

# ANTS X Proceedings of the Tenth Algorithmic Number Theory Symposium

Deterministic elliptic curve primality proving  
for a special sequence of numbers

Alexander Abatzoglou, Alice Silverberg,  
Andrew V. Sutherland, and Angela Wong





# Deterministic elliptic curve primality proving for a special sequence of numbers

Alexander Abatzoglou, Alice Silverberg,  
Andrew V. Sutherland, and Angela Wong

We give a deterministic algorithm that very quickly proves the primality or compositeness of the integers  $N$  in a certain sequence, using an elliptic curve  $E/\mathbb{Q}$  with complex multiplication by the ring of integers of  $\mathbb{Q}(\sqrt{-7})$ . The algorithm uses  $O(\log N)$  arithmetic operations in the ring  $\mathbb{Z}/N\mathbb{Z}$ , implying a bit complexity that is quasiquadratic in  $\log N$ . Notably, neither of the classical “ $N - 1$ ” or “ $N + 1$ ” primality tests apply to the integers in our sequence. We discuss how this algorithm may be applied, in combination with sieving techniques, to efficiently search for very large primes. This has allowed us to prove the primality of several integers with more than 100,000 decimal digits, the largest of which has more than a million bits in its binary representation. At the time it was found, it was the largest proven prime  $N$  for which no significant partial factorization of  $N - 1$  or  $N + 1$  is known (as of final submission it was second largest).

## 1. Introduction

With the celebrated result of Agrawal, Kayal, and Saxena [3], one can now unequivocally determine the primality or compositeness of any integer in deterministic polynomial time. With the improvements of Lenstra and Pomerance [27], the AKS algorithm runs in  $\tilde{O}(n^6)$  time, where  $n$  is the size of the integer to be tested (in bits). However, it has long been known that for certain special sequences of integers, one can do much better. The two most famous examples are the Fermat numbers  $F_k = 2^{2^k} + 1$ , to which one may apply Pépin’s criterion [35], and the Mersenne numbers  $M_p = 2^p - 1$ , which are subject to the Lucas-Lehmer test [24]. In both cases, the corresponding algorithms are deterministic and run in  $\tilde{O}(n^2)$  time.

---

*MSC2010:* primary 11Y11; secondary 11G05, 14K22.

*Keywords:* primality, elliptic curves, complex multiplication.

In fact, every prime admits a proof of its primality that can be verified by a deterministic algorithm in  $\tilde{O}(n^2)$  time. Pomerance shows in [36] that for every prime  $p > 31$  there exists an elliptic curve  $E/\mathbb{F}_p$  with an  $\mathbb{F}_p$ -rational point  $P$  of order  $2^r > (p^{1/4} + 1)^2$ , which allows one to establish the primality of  $p$  using just  $r$  elliptic curve group operations. Elliptic curves play a key role in Pomerance's proof; the best analogous result using classical primality certificates yields an  $\tilde{O}(n^3)$  time bound (see [38], and compare [9, Theorem 4.1.9]).

The difficulty in applying Pomerance's result lies in finding the pair  $(E, P)$ , a task for which no efficient method is currently known. Rather than searching for suitable pairs  $(E, P)$ , we instead fix a finite set of curves  $E_a/\mathbb{Q}$ , each equipped with a known rational point  $P_a$  of infinite order. To each positive integer  $k$  we associate one of the curves  $E_a$  and define an integer  $J_k$  for which we give a necessary and sufficient condition for primality:  $J_k$  is prime if and only if the reduction of  $P_a$  in  $E_a(\mathbb{F}_p)$  has order  $2^{k+1}$  for every prime  $p$  dividing  $J_k$ . Of course  $p = J_k$  when  $J_k$  is prime, but this condition can easily be checked without knowing the prime factorization of  $J_k$ . This yields a deterministic algorithm that runs in  $\tilde{O}(n^2)$  time (see Algorithm 5.1).

Our results extend the methods used by Gross [20], Denomme and Savin [11], Tsumura [44], and Gurevich and Kunyavskiĭ [22], all of which fit within a general framework laid out by Chudnovsky and Chudnovsky in [8] for determining the primality of integers in special sequences using elliptic curves with complex multiplication (CM). The elliptic curves that we use lie in the family of quadratic twists defined by the equations

$$E_a : y^2 = x^3 - 35a^2x - 98a^3, \quad (1)$$

for squarefree integers  $a$  such that  $E_a(\mathbb{Q})$  has positive rank. Each curve has good reduction outside of 2, 7, and the prime divisors of  $a$ , and has CM by  $\mathbb{Z}[\alpha]$ , where

$$\alpha = \frac{1 + \sqrt{-7}}{2}.$$

For each curve  $E_a$ , we fix a point  $P_a \in E_a(\mathbb{Q})$  of infinite order with  $P_a \notin 2E_a(\mathbb{Q})$ .

For each positive integer  $k$ , let

$$\begin{aligned} j_k &= 1 + 2\alpha^k \in \mathbb{Z}[\alpha], \\ J_k &= j_k \bar{j}_k = 1 + 2(\alpha^k + \bar{\alpha}^k) + 2^{k+2} \in \mathbb{N}. \end{aligned}$$

The integer sequence  $J_k$  satisfies the linear recurrence relation

$$J_{k+4} = 4J_{k+3} - 7J_{k+2} + 8J_{k+1} - 4J_k,$$

with initial values  $J_1 = J_2 = 11$ ,  $J_3 = 23$ , and  $J_4 = 67$ . Then (by Lemma 4.5)  $J_k$  is composite for  $k \equiv 0 \pmod{8}$  and for  $k \equiv 6 \pmod{24}$ . To each other value of

$k$  we assign a squarefree integer  $a$ , based on the congruence class of  $k \pmod{72}$ , as listed in Table 1. Our choice of  $a$  is based on two criteria. First, it ensures that when  $J_k$  is prime, the Frobenius endomorphism of  $E_a \bmod J_k$  corresponds to complex multiplication by  $j_k$  (rather than  $-j_k$ ) and

$$E_a(\mathbb{Z}/J_k\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{k+1}\mathbb{Z}.$$

Second, it implies that when  $J_k$  is prime, the reduction of the point  $P_a$  has order  $2^{k+1}$  in  $E(\mathbb{Z}/J_k\mathbb{Z})$ . The second condition is actually stronger than necessary (in general, one only needs  $P_a$  to have order greater than  $2^{k/2+1}$ ), but it simplifies matters. Note that choosing a sequence of the form  $j_k = 1 + \Lambda_k$  means that  $E_a(\mathbb{Z}[\alpha]/(j_k)) \simeq \mathbb{Z}[\alpha]/\Lambda_k$ , whenever  $J_k$  is prime and  $j_k$  is the Frobenius endomorphism of  $E_a \bmod J_k$  (see Lemma 4.6).

We prove in Theorem 4.1 that the integer  $J_k$  is prime if and only if the point  $P_a$  has order  $2^{k+1}$  on “ $E_a \bmod J_k$ ”. More precisely, we prove that if one applies the standard formulas for the elliptic curve group law to compute scalar multiples  $Q_i = 2^i P_a$  using projective coordinates  $Q_i = [x_i, y_i, z_i]$  in the ring  $\mathbb{Z}/J_k\mathbb{Z}$ , then  $J_k$  is prime if and only if  $\gcd(J_k, z_k) = 1$  and  $z_{k+1} = 0$ . This allows us to determine whether  $J_k$  is prime or composite using  $O(k)$  operations in the ring  $\mathbb{Z}/J_k\mathbb{Z}$ , yielding a bit complexity of  $O(k^2 \log k \log \log k) = \tilde{O}(k^2)$  (see Proposition 5.2 for a more precise bound).

We note that, unlike the Fermat numbers, the Mersenne numbers, and many similar numbers of a special form, the integers  $J_k$  are not amenable to any of the classical “ $N - 1$ ” or “ $N + 1$ ” type primality tests (or combined tests) that are typically used to find very large primes (indeed, the 500 largest primes currently listed in [7] all have the shape  $ab^n \pm 1$  for some small integers  $a$  and  $b$ ).

In combination with a sieving approach described in Section 5, we have used our algorithm to determine the primality of  $J_k$  for all  $k \leq 1.2 \times 10^6$ . The prime values of  $J_k$  are listed in Table 4. At the time it was found, the prime  $J_{1,111,930}$ , which has 334,725 decimal digits, was the largest proven prime  $N$  for which no significant partial factorization of either  $N - 1$  or  $N + 1$  was known [1]. On July 4, 2012 it was superseded by a 377,922 digit prime found by David Broadhurst [6] for which no significant factorization of  $N - 1$  or  $N + 1$  is known; Broadhurst constructed an ECPP primality proof for this prime, but it is not a Pomerance proof.

Generalizations have been suggested to the settings of higher-dimensional abelian varieties with complex multiplication, algebraic tori, and group schemes by Chudnovsky and Chudnovsky [8], Gross [20], and Gurevich and Kunyavskiĭ [21], respectively. In the PhD theses of the first and fourth authors, and in a forthcoming paper, we are extending the results in this paper to a more general framework. In that paper we will also explain why, when restricting to elliptic curves over  $\mathbb{Q}$ , this method requires curves with CM by  $\mathbb{Q}(\sqrt{-D})$  with  $D = 1, 2, 3$ , or  $7$ .

## 2. Relation to prior work

In [8], Chudnovsky and Chudnovsky consider certain sequences of integers  $s_k = \text{Norm}_{K/\mathbb{Q}}(1 + \alpha_0 \alpha_1^k)$ , defined by algebraic integers  $\alpha_0$  and  $\alpha_1$  in an imaginary quadratic field  $K = \mathbb{Q}(\sqrt{D})$ . They give sufficient conditions for the primality of  $s_k$ , using an elliptic curve  $E$  with CM by  $K$ . In our setting,  $D = -7$ ,  $\alpha_0 = 2$ ,  $\alpha_1 = (1 + \sqrt{-7})/2$ , and  $J_k = s_k$ . The key difference here is that we give necessary and sufficient criteria for primality that can be efficiently checked by a deterministic algorithm. This is achieved by carefully selecting the curves  $E_a/\mathbb{Q}$  that we use, so that in each case we are able to prove that the point  $P_a \in E_a(\mathbb{Q})$  reduces to a point of maximal order  $2^{k+1}$  on  $E_a \bmod J_k$ , whenever  $J_k$  is prime. Without such a construction, we know of no way to obtain *any* nontrivial point on  $E \bmod s_k$  in deterministic polynomial time.

Our work is a direct extension of the techniques developed by Gross [20; 45], Denomme and Savin [11], Tsumura [44], and Gurevich and Kunyavskii [22], who use elliptic curves with CM by the ring of integers of  $\mathbb{Q}(i)$  or  $\mathbb{Q}(\sqrt{-3})$  to test the primality of Mersenne, Fermat, and related numbers. However, as noted by Pomerance [37, §4], the integers considered in [11] can be proved prime using classical methods that are more efficient and do not involve elliptic curves, and the same applies to [20; 44; 45; 22]. But this is not the case for the sequence we consider here.

## 3. Background and notation

**3A. Elliptic curve primality proving.** Primality proving algorithms based on elliptic curves have been proposed since the mid-1980s. Bosma [5] and Chudnovsky and Chudnovsky [8] considered a setting similar to the one employed here, using elliptic curves to prove the primality of numbers of a special form; Bosma proposed the use of elliptic curves with complex multiplication by  $\mathbb{Q}(i)$  or  $\mathbb{Q}(\sqrt{-3})$ , while Chudnovsky and Chudnovsky considered a wider range of elliptic curves and other algebraic varieties. Goldwasser and Kilian [16; 17] gave the first general purpose elliptic curve primality proving algorithm, using randomly generated elliptic curves. Atkin and Morain [4; 32] developed an improved version of the Goldwasser-Kilian algorithm that uses the CM method to construct the elliptic curves used, rather than generating them at random (it does rely on probabilistic methods for root-finding). With asymptotic improvements due to Shallit, the Atkin-Morain algorithm has a heuristic expected running time of  $\tilde{O}(n^4)$ , which makes it the method of choice for general purpose primality proving [33]. Gordon [18] proposed a general purpose compositeness test using supersingular reductions of CM elliptic curves over  $\mathbb{Q}$ .

Throughout this paper, if  $E \subset \mathbb{P}^2$  is an elliptic curve over  $\mathbb{Q}$ , we shall write points  $[x, y, z] \in E(\mathbb{Q})$  so that  $x, y, z \in \mathbb{Z}$  and  $\gcd(x, y, z) = 1$ , and we may use

$(x, y)$  to denote the projective point  $[x, y, 1]$ .

We say that a point  $P = [x, y, z] \in E(\mathbb{Q})$  is *zero mod  $N$*  when  $N$  divides  $z$ ; otherwise  $P$  is *nonzero mod  $N$* . Note that if  $P$  is zero mod  $N$  then  $P$  is zero mod  $p$  for all primes  $p$  dividing  $N$ .

**Definition 3.1.** Given an elliptic curve  $E$  over  $\mathbb{Q}$ , a point  $P = [x, y, z] \in E(\mathbb{Q})$ , and  $N \in \mathbb{Z}$ , we say that  $P$  is *strongly nonzero mod  $N$*  if  $\gcd(z, N) = 1$ .

If  $P$  is strongly nonzero mod  $N$ , then  $P$  is nonzero mod  $p$  for every prime  $p$  dividing  $N$ , and if  $N$  is prime, then  $P$  is strongly nonzero mod  $N$  if and only if  $P$  is nonzero mod  $N$ .

We rely on this fundamental result, which can be found in [16; 26; 17]:

**Proposition 3.2.** *Let  $E/\mathbb{Q}$  be an elliptic curve, let  $N$  be a positive integer prime to  $\text{disc}(E)$ , let  $P \in E(\mathbb{Q})$ , and let  $m > (N^{1/4} + 1)^2$ . Suppose  $mP$  is zero mod  $N$  and  $(m/q)P$  is strongly nonzero mod  $N$  for all primes  $q \mid m$ . Then  $N$  is prime.*

To make practical use of Proposition 3.2, one needs to know the prime factorization of  $m$ . For general elliptic curve primality proving this presents a challenge; the algorithms of Goldwasser-Kilian and Atkin-Morain use different approaches to ensure that  $m$  has an easy factorization, but both must then recursively construct primality proofs for the primes  $q$  dividing  $m$ . In our restricted setting we effectively fix the prime factorization of  $m = 2^{k+1}$  ahead of time.

Next we give a variant of Proposition 3.2 that replaces “strongly nonzero” with “nonzero”, at the expense of  $m$  being a prime power with a larger lower bound.

**Proposition 3.3.** *Let  $E/\mathbb{Q}$  be an elliptic curve, let  $p$  be a prime, let  $N$  be an odd positive integer prime to  $p \text{ disc}(E)$ , and let  $P \in E(\mathbb{Q})$ . Suppose  $b$  is a positive integer such that  $p^b > (\sqrt{N/3} + 1)^2$  and  $p^b P$  is zero mod  $N$  and  $p^{b-1} P$  is nonzero mod  $N$ . Then  $N$  is prime.*

*Proof.* Since  $p^{b-1} P$  is nonzero mod  $N$ , there are a prime divisor  $q$  of  $N$  and a positive integer  $r$  such that  $q^r$  exactly divides  $N$  and  $p^{b-1} P$  is nonzero mod  $q^r$ . Let  $E_1(\mathbb{Z}/q^r\mathbb{Z})$  denote the kernel of the reduction map  $E(\mathbb{Z}/q^r\mathbb{Z}) \rightarrow E(\mathbb{F}_q)$ . It follows, for example, from [29, Theorem 4.1] that  $E_1(\mathbb{Z}/q^r\mathbb{Z})$  is a  $q$ -group. Let  $P' \in E(\mathbb{Z}/q^r\mathbb{Z})$  be the reduction of  $P$  mod  $q^r$  and let  $P''$  be the image of  $P'$  in  $E(\mathbb{F}_q)$ . If  $p^{b-1} P'' = 0$  then  $p^{b-1} P' \in E_1(\mathbb{Z}/q^r\mathbb{Z})$ , so  $p^{b-1} P'$  has order a power of  $q$ . But by assumption it has order  $p$ , which is prime to  $N$ . This is a contradiction, so  $P''$  has order  $p^b$ . If  $N$  were composite, then  $q \leq N/3$  since  $N$  is odd, so by the Hasse bound,

$$p^b \leq |E(\mathbb{F}_q)| \leq (\sqrt{q} + 1)^2 \leq (\sqrt{N/3} + 1)^2,$$

contradicting the hypothesis that  $p^b > (\sqrt{N/3} + 1)^2$ . □

**3B. Complex multiplication and Frobenius endomorphism.** For any number field  $F$ , let  $\mathbb{O}_F$  denote its ring of integers. If  $E$  is an elliptic curve over a field  $K$ , and  $\Omega_K$  is the space of holomorphic differentials on  $E$  over  $K$ , then  $\Omega_K$  is a one-dimensional  $K$ -vector space, and there is a canonical ring homomorphism

$$\text{End}_K(E) \rightarrow \text{End}_K(\Omega) = K. \quad (2)$$

Suppose now that  $E$  is an elliptic curve over an imaginary quadratic field  $K$ , and that  $E$  has complex multiplication (CM) by  $\mathbb{O}_K$ , meaning that  $\text{End}_K(E) \simeq \mathbb{O}_K$ . Then the image of the map in (2) is  $\mathbb{O}_K$ . Let  $\psi : \mathbb{O}_K \rightarrow \text{End}_K(E)$  denote the inverse map. Suppose that  $\mathfrak{p}$  is a prime ideal of  $K$  at which  $E$  has good reduction and let  $\tilde{E}$  denote the reduction of  $E \bmod \mathfrak{p}$ . Then the composition

$$\mathbb{O}_K \xrightarrow{\psi} \text{End}_K(E) \hookrightarrow \text{End}_{\mathbb{O}_K/\mathfrak{p}}(\tilde{E}),$$

where the first map is  $\psi$  and the second is induced by reduction mod  $\mathfrak{p}$ , gives a canonical embedding

$$\mathbb{O}_K \hookrightarrow \text{End}(\tilde{E}). \quad (3)$$

The Frobenius endomorphism of  $\tilde{E}$  is  $(x, y) \mapsto (x^q, y^q)$  where  $q = \text{Norm}_{K/\mathbb{Q}}(\mathfrak{p})$ ; under the embedding in (3), the Frobenius endomorphism is the image of a particular generator  $\pi$  of the (principal) ideal  $\mathfrak{p}$ . By abuse of notation, we say that the Frobenius endomorphism is  $\pi$ .

#### 4. Main theorem

In this section we state and prove our main result, Theorem 4.1, which gives a necessary and sufficient condition for the primality of the numbers  $J_k$ .

Fix a particular square root of  $-7$  and let  $K = \mathbb{Q}(\sqrt{-7})$ . Let

$$\alpha = \frac{1 + \sqrt{-7}}{2} \in \mathbb{O}_K,$$

and for each positive integer  $k$ , let

$$j_k = 1 + 2\alpha^k \in \mathbb{Z}[\alpha] \quad \text{and} \quad J_k = \text{Norm}_{K/\mathbb{Q}}(j_k) = j_k \bar{j}_k \in \mathbb{N}.$$

Note that  $J_k$  is prime in  $\mathbb{Z}$  if and only if  $j_k$  is prime in  $\mathbb{O}_K$ . Note also that  $\text{Norm}_{K/\mathbb{Q}}(\alpha) = \alpha \bar{\alpha} = 2$ .

Recall the family of elliptic curves  $E_a$  defined by (1). Lemma 4.5 below shows that  $J_k$  is composite if  $k \equiv 0 \pmod{8}$  or  $k \equiv 6 \pmod{24}$ , so we omit these cases from our primality criterion. For each remaining value of  $k$ , Table 1 lists the twisting parameter  $a$  and the point  $P_a \in E_a(\mathbb{Q})$  we associate to  $k$ . For each of these  $a$ , the elliptic curve  $E_a$  has rank one over  $\mathbb{Q}$ , and the point  $P_a$  is a generator for  $E_a(\mathbb{Q})$  modulo torsion.

$k$	$a$	$P_a$
$k \equiv 0 \text{ or } 2 \pmod{3}$	$-1$	$(1, 8)$
$k \equiv 4, 7, 13, 22 \pmod{24}$	$-5$	$(15, 50)$
$k \equiv 10 \pmod{24}$	$-6$	$(21, 63)$
$k \equiv 1, 19, 49, 67 \pmod{72}$	$-17$	$(81, 440)$
$k \equiv 25, 43 \pmod{72}$	$-111$	$(-633, 12384)$

**Table 1.** The twisting parameters  $a$  and points  $P_a$ .

**Theorem 4.1.** Fix  $k > 1$  such that  $k \not\equiv 0 \pmod{8}$  and  $k \not\equiv 6 \pmod{24}$ . Let  $P_a \in E_a(\mathbb{Q})$  be as in Table 1 (depending on  $k$ ). The following are equivalent:

- (i)  $2^{k+1}P_a$  is zero mod  $J_k$  and  $2^kP_a$  is strongly nonzero mod  $J_k$ ;
- (ii)  $J_k$  is prime.

**Remark 4.2.** Applying Proposition 3.3 with  $N = J_k$ ,  $p = 2$ , and  $b = k + 1$ , we can add an equivalent condition in Theorem 4.1 as long as  $k \geq 6$ , namely:

- (iii)  $2^{k+1}P_a$  is zero mod  $J_k$  and  $2^kP_a$  is nonzero mod  $J_k$ .

We shall prove Theorem 4.1 via a series of lemmas, but let us first outline the proof. One direction is easy: Since  $2^{k+1} > (J_k^{1/4} + 1)^2$  for all  $k > 1$ , if (i) holds then so does (ii), by Proposition 3.2 (where the hypothesis  $\gcd(J_k, \text{disc}(E_a)) = 1$  holds by Lemma 4.5 below).

Now fix  $a$  and  $P_a$  as in Table 1, and let  $\tilde{P}_a$  denote the reduction of  $P_a$  modulo  $j_k$ . We first compute a set  $S_a$  such that if  $k \in S_a$  and  $j_k$  is prime, then  $E_a(\mathbb{O}_K/(j_k)) \simeq \mathbb{O}_K/(2\alpha^k)$  as  $\mathbb{O}_K$ -modules. We then compute a set  $T_a$  such that if  $k \in T_a$  and  $j_k$  is prime, then  $\tilde{P}_a$  does not lie in  $\alpha E_a(\mathbb{O}_K/(j_k))$  if and only if  $k \in T_a$  (note that  $\alpha \in \mathbb{O}_K \hookrightarrow \text{End}(E_a)$ ). For  $k \in S_a \cap T_a$ , the point  $\tilde{P}_a$  has order  $2^{k+1}$  whenever  $J_k$  is prime.

We now fill in the details. Many of the explicit calculations below were performed with the assistance of the Sage computer algebra system [43].

**4A. The linear recurrence sequence  $J_k$ .** As noted in the introduction, the sequence  $J_k$  satisfies the linear recurrence relation

$$J_{k+4} = 4J_{k+3} - 7J_{k+2} + 8J_{k+1} - 4J_k. \quad (4)$$

We now prove this, and also note some periodic properties of this sequence. See [12] or [28, Chapter 6] for basic properties of linear recurrence sequences.

**Definition 4.3.** We call a sequence  $a_k$  (purely) periodic if there exists an integer  $m$  such that  $a_k = a_{k+m}$  for all  $k$ . The minimal such  $m$  is the *period* of the sequence.

**Lemma 4.4.** *The sequence  $J_k$  satisfies (4). If  $p$  is an odd prime and  $\mathfrak{p} \subset \mathbb{O}_K$  is a prime ideal above  $(p)$ , then the sequence  $J_k \bmod p$  is periodic, with period equal to the least common multiple of the orders of 2 and  $\alpha$  in  $(\mathbb{O}_K/\mathfrak{p})^*$ .*

*Proof.* The characteristic polynomial of the linear recurrence in (4) is

$$f(x) = x^4 - 4x^3 + 7x^2 - 8x + 4 = (x-1)(x-2)(x^2 - x + 2),$$

whose roots are 1, 2,  $\alpha$ , and  $\bar{\alpha}$ . It follows that the sequences  $1^k$ ,  $2^k$ ,  $\alpha^k$ , and  $\bar{\alpha}^k$ , and any linear combination of these sequences, satisfy (4). Thus  $J_k$  satisfies (4).

One easily checks that the lemma is true for  $p = 7$ , so assume  $p \neq 7$ . Let  $A$  be the  $4 \times 4$  matrix with  $A_{i,j} = J_{i+j-1}$ . Then  $\det A = -2^{12} \cdot 7$  is nonzero mod  $p$ , hence its rows are linearly independent over  $\mathbb{F}_p$ . It follows from Theorems 6.19 and 6.27 of [28] that the sequence  $J_k \bmod p$  is periodic, with period equal to the lcm of the orders of the roots of  $f$  in  $\overline{\mathbb{F}}_p^*$  (which we note are distinct). These roots all lie in  $\mathbb{O}_K/\mathfrak{p} \simeq \mathbb{F}_{p^d}$ , where  $d \in \{1, 2\}$  is the residue degree of  $\mathfrak{p}$ . Since  $\bar{\alpha} = 2/\alpha$ , the order of  $\bar{\alpha}$  in  $(\mathbb{O}_K/\mathfrak{p})^*$  divides the lcm of the orders of 2 and  $\alpha$ . The lemma follows.  $\square$

When  $p$  is an odd prime, let  $m_p$  denote the period of the sequence  $J_k \bmod p$ . Lemma 4.4 implies that  $m_p$  always divides  $p^2 - 1$ , and it divides  $p - 1$  whenever  $p$  splits in  $K$ .

**Lemma 4.5.**

- (i)  $J_k$  is divisible by 3 if and only if  $k \equiv 0 \pmod{8}$ .
- (ii)  $J_k$  is divisible by 5 if and only if  $k \equiv 6 \pmod{24}$ .
- (iii)  $J_k \equiv 2 \pmod{7}$  if  $k \equiv 0 \pmod{3}$ , and  $J_k \equiv 4 \pmod{7}$  otherwise.
- (iv) For  $k > 1$ , we have  $J_k \equiv 3 \pmod{8}$  if  $k$  is even, and  $J_k \equiv 7 \pmod{8}$  if  $k$  is odd.
- (v)  $J_k$  is divisible by 17 if and only if  $k \equiv 54 \pmod{144}$ .
- (vi)  $J_k$  is not divisible by 37.

*Proof.* Lemma 4.4 allows us to compute the periods  $m_3 = 8$ ,  $m_5 = 24$ ,  $m_7 = 3$ ,  $m_{17} = 144$ , and  $m_{37} = 36$ . It then suffices to check, for  $p = 3, 5, 17$ , and 37, when  $J_k \equiv 0 \pmod{p}$  for  $1 \leq k \leq m_p$ , and to determine the values of  $J_k \pmod{7}$  for  $1 \leq k \leq 3$ .

It is easy to check that  $\alpha^k + \bar{\alpha}^k \equiv 3 \pmod{4}$  for odd  $k > 1$ , and  $\alpha^k + \bar{\alpha}^k \equiv 1 \pmod{4}$  otherwise. Since  $J_k = 1 + 2(\alpha^k + \bar{\alpha}^k) + 2^{k+2}$ , we have (iv).

As an alternative proof for one direction of (i) and (ii), note that  $\alpha$  and  $\bar{\alpha}$  each has order 8 in  $(\mathbb{O}_K/(3))^\times$ . Hence if  $k \equiv 0 \pmod{8}$ , then  $J_k = 1 + 2(\alpha^k + \bar{\alpha}^k) + 2^{k+2} \equiv 1 + 2(1 + 1) + 1 \equiv 0 \pmod{3}$ . Similarly,  $\alpha^6 \equiv 2 \equiv \bar{\alpha}^6 \pmod{5}$ , so  $J_k \equiv 1 + 2(4) + 1 \equiv 0 \pmod{5}$  when  $k \equiv 6 \pmod{24}$ .  $\square$

**4B. The set  $S_a$ .** For each squarefree integer  $a$  we define the set of integers

$$S_a := \left\{ k > 1 : \left( \frac{a}{J_k} \right) \left( \frac{j_k}{\sqrt{-7}} \right) = 1 \right\},$$

where  $(-)$  denotes the (generalized) Jacobi symbol.

If  $j_k$  is prime in  $\mathbb{O}_K$ , then the Frobenius endomorphism of  $E_a$  over the finite field  $\mathbb{O}_K/(j_k)$  corresponds to either  $j_k$  or  $-j_k$ . For elliptic curves over  $\mathbb{Q}$  with complex multiplication, one can easily determine which is the case.

**Lemma 4.6.** *Suppose  $a$  is a squarefree integer,  $k > 1$ , and  $j_k$  is prime in  $\mathbb{O}_K$ . Then:*

- (i)  $k \in S_a$  if and only if the Frobenius endomorphism of  $E_a$  over the finite field  $\mathbb{O}_K/(j_k)$  is  $j_k$ ;
- (ii) if  $k \in S_a$ , then  $E_a(\mathbb{O}_K/(j_k)) \simeq \mathbb{O}_K/(2\alpha^k)$  as  $\mathbb{O}_K$ -modules.

*Proof.* The elliptic curve  $E_a$  is the curve in Theorem 1 of [42, p. 1117], with  $D = -7$  and  $\pi = j_k$ . By [42, p. 1135], the Frobenius endomorphism of  $E_a$  over  $\mathbb{O}_K/(j_k)$  is

$$\left( \frac{a}{J_k} \right) \left( \frac{j_k}{\sqrt{-7}} \right) j_k \in \mathbb{O}_K.$$

Part (i) then follows from the definition of  $S_a$ . For (ii), note that (i) implies that if  $k \in S_a$ , then

$$E_a(\mathbb{O}_K/(j_k)) \simeq \ker(j_k - 1) = \ker(2\alpha^k) \simeq \mathbb{O}_K/(2\alpha^k),$$

which completes the proof.  $\square$

The next lemma follows directly from Lemma 4.5(iv).

**Lemma 4.7.** *Let  $k > 1$ .*

$$(i) \quad \left( \frac{-1}{J_k} \right) = -1. \quad (ii) \quad \left( \frac{2}{J_k} \right) = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ -1 & \text{if } k \text{ is even.} \end{cases}$$

We now explicitly compute the sets  $S_a$  for the values of  $a$  used in Theorem 4.1.

**Lemma 4.8.** *For  $a \in \{-1, -5, -6, -17, -111\}$  the sets  $S_a$  are as in Table 2.*

*Proof.* Since  $j_k = 1 + 2\alpha^k$ , and  $\alpha \equiv 4 \pmod{\sqrt{-7}}$ , and  $2^3 \equiv 1 \pmod{7}$ , we have

$$\left( \frac{j_k}{\sqrt{-7}} \right) = \left( \frac{1 + 2^{2k+1}}{7} \right) = \begin{cases} 1 & \text{if } k \equiv 1 \pmod{3}, \\ -1 & \text{if } k \equiv 0, 2 \pmod{3}. \end{cases}$$

We now need to compute  $\left( \frac{a}{J_k} \right)$  for  $a = -1, -5, -6, -17$ , and  $-111$ . The case  $a = -1$  is given by Lemma 4.7(i). As in the proof of Lemma 4.5, applying Lemma 4.4 to the odd primes  $p = 3, 5, 17, 37$  that can divide  $a$ , we found that

$a$	$m$	$S_a = \{k > 1 : k \bmod m \text{ is as below}\}$
-1	3	0, 2
-5	24	0, 2, 4, 5, 7, 9, 12, 13, 16, 18, 21, 22, 23
-6	24	3, 7, 9, 10, 11, 12, 13, 17, 20, 22
-17	144	0, 1, 5, 7, 9, 10, 13, 14, 15, 18, 19, 20, 22, 23, 27, 30, 31, 33, 34, 36, 42, 43, 44, 45, 49, 50, 53, 56, 61, 62, 63, 66, 67, 68, 70, 71, 72, 73, 75, 76, 78, 79, 80, 81, 82, 83, 90, 91, 92, 93, 97, 99, 100, 104, 106, 108, 110, 111, 112, 114, 117, 118, 121, 122, 123, 125, 126, 128, 129, 133, 135, 136, 137, 138, 139, 141, 143
-111	72	2, 4, 6, 9, 14, 15, 18, 20, 22, 23, 25, 30, 33, 34, 35, 37, 38, 39, 41, 42, 43, 47, 49, 50, 52, 53, 54, 55, 57, 58, 63, 65, 66, 67, 68, 70

**Table 2.** The sets  $S_a$ .

the periods  $m_p$  of the sequences  $J_k \bmod p$  are  $m_3 = 8$ ,  $m_5 = 24$ ,  $m_{17} = 144$ , and  $m_{37} = 36$ . Since  $\left(\frac{-1}{J_k}\right) = -1$ , it follows from quadratic reciprocity that for  $a = -5, -17$ , and  $-111$ , the period of the sequence  $\left(\frac{a}{J_k}\right)$  divides the least common multiple of the periods  $m_p$  for  $p$  dividing  $a$ . For  $a = -6$ , by Lemma 4.7(ii) the period of  $\left(\frac{2}{J_k}\right)$  is 2, which already divides  $m_3 = 8$ . Since the period of the sequence  $\left(\frac{j_k}{\sqrt{-7}}\right)$  is 3, we find the period  $m$  of  $\left(\frac{a}{J_k}\right)\left(\frac{j_k}{\sqrt{-7}}\right)$  listed in Table 2 by taking the least common multiple of 3 and the  $m_p$  for  $p$  dividing  $a$ . To compute  $S_a$ , it then suffices to compute  $\left(\frac{a}{J_k}\right)$  and check when  $\left(\frac{a}{J_k}\right) = \left(\frac{j_k}{\sqrt{-7}}\right)$ , for  $1 < k \leq m + 1$ .  $\square$

**4C. The set  $T_a$ .** We now define the sets  $T_a$ .

**Definition 4.9.** Let  $a$  be a squarefree integer, and suppose that  $P \in E_a(K)$ . Then the field  $K(\alpha^{-1}(P))$  has degree 1 or 2 over  $K$ , so it can be written in the form  $K(\sqrt{\delta_P})$  with  $\delta_P \in K$ . Let

$$T_P := \left\{ k > 1 : \left( \frac{\delta_P}{j_k} \right) = -1 \right\}.$$

For the values of  $a$  listed in Table 1, let  $T_a = T_{P_a}$  and let  $\delta_a = \delta_{P_a}$ .

**Lemma 4.10.** Suppose that  $k > 1$ ,  $j_k$  is prime in  $\mathbb{O}_K$ , and  $a$  is a squarefree integer. Suppose that  $P \in E_a(K)$ , and let  $\tilde{P}$  denote the reduction of  $P \bmod j_k$ . Then  $\tilde{P} \notin \alpha E_a(\mathbb{O}_K/(j_k))$  if and only if  $k \in T_P$ .

*Proof.* Let  $L = K(\alpha^{-1}(P)) = K(\gamma)$  for some  $\gamma \in L$  such that  $\gamma^2 = \delta_P$ . Fix a  $Q \in E_a(\overline{\mathbb{Q}})$  such that  $\alpha Q = P$ . Since  $\ker(\alpha) \subset E_a[2] \subset E_a(K)$ , we have  $K(Q) = L = K(\gamma)$ . Fix a prime ideal  $\mathfrak{p}$  of  $L$  above  $(j_k)$ , let  $\mathbb{F} = \mathbb{O}_K/(j_k)$ , let

$\tilde{Q} \in E_a(\bar{\mathbb{F}})$  be the reduction of  $Q \bmod \mathfrak{p}$ , and let  $\tilde{\gamma}$  be the reduction of  $\gamma \bmod \mathfrak{p}$ . Then  $\mathbb{F}(\tilde{Q}) = \mathbb{F}(\tilde{\gamma})$ .

Now  $\tilde{P} \in \alpha E_a(\mathbb{F})$  if and only if  $\tilde{Q} \in E_a(\mathbb{F})$ . By the above, this happens if and only if  $\tilde{\gamma} \in \mathbb{F}$ , that is, if and only if  $\delta_P$  is a square modulo  $j_k$ .  $\square$

**Lemma 4.11.** *We can take*

$$\delta_{-1} = \alpha, \quad \delta_{-5} = -5\alpha, \quad \delta_{-6} = -3\sqrt{-7}, \quad \delta_{-17} = \alpha, \quad \delta_{-111} = -3.$$

*Proof.* The action of the endomorphism  $\alpha$  on the elliptic curve  $E_a$  and its reductions is as follows (see Proposition II.2.3.1 of [41, p. 111]). For  $(x, y) \in E_a$ , we have

$$\alpha(x, y) = \left( \frac{2x^2 + a(7-s)x + a^2(-7-21s)}{(-3+s)x + a(-7+5s)}, \frac{y(2x^2 + a(14-2s)x + a^2(28+14s))}{-(5+s)x^2 - a(42+2s)x - a^2(77-7s)} \right),$$

where  $s = \sqrt{-7}$ . Solving for  $R$  in  $\alpha R = P_a$  yields  $\delta_a$  in each case.  $\square$

**Lemma 4.12.** *If  $k > 1$  then  $\left(\frac{\alpha}{j_k}\right) = -1$ .*

*Proof.* Let  $M = K(\sqrt{\alpha})$ . By the reciprocity law of global class field theory we have

$$\prod_{\mathfrak{p}} (j_k, M_{\mathfrak{p}}/K_{\mathfrak{p}}) = 1,$$

where  $(j_k, M_{\mathfrak{p}}/K_{\mathfrak{p}})$  is the norm residue symbol.

Let  $f(x) = x^2 - j_k \in \mathbb{O}_{K_{\alpha}}[x]$ . For  $k > 1$  we have

$$|f(1)|_{\alpha} = |2\alpha^k|_{\alpha} = 2^{-(k+1)} < 2^{-2} = |4|_{\alpha} = |f'(1)|_{\alpha}^2,$$

and Hensel's lemma implies that  $f(x)$  has a root in  $\mathbb{O}_{K_{\alpha}}$ . Thus  $j_k$  is a square in  $K_{\alpha}$  and  $(j_k, M_{\alpha}/K_{\alpha}) = 1$ .

Identify  $K_{\bar{\alpha}}$  with  $\mathbb{Q}_2$ . Applying Theorem 1 of [40, p. 20] with  $a = j_k$  and  $b = \alpha$ , and using  $\bar{\alpha}^5 = 5 + \alpha$ , gives  $(j_k, \alpha) = -1$ , where  $(j_k, \alpha)$  is the Hilbert symbol. Thus  $j_k \notin \text{Norm}_{M_{\bar{\alpha}}/K_{\bar{\alpha}}}(M_{\bar{\alpha}}^*)$ , and therefore  $(j_k, M_{\bar{\alpha}}/K_{\bar{\alpha}}) = -1$ .

If  $\mathfrak{p}$  is a prime ideal of  $\mathbb{O}_K$  that does not divide 2, then  $M_{\mathfrak{p}}/K_{\mathfrak{p}}$  is unramified. By local class field theory we then have

$$(j_k, M_{\mathfrak{p}}/K_{\mathfrak{p}}) = \left(\frac{\alpha}{\mathfrak{p}}\right)^{\text{ord}_{\mathfrak{p}}(j_k)}.$$

Since  $j_k$  is prime to 2, we have  $\text{ord}_{\alpha}(j_k) = \text{ord}_{\bar{\alpha}}(j_k) = 0$ , hence

$$\prod_{\mathfrak{p} \nmid 2} (j_k, M_{\mathfrak{p}}/K_{\mathfrak{p}}) = \prod_{\mathfrak{p} \nmid 2} \left(\frac{\alpha}{\mathfrak{p}}\right)^{\text{ord}_{\mathfrak{p}}(j_k)} = \prod_{\text{all } \mathfrak{p}} \left(\frac{\alpha}{\mathfrak{p}}\right)^{\text{ord}_{\mathfrak{p}}(j_k)} = \left(\frac{\alpha}{j_k}\right).$$

Therefore

$$1 = \prod_{\mathfrak{p}} (j_k, M_{\mathfrak{p}}/K_{\mathfrak{p}}) = \left(\frac{\alpha}{j_k}\right) (j_k, M_{\alpha}/K_{\alpha}) (j_k, M_{\bar{\alpha}}/K_{\bar{\alpha}}) = -\left(\frac{\alpha}{j_k}\right),$$

as desired.  $\square$

**Lemma 4.13.** *For  $a \in \{-1, -5, -6, -17, -111\}$  the sets  $T_a$  are as follows:*

$$\begin{aligned} T_{-1} &= \{k > 1\}, \\ T_{-5} &= \{k > 1 : k \equiv 3, 4, 7, 8, 11, 13, 14, 15, 16, 17, 20, 22 \pmod{24}\}, \\ T_{-6} &= \{k > 1 : k \equiv 1, 5, 10, 12, 15, 19, 20, 21, 22, 23 \pmod{24}\}, \\ T_{-17} &= \{k > 1\}, \\ T_{-111} &= \{k > 1 : k \equiv 1, 2, 3, 6 \pmod{8}\}. \end{aligned}$$

*Proof.* We apply Lemma 4.11 and the definition of  $T_a$ . Lemma 4.12 implies that  $T_{-1} = T_{-17} = \{k > 1\}$ . For  $a = -6$  we use quadratic reciprocity in quadratic fields (see Theorem 8.15 of [25, p. 257]) to compute  $\left(\frac{\sqrt{-7}}{j_k}\right)$ . For the remaining cases we compute  $\left(\frac{-3}{j_k}\right) = \left(\frac{-3}{j_k}\right)$  and  $\left(\frac{-5}{j_k}\right) = \left(\frac{-5}{j_k}\right)$  as in the proof of Lemma 4.8, and apply  $\left(\frac{\alpha}{j_k}\right) = -1$  from Lemma 4.12.  $\square$

#### 4D. Proof of Theorem 4.1.

**Lemma 4.14.** *Let  $a$  be a squarefree integer. Suppose that  $P \in E_a(K)$ ,  $k \in S_a \cap T_P$ , and  $j_k$  is prime. Let  $\tilde{P}$  denote the reduction of  $P$  mod  $j_k$ . Then the annihilator of  $\tilde{P}$  in  $\mathbb{O}_K$  is divisible by  $\alpha^{k+1}$ .*

*Proof.* We have  $E_a(\mathbb{O}_K/(j_k)) \simeq \mathbb{O}_K/(2\alpha^k) = \mathbb{O}_K/(\bar{\alpha}\alpha^{k+1})$ , by Lemma 4.6(ii). It then suffices to show  $\tilde{P} \notin \alpha E_a(\mathbb{O}_K/(j_k))$ , which follows from Lemma 4.10.  $\square$

The congruence conditions for  $k$  in Table 1 come from taking  $S_a \cap T_a$ , excluding the cases handled by Lemma 4.5, and adjusting to give disjoint sets.

We now prove Theorem 4.1. Suppose that  $k > 1$ ,  $k \not\equiv 0 \pmod{8}$ ,  $k \not\equiv 6 \pmod{24}$ , and  $J_k$  is prime. Let  $a$  and  $P_a$  be as listed in Table 1. Then  $k \in S_a \cap T_a$ . Let  $\tilde{P}$  denote the reduction of  $P_a$  mod  $j_k$ . We have  $E_a(\mathbb{O}_K/(j_k)) \simeq \mathbb{O}_K/(2\alpha^k)$  by Lemma 4.6(ii), and therefore the annihilator of  $\tilde{P}$  in  $\mathbb{O}_K$  divides  $2\alpha^k$ . By Lemma 4.14, the annihilator of  $\tilde{P}$  in  $\mathbb{O}_K$  is divisible by  $\alpha^{k+1}$ . Since  $2\alpha^k$  divides  $2^{k+1}$  but  $\alpha^{k+1}$  does not divide  $2^k$ , we must have  $2^{k+1}\tilde{P} = 0$  and  $2^k\tilde{P} \neq 0$ . Therefore  $2^{k+1}P_a$  is zero mod  $J_k$  and  $2^kP_a$  is strongly nonzero mod  $J_k$ .

For the converse, note that  $\text{disc}(E_a) = -2^{12} \cdot 7^3 \cdot a^6$ , so Lemma 4.5 shows that  $\gcd(J_k, \text{disc}(E_a)) = 1$  if  $k \not\equiv 0 \pmod{8}$  and  $k \not\equiv 6 \pmod{24}$ . We can therefore apply Proposition 3.2 with  $m = 2^{k+1}$ , noting that

$$2^{k+1} > ((3 \cdot 2^{k+1})^{\frac{1}{4}} + 1)^2 > (J_k^{1/4} + 1)^2$$

for all  $k > 2$ , and for  $k = 2$  we have  $2^{k+1} = 8 > (11^{1/4} + 1)^2 = (J_k^{1/4} + 1)^2$ . This proves Theorem 4.1.

**Remark.** As pointed out by Richard Pinch,  $P_a \in 2E_a(\mathbb{O}_K/(j_k))$  if and only if all  $x(P_a) - e_i$  are squares mod  $j_k$ , where  $E_a$  is  $y^2 = \prod_{i=1}^3 (x - e_i)$  and  $x(P_a)$  is the  $x$ -coordinate. We tested for divisibility by  $\alpha$  instead of by 2, to make it clearer how this approach (as initiated by Gross in [20]) makes use of the  $\mathbb{O}_K$ -module structure of  $E_a(\mathbb{O}_K/(j_k))$ . Such an approach is useful for further generalizations.

## 5. Algorithm

A naïve implementation of Theorem 4.1 is entirely straightforward, but here we describe a particularly efficient implementation and analyze its complexity. We then discuss how the algorithm may be used in combination with sieving to search for prime values of  $J_k$ , and give some computational results.

**5A. Implementation.** There are two features of the primality criterion given by Theorem 4.1 worth noting. First, it is only necessary to perform the operation of adding a point on the elliptic curve to itself (doubling), no general additions are required. Second, testing whether a projective point  $P = [x, y, z]$  is zero or strongly nonzero modulo an integer  $J_k$  only involves the  $z$ -coordinate:  $P$  is zero mod  $J_k$  if and only if  $J_k \mid z$ , and  $P$  is strongly nonzero mod  $J_k$  if and only if  $\gcd(z, J_k) = 1$ .

To reduce the cost of doubling, we transform the curve

$$E_a : y^2 = x^3 - 35a^2x - 98a^3$$

to the Montgomery form [31]

$$E_{A,B} : By^2 = x^3 + Ax^2 + x.$$

Such a transformation is not possible over  $\mathbb{Q}$ , but it can be done over  $\mathbb{Q}(\sqrt{-7})$ . In general, one transforms a short Weierstrass equation  $y^2 = f(x) = x^3 + a_4x + a_6$  into Montgomery form by choosing a root  $\gamma$  of  $f(x)$  and setting  $B = (3\gamma^2 - a_4)^{-1/2}$  and  $A = 3\gamma B$ ; see, for example, [34]. For the curve  $E_a$ , we choose  $\gamma = \frac{1}{2}(-7 + \sqrt{-7})a$ , yielding

$$A = \frac{-15 - 3\sqrt{-7}}{8} \quad \text{and} \quad B = \frac{7 + 3\sqrt{-7}}{56a}.$$

With this transformation, the point  $P_a = (x_0, y_0)$  on  $E_a$  corresponds to the point  $(B(x_0 - \gamma), By_0)$  on the Montgomery curve  $E_{A,B}$ , and is defined over  $\mathbb{Q}(\sqrt{-7})$ .

In order to apply this transformation modulo  $J_k$ , we need a square root of  $-7$  in  $\mathbb{Z}/J_k\mathbb{Z}$ . If  $J_k$  is prime and  $d = 7^{(J_k+1)/4}$ , then

$$d^2 \equiv 7^{(J_k-1)/2} \cdot 7 \equiv \left(\frac{7}{J_k}\right) 7 \equiv -7 \pmod{J_k},$$

since  $J_k \equiv 3 \pmod{4}$  and  $J_k \equiv 2, 4 \pmod{7}$  is a quadratic residue modulo 7. If we find that  $d^2 \not\equiv -7 \pmod{J_k}$ , then we immediately know that  $J_k$  must be composite and no further computation is required.

With the transformation to Montgomery form, the formulas for doubling a point on  $E_a$  become particularly simple. If  $P = [x_1, y_1, z_1]$  is a projective point on  $E_{A,B}$  and  $2P = [x_2, y_2, z_2]$ , we may determine  $[x_2, z_2]$  from  $[x_1, z_1]$  via

$$\begin{aligned} 4x_1z_1 &= (x_1 + z_1)^2 - (x_1 - z_1)^2, \\ x_2 &= (x_1 + z_1)^2(x_1 - z_1)^2, \\ z_2 &= 4x_1z_1((x_1 - z_1)^2 + C(4x_1z_1)), \end{aligned} \tag{5}$$

where  $C = \frac{1}{4}(A + 2) = \frac{1}{32}(1 - 3\sqrt{-7})$ . Note that  $C$  does not depend on  $P$  (or even  $a$ ), and may be precomputed. Thus doubling requires just 2 squarings, 3 multiplications, and 4 additions in  $\mathbb{Z}/J_k\mathbb{Z}$ .

We now present the algorithm, which exploits the transformation of  $E_a$  into Montgomery form. We assume that elements of  $\mathbb{Z}/J_k\mathbb{Z}$  are uniquely represented as integers in  $[0, J_k - 1]$ .

### Algorithm 5.1.

*Input:* Positive integers  $k$  and  $J_k$ .

*Output:* *True* if  $J_k$  is prime and *false* if  $J_k$  is composite.

1. If  $k \equiv 0 \pmod{8}$  or  $k \equiv 6 \pmod{24}$  then return *false*.
2. Compute  $d = 7^{(J_k+1)/4} \pmod{J_k}$ .
3. If  $d^2 \not\equiv -7 \pmod{J_k}$  then return *false*.
4. Determine  $a$  via Table 1, depending on  $k \pmod{72}$ .
5. Compute  $r = (-7 + d)a/2 \pmod{J_k}$ ,  $B = (7 + 3d)/(56a) \pmod{J_k}$ , and  $C = (1 - 3d)/32 \pmod{J_k}$ .
6. Let  $x_1 = B(x_0 - r) \pmod{J_k}$  and  $z_1 = 1$ , where  $P_a = (x_0, y_0)$  is as in Table 1.
7. For  $i$  from 1 to  $k + 1$ , compute  $[x_i, z_i]$  from  $[x_{i-1}, z_{i-1}]$  via (5).
8. If  $\gcd(z_k, J_k) = 1$  and  $J_k \mid z_{k+1}$  then return *true*, otherwise return *false*.

The tests in step 1 rule out cases where  $J_k$  is divisible by 3 or 5, by Lemma 4.5;  $J_k$  is then composite, since  $J_k > 5$  for all  $k$ . This also ensures  $\gcd(a, J_k) = 1$  (see Lemma 4.5), so the divisions in step 5 are all valid ( $J_k$  is never divisible by 2 or 7). By Remark 4.2, for  $k \geq 6$  the condition  $\gcd(z_k, J_k) = 1$  in step 8 can be replaced with  $z_k \not\equiv 0 \pmod{J_k}$ .

**Proposition 5.2.** *Algorithm 5.1 performs  $6k + o(k)$  multiplications and  $4k$  additions in  $\mathbb{Z}/J_k\mathbb{Z}$ . Its time complexity is  $O(k^2 \log k \log \log k)$  and it uses  $O(k)$  space.*

$k$	step 2	step 7	$k$	step 2	step 7	$k$	step 2	step 7
$2^{10} + 1$	0.00	0.01	$2^{14} + 1$	0.88	5.50	$2^{17} + 1$	133	983
$2^{11} + 1$	0.00	0.02	$2^{15} + 1$	5.26	32.2	$2^{18} + 1$	723	5010
$2^{12} + 1$	0.02	0.15	$2^{16} + 1$	27.5	183	$2^{19} + 1$	3310	23600
$2^{13} + 1$	0.15	0.91				$2^{20} + 1$	13700	107000

**Table 3.** Timings for Algorithm 5.1 (CPU seconds on a 3.0 GHz AMD Phenom II 945).

*Proof.* Using standard techniques for fast exponentiation [46], step 2 uses  $k + o(k)$  multiplications in  $\mathbb{Z}/J_k\mathbb{Z}$ . Steps 5–6 perform  $O(1)$  operations in  $\mathbb{Z}/J_k\mathbb{Z}$  and step 7 uses  $5k$  multiplications and  $4k$  additions. The cost of the divisions in step 5 are comparatively negligible, as is the cost of step 8. Multiplications (and additions) in  $\mathbb{Z}/J_k\mathbb{Z}$  have a bit complexity of  $O(M(k))$ , where  $M(k)$  counts the bit operations needed to multiply two  $k$ -bit integers [14, Theorem 9.8]. The bound on the time complexity of Algorithm 5.1 then follows from the Schönhage-Strassen [39] bound:  $M(k) = O(k \log k \log \log k)$ . The space complexity bound is immediate: The algorithm only needs to keep track of two pairs  $[x_i, z_i]$  and  $[x_{i-1}, z_{i-1}]$  at any one time, and elements of  $\mathbb{Z}/J_k\mathbb{Z}$  can be represented using  $O(k)$  bits.  $\square$

Table 3 gives timings for Algorithm 5.1 when implemented using the gmp library [19] for all integer arithmetic, including the gcd computations. We list the times for step 2 and step 7 separately (the time spent on the other steps is negligible). In the typical case, where  $J_k$  is composite, the algorithm is very likely<sup>1</sup> to terminate in step 2, which effectively determines whether  $J_k$  is a strong probable prime base  $-7$ , as in [9, Algorithm 3.5.3]. To obtain representative timings at the values of  $k$  listed, we temporarily modified the algorithm to skip step 2.

We note that the timings for step 7 are suboptimal due to the fact that we used the gmp function `mpz_mod` to perform modular reductions. A lower level implementation (using Montgomery reduction [30], for example) might improve these timings by perhaps 20 or 30 percent.

We remark that Algorithm 5.1 can easily be augmented, at essentially no additional cost, to retain an intermediate point  $Q = [x_s, y_s, z_s]$ , where  $s = k + 1 - r$  is chosen so that the order  $2^r$  of  $Q$  is the least power of 2 greater than  $(J_k^{1/4} + 1)^2$ . The value of  $y_s$  may be obtained as a square root of  $y_s^2 = (x_s^3 + Ax_s^2z_s + x_sz_s^2)/(Bz_s)$  by computing  $(y_s^2)^{(J_k+1)/4}$ . When  $J_k$  is prime, the algorithm can then output a Pomerance-style certificate  $(E_{A,B}, Q, r, J_k)$  for the primality of  $J_k$ . This certificate has the virtue that it can be verified using just  $2.5k + O(1)$  multiplications in  $\mathbb{Z}/J_k\mathbb{Z}$ , versus the  $6k + o(k)$  multiplications used by Algorithm 5.1, by checking that the point  $Q$  has order  $2^r$  on the elliptic curve  $E_{A,B} \bmod J_k$ .

<sup>1</sup> Indeed, we have yet to encounter even a single  $J_k$  that is a strong pseudoprime base  $-7$ .

**5B. Searching for prime values of  $J_k$ .** While one can directly apply Algorithm 5.1 to any particular  $J_k$ , when searching a large range  $1 \leq k \leq n$  for prime values of  $J_k$  it is more efficient to first *sieve* the interval  $[1, n]$  to eliminate values of  $k$  for which  $J_k$  cannot be prime.

For example, as noted in Lemma 4.5, if  $k \equiv 0 \pmod{8}$  then  $J_k$  is divisible by 3. More generally, for any small prime  $\ell$ , one can very quickly compute  $J_k \pmod{\ell}$  for all  $k \leq n$  by applying the linear recurrence (4) for  $J_k$ , working modulo  $\ell$ . If  $\ell < \sqrt{n}$ , then the sequence  $J_k \pmod{\ell}$  will necessarily cycle, but in any case it takes very little time to identify all the values of  $k \leq n$  for which  $J_k$  is divisible by  $\ell$ ; the

$k$	$J_k$	$a$	$k$	$J_k$	$a$	$k$	$J_k$	$a$
2	11	-1	319	427...247	-5	17807	110...799	-1
3	23	-1	375	307...023	-1	18445	125...407	-5
4	67	-5	467	152...727	-1	19318	793...763	-5
5	151	-1	489	639...239	-1	26207	495...799	-1
7	487	-5	494	204...963	-1	27140	359...907	-1
9	2039	-1	543	115...143	-1	31324	116...867	-5
10	4211	-6	643	145...399	-17	36397	155...007	-5
17	524087	-1	684	321...531	-1	47294	327...963	-1
18	1046579	-1	725	706...551	-1	53849	583...567	-1
28	107...427	-5	1129	291...591	-17	83578	122...491	-6
38	109...043	-1	1428	297...011	-1	114730	593...411	-6
49	225...791	-17	2259	425...023	-1	132269	345...831	-1
53	360...711	-1	2734	415...123	-5	136539	864...023	-1
60	461...451	-1	2828	822...787	-1	147647	599...399	-1
63	368...943	-1	3148	175...227	-5	167068	120...027	-5
65	147...007	-1	3230	849...483	-1	167950	388...883	-5
77	604...191	-1	3779	156...127	-1	257298	104...179	-1
84	773...531	-1	5537	254...887	-1	342647	423...399	-1
87	618...703	-1	5759	171...279	-1	414349	120...207	-5
100	507...507	-5	7069	382...207	-5	418033	118...831	-17
109	259...207	-5	7189	508...207	-5	470053	451...407	-5
147	713...023	-1	7540	233...107	-5	475757	536...791	-1
170	598...611	-1	7729	183...591	-111	483244	347...667	-5
213	526...239	-1	9247	168...687	-5	680337	279...759	-1
235	220...519	-17	10484	398...747	-1	810653	295...711	-1
287	994...999	-1	15795	234...023	-1	857637	115...519	-1
						1111930	767...411	-6

**Table 4.** Prime values of  $J_k \approx 2^{k+2}$  for  $k \leq 1.2 \times 10^6$ . The column labeled  $a$  gives the value of the twisting factor.

total time required is just  $\tilde{O}(n \log \ell)$ , versus  $\tilde{O}(n^2)$  if one were to instead apply a trial division by  $\ell$  to each  $J_k$ .

We used this approach to sieve the interval  $[1, n]$  for those  $k$  for which  $J_k$  is not divisible by any prime  $\ell \leq L$ . Of course one still needs to consider  $J_k \leq L$ , but this is a small set consisting of roughly  $\log_2 L$  values, each of which can be tested very quickly. With  $n = 10^6$  and  $L = 2^{35}$ , sieving reduces the number of potentially prime  $J_k$  by a factor of more than 10, leaving 93,707 integers  $J_k$  as candidate primes to be tested with Algorithm 5.1. The prime values of  $J_k$  found by the algorithm are listed in Table 4, along with the corresponding value of  $a$ . As noted in the introduction, we have extended these results to  $n = 1.2 \times 10^6$ , finding one additional prime with  $k = 1,111,930$ , which is also listed in Table 4. The data in Table 4 suggests that prime values of  $J_k$  may be more common than prime values of Mersenne numbers  $M_n$ ; there are 78 primes  $J_k$  with fewer than one million bits, but only 33 Mersenne primes in this range. This can be at least partly explained by the fact that  $M_n$  can be prime only when  $n$  is prime, whereas the values of  $k$  for which  $J_k$  can be prime are not so severely constrained. By analyzing these constraints in detail, it may be possible to give a heuristic estimate for the density of primes in the sequence  $J_k$ , but we leave this to a future article.

### Acknowledgments

We thank Daniel J. Bernstein, François Morain, Carl Pomerance, and Karl Rubin for helpful conversations, and the organizers of ECC 2010, the First Abel Conference, and the AWM Anniversary Conference where useful discussions took place. We thank the reviewers for helpful comments. We also thank Henri Cohen and Richard Pinch for helpful comments given at ANTS-X.

This work was supported by the National Science Foundation under grants CNS-0831004 and DMS-1115455.

### References

- [1] Alexander Abatzoglou, Alice Silverberg, Andrew V. Sutherland, and Angela Wong, *The Prime Database*:  $2^{1111932} + 2 \cdot V(1, 2, 1111930) + 1$ , 2012. [http://primes.utm.edu/en\\_US/primes/page.php?id=106847](http://primes.utm.edu/en_US/primes/page.php?id=106847)
- [2] ACM (ed.), *Proceedings of the Eighteenth Annual ACM Symposium on Theory of Computing (STOC '86)*, New York, Association for Computing Machinery, 1986.
- [3] Manindra Agrawal, Neeraj Kayal, and Nitin Saxena, *PRIMES is in P*, Ann. of Math. (2) **160** (2004), no. 2, 781–793. MR 2006a:11170
- [4] A. O. L. Atkin and F. Morain, *Elliptic curves and primality proving*, Math. Comp. **61** (1993), no. 203, 29–68. MR 93m:11136
- [5] Wieb Bosma, *Primality testing with elliptic curves*, Ph.D. thesis, Mathematisch Instituut, Universiteit van Amsterdam, 1985. <http://www.math.ru.nl/~bosma/pubs/PRITwEC1985.pdf>

- [6] David Broadhurst, *The Prime Database*:  $(935695 \cdot 2^{627694} + 3)^2 + (1123581 \cdot 2^{313839})^2$ , 2012. [http://primes.utm.edu/en\\_US/primes/page.php?id=108157](http://primes.utm.edu/en_US/primes/page.php?id=108157)
- [7] Chris Caldwell, *The prime pages: prime number research, records, and resources*, 2012. <http://primes.utm.edu/>
- [8] D. V. Chudnovsky and G. V. Chudnovsky, *Sequences of numbers generated by addition in formal groups and new primality and factorization tests*, Adv. in Appl. Math. **7** (1986), no. 4, 385–434. MR 88h:11094
- [9] Richard Crandall and Carl Pomerance, *Prime numbers: A computational perspective*, second ed., Springer, New York, 2005. MR 2006a:11005
- [10] Jean-Marie De Koninck and Claude Levesque (eds.), *Théorie des nombres: Proceedings of the International Conference held at the Université Laval, Québec, July 5–18, 1987*, Berlin, de Gruyter, 1989. MR 90f:11002
- [11] Robert Denomme and Gordan Savin, *Elliptic curve primality tests for Fermat and related primes*, J. Number Theory **128** (2008), no. 8, 2398–2412. MR 2009c:11208
- [12] Graham Everest, Alf van der Poorten, Igor Shparlinski, and Thomas Ward, *Recurrence sequences*, Mathematical Surveys and Monographs, no. 104, American Mathematical Society, Providence, RI, 2003. MR 2004c:11015
- [13] Victor G. Ganzha, Ernst W. Mayr, and Evgenii V. Vorozhtsov (eds.), *Computer algebra in scientific computing: Proceedings of the 9th International Workshop (CASC 2006) held in Chişinău, September 11–15, 2006*, Lecture Notes in Computer Science, no. 4194, Berlin, Springer, 2006. MR 2007j:68005
- [14] Joachim von zur Gathen and Jürgen Gerhard, *Modern computer algebra*, second ed., Cambridge University Press, Cambridge, 2003. MR 2004g:68202
- [15] Andrew M. Gleason (ed.), *Proceedings of the International Congress of Mathematicians* (Berkeley, 1986), vol. 1, Providence, RI, American Mathematical Society, 1987. MR 89c:00042
- [16] Shafi Goldwasser and Joe Kilian, *Almost all primes can be quickly certified*, in ACM [2], 1986, pp. 316–329.
- [17] ———, *Primality testing using elliptic curves*, J. ACM **46** (1999), no. 4, 450–472. MR 2002e:11182
- [18] Daniel M. Gordon, *Pseudoprimes on elliptic curves*, in De Koninck and Levesque [10], 1989, pp. 290–305. MR 91g:11158
- [19] Torbjörn Granlund and the GMP development team, *GNU MP: The GNU Multiple Precision Arithmetic Library* (version 5.0.1), 2011. <http://gmplib.org/>
- [20] Benedict H. Gross, *An elliptic curve test for Mersenne primes*, J. Number Theory **110** (2005), no. 1, 114–119. MR 2005m:11007
- [21] Alexander Gurevich and Boris Kunyavskiĭ, *Primality testing through algebraic groups*, Arch. Math. (Basel) **93** (2009), no. 6, 555–564. MR 2011g:11235
- [22] ———, *Deterministic primality tests based on tori and elliptic curves*, Finite Fields Appl. **18** (2012), no. 1, 222–236. MR 2874918
- [23] Hideki Imai and Yuliang Zheng (eds.), *Public key cryptography: Proceedings of the 3rd International Workshop on Practice and Theory in Public Key Cryptosystems (PKC 2000) held in Melbourne, January 18–20, 2000*, Lecture Notes in Computer Science, no. 1751, Berlin, Springer, 2000. MR 2002f:94052

- [24] D. H. Lehmer, *An extended theory of Lucas' functions*, Ann. of Math. (2) **31** (1930), no. 3, 419–448. MR 1502953
- [25] Franz Lemmermeyer, *Reciprocity laws: From Euler to Eisenstein*, Springer, Berlin, 2000. MR 2001i:11009
- [26] H. W. Lenstra, Jr., *Elliptic curves and number-theoretic algorithms*, in Gleason [15], 1987, pp. 99–120. <http://www.mathunion.org/ICM/ICM1986.1/Main/icm1986.1.0099.0120.ocr.pdf> MR 89d:11114
- [27] H. W. Lenstra, Jr. and Carl Pomerance, *Primality testing with Gaussian periods*, preprint, 2011. <http://www.math.dartmouth.edu/~carlp/aks041411.pdf>
- [28] Rudolf Lidl and Harald Niederreiter, *Introduction to finite fields and their applications*, Cambridge University Press, Cambridge, 1994, Revision of the 1986 first edition. MR 95f:11098
- [29] J. S. Milne, *Elliptic curves*, BookSurge, Charleston, SC, 2006. MR 2007h:14044
- [30] Peter L. Montgomery, *Modular multiplication without trial division*, Math. Comp. **44** (1985), no. 170, 519–521. MR 86e:11121
- [31] ———, *Speeding the Pollard and elliptic curve methods of factorization*, Math. Comp. **48** (1987), no. 177, 243–264. MR 88e:11130
- [32] François Morain, *Elliptic curves, primality proving and some titanic primes*, Journées Arithmétiques (Luminy, 1989), Astérisque, vol. 198-200, 1991, pp. 245–251. MR 92m:11147
- [33] ———, *Implementing the asymptotically fast version of the elliptic curve primality proving algorithm*, Math. Comp. **76** (2007), no. 257, 493–505. MR 2007m:11167
- [34] Katsuyuki Okeya, Hiroyuki Kurumatani, and Kouichi Sakurai, *Elliptic curves with the Montgomery-form and their cryptographic applications*, in Imai and Zheng [23], 2000, pp. 238–257. MR 2003h:94045
- [35] Th. Pépin, *Sur la formule  $2^{2^n} + 1$* , C. R. Acad. Sci. Paris **85** (1877), 329–331.
- [36] Carl Pomerance, *Very short primality proofs*, Math. Comp. **48** (1987), no. 177, 315–322. MR 88b:11088
- [37] ———, *Primality testing: variations on a theme of Lucas*, Congr. Numer. **201** (2010), 301–312. MR 2010k:11191
- [38] Vaughan R. Pratt, *Every prime has a succinct certificate*, SIAM J. Comput. **4** (1975), no. 3, 214–220. MR 52 #12395
- [39] A. Schönhage and V. Strassen, *Schnelle Multiplikation großer Zahlen*, Computing (Arch. Elektron. Rechnen) **7** (1971), 281–292. MR 45 #1431
- [40] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, no. 7, Springer, New York, 1973. MR 49 #8956
- [41] Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, no. 151, Springer, New York, 1994. MR 96b:11074
- [42] H. M. Stark, *Counting points on CM elliptic curves*, Rocky Mountain J. Math. **26** (1996), no. 3, 1115–1138. MR 98b:11060
- [43] W. A. Stein et al., *Sage Mathematics Software (version 4.7.1)*, The Sage Development Team, 2011. <http://www.sagemath.org>
- [44] Yu Tsumura, *Primality tests for  $2^p \pm 2^{(p+1)/2} + 1$  using elliptic curves*, Proc. Amer. Math. Soc. **139** (2011), no. 8, 2697–2703. MR 2012e:11210

- [45] Song Y. Yan and Glyn James, *Testing Mersenne primes with elliptic curves*, in Ganzha et al. [13], 2006, pp. 303–312. MR 2007k:11209
- [46] Andrew Chi Chih Yao, *On the evaluation of powers*, SIAM J. Comput. **5** (1976), no. 1, 100–103. MR 52 #16128

ALEXANDER ABATZOGLOU: [aabatzog@math.uci.edu](mailto:aabatzog@math.uci.edu)

*Department of Mathematics, University of California, Irvine, CA 92697, United States*

ALICE SILVERBERG: [asilverb@math.uci.edu](mailto:asilverb@math.uci.edu)

*Mathematics Department, University of California, Irvine, CA 92697-3875, United States*

ANDREW V. SUTHERLAND: [drew@math.mit.edu](mailto:drew@math.mit.edu)

*Department of Mathematics, MIT, Cambridge, MA 02139, United States*

ANGELA WONG: [awong@math.uci.edu](mailto:awong@math.uci.edu)

*Department of Mathematics, University of California, Irvine, CA 92697, United States*

## VOLUME EDITORS

Everett W. Howe  
Center for Communications Research  
4320 Westerra Court  
San Diego, CA 92121-1969  
United States

Kiran S. Kedlaya  
Department of Mathematics  
University of California, San Diego  
9500 Gilman Drive #0112  
La Jolla, CA 92093-0112

---

Front cover artwork based on a detail of  
*Chicano Legacy 40 Años* ©2010 Mario Torero.

The contents of this work are copyrighted by MSP or the respective authors.  
All rights reserved.

Electronic copies can be obtained free of charge from <http://msp.org/obs/1>  
and printed copies can be ordered from MSP ([contact@msp.org](mailto:contact@msp.org)).

The Open Book Series is a trademark of Mathematical Sciences Publishers.

ISSN: 2329-9061 (print), 2329-907X (electronic)

ISBN: 978-1-935107-00-2 (print), 978-1-935107-01-9 (electronic)

First published 2013.

---



**MATHEMATICAL SCIENCES PUBLISHERS**

798 Evans Hall #3840, c/o University of California, Berkeley CA 94720-3840  
[contact@msp.org](mailto:contact@msp.org)

<http://msp.org>

## Tenth Algorithmic Number Theory Symposium

The Algorithmic Number Theory Symposium (ANTS), held biennially since 1994, is the premier international forum for research in computational number theory. ANTS is devoted to algorithmic aspects of number theory, including elementary, algebraic, and analytic number theory, the geometry of numbers, arithmetic algebraic geometry, the theory of finite fields, and cryptography.

This volume is the proceedings of the tenth ANTS meeting, held July 9–13, 2012, at the University of California, San Diego. It includes revised and edited versions of the 25 refereed papers presented at the conference, together with extended abstracts of two of the five invited talks.

## TABLE OF CONTENTS

Deterministic elliptic curve primality proving for a special sequence of numbers — Alexander Abatzoglou, Alice Silverberg, Andrew V. Sutherland, and Angela Wong	1
Imaginary quadratic fields with isomorphic abelian Galois groups — Athanasios Angelakis and Peter Stevenhagen	21
Iterated Coleman integration for hyperelliptic curves — Jennifer S. Balakrishnan	41
Finding ECM-friendly curves through a study of Galois properties — Razvan Bărbulescu, Joppe W. Bos, Cyril Bouvier, Thorsten Kleinjung, and Peter L. Montgomery	63
Two grumpy giants and a baby — Daniel J. Bernstein and Tanja Lange	87
Improved techniques for computing the ideal class group and a system of fundamental units in number fields — Jean-François Biasse and Claus Fieker	113
Conditionally bounding analytic ranks of elliptic curves — Jonathan W. Bober	135
A database of elliptic curves over $\mathbb{Q}(\sqrt{5})$ : a first report — Jonathan Bober, Alyson Deines, Ariah Klages-Mundt, Benjamin LeVeque, R. Andrew Ohana, Ashwath Rabindranath, Paul Sharaba, and William Stein	145
Finding simultaneous Diophantine approximations with prescribed quality — Wieb Bosma and Ionica Smeets	167
Success and challenges in determining the rational points on curves — Nils Bruin	187
Solving quadratic equations in dimension 5 or more without factoring — Pierre Castel	213
Counting value sets: algorithm and complexity — Qi Cheng, Joshua E. Hill, and Daqing Wan	235
Haberland's formula and numerical computation of Petersson scalar products — Henri Cohen	249
Approximate common divisors via lattices — Henry Cohn and Nadia Heninger	271
Explicit descent in the Picard group of a cyclic cover of the projective line — Brendan Creutz	295
Computing equations of curves with many points — Virgile Ducet and Claus Fieker	317
Computing the unit group, class group, and compact representations in algebraic function fields — Kirsten Eisenträger and Sean Hallgren	335
The complex polynomials $P(x)$ with $\text{Gal}(P(x) - t) \cong M_{23}$ — Noam D. Elkies	359
Experiments with the transcendental Brauer-Manin obstruction — Andreas-Stephan Elsenhans and Jörg Jahnel	369
Explicit 5-descent on elliptic curves — Tom Fisher	395
On the density of abelian surfaces with Tate-Shafarevich group of order five times a square — Stefan Keil and Remke Kloosterman	413
Improved CRT algorithm for class polynomials in genus 2 — Kristin E. Lauter and Damien Robert	437
Fast computation of isomorphisms of hyperelliptic curves and explicit Galois descent — Reynald Lercier, Christophe Ritzenthaler, and Jeroen Sijsling	463
Elliptic factors in Jacobians of hyperelliptic curves with certain automorphism groups — Jennifer Paulhus	487
Isogeny volcanoes — Andrew V. Sutherland	507
On the evaluation of modular polynomials — Andrew V. Sutherland	531
Constructing and tabulating dihedral function fields — Colin Weir, Renate Scheidler, and Everett W. Howe	557