# SUPPLEMENTARY MATERIAL

The explicit formulas presented on the following pages were typeset using latex source generated by an automated script that reads an executable version of verified source code; they should thus be free of the typos that unfortunately plague many of the formulas one finds in the literature. Magma source code for the formulas and an implementation of all the algorithms in this article can be found at the author's website, along with scripts that verify their correctness.

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TypicalAddition: $\mathrm{div}[u_5, v_5, n_5] \sim \mathrm{div}[u_1, v_1, 0] + \mathrm{div}[u_2, v_2, 0]$ with $\gcd(u_1, u_2) = 1$.
1. Compute $r := \mathrm{Res}(u_1, u_2)$ and $i(x) = i_2x^2 + i_1x + i_0 := ru_1^{-1} \bmod u_2$ (and $w_0 := u_{11} - u_{12}$).   [15M+12A]
    $t_1 := u_{10} - u_{20}$;   $t_2 := u_{11} - u_{21}$;   $w_0 := u_{12} - u_{22}$;   $t_3 := t_2 - u_{22}w_0$;
    $t_4 := t_1 - u_{21}w_0$;   $t_5 := u_{22}t_3 - t_4$;   $t_6 := u_{20}w_0 + u_{21}t_3$;
    $i_0 := t_4t_5 - t_3t_6$;   $i_1 := w_0t_6 - t_2t_5$;   $i_2 := w_0t_4 - t_2t_3$;
    $r := t_1i_0 - u_{20}(t_3i_2 + w_0i_1)$;
2. Compute $q(x) = q_2x^2 + q_1x + q_0 := r(v_2 - v_1)\,u_1^{-1} \bmod u_2$.   [10M+30A]
    $t_1 := v_{20} - v_{10}$;   $t_2 := v_{11} - v_{21}$;   $t_3 := v_{12} - v_{22}$;   $t_4 := t_2i_1$;   $t_5 := t_1i_0$;   $t_6 := t_3i_2$;   $t_7 := u_{22}t_6$;
    $t_8 := t_4 + t_6 + t_7 - (t_2 + t_3)(i_1 + i_2)$;   $t_9 := u_{20} + u_{22}$;   $t_{10} := (t_9 + u_{21})(t_8 - t_6)$;   $t_{11} := (t_9 - u_{21})(t_8 + t_6)$;
    $q_0 := t_5 - u_{20}t_8$;
    $q_1 := t_4 - t_5 + (t_{11} - t_{10})/2 - t_7 + (t_1 - t_2)(i_0 + i_1)$;
    $q_2 := t_6 - q_0 - t_4 + (t_1 - t_3)(i_0 + i_2) - (t_{10} + t_{11})/2$;
3. Compute $t_1 := rq_2\tilde{v}_{43}$ via (1), and $w_1 := c^{-1} = q_2/r$,  $w_2 := c = r/q_2$,  $w_3 := c^2$,  $w_4 := (2\tilde{v}_{43})^{-1}$.
   Then compute $s(x) = x^2 + s_1x + s_0 := c(v_2 - v_1)\,u_1^{-1} \bmod u_2$ and $\tilde{v}_{43}$.   [I+18M+5A]
    $t_1 := (r + q_1)^2 + q_2(rw_0 + q_2u_{21} - q_1u_{22} - q_0)$;   $t_2 := 2t_1$;   $t_3 := rq_2$;
    If $t_2 = 0$ or $t_3 = 0$ then abort (revert to Addition).
    $t_4 := 1/(t_2t_3)$;   $t_5 := t_2t_4$;   $t_6 := rt_5$;
    $w_1 := t_5q_2^2$;   $w_2 := rt_6$;   $w_3 := w_2^2$;   $w_4 := t_3^2t_4$;
    $s_0 := t_6q_0$;   $s_1 := t_6q_1$;
    $\tilde{v}_{43} := t_1t_5$;
4. Compute $z(x) = x^5 + z_4x^4 + z_3x^3 + z_2x^2 + z_1x + z_0 := su_1$.   [4M+15A]
    $t_6 := s_0 + s_1$;   $t_1 := u_{10} + u_{12}$;   $t_2 := t_6(t_1 + u_{11})$;   $t_3 := (t_1 - u_{11})(s_0 - s_1)$;   $t_4 := u_{12}s_1$;
    $z_0 := u_{10}s_0$;   $z_1 := (t_2 - t_3)/2 - t_4$;   $z_2 := (t_2 + t_3)/2 - z_0 + u_{10}$;   $z_3 := u_{11} + s_0 + t_4$;   $z_4 := u_{12} + s_1$;
5. Compute $u_4(x) = x^4 + u_{43}x^3 + u_{42}x^2 + u_{41}x + u_{40} := (s(z + 2cv_1) - c^2(f - v_1^2)/u_1)/u_2$.   [14M+31A]
    $u_{43} := z_4 + s_1 - u_{22}$;
    $t_0 := s_1z_4$;   $t_1 := u_{22}u_{43}$;
    $u_{42} := z_3 + t_0 + s_0 - w_3 - u_{21} - t_1$;
    $t_2 := u_{21}u_{42}$;   $t_3 := (u_{21} + u_{22})(u_{42} + u_{43}) - t_1 - t_2$;   $t_4 := 2w_2$;
    $t_5 := t_4v_{12}$;   $t_6 := s_0z_3$;   $t_7 := (s_0 + s_1)(z_3 + z_4) - t_0 - t_6$;
    $u_{41} := z_2 + t_7 + t_5 + w_3u_{12} - u_{20} - t_3$;
    $u_{40} := z_1 + s_1(t_5 + z_2) + t_6 + t_4v_{11} - w_3(f_6 + u_{12}^2 - u_{11}) - u_{20}u_{43} - t_2 - u_{22}u_{41}$;
6. Compute $\tilde{v}_4(x) = x^4 + \tilde{v}_{43}x^3 + \tilde{v}_{42}x^2 + \tilde{v}_{41}x + \tilde{v}_{40} := -\hat{v}_4 = v_1 + u_4 + c^{-1}(z \bmod u_4)$.   [6M+10A]
    $t_1 := u_{43} - z_4 + w_2$;
    $\tilde{v}_{40} := v_{10} + w_1(z_0 + u_{40}t_1)$;
    $\tilde{v}_{41} := v_{11} + w_1(z_1 - u_{40} + u_{41}t_1)$;
    $\tilde{v}_{42} := v_{12} + w_1(z_2 - u_{41} + u_{42}t_1)$;
7. Compute $u_5(x) = x^3 + u_{52}x^2 + u_{51}x + u_{50} := (2\tilde{v}_{43})^{-1}(\tilde{v}_4^2 - f)/u_4$.   [9M+17A]
    $u_{52} := \tilde{v}_{43}/2 + w_4(2\tilde{v}_{42} - f_6) - u_{43}$;
    $u_{51} := w_4(2(\tilde{v}_{41} + \tilde{v}_{43}\tilde{v}_{42}) - f_5) - u_{52}u_{43} - u_{42}$;
    $u_{50} := w_4(\tilde{v}_{42}^2 + 2(\tilde{v}_{40} + \tilde{v}_{43}\tilde{v}_{41}) - f_4) - u_{51}u_{43} - u_{52}u_{42} - u_{41}$;
8. Compute $v_5(x) = v_{52}x^2 + v_{51}x + v_{50} := \tilde{v}_4 \bmod u_5$.   [3M+6A]
    $t_1 := u_{52} - \tilde{v}_{43}$;
    $v_{50} := \tilde{v}_{40} + t_1u_{50}$;
    $v_{51} := \tilde{v}_{41} - u_{50} + t_1u_{51}$;
    $v_{52} := \tilde{v}_{42} - u_{51} + t_1u_{52}$;
9. Output $\mathrm{div}[u_5, v_5, 3 - \deg u_5]$.   [Total: I+79M+126A]
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TypicalDoubling: $\mathrm{div}[u_5, v_5, n_4] \sim 2\,\mathrm{div}[u_1, v_1, 0]$ with $\gcd(u_1, v_1) = 1$.
1. Compute $r := \mathrm{Res}(u_1, v_1)$ and $i(x) = i_2x^2 + i_1x + i_0 := rv_1^{-1} \bmod u_1$ $(w_0 := v_{11} - u_{12}v_{12})$.   [15M+9A]
    $w_0 := v_{11} - u_{12}v_{12}$;   $t_2 := v_{10} - u_{11}v_{12}$;   $t_3 := u_{12}w_0 - t_2$;   $t_4 := u_{10}v_{12} + u_{11}w_0$;
    $i_0 := w_0t_4 - t_2t_3$;   $i_1 := v_{11}t_3 - v_{12}t_4$;   $i_2 := v_{11}w_0 - v_{12}t_2$;
    $r := v_{10}i_0 - u_{10}(w_0i_2 + v_{12}i_1)$;
2. Compute $p(x) = p_2x^2 + p_1x + p_0 := \overline{w} := (f - v_1^2)/u_1 \bmod u_1$ $(w_1 := u_{12}^2,\ w_2 := w_1 + f_6)$.   [11M+24A]
    $w_1 := u_{12}^2$;   $t_2 := 2u_{10}$;   $t_3 := 3u_{11}$;   $w_2 := w_1 + f_6$;   $t_5 := 2t_2 - f_5$;   $t_6 := 2u_{12}$;   $t_7 := t_3 - w_2$;
    $p_2 := f_5 + t_6(t_7 - w_1) - t_2$;
    $p_1 := f_4 + u_{12}t_5 - v_{12}^2 - u_{11}(2f_6 - t_3) - w_1(t_7 + t_3)$;
    $p_0 := f_3 - u_{11}(w_1t_6 - t_5) - t_2w_2 - u_{12}p_1 - 2v_{11}v_{12}$;
3. Compute $q(x) = q_2x^2 + q_1x + q_0 := r((f - v_1^2)/u_1)v_1^{-1} \bmod u_1$.   [10M+28A]
   $(w_3 := u_{10} + u_{11} + u_{12},\ \ w_4 := u_{10} - u_{11} + u_{12})$
    $t_1 := i_1p_1$;   $t_2 := i_0p_0$;   $t_3 := i_2p_2$;   $t_4 := u_{12}t_3$;   $t_5 := (i_1 + i_2)(p_1 + p_2) - t_1 - t_3 - t_4$;   $t_6 := u_{10}t_5$;
    $t_7 := u_{10} + u_{12}$;   $w_3 := t_7 + u_{11}$;   $w_4 := t_7 - u_{11}$;   $t_{10} := w_3(t_3 + t_5)$;   $t_{11} := w_4(t_5 - t_3)$;
    $q_0 := t_2 - t_6$;
    $q_1 := t_4 + (i_0 + i_1)(p_0 + p_1) + (t_{11} - t_{10})/2 - t_1 - t_2$;
    $q_2 := t_1 + t_6 + (i_0 + i_2)(p_0 + p_2) - t_2 - t_3 - (t_{10} + t_{11})/2$;
4. Compute $t_3 := 2rq_2\tilde{v}_{43}$ via (2), and $w_5 := 1/c$,  $w_6 := c, w_7 := 1/\tilde{v}_{43}$.   [I+18M+7A]
   Then compute $s(x) = x^2 + s_1x + s_0 := q/(2r)$ made monic and $\tilde{v}_{43}$.
    $t_0 := 2r$;   $t_1 := t_0^2$;   $t_2 := q_2^2$;   $t_3 := t_1 - q_0q_2 + q_1(2t_0 + q_1 - q_2u_{12}) + t_2u_{11}$;
    If $q_2 = 0$ or $t_3 = 0$ then abort (revert to Addition).
    $t_4 := 1/(t_0q_2t_3)$;   $t_5 := t_3t_4$;   $t_6 := t_0t_5$;
    $w_5 := t_2t_5$;   $w_6 := t_1t_5$;   $w_7 := t_1t_2t_4$;
    $s_0 := t_6q_0$;   $s_1 := t_6q_1$;   $\tilde{v}_{43} := t_3t_5$;
5. Compute $z(x) = x^5 + z_4x^4 + z_3x^3 + z_2x^2 + z_1x + z_0 := su_1$.   [4M+12A]
    $t_1 := w_3(s_0 + s_1)$;   $t_2 := w_4(s_0 - s_1)$;   $t_3 := u_{12}s_1$;
    $z_0 := s_0u_{10}$;   $z_1 := (t_1 - t_2)/2 - t_3$;   $z_2 := (t_1 + t_2)/2 - z_0 + u_{10}$;   $z_3 := u_{11} + s_0 + t_3$;   $z_4 := u_{12} + s_1$;
6. Compute $u_4(x) = x^4 + u_{43}x^3 + u_{42}x^2 + u_{41}x + u_{40} := s^2 - (c^2(f - v_1^2)/u_1 - 2csv_1)/u_1$.   [8M+14A]
    $t_1 := v_{12}w_6$;   $t_2 := w_6^2$;
    $u_{43} := 2s_1$;
    $u_{42} := 2s_0 + s_1^2 - t_2$;
    $u_{41} := 2(s_0s_1 + u_{12}t_2 + t_1)$;
    $u_{40} := s_0^2 + 2(w_0w_6 + s_1t_1) - t_2(w_2 + 2(w_1 - u_{11}))$;
7. $\tilde{v}_4(x) = \tilde{v}_{43}x^3 + \tilde{v}_{42}x^2 + \tilde{v}_{41}x + \tilde{v}_{40} := -\hat{v}_4 = v_1 + u_4 + c^{-1}(z \bmod u_4)$.   [6M+10A]
    $t_1 := u_{43} - z_4 + w_6$;
    $\tilde{v}_{40} := v_{10} + w_5(z_0 + u_{40}t_1)$;
    $\tilde{v}_{41} := v_{11} + w_5(z_1 - u_{40} + u_{41}t_1)$;
    $\tilde{v}_{42} := v_{12} + w_5(z_2 - u_{41} + u_{42}t_1)$;
8. $u_5(x) = x^3 + u_{52}x^2 + u_{51}x + u_{50} := (2\tilde{v}_{43})^{-1}(\tilde{v}_4^2 - f)/u_4$.   [7M+17A]
    $u_{52} := \tilde{v}_{43}/2 + w_7(\tilde{v}_{42} - f_6/2) - u_{43}$;
    $u_{51} := \tilde{v}_{42} + w_7(\tilde{v}_{41} - f_5/2) - u_{52}u_{43} - u_{42}$;
    $u_{50} := \tilde{v}_{41} + w_7((\tilde{v}_{42}^2 - f_4)/2 + \tilde{v}_{40}) - u_{51}u_{43} - u_{52}u_{42} - u_{41}$;
9. $v_5(x) = v_{52}x^2 + v_{41}x + v_{50} := \tilde{v}_4 \bmod u_5$.   [3M+6A]
    $t_1 := u_{52} - \tilde{v}_{43}$;
    $v_{50} := \tilde{v}_{40} + t_1u_{50}$;
    $v_{51} := \tilde{v}_{41} - u_{50} + t_1u_{51}$;
    $v_{52} := \tilde{v}_{42} - u_{51} + t_1u_{52}$;
10. Output $\mathrm{div}[u_4, v_4, 3 - \deg u_4]$.   [Total: I+82M+127A]
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TypicalNegation: $\mathrm{div}[u_2, v_2, 0] \sim -\mathrm{div}[u_1, v_1, 0]$.
1. Compute $\tilde{v}_1(x) = -x^4 + \tilde{v}_{12}x^2 + \tilde{v}_{11}x + \tilde{v}_{10} := v_1 - V + (V \bmod u_1)$.    [3M+5A]
    $\tilde{v}_{12} := v_{12} - u_{11} + u_{12}^2$;
    $\tilde{v}_{11} := v_{11} - u_{10} + u_{11}u_{12}$;
    $\tilde{v}_{10} := v_{10} + u_{10}u_{12}$;
2. Compute $u_2(x) = x^3 + u_{22}x^2 + u_{21}x + u_{20} := (f_6 + 2\tilde{v}_{12})^{-1}(f - \tilde{v}_1^2)/u_1$.    [I+8M+14A]
    $t_1 := 2\tilde{v}_{12}$;    $t_2 := f_6 + t_1$;
    If $t_1 = 0$ then abort (revert to Negation).
    $t_3 := 1/t_2$;
    $u_{22} := t_3(f_5 + 2\tilde{v}_{11}) - u_{12}$;
    $u_{21} := t_3(f_4 + 2\tilde{v}_{10} - \tilde{v}_{12}^2) - u_{11} - u_{12}u_{22}$;
    $u_{20} := t_3(f_3 - t_1\tilde{v}_{11}) - u_{10} - u_{11}u_{22} - u_{12}u_{21}$;
3. Compute $v_2(x) = v_{22}x^2 + v_{21}x + v_{20} := \tilde{v}_1 \bmod u_2$.    [3M+5A]
    $v_{22} := \tilde{v}_{12} - u_{22}^2 + u_{21}$;
    $v_{21} := \tilde{v}_{11} - u_{21}u_{22} + u_{20}$;
    $v_{20} := \tilde{v}_{10} - u_{20}u_{22}$;
4. Output $\mathrm{div}[u_2, v_2, 0]$.    [Total: I+14M+24A]
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