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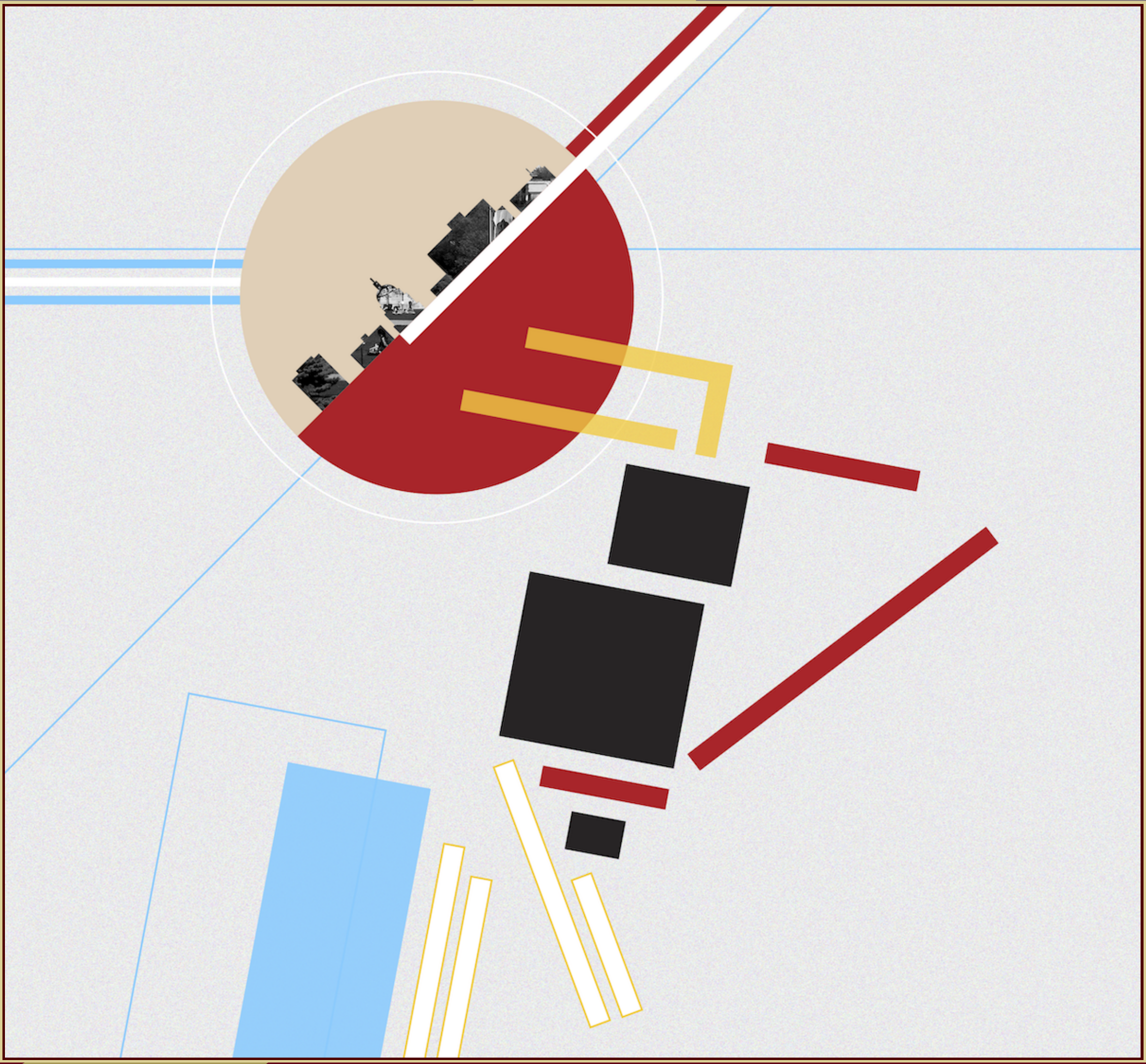

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Computing normalizers of tiled orders in $M_n(k)$

Angelica Babei

Tiled orders are a class of orders in matrix algebras over a non-Archimedean local field generalizing maximal and hereditary orders. Normalizers of tiled orders contain valuable information for finding type numbers of associated global orders. We describe an algorithm for computing normalizers of tiled orders in matrix algebras.

1. Introduction

Let k be a non-Archimedean local field, R its valuation ring with maximal ideal $\mathfrak{p}$, and $B = M_n(k)$. An order Γ in B is a full R-lattice that is also a subring containing 1_B such that $\Gamma \otimes_R k = B$. Orders of the form $\Gamma = (\mathfrak{p}^{\nu_{ij}}) \subseteq M_n(k)$ containing a conjugate of $\operatorname{diag}(R, R, \dots, R)$ have been of interest in many contexts. Such orders generalize maximal and hereditary orders and are known as the graduated orders studied by Plesken in [11], the tiled orders studied by Fujita and Yoshimura in [2; 4], or the split orders studied by Hijikata in [7] and Shemanske in [13]. We will use the term "tiled order" for the rest of the paper.

The goal of this paper is finding ways to compute the normalizer $\mathcal{N}(\Gamma) = \{\xi \in \mathrm{GL}_n(k) \mid \xi\Gamma\xi^{-1} = \Gamma\}$. Clearly $k^\times\Gamma^\times \subseteq \mathcal{N}(\Gamma)$, and the question we address in this paper is how to describe $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ as a subgroup of S_n.

When $n = 2$, Hijikata [7] used knowledge of $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ to compute the trace formula of Hecke operators. Analogously, when one derives a trace formula for Brandt matrices [10], one obtains as a byproduct a means to compute class numbers of certain orders in quaternion algebras, some of whose localizations are tiled orders. More generally, given a central simple algebra over a global field and an order Γ in such an algebra, one can use information about the normalizer $\mathcal{N}(\Gamma_\nu)$ at each of the completions to compute the type number of the global order.

There has been some work describing the normalizer of tiled orders. In particular, for a tiled order Γ, Haefner and Pappacena [6] describe $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ as a subgroup of the automorphisms of a directed multigraph. We will give a more complete description of the normalizer as the group of automorphisms of a certain valued quiver, as described by Roggenkamp and Wiedemann in [15], with an equivalent definition by Müller in [9].

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Our algorithm for finding the normalizer of a tiled order Γ consists of five parts. First, we associate to Γ a new "centered" tiled order Γ_0, which reveals the structure of the normalizer more transparently. Second, we compute the valued quiver $Q_v(\Gamma_0)$ for the centered tiled order Γ_0 and we identify $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ with the automorphism group $\mathrm{Aut}(Q_v(\Gamma_0))$. We then partition the vertices of the valued quiver $Q_v(\Gamma_0)$ into sets with the same weights for incoming and outgoing arrows. This partition allows us to embed the automorphism group of the valued quiver in a product of symmetric groups $S_{l_1} \times S_{l_2} \times \cdots \times S_{l_r} \subseteq S_n$. Finally, the normalizer $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ is given by the elements in this product that permute the weights of the arrows.

2. Preliminaries

As we have said, an order Γ in $B = M_n(k)$ is a full R-lattice that is also a subring containing 1_B such that $\Gamma \otimes_R k = B$. It is known [12, Theorem 17.3] that every maximal order Λ in B is conjugate by an element in $B^\times$ to $M_n(R)$. The orders we are interested in are defined as follows.

Definition 1. We say Γ is a tiled order if it contains a conjugate of the ring $\mathrm{diag}(R, R, \ldots, R)$.

We want to introduce a geometric framework in which a tiled order Γ is realized as a convex polytope C_Γ in a Euclidean space. This geometric realization will give a correspondence between the symmetries of the polytope C_Γ and elements of the normalizer. To do so, we now introduce a bit of the theory of affine buildings and how it relates to tiled orders as described in [13]. For further details the reader may wish to consult [1; 5].

Let V be an n-dimensional vector space over k; so we identify B with $\mathrm{End}_k(V)$. Fixing a basis $\{e_1, e_2, \ldots, e_n\}$ for V and letting L_0 be the free R-lattice generated by this basis, we can identify $\mathrm{End}_R(L_0)$ with the maximal order $\Lambda_0 = M_n(R)$. For any maximal order Λ, we have $\Lambda = \xi\Lambda_0\xi^{-1}$ for some $\xi \in B^\times$, so we can identify Λ with $\mathrm{End}_R(\xi L_0)$.

We say that two full R-lattices L_1 and L_2 in V are homothetic if $L_1 = aL_2$ for some $a \in k^\times$. Homothety of lattices is an equivalence relation, and we denote the homothety class of L by $[L]$. It is easy to see that $[L_1] = [L_2]$ if and only if $\mathrm{End}_R(L_1) = \mathrm{End}_R(L_2)$, so we can identify each homothety class of a lattice with a maximal order.

We construct the affine building for $\mathrm{SL}_n(k)$ as follows. The vertices are the homothety classes of lattices, so by the remarks above we have identified homothety classes of lattices, vertices in the building, and maximal orders in B. Fixing a uniformizer $\pi \in R$, there is an edge between two vertices if there are lattices L_1 and L_2 in their respective homothety classes such that $\pi L_1 \subsetneq L_2 \subsetneq L_1$. The vertices of an m-simplex correspond to chains of lattices of the form $\pi L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{m+1} \subsetneq L_1$. The maximal $(n-1)$-simplices are called chambers.

Given a basis $\{e_1, e_2, \ldots, e_n\}$ as above, we have an associated subcomplex of the affine building for $\mathrm{SL}_n(k)$, called an *apartment*. The vertices of the apartment are homothety classes of lattices of the form $L = R\pi^{m_1}e_1 \oplus R\pi^{m_2}e_2 \oplus \cdots \oplus R\pi^{m_n}e_n$, $m_i \in \mathbb{Z}$, which we encode by $[L] = [m_1, m_2, m_3, \ldots, m_n] = [0, m_2 - m_1, \ldots, m_n - m_1]$. Each apartment is an $(n-1)$-complex and a tessellation of $\mathbb{R}^{n-1}$.

Note that while conjugation changes bases and therefore the apartment we are working with, it doesn't change the structure of the normalizer. Conjugating if necessary, from now on we may and will assume that Γ actually contains $\operatorname{diag}(R, R, \ldots, R)$ and that we are in the apartment where $[0, 0, \ldots, 0]$ corresponds to $\Lambda_0 = M_n(R)$. In this case, by Proposition 2.1 in [13], $\Gamma = (\mathfrak{p}^{\nu_{ij}})$, where

$$\nu_{ij} + \nu_{jk} \geq \nu_{ik} \quad \text{for all } i, j, k \leq n, \qquad \nu_{ii} = 0. \tag{1}$$

We denote by $M_\Gamma = (\nu_{ij})$ the *exponent matrix* of Γ. Let $[P_i] = [\nu_{1i}, \nu_{2i}, \ldots, \nu_{ni}]$ be the homothety class with entries the i-th column of M_Γ. By [11, Remark II.4], the set $\{P_i\}_{i=1}^n$ represents a complete set of isomorphism classes of projective indecomposable left Γ-lattices. Similarly, define $[R_i] = [-\nu_{i1}, -\nu_{i2}, \ldots, -\nu_{in}]$, the homothety class with entries the i-th row of $-M_\Gamma$. Analogously, the set $\{R_i\}_{i=1}^n$ is a complete set of injective indecomposable Γ-lattices. We will observe this duality in other instances later in the paper.

For the sake of brevity, for the majority of the paper we will consider *nondegenerate* tiled orders, that is, tiled orders whose n columns correspond to n different homothety classes. The algorithm for finding elements of the normalizers for other orders is almost identical, and we will mention the modifications at the end of Section 4.

Recall our setting, where $\Gamma = (\mathfrak{p}^{\nu_{ij}})$ is a tiled order containing $\operatorname{diag}(R, R, \ldots, R)$. We associate to Γ a polytope C_Γ in the apartment in the following way. The equations of the form $x_i - x_j = \nu \in \mathbb{Z}$, $1 \leq i, j \leq n$, determine hyperplanes in $\mathbb{R}^{n-1}$, and the hyperplanes $H_{ij} := x_i - x_j = \nu_{ij}$ with ν_{ij} given by the exponents of the tiled order are the bounding hyperplanes of a convex polytope, which we denote by C_Γ. In addition, the vertices given by $P_1, P_2, \ldots, P_n$ defined above are extremal points on C_Γ, and they uniquely determine Γ [14, Proposition 2.2]. From now on, we will refer to the homothety classes $[P_i] = [\nu_{1i}, \nu_{2i}, \ldots, \nu_{ni}] = [0, \nu_{2i} - \nu_{1i}, \ldots, \nu_{ni} - \nu_{1i}]$ as the *distinguished vertices* of C_Γ.

Example 1. Let Γ be the tiled order with exponent matrix

$$M_\Gamma = \begin{bmatrix} 0 & 1 & 4 \\ 2 & 0 & 3 \\ 2 & 2 & 0 \end{bmatrix}.$$

In Figure 1 we see the associated convex polytope C_Γ as determined by

$$-2 \leq x_1 - x_2 \leq 1, \quad -2 \leq x_1 - x_3 \leq 4, \quad -2 \leq x_2 - x_3 \leq 3,$$

or also as the convex hull of its distinguished vertices

$$[P_1] = [0, 2, 2], \quad [P_2] = [0, -1, 1] = [1, 0, 2], \quad [P_3] = [0, -1, -4] = [4, 3, 0].$$

Likewise, C_Γ is also the convex hull of the vertices given by the negatives of the rows:

$$[R_1] = [0, -1, -4], \quad [R_2] = [0, 2, -1] = [-2, 0, 3], \quad [R_3] = [0, 0, 2] = [-2, -2, 0].$$

As described in [13] and expanded in [14], the vertices in C_Γ give an additional description of Γ:

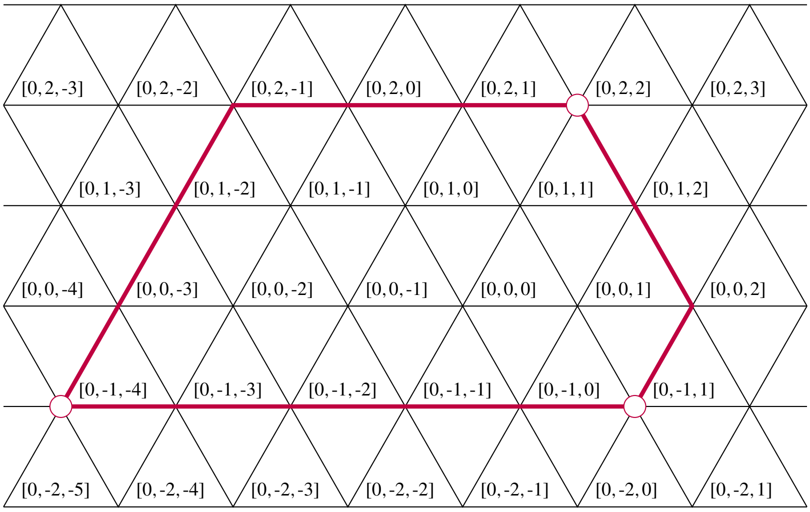

Figure 1. Convex polytope C_Γ from Example 1.

Lemma 2 (Shemanske, [13; 14]). *Let* Γ *be a tiled order with convex polytope* C_Γ. *Then* Γ *is the intersection* $\Gamma = \bigcap_{v \in C_\Gamma} \Lambda_v$ *of maximal orders corresponding to the vertices in* C_Γ. *In addition,* $\Gamma = \bigcap_{i=1}^n \Lambda_i$, *where* Λ_i *are the maximal orders corresponding to the distinguished vertices of* C_Γ.

Proof. For the first assertion, see [13]. For a fixed $\ell \leq n$, we get $[P_\ell] = [\nu_{1\ell}, \nu_{2\ell}, \ldots, \nu_{n\ell}]$ and the associated maximal order is $\Lambda_\ell = (\mathfrak{p}^{\nu_{i\ell} - \nu_{j\ell}})$ by [13, Corollary 2.3]. Since $\nu_{ij} + \nu_{j\ell} \geq \nu_{i\ell}$, we can easily check that indeed $\Gamma = \bigcap_{i=1}^n \Lambda_i$. $\square$

Therefore, given a tiled order Γ, we can obtain its convex polytope C_Γ, and in small enough dimensions, we can visualize it and use geometric intuitions to find elements of the normalizer. We summarize the arguments in [14, Sections 2 and 3] that describe $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ as the symmetries of C_Γ in the following:

Proposition 3 (Shemanske, [14]). *There is a homomorphism* $\phi : \mathcal{N}(\Gamma) \to S_n$ *with* $\ker(\phi) = k^\times\Gamma^\times$.

Proof. By [11, Remark II.4], the columns of Γ are a complete set of isomorphism classes of indecomposable projective left Γ-lattices. Therefore, $\xi \in \mathcal{N}(\Gamma)$ will permute them, so $\mathcal{N}(\Gamma)$ acts on the set of n distinguished vertices. This gives the homomorphism $\phi : \mathcal{N}(\Gamma) \to S_n$. Next we show that $\ker(\phi) = k^\times\Gamma^\times$. It follows easily from Lemma 2 that $k^\times\Gamma^\times \subseteq \ker\phi$.

On the other hand, if ξ fixes each distinguished vertex $P_1, P_2, \ldots, P_n$, then ξ normalizes each maximal order $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ corresponding to each distinguished vertex, so $\xi \in \bigcap_{i=1}^n \mathcal{N}(\Lambda_i) = \bigcap_{i=1}^n k^\times\Lambda_i^\times$. We claim that $\bigcap_{i=1}^n k^\times\Lambda_i^\times = k^\times \bigcap_{i=1}^n \Lambda_i^\times = k^\times\Gamma^\times$, with the latter equality following from $\Gamma^\times = \left(\bigcap_{i=1}^n \Lambda_i\right)^\times$.

We proceed to prove the first equality. Clearly $k^\times \bigcap_{i=1}^n \Lambda_i^\times \subseteq \bigcap_{i=1}^n k^\times\Lambda_i^\times$. To show the nontrivial containment, suppose $\xi \in \bigcap_{i=1}^n k^\times\Lambda_i^\times$. Then we can write $\xi = \pi^{\nu_1}\lambda_1 = \pi^{\nu_2}\lambda_2 = \cdots = \pi^{\nu_n}\lambda_n$, where each $\lambda_i \in \Lambda_i^\times$. Taking the reduced norm, we get $N(\lambda_i) = 1$ for all $i \leq n$. Therefore $N(\xi) = \pi^{n\nu_1} = \pi^{n\nu_2} = \cdots = \pi^{n\nu_n}$, so $\xi = \pi^\nu\lambda$, where $\nu := \nu_1 = \nu_2 = \cdots = \nu_n$ and $\lambda \in \bigcap_{i=1}^n \Lambda_i^\times$. Then $\bigcap_{i=1}^n k^\times\Lambda_i^\times \subseteq k^\times \bigcap_{i=1}^n \Lambda_i^\times$ and the proposition holds. $\square$

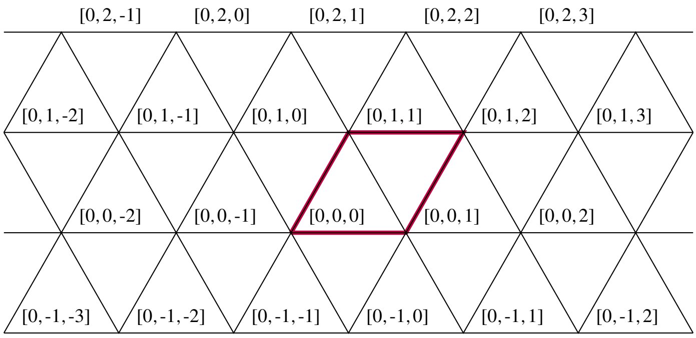

Figure 2. Polytope from Example 2.

By Proposition 3 we may view $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ as a subgroup of S_n. Moreover, Fujita and Yoshimura show in the proof of their main theorem in [4] that every coset in $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ has a monomial representative. Their argument goes as follows.

Let $\{e_{ii} \mid 1 \le i \le n\}$ be the set of n primitive orthogonal idempotents of Γ, where e_{ii} is the $n \times n$ matrix with 1 in the (i,i) position and zero everywhere else. Given an automorphism $\varphi : \Gamma \to \Gamma$ acting by conjugation, i.e., $\varphi(x) = \xi x \xi^{-1}$ for some $\xi \in M_n(k)$, by [8, Proposition 3, p. 77] there exists a unit $u \in \Gamma^\times$ and a permutation matrix w such that $\xi e_{ii} \xi^{-1} = (uw)e_{ii}(uw)^{-1}$. Fujita and Yoshimura then proceed to find a diagonal matrix d such that $(dw)\Gamma(dw)^{-1} = \Gamma$, where ξ and dw represent the same coset in $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$.

Therefore, each coset in $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ has a monomial representative. Geometrically, conjugation by this monomial matrix corresponds to a product of reflections of the convex polytope C_Γ across hyperplanes in the apartment, so each element of $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ permutes the distinguished vertices of C_Γ by rigid motions. We will refer to elements of $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ as the "symmetries of C_Γ", and we associate to $\xi \in \mathcal{N}(\Gamma)/k^\times\Gamma^\times$ the element $\sigma_\xi := \phi(\xi) \in S_n$.

For $n = 3$, C_Γ is 2-dimensional with symmetries a subgroup of S_3 as illustrated below.

Example 2. For

$$M_{\Gamma_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

we have the polytope in Figure 2. We see the symmetries correspond to a fold, so $\mathcal{N}(\Gamma_1)/k^\times{\Gamma_1}^\times \cong \mathbb{Z}/2\mathbb{Z}$.

Example 3. For

$$M_{\Gamma_2} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix}$$

we have the polytope in Figure 3. We see the symmetries correspond to a group of rotations of order 3, so $\mathcal{N}(\Gamma)/k^\times\Gamma^\times \cong A_3 \cong \mathbb{Z}/3\mathbb{Z}$.

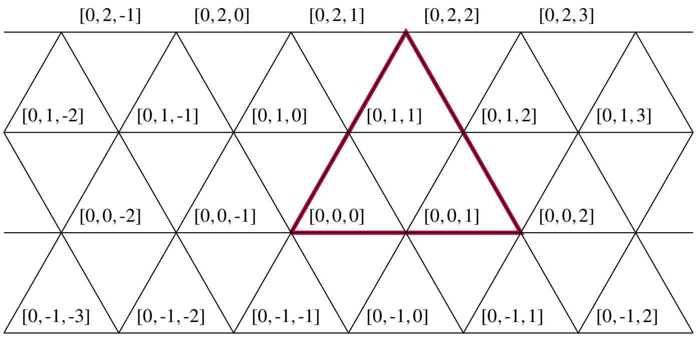

Figure 3. Polytope from Example 3.

In unpublished work [16] (see [11]), Zassenhaus introduced a set of *structural invariants* for tiled orders, defined by

$$m_{ijk} = \nu_{ij} + \nu_{jk} - \nu_{ik} \quad \text{for } 1 \le i, j, k \le n.$$

Note that since for any tiled order $\nu_{ij} + \nu_{jk} \ge \nu_{ik}$, structural invariants are nonnegative. In [14], these structural invariants encode the geometry of the convex polytope C_Γ. When $n = 3$, they correspond to side lengths of C_Γ and gaps between opposite sides; for instance, in Example 2 we get $m_{231} = m_{312} = m_{132} = m_{213}$ are the four sides of length 1, and in Example 3 we see that $m_{213} = m_{321} = m_{132} = 2$ gives us the three sides of length 2.

In the general case, the m_{ijk} still encode geometric data.

Lemma 4. *Fix $1 \le i \le n$. Then the distinguished vertex P_i is at the intersection of the (affine) hyperplanes $\bigcap_{j \ne i} H_{ji}$, where H_{ji} is given by the equation $x_j - x_i = \nu_{ji}$ when $j \ne i$.*

Proof. Since $P_i = [\nu_{1i}, \nu_{2i}, \dots, \nu_{ni}] \sim [0, \nu_{2i} - \nu_{1i}, \dots, \nu_{ni} - \nu_{1i}]$, P_i lies on each of the hyperplanes $x_i - x_j = (\nu_{ii} - \nu_{1i}) - (\nu_{ji} - \nu_{1i}) = -\nu_{ji}$, which are exactly our H_{ji}. □

Proposition 5. *For $i \ne j$, m_{ijk} is the number of hyperplanes between the vertex P_k and H_{ij}.*

Proof. Fix $i, j \le n$. P_k is on the hyperplane $x_i - x_j = (\nu_{ik} - \nu_{1k}) - (\nu_{jk} - \nu_{1k}) = \nu_{ik} - \nu_{jk}$. Since Γ is an order, we have $\nu_{ij} + \nu_{jk} \ge \nu_{ik}$, so $\nu_{ik} - \nu_{jk} \le \nu_{ij}$. Thus, the number of hyperplanes between H_{ij} (given by $x_i - x_j = \nu_{ij}$) and P_k is $\nu_{ij} - (\nu_{ik} - \nu_{jk}) = \nu_{ij} - \nu_{ik} + \nu_{jk} = m_{ijk}$.

In particular, if $j = k$ then $m_{ijk} = 0$, and by Lemma 4 P_k already is on H_{ik}, so the claim holds. □

Since the structural invariants encode geometric data, they determine the "shape" of the polytope, and in fact, in [16] (see [11, Proposition II.6]), Zassenhaus shows that the structural invariants (and therefore the "shape" of C_Γ) also encode the isomorphism class of the tiled order. Two tiled orders are isomorphic if they have the same structural invariants up to a permutation in S_n:

Proposition 6 (Zassenhaus, [16]). *Let Γ, Γ' be two tiled orders containing* $\operatorname{diag}(R, R, \dots, R)$, *and let m_{ijk} and respectively m'_{ijk} be their structural invariants. Then Γ and Γ' are isomorphic if and only if there exists $\sigma \in S_n$ such that $m'_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)}$ for all $1 \le i, j, k \le n$.*

Proof. This result is a particular case of Zassenhaus' result as described in [11, Proposition II.6]. Suppose the two orders are isomorphic. By the main theorem in [4, p. 107], there exists a monomial matrix $\xi \in B^\times$ with $\Gamma' = \xi\Gamma\xi^{-1}$, where $\xi = (\pi^{\alpha_i}\delta_{\sigma(i)j})$ for some $\sigma \in S_n$, $\alpha_i \in \mathbb{Z}$, and δ_{ij} the Kronecker delta. Let $\Gamma = (\mathfrak{p}^{\nu_{ij}})$, $\Gamma' = (\mathfrak{p}^{\nu'_{ij}})$. Conjugating by ξ we deduce

$$\nu'_{ij} = \alpha_i - \alpha_j + \nu_{\sigma(i)\sigma(j)}.$$

Therefore,

$$\begin{aligned} m'_{ijk} &= \nu'_{ij} + \nu'_{jk} - \nu'_{ik} \\ &= \alpha_i - \alpha_j + \nu_{\sigma(i)\sigma(j)} + \alpha_j - \alpha_k + \nu_{\sigma(j)\sigma(k)} - \alpha_i + \alpha_k - \nu_{\sigma(i)\sigma(k)} \\ &= m_{\sigma(i)\sigma(j)\sigma(k)}. \end{aligned}$$

Conversely, suppose we have $\tau \in S_n$ such that $m'_{ijk} = m_{\tau(i)\tau(j)\tau(k)}$ for all $1 \le i, j, k \le n$. Then let $\alpha_i = \nu'_{i1} - \nu_{\tau(i)\tau(1)}$, $i \le n$. Note that $\alpha_1 = 0$, and that also $\alpha_i = \nu_{\tau(1)\tau(i)} - \nu'_{1i}$, since

$$\nu'_{i1} + \nu'_{1i} = m'_{i1i} = m_{\tau(i)\tau(1)\tau(i)} = \nu_{\tau(i)\tau(1)} + \nu_{\tau(1)\tau(i)}.$$

If we let $\xi_{ij} = \pi^{\alpha_i}\delta_{\tau(i)j}$, then the exponents of $\xi\Gamma\xi^{-1}$ are $\alpha_i - \alpha_j + \nu_{\tau(i)\tau(j)}$, which gives

$$\begin{aligned} \alpha_i - \alpha_j + \nu_{\tau(i)\tau(j)} &= \nu'_{i1} - \nu_{\tau(i)\tau(1)} - \nu_{\tau(1)\tau(j)} + \nu'_{1j} + \nu_{\tau(i)\tau(j)} \\ &= \nu'_{i1} + \nu'_{1j} - \nu'_{ij} + \nu'_{ij} - \nu_{\tau(i)\tau(1)} - \nu_{\tau(1)\tau(j)} + \nu_{\tau(i)\tau(j)} \\ &= m'_{i1j} + \nu'_{ij} - m_{\tau(i)\tau(1)\tau(j)} \\ &= \nu'_{ij}, \end{aligned}$$

and therefore $\xi\Gamma\xi^{-1} = \Gamma'$, so the two orders are isomorphic. □

Therefore, the structural invariants determine the isomorphism class of an order. We can find the symmetries of a given isomorphism class, and more specifically the representatives of these symmetries in the normalizer for a given tiled order from its structural invariants as follows:

Proposition 7. *Let $\Gamma = (\mathfrak{p}^{\nu_{ij}})$ be a tiled order, $\{m_{ijk} \mid i, j, k \le n\}$ its set of structural invariants, and $\phi : \mathcal{N}(\Gamma) \to S_n$ the homomorphism defined earlier. If, for some $\sigma \in S_n$, $m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)}$ for all $i, j, k \le n$, then $\xi_\sigma = (\pi^{\alpha_i}\delta_{\sigma(i)j}) \in \mathcal{N}(\Gamma)$, where δ_{ij} is the Kronecker delta and $\alpha_i = \nu_{i1} - \nu_{\sigma(i)\sigma(1)}$. Furthermore, $\phi(\xi_\sigma) = \sigma$.*

Conversely, given $\xi \in \mathcal{N}(\Gamma)$, we have $m_{ijk} = m_{\sigma_\xi(i)\sigma_\xi(j)\sigma_\xi(k)}$ for all $i, j, k \le n$ where $\sigma_\xi := \phi(\xi)$.

Proof. Suppose that for some $\sigma \in S_n$, we have $m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)}$ for all $i, j, k \le n$. Setting $\xi_\sigma = (\pi^{\alpha_i}\delta_{\sigma(i)j})$, where $\alpha_i = \nu_{i1} - \nu_{\sigma(i)\sigma(1)}$, we get $\xi_\sigma\Gamma\xi_\sigma^{-1} = (\mathfrak{p}^{\nu'_{ij}})$, where $\nu'_{ij} = \alpha_i - \alpha_j + \nu_{\sigma(i)\sigma(j)}$. The second step in the proof of Proposition 6 then gives

$$\nu'_{ij} = \alpha_i - \alpha_j + \nu_{\sigma(i)\sigma(j)} = \nu_{ij},$$

so indeed $\Gamma = \xi_\sigma \Gamma \xi_\sigma^{-1}$ and $\xi_\sigma \in \mathcal{N}(\Gamma)$. Since ξ_σ is a monomial matrix, the action of ξ_σ on the distinguished vertices of the convex polytope C_Γ is determined by reflections across affine hyperplanes as determined by σ, and therefore $\phi(\xi_\sigma) = \sigma$.

Now suppose $\xi \in \mathcal{N}(\Gamma)$. By the discussion after Proposition 3, we have a monomial matrix $\eta \in \mathcal{N}(\Gamma)$ that permutes the n distinguished vertices the same way ξ does, so define $\sigma = \sigma_\xi := \phi(\xi) = \phi(\eta)$. Since η is monomial, we can write it as $\eta = (\pi^{\alpha_i} \delta_{\sigma(i)j})$, where δ_{ij} is the Kronecker delta and $\alpha_i \in \mathbb{Z}$. The calculation in the first step in the proof of Proposition 6 shows that $m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)}$ for all $i, j, k \le n$. □

Naive algorithm. Based on Proposition 7, a naive algorithm to find elements in the normalizer is to test each element of S_n to see whether $m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)}$ for all $i, j, k \le n$. However, this method doesn't reveal much about the structure of the normalizer, or which subgroups of S_n are realizable as the normalizer $\mathcal{N}(\Gamma)/k^\times \Gamma^\times$. Our goal for the remainder of the paper is to develop an algorithm which in addition to computing elements of the normalizer, also reveals information about the structure of $\mathcal{N}(\Gamma)/k^\times \Gamma^\times$ as a subgroup of S_n.

3. Centered orders

We now proceed to refine the naive algorithm above. We begin by introducing some geometric motivation for the construction of the tiled centered order Γ_0 in Theorem 8.

Suppose we have $\sigma \in S_n$ a symmetry of C_Γ for a tiled order $\Gamma = (\mathfrak{p}^{\nu_{ij}})$. As discussed in the previous section, defining $\xi_\sigma = (\pi^{\alpha_i} \delta_{\sigma(i)j})$, where $\alpha_i = \nu_{i1} - \nu_{\sigma(i)\sigma(1)}$, gives ξ_σ a monomial matrix such that $\Gamma = \xi_\sigma \Gamma \xi_\sigma^{-1}$. Since ξ_σ is monomial, ξ_σ has a decomposition $\xi_\sigma = d \cdot w_\sigma$, where d is a diagonal matrix and w_σ is a permutation matrix. Geometrically, conjugation of Γ by a diagonal matrix amounts to a translation of C_Γ, while conjugation by a permutation matrix corresponds to a product of reflections of C_Γ across hyperplanes going through the origin $[0, 0, \dots, 0]$.

Suppose we have a tiled order Γ_0 such that each monomial representative $\xi_\sigma \in \mathcal{N}(\Gamma_0)/k^\times \Gamma_0^\times$ has a decomposition $\xi = d \cdot w_\sigma$ with $d \in k^\times$ a scalar. By the discussion above, the origin $[0, 0, \dots, 0]$ is fixed under each symmetry of C_{Γ_0}, in which case we say that Γ_0 is *centered*. The following theorem shows how given any tiled order Γ, we can associate to it a centered tiled order Γ_0 whose convex polytope C_{Γ_0} has the same symmetries in S_n as C_Γ. The advantage of this choice of centered tiled order is that we need only check relations between exponents in M_{Γ_0} instead of checking relations between the n^3 structural invariants $\{m_{ijk}\}$ to find all the symmetries. Since there are only $n^2 - n$ off-diagonal exponents, this will be a small step refining our algorithm.

Theorem 8. *Given a tiled order $\Gamma = (\mathfrak{p}^{\nu_{ij}})$ with structural invariants $\{m_{ijk} = \nu_{ij} + \nu_{jk} - \nu_{ik} \mid 1 \le i, j, k \le n\}$, define $\Gamma_0 = (\mathfrak{p}^{\mu_{ij}})$ where $\mu_{ij} = \sum_{k=1}^n m_{ijk}$. Then Γ_0 is a centered tiled order with structural invariants $\tilde{m}_{ijk} = n \cdot m_{ijk}$ for all $1 \le i, j, k \le n$, and $\sigma \in S_n$ is a symmetry of C_Γ if and only if $\mu_{ij} = \mu_{\sigma(i)\sigma(j)}$.*

Proof. First we show Γ_0 is also a tiled order. Note that

$$\mu_{ii} = \sum_{k=1}^n m_{iik} = \sum_{k=1}^n (\nu_{ii} + \nu_{ik} - \nu_{ik}) = 0.$$

Γ_0 has structural invariants $\{\tilde{m}_{ijl} \mid 1 \le i, j, l \le n\}$ given by

$$\begin{aligned}\tilde{m}_{ijl} = \mu_{ij} + \mu_{jl} - \mu_{il} &= \sum_{k=1}^{n} m_{ijk} + \sum_{k=1}^{n} m_{jlk} - \sum_{k=1}^{n} m_{ilk} = \sum_{k=1}^{n} (m_{ijk} + m_{jlk} - m_{ilk}) \\ &= \sum_{k=1}^{n} (\nu_{ij} + \nu_{jk} - \nu_{ik} + \nu_{jl} + \nu_{lk} - \nu_{jk} - \nu_{il} - \nu_{lk} + \nu_{ik}) \\ &= \sum_{k=1}^{n} (\nu_{ij} + \nu_{jl} - \nu_{il}) = n \cdot m_{ijl} \ge 0,\end{aligned}$$

and since Γ itself is a tiled order and $m_{ijl} \ge 0$, it follows that Γ_0 is also a tiled order.

Next, we establish the bijection between the symmetries of C_Γ and the elements in S_n such that $\mu_{ij} = \mu_{\sigma(i)\sigma(j)}$. By Proposition 7 we need to show that

$$m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)} \text{ for all } i, j, k \le n \iff \mu_{ij} = \mu_{\sigma(i)\sigma(j)} \text{ for all } i, j \le n.$$

Suppose $\sigma \in S_n$ such that $m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)}$ for all $i, j, k \le n$. Then

$$\mu_{\sigma(i)\sigma(j)} = \sum_{k=1}^{n} m_{\sigma(i)\sigma(j)k} = \sum_{\sigma(k)=1}^{n} m_{\sigma(i)\sigma(j)\sigma(k)} = \sum_{\sigma(k)=1}^{n} m_{ijk} = \sum_{k=1}^{n} m_{ijk} = \mu_{ij}.$$

Conversely, if $\mu_{ij} = \mu_{\sigma(i)\sigma(j)}$, then

$$n \cdot m_{ijk} = \tilde{m}_{ijk} = \tilde{m}_{\sigma(i)\sigma(j)\sigma(k)} = n \cdot m_{\sigma(i)\sigma(j)\sigma(k)},$$

so by Proposition 7, σ is a symmetry of C_Γ.

Finally, we show that Γ_0 is centered. To show that the origin $[0, 0, \ldots, 0]$ is within the convex polytope C_{Γ_0} is almost immediate, since the origin sits on each hyperplane $x_i - x_j = 0$. Each such hyperplane satisfies the condition $-\mu_{ji} \le x_i - x_j \le \mu_{ij}$ because $\mu_{ij}, \mu_{ji} \ge 0$ as sums of nonnegative structural invariants.

Now we want to show that each symmetry of C_{Γ_0} fixes the origin. Note that since $\tilde{m}_{ijk} = n \cdot m_{ijk}$, the symmetries of C_Γ are the same as the symmetries of C_{Γ_0}. Given a symmetry $\sigma \in S_n$ of C_{Γ_0}, by Proposition 7 we obtain a representative $\xi_\sigma \in \mathcal{N}(\Gamma_0)$, where $\xi_\sigma = (\pi^{\alpha_i} \delta_{\sigma(i)j})$ and $\alpha_i = \mu_{i1} - \mu_{\sigma(i)\sigma(1)} = 0$, since we have just shown that $\mu_{ij} = \mu_{\sigma(i)\sigma(j)}$ for all $i, j \le n$. Therefore, ξ_σ is a permutation matrix and $\xi \in M_n(R)^\times$. Hence by [12], $\xi_\sigma \in M_n(R)^\times \subseteq k^\times M_n(R)^\times = \mathcal{N}(M_n(R))$. Conjugation by ξ_σ will fix $M_n(R)$, and therefore the σ will fix the vertex $[0, 0, \ldots, 0]$ associated to $M_n(R)$. Since this holds for every symmetry of C_{Γ_0}, we know Γ_0 is by definition centered. □

Example 4. Let Γ be the tiled order with

$$M_\Gamma = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -2 \\ 3 & 3 & 0 \end{bmatrix},$$

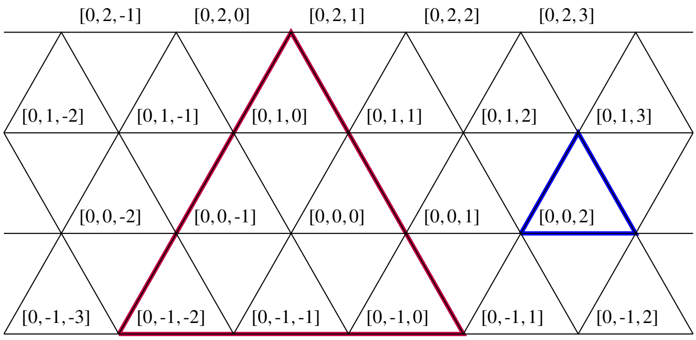

Figure 4. Polytopes from Example 4, with C_Γ shown in blue and C_{Γ_0} in red.

with C_Γ depicted in Figure 4 in blue. By Proposition 7, a representative ξ_σ in the normalizer of Γ is

$$\xi_\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \pi^{-2} \\ \pi^3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pi^{-2} & 0 \\ 0 & 0 & \pi^3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The associated tiled order is Γ_0 with

$$M_{\Gamma_0} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix},$$

with convex polytope depicted in Figure 4 in red. Since $\nu_{12} = \nu_{23} = \nu_{31}$, $\nu_{13} = \nu_{21} = \nu_{32}$, we get a representative of the normalizer

$$\xi_\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

For $n = 2$, Hijikata [7] showed that if Γ is nonmaximal, then $\mathcal{N}(\Gamma)/k^\times\Gamma^\times \cong \mathbb{Z}/2\mathbb{Z}$. For $n = 3$, $\mathcal{N}(\Gamma)/k^\times\Gamma^\times \subseteq S_3$ and in fact all subgroups of S_3 are realizable as symmetry groups of convex polytopes of tiled orders; we have seen two such subgroups in Examples 2 and 3. As n increases, there are however a number of subgroups of S_n that are not realizable as the symmetry group of C_Γ. In particular, we have the following easy corollary to Theorem 8:

Corollary 9. *Suppose we have a tiled order Γ and $\phi : \mathcal{N}(\Gamma) \to S_n$ the homomorphism defined earlier. If H is a 2-transitive subgroup of S_n, then $H \subseteq \phi(\mathcal{N}(\Gamma))$ implies $\mathcal{N}(\Gamma)/k^\times\Gamma^\times \cong S_n$.*

Proof. Let $\Gamma_0 = (\mathfrak{p}^{\mu_{ij}})$ be the associated centered order of Γ. H being 2-transitive means that given any pairs (i, j), (k, l) with $i \neq j$ and $k \neq l$, there exists $\sigma \in H$ such that $\sigma(i) = k$ and $\sigma(j) = l$. Since H is contained in the image of the normalizer, $\mu_{ij} = \mu_{\sigma(i)\sigma(j)} = \mu_{kl}$. Therefore, all of the off-diagonal exponents are equal and $\mu_{ij} = \mu_{\sigma(i)\sigma(j)}$ for all $\sigma \in S_n$, so $\mathcal{N}(\Gamma_0)/k^\times\Gamma_0^\times \cong \mathcal{N}(\Gamma)/k^\times\Gamma^\times \cong S_n$. □

In this section we have shown that we can completely determine the normalizer of a tiled order by examining the exponents of its associated centered order. By Theorem 8, a refined algorithm to find elements in the normalizer is to go through the $n^2 - n$ off-diagonal elements to check for which $\sigma \in S_n$ we have $\mu_{ij} = \mu_{\sigma(i)\sigma(j)}$. This is of course a very small improvement, since we still have to check the above relations for elements $\sigma \in S_n$. In the next section we realize the symmetries of C_Γ as the automorphism group of a directed valued multigraph, which can add to the efficiency of the above algorithm.

4. The normalizer as the automorphisms of a valued quiver

For our main algorithm, we will make use of a realization of the normalizer of a centered tiled order as the automorphism group of a certain valued directed multigraph, also known as a valued quiver.

We construct the *link graph* of $\Gamma = (\mathfrak{p}^{\nu_{ij}})$ as defined by Müller in [9]. Let $M_1, M_2, \ldots, M_n$ be the maximal 2-sided ideals of Γ. It can be shown that $M_\ell = (\mathfrak{p}^{r_{ij}})$, where $r_{ij} = \nu_{ij}$ if $\ell \neq i, j$, and $r_{\ell\ell} = 1$. The vertices of the link graph are labeled by the set $\{1, 2, \ldots, n\}$, and there is an arrow $\alpha : i \to j$ when $M_j M_i \neq M_j \cap M_i$ and the value associated to the arrow α is $v(\alpha) = \nu_{ij}$.

Remark. There is an equivalent way to define the link graph, by projective covers of the Jacobson radicals for the projective left Γ-lattices. However, these directed multigraphs have the arrows pointed the opposite direction. For more information, see [15] for the construction of the graphs and [2] for the proof of the equivalence of the two constructions.

To compute the link graph, we reproduce the following result from [3]:

Lemma 10 (Fujita, Oshima [3]). *Given a tiled order Γ, there is an arrow $i \to j$ in $Q_v(\Gamma)$ if $m_{jki} > 0$ for all $k \neq i, j$, and there is an arrow $i \to i$ if $m_{iki} > 1$ for all $k \neq i$.*

Proof. See [3, p. 578]. However, note that Fujita and Oshima follow a convention where the arrows are pointed in the opposite direction. □

In [6, Lemmas 1 and 3], Haefner and Pappacena identify a subgroup of the automorphisms of the unvalued quiver $Q(\Gamma)$ with monomial representatives of $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$, which we have already found to be in bijection with the symmetries of C_Γ. In [6, Theorem 5], they prove that $\sigma \in \mathrm{Aut}(Q(\Gamma)) \subseteq S_n$ is liftable to a symmetry of C_Γ if the system

$$x_i - x_j = \nu_{ij} - \nu_{\sigma(i)\sigma(j)}, \quad i < j,$$

has a solution $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$.

We can instead consider automorphisms of the valued quiver $Q_v(\Gamma)$. While as shown in [6, Example 2], the symmetries of C_Γ don't always give us an automorphism of $Q_v(\Gamma)$, they do when we have a centered tiled order:

Theorem 11. *Given a centered tiled order $\Gamma_0 = (\mathfrak{p}^{\nu_{ij}})$, there is a bijection between* $\mathrm{Aut}(Q_v(\Gamma_0))$ *and the symmetries of C_{Γ_0}.*

Proof. Let $\sigma \in \mathrm{Aut}(Q_v(\Gamma_0))$. Then $\sigma \in \mathrm{Aut}(Q(\Gamma_0))$ is also an automorphism of the unvalued quiver, and Haefner and Pappacena have shown in [6, Theorem 5] that σ is liftable to a symmetry of C_{Γ_0} if and only if the linear system

$$x_i - x_j = \nu_{ij} - \nu_{\sigma(i)\sigma(j)}, \quad i < j,$$

has a solution $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$. Since $\sigma \in \mathrm{Aut}(Q_v(\Gamma_0))$, for any valued arrow $\alpha : i \to j$, there is an arrow $\beta : \sigma(i) \to \sigma(j)$, and its value is $v(\beta) = v(\alpha)$. Since $v(\alpha) = \nu_{ij}$ and $v(\beta) = \nu_{\sigma(i)\sigma(j)}$, this implies $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \le n$, so the system above has a solution $(0, 0, \dots, 0) \in \mathbb{Z}^n$. Therefore, by [6], σ lifts to a symmetry of C_{Γ_0}.

Now suppose σ is a symmetry of C_{Γ_0}. By [6, Lemma 1], σ is an automorphism of the unvalued quiver $Q(\Gamma_0)$, so for a given arrow $\alpha : i \to j$, there is an arrow $\beta : \sigma(i) \to \sigma(j)$. To show that σ is also an automorphism of the valued quiver $Q_v(\Gamma_0)$, we need in addition that $v(\beta) = v(\alpha)$. Since Γ_0 is centered, we have from Theorem 8 that $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$. But $v(\alpha) = \nu_{ij}$ and $v(\beta) = \nu_{\sigma(i)\sigma(j)}$, so the result follows. □

As described in Theorem 11, given the valued quiver of a centered order, we can determine the symmetries of C_Γ and therefore representatives of the normalizer $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ by finding the automorphisms of the valued quiver. First note that given two vertices i and j, an automorphism of the quiver can only permute them if the incoming arrows of i have the same values as the incoming arrows of j, and the same holds for outgoing arrows. Therefore, for each vertex i, we can associate two multisets, one with values for incoming arrows, and the other with values for outgoing arrows. Then we need only look for elements in S_n that would permute both multisets when permuting vertices.

Finally, we summarize the algorithm, where given a tiled order $\Gamma = (\mathfrak{p}^{\nu_{ij}}) \subseteq M_n(k)$, we find $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ as a subgroup of S_n. We illustrate each step with an example.

Algorithm. (1) For a given tiled order Γ with exponent matrix $M_\Gamma = (\nu_{ij})$ and structural invariants $\{m_{ijk}\}_{1\le i,j,k\le n}$, compute its associated centered order Γ_0 with exponent matrix $M_{\Gamma_0} = (\mu_{ij})$, where $\mu_{ij} = \sum_{k=1}^n m_{ijk}$.

Example 5. Let Γ have exponent matrix

$$M_\Gamma = \begin{bmatrix} 0 & 1 & 3 & 3 & 1 \\ 2 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

Γ_0 is given by

$$M_{\Gamma_0} = (\mu_{ij}) = \begin{bmatrix} 0 & 5 & 10 & 10 & 5 \\ 10 & 0 & 5 & 10 & 5 \\ 5 & 5 & 0 & 10 & 10 \\ 5 & 10 & 5 & 0 & 10 \\ 10 & 10 & 5 & 5 & 0 \end{bmatrix}.$$

(2) Find the valued quiver of Γ_0:

(a) The vertices are $1, 2, \dots, n$.

(b) Let $\tilde{m}_{ikj} = \mu_{ik} + \mu_{kj} - \mu_{ij} = n \cdot m_{ikj}$. For $i \neq j$, there is an arrow $\alpha : i \to j$ if $\tilde{m}_{ikj} > 0$ for all $k \neq i, j$. There is an arrow $\alpha : i \to i$ if $\tilde{m}_{iki} > 1$ for all $k \neq i$.

(c) Given an arrow $\alpha : i \to j$, set $v(\alpha) = \mu_{ij}$.

We can represent $Q_v(\Gamma_0)$ by the $n \times n$ matrix $M(Q_v(\Gamma_0)) = (a_{ij})$, where a_{ij} is blank if there is no arrow from i to j, and $a_{ij} = \mu_{ij}$ if there is an arrow $\alpha : i \to j$ and $v(\alpha) = \mu_{ij}$.

Example 5 (continued). The quiver $Q_v(\Gamma_0)$ is given by the matrix

$$\begin{bmatrix} 0 & & 10 & 10 & \\ 10 & 0 & 5 & & \\ & 5 & 0 & 10 & 10 \\ & & 5 & 0 & 10 \\ 10 & 10 & & & 0 \end{bmatrix}.$$

(3) For each vertex i, let I_i be the multiset of incoming arrow values and O_i be the multiset of outgoing arrow values. Partition the sets $\{I_i\}_{i=1}^n$ and $\{O_i\}_{i=1}^n$ into equal multisets.

Example 5 (continued). The multisets I_i are the columns of the above matrix, and the multisets O_i are the rows. Note that

$$I_1 = I_4 = I_5 = \{0, 10, 10\}, \quad I_2 = \{0, 5, 10\} \quad \text{and} \quad I_3 = \{0, 5, 5, 10\},$$

so we partition the I_i's into $\{I_1, I_4, I_5\}$, $\{I_2\}$ and $\{I_3\}$. Since

$$O_1 = O_5 = \{0, 10, 10\}, \quad O_2 = O_4 = \{0, 5, 10\} \quad \text{and} \quad O_3 = \{0, 5, 10, 10\},$$

we partition the O_i's into $\{O_1, O_5\}$, $\{O_2, O_4\}$ and $\{O_3\}$.

(4) Consider the partition of the I_i's. An automorphism of the quiver can only permute vertices with the same values for incoming arrows, so if the sets in the partition have lengths $l_1, l_2, \dots, l_q$, $\sum_{j=1}^{q} l_j = n$, then the normalizer $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ is a subgroup of $S_{l_1} \times S_{l_2} \times \cdots \times S_{l_q} \subseteq S_n$, where each S_{l_i} is a copy of the symmetric group on l_i elements.

Now consider the partition of the O_i's. Similarly, an automorphism of the quiver can only permute vertices with the same values for outgoing arrows, so $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ is a subgroup of $S_{t_1} \times S_{t_2} \times \cdots \times S_{t_r} \subseteq S_n$, where $t_1, t_2, \dots, t_r$ are the lengths of the sets in the partition of the O_i's and $\sum_{j=1}^{r} t_j = n$.

Therefore, the normalizer $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ is a subgroup of $(S_{l_1} \times S_{l_2} \times \cdots \times S_{l_q}) \cap (S_{t_1} \times S_{t_2} \times \cdots \times S_{t_r})$.

Example 5 (continued). From the partitions in (4), we get that $\mathcal{N}(\Gamma)/k^\times\Gamma^\times \subseteq S_{\{1,4,5\}}$, where as usual $S_{\{1,4,5\}}$ is the symmetric group on the set $\{1, 4, 5\}$. Similarly, we get $\mathcal{N}(\Gamma)/k^\times\Gamma^\times \subseteq S_{\{1,5\}} \times S_{\{2,4\}}$. Therefore,

$$\mathcal{N}(\Gamma)/k^\times\Gamma^\times \subseteq S_{\{1,4,5\}} \cap (S_{\{1,5\}} \times S_{\{2,4\}}) = S_{\{1,5\}}.$$

(5) Check for which elements σ in the intersection found in (4) we have $a_{ij} = a_{\sigma(i)\sigma(j)}$, where a_{ij} are the entries in the matrix defined in (2). The union of these elements is the normalizer $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$.

Example 5 (continued). We need to check whether $a_{ij} = a_{\sigma(i)\sigma(j)}$ for $\sigma = (1, 5)$. Since a_{15} is blank, but $a_{51} = 10$, we conclude that σ is not a symmetry of C_Γ, so $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ has to be trivial.

Remark. We now describe the modifications of our algorithm when Γ is degenerate, so some of the columns correspond to the same homothety classes of lattices. When we create the valued quiver in step (2), if columns P_i and P_j correspond to the same homothety class, we identify vertices i and j together. Therefore, after computing in step (5) the subgroup $H = \{\sigma \in S_n \mid a_{ij} = a_{\sigma(i)\sigma(j)}\}$, where a_{ij} are the entries in $M(Q_v(\Gamma_0))$, the normalizer $\mathcal{N}(\Gamma)/k^\times\Gamma^\times$ is the quotient of H by the product of the symmetric groups that permute each set of equivalent vertices.

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