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Gauge Theory and Low-Dimensional Topology: Progress and Interaction

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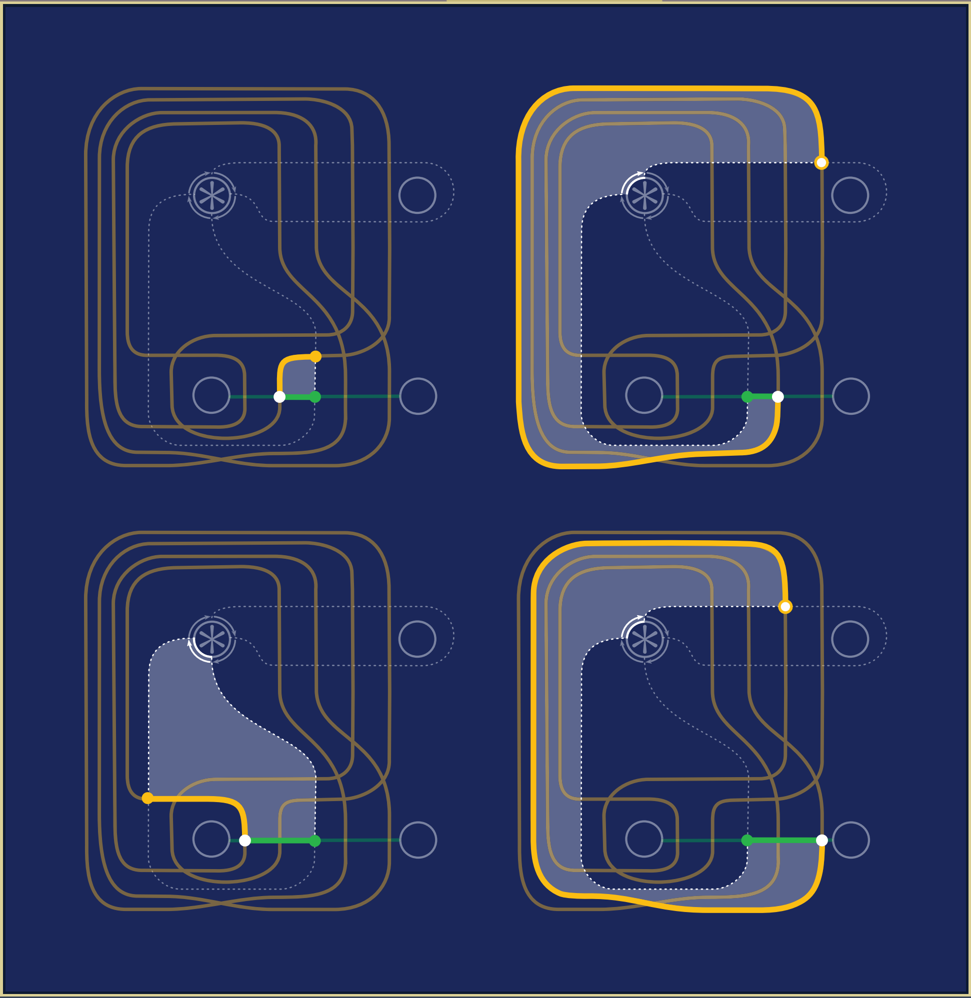

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We give a new, conceptually simpler proof of the fact that knots in S^3 with positive L-space surgeries are fibered and strongly quasipositive. Our motivation for doing so is that this new proof uses comparatively little Heegaard Floer-specific machinery and can thus be translated to other forms of Floer homology. We carried this out for instanton Floer homology in our article "Instantons and L-space surgeries" and used it to generalize Kronheimer and Mrowka's results on SU(2) representations of fundamental groups of Dehn surgeries.

The hat version $\widehat{HF}(Y)$ of Heegaard Floer homology, which we will take with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout, carries an absolute $\mathbb{Z}/2\mathbb{Z}$ grading such that

$$\chi(\widehat{HF}(Y, \mathfrak{s})) = \begin{cases} 1 & \text{if } b_1(Y) = 0, \\ 0 & \text{if } b_1(Y) \geq 1, \end{cases} \tag{1}$$

for all $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ [13, Proposition 5.1]. Thus for any rational homology 3-sphere Y, we have

$$\dim \widehat{HF}(Y) \geq \chi(\widehat{HF}(Y)) = |H_1(Y; \mathbb{Z})|.$$

A rational homology 3-sphere Y is an *L-space* if

$$\dim \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|.$$

Theorem 1 [8; 10; 15; 17]. *If $S^3_r(K)$ is an L-space for some rational slope $r > 0$, then K is fibered and strongly quasipositive*, *and $r \geq 2g(K) - 1$.*

All proofs of Theorem 1 in the literature use at least some of the following tools: the doubly-filtered Heegaard Floer complex associated to a knot, the large integer surgery formula, the $(\infty, 0, n)$-surgery exact triangle for $n > 1$, and the Spin^c decomposition of $\widehat{HF}(Y)$ for Y a rational homology sphere. This presents a major difficulty if one wishes to port this theorem to the instanton Floer setting, where none of this machinery is available.

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Remark 2. A primary motivation for proving an analogue of Theorem 1 in the instanton Floer setting in particular is that such an analogue can be used to prove new results about the SU(2) representation varieties of fundamental groups of 3-manifolds obtained by Dehn surgeries on knots in the 3-sphere, about which relatively little is known; see [4].

Remark 3. Some of the structure mentioned above is known to exist in monopole Floer homology, though not enough of it to translate previous proofs of Theorem 1 to that setting. The new proof of Theorem 1 presented in this article (see below) *can* be adapted directly to monopole Floer homology, with the caveat in Remark 4, to give a proof of the monopole Floer analogue of Theorem 1 which does not rely on an isomorphism between monopole Floer homology and Heegaard Floer homology.

Our goal here is to give a proof of Theorem 1 using instead: the $(\infty, 0, 1)$-surgery exact triangle, the blow-up formula for cobordism maps, the adjunction inequality for cobordism maps, the Spin^c decomposition of the maps associated to 2-handle cobordisms, and Ozsváth and Szabó's description of the contact invariant $c^+(\xi)$ as the image of a certain class under the 2-handle cobordism map

$$HF^+(-S^3_0(K)) \to HF^+(-S^3),$$

where K is a fibered knot supporting the contact structure ξ on S^3. The first four of these tools will be used to show that an L-space knot is fibered, while the last will be used to prove that an L-space knot supports the tight contact structure on S^3 and is therefore strongly quasipositive, by Hedden [8]. Strong quasipositivity will then be used to prove the $2g(K)-1$ bound on L-space surgery slopes.

Remark 4. Ozsváth and Szabó do not prove that $c^+(\xi)$ is well-defined (and hence that it certifies that ξ is tight) directly from its description in terms of the cobordism map associated to 0-surgery on the supporting fibered knot (and it is unclear how to do so—this is an interesting problem!). They instead use the knot filtration for this, which poses a challenge for translating the strong quasipositivity argument presented here to framed instanton homology. We discovered [4] a workaround in that setting, however, by a significantly more complicated argument which involves cabling and our framed instanton contact invariant [2]. We then used that instanton contact class to prove the $r \geq 2g(K)-1$ bound, in a manner very similar to the proof of Proposition 15 here (also using results from [3] and [9]). The same difficulties and solutions apply in monopole Floer homology, using our contact invariant from [1].[1]

[1] It is reasonable to expect that Kronheimer and Mrowka's monopole Floer contact class can be characterized in terms of the 0-surgery cobordism map as above, based on Echeverria's work [5], which would allow one to circumvent the more complicated strong quasipositivity argument we have in mind.

Figure 1. An outline of the proof of Proposition 5.

In our proof of Theorem 1, we will assume that K has genus at least 2 everywhere until Proposition 15, so that we can apply Theorem 6 below to detect whether $S^3_0(K)$ is fibered. The following proposition uses a cabling trick (see Figure 1) to show that we can still conclude Theorem 1 in full generality, and moreover that it suffices to prove the theorem for integral slopes.

Proposition 5. *If Theorem 1 holds for all knots of genus at least* 2 *and integral slopes* $r \in \mathbb{Z}$, *then it is also true for knots of genus* 1 *and* $r \in \mathbb{Q}$.

Proof. The claim in Theorem 1 about the set of positive integral L-space slopes is proved in Proposition 15 without any restrictions on $g(K)$, so we will only address the other claims of Theorem 1 here.

We suppose first that K is an arbitrary nontrivial knot and that some surgery on K of nonintegral slope $r > 0$ is an L-space. We write $r = \frac{p}{q}$ for some positive integers p and $q \geq 2$. By applying [6, Corollary 7.3], we see that

$$S^3_{pq}(K_{p,q}) \cong S^3_{p/q}(K) \# S^3_{q/p}(U),$$

where $K_{p,q}$ is the cable represented by the peripheral element $\mu^p\lambda^q$ in $\pi_1(\partial N(K))$. The two summands on the right are both L-spaces; hence the Künneth formula for $\widehat{HF}$ says that pq-surgery on $K_{p,q}$ is also an L-space. We observe that

$$g(K_{p,q}) = \frac{(p-1)(q-1)}{2} + q \cdot g(K),$$

which implies that $g(K_{p,q}) \geq q \geq 2$ and which is also equivalent to

$$2g(K_{p,q}) - 1 = pq + q\Big(2g(K) - 1 - \frac{p}{q}\Big). \tag{2}$$

We can now apply the assumed case of Theorem 1 to pq-surgery on $K_{p,q}$ to conclude that $pq \geq 2g(K_{p,q}) - 1$, hence $r = \frac{p}{q} \geq 2g(K) - 1$ by (2); and that $K_{p,q}$

is fibered and strongly quasipositive. Since $K_{p,q}$ is fibered, K must be as well. The strong quasipositivity of K then follows from two facts:

(1) a fibered knot is strongly quasipositive if and only if its corresponding open book decomposition supports the tight contact structure on S^3 [8], and

(2) the knots K and $K_{p,q}$ support the same contact structure [7].

This concludes the proof in all cases except when K has genus 1 and r is a positive integer, say $r = n$. In this case it is automatic that $r \geq 2g(K) - 1 = 1$. Moreover, repeated application of [15, Proposition 2.1], which follows easily from the surgery exact triangle for $\widehat{HF}$, says that $S^3_{(2n+1)/2}(K)$ is an L-space. (In fact, it says that $S^3_s(K)$ is an L-space for all rational $s \geq n$.) Since $\frac{1}{2}(2n+1) \notin \mathbb{Z}$, the fiberedness and strong quasipositivity of K follow exactly as above. □

We will suppose henceforth that $K \subset S^3$ is a knot of genus $g \geq 2$.

Let Σ_0 denote the genus g surface in $S^3_0(K)$ obtained by capping off a minimal genus Seifert surface for K. Let $\mathfrak{s}_i$ be the unique Spin^c structure on $S^3_0(K)$ satisfying

$$\langle c_1(\mathfrak{s}_i), [\Sigma_0] \rangle = 2i.$$

The adjunction inequality [13, Theorem 7.1] implies that

$$HF^+(S^3_0(K), \mathfrak{s}_i) = \widehat{HF}(S^3_0(K), \mathfrak{s}_i) = 0$$

for $|i| > g - 1$. Moreover, by [11], we have

$$HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \neq 0$$

and Ni proved [10] (see also [12, Corollary 4.5]) the following.

Theorem 6. *K is fibered if and only if $HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \cong \mathbb{F}$.*

Recall that there is an exact triangle

$$\begin{tikzcd} HF^+(Y, \mathfrak{s}) \arrow[rr, "\cdot U"] & & HF^+(Y, \mathfrak{s}) \arrow[dl, "j"] \\ & \widehat{HF}(Y, \mathfrak{s}) \arrow[ul, "i"] & \end{tikzcd} \tag{3}$$

where i and multiplication by U preserve the $\mathbb{Z}/2\mathbb{Z}$ grading and j shifts it by 1. Moreover, we claim the following.

Proposition 7. *U acts trivially on $HF^+(S^3_0(K), \mathfrak{s}_{g-1})$.*

Proof. Let

$$Z : (\Sigma_0 \times S^1) \sqcup S^3_0(K) \to S^3_0(K)$$

be the cobordism obtained from $S^3_0(K) \times I$ by removing a neighborhood of Σ_0 from the interior. The Spin^c structure $\mathfrak{s}_{g-1}$ on $S^3_0(K)$ extends to a product Spin^c structure on $S^3_0(K) \times I$, and we let $\mathfrak{t}$ denote the restriction of the latter to Z. Then the induced map

$$F_{Z,\mathfrak{t}} : HF^+(\Sigma_0 \times S^1, \mathfrak{t}|_{\Sigma_0 \times S^1}) \otimes HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \to HF^+(S^3_0(K), \mathfrak{s}_{g-1})$$

is surjective. Since $\Sigma_0 \times \{\mathrm{pt}\} \subset \Sigma_0 \times S^1$ is homologous in Z to $\Sigma_0 \subset S^3_0(K) \times \{0\}$, we must have

$$\langle c_1(\mathfrak{t}), [\Sigma_0 \times \{\mathrm{pt}\}] \rangle = \langle c_1(\mathfrak{t}|_{S^3_0(K) \times \{0\}}), [\Sigma_0] \rangle = 2g - 2,$$

and evidently $HF^+(\Sigma_0 \times S^1, \mathfrak{t}|_{\Sigma_0 \times S^1})$ is nonzero. Thus $\mathfrak{t}|_{\Sigma_0 \times S^1}$ must be the unique Spin^c structure, which we also denote by $\mathfrak{s}_{g-1}$, satisfying

$$\langle c_1(\mathfrak{s}_{g-1}), [\Sigma_0 \times \{\mathrm{pt}\}] \rangle = 2g - 2 \quad \text{and} \quad \langle c_1(\mathfrak{s}_{g-1}), [\gamma \times S^1] \rangle = 0$$

for all closed curves $\gamma \subset \Sigma_0$, and we have

$$HF^+(\Sigma_0 \times S^1, \mathfrak{s}_{g-1}) \cong \mathbb{F}. \tag{4}$$

For details, see [12, Theorem 9.3] and the discussion preceding it.

It follows from (4) and the surjectivity of $F_{Z,\mathfrak{t}}$ that the cobordism map

$$F_{Z,\mathfrak{t}} : HF^+(\Sigma_0 \times S^1, \mathfrak{s}_{g-1}) \otimes HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \to HF^+(S^3_0(K), \mathfrak{s}_{g-1})$$

is in fact an isomorphism. Moreover, it satisfies

$$F_{Z,\mathfrak{t}}(a \otimes Ub) = F_{Z,\mathfrak{t}}(Ua \otimes b).$$

The U-action on (4) is clearly trivial, which implies the same for $HF^+(S^3_0(K), \mathfrak{s}_{g-1})$ by the relation above. □

Proposition 7 together with the exact triangle in (3) implies that

$$\widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1}) \cong HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \oplus HF^+(S^3_0(K), \mathfrak{s}_{g-1})[1].$$

In particular, we have the following.

Corollary 8. *If K is fibered then*

$$\widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1}) \cong \mathbb{F}_0 \oplus \mathbb{F}_1,$$

where the subscripts on the right denote the $\mathbb{Z}/2\mathbb{Z}$ grading. If K is not fibered then

$$\dim \widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1}) \geq 4.$$

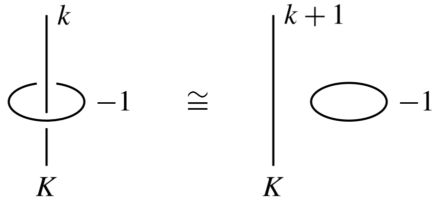

Figure 2. A handleslide showing that $W_{k+1} \circ X_k \cong X_{k+1} \# \overline{\mathbb{CP}}^2$.

We now consider the natural 2-handle cobordisms

$$S^3 \xrightarrow{X_k} S^3_k(K) \xrightarrow{W_{k+1}} S^3_{k+1}(K)$$

for each integer $k \geq 0$, where the 2-handle W_{k+1} is attached along a -1-framed meridian of K. We observe in Figure 2 that

$$W_{k+1} \circ X_k = X_k \cup_{S^3_k(K)} W_{k+1} \cong X_{k+1} \# \overline{\mathbb{CP}}^2,$$

and hence if we write

$$V_k = W_k \circ W_{k-1} \circ \cdots \circ W_1 : S^3_0(K) \to S^3_k(K)$$

then the composition

$$Z_k = V_k \circ X_0 : S^3 \to S^3_k(K)$$

is a k-fold blow-up of X_k, i.e.,

$$\begin{aligned} X_k \# k\overline{\mathbb{CP}}^2 &\cong W_k \circ (X_{k-1} \# (k-1)\overline{\mathbb{CP}}^2) \\ &\cong W_k \circ W_{k-1} \circ (X_{k-2} \# (k-2)\overline{\mathbb{CP}}^2) \\ &\cong \cdots \\ &\cong (W_k \circ \cdots \circ W_1) \circ X_0 = Z_k. \end{aligned}$$

The maps induced by X_k and W_{k+1} fit into an $(\infty, 0, 1)$-surgery exact triangle,

$$\begin{tikzcd} \widehat{HF}(S^3) \arrow[rr, "F_{X_k}"] & & \widehat{HF}(S^3_k(K)) \arrow[dl, "F_{W_{k+1}}"] \\ & \widehat{HF}(S^3_{k+1}(K)) \arrow[ul] & \end{tikzcd} \tag{5}$$

A Spin^c structure on X_0 is determined by its restriction to $S^3_0(K)$, or, equivalently, by the evaluation of its first Chern class on $[\Sigma_0]$. Let $\mathfrak{t}_i$ denote the unique Spin^c structure on X_0 with

$$\langle c_1(\mathfrak{t}_i), [\Sigma_0] \rangle = 2i.$$

Define

$$y_i := F_{X_0,\mathfrak{t}_i}(\mathbf{1}) \in \widehat{HF}(S^3_0(K), \mathfrak{s}_i),$$

where $\mathbf{1}$ denotes the generator of $\widehat{HF}(S^3) \cong \mathbb{F}$.

Let Σ_k denote the capped off Seifert surface in X_k, with

$$\Sigma_k \cdot \Sigma_k = k.$$

A Spin^c structure on X_k is determined by the evaluation of its first Chern class on $[\Sigma_k]$. Such Chern classes are characteristic elements, so this evaluation agrees with $k \pmod 2$. Let $\mathfrak{t}_{k,i}$ denote the unique Spin^c structure on X_k satisfying

$$\langle c_1(\mathfrak{t}_{k,i}), [\Sigma_k]\rangle + k = 2i.$$

The adjunction inequality [16, Proof of Theorem 1.5] implies that the map

$$F_{X_k,\mathfrak{t}_{k,i}} : \widehat{HF}(S^3) \to \widehat{HF}(S^3_k(K))$$

is nontrivial only if

$$|\langle c_1(\mathfrak{t}_{k,i}), [\Sigma_k]\rangle| + k \le 2g-2,$$

or equivalently $1-g+k \le i \le g-1$.

Lemma 9. *Let $x_{k,i} = F_{X_k,\mathfrak{t}_{k,i}}(\mathbf{1})$ for all $k \ge 1$ and all i. Then*

$$F_{V_k}(y_i) = x_{k,i} + \binom{k}{1} x_{k,i+1} + \binom{k}{2} x_{k,i+2} + \cdots + \binom{k}{g-i-1} x_{k,g-1}$$

as elements of $\widehat{HF}(S^3_k(K))$.

Proof. Let $E_1, \ldots, E_k \subset Z_k$ denote the exceptional spheres in $Z_k \cong X_k \# k\overline{\mathbb{CP}}^2$, and $e_1, \ldots, e_k$ their Poincaré duals in $H^2(Z_k)$. Note that in Z_k, the surface Σ_0 is given by

$$\Sigma_0 = \Sigma_k - E_1 - \cdots - E_k.$$

In particular,

$$\langle c_1(\mathfrak{t}_{k,i} + a_1e_1 + \cdots + a_ke_k), [\Sigma_0]\rangle = 2i - k + a_1 + \cdots + a_k \tag{6}$$

in Z_k. We will evaluate F_{Z_k} by applying the blow-up formula for cobordism maps [16, Theorem 3.7], which says that for a Spin^c cobordism

$$(W, \mathfrak{t}) : (Y_1, \mathfrak{s}_1) \to (Y_2, \mathfrak{s}_2)$$

with blow-up $\widehat{W} = W \# \overline{\mathbb{CP}}^2$ and exceptional sphere E,

$$F_{\widehat{W},\mathfrak{t}\pm(2\ell+1)PD(E)} = \begin{cases} F_{W,\mathfrak{t}} & \text{if } \ell = 0, \\ 0 & \text{if } \ell \neq 0, \end{cases}$$

as maps on $\widehat{HF}$ for any $\ell \ge 0$.

Let F_i denote the component of $F_{Z_k} = F_{V_k} \circ F_{X_0}$ that factors through $\widehat{HF}(S^3_0(K), \mathfrak{s}_i)$. On the one hand, we have

$$F_i = F_{V_k} \circ F_{X_0,\mathfrak{t}_i} : \widehat{HF}(S^3) \to \widehat{HF}(S^3_k(K)).$$

On the other hand, if we let $e = e_1 + \cdots + e_k$, then for each i we have

$$F_i = F_{Z_k,\mathfrak{t}_{k,i}+e} + \sum_{j_1} F_{Z_k,\mathfrak{t}_{k,i+1}+e-2e_{j_1}} + \sum_{j_1<j_2} F_{Z_k,\mathfrak{t}_{k,i+2}+e-2e_{j_1}-2e_{j_2}} + \cdots + \sum_{j_1<\cdots<j_{g-i-1}} F_{Z_k,\mathfrak{t}_{k,g-1}+e-2e_{j_1}+\cdots-2e_{j_{g-i-1}}},$$

by the formula (6). From the blow-up formula, we have

$$F_{Z_k,\mathfrak{t}_{k,j}\pm e_1\pm\cdots\pm e_k} = F_{X_k,\mathfrak{t}_{k,j}},$$

so the expression for F_i above becomes

$$F_i = F_{X_k,\mathfrak{t}_{k,i}} + \binom{k}{1} F_{X_k,\mathfrak{t}_{k,i+1}} + \binom{k}{2} F_{X_k,\mathfrak{t}_{k,i+2}} + \cdots + \binom{k}{g-i-1} F_{X_k,\mathfrak{t}_{k,g-1}}.$$

We conclude by evaluating both sides on the element $\mathbf{1} \in \widehat{HF}(S^3)$. $\square$

Proposition 10. *For all integers $k \geq 1$, we have*

$$\ker\bigl(F_{V_k} : \widehat{HF}(S^3_0(K)) \to \widehat{HF}(S^3_k(K))\bigr) \subset \mathrm{Span}_{\mathbb{F}}(y_{1-g}, \ldots, y_{g-1}).$$

This inclusion is an equality for all $k \geq 2g-1$.

Proof. When $k = 1$, the exact triangle (5) says that

$$\ker(F_{V_1}) = \ker(F_{W_1}) = \mathrm{Im}(F_{X_0}) = \mathrm{Span}_{\mathbb{F}}\Bigl(\sum_{i=1-g}^{g-1} y_i\Bigr).$$

We prove the inclusion in general by induction on k.

Suppose that $k \geq 1$, and fix an element $z \in \ker(F_{V_{k+1}})$. Then

$$F_{W_{k+1}}\bigl(F_{V_k}(z)\bigr) = 0$$

by definition, so the exact triangle (5) tells us that $F_{V_k}(z) \in \mathrm{Im}(F_{X_k})$, or equivalently

$$F_{V_k}(z) = c \cdot F_{X_k}(\mathbf{1}) \tag{7}$$

for some $c \in \mathbb{F}$. Lemma 9 says that each element

$$x_{k,i} = F_{X_k,\mathfrak{t}_{k,i}}(\mathbf{1}) \in \widehat{HF}(S^3_k(K))$$

is a linear combination of the various $F_{V_k}(y_i)$, since the matrix of the coefficients of the system of linear equations relating $(F_{V_k}(y_i))_i$ to $(x_{k,i})_i$ is triangular and clearly invertible. In particular, summing over all i reveals that

$$F_{X_k}(\mathbf{1}) \in \mathrm{Span}_{\mathbb{F}}\bigl(F_{V_k}(y_{1-g}), \ldots, F_{V_k}(y_{g-1})\bigr). \tag{8}$$

Combining (7) and (8), there are coefficients $a_j \in \mathbb{F}$ such that

$$F_{V_k}(z) = c \cdot \sum_{j=1-g}^{g-1} a_j F_{V_k}(y_j),$$

or equivalently

$$z - \sum_{j=1-g}^{g-1} ca_j \cdot y_j \in \ker F_{V_k}. \tag{9}$$

By induction the left side of (9) lies in $\mathrm{Span}_{\mathbb{F}}(y_{1-g}, \dots, y_{g-1})$; hence the same is true of z. Since z was an arbitrary element of $\ker F_{V_{k+1}}$, this completes the inductive step.

To see that equality holds when $k \geq 2g-1$, we observe from Lemma 9 that $F_{V_k}(y_i)$ is a linear combination of various elements $x_{k,j} = F_{X_k, \mathfrak{t}_{k,j}}(\mathbf{1})$. Using the adjunction inequality, we have already noted that $x_{k,j} = 0$ unless

$$1 - g + k \leq j \leq g - 1,$$

so for $k \geq 2g-1$ the elements $x_{k,j}$ and hence the $F_{V_k}(y_i)$ are all zero. $\square$

Proposition 11. *Suppose that $S^3_n(K)$ is an L-space for some positive integer n. Then*

$$\widehat{HF}(S^3_0(K), \mathfrak{s}_j) = \begin{cases} \mathbb{F}_0 \oplus \mathbb{F}_1 & \text{if } y_j \neq 0, \\ 0 & \text{if } y_j = 0, \end{cases}$$

for all j, where the subscripts on each copy of $\mathbb{F}$ denote the $\mathbb{Z}/2\mathbb{Z}$ grading.

Proof. We observe from (5) that

$$\dim_{\mathbb{F}} \widehat{HF}(S^3_{k+1}(K)) = \dim_{\mathbb{F}} \widehat{HF}(S^3_k(K)) + \begin{cases} 1 & \text{if } F_{X_k} = 0, \\ -1 & \text{if } F_{X_k} \neq 0, \end{cases} \tag{10}$$

for all $k \geq 0$. If m denotes the number of $k \in \{0, 1, \dots, n-1\}$ such that $F_{X_k} \neq 0$, then

$$n = \dim_{\mathbb{F}} \widehat{HF}(S^3_n(K)) = \dim_{\mathbb{F}} \widehat{HF}(S^3_0(K)) + (n-m) - m,$$

which simplifies to

$$\dim_{\mathbb{F}} \widehat{HF}(S^3_0(K)) = 2m. \tag{11}$$

Our goal is thus to compute m.

Supposing that $F_{X_k} \neq 0$ for some $k \geq 0$, then $F_{X_k}(\mathbf{1})$ is a nonzero element which spans $\ker(W_{k+1})$, and from (8) it has the form

$$F_{X_k}(\mathbf{1}) = F_{V_k}\left(\sum_{j=1-g}^{g-1} a_j y_j\right)$$

for some coefficients $a_j \in \mathbb{F}$. The sum $\sum a_j y_j$ is thus not in $\ker(F_{V_k})$, but it is in

$$\ker(F_{V_{k+1}}) = \ker(F_{W_{k+1}} \circ F_{V_k}),$$

so we have $\dim\ker(F_{V_{k+1}}) > \dim\ker(F_{V_k})$. This implies that

$$\dim\ker(F_{V_n}) \geq m.$$

Proposition 10 then implies that

$$m \leq \dim \operatorname{Span}_{\mathbb{F}}(y_j). \tag{12}$$

But the nonzero y_j are all linearly independent, since they belong to different summands $\widehat{HF}(S^3_0(K), \mathfrak{s}_j)$ of $\widehat{HF}(S^3_0(K))$, so by combining (11) and (12) we conclude that

$$\dim_{\mathbb{F}} \widehat{HF}(S^3_0(K)) \leq 2 \cdot \#\{j \mid y_j \neq 0\}. \tag{13}$$

If $y_j \neq 0$ then $\widehat{HF}(S^3_0(K), \mathfrak{s}_j)$ is nonzero, and its Euler characteristic is zero by (1), so

$$\mathbb{F}_0 \oplus \mathbb{F}_1 \subset \widehat{HF}(S^3_0(K), \mathfrak{s}_j) \quad \text{if } \ y_j \neq 0.$$

Thus the inequality in (13) must be an equality, and each nonzero $\widehat{HF}(S^3_0(K), \mathfrak{s}_j)$ must have the form $\mathbb{F}_0 \oplus \mathbb{F}_1$, completing the proof. □

Proposition 12. *If $S^3_n(K)$ is an L-space for some integer $n > 0$ then K is fibered.*

Proof. Corollary 8 and Proposition 11 tell us that

$$2 \leq \dim \widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1}) \leq 2, \tag{14}$$

and that equality on the left holds if and only if K is fibered, so K must be fibered. □

Proposition 13. *If $S^3_n(K)$ is an L-space for some integer $n > 0$ then K is strongly quasipositive.*

Proof. We already have seen in (14) that $\widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1})$ is nonzero; hence $y_{g-1} \neq 0$ by Proposition 11. Equivalently, the map

$$\widehat{HF}(S^3) \to \widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1}) \tag{15}$$

induced by X_0 is nonzero. Now, we can also view X_0 as a cobordism

$$X_0 : -S^3_0(K) \to -S^3,$$

in which case the induced map

$$\widehat{HF}(-S^3_0(K), \mathfrak{s}_{1-g}) \to \widehat{HF}(-S^3)$$

is dual to that in (15). In particular, this map is also nonzero. The commutativity of

$$\begin{tikzcd}
\widehat{HF}(-S^3_0(K), \mathfrak{s}_{1-g}) \arrow[r, "i"] \arrow[d] & HF^+(-S^3_0(K), \mathfrak{s}_{1-g}) \arrow[d] \\
\widehat{HF}(-S^3) \arrow[r, "i"] & HF^+(-S^3)
\end{tikzcd}$$

where the vertical maps are those induced by X_0, together with the facts that $\widehat{HF}(-S^3) \cong \mathbb{F}$ and the bottom horizontal map is nonzero, implies that the rightmost vertical map

$$HF^+(-S_0^3(K), \mathfrak{s}_{1-g}) \to HF^+(-S^3) \tag{16}$$

is nonzero as well. But Proposition 12 says that K is fibered; hence

$$HF^+(-S_0^3(K), \mathfrak{s}_{1-g}) \cong \mathbb{F},$$

and the image of its generator under the map in (16) is the contact invariant $c^+(\xi_K)$ [14], where ξ_K is the contact structure corresponding to K. Thus, $c^+(\xi_K)$ is nonzero, which implies that ξ_K is the tight contact structure on S^3. It follows that K is strongly quasipositive, by work of Hedden [8, Proposition 2.1]. □

We will now use the fact that L-space knots are strongly quasipositive to determine the range of L-space slopes for any such knot. We begin with the following general lemma.

Lemma 14. *Let Y be a rational homology sphere with $|H_1(Y;\mathbb{Z})| = n$. Suppose that*

$$\ker\big(U : HF^+(Y) \to HF^+(Y)\big)$$

has dimension $n+k$. Then $\dim \widehat{HF}(Y) = n + 2k$.

Proof. The exact triangle (3) involving the U-action on $HF^+(Y)$ produces a short exact sequence

$$0 \to \operatorname{coker}(U) \to \widehat{HF}(Y) \to \ker(U) \to 0.$$

Thus it will suffice to show that $\dim\operatorname{coker}(U) = k$.

Since each Spin^c structure on Y is torsion, we have a short exact sequence

$$0 \to (\mathcal{T}^+)^{\oplus n} \to HF^+(Y) \xrightarrow{\pi} HF_{\mathrm{red}}(Y) \to 0,$$

of $\mathbb{F}[U]$-modules, where $\mathcal{T}^+ \cong \mathbb{F}[U, U^{-1}]/U F[U]$. The quotient $HF_{\mathrm{red}}(Y)$ is defined as $HF^+(Y)/\mathrm{Im}(U^d)$ for $d \gg 0$; it is finitely generated over $\mathbb{F}[U]$ and over $\mathbb{F}$, and every element is U-torsion, so it has a decomposition

$$HF_{\mathrm{red}}(Y) \cong \bigoplus_{i=1}^{r} \mathbb{F}[U]/\langle U^{n_i}\rangle,$$

with each $n_i \geq 1$. Moreover this sequence can be shown to split, so that

$$HF^+(Y) \cong (\mathcal{T}^+)^{\oplus n} \oplus \bigoplus_{i=1}^{r} \mathbb{F}[U]/\langle U^{n_i}\rangle.$$

But then it is clear that $\ker(U) \cong \mathbb{F}^{n+r}$, so that $r = k$, and then that $\operatorname{coker}(U) \cong \mathbb{F}^r = \mathbb{F}^k$, and the lemma follows immediately. □

The following proposition completes our proof of Theorem 1. The proof below is partly inspired by the work of Lidman, Pinzón-Caicedo, and Scaduto [9].

Proposition 15. *If K has genus $g \geq 1$ and $S^3_n(K)$ is an L-space for some positive integer n, then $S^3_n(K)$ is an L-space for an arbitrary integer n if and only if $n \geq 2g-1$.*

Proof. Since K is strongly quasipositive, its maximal self-linking number is $\overline{sl}(K) = 2g-1$. We take a Legendrian representative Λ of K in the standard contact S^3 with classical invariants

$$(tb(\Lambda), r(\Lambda)) = (\tau_0, r_0), \quad \tau_0 - r_0 = 2g-1,$$

and for $n \geq 1-\tau_0$, we can positively stabilize this k times and negatively stabilize it $\tau_0 + n - 1 - k$ times to get a Legendrian representative with

$$(tb, r) = (1-n, 2-2g-n+2k), \quad 0 \leq k \leq \tau_0 + n - 1.$$

For odd $n \gg 0$, these values of r include every positive odd number between 1 and $n+2g-2$.

Fixing such a large value of n, we perform Legendrian surgery on these knots Λ_i with

$$(tb(\Lambda_i), r(\Lambda_i)) = (1-n, 2i-1), \quad 1 \leq i \leq \frac{n+2g-1}{2},$$

to get contact structures

$$\xi_1, \dots, \xi_{(n+2g-1)/2}$$

on $S^3_{-n}(K)$. If $X_{-n}(K)$ is the trace of this $-n$-surgery, and $\widehat{\Sigma} \subset X_{-n}(K)$ the union of a Seifert surface for K with the core of the 2-handle, then each ξ_i admits a Stein filling $(X_{-n}(K), J_i)$ with

$$\langle c_1(J_i), [\widehat{\Sigma}] \rangle = r(\Lambda_i) = 2i-1.$$

We can also take contact structures

$$\bar{\xi}_i = T(S^3_{-n}(K)) \cap \bar{J}_i T(S^3_{-n}(K)), \quad 1 \leq i \leq \frac{n+2g-1}{2},$$

which are filled by $X_{-n}(K)$ with the conjugate Stein structure $\bar{J}_i$ for each i. These satisfy $\langle c_1(\bar{J}_i), [\widehat{\Sigma}] \rangle = -(2i-1)$, so we have exhibited $n+2g-1$ Stein structures

$$J_1, J_2, \dots, J_{(n+2g-1)/2}, \bar{J}_1, \bar{J}_2, \bar{J}_{(n+2g-1)/2}$$

on $X_{-n}(K)$ which are all distinguished by their first Chern classes.

A theorem of Plamenevskaya [18, Theorem 4] now tells us that the corresponding contact invariants

$$c^+(\xi_1), \dots, c^+(\xi_{(n+2g-1)/2}), c^+(\bar{\xi}_1), \dots, c^+(\bar{\xi}_{(n+2g-1)/2}) \in HF^+(-S^3_{-n}(K))$$

are linearly independent. These elements lie in $\ker(U)$, as can be seen, for example, from the fact that they are by defined by maps of the form (16) whose domains have trivial U action. Thus

$$\dim \ker(U) \geq n + 2g - 1,$$

and it follows from Lemma 14 that

$$\dim \widehat{HF}(S^3_{-n}(K)) = \dim \widehat{HF}(-S^3_{-n}(K)) \geq n + 4g - 2.$$

This same argument applies for any larger odd value of n as well, and the conclusion also holds for even values of n after making only cosmetic changes to the argument, so that $S^3_{-m}(K)$ cannot be an L-space for any $m \geq n$.

We now repeatedly apply the surgery exact triangle (5) to see that

$$\dim \widehat{HF}(S^3_{-m}(K)) \geq m + 4g - 2, \quad 0 \leq m \leq n,$$

and then that

$$\dim \widehat{HF}(S^3_{m}(K)) \geq 4g - 2 - m \geq m + 2, \quad 0 \leq m \leq 2g - 2.$$

Thus $S^3_m(K)$ cannot be an L-space for any integer $m < 2g(K) - 1$. On the other hand, equation (10) says that

$$\dim \widehat{HF}(S^3_{2g-1+n}(K)) = \dim \widehat{HF}(S^3_{2g-1}(K)) + n$$

for all $n \geq 0$, since the maps $F_{X_{2g-1}}, \dots, F_{X_{2g-2+n}}$ are all zero by the adjunction inequality. Thus $S^3_{2g-1+n}(K)$ is an L-space if and only if $S^3_{2g-1}(K)$ is, and this completes the proof. □

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Gauge Theory and Low-Dimensional Topology: Progress and Interaction

This volume is a proceedings of the 2020 BIRS workshop *Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4*. This was the 6th iteration of a recurring workshop held in Banff. Regrettably, the workshop was not held onsite but was instead an online (Zoom) gathering as a result of the Covid-19 pandemic. However, one benefit of the online format was that the participant list could be expanded beyond the usual strict limit of 42 individuals. It seemed to be also fitting, given the altered circumstances and larger than usual list of participants, to take the opportunity to put together a conference proceedings.

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