The same kind of ideas are explained in the author's PhD thesis [Hossain 2024, Remark 2.1.7] in order to describe independence in valued groups. They may prove enlightening to the readers who are familiar with valuation theory.

Let us now discuss the second direction. It should be easy to see that A-translations, and multiplication by a nonzero rational number, do not change whether a point is Archimedean or ramified, and which group G corresponds to its cut over A. The real challenge consists in understanding how things are affected by addition: given $\alpha, \beta \in \{\text{Archimedean, ramified}\} \times \{A\text{-type-definable convex subgroups}\}$ the descriptions of the respective cuts of c, d over A, what is that of $c+d$? After some computations, the reader may realize that the answer is only uncertain (it depends on c, d) when $\alpha = \beta$, else the answer is always the same, and it is one of the two descriptions α, β. One may recognize here the behavior of a valuation, and indeed, it turns out that there exists some total ordering of the descriptions for which the map $M \to \{\text{Archimedean, ramified}\} \times \{A\text{-type-definable convex subgroups}\}$ is an A-valuation (it corresponds to val^2_A). Now, if we manage to reduce to the study of a *separated* tuple with respect to this valuation, we will have total control over the type of this tuple, and over its global invariant extensions. It should now be more clear to the reader why valuations are so heavily involved in this paper.

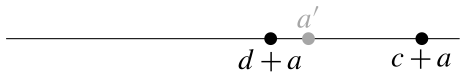
Now we establish the statements that we made in a way which is more suitable to formal proofs. We work with Assumptions 1.18, i.e., M is a monster model of DOAG, $C \geqslant A \leqslant B$ are $\mathbb{Q}$-vector subspaces of M, $C \mathop{\perp}^{\mathbf{cut}}_A B$, and $\vec{c} = (c_1, \dots, c_n)$ is a tuple from C.

2.1. *Basic definitions and classification of the cuts.*

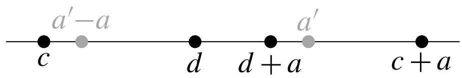
Lemma 2.1. *Let $c, d \in M$, and let $a \in A$, such that* $\operatorname{ct}(c/A) = \operatorname{ct}(d/A)$. *Then*

$$\operatorname{ct}(c + a/A) = \operatorname{ct}(d + a/A).$$

Proof. If not, then there exists $a' \in A$ which lies strictly between $c + a$ and $d + a$.



Then $a' - a \in A$ lies strictly between c and d, contradicting the hypothesis.



□

We hope the figures make the proofs easier to understand. However, we do not want them to be misleading, so we would like to say that they may not cover all the possible cases. For instance, if a were negative, then the correct picture would be reversed.

Definition 2.2. By Lemma 2.1 (and by saturation), A acts by translation over the set of all the cuts of M over A. For $c \in M$, denote by $\operatorname{Stab}(c/A)$ the stabilizer of $\operatorname{ct}(c/A)$.

Note that such a stabilizer is only a subgroup of A, which is clearly convex in A, but not in M.

Definition 2.3. Let $d \in M$. If $d \notin A$, then we define

$$G(d/A) = \bigcap \big\{]-|a|, |a|[: a \in A \setminus \operatorname{Stab}(d/A) \big\},$$

else we define $G(d/A) = \{0\}$. Either way, we also define

$$H(d/A) = \bigcup \big\{ [-|a|, |a|] : a \in \operatorname{Stab}(d/A) \big\}.$$

We view $G(d/A)$ as an A-type-definable set, and $H(d/A)$ as an A-$\vee$-definable set. They have the same points in A, however they do not have the same points in M when $d \notin A$. They are A-(type/$\vee$)-definable convex subgroups. By convention, if $\operatorname{Stab}(d/A) = A$, then $G(d/A)$ is the definable convex subgroup $]-\infty, +\infty[$ of all elements.

Example 2.4. In Example 1.33, $G(c_2/A) = G(c_3/A) = G$, and $H(c_2/A) = H(c_3/A) = H'$. However, $G(c_1/A)$ is the group of elements which are infinitesimal with respect to A, and it is distinct from G. As for $H(c_1/A)$, it is trivial.

Likewise, for arbitrary A, if $\Delta(0) < \Delta(d) < \Delta(A \setminus \{0\})$ (i.e., d is infinitesimal with respect to A), then $H(d/A)$ is trivial, and $G(d/A)$ is the type-definable group of elements which are infinitesimal with respect to A.

If $\Delta(d) > \Delta(A)$, then $H(d/A)$ is the convex subgroup generated by A, whereas $G(d/A)$ is the whole group.

Remark 2.5. We always have $G(d/A) \geqslant H(d/A)$. By definition, $H(d/A)$ is the convex subgroup generated by $\mathrm{Stab}(d/A)$, i.e., the least A-$\vee$-definable convex subgroup containing $\mathrm{Stab}(d/A)$. In case $d \notin A$, $G(d/A)$ is the largest A-type-definable convex subgroup disjoint from $A \setminus \mathrm{Stab}(d/A)$. For all $d, d' \in M$, we have in fact

$$G(d/A) = G(d'/A) \quad \Longrightarrow \quad H(d/A) = H(d'/A),$$

the only case where the implication is not an equivalence being when one point is in A and some A-translate of the other is infinitesimal with respect to A. We also have

$$\mathrm{Stab}(d/A) = \mathrm{Stab}(d'/A) \quad \Longleftrightarrow \quad H(d/A) = H(d'/A).$$

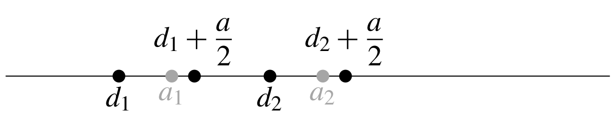
Proposition 2.6. *Let $d, d' \in M$. If $G(d/A) < G(d'/A)$, then $G(d/A) < H(d'/A)$.*

Proof. By definition of $G(d/A)$, there must exist $a \in A \setminus \mathrm{Stab}(d/A)$ such that $a \in G(d'/A)$. By Remark 2.5, $a \in H(d'/A) \setminus G(d/A)$. □

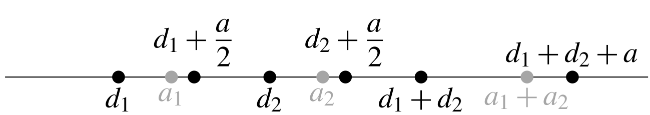
Let us now build val^1_A. Recall that convex subgroups are totally ordered by inclusion.

Lemma 2.7. *For all $d_1, d_2 \in M$, we have $G(d_1+d_2/A) \leqslant \max(G(d_1/A), G(d_2/A))$.*

Proof. Suppose for a contradiction that $G(d_1+d_2/A) > \max(G(d_1/A), G(d_2/A))$. Then there must exist $a \in \mathrm{Stab}(d_1+d_2/A)$ such that $a \notin \mathrm{Stab}(d_i/A)$. As $\mathrm{Stab}(d_i/A)$ is a convex subgroup of A, we also have $a/2 \notin \mathrm{Stab}(d_i/A)$, so there exists $a_i \in A$ which lies strictly between d_i and $d_i + a/2$.



Then $a_1 + a_2$ lies strictly between $d_1 + d_2$ and $d_1 + d_2 + a$, contradicting the fact that $a \in \mathrm{Stab}(d_1 + d_2/A)$.



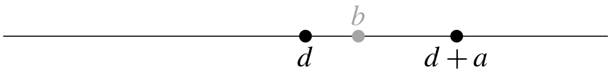
□

Definition 2.8. By Lemma 2.7, the map $x \mapsto G(x/A)$ is an A-valuation, so we set $\mathrm{val}^1_A(x) = G(x/A)$.

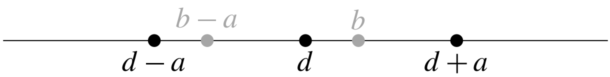
Note that, by definition, we always have $G(d/B) \leqslant G(d/A)$ for all $d \in M$. Furthermore:

Proposition 2.9. *For all $d \in M$, if $d \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{cut}}_A B$, then $H(d/B) \geqslant H(d/A)$.*

Proof. Suppose $H(d/B) < H(d/A)$. Then there exists $a \in \mathrm{Stab}(d/A)$ such that $a \notin \mathrm{Stab}(d/B)$. Thus we can find $b \in B$ which lies strictly between d and $d + a$.



On one hand, $b - a$ lies strictly between d and $d - a$.



As $\mathrm{Stab}(d/A)$ is a group, we have on the other hand $-a \in \mathrm{Stab}(d/A)$, that is, $\mathrm{ct}(d + a/A) = \mathrm{ct}(d/A) = \mathrm{ct}(d - a/A)$. In particular, no point from A lies in the closed interval with bounds b and $b - a$. On the other hand, d lies strictly between b and $b - a$, which implies $d \not\perp\!\!\!\perp^{\mathbf{cut}}_A B$. $\square$

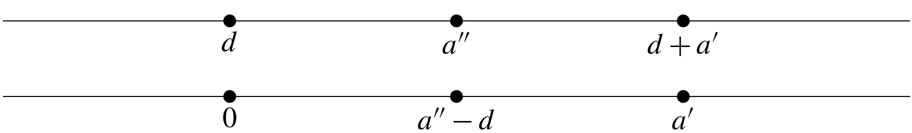
In the cut-independent setting, one may intuitively see $G(c/A)$ as a distance that approximates the position of c "from the top", while $H(c/A)$ approximates c "from the bottom". Then, when we go to a larger parameter set B, we get a finer approximation.

Proposition 2.10. *Let $a \in A$, and $d \in M$. Then the following conditions are equivalent*:

- $d - a \in G(d/A)$ *and* $d \notin A$.
- $\Delta(d - a) \notin \Delta(A)$.

Proof. Suppose $d - a \in G(d/A)$. If we had $\Delta(d - a) \in \Delta(A)$, then there would be $a' \in A$ such that $\Delta(d - a) = \Delta(a')$. This would imply $a' \in (G(d/A) \cap A) = \mathrm{Stab}(d/A)$. Since, for some n, $d - n|a'| \leqslant a \leqslant d + n|a'|$, as $na' \in \mathrm{Stab}(d/A)$, it follows that $\mathrm{ct}(d/A) = \mathrm{ct}(a/A)$, and thus $d = a$, proving the first direction.

Conversely, suppose $\Delta(d - a) \notin \Delta(A)$. By definition of $G(d/A)$, in order to show that $a \in (d + G(d/A))$, it suffices to show that $a \in \,]d - |a'|, d + |a'|[$ (i.e., $|d - a| \leqslant |a'|$) for every $a' \in A \setminus \mathrm{Stab}(d/A)$. Suppose towards a contradiction this fails for some a'. Then $\Delta(d - a) \geqslant \Delta(a')$, and this inequality is strict by the hypothesis. As $a' \notin \mathrm{Stab}(d/A)$, let $a'' \in A$ lying strictly between d and $d + a'$.



Then $|d - a''| \leqslant |a'|$, and thus $\Delta(d - a'') < \Delta(d - a)$. We then apply Remark 1.21 to get $\Delta(d - a) = \Delta(d - a'' - d + a) = \Delta(a - a'') \in \Delta(A)$, a contradiction. $\square$

Definition 2.11. If a satisfies the conditions of Proposition 2.10, then we say that a is a *ramifier of d over A*. We write $\mathrm{ram}(d/A)$ for the set of those ramifiers, and we say that d is *ramified over A* if this set is nonempty. We say that d is *Archimedean over A* whenever it is not ramified over A.

In Example 1.33, c_1 and c_2 are Archimedean over A, c_3 is ramified over A, and 0 is a ramifier.

Remark 2.12. If $d \in M$, then d is ramified over A if and only if it satisfies the following equivalent conditions:

(1) The coset $d + G(d/A)$ has a point in A, and $d \notin A$.

(2) $\Delta(A + \mathbb{Q}d) \supsetneq \Delta(A)$.

Moreover, as $A \cap G(d/A) = \mathrm{Stab}(d/A)$, $\mathrm{ram}(d/A)$ is a coset from the quotient $A/\mathrm{Stab}(d/A)$, and thus all its elements lie in the same coset of $H(d/A)$. It turns out that d does not belong to that coset:

Lemma 2.13. *Let $d \in M \setminus A$ and $a \in A$. Then $d - a \notin H(d/A)$.*

Proof. Suppose towards a contradiction that there exists $a' \in \mathrm{Stab}(d/A)$, with $|d - a| \leqslant |a'|$. Then $a \in \mathrm{ram}(d/A)$, thus $\Delta(d - a) \notin \Delta(A)$, and in particular $\Delta(d - a) < \Delta(a')$. By Remark 1.21, $\Delta(d - (a + a')) = \Delta(a') \in \Delta(A)$, and thus $a + a' \notin \mathrm{ram}(d/A)$. This contradicts the fact that $a \in \mathrm{ram}(d/A)$, and the fact that $a' \in \mathrm{Stab}(d/A)$. □

Corollary 2.14. *Let $d \in M$ be ramified over A, and consider two elements a and a' in $\mathrm{ram}(d/A)$. Then $\Delta(d - a) = \Delta(d - a')$.*

Proof. We have $d - a \notin H(d/A) \ni a - a'$, thus $\Delta(d - a) > \Delta(a - a')$, and by Remark 1.21 we have $\Delta(d - a) = \Delta(d - a + a - a') = \Delta(d - a')$. □

Definition 2.15. Let $d \in M$ be ramified over A. We define $\delta(d/A)$ as $\Delta(d - a)$ for some $a \in \mathrm{ram}(d/A)$. This definition does not depend on the choice of the ramifier a by Corollary 2.14.

Note that $\delta(d/A)$ is the unique element of $\Delta(A + \mathbb{Q} \cdot d) \setminus \Delta(A)$.

Remark 2.16. Let d be ramified over A, and $a \in \mathrm{ram}(d/A)$. Then, as $G(d/A)$ and $H(d/A)$ have the same points in A, one can easily compute $\mathrm{ct}(\delta/\Delta(A))$. A convex subgroup is induced by its Archimedean classes, so by definition of the $\vee$-definable convex subgroup $H = H(d/A)$, for any model N containing A, $\Delta(H(N))$ is the least initial segment of $\Delta(N)$ containing $\Delta(\mathrm{Stab}(d/A))$. Now, as $\delta \in \Delta(G(d/A)) \setminus \Delta(H)$, we clearly have $\mathrm{ct}(\delta/\Delta(A)) = \mathrm{ct}_{>}(\Delta(\mathrm{Stab}(d/A))/\Delta(A))$.

Many notions that we manipulate behave very differently with ramified and Archimedean points, often leading to case disjunctions. For instance, the following statements show us that cuts do not look the same for Archimedean and ramified points:

Lemma 2.17. *Let $d \in M$. Then* $\mathrm{ct}(d/A) \subseteq (d + G(d/A))$ (*as type-definable sets*).

Proof. By contraposition, let $d' \in M$ be such that $d - d' \notin G(d/A)$. Then, by definition, there exists $a \in A \setminus \mathrm{Stab}(d/A)$ such that $|d - d'| \geqslant |a|$. Let $a' \in A$ lie strictly between d and $d + a$. As either $d + a$ or $d - a$ lies strictly between d and d', either a' or $a' - a$ lies strictly between d and d'. Thus $\mathrm{ct}(d/A) \neq \mathrm{ct}(d'/A)$. $\square$

Proposition 2.18. *Let $d \in M$ be Archimedean over A. Then the A-type-definable set* $\mathrm{ct}(d/A)$ *coincides with $d + G(d/A)$.*

Proof. Suppose $d \notin A$, else the proposition is trivial. We have the inclusion $d + G(d/A) \subseteq \mathrm{ct}(d/A)$, as $d + G(d/A)$ is a convex set containing d and disjoint from A. The other direction follows from Lemma 2.17. $\square$

Proposition 2.19. *Let $d \in M$ be ramified over A. Then* $\mathrm{ct}(d/A)$ *can be written as an A-translate of one of the two connected components of $G(d/A) \setminus H(d/A)$.*

These translates are exactly the two thick black segments in the last line of Figure 2. The connected components of some set will always refer to its maximal convex subsets.

Proof. Let $a \in \mathrm{ram}(d/A)$. By Lemma 2.17, we have

$$\mathrm{ct}(d/A) \subseteq d + G(d/A) = a + G(d/A).$$

By Lemma 2.13, $d \notin (a + H(d/A))$, which is the convex closure of $\mathrm{ram}(d/A) \subseteq A$, and thus $a + H(d/A)$ must be disjoint from $\mathrm{ct}(d/A)$. It follows that $\mathrm{ct}(d/A)$ is included in the connected component of $(a + G(d/A)) \setminus (a + H(d/A))$ containing d.

Let $\mathcal{C}$ be that connected component, and $d' \in \mathcal{C}$. There does not exist $a' \in A$ lying between d and d', for such a' would be in

$$(a + G(d/A)) \cap A = \mathrm{ram}(d/A) \subseteq a + H(d/A),$$

and a' would also belong to $\mathcal{C}$ as $\mathcal{C}$ is convex, contradicting the fact that $\mathcal{C}$ is disjoint from $a + H(d/A)$. $\square$

Corollary 2.20. *With the hypothesis from Proposition 2.19, if a is in* $\mathrm{ram}(d/A)$, *and d' is another point from M, then $d' \equiv_A d$ if and only if the following conditions hold*:

- $\mathrm{ct}(\Delta(d' - a)/\Delta(A)) = \mathrm{ct}(\Delta(d - a)/\Delta(A))$ (*hence $\Delta(d' - a)$ is not in $\Delta(A)$*).
- $d' < a \Longleftrightarrow d < a$.

Proof. The first condition is equivalent to $d' - a \in G(d/A) \setminus H(d/A)$. The second condition ensures that d' lies in the correct connected component. $\square$

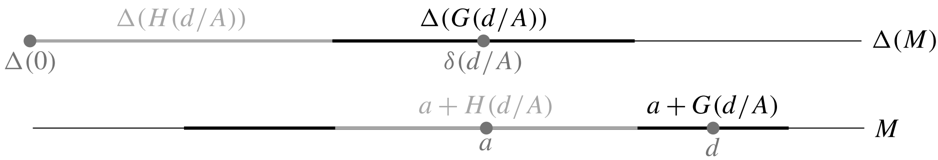


Figure 2. A ramified point, its corresponding Archimedean class, and their cuts.

The two statements Proposition 2.18 and Proposition 2.19 yield a simple classification of the cuts over A. In order to consider nonforking extensions of types, we now have to deal with ways to refine those cuts to cuts over a larger parameter set B. As we classified the cuts over A by looking at $\mathrm{Aut}(M/A)$-invariant convex subgroups, we have to compute what are the $\mathrm{Aut}(M/B)$-invariant convex subgroup corresponding to their refinements.

Definition 2.21. If G is an A-type-definable convex subgroup, then we define the A-$\vee$-definable convex subgroup

$$\underline{G}_A = \bigcup_{a \in A \cap G} [-|a|, |a|].$$

If H is an A-$\vee$-definable convex subgroup, then we define the A-type-definable convex subgroup

$$\overline{H}^A = \bigcap_{a \in A \setminus H}]-|a|, |a|[.$$

Remark 2.22. Let $d \in M$. We have

$$H(d/A) \leqslant \overline{H(d/A)}^M \leqslant \overline{H(d/A)}^B \leqslant \underline{G(d/A)}_B \leqslant \underline{G(d/A)}_M \leqslant G(d/A)$$

and

$$\underline{G(d/A)}_A = H(d/A), \quad \overline{H(d/A)}^A = G(d/A).$$

Moreover, if $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_A^{\mathbf{cut}} B$, then by Proposition 2.9 we recall

$$H(d/A) \leqslant H(d/B) \leqslant G(d/B) \leqslant G(d/A).$$

2.2. ***Weak orthogonality.*** Now we start to deal with global invariant extensions, in order to prove (GL1).

Lemma 2.23. *Let $d \in M \setminus A$ be Archimedean over A, with $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_A^{\mathbf{cut}} B$. Let p be a global* $\mathrm{Aut}(M/A)$*-invariant extension of* $\mathrm{tp}(d/B)$*. Then $G(d/B) = G(d/A)$ (as B-type-definable sets), and*

$$p(x) \models x - d \in G(d/A) \setminus \underline{G(d/A)}_M.$$

Note that the statement holds even if the element d is ramified over B.

Proof. Suppose towards a contradiction that $G(d/B) < G(d/A)$. Then there exists b inside $B \setminus \mathrm{Stab}(d/B)$ such that $b \in G(d/A)$. Let $b_1 \in B$ lie strictly between d and $d+b$. Likewise, as $-b$ is also in $G(d/A) \setminus \mathrm{Stab}(d/B)$, we find $b_2 \in B$ which lies strictly between d and $d-b$. Then the closed interval with bounds b_1, b_2 contains d and is strictly included in $d + G(d/A) = \mathrm{ct}(d/A)$, which contradicts cut-independence.

By Proposition 2.18, as p extends $\mathrm{tp}(d/A)$, we necessarily have

$$p(x) \models x - d \in G(d/A).$$

Let $c \models p$ in some elementary extension. As p is $\mathrm{Aut}(M/A)$-invariant, we have $c \mathop{\smile\!\!\!\!\!\mid}_A^{\mathbf{cut}} M$, and thus $G(c/M) = G(d/A)$ by the above paragraph (c is of course Archimedean over A). As $c - d \in G(c/M)$, and $d \in M \not\ni c$, we have $d \in \mathrm{ram}(c/M)$, and thus $c - d \notin H(c/M) = \underline{G(c/M)}_M = \underline{G(d/A)}_M$. We conclude $p(x) \models x - d \notin \underline{G(d/A)}_M$. □

Lemma 2.24. *Let $d \in M$ be ramified over A, with $d \mathop{\smile\!\!\!\!\!\mid}_A^{\mathbf{cut}} B$, and let $a \in \mathrm{ram}(d/A)$. Then $\Delta(d-a) \notin \Delta(B)$.*

Moreover, if p is a global $\mathrm{Aut}(M/A)$-invariant extension of $\mathrm{tp}(d/B)$, then either

$$G(d/B) = G(d/A) \quad \text{and} \quad p(x) \models x - a \in G(d/A) \setminus \underline{G(d/A)}_M$$

or

$$H(d/B) = H(d/A) \quad \text{and} \quad p(x) \models x - a \in \overline{H(d/A)}^M \setminus H(d/A).$$

Proof. Recall that, by definition of $\mathrm{ram}(d/A)$ (see Proposition 2.10 and the definition that follows), $\Delta(d-a)$ is not in $\Delta(A)$. Suppose we have $\Delta(d-a) = \Delta(b)$ for some $b \in B$. Let $n > 0$ such that $|b|/n \leqslant |d-a| \leqslant n|b|$. Then we have $d \in [a - n|b|, a - |b|/n] \cup [a + b/n, a + n|b|]$. By cut-independence, one of those two intervals has a point in A, say, a'. We have $|b|/n \leqslant |a - a'| \leqslant n|b|$, and therefore $\Delta(b) = \Delta(a - a') \in \Delta(A) \not\ni \Delta(d-a) = \Delta(b)$, a contradiction.

As d is ramified over A, and p extends $\mathrm{tp}(d/A)$, it follows from Proposition 2.19 that $p(x) \models x - a \in G(d/A) \setminus H(d/A)$.

Suppose towards a contradiction that $p(x) \models x - a \in \underline{G(d/A)}_M \setminus \overline{H(d/A)}^M$. Then there exist $m_1, m_2 \in M$ such that $m_1 \notin H(d/A)$, $m_2 \in \underline{G(d/A)}$, and $p(x) \models |m_1| \leqslant |x - a| \leqslant |m_2|$, i.e., $p(x) \models x \in [a - |m_2|, a - |m_1|] \cup [a + |m_1|, a + |m_2|]$. As $m_1 \notin H(d/A)$ and $m_2 \in G(d/A)$, each of those two intervals is included in each connected component of $a + (G(d/A) \setminus H(d/A))$. As a result, none of those intervals belongs to p by Corollary 1.8, a contradiction.

By Lemma 2.23 and Remark 2.22, if we had $p(x) \models x - a \in \underline{G(d/A)}_B$ (resp. $p(x) \models x - a \notin \overline{H(d/A)}^B$), then we would have

$$p(x) \models x - a \in \overline{H(d/A)}^M \setminus H(d/A) \quad (\text{resp. } p(x) \models x - a \in G(d/A) \setminus \underline{G(d/A)}_M).$$

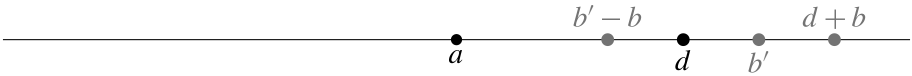
To conclude, it now suffices to show the two following properties:

(1) If $G(d/B) < G(d/A)$, then $\mathrm{tp}(d/B) \models x - a \in \underline{G(d/A)}_B$.

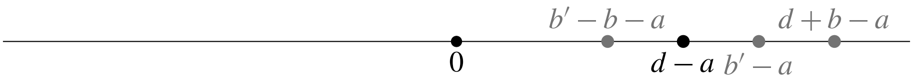
(2) If $H(d/B) > H(d/A)$, then $\mathrm{tp}(d/B) \models x - a \notin \overline{H(d/A)}^B$.

Let us proceed:

(1) By hypothesis, let $b \in B \setminus \mathrm{Stab}(d/B)$ such that $b \in G(d/A)$. Let $b' \in B$ lie strictly between d and $d + b$.



Then the interval I with bounds $b' - a$, $b' - b - a$ is included in $G(d/A)$, and $\mathrm{tp}(d/B) \models x - a \in I$.



(2) By hypothesis, let $b \in \mathrm{Stab}(d/B)$ such that $b \notin H(d/A)$. Then we have $\mathrm{ct}(d - b/B) = \mathrm{ct}(d/B) = \mathrm{ct}(d + b/B)$, which implies $a \notin [d - |b|, d + |b|]$. As a result, $\mathrm{tp}(d/B) \models x - a \notin [-|b|, |b|]$. □

Lemmas 2.23 and 2.24 imply:

Corollary 2.25. *Let $d \in M$ be such that $d \mathop{\smile\!\!\!\!\!\!\!\;\vert}^{\mathbf{cut}}_{A} B$. Let $p \in S(M)$ be a global* $\mathrm{Aut}(M/A)$*-invariant extension of* $\mathrm{tp}(d/B)$. *Assume p is not realized, i.e., $d \notin A$. Then there exists f an A-definable map such that one of the two following conditions hold*:

- *The cut over M induced by p is one of the two connected components of the translation by $f(d)$ of $G(d/A) \setminus \underline{G(d/A)}_M$. In that case, $G(d/B) = G(d/A)$.*
- *The cut over M induced by p is one of the two connected components of the translation by $f(d)$ of $\overline{H(d/A)}^M \setminus H(d/A)$. In that case, $H(d/B) = H(d/A)$.*

Proof. If d is Archimedean over A, then the first condition holds with $f = \mathrm{id}$, as stated in Lemma 2.23. If d is ramified over A, then either the first or the second condition holds with $f : x \mapsto x - a$ for some $a \in \mathrm{ram}(d/A)$, by Lemma 2.24. □

Remark 2.26. If $d \notin A$, while only one of the conditions from Corollary 2.25 holds, we might still have $G(d/B) = G(d/A)$ and $H(d/B) = H(d/A)$ at the same time. For instance, this is the case whenever $\Delta(d) < \Delta(B \setminus \{0\})$, and B has no element that is infinitesimal compared to A.

In fact, one can see that p is the type of an M-ramified point which adds an Archimedean class δ which is cut-independent over $\Delta(A)$: the two conditions of the statement expresses that δ either leans left or right with respect to $\Delta(A)$ and $\Delta(M)$.

Remark 2.27. For all $d \in M$, recall from Remark 2.22 that $G(d/B) = \overline{H(d/B)}^B$. As a result, if $H(d/B) = H(d/A)$, then $G(d/B) = \overline{H(d/A)}^B$. In particular, we see that in either case of Corollary 2.25, $p(x)$ implies $x - f(d) \in G(d/B)$. Likewise:

Lemma 2.28. *Let $d \in M$ such that $d \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathbf{cut}}_A B$, and $d \notin A$. Let p be a global* $\mathrm{Aut}(M/A)$*-invariant extension of* $\mathrm{tp}(d/B)$. *Then $p(x)$ implies $x \notin (M + H(d/B))$.*

Proof. Let f be the A-definable map witnessing Corollary 2.25.

As $G(d/A) \geqslant G(d/B)$, we have

$$\underline{G(d/A)}_M \geqslant \underline{G(d/B)}_M \geqslant H(d/B).$$

If the second condition holds, let $G = G(d/B)$ and $H = \underline{G(d/B)}_M$, else let $G = \overline{H(d/B)}^M$ and $H = H(d/B)$. Either way, G and H have the same points in M, $p(x) \models x - f(d) \in G \setminus H$, and $H \geqslant H(d/B)$. Suppose towards a contradiction that there is $m \in M$ such that $p(x) \models x - m \in H(d/B)$. Then, as $p(x) \models x - f(d) \in G$, we have $m - f(d) \in G$, yet $m - f(d) \in M$. Therefore $m - f(d) \in H$. This is a contradiction, as we cannot have at the same time $p(x) \models x - f(d) \notin H$, $p(x) \models x - m \in H$ and $f(d) - m \in H$. □

We established that realizations of unary global invariant types belong to some cosets of convex subgroups, and not others. Just as in Lemma 1.32, we may relate those statements to orthogonality in order to prove (GL1).

Lemma 2.29. *Let $\vec{d}_1, \dots, \vec{d}_n$ be tuples of M such that, for all i,* $\mathrm{tp}(\vec{d}_i/A)$ *and* $\mathrm{tp}(\vec{d}_{<i}/A)$ *are weakly orthogonal. Then the tuple of types* $(\mathrm{tp}(\vec{d}_1/A), \dots, \mathrm{tp}(\vec{d}_n/A))$ *is weakly orthogonal with respect to Definition 1.31.*

Proof. We show by induction on i that $(\mathrm{tp}(\vec{d}_1/A), \dots, \mathrm{tp}(\vec{d}_i/A))$ is weakly orthogonal. The statement is trivial if $i = 1$. By the induction hypothesis, we have $\bigcup_{j<i} \mathrm{tp}(\vec{d}_j/A) \models \mathrm{tp}(\vec{d}_{<i}/A)$. By hypothesis, $\mathrm{tp}(\vec{d}_i/A) \cup \mathrm{tp}(\vec{d}_{<i}/A) \models \mathrm{tp}(\vec{d}_{\leqslant i}/A)$, and we conclude by induction. □

Lemma 2.30. *Let $\vec{d}_1, \vec{d}_2$ be two tuples from M. Assume for all f in* $\mathrm{LC}(\mathbb{Q})$ *that* $\mathrm{tp}(f(\vec{d}_2)/A)$ *is weakly orthogonal to* $\mathrm{tp}(\vec{d}_1/A)$. *Then* $\mathrm{tp}(\vec{d}_1/A)$ *is weakly orthogonal to* $\mathrm{tp}(\vec{d}_2/A)$.

Proof. Let $\vec{d} \equiv_A \vec{d}_2$. It suffices to show that $\vec{d} \equiv_{A\vec{d}_1} \vec{d}_2$. By quantifier elimination, this holds if and only if, for every $f \in \mathrm{LC}(\mathbb{Q})$, we have $f(\vec{d}) \equiv_{A\vec{d}_1} f(\vec{d}_2)$, which is exactly the hypothesis of the statement. □

Corollary 2.31. *Let $d_1 \in M$ be a singleton, and let $\vec{d}_2 \in M$ be a tuple. Let V be the $\mathbb{Q}$-vector space* $\mathrm{dcl}(A\vec{d}_2)$, *and let G be some* $\mathrm{Aut}(M/A)$*-invariant convex subgroup of M. Suppose we have $V \subseteq (A + G) \not\ni d_1$. Then* $\mathrm{tp}(d_1/A)$ *and* $\mathrm{tp}(\vec{d}_2/A)$ *are weakly orthogonal.*

Proof. This follows immediately from Lemmas 1.32 and 2.30. □

Let us now prove (GL1):

Proposition 2.32. *Let $\vec{c}$ be a finite tuple from M that is* val^1_B*-separated, and let* $(\vec{c}_i)_i$ *be its* val^1_A*-blocks. Suppose we have* $(p_i(\vec{x}_i))_i$*, a weak* val^1_B*-block extension of* $\vec{c}$*. Then* $(p_i)_i$ *is weakly orthogonal.*

Recall from Definition 1.22 that types in a block extension are $\mathrm{Aut}(M/A)$-invariant. In particular, we have $\vec{c}_i \mathop{\smile\!\!\!\!\!\!\;|}^{\mathbf{cut}}_A B$ for each i.

Proof. Enumerate the $\vec{c}_i$ in increasing order of their values. Let N be some $|M|^+$-saturated, strongly $|M|^+$-homogeneous elementary extension of M, and let $\vec{d}_i \in N$ be a realization of p_i. By Lemmas 2.29 and 2.30, it suffices to show that for all i, for all $f \in \mathrm{LC}(\mathbb{Q})$, $\mathrm{tp}(f(\vec{c}_i)/M)$ is weakly orthogonal to $\mathrm{tp}(\vec{c}_{<i}/M)$. This is trivial if $f = 0$.

Suppose $f \neq 0$. By val^1_B-separatedness, we have $\mathrm{val}^1_B(f(\vec{c}_i)) = \mathrm{val}^1_B(\vec{c}_i)$. Let $H = H(f(\vec{c}_i)/B)$. As H is a B-$\vee$-definable group, $H(N)$ is $\mathrm{Aut}(M/N)$-invariant. As $\mathrm{tp}(f(\vec{d}_i)/M)$ is a global $\mathrm{Aut}(M/A)$-invariant extension of $\mathrm{tp}(f(\vec{c}_i)/M)$, we have $f(\vec{d}_i) \notin (M + H(N))$ by Lemma 2.28. Let $V = \mathrm{dcl}(M\vec{d}_{<i})$. By Corollary 2.31, it suffices to show that $V \subseteq (M + H(N))$. Let $v \in V$. Let $g \in \mathrm{LC}(\mathbb{Q})$ and $m \in M$ be such that $v = m + g(\vec{d}_{<i})$. Let h be the A-definable map witnessing Corollary 2.25 applied to $g(\vec{d}_{<i})$ (which is not in A by separatedness). As val^1_B is a B-valuation, and the values of the blocks before $\vec{c}_i$ are strictly smaller than that of $\vec{c}_i$, we have $G(g(\vec{c}_{<i})/B) < G(\vec{c}_i/B)$. Thus $G(g(\vec{c}_{<i})/B) < H$ by Proposition 2.6. By Remark 2.27,

$$g(\vec{d}_{<i}) - h \circ g(\vec{c}_{<i}) \in G(g(\vec{c}_{<i})/B) < H.$$

It follows that $v = m + h \circ g(\vec{c}_{<i}) + (g(\vec{d}_{<i}) - h \circ g(\vec{c}_{<i})) \in (M + H(N))$, concluding the proof. □

Let us now build val^2_A:

Lemma 2.33. *Let $d, d' \in M$ be such that $G(d/A) = G(d'/A)$, d is Archimedean over A and d' is ramified over A. Then $G(d + d'/A) = G(d/A)$ and $d + d'$ is Archimedean over A.*

Proof. Let $a \in \mathrm{ram}(d'/A)$, and $G = G(d/A) = G(d'/A)$. Then the coset $((d+a)+G)$ has no point in A and contains $d + d'$. In particular, $d + d' + G \subseteq \mathrm{ct}(d + d'/A)$. By the ultrametric inequality for the valuation val^1_A, $G(d+d'/A) \leqslant G$. By Lemma 2.17, $\mathrm{ct}(d + d'/A)$ is contained in $d + d' + G(d + d'/A)$. Therefore, we have

$$d + d' + G \subseteq \mathrm{ct}(d + d'/A) \subseteq ((d + d') + G(d + d'/A)) \subseteq ((d + d') + G).$$

As a result, $G(d+d'/A) = G$. This concludes the proof, as $((d+d')+G(d+d'/A))$ has no point in A. □

Corollary 2.34. *The following satisfies the conditions of Lemma 1.25: $x \leqslant y$ if and only if either $\mathrm{val}^1_A(x) < \mathrm{val}^1_A(y)$, or they have the same value and x is ramified over A, or they have the same value and y is Archimedean over A.*

Definition 2.35. We define val^2_A to be the valuation induced by the above preorder. It is clearly an A-valuation which refines val^1_A.

This new valuation splits the val^1_A-blocks in two, putting the A-ramified points below the others. In Example 1.33, we have $\mathrm{val}^2_A(c_1) < \mathrm{val}^2_A(c_3) < \mathrm{val}^2_A(c_2)$.

Let us now prove (GL2):

Proposition 2.36. *Let $\vec{c}$ be a finite tuple from M that is val^2_A-separated, and let $(\vec{c}_i)_i$ be its val^2_A-blocks. Then $(\mathrm{tp}(\vec{c}_i/A))_i$ is weakly orthogonal.*

Note that the weakly orthogonal types are types over A here, as opposed to the types in Proposition 2.32, which were global types. The respective restrictions over A of non-weakly orthogonal global types can very well be weakly orthogonal.

Proof. Just like in Proposition 2.32, enumerate the blocks in increasing order of their value by val^2_A. Choose i and $f \in \mathrm{LC}(\mathbb{Q})$ with $f \neq 0$. As in the proof of Proposition 2.32, it suffices to find G an $\mathrm{Aut}(M/A)$-invariant convex subgroup of M such that $f(\vec{c}_i) \notin (A+G)$, and for all $g \in \mathrm{LC}(\mathbb{Q})$, $g(\vec{c}_{<i}) \in (A+G)$.

- Suppose the coordinates of $\vec{c}_i$ are ramified over A. Then we choose $G = H(\vec{c}_i/A)$. By separatedness, $G = H(f(\vec{c}_i)/A)$. Now, for any $g \in \mathrm{LC}(\mathbb{Q})$, by Proposition 2.6 we have $G(g(\vec{c}_{<i})/A) < G$, and hence there exists some a in $A \setminus \mathrm{Stab}(g(\vec{c}_{<i})/A)$ such that $a \in G$. Let $a' \in A$ be such that $a' \in \,]g(\vec{c}_{<i}) - |a|, g(\vec{c}_{<i}) + |a|[$. Then $g(\vec{c}_{<i}) \in \,]a' - |a|, a' + |a|[\, \subseteq (A+G)$.

- Suppose the coordinates of $\vec{c}_i$ are Archimedean over A, and let $g \in \mathrm{LC}(\mathbb{Q})$. This time we let $G = G(\vec{c}_i/A)$. By definition of being Archimedean, $f(\vec{c}_i) \notin (A+G)$. Either $G(g(\vec{c}_{<i})/A) < G$, in which case we have $g(\vec{c}_{<i}) \in (A + H(\vec{c}_i/A)) \subseteq (A+G)$ just as in the above item, or $G(g(\vec{c}_{<i})/A) = G$, in which case $g(\vec{c}_{<i})$ is ramified over A as $\mathrm{val}^2_A(g(\vec{c}_{<i})) < \mathrm{val}^2_A(\vec{c}_i)$, and therefore $g(\vec{c}_{<i}) \in (A+G)$.

This concludes the proof. $\square$

2.3. ***Tensor product.*** We saw that the blocks of a block extension for the coarsest valuation are weakly orthogonal, and thus can be glued by just taking the union. This will no longer be the case for finer valuations: the tensor product of the blocks will no longer commute, and we will have to choose the order of the factors. For the third valuation, which is yet undefined, this choice will have to be taken with care, because some choices may give a global type which does not extend our type over B.

The following proposition shows us how to properly glue the elements of a val^2_B-block extension.

Proposition 2.37. *Let $\vec{c}$ be a finite tuple from M such that $\vec{c} \downarrow^{\mathbf{cut}}_A B$. Suppose $\vec{c}$ is a val^1_B-block (i.e., a tuple having only one block) that is val^2_B-separated, and let $\vec{c}_1$ (resp. $\vec{c}_2$) be the block of B-ramified (resp. Archimedean) elements of $\vec{c}$. Suppose (p_1, p_2) is a weak val^2_B-block extensions of $\vec{c}$. Then $\vec{c}$ satisfies the restriction to B of the tensor product of p_1, p_2 in any order.*

Note that, although the restrictions to B of $p_1 \otimes p_2$ and $p_2 \otimes p_1$ coincide, these two global types are distinct, i.e., (p_1, p_2) is not weakly orthogonal. This will be explained in the next section.

Proof. By Proposition 2.36, $(\mathrm{tp}(\vec{c}_i/B))_{i\in\{1,2\}}$ is weakly orthogonal, so $\mathrm{tp}(\vec{c}_1/B)$ is complete in $S(B + \mathbb{Q}\vec{c}_2)$, and it coincides in particular with $(p_1)_{|B+\mathbb{Q}\vec{c}_2}$. It follows that $\vec{c} \models (p_1 \otimes p_2)_{|B}$, and the same reasoning can be applied to $p_2 \otimes p_1$. □

Let us now build val^3_B:

Lemma 2.38. *Let $d_1, d_2 \in M$ be two points which are ramified over A and have the same val^2_A-value. Suppose $\delta(d_1/A) < \delta(d_2/A)$. Then $\mathrm{val}^2_A(d_1 + d_2) = \mathrm{val}^2_A(d_2)$ and $\delta(d_1 + d_2/A) = \delta(d_2/A)$ (as defined in Definition 2.15).*

Proof. Let $a_i \in \mathrm{ram}(d_i/A)$. Then, by Remark 1.21, $\Delta(d_1+d_2-a_1-a_2) = \delta(d_2/A)$. Therefore, d_1+d_2 is ramified over A, and $\delta(d_1+d_2/A) = \delta(d_2/A)$. By Remark 2.16, we have

$$\mathrm{ct}_>\big(\Delta(\mathrm{Stab}(d_2/A))/\Delta(A)\big) = \mathrm{ct}_>\big(\Delta(\mathrm{Stab}(d_1 + d_2/A))/\Delta(A)\big).$$

However, those stabilizers are convex subgroups of A. Thus the sets of their Archimedean classes are initial segments of $\Delta(A)$, so this equality of cuts implies

$$\Delta(\mathrm{Stab}(d_2/A)) = \Delta(\mathrm{Stab}(d_1 + d_2/A)).$$

It follows that $\mathrm{Stab}(d_2/A) = \mathrm{Stab}(d_1 + d_2/A)$. Thus $H(d_2/A) = H(d_1 + d_2/A)$, and thus $\mathrm{val}^1_A(d_2) = \mathrm{val}^1_A(d_1 + d_2)$. This concludes the proof, as d_2, $d_1 + d_2$ are both ramified over A. □

Corollary 2.39. *The following satisfies the conditions of Lemma 1.25:*

$x \leqslant y \iff$ *either $\mathrm{val}^2_A(x) < \mathrm{val}^2_A(y)$, or they have the same value and are both Archimedean over A, or they have the same value, are both ramified over A, and $\delta(x/A) \leqslant \delta(y/A)$.*

Definition 2.40. We define val^3_A as the valuation induced by the above preorder. It is clearly an A-valuation which refines val^2_A.

The valuation splits the ramified val^2_A-blocks in a way which depends on the Archimedean classes δ added by their elements. Note that, while for $i \in \{1, 2\}$, val^i_A factors through the quotient by A-elementary equivalence (these are model-theoretic objects), val^3_A is an *algebraic* object that is no longer refined by types.

We need to define additional notions in order to better understand val^3_B-block extensions. Contrary to val^2_B and val^1_B, a weak val^3_B-block extension of some tuple might not be strong. We need to find necessary and sufficient conditions for such a block extension to be strong.

Proposition 2.41. *Let $\vec{c}$ be a finite val^2_B-block of B-ramified points from M that is val^3_B-separated, and let $(\vec{c}_i)_i = ((\dots, c_{ij}, \dots))_i$ be the val^3_B-blocks of $\vec{c}$. Suppose the indices i are ordered such that $i < k$ if and only if $\delta(\vec{c}_i/B) < \delta(\vec{c}_k/B)$. Let $\vec{d} \in M^{|\vec{c}|}$ be another tuple, and let $\vec{d}_i$ be the subtuple of $\vec{d}$ corresponding to $\vec{c}_i$ in $\vec{c}$. Then $\vec{c} \equiv_B \vec{d}$ if and only if the following conditions hold:*

(1) *For each i, we have $\vec{d}_i \equiv_B \vec{c}_i$. In particular, each $\vec{d}_i$ is a val^3_B-block of B-ramified points that has the same val^2_B-value as $\vec{c}$.*

(2) *For every i and k, $i < k$ if and only if $\delta(\vec{d}_i/B) < \delta(\vec{d}_k/B)$.*

Proof. For each i and j, let $b_{ij} \in \mathrm{ram}(c_{ij}/B)$.

Suppose $\vec{c} \equiv_B \vec{d}$ (then (1) holds). Let i, k, j, l be indices. If $i < k$, then

$$\mathrm{tp}(\vec{c}/B) \models |x_{ij} - b_{ij}| < |x_{kl} - b_{kl}|/n \quad \text{for every } n.$$

Hence $\Delta(d_{ij} - b_{ij}) < \Delta(d_{kl} - b_{kl})$. Moreover, by Corollary 2.20, we have $\delta(d_{ij}/B) = \Delta(d_{ij} - b_{ij})$ (and similarly for kl), and (2) holds.

Conversely, suppose that (1) and (2) hold. Let f_i be in $\mathrm{LC}^{|\vec{c}_i|}(\mathbb{Q})$, and let $b' = \sum_i f_i(\vec{b}_i)$. We have to prove that $c' = \sum_i f_i(\vec{c}_i)$ and $d' = \sum_i f_i(\vec{d}_i)$ have the same cut over B. This is trivial if each f_i is zero, else let k be the maximal index for which f_k is nonzero. By val^3_B-separatedness, $\mathrm{val}^3_B(f_k(\vec{c}_k)) = \mathrm{val}^3_B(c_{kj})$ for any j. Moreover, $f_k((c_{kj} - b_{kj})_j)$ is a linear combination of elements of $G(\vec{c}_k/B)$, so it belongs to $G(\vec{c}_k/B)$, and $f_k(\vec{b}_k)$ is a ramifier of $f_k(\vec{c}_k)$. Likewise, we have $f_i(\vec{c}_i) - f_i(\vec{b}_i) \in G(\vec{c}_i/B)$ for any i. It follows by maximality of k that

$$\delta(\vec{c}_k/B) = \Delta(f_k(\vec{c}_k) - f_k(\vec{b}_k)) > \Delta(f_i(\vec{c}_i) - f_i(\vec{b}_i)) \tag{3}$$

for each $i \neq k$. By the conditions of the list, this also holds for $\vec{d}$. Choose an arbitrary index j. There exists $n \in \omega_{>0}$ and $\varepsilon \in \{\pm 1\}$ such that $f_k(\vec{c}_k) - f_k(\vec{b}_k)$ lies between $\varepsilon(c_{kj} - b_{kj})/n$ and $\varepsilon n(c_{kj} - b_{kj})$. By the first condition, this must hold for $\vec{c}$ replaced with $\vec{d}$ (with the same ε, n). Moreover, as (3) holds for $\vec{d}$, for all $i \neq k$ and $N > 0$, we have $|f_i(\vec{d}_i) - f_i(\vec{b}_i)| < |d_{kj} - b_{kj}|/N$. By choosing N large enough, and by summing everything, we get that $d' - b'$ lies between $\varepsilon(d_{kj} - b_{kj})/(N+1)$ and $\varepsilon(N+1)(d_{kj} - b_{kj})$, and we get the same property with $\vec{d}$ replaced by $\vec{c}$. Then

$$\begin{aligned}\mathrm{ct}(\Delta(c' - b')/\Delta(B)) &= \mathrm{ct}(\Delta(c_{kj} - b_{kj})/\Delta(B)) \\ &= \mathrm{ct}(\Delta(d_{kj} - b_{kj})/\Delta(B)) \\ &= \mathrm{ct}(\Delta(d' - b')/\Delta(B)).\end{aligned}$$

The second equality is a consequence of Corollary 2.20 applied to c_{kj}, d_{kj} (by the hypothesis $\vec{c}_k \equiv_B \vec{d}_k$). The other equalities follow from the inequalities of the previous paragraph. We also have

$$d' < b' \iff \varepsilon = -1 \iff c' < b'.$$

As a result, by Corollary 2.20, we have $c' \equiv_B d'$, concluding the proof. □

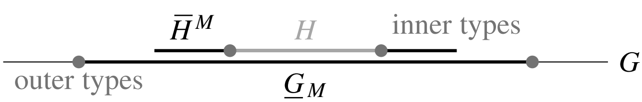
Remark 2.42. We recall some facts from previous subsections. Let $\vec{c}$ be a finite val^2_B-block of B-ramified points from M that is val^3_B-separated, and let $(\vec{c}_i)_i$ be its val^3_B-blocks. Let $(p_i)_i$ be a weak val^3_B-block extension of $\vec{c}$. By Corollary 2.25, and by separatedness, there exists G (resp. H) a unique A-type-definable (resp. A-$\vee$-definable) convex subgroup such that $H < G$, and exactly one of the following conditions holds for each i:

(1) For every nonzero $f \in \mathrm{LC}^{|\vec{c}_i|}(\mathbb{Q})$, there exists $b \in B$ such that we have
$$p_i(\vec{x}_i) \models f(\vec{x}_i) - b \in G \setminus \underline{G}_M.$$
(2) For every nonzero $f \in \mathrm{LC}^{|\vec{c}_i|}(\mathbb{Q})$, there exists $b \in B$ such that we have
$$p_i(\vec{x}_i) \models f(\vec{x}_i) - b \in \overline{H}^M \setminus H.$$

Of course we know exactly what G and H are, but this abstraction will simplify the definitions and proofs.

The following notion will mostly be used to understand strong val^3_B-block extensions, and to prove (GL3):

Definition 2.43. If the first condition of Remark 2.42 holds, then we say that p_i is *outer*, else p_i is *inner*.



Lemma 2.44. *Let $\vec{c}$ be a finite val^2_B-block of B-ramified points from M that is val^3_B-separated, and let $(\vec{c}_i)_i = ((\ldots, c_{ij}, \ldots))_i$ be its val^3_B-blocks. Suppose the indices i are ordered such that $i < k$ if and only if $\delta(\vec{c}_i/B) < \delta(\vec{c}_k/B)$. Let $(p_i)_i$ be a weak val^3_B-block extension of $\vec{c}$. Let I (resp. O) be the set of indices i such that p_i is inner (resp. outer). Then the following are equivalent*:

- *$(p_i)_i$ is a strong val^3_B-block extension of $\vec{c}$.*
- *For every $i \in I$, $o \in O$, we have $i < o$.*

Proof. The realizations of $\bigcup_i p_i$ satisfy the first condition of Proposition 2.41, so we are trying to characterize when some of them satisfy the second condition. Let $H = H(\vec{c}/A) < G(\vec{c}/A) = G$ be the convex subgroups witnessing Remark 2.42.

Suppose we have $i \in I$ and $o \in O$ such that $o < i$. Let $\vec{d} \models \bigcup_j p_j$. Then, as p_i is inner, we have $\delta(\vec{d}_i/M) \in \Delta(\overline{H}^M)$. Likewise, $\delta(\vec{d}_o/M) \notin \Delta(\underline{G}_M)$. Therefore, $\delta(\vec{d}_i/M) < \delta(\vec{d}_o/M)$, so $\vec{d} \not\equiv_B \vec{c}$, and we proved the top-to-bottom implication.

Conversely, suppose we have $i < o$ for every $i \in I$, $o \in O$. In order to satisfy the second condition of Proposition 2.41, it suffices to prove that the following partial type is consistent:

$$q(\vec{x}) = \bigcup_i p_i(\vec{x}_i) \cup \left\{ |x_{if(i)} - b_i| < \frac{1}{n}|x_{kf(k)} - b_k| : i < k, n > 0 \right\}$$

with $f(i)$ an arbitrary choice of a coordinate of $\vec{c}_i$ for each i, and with b_i a ramifier in $\mathrm{ram}(c_{if(i)}/B)$. Let us build a realization $\vec{\beta}$ of q.

Let $(\vec{\alpha}_i)_{i\in I} \models \bigcup_{i\in I} p_i$. We define by induction, for each $i \in I$, a tuple $\vec{\beta}_i \models p_i$ such that for $i < k$ in I, we have $\Delta(\beta_{if(i)} - b_i) < \Delta(\beta_{kf(k)} - b_k)$. Let N be an $|M|^+$-saturated, strongly $|M|^+$-homogeneous elementary extension of M containing $(\vec{\alpha}_i)_{i\in I}$. Let $k \in I$, and suppose we defined $\vec{\beta}_i \in N^{|\vec{c}_i|}$ for every $i < k$. By saturation, let $\gamma \in N$ such that $\gamma \in \overline{H}^M \setminus H$, and $\Delta(\gamma) > \Delta(\beta_{if(i)} - b_i)$ for all $i < k$, which means that $\gamma \in \overline{H}^M \setminus \underline{(\overline{H}^M)}_{M+\sum_{i<k}\mathbb{Q}\beta_{if(i)}}$. Such a γ exists, as we are only dealing with indices of inner types. Then, by Corollary 2.20, there exists $\varepsilon \in \{\pm 1\}$ such that $b_k + \varepsilon\gamma \equiv_M \alpha_{kf(k)}$. By strong homogeneity, there exists $\sigma \in \mathrm{Aut}(N/M)$ such that $\sigma(\alpha_{kf(k)}) = b_k + \varepsilon\gamma$. We set $\vec{\beta}_k = \sigma(\vec{\alpha}_k)$.

We can similarly define $(\vec{\beta}_o)_{o\in O}$. Then $(\vec{\beta}_i)_{i\in I\cup O}$ satisfies q, concluding the proof. ☐

Now we can prove (GL3):

Proposition 2.45. *Let $\vec{c}$ be a finite val^2_B-block of B-ramified points from M that is val^3_B-separated, and let $(\vec{c}_i)_i$ be its val^3_B-blocks. Let $(p_i)_i$ be a strong val^3_B-block extension of $\vec{c}$. Then some tensor product of the p_i is consistent with $\mathrm{tp}(\vec{c}/B)$.*

Proof. Suppose the indices i are ordered such that $i < k$ if and only if we have $\delta(\vec{c}_i/B) < \delta(\vec{c}_k/B)$. Define I, O, G, H just as in Lemma 2.44. Let

$$\vec{\alpha}_I \models p_I = \bigcup_{i\in I} p_i, \qquad \vec{\alpha}_O \models p_O = \bigcup_{o\in O} p_o.$$

As the conditions of Lemma 2.44 are satisfied, if we had $\vec{\alpha}_I \equiv_B (\vec{c}_i)_{i\in I}$ and $\vec{\alpha}_O \equiv_B (\vec{c}_o)_{o\in O}$, then by Proposition 2.41 we would clearly have $\vec{\alpha}_I\vec{\alpha}_O \equiv_B \vec{c}$. Moreover,

$$G(\vec{\alpha}_I/M) = \overline{H}^M < \underline{G}_M = H(\vec{\alpha}_O/M).$$

Thus, one clearly sees that $\vec{\alpha}_I$ and $\vec{\alpha}_O$ are val^2_M-blocks of distinct values, so their types over M are weakly orthogonal by Proposition 2.36. Therefore, if the respective types over M of $\vec{\alpha}_I$ and $\vec{\alpha}_O$ are tensor products of the p_i, then so is the type of $\vec{\alpha}_I\vec{\alpha}_O$. As a result, we can assume that either I or O is empty.

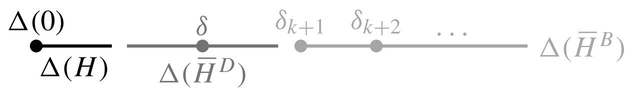
Suppose $O = \varnothing$. For $k \in I$, and $\vec{c}' = (\vec{c}_i)_{i>k}$, let us show that $\vec{c}_k \models p_{k|D}$, with D the $\mathbb{Q}$-vector subspace generated by $B\vec{c}'$. This implies that $\mathrm{tp}(\vec{c}/B)$ is consistent with the tensor product of the p_i in increasing order. Let f be nonzero in $\mathrm{LC}^{|\vec{c}_k|}(\mathbb{Q})$. By val^3_B-separatedness, we have $\delta(f(\vec{c}_k)/B) = \delta(\vec{c}_k/B)$ (call this Archimedean class δ). Let b be in $\mathrm{ram}(f(\vec{c}_k)/B)$. As $i \in I$, we have $p_k(\vec{x}_k) \models f(\vec{x}_k) - b \in \overline{H}^M \setminus H$. It is enough to show that $f(\vec{c}_k) - b \in \overline{H}^D$, for the cut over M corresponding to the pushforward of p_k by f would be included in $\mathrm{ct}(f(\vec{c}_k)/D)$, and we could conclude. For $i > k$, let $\delta_i = \delta(\vec{c}_i/B)$. By val^3_B-separatedness, $\Delta(D) = \Delta(B) \cup \{\delta_i \mid i > k\}$. By definition of I, we have $\delta < \delta_i$ for every $i > k$. Moreover, we have by Remark 2.16

$$\mathrm{ct}(\delta_i/\Delta(B)) = \mathrm{ct}_>(\Delta(H) \cap \Delta(B)/\Delta(B)).$$

As a result, $\Delta(H) \cap \Delta(D) = \Delta(H) \cap \Delta(B)$ and

$$\mathrm{ct}(\delta/\Delta(D)) = \mathrm{ct}_>(\Delta(H) \cap \Delta(D)/\Delta(D)).$$

Thus $\delta \in \Delta(\overline{H}^D)$, concluding the proof.



Now, if instead $I = \varnothing$, then a similar proof would show that the tensor product of the p_o in decreasing order would be consistent with $\mathrm{tp}(\vec{c}/B)$. □

Corollary 2.46. *The property (GL3) holds.*

Proof. Proposition 2.45 deals with the particular case of a val^2_B-block. Then, given the global invariant extensions of the types of each val^2_B-block from Proposition 2.45, some tensor product of those global types witnesses the statement, by Propositions 2.37 and 2.32. □

3. How to build a block extension

In the previous section, we defined the key valuations val^i_A, and we proved the properties (GL3), (GL2), (GL1). In this section, we give a proof of (BE3). The main technical goal is to show that $\mathop{\smile\!\!\!\!\!\mid}^{\mathbf{cut}}_A B$ coincides with $\mathop{\smile\!\!\!\!\!\mid}^{\mathbf{inv}}_A B$, when what we put on the left is a val^3_B-separated val^3_B-block. This will show the existence of a weak block extension, and then it will not be hard to show that such a block extension can be chosen strong. We work in this section with Assumptions 1.18, i.e., M is a monster model of DOAG, $C \geqslant A \leqslant B$ are $\mathbb{Q}$-vector subspaces of M, $C \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{cut}}_A B$, and $\vec{c} = (c_1, \dots, c_n)$ is a tuple from C.

3.1. ***The Archimedean case.*** In this subsection, we deal with the case where $\vec{c}$ is a val^3_B-separated block of B-Archimedean points from M.

Remark 3.1. Suppose $\vec{c}=(c_1,\dots,c_n)$ is a val^3_B-separated block of B-Archimedean points from M. As $\vec{c} \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{cut}}_A B$, the c_i must be Archimedean over A. Indeed, by contraposition, if c_i is ramified over A for some i, then we cannot have $\delta(c_i/A) \in \Delta(B)$, otherwise there would exist $b \in B$, $a \in A$ such that $\delta(c_i/A) = \Delta(c_i - a) = \Delta(b)$, and thus c_i would lie between $a + b/n$ and $a + nb$ for some $n \in \mathbb{Z}$, which would contradict $\vec{c} \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{cut}}_A B$. Now, as $\Delta(c_i - a) \notin \Delta(B)$ we have a contradiction with the hypothesis that all the c_i are Archimedean over B.

Assumptions 3.2. On top of Assumptions 1.18 (M a monster model of DOAG, $C \geqslant A \leqslant B$ $\mathbb{Q}$-vector subspaces of M, $C \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{cut}}_A B$, and $\vec{c} = (c_1,\dots,c_n)$ a tuple from C), we assume that $\vec{c}$ is a val^3_B-separated block of B-Archimedean points from M. By Remark 3.1 and Lemma 2.23, let us also fix $G = G(\vec{c}/A) = G(\vec{c}/B)$.

Lemma 3.3. *Let $p_i \in S(M)$ be a global $\mathrm{Aut}(M/A)$-invariant extension of $\mathrm{tp}(c_i/B)$. Suppose q is a complete global extension of $\mathrm{tp}(\vec{c}/B)$ which extends $\bigcup_i p_i$. If we have $q(\vec{x}) \models f(\vec{x}) - f(\vec{c}) \notin \underline{G}_M$ for all nonzero $f \in \mathrm{LC}^n(\mathbb{Q})$, then q is $\mathrm{Aut}(M/A)$-invariant.*

Proof. Suppose towards a contradiction q is not $\mathrm{Aut}(M/A)$-invariant. Then there exists a nonzero $f \in \mathrm{LC}^n(\mathbb{Q})$, and a cut X over M, such that $q(\vec{x}) \models f(\vec{x}) \in X$, and the global 1-type induced by X is not $\mathrm{Aut}(M/A)$-invariant. As $\vec{c} \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{cut}}_A B$, we have $f(\vec{c}) \mathop{\smile\!\!\!\!\!\!|}^{\mathbf{inv}}_A B$. By Lemma 2.23, we clearly see that the two global $\mathrm{Aut}(M/A)$-invariant extensions of $\mathrm{tp}(f(\vec{c})/B)$ correspond to the translates by $f(\vec{c})$ of the two connected components of $G \setminus \underline{G}_M$. We obtain a contradiction by showing that X coincides with one of those. It is enough to show that $q(\vec{x}) \models f(\vec{x}) - f(\vec{c}) \in G \setminus \underline{G}_M$. By hypothesis, we just need to have $q(\vec{x}) \models f(\vec{x}) - f(\vec{c}) \in G$, which follows from the fact that $\vec{c}$ is a val^3_B-separated block. Indeed, $\mathrm{val}^3_B(f(\vec{c})) = \mathrm{val}^3_B(\vec{c})$, and therefore $f(\vec{c})$ is Archimedean over B, and $G(f(\vec{c})/B) = G$. This implies by Proposition 2.18 that $\mathrm{tp}(f(\vec{c})/B) \models y - f(\vec{c}) \in G$, concluding the proof. □

It turns out that the space of global $\mathrm{Aut}(M/A)$-invariant extensions of $\mathrm{tp}(\vec{c}/B)$ in the Archimedean case has a very simple description:

Proposition 3.4. *Let N be some $|M|^+$-saturated and strongly $|M|^+$-homogeneous elementary extension of M. Let $H = (\underline{G}_M)(N)$ be the convex subgroup of N generated by $G(M)$, and let N' be the ordered abelian group ${}^{N}\!/_{H}$, which is a model of* DOAG *by saturation. Let $F_1 \subseteq S^{|\vec{c}|}(M)$ be the closed set of* $\mathrm{Aut}(M/A)$*-invariant extensions of* $\mathrm{tp}(\vec{c}/B)$*, and $F_2 \subseteq S^{|\vec{c}|}(\{0\})$ be the closed set of types of $\mathbb{Q}$-free tuples. Then the map*

$$g : \mathrm{tp}^N(\vec{\alpha}/M) \mapsto \mathrm{tp}^{N'}((\alpha_i - c_i + H)_i/\{0\})$$

is a well-defined homeomorphism $F_1 \to F_2$.

Proof. Let $f \in \mathrm{LC}^n(\mathbb{Q})$ be nonzero, and let $\vec{\alpha} \in N^{|\vec{c}|}$ realize a global $\mathrm{Aut}(M/A)$-invariant extension of $\mathrm{tp}(\vec{c}/B)$. Then, by separatedness, $f(\vec{c})$ is Archimedean over A, and $G(f(\vec{c})/A) = G$. Thus $f(\vec{c}) - f(\vec{\alpha}) \notin \underline{G}_M$ by Lemma 2.23, i.e., $f(\vec{c}) - f(\vec{\alpha}) \notin H$. No nontrivial linear combination of $(c_i - \alpha_i)_i$ belongs to H, and hence $(c_i - \alpha_i + H)_i$ is $\mathbb{Q}$-free.

Moreover, if $\vec{\beta} \equiv_M \vec{\alpha}$, then we have $f((\beta_i - c_i + H)_i) > 0$ in N' if and only if $f((\beta_i - c_i)_i) \in \bigcap_{m \in H}]m, +\infty[$ in N, if and only if the same holds for $\vec{\alpha}$. Therefore $(\beta_i - c_i + H)_i \equiv_{\{0\}} (\alpha_i - c_i + H)_i$ in N'. It follows that g is well-defined, and it is continuous by definition (and quantifier elimination).

Let $p_1, p_2 \in F_1$ be distinct. Then there exists two distinct cuts X_1, X_2 over M, and $f \in \mathrm{LC}^n(\mathbb{Q})$ nonzero, such that $p_i(\vec{x}) \models f(\vec{x}) \in X_i$. However, by Lemma 2.23, X_i must be the translate by $f(\vec{c})$ of a connected component of $G \setminus \underline{G}_M$. As a result, one of the X_i (say, X_1) is the leftmost connected component, and the other is the rightmost one. We have

$$p_1(\vec{x}) \models f(\vec{x}) - f(\vec{c}) < \underline{G}_M.$$

Thus $g(p_1) \models f(\vec{x}) < 0$. With the same reasoning, $g(p_2) \models f(\vec{x}) > 0$, and we conclude that $g(p_1) \neq g(p_2)$, proving that g is injective.

Now let us prove surjectivity. Let $\vec{\beta}'$ be a $\mathbb{Q}$-free tuple from N'. Let $\vec{\beta}$ be a preimage of $\vec{\beta}'$ by $\pi : N \to N'$. Let $d_1 \in N$ such that $\Delta(d_1) > \Delta(\beta_i)$ for every i. Let $d_2 \in N$ such that $\Delta(d_2) \in \Delta(G) \setminus \Delta(\underline{G}_M)$. Then $|d_1|$ and $|d_2|$ have the same cut over $G(M)$: $+\infty$, the cut of positive elements larger than all the elements of $G(M)$. By strong homogeneity, there is some $\sigma \in \mathrm{Aut}(N/G(M))$ such that $\sigma(|d_1|) = |d_2|$. Since σ fixes $G(M)$, $\sigma(H) = H$. Thus $\sigma' : \pi(x) \mapsto \pi(\sigma(x))$ is a well-defined automorphism of the ordered group N'. As $\vec{\beta}'$ is $\mathbb{Q}$-free, so is $\sigma'(\vec{\beta}')$. As $\sigma(\vec{\beta})$ is a preimage of $\sigma'(\vec{\beta}')$, none of its elements is in H. Moreover, $|\sigma(\beta_i)| \leqslant \sigma(|d_1|) = |d_2| \in G$ for every i. Thus $\sigma(\beta_i) \in G$. As $\mathrm{tp}^{N'}(\vec{\beta}'/\{0\}) = \mathrm{tp}^{N'}(\sigma'(\vec{\beta}')/\{0\})$, and we are looking for a preimage of this type by g, we may assume $\sigma = \mathrm{id}$, and thus, by $\mathbb{Q}$-freeness of $\vec{\beta}'$, $f(\vec{\beta}) \in G \setminus \underline{G}_M$ for every nonzero $f \in \mathrm{LC}^n(\mathbb{Q})$. Let $p = \mathrm{tp}((c_i + \beta_i)_i/M)$. As $\beta_i \in G$ for all i, p extends $\mathrm{tp}(\vec{c}/B)$, and it is $\mathrm{Aut}(M/A)$-invariant by Lemma 3.3. Thus $p \in F_1$, and it is a preimage of $\mathrm{tp}(\vec{\beta}'/\{0\})$ by g.

We showed that g is a continuous bijection, and we conclude by compactness and separation. □

Since F_2 is nonempty, we have in particular proved the following:

Corollary 3.5. *If $\vec{c}$ is a val^3_B-separated block of B-Archimedean points from M, then we have $\vec{c} \mathop{\smile\!\!\!\!\!\!|}\,^{\mathbf{inv}}_A B$.*

The next subsection gives tools to understand what F_2 looks like. For the most part, it is used to deal with the ramified blocks.

3.2. ***Archimedean groups.*** We need to define algebraic concepts in Archimedean groups in order to better understand the global invariant extensions of the types of ramified blocks. Let us start with well-known facts:

Proposition 3.6. *Let G be an ordered abelian group. Then G has at most two Archimedean classes* (*including* $\Delta(0)$) *if and only if it embeds into* $(\mathbb{R}, +, <)$. *In that case*, *for all* $x \in G_{>0}$, *for all* $\mu \in \mathbb{R}_{>0}$, *there exists a unique embedding of ordered groups* $\sigma : G \to \mathbb{R}$ *sending x to μ.*

The idea of the proof is to send each $y \in G$ to $\mu \cdot (\sup\{\lambda \in \mathbb{Q} \mid \lambda y < x\})^{-1}$.

Remark 3.7. In the special case $G = \mathbb{R}$, we see that an embedding of ordered groups $\sigma : \mathbb{R} \to \mathbb{R}$ is uniquely determined by the choice of $\sigma(1)$, and hence σ actually coincides with the automorphism $x \mapsto \sigma(1) \cdot x$. This establishes an easy description of the group of ordered group automorphisms of $\mathbb{R}$, which is naturally isomorphic to the multiplicative group $\mathbb{R}_{>0}$.

Definition 3.8. We define $\mathbb{P}^+_n$, the space of half-lines in dimension n (plus the origin) as the set of orbits of $\mathbb{R}^n$ under the automorphisms of the ordered group $\mathbb{R}$. We call $\mathbb{P}^+$ the canonical map $\mathbb{R}^n \to \mathbb{P}^+_n$.

Suppose G is an Archimedean ordered abelian group, and let $\vec{u}$ be a finite tuple from G. By the universal property of Proposition 3.6, we see that $\mathbb{P}^+(\sigma(\vec{u}))$ is always the same for any ordered group embedding $\sigma : G \to \mathbb{R}$. As a result, we can define $\mathbb{P}^+(\vec{u}) \in \mathbb{P}^+_{|\vec{u}|}$ as this unique class.

Remark 3.9. If $f \in \mathrm{LC}^n(\mathbb{Q})$, then f commutes with any automorphism. Thus one can easily show that $f(\mathbb{P}^+(\vec{u})) = \mathbb{P}^+(f(\vec{u})) \in \mathbb{P}^+_1$ for any $u \in \mathbb{R}^n$. However, $\mathbb{P}^+_1$ is not very complicated to describe: $f(\mathbb{P}^+(\vec{u}))$ is either $\mathbb{R}_{>0}$, $\mathbb{R}_{<0}$ or $\{0\}$.

Definition 3.10. Let $\vec{u} \in \mathbb{R}^n$. We say that $\mathbb{P}^+(\vec{u})$ is $\mathbb{Q}$-*free* if $\vec{u}$ is $\mathbb{Q}$-free. By Remark 3.9, this definition does not depend on the choice of the representative $\vec{u}$.

Proposition 3.11. *Let G be an ordered abelian group. Then for any subgroup H of G*, *and* $\Delta(0) \neq \delta \in \Delta(G)$, *the quotient*

$$H_\delta = H_{\leqslant\delta}/H_{<\delta}$$

as defined in Definition 1.30 is Archimedean.

In the literature, the $(H_\delta)_{\delta\in\Delta(H)}$ are called the "components" of H.

Definition 3.12. Let $\delta \in \Delta(M)$, and $\vec{u}$ a tuple from $M_{=\delta}$. By Proposition 3.11, we can define $\mathbb{P}^+(\vec{u}) = \mathbb{P}^+(\vec{u} + M_{<\delta})$ with respect to the Definition 3.8 in the Archimedean group $M_{\leqslant\delta}/M_{<\delta}$.

Suppose that $\vec{d} = (d_1, \ldots, d_n)$ is a val^3_B-block of B-ramified elements of M, and define $\delta = \delta(d_i/B) \in \Delta(M)$. Let $b_i \in \mathrm{ram}(d_i/B)$.

Then we can define $\mathbb{P}^+(\vec{d}/B) = \mathbb{P}^+((d_i - b_i)_i)$. As $H(d_i/B) \leqslant M_{<\delta}$, by Remark 2.12, this definition does not depend on the choice of the b_i.

Proposition 3.13. *Let $\vec{d} = (d_1, \dots, d_n)$ be a finite val_B^3-block of B-ramified points from M. Then $\vec{d}$ is val_B^3-separated if and only if $\mathbb{P}^+(\vec{d}/B)$ is $\mathbb{Q}$-free.*

Proof. Let $b_i \in \mathrm{ram}(d_i/B)$, and let $\delta = \Delta(\vec{d}/B)$. Then $\vec{d}$ is not val_B^3-separated if and only if $\mathrm{val}_B^3(f(\vec{d})) < \mathrm{val}_B^3(\vec{d})$ for some nonzero $f \in \mathrm{LC}^n(\mathbb{Q})$, if and only if there exists such an f with $\Delta(f(\vec{d}) - f(\vec{b})) < \delta$, if and only if we have f such that $\{0\} = f(\mathbb{P}^+((d_i - b_i)_i)) = f(\mathbb{P}^+(\vec{d}/B))$, if and only if $\mathbb{P}^+(\vec{d}/B)$ is not $\mathbb{Q}$-free. □

3.3. *The ramified case.*

Assumptions 3.14. In this subsection, on top of Assumptions 1.18 (M a monster model of DOAG, $C \geqslant A \leqslant B$ $\mathbb{Q}$-vector subspaces of M, $C \mathop{\smile\!\!\!\!\!\!\mid}_A^{\mathbf{cut}} B$, and $\vec{c} = (c_1, \dots, c_n)$ a tuple from C), we assume $\vec{c} = (c_i)_i$ is a val_B^3-separated val_B^3-block of B-ramified points. Fix $G = G(\vec{c}/B)$, $H = H(\vec{c}/B)$, $\delta = \delta(\vec{c}/B)$. For each i, let b_i be in $\mathrm{ram}(c_i/B)$, and write $\vec{b} = (b_i)_i$.

Recall that Corollary 2.20 characterizes the type of a ramified singleton in terms of cuts of Archimedean classes. We can now extend this characterization to the type of our val_B^3-separated ramified val_B^3-block:

Proposition 3.15. *Let $\vec{d} = (d_i)_i$ be a tuple from M. Then we have $\vec{d} \equiv_B \vec{c}$ if and only if the following conditions hold*:

- *$d_i \equiv_B c_i$ for every i.*
- *$\vec{d}$ is a val_B^3-block, i.e., $\delta(d_i/B) = \delta(d_j/B)$ for every i, j.*
- *$\mathbb{P}^+(\vec{d}/B) = \mathbb{P}^+(\vec{c}/B)$.*

Proof. Suppose the conditions of the list hold. Let $f \in \mathrm{LC}^n(\mathbb{Q})$ be nonzero. Then, as $\vec{c}$ is val_B^3-separated, $\mathbb{P}^+(\vec{c}/B)$ is $\mathbb{Q}$-free. Therefore $\Delta(f(\vec{c}) - f(\vec{b})) = \delta$. As a result, $\mathrm{ct}(f(\vec{c})/B)$ is the translate by $f(\vec{b})$ of one of the connected components of $G \setminus H$. The same holds for $\vec{d}$, but we need to make sure that $f(\vec{c})$ and $f(\vec{d})$ lie in the same connected component. Notice that $f(\vec{c})$ lies in the rightmost one if and only if $(f(\vec{c}) - f(\vec{b}) + M_{<\delta}) > 0$ in the ordered group $M_{\leqslant\delta}/M_{<\delta}$, if and only if $\mathbb{P}^+(f(\vec{c}) - f(\vec{b})) = \mathbb{R}_{>0}$, if and only if $\mathbb{R}_{>0} = f(\mathbb{P}^+((c_i - b_i)_i)) = f(\mathbb{P}^+(\vec{c}/B))$. Now, by the hypothesis $\mathbb{P}^+(\vec{d}/B) = \mathbb{P}^+(\vec{c}/B)$, those conditions are equivalent to $f(\vec{d})$ lying in the rightmost connected component, and we get the bottom-to-top direction.

Conversely, suppose now $\vec{d} \equiv_B \vec{c}$, in particular the first condition holds. As for the second, $\vec{d}$ must be a val_B^3-block; otherwise it would be witnessed in its type by formulas $(|x_i - b_i| > n|x_j - b_j|)_n$ for some $i \neq j$, which would contradict the hypothesis on $\vec{c}$. By strong homogeneity of M, let $\sigma \in \mathrm{Aut}(M/B)$ be such that $\sigma(\vec{c}) = \vec{d}$. The equivalence relation $\Delta(x) = \Delta(y)$ is an $\vee$-definable subset of $S^2(\varnothing)$, and therefore it is invariant by σ. As a result, the map

$$\sigma' : (x + M_{<\delta}) \mapsto (\sigma(x) + M_{<\delta(\vec{d}/B)})$$

is a well-defined automorphism of ordered groups from the quotient $M_{\leqslant\delta}/M_{<\delta}$ to $M_{\leqslant\delta(\vec{d}/B)}/M_{<\delta(\vec{d}/B)}$, which sends each $c_i - b_i$ to $d_i - b_i$. Let τ be an embedding of ordered groups $M_{\leqslant\delta(\vec{d}/B)}/M_{<\delta(\vec{d}/B)} \to \mathbb{R}$. Then

$$\mathbb{P}^+(\vec{c}/B) \ni \tau \circ \sigma'((c_i - b_i)_i) = \tau((d_i - b_i)_i) \in \mathbb{P}^+(\vec{d}/B).$$

Thus the last condition of the list holds, concluding the proof. □

Lemma 3.16. *Choose an arbitrary index i. Assume p is a global* $\mathrm{Aut}(M/A)$-*invariant extension of* $\mathrm{tp}(c_i/B)$. *Then $q = p \cup \mathrm{tp}(\vec{c}/B)$ is a complete type in $S(M)$.*

Note that q may not be $\mathrm{Aut}(M/A)$-invariant. We will see an example soon.

Proof. Proving that q is consistent is easy: choose d a realization of p in some strongly $|B|^+$-homogeneous elementary extension N of M, choose σ in $\mathrm{Aut}(N/B)$ such that $\sigma(c_i) = d$, then $\sigma(\vec{c})$ is a realization of q.

Let $\vec{\alpha} \models q$, and $\delta' = \Delta(\alpha_i - b_i)$. As $\alpha_i \models p$, we have

$$\delta' \in (\Delta(G) \setminus \Delta(\underline{G}_M)) \cup (\Delta(\overline{H}^M) \setminus \Delta(H)).$$

In particular, it does not belong to $\Delta(M)$. For every $j \neq i$, not only do we have $\Delta(\alpha_j - b_j) \notin \Delta(B)$ as $\alpha_j \equiv_B c_j$, but in fact we have $\Delta(\alpha_j - b_j) = \delta'$ by the second condition of Proposition 3.15. In particular, $\vec{\alpha}$ is a val^3_M-block, and we have

$$\mathbb{P}^+(\vec{\alpha}/M) = \mathbb{P}^+((\alpha_j - b_j)_j) = \mathbb{P}^+(\vec{\alpha}/B) = \mathbb{P}^+(\vec{c}/B).$$

The last equality follows from Proposition 3.15, and the others from the definitions. We established that, for all $\vec{\beta} \models q$, $\vec{\beta}$ is a val^3_M-block, and $\mathbb{P}^+(\vec{\beta}/M) = \mathbb{P}^+(\vec{\alpha}/M)$. By Proposition 3.15 (with $\vec{c}$, B replaced by $\vec{\alpha}$, M), in order to check that q is complete, it is enough to show that for every j, the pushforward q_j of q by the j-th projection is a complete type in $S(M)$.

Let $f \in \mathrm{LC}^N(\mathbb{Q})$ be the projection on the j-th coordinate.

- On one hand, $q_j(x_j) \models x_j > b_j$ if and only if $f(\mathbb{P}^+(\vec{c}/B)) > 0$.
- On the other hand, we have

$$q(\vec{x}) \models \mathrm{ct}(\Delta(x_j - b_j)/\Delta(M)) = \mathrm{ct}(\Delta(x_i - b_i)/\Delta(M)) = \mathrm{ct}(\delta'/\Delta(M)).$$

The first equality follows from the fact that the realizations of q are val^3_M-blocks, and the last equality follows from Corollary 2.20, as p is complete. In particular,

$$q_j(x_j) \models \mathrm{ct}(\Delta(x_j - b_j)/\Delta(M)) = \mathrm{ct}(\delta'/\Delta(M)).$$

Then we can apply Corollary 2.20 to show that q_j, and thus q, is complete. □

Proposition 3.17. *Let i (if it exists) be such that c_i is Archimedean over A. Then* $\mathrm{tp}(c_i/B)$ *has exactly one global* $\mathrm{Aut}(M/A)$-*invariant extension p, and $p \cup \mathrm{tp}(\vec{c}/B)$ is a complete,* $\mathrm{Aut}(M/A)$-*invariant global type in $S(M)$.*

Proof. Suppose for simplicity $c_i > b_i$ (else apply the proposition with c_i, b_i replaced by $-c_i, -b_i$). Recall that $\vec{c} \mathop{\downarrow}^{\mathbf{cut}}_A B$, and thus $\mathrm{tp}(c_i/B)$ must have at least one global $\mathrm{Aut}(M/A)$-invariant extension. By Lemma 2.23, $G(c_i/A) = G$, and the types $p_>, p_<$ over M corresponding respectively to the cuts

$$\mathrm{ct}_>(\mathrm{ct}(c_i/A)/M) = \mathrm{ct}_>(b_i + G(M)/M),$$
$$\mathrm{ct}_<(\mathrm{ct}(c_i/A)/M) = \mathrm{ct}_<(b_i + G(M)/M)$$

are clearly global $\mathrm{Aut}(M/A)$-invariant extensions of $\mathrm{tp}(c_i/A)$. These two types are the only ones implying that $x - b_i \in G \setminus \underline{G}_M$. As $c_i > b_i$, only $p_>$ extends $\mathrm{tp}(c_i/B)$. By Lemma 2.23, there are no other global $\mathrm{Aut}(M/A)$-invariant extensions of $\mathrm{tp}(c_i/B)$.

Let $q = p_> \cup \mathrm{tp}(\vec{c}/B)$, which is complete by Lemma 3.16. Let us show that q is $\mathrm{Aut}(M/A)$-invariant. As $\mathbb{P}^+(\vec{c}/B)$ is $\mathbb{Q}$-free, the realizations of q are val^3_M-separated by Proposition 3.13. As a result, for every nonzero $f \in \mathrm{LC}^N(\mathbb{Q})$,

$$q(\vec{x}) \models f(\vec{x}) - f(\vec{b}) \in G \setminus \underline{G}_M.$$

As $G = G(c_i/A)$ and $\underline{G}_M$ are $\mathrm{Aut}(M/A)$-invariant, we only have to show that $X = f(\vec{b}) + \underline{G}_M$ is $\mathrm{Aut}(M/A)$-invariant as a subset of M, because

$$q(\vec{x}) \models \mathrm{ct}(f(\vec{x})/M) \in \{\mathrm{ct}_>(X/M), \mathrm{ct}_<(X/M)\}.$$

However, as a subset of M, X coincides with $f(\vec{b}) + G(M)$. If $f(\vec{c})$ is Archimedean over A, then $X = \mathrm{ct}(f(\vec{c})/A)(M)$, which is invariant under $\mathrm{Aut}(M/A)$. Else, by definition, there exists $a \in A$ such that $f(\vec{c}) - a \in G(M)$. However, we also have $f(\vec{c}) - f(\vec{b}) \in G(M)$, and therefore $X = a + G(M)$, which is $\mathrm{Aut}(M/A)$-invariant. This concludes the proof. □

Note that $p_>$, and thus q, is outer (with respect to Definition 2.43).

Remark 3.18. If $j \neq i$, and c_j is ramified over A, then $\mathrm{tp}(c_j/B)$ might have two global $\mathrm{Aut}(M/A)$-invariant extensions, and the union of each of those extensions with $\mathrm{tp}(\vec{c}/B)$ would be a consistent, complete type in $S(M)$. However, one of the two will not be $\mathrm{Aut}(M/A)$-invariant, as it will not be consistent with $p_>$. In the above proof, we would encounter a problem where we would want to show that X is $\mathrm{Aut}(M/A)$-invariant, because we would have to deal with the case where $f(\vec{c})$ is Archimedean over A, and $X = f(\vec{b}) + H$.

This is exactly what happens in the following example:

Example 3.19. Let $A = \mathbb{Q}$, $B = \mathbb{Q} + \mathbb{Q}\sqrt{2}$, ε a positive infinitesimal element, and let $c_1 = \sqrt{2} + \varepsilon$ and $c_2 = \varepsilon \cdot \sqrt{2}$. Then $c_1 c_2 \mathop{\downarrow}^{\mathbf{cut}}_A B$, and $c_1 c_2$ is a val^3_B-separated block. Moreover, $\mathrm{tp}(c_1/B)$ has exactly one global $\mathrm{Aut}(M/A)$-invariant extension, and $\mathrm{tp}(c_2/B)$ has two, one of which is p, the type of a positive element that is infinitesimal with respect to M. Then $q = p \cup \mathrm{tp}(c_1c_2/B)$ is complete and consistent in $S(M)$, but not $\mathrm{Aut}(M/A)$-invariant. The reason is that

$$q(x, y) \models \sqrt{2} \leqslant x \leqslant \sqrt{2} + \varepsilon.$$

Proposition 3.20. *Suppose c_i is ramified over A for every i. Choose an index i. Let $H' = H(\vec{c}/A)$ and $G' = G(\vec{c}/A)$. If $H \neq H'$ or $G \neq G'$, then $\mathrm{tp}(c_i/B)$ has exactly one global $\mathrm{Aut}(M/A)$-invariant extension, else it has exactly two. Moreover, if p is such an extension, then $p \cup \mathrm{tp}(\vec{c}/B)$ is complete and $\mathrm{Aut}(M/A)$-invariant in $S(M)$.*

Proof. We can assume $b_j \in A$ for every j. Suppose $c_i > b_i$. As $\vec{c} \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{cut}}_A B$, recall that $H' \leqslant H \leqslant G \leqslant G'$. Let $p_<, p_>$ be the global 1-types corresponding to the respective cuts $\mathrm{ct}_<(\mathrm{ct}(c_i/A)/M) = \mathrm{ct}_>(b_i + H'/M)$, $\mathrm{ct}_>(\mathrm{ct}(c_i/A)/M) = \mathrm{ct}_>(b_i + \underline{G'}_M/M)$. We note that $p_<, p_>$ are the only global $\mathrm{Aut}(M/A)$-invariant extensions of $\mathrm{tp}(c_i/A)$. Moreover, $p_<$ (resp. $p_>$) is consistent with $\mathrm{tp}(c_i/B)$ if and only if $H = H'$ (resp. $G = G'$).

Now, let p be a global $\mathrm{Aut}(M/A)$-invariant extension of $\mathrm{tp}(c_i/B)$, and let $q = p \cup \mathrm{tp}(\vec{c}/B)$. Then q is complete by Lemma 3.16. Let us show that q is $\mathrm{Aut}(M/A)$-invariant.

Let $f \in \mathrm{LC}^n(\mathbb{Q})$ be nonzero. One just has to show that the cosets $f(\vec{b}) + H'(M)$ and $f(\vec{b}) + G'(M)$ are both $\mathrm{Aut}(M/A)$-invariant. However, $H'(M)$ and $G'(M)$ are both $\mathrm{Aut}(M/A)$-invariant, and the b_i were now chosen in A. This concludes the proof. $\square$

Note that $p_<$ (thus $p_< \cup \mathrm{tp}(\vec{c}/B)$) is inner, while $p_>$ (thus $p_> \cup \mathrm{tp}(\vec{c}/B)$) is outer.

Remark 3.21. The groups H' and G' have the same points in A. By quantifier elimination, it is not hard to show that G' does not admit an A-($\vee$/type)-definable proper convex subgroup that strictly contains H'. Recall that $H' \leqslant H \leqslant G \leqslant G'$ by cut-independence. As a result, if $G \neq G'$, then G cannot be A-type-definable. Similarly, if $H \neq H'$, then H is not A-$\vee$-definable.

Corollary 3.22. *If one of the c_i is Archimedean over A, or H is not A-$\vee$-definable, or G is not A-type-definable, then $\mathrm{tp}(\vec{c}/B)$ has exactly one global $\mathrm{Aut}(M/A)$-invariant extension. Else, it has exactly two. More precisely, if G is A-type-definable, then $\mathrm{tp}(\vec{c}/B)$ admits a global $\mathrm{Aut}(M/A)$-invariant extension that is outer, while if H is A-$\vee$-definable and none of the c_i is Archimedean over A, then $\mathrm{tp}(\vec{c}/B)$ has a global $\mathrm{Aut}(M/A)$-invariant extension that is inner.*

3.4. ***Gluing everything together.*** Recall that we adopt Assumptions 1.18: M is a monster model of DOAG, $C \geqslant A \leqslant B$ are $\mathbb{Q}$-vector subspaces of M, $C \mathop{\smile\!\!\!\!\!\mid}^{\mathbf{cut}}_A B$, and $\vec{c} = (c_1, \ldots, c_n)$ is a tuple from C.

We now have enough tools to prove (BE3). We can actually prove the following more precise statement:

Proposition 3.23. *Suppose $\vec{c}$ is a finite val^2_B-block of B-ramified points from M that is val^3_B-separated, and let $(\vec{c}_i)_{i\in E} = ((\dots, c_{ik}, \dots))_{i\in E}$ be its val^3_B-blocks. Let $G = G(\vec{c}/B)$ and $H = H(\vec{c}/B)$. Suppose the indices $i \in E$ are ordered such that $i < l$ if and only if we have $\delta(\vec{c}_i/B) < \delta(\vec{c}_l/B)$. Define I, $O \subseteq E$ as follows:*

- *If H is not A-$\vee$-definable, then $I = \varnothing$, $O = E$.*
- *If G is not A-type-definable, then $I = E$, $O = \varnothing$.*
- *Else, $I = \varnothing$, and O is the least final segment of E containing the set of indices $o \in E$ such that a coordinate of $\vec{c}_o$ is Archimedean over A.*

Either way, let $J = E \setminus (I \cup O)$. Then the following conditions hold:

(1) *For all $i \in I$, $\mathrm{tp}(\vec{c}_i/B)$ has exactly one $\mathrm{Aut}(M/A)$-invariant global extension p_i, and it is inner.*

(2) *For all $o \in O$, $\mathrm{tp}(\vec{c}_o/B)$ has at least one $\mathrm{Aut}(M/A)$-invariant global extension, of which exactly one is outer. Denote it by q_o.*

(3) *For all $j \in J$, $\mathrm{tp}(\vec{c}_j/B)$ has exactly two $\mathrm{Aut}(M/A)$-invariant global extensions: r_j, which is inner, and s_j, which is outer.*

(4) *The map $K \mapsto (p_i)_{i\in I}, (r_j)_{j\in K}, (s_j)_{j\in J\setminus K}, (q_o)_{o\in O}$ is a bijection from the set of initial segments of J to the set of strong val^3_B-block extensions of $\vec{c}$. In particular, $\vec{c}$ admits exactly $|J|+1$ strong val^3_B-block extensions.*

Proof. Note that I, O, J are pairwise-disjoint convex subsets of E which cover E. In fact, I is an initial segment of E, O is a final segment, and J is in-between. The first two conditions are easy consequences of Corollary 3.22. The third condition also easily follows from the fact that for all $j \in J$, none of the c_{jk} is Archimedean over A. It remains to prove the last condition.

Suppose we have $o \in O$ such that $\mathrm{tp}(\vec{c}_o/B)$ has a global $\mathrm{Aut}(M/A)$-invariant extension q' which is inner. Let us show that q' cannot be extended to a strong val^3_B-block extension of $\vec{c}$. Let $(q'_e)_{e\in E}$ be some strong block extension of $\vec{c}$. By definition of O, there exists o', k such that $o' \leqslant o$ and $c_{o'k}$ is Archimedean over A. As a result, $q_{o'}$ is the unique global $\mathrm{Aut}(M/A)$-invariant extension of $\mathrm{tp}(\vec{c}_{o'}/B)$ by Corollary 3.22. Therefore, $q'_{o'} = q_{o'}$. As $o' \leqslant o$, by Lemma 2.44, q'_o must be outer. Thus $q'_o \neq q'$, and we are done.

As a result, all the strong block extensions of $\vec{c}$ extend $(p_i)_{i\in I}, (q_o)_{o\in O}$. Then, each such extension $(q'_e)_e$ identifies with a subset of J: the set of all j such that $q'_j = r_j$. This concludes the proof, as Lemma 2.44 tells us that the valid choices are exactly the initial segments of J. □

Corollary 3.24. *If $\vec{c}$ is a finite val^3_B-separated tuple from M such that $\vec{c} \mathop{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}}^{\mathbf{cut}}_A B$, then $\vec{c}$ admits a strong val^3_B-block extension.*

Proof. By Proposition 2.36, it is enough to show that each val^2_B-block of $\vec{c}$ has a val^3_B-block extension. The Archimedean val^2_B-blocks of $\vec{c}$ coincide with its Archimedean val^3_B-blocks, by definition of val^3_B. We find a val^3_B-block extension of those blocks by Corollary 3.5 (the valid choices are built explicitly in Proposition 3.4), and we find a block extension of the B-ramified val^2_B-blocks by Proposition 3.23. □

This concludes the proof of property (BE3).

4. Normal forms

We built in the last two sections a nice framework to get a fine understanding of a particular class of tuples, the val^3_B-separated tuples. What remains for us to do in order to prove Theorem 1.16 is to prove (NF3), which allows us to always reduce to the case where we deal with such a nice tuple.

Assumptions 4.1. We work with Assumptions 1.18, i.e., M is a monster model of DOAG, $C \geqslant A \leqslant B$ are $\mathbb{Q}$-vector subspaces of M, $C \mathop{\perp\!\!\!\!\perp}^{\mathbf{cut}}_A B$, and $\vec{c} = (c_1, \dots, c_n)$ is a tuple from C. On top of that, by Remark 1.24, we may assume that $\vec{c}$ is a lift of a $\mathbb{Q}$-free tuple from $C/_A$.

Remark 4.2. Now, if $\vec{d}$ is another lift of a $\mathbb{Q}$-free tuple from $C/_A$, then $\vec{c}$ and $\vec{d}$ are A-interdefinable if and only if they have the same size n, and there exists some $f \in \mathrm{GL}_n(\mathbb{Q})$ such that the i-th component of $\vec{d}$ is an A-translate of the i-th component of $f(\vec{c})$ for each i.

We have to show that $\vec{c}$ is A-interdefinable with a val^3_B-separated tuple $\vec{d}$, which is cut-independent by Remark 1.24. In fact, $\vec{d}$ will also be val^3_A-separated. This could be useful, as we saw in Proposition 3.23 that we have to keep track of which points from $\vec{d}$ are Archimedean or ramified over A in order to have a good understanding of the global $\mathrm{Aut}(M/A)$-invariant extensions of its type. By Remark 4.2, $\vec{d}$ is the image of $\vec{c}$ by a composition of A-translations, and a map from $\mathrm{GL}_n(\mathbb{Q})$. In order to build $\vec{d}$, we start with a particular basis of $\mathrm{dcl}(A\vec{c})$ (remember that the definable closure of a set is the $\mathbb{Q}$-vector subspace generated by said set in DOAG), then we go through a process of several steps, where we apply A-translations and maps from $\mathrm{GL}_n(\mathbb{Q})$ to this basis. At the end of each step, our current tuple satisfies an additional property from a list of conditions, the conjunction of which is a sufficient condition to be a val^3_B- and val^3_A-separated tuple.

Remark 4.3. Recall that if $d \in M$ is a B-Archimedean singleton such that $d \mathop{\perp\!\!\!\!\perp}^{\mathbf{cut}}_A B$, then d is Archimedean over A (see Remark 3.1). In particular, $G(d/B) = G(d/A)$. Moreover, once again by Remark 3.1 and by cut-independence, if d is ramified over A, then $\delta(d/B) = \delta(d/A)$. In particular, we have

$$\Delta(C + A) \setminus \Delta(A) = \Delta(C + B) \setminus \Delta(B).$$

The following definition allows us to enumerate our tuples in a way that helps us to simultaneously consider their val_B^3-blocks and their val_A^3-blocks. The elements of our tuples are indexed depending on the subgroups $G(-/B)$, and the eventual Archimedean classes $\delta(-/B)$ that they bring. We use the prime $'$ to refer to elements that are Archimedean over A and ramified over B. We use the tilde $\tilde{}$ to refer to elements that are ramified over A (and thus over B if the tuple is cut-independent from B over A). By elimination, the other elements are those that are Archimedean over B and A.

Definition 4.4. Let I, J be totally ordered sets of indices, $(G_i)_{i\in I}$ an enumeration of the B-type definable convex subgroups, and $(\delta_j)_{j\in J}$ an enumeration of the Archimedean classes of $\Delta(C+B)\setminus\Delta(B)$. Let those enumerations be built such that $G_i \subsetneq G_k$ if and only if $i<k$, and $\delta_j<\delta_l$ if and only if $j<l$.

We say that a tuple

$$\vec{d}\vec{d}'\vec{\tilde{d}} = (d_{ik})_{ik}(d'_{i'j'k'})_{i'j'k'}(\tilde{d}_{\tilde{i}\tilde{j}\tilde{k}})_{\tilde{i}\tilde{j}\tilde{k}}$$

of elements of C is *under normal enumeration* (with respect to A and B) if the following conditions hold:

- All the d_{ik} are Archimedean over B.
- All the $d'_{i'j'k'}$ are Archimedean over A and ramified over B.
- All the $\tilde{d}_{\tilde{i}\tilde{j}\tilde{k}}$ are ramified over A.
- For all i, k, we have $G(d_{ik}/B)=G_i$.
- For all i', j', k', we have $G(d'_{i'j'k'}/B)=G_{i'}$ and $\delta(d'_{i'j'k'}/B)=\delta_{j'}$.
- For all $\tilde{i}$, $\tilde{j}$, $\tilde{k}$, we have $G(\tilde{d}_{\tilde{i}\tilde{j}\tilde{k}}/B)=G_{\tilde{i}}$, and $\delta(\tilde{d}_{\tilde{i}\tilde{j}\tilde{k}}/A)=\delta_{\tilde{j}}$.

In this tuple, the ramified val_B^3-blocks are the $((\dots,d'_{i'j'k'},\dots,\tilde{d}_{i'j'\tilde{k}},\dots))_{i'j'}$, its Archimedean val_B^3-blocks the $((\dots,d_{ik},\dots))_i$, its ramified val_A^3-blocks the $((\dots,\tilde{d}_{\tilde{i}\tilde{j}\tilde{k}},\dots))_{\tilde{i}\tilde{j}}$, and its Archimedean val_A^3-blocks the $((\dots,d_{ik},\dots,d'_{ij'k'},\dots))_i$.

Definition 4.5. Let $\vec{d}\vec{d}'\vec{\tilde{d}}$ be a tuple under normal enumeration.

- For each i, define $\vec{d}_i$ as the tuple $(\dots,d_{ik},\dots)$ whose elements vary with k.
- For each i', define $\vec{d}'_{i'}$ as the tuple $(\dots,d'_{i'j'k'},\dots)$ whose elements vary with $j'k'$.
- For each $i'j'$, define $\vec{d}'_{i'j'}$ as the tuple $(\dots,d'_{i'j'k'},\dots)$ whose elements vary with k'.
- For each $\tilde{i}\tilde{j}$, define $\vec{\tilde{d}}_{\tilde{i}\tilde{j}}$ as the tuple $(\dots,\tilde{d}_{\tilde{i}\tilde{j}\tilde{k}},\dots)$ whose elements vary with $\tilde{k}$.

One can note that the presence of the elements of $\vec{d}'$ is the reason why none of the valuations val_A^3, val_B^3 refines the other.

Let us now give the list of properties that we want to satisfy:

Definition 4.6. Let $\vec{d}\vec{d}'\vec{\vec{d}}$ be a tuple under normal enumeration with respect to A and B. We say that $\vec{d}\vec{d}'\vec{\vec{d}}$ is *under normal form* with respect to A and B if it is a lift of a $\mathbb{Q}$-free tuple from ${C}/{B}$, and the following conditions hold:

(P1) For all $\tilde{i}$ and $\tilde{j}$, $\vec{\vec{d}}_{\tilde{i}\tilde{j}}$ is a lift of a $\mathbb{Q}$-free tuple from ${M_{\leqslant\delta_{\tilde{j}}}}/{M_{<\delta_{\tilde{j}}}}$.

(P2) Nontrivial linear combinations of $\vec{d}\vec{d}'$ are Archimedean over A.

(P3) For all i, all the nontrivial linear combinations e of $\vec{d}_i\vec{d}'_i$ satisfy $G(e/A) = G_i$.

(P4) For all i, any nontrivial linear combination c of the tuple $\vec{d}_i$ is Archimedean over B (which implies $G(c/B) = G_i$ if the tuple is cut-independent, by (P3) and Remark 4.3).

(P5) For all i' and j', all the nontrivial linear combinations e' of the tuple $\vec{d}'_{i'j'}\vec{\vec{d}}_{i'j'}$ are ramified over B, and satisfy $\delta(e'/B) = \delta_{j'}$.

Remark 4.7. Condition (P1) implies that each ramified val^3_A-block is val^3_A-separated. The conjunction of (P2) and (P3) is equivalent to each Archimedean val^3_A-block being val^3_A-separated. Likewise, (P4) (resp. (P5)) is equivalent to each Archimedean (resp. ramified) val^3_B-block being val^3_B-separated. As a result, if we manage to prove that $\vec{c}$ is A-interdefinable with a tuple under normal form, then this tuple would be simultaneously val^3_A-separated and val^3_B-separated, and (NF3) would follow.

Remark 4.8. From the definition of the normal form, if $\vec{d}\vec{d}'\vec{\vec{d}}$ is under normal form with respect to A and B, then one can note that any subtuple of $\vec{d}\vec{d}'\vec{\vec{d}}$ is also under normal form with respect to A and B.

Definition 4.9. Define C to be the $\mathbb{Q}$-vector space generated by $A\vec{c}$. We define a *basis* (resp. *free tuple*) *of C over A* as a lift of a basis (resp. free tuple) of ${C}/{A}$.

Remark 4.10. Remember that $\dim\left({C}/{A}\right)$ is finite. It is easy to see that any concatenation of lifts of bases of ${C_{\leqslant\delta}}/{C_{<\delta}}$, with $\delta \in \Delta(C) \setminus \Delta(A)$, is a free tuple over A, and hence a finite tuple. In particular, $\Delta(C) \setminus \Delta(A)$ is finite.

Lemma 4.11. *For each $\tilde{j}$, let $\vec{\vec{d}}_{\tilde{j}}$ be a lift of a basis of ${C_{\leqslant\delta_{\tilde{j}}}}/{C_{<\delta_{\tilde{j}}}}$, and let $\vec{\vec{d}}$ be the concatenation of all of the $\vec{\vec{d}}_{\tilde{j}}$. Then for all e in $C \setminus \mathrm{dcl}(A\vec{\vec{d}})$, there exists $\tilde{e} \in \mathrm{dcl}(A\vec{\vec{d}})$ such that $e + \tilde{e}$ is Archimedean over A.*

Moreover, if e is ramified over A, then we can choose $\tilde{e}$ such that there exists $\delta \in \Delta(C) \setminus \Delta(A)$ for which $\Delta(e + \tilde{e}) < \delta \leqslant \Delta(e)$.

Proof. Let us build $\tilde{e}$ by induction. Let $\tilde{e}_0 = 0 \in \mathrm{dcl}(A\tilde{d})$. If $e+\tilde{e}_n$ is not Archimedean over A, then let $a \in \mathrm{ram}(e + \tilde{e}_n/A)$, and let $\tilde{j}_n$ be such that $\delta_{\tilde{j}_n} = \delta(e + \tilde{e}_n/A)$. By definition of $\vec{\vec{d}}$, there exists v a linear combination of $\vec{\vec{d}}_{\tilde{j}_n}$ for which we have $\Delta(e + \tilde{e}_n - a - v) < \delta_{\tilde{j}_n}$, so let us set $\tilde{e}_{n+1} = \tilde{e}_n - a - v$.

By finiteness of $\Delta(C) \setminus \Delta(A)$, the sequence of the $\tilde{j}_n$, and hence that of the $\tilde{e}_n$, must eventually stop. The ultimate value $\tilde{e}$ of the $\tilde{e}_n$ witnesses the first part of the statement. If e is ramified over A, then $\tilde{e} \neq 0 = \tilde{e}_0$, so $\tilde{j}_0$ exists and witnesses the second part of the statement. □

Lemma 4.12. *For each $\tilde{j}$, let $\vec{\tilde{d}}_{\tilde{j}}$ be a lift of a basis of $C_{\leqslant\delta_{\tilde{j}}}/C_{<\delta_{\tilde{j}}}$, and let $\vec{\tilde{d}}$ be the concatenation of all of the $\vec{\tilde{d}}_{\tilde{j}}$. Let $(\vec{v}_n)_n$ be a sequence of tuples such that $\vec{\tilde{d}}\vec{v}_n$ is a basis of C over A for each n. Suppose that, for all n, there exists v'_1 a term from $\vec{v}_n$, v'_2 a term from $\vec{v}_{n+1}$, and $\delta \in \Delta(C) \setminus \Delta(A)$, such that $\Delta(v'_2) < \delta \leqslant \Delta(v'_1)$, and $\vec{v}_{n+1}$ is obtained from $\vec{v}_n$ by replacing v'_1 by v'_2.*

Then the sequence $(\vec{v}_n)_n$ must eventually stop.

Proof. Let $N = |\Delta(C) \setminus \Delta(A)| < \omega$. We may assume that the enumeration $(\delta_j)_j$ is indexed by $0, \dots, N-1$. Let $\mathcal{I}$ be the set of intervals

$$\big\{]-\infty, \delta_0[, [\delta_0, \delta_1[, [\delta_1, \delta_2[, \dots, [\delta_{N-2}, \delta_{N-1}[, [\delta_{N-1}, +\infty[\big\}.$$

Define on $\mathcal{I}$ the only total order making the canonical projection $f : \Delta(C) \to \mathcal{I}$ an order-preserving map. With respect to that order, let g be the unique order-isomorphism $\mathcal{I} \to \{0, 1, \dots, N\}$. For each j between 0 and N, let $F(n, j)$ be the number of terms v from $\vec{v}_n$ such that $g(f(\Delta(v))) = j$. Finally, let ρ be the function

$$n \mapsto F(n, N) \times \omega^N + F(n, N-1) \times \omega^{N-1} + \cdots + F(n, 1) \times \omega + F(n, 0).$$

Then ρ has its values in the well-ordered set ω^{N+1}. By hypothesis, for each n, there exists $m < m'$ such that $F(n, m') = F(n+1, m') + 1$, $F(n, m) + 1 = F(n+1, m)$, and $F(n, j) = F(n+1, j)$ for every $j \neq m, m'$. As a result, ρ is strictly decreasing, so the well-ordering forces the sequence $(\vec{v}_n)_n$ to stop eventually. □

Lemma 4.13. *For each $\tilde{j}$, let $\vec{u}_{\tilde{j}}$ be a lift of a basis of $C_{\leqslant\delta_{\tilde{j}}}/C_{<\delta_{\tilde{j}}}$, and let $\vec{u}$ be the concatenation of all the $\vec{u}_{\tilde{j}}$. Then there exists $\vec{d}\vec{d}'\vec{\tilde{d}}$ a basis of C over A, under normal enumeration with respect to A and B, for which (P2) holds, and $\vec{\tilde{d}} = \vec{u}$ (so (P1) holds as well).*

Proof. Let $\vec{w}_0$ be some basis of C over A containing $\vec{u}$. Note that $\vec{w}_n = \vec{u}\vec{v}_n$ does not witness the statement if and only if there exists a nontrivial linear combination v' of $\vec{v}_n$ that is ramified over A. In that case, let v be some term of $\vec{v}_n$ that appears in v', chosen such that $\Delta(v)$ is maximal. By valuation inequality, we must have $\Delta(v) \geqslant \Delta(v')$. Apply Lemma 4.11 to find $u' \in \mathrm{dcl}(A\vec{u})$ and $\delta \in \Delta(C) \setminus \Delta(A)$ for which $\Delta(v' + u') < \delta \leqslant \Delta(v') \leqslant \Delta(v)$. Replace v by $v' + u'$ in $\vec{w}_n$, and let $\vec{w}_{n+1}$ be the new tuple obtained. Now, $\vec{w}_{n+1}$ is also a basis of C over A containing $\vec{u}$.

The sequence $(\vec{v}_n)_n$ witnesses the hypothesis of Lemma 4.12, so it must eventually stop, and its last element witnesses the statement. □

Remark 4.14. If $\vec{w} = \vec{u}\vec{v}$ witnesses Lemma 4.13, and M is an invertible matrix of size $|\vec{v}|$ with coefficients in $\mathbb{Q}$, then $\vec{u}(M\vec{v})$ also witnesses Lemma 4.13.

Lemma 4.15. *Let $\vec{d}\vec{d}'\vec{\vec{d}} = \vec{v}\vec{\vec{d}}$ be a free tuple over A that witnesses (P1) and (P2), and $N = |\vec{v}|$. Then there exists $M \in \mathrm{GL}_N(\mathbb{Q})$ such that $(M\vec{v})\vec{\vec{d}}$ witnesses (P3) (and thus (P1), (P2) by Remark 4.14).*

Proof. This is by induction over N. Note that (P3) trivially holds if $N = 0, 1$.

For every i, let $\vec{v}_i$ be the val^1_A-block of $\vec{v}$ of value G_i. Let i be maximal such that $\vec{v}_i$ is not val^3_A-separated. As long as $\vec{v}_i$ is not val^3_A-separated, there must exist e a nontrivial linear combination of $\vec{v}_i$ such that $G(e/A) < G_i$, then replace some term of $\vec{v}_i$ that appears in e by e (making that replacement corresponds to replacing $\vec{v}$ by $M\vec{v}$ for some $M \in \mathrm{GL}_N(\mathbb{Q})$). As we have $G(e/A) < G_i$, these replacements always decrease the size of $\vec{v}_i$, and leave $\vec{v}_{>i}$ untouched as the value of e is smaller, so the val^1_A-blocks of value larger than G_i remain separated. Thus we keep the maximality of i. As a result, these replacements eventually stop, and they stop before we removed all the terms from $\vec{v}_i$, because if there is only one term left in $\vec{v}_i$, then $\vec{v}_i$ is clearly separated.

By induction hypothesis, we can replace $\vec{v}_{<i}$ by $M\vec{v}_{<i}$ such that the tuple $(M\vec{v}_{<i})\vec{\vec{d}}$ witnesses (P3). It is now clear that $(M\vec{v}_{<i})\vec{v}_{\geqslant i}\vec{\vec{d}}$ witnesses (P3). The only changes we made to the original tuple $\vec{v}$ are operations from $\mathrm{GL}_N(\mathbb{Q})$, concluding the proof. □

Remark 4.16. Let $\vec{w} = \vec{d}\vec{d}'\vec{\vec{d}} = \vec{v}\vec{\vec{d}}$ be a free tuple over A that witnesses (P1), (P2) and (P3). Let $\vec{v}_i$ be the val^1_A-block of $\vec{v}$ of value G_i. If we replace $\vec{v}_i$ by $M\vec{v}_i$ in $\vec{w}$ for some invertible matrix M, then the new tuple $\vec{w}'$ still witnesses (P1), (P2) and (P3).

Lemma 4.17. *There exists a basis of C over A witnessing (P1)–(P4).*

Proof. Let $\vec{w} = \vec{d}\vec{d}'\vec{\vec{d}}$ be a basis of C over A that witnesses (P1), (P2) and (P3). For each i, as long as there is a nontrivial linear combination e of $\vec{d}_i$ that is not Archimedean over B, replace some term of $\vec{d}_i$ that appears in e by e. Each of these replacements removes a term from $\vec{d}_i$ and adds a new term to $\vec{d}'_i$. This eventually stops as the number of terms of $\vec{d}_i$ is finite and strictly decreases at each step.

By Remark 4.16, the new tuple $\vec{w}'$ that we obtain after all these replacements still witnesses (P1), (P2) and (P3). We perform these replacements for each i, and the new tuple obtained clearly witnesses (P4). □

Remark 4.18. Let $\vec{w} = \vec{d}\vec{d}'\vec{\vec{d}}$ be a free tuple over A that witnesses (P1)–(P4), and i' some index. Then, if we replace $\vec{d}'_{i'}$ by $M\vec{d}'_{i'}$ in $\vec{w}$ for some invertible matrix M, it is clear that the new tuple still witnesses (P1)–(P4).

Lemma 4.19. *Let $\vec{w} = \vec{d}\vec{d}'\vec{\vec{d}}$ be a basis of C over A that witnesses (P1)–(P4). Let $d'_{i'j'k'}$ be a term from $\vec{d}'$, and $\tilde{e}$ be a linear combination of $\vec{\vec{d}}_{i'}$. If we replace $d'_{i'j'k'}$ by $e = d'_{i'j'k'} + \tilde{e}$ in w, then the new tuple obtained is still a basis of C over A that witnesses (P1)–(P4).*

Proof. First of all, as $\mathrm{val}^2_A(d'_{i'j'k'}) > \mathrm{val}^2_A(\tilde{e})$, e is Archimedean over A, and $G(e/A) = G_{i'}$. As $\mathrm{val}^2_B(e) \leqslant \max(\mathrm{val}^2_B(d'_{i'j'k'}), \mathrm{val}^2_B(\tilde{e}))$, e cannot be Archimedean over B. This implies that the A-ramified and B-Archimedean blocks of our new tuple are left untouched, and hence we keep the properties (P1), (P4). Let $\vec{u}'$ be the new block of A-Archimedean and B-ramified points in the normal enumeration. Then e lies in the block $\vec{u}'_{i'}$. In order to show that (P2) and (P3) hold in the new tuple $\vec{d}\vec{u}'\vec{\vec{d}}$, we just have to show that the val^3_A-block $\vec{d}_{i'}\vec{u}'_{i'}$ is val^3_A-separated. Indeed, it would follow that the whole $\vec{d}\vec{u}'$ is val^3_A-separated, which is clearly equivalent to (P2) and (P3) being true in $\vec{d}\vec{u}'\vec{\vec{d}}$.

Any nontrivial linear combination of $\vec{d}_{i'}\vec{u}'_{i'}$ can be written $u + \lambda \cdot \tilde{e}$, with u a nontrivial linear combination of $\vec{d}_{i'}\vec{d}'_{i'}$, and $\lambda \in \mathbb{Q}$. As (P2) and (P3) hold in $\vec{d}\vec{d}'\vec{\vec{d}}$, we have $G(u/A) = G_{i'}$, and u is Archimedean over A. Then, we have $\mathrm{val}^3_A(u) > \mathrm{val}^3_A(\tilde{e})$, and hence $\mathrm{val}^3_A(u + \lambda \cdot \tilde{e}) = \mathrm{val}^3_A(u)$. This value equals that of the corresponding block, $\mathrm{val}^3_A(\vec{d}_{i'}\vec{d}'_{i'})$, as (P2) and (P3) hold in $\vec{d}\vec{d}'\vec{\vec{d}}$. Thus it equals $\mathrm{val}^3_A(\vec{d}_{i'}\vec{u}'_{i'})$, concluding the proof. □

Lemma 4.20. *Let $\vec{w} = \vec{d}\vec{d}'\vec{\vec{d}}$ be a basis of C over A that witnesses (P1)–(P4), and not (P5). Then there exists i', j', and two $\mathbb{Q}$-linear maps $f, g \in \mathrm{LC}(\mathbb{Q})$, such that $v = f(\vec{d}'_{i'j'}) + g(\vec{\vec{d}}_{i'j'})$ is ramified over B, and $\delta(v/B) < \delta_{j'}$.*

Moreover, if some $d'_{i'j'l'}$ appears in v, then replacing it by v in $\vec{w}$ gives a new tuple $\vec{w}'$ which is still a basis of C over A that witnesses (P1)–(P4).

Proof. By definition of (P5), there exists i', j', and $\mathbb{Q}$-linear maps $f, g \in \mathrm{LC}(\mathbb{Q})$ so that either $v = f(\vec{d}'_{i'j'}) + g(\vec{\vec{d}}_{i'j'})$ is not ramified over B, or $\delta(v/B) \neq \delta_{j'}$.

Note that f must be nonzero, otherwise the above hypothesis fails by (P1). By (P1), (P2), (P3), $\vec{d}\vec{d}'\vec{\vec{d}}$ is val^3_A-separated. As f is nonzero, $\mathrm{val}^3_A(v) = \mathrm{val}^3_A(\vec{d}'_{i'})$, and thus v is Archimedean over A, and $G(v/A) = G_{i'} = G(v/B)$. For all k', $\tilde{k}$, we have $d'_{i'j'k'}, \tilde{d}_{i'j'\tilde{k}} \in B + G_{i'}$ (by the definition of being ramified over B), so $v \in B + G_{i'} = B + G(v/B)$. Therefore v must be ramified over B, and by hypothesis we have $\delta(v/B) \neq \delta_{j'}$. By valuation inequality (for val^3_B), we must have $\delta(v/B) < \delta_{j'}$, and we get the first part of the statement.

Suppose that some $d'_{i'j'l'}$ appears in v. Then replacing $d'_{i'j'l'}$ by $v' = f(\vec{d}'_{i'j'})$ is just a replacement of $\vec{d}'_{i'}$ by $M\vec{d}'_{i'}$ for some invertible matrix M. By Remark 4.18, the new tuple $\vec{w}''$ obtained by this replacement is still a basis of C over A that witnesses (P1)–(P4). By (P2) and (P3), v' is Archimedean over A, and $G(v'/A) = G_{i'}$. Moreover, as all the coordinates of $\vec{d}'_{i'j'}$ are in $B + G_{i'}$, v' must be ramified over B. Then, $\vec{w}'$ is obtained by replacing v' by v in (a normal enumeration of) $\vec{w}''$. This is exactly the operation described in Lemma 4.19, concluding the proof. □

Theorem 4.21. *There exists $\vec{w} = \vec{d}\vec{d}'\vec{\vec{d}}$ a basis of C over A that is under normal form with respect to A and B.*

Proof. Let $\vec{w}_0 = \vec{d}\vec{d}'^0\vec{\tilde{d}}$ be a basis of C over A that witnesses (P1)–(P4).

Suppose $\vec{w}_n = \vec{d}\vec{d}'^n\vec{\tilde{d}}$ does not witness (P5). Then the hypothesis of Lemma 4.20 holds in $\vec{w}_n$, so we do the replacement from this lemma. Let $\vec{w}_{n+1}$ be the new tuple obtained.

To prove the theorem, we have to prove that the sequence $(\vec{w}_n)_n$ must stop.

To do this, we once again build a strictly decreasing map with values in ω^ω. Define $F(n, j)$ as the number of terms c from $\vec{d}'^n$ for which $\delta(c/B) = \delta_j$. Let $N = |\Delta(C) \setminus \Delta(A)|$. We may assume that the enumeration $(\delta_j)_j$ is indexed by $0, \dots, N-1$. We define g as the map

$$n \mapsto F(n, N-1) \times \omega^{N-1} + F(n, N-2) \times \omega^{N-2} + \cdots + F(n, 1) \times \omega + F(n, 0).$$

For each n, the tuple $\vec{w}_{n+1}$ is obtained from $\vec{w}_n$ by replacing some term v from $\vec{d}'^n$ by a point v' from $\vec{d}'^{n+1}$ for which $\delta(v'/B) < \delta(v/B)$, so g is strictly decreasing. This concludes the proof. $\square$

With that, we proved Theorem 1.16, the main technical theorem of this paper stating that $\mathop{\perp\!\!\!\!\perp}^{\mathbf{cut}} = \mathop{\perp\!\!\!\!\perp}^{\mathbf{inv}}$ in DOAG. The reasoning is explained in Section 1.2, and the different steps of the proof are carried out in Proposition 2.32, Proposition 2.36, Proposition 2.45, Corollary 3.24 and Theorem 4.21.

5. Regular ordered abelian groups

In this section, we define regular groups as first-order structures in the Presburger language, and we state some key properties that they satisfy.

5.1. *Quantifier-free types in the Presburger language.*

Definition 5.1. The *Presburger language* is defined as

$$\mathcal{L}_P = \{+, -, <, 0, \mathbb{1}, (\mathfrak{d}_{l^n})_{l \text{ prime}, n>0}\}.$$

Given an ordered abelian group G, we see G as an $\mathcal{L}_P$-structure by setting $\mathfrak{d}_{l^n}(x)$ as the predicate $\exists y,\ l^n y = x$, and setting $\mathbb{1} = \min(]0, +\infty[)$ if G is discrete, else we interpret $\mathbb{1}$ as the only $\varnothing$-definable choice, $\mathbb{1} = 0$ (this is the standard interpretation of the language). A *special* subgroup of G is a pure subgroup (that is a relatively divisible subgroup) $A \leqslant G$ containing $\mathbb{1}$. Special subgroups will correspond to definably closed sets in the groups we are interested in (see Corollary 5.11).

Define $S^m_{\mathrm{qf}}(A)$ to be the Stone space of quantifier-free types over A in m variables. Given an m-tuple $\vec{c}$, we write $\mathrm{tp}_l(\vec{c}/A)$ as the partial type generated by the set of quantifier-free formulas satisfied by $\vec{c}$, with parameters in A, which only involve the predicates $(\mathfrak{d}_{l^n})_{n>0}$ (equality does not appear in these formulas either). We write $S^m_l(A)$ for the Stone space of all such partial types, and we call its elements the *l-types*. We define similarly $\mathrm{tp}_<(\vec{c}/A)$, $S^m_<(A)$ for the quantifier-free formulas which only involve the predicates $=$, $<$.

For ease of notation, define the set of indices J as the union of the set of primes with $\{<\}$. For every $j \in J$, we have natural restriction maps $S^m_{\mathrm{qf}}(A) \to S^m_j(A)$, and they are all surjective. We write $(\pi_j)_j$ for those maps.

Assumptions 5.2. We fix an ordered abelian group M, and a special subgroup $A \leqslant M$, such that M is $|A|^+$-saturated and strongly $|A|^+$-homogeneous.

Lemma 5.3 (standard variant of the Chinese remainder theorem). *Let L be a finite set of primes, and $N > 0$. Then for all $(a_l)_l \in M^L$, there exists $b \in M$ such that $M \models \mathfrak{d}_{l^N}(a_l - b)$ for all $l \in L$.*

Corollary 5.4. *The natural map $S^m_{\mathrm{qf}}(A) \to \prod_{l \text{ prime}} S^m_l(A)$ is surjective.*

Proof. For each l, let $\vec{a_l} = (\dots, a_{il}, \dots) \in M^m$. By Lemma 5.3 and compactness, for each i, there must exist $b_i \in M$ such that $\mathfrak{d}_{l^N}(b_i - a_{il})$ for all l, N. In particular, $\mathrm{tp}_l(\vec{b}/A) = \mathrm{tp}_l(\vec{a_l}/A)$ for all l, which concludes the proof. $\square$

Remark 5.5. The map $\pi : S^m_{\mathrm{qf}}(A) \to \prod_{j\in J} S^m_j(A)$ is obviously injective. Note that $S^m_<(A)$ is not a factor of the product of Lemma 5.3, so π may not be surjective.

5.2. *Basic properties of regular groups.* The class of regular ordered abelian groups (ROAG) has several equivalent definitions. The ones that are necessary for us are quantifier elimination in $\mathcal{L}_P$, and some compatibility conditions between the $<$-types and the l-types. The definitions involving Archimedean groups and definable convex subgroups are relevant, because they give a motivation as to why we are interested in ROAG.

The equivalence between these different definitions is folklore. In this subsection, we prove the easy implications between these equivalent definitions, and we give references for the harder implications.

Definition 5.6. Let G be an ordered abelian group. We say that G is *regular* if, for every positive integer n, every interval of G that contains at least n elements intersects nG.

For any ordered abelian group G, we recall that $\mathrm{div}(G)$ refers to the divisible closure of G, which we see as an ordered group to which the order on G naturally extends.

Lemma 5.7. *Let $G \models$ ROAG, and let A be a special subgroup of G. Let F be the closed subspace of $S^m_{\mathrm{qf}}(A)$ of types of m-tuples that are $\mathbb{Q}$-free over A (i.e., lifts in G^m of $\mathbb{Q}$-free tuples from $\mathrm{div}(G)/\mathrm{div}(A)$). Then the map $F \to \pi_<(F) \times \prod_{l \text{ prime}} S^m_l(A)$ is a homeomorphism.*

In particular, $\pi_l(F) = S^m_l(A)$ for every prime l.

Proof. Two elements having the same type must have the same j-type for all $j \in J$. Therefore, the map is injective, and it is clearly continuous. Let us prove that it is surjective.

Let $\vec{p} = (p_j)_j$ be in the direct product. Let $\vec{c} = (c_i)_i$ be a realization of $p_<$ which is $\mathbb{Q}$-free over A. By Corollary 5.4, let $\vec{c}'$ be some tuple which realizes simultaneously all the (p_l) for l prime ($\vec{c}'$ might not be $\mathbb{Q}$-free over A, but it does not matter). Let us show that there exist $\vec{d} = (d_i)_{i<m}$ such that $d_i - c'_i \in \bigcap_{N>0} NG$ for all i, and $\vec{d} \models p_<$ (in particular, $\vec{d}$ will be $\mathbb{Q}$-free over A). The existence of $\vec{d}$ will be established by a disjunction of two cases.

Suppose G is discrete. Let $N > 0$. Then, by regularity of G applied to the interval $[c_i - c'_i, c_i - c'_i + N \cdot \mathbb{1}]$, there must exist an integer k_i between 0 and N such that $c_i - c'_i + k_i \cdot \mathbb{1} \in NG$. Let us show that $\vec{d}' = (d'_i)_i = (c_i + k_i \cdot \mathbb{1})_i \models p_<$. If not, then there must exist some atomic formula with predicate $=$ or $<$ which is satisfied by $\vec{c}$ and not $\vec{d}'$, or vice versa. Without loss, there must exist $a \in A$, and some nonzero $f \in \mathrm{LC}^m(\mathbb{Z})$ such that $f(\vec{c}) < a \leqslant f((c_i + k_i \cdot \mathbb{1})_i)$. By subtracting $f((k_i \cdot \mathbb{1})_i)$ to the second inequality, $f(\vec{c})$ belongs to the interval $[a - f((k_i \cdot \mathbb{1})_i), a]$, which is included in A as A is special, contradicting $\mathbb{Q}$-freeness of $\vec{c}$ over A. It follows that $\vec{d}' \models p_<$, and we conclude by compactness that $\vec{d}$ exists.

Now, suppose G is dense. Let D be the special subgroup generated by A, and all the $(c_i)_i$. For every positive element a of D, the open interval $]c_i - c'_i, c_i - c'_i + a[$ is infinite, and thus contains an element of NG by regularity, for any $N > 0$. By compactness, there exists, for each i, an element d'_i such that $d'_i \in]c_i - c'_i, c_i - c'_i + a[$ for every positive $a \in D$, and $d'_i \in \bigcap_{N>0} NG$. Let $d_i = d'_i + c'_i$. Notice that $d_i - c_i$ is a positive element which is infinitesimal with respect to D. It remains to show that $\vec{d} \models p_<$. Let $f \in \mathrm{LC}^m(\mathbb{Z})$ be nonzero, and $a \in A$, such that $f(\vec{c}) > a$. As each $d_i - c_i$ is infinitesimal with respect to D, this is also the case for $f(\vec{d} - \vec{c}) = f(\vec{d}) - f(\vec{c})$, thus $|f(\vec{d}) - f(\vec{c})| < f(\vec{c}) - a$, thus $f(\vec{c}) - f(\vec{d}) < f(\vec{c}) - a$, and we conclude that $f(\vec{d}) > a$, proving that $\vec{d} \models p_<$.

In both cases, we have $\vec{d} \models p_<$. As $d_i - c'_i \in l^N G$ for every l prime and $N > 0$, we have $\vec{d} \models p_l$ for every l, which concludes the proof. □

Remark 5.8. Note that, in this proof, d_i can always be chosen so that $d_i > c_i$ for each i (in the discrete case, replace k_i by $k_i + N > 0$). In particular, if we have $\vec{c} = \vec{c}'$ in the construction of the proof, then the tuple $\vec{d}$ that we build is distinct from $\vec{c}$, and reiterating this operation generates infinitely many pairwise-distinct tuples having the same quantifier-free type as $\vec{d}$. It follows that any consistent quantifier-free type over A whose realizations are $\mathbb{Q}$-free over A has infinitely many realizations.

Definition 5.9. Let G be an ordered abelian group, $n < \omega$ and $g \in G$. Define $H_n(g)$ to be the largest convex subgroup of G for which we have $(g + H_n(g)) \cap nG = \varnothing$

(by convention, if $g \in nG$, then $H_n(g) = \{0\}$). This convex subgroup is definable:

$$H_n(g) = \{0\} \text{ or } \big\{x \in G \mid \forall y \in G,\ (|y| \leqslant n|x| \Rightarrow g + y \notin nG)\big\},$$

for if $x \in G$, $z \in G$ satisfy $\Delta(x) = \Delta(z)$ (Δ is defined in Definition 1.28) and $z + g \in nG$, then $x \notin H_n(g)$ is witnessed by choosing $y = z \pm mn|x|$, with m the least natural integer for which $|z| - mn|x| \leqslant n|x|$.

In the literature, the set of the $H_n(g)$ for all $g \in G$ is called the *n-spine* of G. This is a definable family of definable convex subgroups. One of the most general "complexity classes" of ordered abelian groups that is still considered rather "nice" is the class of ordered abelian groups with finite spines. This class contains in particular the dp-finite ordered abelian groups. For reference, Section 2 of [Farré 2017] gives nice characterizations and a quantifier elimination result for this class.

Proposition 5.10. *Let G be an ordered abelian group. Then the following conditions are equivalent*:

(1) *G does not have any proper nontrivial definable convex subgroups.*

(2) *For all $g \in G$ and all $n < \omega$, $H_n(g) = \{0\}$.*

(3) *G is regular.*

(4) *The theory of G eliminates quantifiers in the Presburger language.*

(5) *There exists an Archimedean ordered abelian group elementarily equivalent to G.*

Proof. We establish the equivalence by proving (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) $\Rightarrow$ (1) and (3) $\Rightarrow$ (5) $\Rightarrow$ (1).

The implications (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1) are trivial, (3) $\Rightarrow$ (4) is due to [Weispfenning 1981, Theorems 2.3 and 2.6], and Presburger in the discrete case, and (3) $\Rightarrow$ (5) follows from [Zakon 1961, Theorem 2.5; Robinson and Zakon 1960, Theorem 4.7].

Let us show (2) $\Rightarrow$ (3) by contraposition. We have an interval I of G and an integer n such that $|I| \geqslant n$ and $I \cap nG = \varnothing$. If $|I| < 2n + 1$, then G is discrete, and the points from $I + n \cdot \mathbb{1}$ have the same cosets of nG as those of I, so we can assume $|I| \geqslant 2n + 1$ by replacing I by $I \cup (I + n \cdot \mathbb{1}) \cup (I + 2n \cdot \mathbb{1})$. Let f be a strictly increasing map $\{0, \dots, 2n\} \to I$, and $h = \min\{f(i+1) - f(i) \mid 0 \leqslant i \leqslant 2n - 1\}$. As f is strictly increasing, we have $h \neq 0$. As h is minimal, we can sum n many inequalities to get $n \cdot h \leqslant \sum_{i=0}^{n-1} (f(i+1) - f(i)) = f(n) - f(0)$. With the same argument, $n \cdot h \leqslant f(2n) - f(n)$, and thus $[f(n) - n \cdot h,\ f(n) + n \cdot h] \subseteq I$, which is disjoint from nG. It follows that $h \in H_n(f(n)) \setminus \{0\}$, so (2) fails.

Let us prove (4) $\Rightarrow$ (1) by contraposition. Let G be an ordered abelian group with a proper nontrivial definable convex subgroup H. Let A be a definably closed subset of G such that H is A-definable. Let $A_1 = A_{\geqslant 0} \cap H$ and $A_2 = (A_{>0} \setminus H) \cup \{+\infty\}$.

Let $X_< = \left(\bigcap_{a\in A_1, b\in A_2}]a, b[\right)$, an A-type definable set which is nonempty by compactness. For each prime l, let $X_l = \bigcap_N l^N G$, which corresponds to $\mathrm{tp}_l(0/A)$. Let us show that the partial type defined as $q(x) : x \in X_< \cap \bigcap_l X_l$ is consistent with H and $\neg H$. Let $a \in A_1$, $b \in A_2$, $N > 0$. By compactness, it suffices to show that $Y =]a, b[\cap NG$ intersects H and $\neg H$, which is witnessed by $N \cdot a \in Y \cap H$, and $N \cdot (b - a) \in Y \setminus H$. Now let $h \in H$, $g \notin H$ be two realizations of q. Then $\mathrm{qftp}(h/A) = q = \mathrm{qftp}(g/A)$, but $\mathrm{tp}(h/A) \neq \mathrm{tp}(g/A)$, and the theory of G does not eliminate quantifiers in $\mathcal{L}_P$.

□

Note that in particular, the domain of the homeomorphism given by Lemma 5.7 is in fact a space of *complete types*.

Corollary 5.11. *The definable closure of a parameter set A coincides with the special subgroup it generates.*

Proof. The special subgroup generated by A is clearly included in $\mathrm{dcl}(A)$. Conversely, if A is a special subgroup, and $c \notin A$, then the 1-tuple c is $\mathbb{Q}$-free over A, and thus $c \notin \mathrm{acl}(A)$ by Remark 5.8. □

There is another important corollary which allows us to better understand independence:

Corollary 5.12. *Let $M \models \mathrm{ROAG}$, let $A \leqslant B$ be special subgroups of M such that M is $|B|^+$ saturated and strongly $|B|^+$-homogeneous, and let $\vec{c} = (c_1, \dots, c_n) \in M^n$. Suppose $\vec{c}$ is $\mathbb{Q}$-free over B, and let F be the closed subspace of $S^n(M)$ of tuples that are $\mathbb{Q}$-free over M. For each $j \in J$, consider the action of* $\mathrm{Aut}(M/A)$ *on $\pi_j(F)$. Then the following conditions are equivalent*:

(1) $\vec{c} \mathop{\downarrow}^{\mathbf{inv}}_A B$ (*resp.* $\vec{c} \mathop{\downarrow}^{\mathbf{bo}}_A B$).

(2) *For each $j \in J$,* $\mathrm{tp}_j(\vec{c}/B)$, *which is a partial type over B, can be extended to an* $\mathrm{Aut}(M/A)$*-invariant* (*resp. of bounded orbit*) *element of $\pi_j(F)$.*

Proof. First of all, note that the global extensions of $\mathrm{tp}(\vec{c}/B)$ realized by tuples that are not $\mathbb{Q}$-free over M divide over A, which implies that they have an unbounded orbit (see, for instance, [Tent and Ziegler 2012, Exercise 7.1.5]), so we do have to restrict ourselves to F.

For each $j \in J$, $F \to \pi_j(F)$ is an equivariant surjection, and thus the stabilizer of a point of F is a subgroup of that of its image. So the cardinal of its orbit is larger than the supremum of the cardinals of the orbits of each of its images, and we get the top-to-bottom direction by contraposition.

Suppose that for each j, we have $p_j \in \pi_j(F)$ a witness of the second condition: it extends $\mathrm{tp}_j(\vec{c}/B)$, and its orbit is a singleton (resp. bounded). Note that the orbit of $(p_j)_j$ under $\mathrm{Aut}(M/A)$ is contained in the Cartesian product of the orbits of the $(p_j)_j$, and therefore it is also a singleton (resp. bounded). Now, by Lemma 5.7 and quantifier elimination, the map $F \to \prod_j \pi_j(F)$ is an equivariant bijection, so the consistent global type corresponding to $(p_j)_j$ witnesses the first condition, and we get the bottom-to-top direction. □

Note that, given special subgroups $A \leqslant B$, a tuple $\vec{c}$ will always be A-interdefinable with a subtuple $\vec{d}$ which is $\mathbb{Q}$-free over A (choose any of those subtuples, of maximal length). Thus there is a natural equivariant homeomorphism between the space of global extensions of $\mathrm{tp}(\vec{c}/B)$ and that of $\mathrm{tp}(\vec{d}/B)$. Corollary 5.12 completes the picture, and gives us easy conditions on $\vec{d}$ to check whether $\vec{c} \mathop{\smile\!\!\!\!\!|}_A^{\mathbf{inv}} B$ or $\vec{c} \mathop{\smile\!\!\!\!\!|}_A^{\mathbf{bo}} B$.

Remark 5.13. Corollary 5.12 raises one interesting question: does the orbit of an element of $\prod_j \pi_j(F)$ coincide with the Cartesian product of the orbits of its components? In other words, for $p_j \in \pi_j(F)$, is the natural (injective) map

$$\mathrm{Aut}(M/A)/\bigcap_j \mathrm{Stab}(p_j) \to \prod_j \mathrm{Aut}(M/A)/\mathrm{Stab}(p_j)$$

a bijection?

6. Invariance and boundedness of the global extensions of the partial types

Assumptions 6.1. Let $M \models \mathrm{ROAG}$, let $A \leqslant B$ be special subgroups of M, let $\lambda = \max(|B|, 2^{\aleph_0})^+$, let $\vec{c} = (c_1, \ldots, c_n)$ be a tuple from M, and suppose that M is λ-saturated and strongly λ-homogeneous. Suppose $\vec{c}$ is $\mathbb{Q}$-free over B, and let F be the closed space of global types whose realizations are $\mathbb{Q}$-free over M. For each $j \in J$, let $p_j = \mathrm{tp}_j(\vec{c}/B)$. A set of cardinality κ is called *small* if M is κ^+-saturated and strongly κ^+-homogeneous, else it is *large*.

By Corollary 5.12, in order to understand the global extensions of $\mathrm{tp}(\vec{c}/B)$ which are invariant or have a bounded orbit under the action of $\mathrm{Aut}(M/A)$, one has to understand for each $j \in J$ the global extensions in $\pi_j(F)$ of p_j which are invariant or have a bounded orbit.

6.1. *Partial types using equality and order.*

Assumptions 6.2. On top of Assumptions 6.1 we fix $\overline{M}$ some $|M|^+$-saturated, strongly $|M|^+$-homogeneous elementary extension of $\mathrm{div}(M) \models \mathrm{DOAG}$.

Remark 6.3. Let $\vec{d}$ be some tuple from M. Then $\vec{d} \models p_<$ if and only if $\vec{c}$ and $\vec{d}$ have the same type over B in $\overline{M}$.

Lemma 6.4. *Let D be some small special subgroup of M. Let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{m-1})$ be a tuple from $\overline{M}$ which is $\mathbb{Q}$-free over D. Suppose for all nonzero $f \in \mathrm{LC}^m(\mathbb{Q})$,* *and for all $d \in D$, we have $f(\vec{\alpha}) \notin]d, d+\mathbb{1}[$ (this interval of $\overline{M}$ being by convention empty when $\mathbb{1} = 0$). Then there exists $\sigma \in \mathrm{Aut}(\overline{M}/D)$ such that $\sigma(\vec{\alpha})$ is a tuple from M.*

Proof. Suppose by induction that we have $\sigma_i \in \mathrm{Aut}(\overline{M}/D)$ sending $\vec{\alpha}_{<i}$ to a tuple from M for some $i < m$ ($\sigma_0 = \mathrm{id}$ will do for $i = 0$). Let D_i be the special subgroup of M generated by $D\sigma_i(\vec{\alpha}_{<i})$, which is exactly the relative divisible closure in M of $D + \sum_{k<i} \mathbb{Z} \cdot \sigma_i(\alpha_k)$. Then, for all nonzero $f \in \mathrm{LC}^{n-i}(\mathbb{Q})$, for all $d \in D_i$, we have by hypothesis $f(\vec{\alpha}_{\geqslant i}) \notin]d, d+\mathbb{1}[$. Let us find $\tau \in \mathrm{Aut}(\overline{M}/D_i)$ such that $\tau(\sigma_i(\alpha_i)) \in M$. Then we could set $\sigma_{i+1} = \tau \circ \sigma_i$, and conclude by induction.

By strong homogeneity of $\overline{M}$, it is enough to show that any interval of $\overline{M}$ with bounds in $\mathrm{div}(D_i) \cup \{\pm\infty\}$ containing $\sigma_i(\alpha_i)$ has a point β in M. Let I be such an interval. If either the lower or the upper bound of I is in $\{\pm\infty\}$, then β clearly exists, one may choose some multiple of d with large enough absolute value if $d \neq 0$, else choose any nontrivial element with correct sign. Now, suppose $I =]d_1/N_1, d_2/N_2[$, with $d_k \in D_i$, $N_k > 0$ (by the $\mathbb{Q}$-freeness assumption on $\vec{\alpha}$, $\sigma_i(\alpha_i) \notin \mathrm{div}(D_i)$, thus it does not matter whether the bounds of I belong to I). Then I has a point in M if and only if $]N_2 d_1, N_1 d_2[$ has a point in $N_1 N_2 M$. If M is dense, then I has infinitely many points in M, and we can use the axioms of ROAG to conclude.

It remains to deal with the case where M is discrete. We established earlier that the $\mathbb{Q}$-linear combinations of $\sigma_i(\vec{\alpha}_{\geqslant i})$ do not belong to $]d, d+\mathbb{1}[$ for any $d \in D_i$. In particular, for every $N \in \mathbb{Z}$,

$$N_1 N_2 \sigma_i(\alpha_i) \notin \bigcup_{N \in \mathbb{Z}} \big]N_2 d_1 + N \cdot \mathbb{1}, N_2 d_1 + (N+1) \cdot \mathbb{1}\big[$$

and by $\mathbb{Q}$-freeness we also have $N_1 N_2 \sigma_i(\alpha_i) \notin \{N_2 d_1 + N \cdot \mathbb{1} \mid N \in \mathbb{Z}\}$. As a result, $N_1 N_2 \sigma_i(\alpha_i) \in]N_2 d_1, N_1 d_2[\setminus]N_2 d_1, N_2 d_1 + (N_1 N_2 + 2) \cdot \mathbb{1}]$, which must imply that $N_2 d_1 + (N_1 N_2 + 1) \cdot \mathbb{1} < N_1 d_2$, so I has at least $N_1 N_2 + 1$ (in fact, infinitely many) points in M, and we can also conclude with the axioms of ROAG. □

Lemma 6.5. *Let $\vec{\alpha} \in M^n$ be a tuple which is $\mathbb{Q}$-free over B, and such that $p = \mathrm{tp}^{\mathrm{DOAG}}(\vec{\alpha}/B)$ does not fork over A. Let D be a small special subgroup of M containing B, and let $q \in S^n_{\mathrm{DOAG}}(\mathrm{div}(D))$ be an extension of p which does not fork over A. Then q has a realization in M^n.*

Proof. First of all, the realizations of q are $\mathbb{Q}$-free over D. Let $d \in D$, and let f be a nonzero element of $\mathrm{LC}^n(\mathbb{Q})$. By Lemma 6.4, one just has to prove that $q(\vec{x}) \models f(\vec{x}) \notin]d, d+\mathbb{1}[$. If not, then by Theorem 1.16, the interval $[d, d+\mathbb{1}]$ must have a point in $\mathrm{div}(A)$. By multiplying everything by a sufficiently large $N > 0$, the interval $]Nd, Nd + N \cdot \mathbb{1}[$ has a point in A, and $Nd \in D$, so $Nd \in A$. This implies $q_{|A}(\vec{x}) \models Nf(\vec{x}) \in]Nd, Nd + N \cdot \mathbb{1}[$, thus $p(\vec{x}) \models Nf(\vec{x}) \in]Nd, Nd + N \cdot \mathbb{1}[$, and $Nf(\vec{\alpha}) \in]Nd, Nd + N \cdot \mathbb{1}[$. As Nd and $\vec{\alpha}$ are in A, we have $Nf(\vec{\alpha}) \in A$, contradicting $\mathbb{Q}$-freeness of $\vec{\alpha}$. □

Proposition 6.6. *By Remark 6.3, let h be the natural injection*

$$\pi_<(F) \to S^n_{\mathrm{DOAG}}(\mathrm{div}(M)).$$

Consider the action of $\mathrm{Aut}(M/A)$ *on $Q = \pi_<(F)$. Then the following conditions are equivalent*:

(1) *Some element of Q is invariant and extends $p_<$.*
(2) *Some element of Q has a bounded orbit and extends $p_<$.*
(3) *The partial type $p_<$ does not fork over A.*
(4) *The partial type $p_<$ does not divide over A.*
(5) *Every closed bounded interval with bounds in B containing a $\mathbb{Z}$-linear combination of $\vec{c}$ also has a point in* $\mathrm{div}(A)$.
(6) *For some $p \in Q$ extending $p_<$, $h(p)$ extends to some* $\mathrm{Aut}(\overline{M}/A)$*-invariant type over $\overline{M}$.*

Proof. The directions (1) ⇒ (2) and (3) ⇒ (4) are immediate.

Let us prove (2) ⇒ (3). Let $q_< \in Q$ witness (2). For each prime l, let $q_l(\vec{x})$ in $S^n_l(M)$ be the partial type $\{\mathfrak{d}_{l^N}(f(\vec{x})) \mid n > 0,\ f \in \mathrm{LC}(\mathbb{Z}),\ f \neq 0\} = \mathrm{tp}_l(0/M)$. Note that q_l is clearly invariant. Then the complete global type in F corresponding to $(q_<, (q_l)_l)$ (recall $F \to \pi_<(F) \times \prod_{l \text{ prime}} S^n_l(M)$ is a homeomorphism by Lemma 5.7, so this type is consistent) has a bounded orbit, and thus does not fork over A, and we get (3).

Let us prove (4) ⇒ (5). Suppose we have $b_k \in B$, $f \in \mathrm{LC}(\mathbb{Z})$ such that the formula $f(\vec{c}) \in [b_1, b_2]$ witnesses the failure of (5). Let us show that the formula $f(\vec{x}) \in [b_1, b_2]$ divides over A, witnessing the failure of (4). As in the proof of Corollary 1.8, it is enough to find $d \equiv_A b_1$ such that $d > b_2$. As $[b_1, b_2]$ has no point in $\mathrm{div}(A)$, we have $b_k \notin A$, and thus the singletons b_1 and b_2 are both $\mathbb{Q}$-free over A. Now, by Lemma 5.7, one just has to show that $\mathrm{tp}_<(b_1/A)$ is consistent with $]b_2, +\infty[$. If not, then all the elements in M of $]b_2, +\infty[$ have a type over A in $\overline{M}$ which is distinct from that of b_1. As a result, by the characterization of 1-types in DOAG, there must exist in $\overline{M}$ a point in $\mathrm{div}(A) \cap]b_1, b_2]$, a contradiction.

Let us show (5) ⇒ (6). Suppose (6) fails. Then, by Theorem 1.16, $p_<$ is inconsistent with the partial type $\{f(\vec{x}) \notin I \mid I \in \mathcal{I},\ f \in \mathrm{LC}(\mathbb{Z})\}$, with $\mathcal{I}$ the set of closed bounded intervals with bounds in M that have no points in $\mathrm{div}(A)$. By compactness, there exist finite subsets $\mathcal{I}' \subseteq \mathcal{I}$, $G \subseteq \mathrm{LC}^n(\mathbb{Z})$ such that $p_<(\vec{x}) \models \bigvee_{I \in \mathcal{I}', g \in G} g(\vec{x}) \in I$. Let D be the special subgroup of M generated by B and the bounds of the elements of $\mathcal{I}'$. Then, for all $p \in Q$ extending $p_<$, the restriction of p to D corresponds to a type in $S^n_{\mathrm{DOAG}}(D)$ which forks over A in DOAG.

Suppose now, towards a contradiction, that (5) holds. Let q be the type of $S^n_{\mathrm{DOAG}}(B)$ corresponding to $p_<$. Then q does not fork over A. By extension, let $q' \in S^n_{\mathrm{DOAG}}(D)$ be an extension of q which does not fork over A. By Lemma 6.5, q' must admit a realization $\vec{\beta}$ in M^n (in particular $\vec{\beta} \models p_<$). As q' does not fork over A, we have $g(\vec{\beta}) \notin I$ for all $g \in G$, $I \in \mathcal{I}'$. As a result, $q_< = \mathrm{tp}_<(\vec{\beta}/D)$ cannot be extended to any element of F. This means that in any elementary extension of M, no realization of $q_<$ is $\mathbb{Q}$-free over M. By compactness, this means that there exist finite subsets $P \subseteq M$, $G' \subseteq \mathrm{LC}^n(\mathbb{Q}) \setminus \{0\}$ such that $q_<(\vec{x}) \models \bigvee_{m\in P, g\in G'} g(\vec{x}) = m$.

Then we can reach a contradiction by extending q' once more: let $\widetilde{D}$ be the special subgroup of M generated by $D \cup P$, and let $\tilde{q} \in S^n_{\mathrm{DOAG}}(\widetilde{D})$ be an extension of q' which does not fork over A. Then with the same argument, $\tilde{q}$ has a realization $\vec{\gamma} \in M^n$, and by hypothesis $\vec{\gamma}$ is not $\mathbb{Q}$-free over $\widetilde{D}$, a contradiction with the fact that $\tilde{q}$ does not fork over A.

Suppose $p \in Q$ witnesses (6), and let us show that p is $\mathrm{Aut}(M/A)$-invariant, which would allow us to conclude the whole proof with the direction (6) ⇒ (1). Let $q \in S^n_{\mathrm{DOAG}}(\overline{M})$ be some global $\mathrm{Aut}(\overline{M}/A)$-invariant extension of $h(p)$, and $\sigma \in \mathrm{Aut}(M/A)$. Then σ extends uniquely to σ', an automorphism of the ordered group $\mathrm{div}(M)$, which fixes A pointwise. We clearly have $h(\sigma(p)) = \sigma'(h(p))$ (look at the atomic formulas with predicate $=$, $<$ that belong to $X_<$, check that their image by σ is satisfied by the realizations of $\sigma'(h(p))$). By quantifier elimination in DOAG, σ' is a partial elementary map in $\overline{M}$. By strong homogeneity of $\overline{M}$, σ' extends to some $\tilde{\sigma} \in \mathrm{Aut}(\overline{M})$. As σ' fixes A pointwise, so does $\tilde{\sigma}$, and thus $\tilde{\sigma}(q) = q$. Now, we conclude that

$$\begin{aligned}\sigma(p) = h^{-1}(\sigma'(h(p))) &= h^{-1}(\sigma'(q_{|\mathrm{div}(M)})) = h^{-1}(\tilde{\sigma}(q_{|\mathrm{div}(M)}))\\ &= h^{-1}(\tilde{\sigma}(q)_{|\tilde{\sigma}(\mathrm{div}(M))}) = h^{-1}(q_{|\mathrm{div}(M)}) = h^{-1}(h(p)) = p. \qquad \square\end{aligned}$$

The fifth condition of Proposition 6.6 is a very simple geometric condition, the kind of statement that would be very satisfactory for a characterization of forking. In the subsections 6.2 and 6.3, we look for similar conditions for the $(p_l)_l$.

6.2. ***Partial types using a prime of finite index.*** For the rest of this section, we fix a prime l.

Proposition 6.7. *Let G be a torsion-free abelian group. Then for all $N > 0$, we have in G^{eq} a $\varnothing$-definable group isomorphism between the two definable groups G/lG and $l^N G/l^{N+1}G$.*

Proof. Let $x, y \in G$, and $N > 0$. Suppose $l^N x - l^N y \in l^{N+1}G$. Then, as G is torsion-free, we have $x - y \in lG$. Thus the map

$$f : l^N x + l^{N+1}G \mapsto x + lG$$

is a well-defined group homomorphism which is clearly surjective. Now, if $x \notin lG$, then $l^N x \notin l^{N+1}G$, so the map is injective. $\square$

Corollary 6.8. *If lG has a finite index d, then $\bigcap_N l^N G$ has index less than $2^{\aleph_0}$.*

Proof. Define a tree structure on $\bigcup_{N\geqslant 0} G/l^N G$ of root G, such that the parent of some node $x + l^{N+1}G$ is $x + l^N G$. Then every node has d children and the tree has $\aleph_0$ levels, and thus it admits at most $2^{\aleph_0}$ branches. To conclude, the map $x + \bigcap_N l^N G \mapsto (x + l^N G)_N$ is a natural bijection between $G/\bigcap_N l^N G$ and the set of the branches. $\square$

Lemma 6.9. *Let D be a small special subgroup of M. Suppose we have $e_1, e_2 \in M$ such that $e_1, e_2 \notin D + l^N M$ and $e_1 - e_2 \in l^{N-1}M$ for some $N > 0$. Then we have $\mathrm{tp}_l(e_1/D) = \mathrm{tp}_l(e_2/D)$.*

Proof. Any atomic formula with parameters in D and predicate in $\{\mathfrak{d}_{l^m} \mid m > 0\}$ can clearly be written $\mathfrak{d}_{l^m}(\lambda x - d)$, with $\lambda \in \mathbb{Z}$, $m > 0$, $d \in D$. We have several cases:

- $\lambda \in l\mathbb{Z}$, $d \notin lM$, in which case the formula always fails.
- $\lambda \in l\mathbb{Z}$, $d \in lM$, $m = 1$, in which case the formula always holds.
- $\lambda \notin l\mathbb{Z}$, in which case λ and l are coprime. Therefore, by Bézout's identities, the formula is equivalent to $\mathfrak{d}_{l^m}(x - d')$ for some $d' \in D$. By hypothesis on the e_i, this formula is satisfied by e_1 if and only if it is satisfied by e_2.
- $\lambda \in l\mathbb{Z}$, $d \in lM$, $m > 1$, in which case the formula is equivalent to the formula $\mathfrak{d}_{l^{m-1}}((\lambda/l)x - (d/l))$, and we reduce by induction to one of the above three cases.

Either way, we clearly see that those formulas are satisfied by e_1 if and only if they are satisfied by e_2, which concludes the proof. $\square$

Corollary 6.10. *Let $\vec{d}$ be a tuple from M, which is $\mathbb{Q}$-free over B. Then $\vec{d} \models p_l$ if and only if for each $f \in \mathrm{LC}^n(\mathbb{Z})$ and each $N > 0$, either we have $f(\vec{c}) - f(\vec{d}) \in l^N M$, or neither $f(\vec{c})$ nor $f(\vec{d})$ are in $B + l^N M$.*

Proof. Any atomic formula $\varphi(x_1, \ldots, x_n)$ with predicate in $\{\mathfrak{d}_{l^N} \mid N > 0\}$ can be written $\psi(f(\vec{x}))$, with $f \in \mathrm{LC}(\mathbb{Z})$, and ψ an atomic formula on the same predicate with one variable. As a result, $\vec{d} \models p_l$ if and only if $\mathrm{tp}_l(f(\vec{d})/B) = \mathrm{tp}_l(f(\vec{c})/B)$ for all $f \in \mathrm{LC}(\mathbb{Z})$. Fix $f \in \mathrm{LC}^n(\mathbb{Z})$.

Suppose that we have $f(\vec{c}) \in B + l^N M$ for all $N > 0$. For each N, let $b_N \in B$ such that $f(\vec{c}) - b_N \in l^N M$. Then $\mathrm{tp}_l(f(\vec{c})/B)$ contains $\{\mathfrak{d}_{l^N}(x - b_N) \mid N > 0\}$. This inclusion is an equality, for if $\mathfrak{d}_{l^N}(f(\vec{d}) - b_N)$ for all N, then $f(\vec{d}) - f(\vec{c}) \in l^N M$ for all N, which clearly implies $\mathrm{tp}_l(f(\vec{c})/B) = \mathrm{tp}_l(f(\vec{d})/B)$. In particular, $\mathrm{tp}_l(f(\vec{d})/B) = \mathrm{tp}_l(f(\vec{c})/B)$ if and only if $f(\vec{d}) - f(\vec{c}) \in \bigcap_N l^N M$.

Suppose now that there exists $N > 0$ such that $f(\vec{c}) \notin B + l^N M$. The set of all such N is a final segment of ω. Choose N its least element. Let $b \in B$ such that

$f(\vec{c})-b\in l^{N-1}M$ (if $N=1$, then $b=0$ will do). Then, by Lemma 6.9, $\mathrm{tp}_l(f(\vec{c})/B)$ is generated by the partial type

$$\{\mathfrak{d}_{l^{N-1}}(x-b)\}\cup\{\neg\mathfrak{d}_{l^N}(x-b')\mid b'\in B\}.$$

This concludes the proof. □

Remark 6.11. If $[M:lM]$ is finite, then by Corollary 6.8 the $\varnothing$-type-definable equivalence relation $\{\mathfrak{d}_{l^N}(x-y)\mid N>0\}$ is bounded (has a small number of classes) and finer than the equivalence relation of having the same l-type over any parameter set. Thus $S_l^n(M)$ is small, and each orbit of $S_l^n(M)$ under $\mathrm{Aut}(M/A)$ is obviously bounded.

Moreover, the classes of this $\varnothing$-type-definable equivalence relation are all invariant under $\mathrm{Aut}(M^{\mathrm{eq}}/\operatorname{acl}^{\mathrm{eq}}(\varnothing))$. Thus $\mathrm{Aut}(M^{\mathrm{eq}}/\operatorname{acl}^{\mathrm{eq}}(\varnothing))$ acts trivially over $S_l^n(M)$.

Let us recall that we adopted Assumptions 6.1 in this section: we have M a monster model of ROAG, $A\leqslant B$ special subgroups of M, $\vec{c}=(c_1,\dots,c_n)$ a tuple from M which is $\mathbb{Q}$-free over B, F the space of global types of tuples which are $\mathbb{Q}$-free over M, and $p_j=\mathrm{tp}_j(\vec{c}/B)$.

Proposition 6.12. *Suppose* $[M:lM]$ *is finite. Consider the action of* $\mathrm{Aut}(M/A)$ *on* $Q=S_l^n(M)$. *Then* p_l *has an invariant extension in* Q *if and only if every* $\mathbb{Z}$*-linear combination of* $\vec{c}$ *belongs to* $\bigcap_N(A+l^NM)$.

Proof. Suppose every $\mathbb{Z}$-linear combination of $\vec{c}$ belongs to $\bigcap_N(A+l^NM)$. Then $p_l(M)$ can be written as the intersection of A-definable sets of the form

$$\{\vec{x}\mid\mathfrak{d}_{l^N}(f(\vec{x})-a)\},$$

with $N>0, a\in A$, $f\in\mathrm{LC}(\mathbb{Z})$. As a result, $p_l(M)$ is A-type definable (thus $\mathrm{Aut}(M/A)$-invariant), and its realizations $\vec{d}\in M^n$ satisfy $f(\vec{c})-f(\vec{d})\in\bigcap_N l^NM$ for all $f\in\mathrm{LC}^n(\mathbb{Z})$. By Corollary 6.10, p_l is complete as an element of Q, so we get the right-to-left direction.

Suppose now that we have $f\in\mathrm{LC}^n(\mathbb{Z})$ and $k>0$ such that $f(\vec{c})$ does not belong to $A+l^kM$. Let $p\in Q$ be an extension of p_l. By compactness and finiteness of $[M:lM]$, let $\alpha\in M$ be such that $p(\vec{x})\models\mathfrak{d}_{l^N}(f(\vec{x})-\alpha)$ for all $N>0$. As $[M:A+l^kM]>1$, we must have by Proposition 6.7 the inequalities $1<[M:l^kM]=[M:lM]^k$. Thus $1<[M:lM]=[l^kM:l^{k+1}M]$, so let $m\in l^kM\setminus l^{k+1}M$. Then, by Lemma 6.9, $\mathrm{tp}_l(\alpha/A)=\mathrm{tp}_l(\alpha+m/A)$. Let D be a special subgroup of M of size $\leqslant\max(2^{\aleph_0},|A|)$ containing A, and a system of representatives of the cosets of $\bigcap_N l^NM$. By saturation of M, there is $\beta\in M$ such that $\mathrm{tp}_l(\beta/D)=\mathrm{tp}_l(\alpha-m/D)$ (which implies $\beta-\alpha-m\in\bigcap_N l^NM$), and $\mathrm{tp}_j(\beta/D)=\mathrm{tp}_j(\alpha/D)$ for all $j\neq l$ (which implies $\mathrm{tp}_j(\beta/A)=\mathrm{tp}_j(\alpha/A)$). Then, by strong homogeneity of M, there exists $\sigma\in\mathrm{Aut}(M/A)$ such that $\sigma(\alpha)=\beta$. Now, we have $\sigma(p)(\vec{x})\models\mathfrak{d}_{l^{k+1}}(f(\vec{x})-\alpha-m)$, a formula that is inconsistent with $\mathfrak{d}_{l^{k+1}}(f(\vec{x})-\alpha)$. As a result, $\sigma(p) \neq p$, concluding the proof. □

6.3. ***Partial types using a prime of infinite index.*** We still fix a prime l, and we assume here that $[M : lM]$ is infinite.

Lemma 6.13. *Let D be a small special subgroup of M, and $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ a finite tuple from M which is $\mathbb{Q}$-free over D. Let $\vec{\beta} = (\beta_1, \dots, \beta_m)$ be another tuple from M such that $\mathrm{tp}_l(\vec{\beta}/D) = \mathrm{tp}_l(\vec{\alpha}/D)$. Then there exists $\vec{\gamma} = (\gamma_1, \dots, \gamma_m) \equiv_D \vec{\alpha}$ such that $\gamma_i - \beta_i \in \bigcap_N l^N M$ for all i.*

Proof. As $\bigcap_N l^N M$ is large, choose by induction $\varepsilon_i \in \bigcap_N l^N M$ such that ε_i is not in the special subgroup generated by $D\vec{\alpha}\vec{\beta}\varepsilon_{<i}$, and let $\beta'_i = \beta_i + \varepsilon_i$. Then we have $\beta_i - \beta'_i \in \bigcap_N l^N M$, and $\vec{\varepsilon}$ is $\mathbb{Q}$-free over the special subgroup generated by $D\vec{\alpha}\vec{\beta}$. As a result, $\vec{\alpha}$ (and hence $(\alpha_i - \beta'_i)_i$) is $\mathbb{Q}$-free over D', the special subgroup generated by $D\vec{\beta}'$. By Lemma 5.7, there exists in M a tuple $\vec{e} = (e_1, \dots, e_m)$ such that $e_i \in \bigcap_N l^N M$ for all i, and $\mathrm{tp}_j(\vec{e}/D') = \mathrm{tp}_j((\alpha_i - \beta'_i)_i/D')$ for all $j \neq l$. Let $\gamma_i = \beta'_i + e_i$. Then, for all $j \neq l$, we have $\mathrm{tp}_j(\vec{\gamma}/D') = \mathrm{tp}_j(\vec{\alpha}/D')$. Moreover, we have $\gamma_i - \beta'_i \in \bigcap_N l^N M$ for all i. Thus $\mathrm{tp}_l(\vec{\gamma}/D) = \mathrm{tp}_l(\vec{\beta}'/D) = \mathrm{tp}_l(\vec{\beta}/D) = \mathrm{tp}_l(\vec{\alpha}/D)$, so we conclude that $\vec{\gamma} \equiv_D \vec{\alpha}$. □

Remark 6.14. Recall that any discrete model of ROAG is an elementary extension of $\mathbb{Z}$. In particular, if $[M : lM]$ is infinite for some l, then M must be dense. In that case, the special subgroups of M (which are exactly the definably closed sets) are its pure subgroups.

Lemma 6.15. *Let $(\alpha_i)_{i\in I}$ be a finite tuple from M which is $\mathbb{Q}$-free over B. Suppose that for each $i \in I$, there exists $N > 0$ such that $\alpha_i \notin B + l^N M$, and choose N_i the least of those integers for each $i \in I$. Then there exists $(\beta_i)_i \equiv_B (\alpha_i)_i$ such that $\beta_i - \alpha_k \notin l^{N_i} M$ for each $i, k \in I$.*

Proof. Note that by Remark 6.14, the special subgroups of M are exactly its pure subgroups.

Let $b_i \in B$ such that $b_i - \alpha_i \in l^{N_i - 1} M$. Suppose we can find a witness $(\gamma_i)_i$ of the statement with $(\alpha_i)_i$ replaced by $((\alpha_i - b_i)/l^{N_i - 1})_i$ (N_i will be replaced by 1). Then we can choose $\beta_i = l^{N_i - 1}\gamma_i + b_i$. For the remainder of the proof, we can suppose without loss that $N_i = 1$ for every i (in particular, $b_i = 0$).

Let V be the large $\mathbb{F}_l$-vector space M/lM, and $U \leqslant V$ the $\mathbb{F}_l$-vector subspace $(B + lM)/lM$. Note that U is small: it is the image by B of $M \to M/lM$. Note also that we have $\alpha_i \in V \setminus U$ for every $i \in I$. Now, let $I_0 \subseteq I$ such that $(\alpha_i)_{i\in I_0}$ is a lift of an $\mathbb{F}_l$-basis of the image in V/U of $U + \sum_{i\in I} \mathbb{F}_l \cdot (\alpha_i + lM)$. Let D be the special subgroup of M generated by $B(\alpha_i)_{i\in I}$, and let $U' = (D + lM)/lM$, a small vector subspace of V containing U. As V/U' is large, one can choose an arbitrary tuple $(\beta_i)_{i\in I_0} \in M^{I_0}$ whose image in V/U' is $\mathbb{F}_l$-free. Now, for each nonzero $f \in \mathrm{LC}^{|I_0|}(\mathbb{Z})$, if m is the largest integer such that all the coefficients of f lie in $l^m\mathbb{Z}$, then we have

$$f((\beta_i)_{i\in I_0}) \in l^m M \ni f((\alpha_i)_{i\in I_0}), \qquad f((\beta_i)_{i\in I_0}) \notin B + l^{m+1}M \not\ni f((\alpha_i)_{i\in I_0}).$$

Thus, $\mathrm{tp}_l((\alpha_i)_{i\in I_0}/B) = \mathrm{tp}_l((\beta_i)_{i\in I_0}/B)$ by the same reasoning as in Corollary 6.10. For each $i \in I_0$, as $\beta_i + lM \notin U'$, we have $\beta_i \notin D + lM$, and thus $\beta_i - \alpha_k \notin lM$ for each $k \in I$. By Lemma 6.13, we can suppose without loss that $(\beta_i)_{i\in I_0} \equiv_B (\alpha_i)_{i\in I_0}$.

Choose $\sigma \in \mathrm{Aut}(M/B)$ such that $\sigma(\alpha_i) = \beta_i$ for every $i \in I_0$. Let $\beta_i = \sigma(\alpha_i)$ for each $i \in I \setminus I_0$. Then we have $(\beta_i)_{i\in I} \equiv_B (\alpha_i)_{i\in I}$. In order to conclude, we need to show that $\beta_i - \alpha_k \notin lM$ for each $i \in I \setminus I_0$, $k \in I$. Choose $i \in I \setminus I_0$. Then there must exist $f_i \in \mathrm{LC}(\mathbb{Z})$ such that $\alpha_i - f_i((\alpha_k)_{k\in I_0}) \in B + lM$. Let $e_i \in B$ be such that $\alpha_i - f_i((\alpha_k)_{k\in I_0}) - e_i$ lies in lM. Then $\beta_i - f_i((\beta_k)_{k\in I_0}) - e_i \in lM$, so we just have to show that $f_i((\beta_k)_{k\in I_0}) \notin D + lM$. As $\alpha_i \notin B + lM$, some coefficient of f_i is coprime with l, which concludes the proof as $(\beta_k + lM)_{k\in I_0}$ is $\mathbb{F}_l$-free over U'. □

Lemma 6.16. *Let $(\alpha_i)_i$ be some tuple from M. Then there exists in M a tuple $(\beta_i)_i$ which is $\mathbb{Q}$-free over B, and for which $\alpha_i - \beta_i \in \bigcap_N l^N M$ for every i.*

Proof. One merely has to define by induction $\varepsilon_i \in \bigcap_N l^N M$ which does not lie in the small special subgroup generated by $B(\alpha_k)_k\varepsilon_{<i}$, and choose $\beta_i = \alpha_i + \varepsilon_i$. □

Lemma 6.17. *Let $N > 0$. Suppose we have $f_1, \dots, f_N \in \mathrm{LC}^n(\mathbb{Z})$, as well as $Y_1, \dots, Y_N \in \bigcup_m M/l^m M$, such that every tuple of M which realizes p_l satisfies the formula $\bigvee_i f_i(\vec{x}) \in Y_i$. Then at least one of the Y_i must have a point in B.*

Proof. Let $\alpha_i \in Y_i$ for each i. By Lemma 6.16, we can assume without loss that $(\alpha_i)_i$ is $\mathbb{Q}$-free over B. Suppose towards a contradiction that none of the Y_i has a point in B. Then we can apply Lemma 6.15 to $(\alpha_i)_i$, we find $(\beta_i)_i \equiv_B (\alpha_i)_i$ such that, for all $m > 0$ and for all i, k, if $\alpha_i \notin B + l^m M$, then $\beta_i - \alpha_k \notin l^m M$. In particular, as Y_k does not intersect B, we have $\beta_i \notin Y_k$ for all i, k. Let $\sigma \in \mathrm{Aut}(M/B)$ such that $\sigma(\alpha_i) = \beta_i$ for each i. Then we must have $\left(\bigcup_i Y_i\right) \cap \left(\bigcup_i \sigma(Y_i)\right) = \varnothing$. However, as p_l is $\mathrm{Aut}(M/B)$-invariant, we must have $p_l(x) \models \bigvee_i f_i(\vec{x}) \in \sigma(Y_i)$, a contradiction. □

Let us again recall that we adopted Assumptions 6.1 in this section: we have M a monster model of ROAG, $A \leqslant B$ special subgroups of M, $\vec{c} = (c_1, \dots, c_n)$ a tuple from M which is $\mathbb{Q}$-free over B, F the space of global types of tuples which are $\mathbb{Q}$-free over M, and $p_j = \mathrm{tp}_j(\vec{c}/B)$.

Proposition 6.18. *Consider the action of* $\mathrm{Aut}(M/A)$ *on $Q = S^n_l(M)$. Let C be the group of all $\mathbb{Z}$-linear combinations of $\vec{c}$. Then the following conditions are equivalent*:

(1) *Some invariant element of Q extends p_l.*

(2) *Some element of Q of bounded orbit extends p_l.*

(3) *p_l does not fork over A.*

(4) *p_l does not divide over A.*

(5) *For each $N > 0$, we have $(C + l^N M) \cap (B + l^N M) = A + l^N M$.*

Proof. The directions (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial.

Let us prove (2) $\Rightarrow$ (3). For each prime $l' \neq l$, we can define $q_{l'}$ just as in Proposition 6.6. It remains to show that there exists $q_< \in \pi_<(F)$ whose orbit is $\mathrm{Aut}(M/A)$-invariant. In some elementary extension of M, define by induction $(a_i)_i$, $(D_i)_i$ such that D_i is the special subgroup generated by $M\vec{a}_{<i}$, and $a_i \in \bigcap_{b\in D_i}]b, +\infty[$. Define $q_< = \mathrm{tp}_<(\vec{a}/M)$. Then, for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ and $b \in M$, whether $\sum_i \lambda_i a_i > b$ depends uniquely on the sign of $(\lambda_i)_i$ in the antilexicographic sum $\mathbb{Z}^n$, and it does not depend on b. Moreover, $(a_i)_i$ is clearly $\mathbb{Q}$-free over M. As a result, $q_<$ is an $\mathrm{Aut}(M)$-invariant (and hence $\mathrm{Aut}(M/A)$-invariant) element of $\pi_<(F)$, and we conclude.

Let us prove (4) $\Rightarrow$ (5). Let $f \in \mathrm{LC}(\mathbb{Z})$ and $N > 0$ be such that $f(\vec{c})$ is in $B + l^N M \setminus A + l^N M$. Let $b \in B$ such that $f(\vec{c}) - b \in l^N M$ (we know that $b \notin A + l^N M$ by hypothesis, and in particular it is $\mathbb{Q}$-free over A as a singleton). Let us show that the formula $\varphi(\vec{x}, b) := \mathfrak{D}_{l^N}(f(\vec{x}) - b)$ divides over A, which implies that (4) fails. We can repeatedly apply Lemma 6.15 to find $(b_i)_{i<\omega}$ such that $b_0 = b$, $b_i \equiv_B b_{i+1}$, and $b_i - b_j \notin l^N M$ for all $j \neq i$. Then the set of formulas $\{\varphi(\vec{x}, b_i) \mid i < \omega\}$ is clearly 2-inconsistent, so we can conclude.

Let us prove (5) $\Rightarrow$ (1). Suppose (5) holds, and let us build an explicit invariant element of Q which extends p_l, witnessing (1). Define

$$F = \{(f, N) \in \mathrm{LC}^n(\mathbb{Z}) \times \omega \mid f(\vec{c}) \in A + l^N M\}.$$

For each $(f, N) \in F$, let $a_{f,N} \in A$ such that $f(\vec{c}) - a_{f,N} \in l^N M$. Define the partial type

$$p(\vec{x}) = \{\mathfrak{D}_{l^N}(f(\vec{x}) - a_{f,N}) \mid (f, N) \in F\} \cup \{\neg\mathfrak{D}_{l^N}(f(\vec{x}) - e) \mid (f, N) \notin F, e \in M\}.$$

Suppose towards a contradiction that p is not consistent with p_l. Then, by compactness, there exists a finite tuple $(f_i, N_i, e_i)_i$ such that, for every i, $(f_i, N_i) \notin F$, and $p_l(\vec{x}) \models \bigvee_i \mathfrak{D}_{l^{N_i}}(f_i(\vec{x}) - e_i)$. By Lemma 6.17 applied to $Y_i = e_i + l^{N_i} M$, there must exist i such that $e_i \in B + l^{N_i} M$. As $(f_i, N_i) \notin F$, we must have $e_i \notin A + l^{N_i} M$, so $e_i \in B + l^{N_i} M \setminus A + l^{N_i} M$. By hypothesis (5), we have $p_l(\vec{x}) \models \neg\mathfrak{D}_{l^{N_i}}(f_i(\vec{x}) - e_i)$, which implies that $p_l(\vec{x}) \models \bigvee_{k\neq i} \mathfrak{D}_{l^{N_k}}(f_k(\vec{x}) - e_k)$. We keep decreasing by induction the size of this disjunction, until we inevitably reach a contradiction. As a result, p must be consistent with p_l, and it is clearly $\mathrm{Aut}(M/A)$-invariant. This partial type is of course complete as an element of Q, because every atomic formula with predicate in $\{\mathfrak{D}_{l^N} \mid N > 0\}$ and parameters in M either lies in p, or its negation lies in p. $\square$

7. Computation of forking

Let us prove Theorem 2 and Theorem 3.

Theorem 7.1. *Let $M \models$ ROAG, let $A, B \subseteq M$, let $\kappa = \max(|AB|, 2^{\aleph_0})^+$, and let $\vec{c} = (c_1, \dots, c_n) \in M^n$. Suppose M is κ-saturated and strongly κ-homogeneous. Let C be the subgroup of M generated by $\vec{c}$, and A', B' the special subgroups generated by A, AB. Then the following conditions are equivalent*:

- $\vec{c} \mathop{\perp\!\!\!\!\perp}^{\mathbf{f}}_A B$.
- $\vec{c} \mathop{\perp\!\!\!\!\perp}^{\mathbf{d}}_A B$.
- $\vec{c} \mathop{\perp\!\!\!\!\perp}^{\mathbf{bo}}_A B$.
- $\operatorname{tp}(\vec{c}/AB)$ *has a global* $\operatorname{Aut}(M^{\mathrm{eq}}/A \cup \operatorname{acl}^{\mathrm{eq}}(\varnothing))$*-invariant extension.*
- *The following conditions hold*:
  1. *Every closed bounded interval of B' that has a point in C already has a point in* $\operatorname{div}(A')$.
  2. *For all primes l, if $[M : lM]$ is infinite, then for all $N > 0$ we have $(C + l^N M) \cap (B' + l^N M) = A' + l^N M$.*

Moreover, $\vec{c} \mathop{\perp\!\!\!\!\perp}^{\mathbf{inv}}_A B$ if and only if the above conditions hold, and, additionally, for every prime l for which $[M : lM]$ is finite, we have $C \subseteq A' + \bigcap_{N>0} l^N M$.

Note that, if $\vec{c} \mathop{\not\!\perp\!\!\!\!\perp}^{\mathbf{d}}_A B$, then $\operatorname{tp}(\vec{c}/AB)$ divides over $\operatorname{acl}^{\mathrm{eq}}(A)$. Thus it does not admit any global $\operatorname{Aut}(M^{\mathrm{eq}}/A \cup \operatorname{acl}^{\mathrm{eq}}(\varnothing))$-invariant extension.

Proof. Firstly, we have $A' = \operatorname{dcl}(A)$ and $B' = \operatorname{dcl}(AB)$ by Corollary 5.11. Thus we have $\vec{c} \mathop{\perp\!\!\!\!\perp}_A B$ if and only if $\vec{c} \mathop{\perp\!\!\!\!\perp}_{A'} B'$ for every $\mathop{\perp\!\!\!\!\perp} \in \{\mathop{\perp\!\!\!\!\perp}^{\mathbf{f}}, \mathop{\perp\!\!\!\!\perp}^{\mathbf{d}}, \mathop{\perp\!\!\!\!\perp}^{\mathbf{bo}}, \mathop{\perp\!\!\!\!\perp}^{\mathbf{inv}}\}$. Secondly, given $\vec{c}' = (c'_1, \dots, c'_m)$ a maximal subtuple of $\vec{c}$ which is $\mathbb{Q}$-free over A', $\vec{c}$ and $\vec{c}'$ are A-interdefinable. Thus one can show that for all of those independence notions we have $\vec{c} \mathop{\perp\!\!\!\!\perp}_{A'} B'$ if and only if $\vec{c}' \mathop{\perp\!\!\!\!\perp}_{A'} B'$. Likewise, $\operatorname{tp}(\vec{c}/AB)$ has an $\operatorname{Aut}(M^{\mathrm{eq}}/A \cup \operatorname{acl}^{\mathrm{eq}}(\varnothing))$-invariant extension if and only if $\operatorname{tp}(\vec{c}'/B')$ has an $\operatorname{Aut}(M^{\mathrm{eq}}/A' \cup \operatorname{acl}^{\mathrm{eq}}(\varnothing))$-invariant extension.

If $\vec{c}'$ was not $\mathbb{Q}$-free over B', then one could show that $\vec{c}' \mathop{\not\!\perp\!\!\!\!\perp}^{\mathbf{d}}_{A'} B'$ (hence $\vec{c}' \mathop{\not\!\perp\!\!\!\!\perp}^{\mathbf{f}}_{A'} B'$, $\vec{c}' \mathop{\not\!\perp\!\!\!\!\perp}^{\mathbf{bo}}_{A'} B'$, and $\operatorname{tp}(\vec{c}/AB)$ does not admit any global $\operatorname{Aut}(M^{\mathrm{eq}}/A \cup \operatorname{acl}^{\mathrm{eq}}(\varnothing))$-invariant extension). Moreover, condition (1) would fail on some singleton of $B \setminus A$ witnessing the fact that $\vec{c}'$ is not $\mathbb{Q}$-free over B.

Suppose $\vec{c}'$ is $\mathbb{Q}$-free over B'. For each $j \in J$, let $p_j = \operatorname{tp}_j(\vec{c}'/B')$, and let F_j be the image by π_j of the space of complete global types with realizations that are $\mathbb{Q}$-free over M.

If condition (1) fails, then $p_<$ divides over A' by Proposition 6.6, and if condition (2) fails for some l, then p_l divides over A' by Proposition 6.18. Either way, we have $\vec{c}' \mathop{\not\!\perp\!\!\!\!\perp}_{A'} B'$ for every $\mathop{\perp\!\!\!\!\perp} \in \{\mathop{\perp\!\!\!\!\perp}^{\mathbf{f}}, \mathop{\perp\!\!\!\!\perp}^{\mathbf{d}}, \mathop{\perp\!\!\!\!\perp}^{\mathbf{bo}}, \mathop{\perp\!\!\!\!\perp}^{\mathbf{inv}}\}$.

Suppose conditions (1) and (2) hold. Then $p_<$ extends to an $\mathrm{Aut}(M/A')$-invariant (thus $\mathrm{Aut}(M^{\mathrm{eq}}/A' \cup \mathrm{acl}^{\mathrm{eq}}(\varnothing))$-invariant) element of $F_<$ by Proposition 6.6, and, for every prime l for which $[M : lM]$ is infinite, p_l extends to some $\mathrm{Aut}(M/A')$-invariant extension of F_l by Proposition 6.18. By Remark 6.11, for each prime l of finite index, any global extension of p_l is automatically $\mathrm{Aut}(M^{\mathrm{eq}}/A' \cup \mathrm{acl}^{\mathrm{eq}}(\varnothing))$-invariant, and thus it follows that $\mathrm{tp}(\vec{c}\,'/A'B')$ has a global $\mathrm{Aut}(M^{\mathrm{eq}}/A' \cup \mathrm{acl}^{\mathrm{eq}}(\varnothing))$-invariant extension. Moreover, by Remark 6.11, we know that p_l extends to some element of F_l of bounded orbit under $\mathrm{Aut}(M/A')$ for every prime l for which $[M : lM]$ is finite. By Corollary 5.12, $\vec{c}\,' \mathop{\smile\!\!\!\!\!|}^{\mathbf{bo}}_{A'} B'$, so $\vec{c}\,' \mathop{\smile\!\!\!\!\!|}^{\mathbf{f}}_{A'} B'$ and $\vec{c}\,' \mathop{\smile\!\!\!\!\!|}^{\mathbf{d}}_{A'} B'$. Moreover, by Corollary 5.12, we have $\vec{c}\,' \mathop{\smile\!\!\!\!\!|}^{\mathbf{inv}}_{A'} B'$ if and only if, for every prime l for which $[M : lM]$ is finite, p_l extends to some $\mathrm{Aut}(M/A')$-invariant element of F_l. Then we can conclude using Proposition 6.12. □

Corollary 7.2. *We have* $\mathop{\smile\!\!\!\!\!|}^{\mathbf{f}} = \mathop{\smile\!\!\!\!\!|}^{\mathbf{Sh}}$ *in* ROAG, *where* $\vec{c} \mathop{\smile\!\!\!\!\!|}^{\mathbf{Sh}}_{A} B$, *Shelah-independence, is equivalent to* $\mathrm{tp}(\vec{c}/AB)$ *having a global extension which is* $\mathrm{Aut}(M^{\mathrm{eq}}/\mathrm{acl}^{\mathrm{eq}}(A))$-*invariant.*

Proof. Shelah-invariance is a weaker condition than being $\mathrm{Aut}(M^{\mathrm{eq}}/A \cup \mathrm{acl}^{\mathrm{eq}}(\varnothing))$-invariant, which is weaker than nonforking in general. The other direction follows from the theorem. □

We know (see Remark 1.5) that in any NIP theory, $\mathop{\smile\!\!\!\!\!|}^{\mathbf{f}}$ coincides with $\mathop{\smile\!\!\!\!\!|}^{\mathbf{inv}}$ over models. We can now refine that result for ROAG:

Corollary 7.3. *Let A' be the special subgroup generated by A. Then* $\mathop{\smile\!\!\!\!\!|}^{\mathbf{f}}_{A} = \mathop{\smile\!\!\!\!\!|}^{\mathbf{inv}}_{A}$ *if and only if* $A' + \bigcap_N l^N M = M$ *for all primes l of finite index.*

Moreover, $\mathop{\smile\!\!\!\!\!|}^{\mathbf{f}} = \mathop{\smile\!\!\!\!\!|}^{\mathbf{inv}}$ *if and only if for each prime l, either M is l-divisible, or the index* $[M : lM]$ *is infinite.*

References

[Chernikov and Kaplan 2012] A. Chernikov and I. Kaplan, "Forking and dividing in NTP_2 theories", *J. Symbolic Logic* **77**:1 (2012), 1–20. MR

[Dolich 2004] A. Dolich, "Forking and independence in o-minimal theories", *J. Symbolic Logic* **69**:1 (2004), 215–240. MR

[Farré 2017] R. Farré, "Strong ordered Abelian groups and dp-rank", preprint, 2017. arXiv 1706.05471

[Gravett 1956] K. A. H. Gravett, "Ordered abelian groups", *Quart. J. Math. Oxford Ser.* (2) **7** (1956), 57–63. MR

[Haskell et al. 2008] D. Haskell, E. Hrushovski, and D. Macpherson, *Stable domination and independence in algebraically closed valued fields*, Lecture Notes in Logic **30**, Cambridge Univ. Press, 2008. MR

[Hils and Mennuni 2024] M. Hils and R. Mennuni, "The domination monoid in henselian valued fields", *Pacific J. Math.* **328**:2 (2024), 287–323. MR

[Hossain 2024] A. Hossain, "Forking in valued fields and related structures", preprint, 2024. arXiv 2409.16393

[Hrushovski and Pillay 2011] E. Hrushovski and A. Pillay, "On NIP and invariant measures", *J. Eur. Math. Soc.* (*JEMS*) **13**:4 (2011), 1005–1061. MR

[Mennuni 2022] R. Mennuni, "The domination monoid in o-minimal theories", *J. Math. Log.* **22**:1 (2022), art. id. 2150030. MR

[Robinson and Zakon 1960] A. Robinson and E. Zakon, "Elementary properties of ordered abelian groups", *Trans. Amer. Math. Soc.* **96** (1960), 222–236. MR

[Simon 2011] P. Simon, "On dp-minimal ordered structures", *J. Symbolic Logic* **76**:2 (2011), 448–460. MR

[Simon 2013] P. Simon, "Distal and non-distal NIP theories", *Ann. Pure Appl. Logic* **164**:3 (2013), 294–318. MR

[Starchenko 2008] S. Starchenko, "A note on Dolich's paper", preprint, 2008, available at https://modnet.imj-prg.fr/preprints/papers/archive/131.pdf.

[Tent and Ziegler 2012] K. Tent and M. Ziegler, *A course in model theory*, Lecture Notes in Logic **40**, Cambridge Univ. Press, 2012. MR

[Weispfenning 1981] V. Weispfenning, "Elimination of quantifiers for certain ordered and lattice-ordered abelian groups", *Bull. Soc. Math. Belg. Sér. B* **33**:1 (1981), 131–155. MR

[Zakon 1961] E. Zakon, "Generalized archimedean groups", *Trans. Amer. Math. Soc.* **99** (1961), 21–40. MR

Received 8 Nov 2024. Revised 17 Jul 2025.

AKASH HOSSAIN:

akashoss42@gmail.com
Institut de Mathématiques d'Orsay, Faculté des Sciences d'Orsay, Université Paris-Saclay, Orsay, France
Current address: CEA-LIST, Site Nano-INNOV, Palaiseau, France

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the submission page.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles are usually in English or French, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not refer to bibliography keys. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and a Mathematics Subject Classification for the article, and, for each author, affiliation (if appropriate) and email address.

Format. Authors are encouraged to use LaTeX and the standard amsart class, but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should normally be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages — Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc. — allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with as many details as you can about how your graphics were generated.

Bundle your figure files into a single archive (using zip, tar, rar or other format of your choice) and upload on the link you been provided at acceptance time. Each figure should be captioned and numbered so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript with all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables.

White Space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Model Theory

no. 3 vol. 4 2025