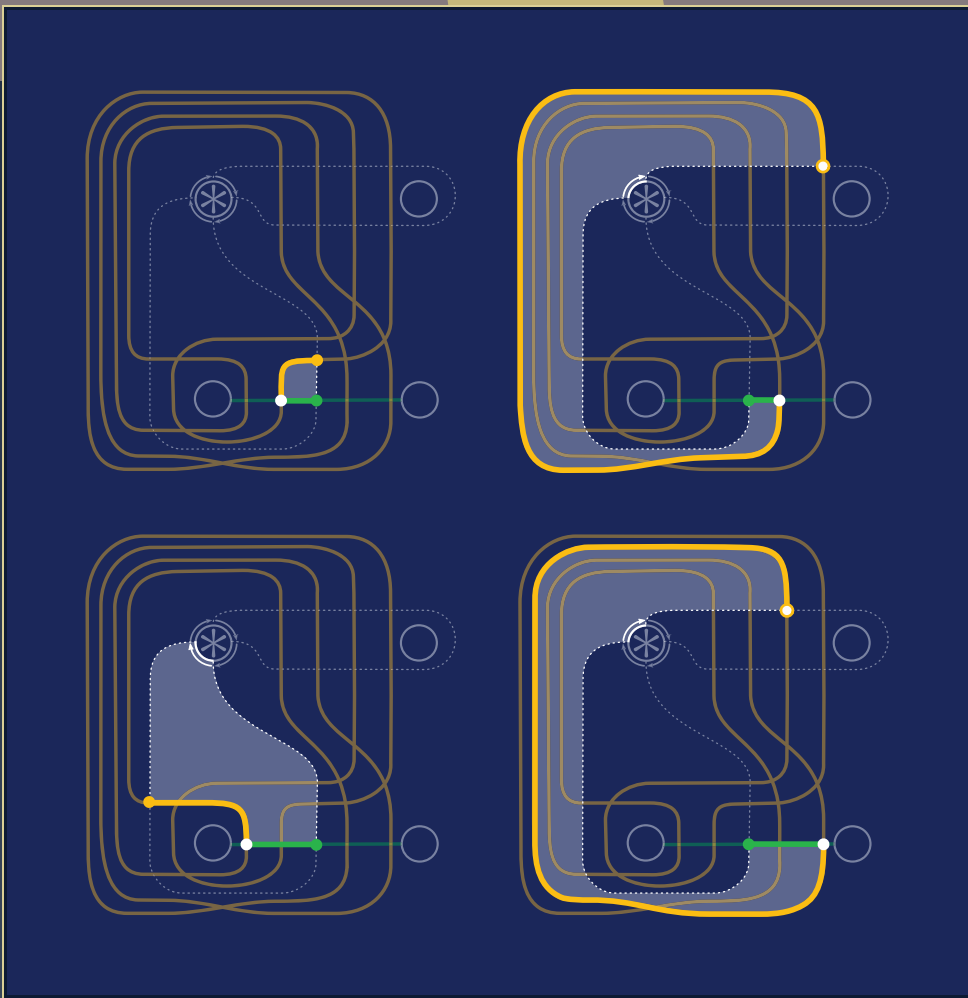# Gauge Theory and Low-Dimensional Topology: Progress and Interaction

Tangles, relative character varieties, and
holonomy perturbed traceless flat moduli spaces

Guillem Cazassus, Chris Herald and Paul Kirk

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Tangles, relative character varieties, and holonomy perturbed traceless flat moduli spaces

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We prove that the restriction map from the subspace of regular points of the holonomy perturbed SU(2) traceless flat moduli space of a tangle in a 3-manifold to the traceless flat moduli space of its boundary marked surface is a Lagrangian immersion. A key ingredient in our proof is the use of composition in the Weinstein category, combined with the fact that SU(2) holonomy perturbations in a cylinder induce Hamiltonian isotopies. In addition, we show that $(S^2, 4)$, the 2-sphere with four marked points, is its own traceless flat SU(2) moduli space.

1. Introduction

We gather together some of the key symplectic properties of character varieties and traceless character varieties, as well as variants which correspond to perturbed flat moduli spaces that arise in the gauge theoretic study of 3-manifolds. Some of these results are well known to the experts, but the proofs in the literature are framed in contexts that include gauge theory, Hodge theory, and symplectic reduction. In the present exposition, we provide a general proof of the fact that, roughly, the character variety of a 3-manifold provides an immersed Lagrangian in the character variety of its boundary surface, for any compact Lie group G, whether the 3-manifold and its boundaries include tangles, or whether there are trace or other conjugacy restrictions on some meridional generators, and furthermore we extend the result to the holonomy perturbed situation. Moreover, we clarify why different natural definitions of symplectic structures on the pillowcase, arising as the character variety of the torus or the 2-sphere with four marked points, are equivalent. The results are proved using only the Poincaré-Lefschetz duality theorem, basic algebraic topology, and the notion of composition in the Weinstein category.

Let $(X, \mathcal{L})$ be a tangle in a compact, oriented 3-manifold X; that is, assume that $\mathcal{L}$ is a properly embedded, compact 1-manifold. For our initial discussion, we consider $G = \mathrm{SU}(2)$. Let π denote some holonomy perturbation data supported in a finite disjoint union of solid tori in the interior of $X \setminus \mathcal{L}$.

This data determines two moduli spaces,

$$\mathcal{M}_\pi(X, \mathcal{L}),$$

the moduli space of π *holonomy-perturbed flat* $\mathrm{SU}(2)$ *connections on* $X \setminus \mathcal{L}$ *with traceless holonomy on small meridians of* $\mathcal{L}$, and the (well studied) moduli space

$$\mathcal{M}(\partial X, \partial \mathcal{L})$$

of *flat* $\mathrm{SU}(2)$ *connections on the punctured surface* $\partial X \setminus \partial \mathcal{L}$ *with traceless holonomy around the marked points* $\partial \mathcal{L}$.

Holonomy perturbed flat connections on a 3-manifold are flat near the boundary, and restriction to the boundary defines a map

$$r : \mathcal{M}_\pi(X, \mathcal{L}) \to \mathcal{M}(\partial X, \partial \mathcal{L}). \tag{1-1}$$

The moduli space $\mathcal{M}(\partial X, \partial \mathcal{L})$ is the cartesian product of the moduli spaces of its path components. The flat $\mathrm{SU}(2)$ moduli space of an oriented connected surface of genus g with k marked points is known to be a singular variety with smooth top stratum carrying a symplectic form called the *Atiyah–Bott–Goldman form* [3; 12]. Thus the smooth top stratum of the cartesian product $\mathcal{M}(\partial X, \partial \mathcal{L})$, denoted $\mathcal{M}(\partial X, \partial \mathcal{L})^*$, is endowed with the product symplectic form.

The main result of this article is the following theorem (Theorem 6.9 below), concerning the regular points (see Section 4C) of the perturbed moduli space.

Theorem A. *Suppose* $A \in \mathcal{M}_\pi(X, \mathcal{L})$ *is a regular point. Then A has a neighborhood U so that the restriction* $r|_U : U \subset \mathcal{M}_\pi(X, \mathcal{L}) \to \mathcal{M}(\partial X, \partial \mathcal{L})^*$ *is a Lagrangian immersion.*

This is not a surprising result; indeed, many special cases are known, for example, when $\mathcal{L}$ is empty this result is proven in [16]. Our primary aim is to provide details of the assertion that well-known arguments in the flat case extend to the holonomy-perturbed flat case when $\mathcal{L}$ is nonempty, in support of one claim of the main result of [7]. In that article it is shown that a certain process introduced by Kronheimer and Mrowka to ensure admissibility of bundles in instanton homology [21] manifests itself on the symplectic side of the Atiyah–Floer conjecture [1] (i.e., Lagrangian Floer theory of character varieties or flat moduli spaces) as a certain Lagrangian immersion of a smooth closed genus 3 surface into the smooth stratum $\mathbb{P}^* \times \mathbb{P}^*$ of the product of two pillowcases (cf. (1-3)). What is proved in [7] is that this genus 3 surface satisfies the hypotheses of Theorem A at every point.

Since it causes no extra work, we take the opportunity to provide an elementary algebraic topology proof of Theorem A *in the flat case*, for any compact Lie group G. The statement can be found in Corollary 4.5. We emphasize that the flat case of Theorem A, when $\mathcal{L}$ is nonempty, is known (e.g., to those who attended the appropriate Oxford seminars in the 1980s) and indeed discussed on pages 15–16 of Atiyah's monograph [2].

Having stated Theorem A and described its relation to the gauge theoretic literature using the language of flat and perturbed flat moduli spaces, we note that these spaces can also be identified with certain character varieties, the definitions of which do not require any of the analytical machinery of gauge theory; the proofs in this article are most simply explained without it, so we shall henceforth revert to the character variety terminology in our exposition. In the simplest situation of a connected 2- or 3-manifold M with base point x_0, for a fixed compact Lie group G, each flat G connection determines a holonomy representation from $\pi_1(M, x_0)$ to G, and this correspondence induces a bijection between the moduli space of flat connections and the set of G representations of π_1 modulo conjugation, known as the *character variety*. We describe various extensions of the notion of the character variety corresponding to traceless and perturbed flat moduli spaces below.

We also take this opportunity to provide an elementary exposition of holonomy perturbations from the perspective of fundamental groups and character varieties, in the language of composition of Lagrangian immersions. Algebraic topology arguments simplify the task of explaining how to understand the extension from the flat to the holonomy perturbed flat situation algebraically. Other arguments, which instead appeal to Hodge and elliptic theory of perturbed flat bundles over 3-manifolds with boundary, can be made when $\mathcal{L}$ is empty, and can be found in detail in [16; 27]. But when $\mathcal{L}$ is nonempty, proper treatment of the traceless condition requires more subtle analytic tools.

Our focus on holonomy perturbations is motivated by the fact that they are compatible with the analytical framework of the instanton gauge theory side of the Atiyah–Floer conjecture [10; 27]. Holonomy perturbations modify the flatness condition (i.e., the nonlinear PDE *Curvature* $= 0$) in a specific way on a collection of solid tori, as described in Lemma 8.1 of Taubes [27] (see also Lemma 16 of [16]). We translate this result into the language of character varieties in Section 6B.

In addition to proving Theorem A, we prove Theorem B, whose statement roughly says *the four punctured 2-sphere is its own traceless* SU(2) *moduli space*. This is a variant, for $(S^2, 4)$, of the ubiquitous mathematical statement that a torus is isomorphic to its Jacobian. We now set some notation in preparation for the formal statement. First, denote $U(1) \times U(1)$ by $\mathbb{T}$. The group SL$(2, \mathbb{Z})$ acts on $\mathbb{T}$ via

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} \cdot (e^{x\mathbf{i}}, e^{y\mathbf{i}}) = (e^{(px+ry)\mathbf{i}}, e^{(qx+sy)\mathbf{i}}). \tag{1-2}$$

The 2-form $dx \wedge dy$ defines the standard symplectic structure on $\mathbb{T}$ and is invariant under the SL$(2, \mathbb{Z})$ action. Note that the action of the central element $-1 \in \mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{T}$ defines the *elliptic involution*, denoted by $\iota(e^{ix}, e^{iy}) = (e^{-ix}, e^{-iy})$; this involution has four fixed points $(\pm 1, \pm 1)$. Denote by $\mathbb{T}^* \subset \mathbb{T}$ the complement of the four fixed points, on which ι acts freely. Then set

$$\mathbb{P} = \mathbb{T}/\iota \quad \text{and} \quad \mathbb{P}^* = \mathbb{T}^*/\iota. \tag{1-3}$$

The quotient PSL$(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\{\pm 1\}$ acts on the smooth locus $\mathbb{P}^*$, a 4-punctured 2-sphere, preserving the symplectic form $dx \wedge dy$.

Next, we consider the relative character variety (defined below) of the 2-sphere with four marked points $\chi_{\mathrm{SU}(2), J}(S^2, 4)$, where the J subscript indicates the four meridians are sent to the conjugacy class J of traceless elements. This character variety is equipped with its relative Atiyah–Bott–Goldman 2-form $\omega_{(S^2, 4)}$ (defined below).
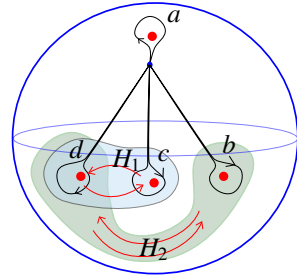
Figure 1. Half Dehn twists along the boundaries of H_1 and H_2.

From the presentation $\pi_1(S^2 \backslash 4) = \langle a, b, c, d \mid abcd = 1 \rangle$, we see that this group is freely generated by a, b, c. Hence a representation is uniquely defined by where it sends the generators a, b, c. Define the function (identifying SU(2) with the group of unit quaternions; see Section 2A2):

$$\hat{\rho} : \mathbb{P} \to \chi_{\mathrm{SU}(2), J}(S^2, 4), \quad [e^{ix}, e^{iy}] \mapsto [a \mapsto \boldsymbol{j}, b \mapsto e^{ix}\boldsymbol{j}, c \mapsto e^{iy}\boldsymbol{j}].$$

Theorem B. *Half Dehn twists in the two twice-punctured disks indicated in Figure 1 generate a* PSL$(2, \mathbb{Z})$ *action on* $\chi(S^2, 4)$ *which preserves the Atiyah–Bott–Goldman symplectic form* $\widehat{\omega}_{(S^2, 4)}$. *For some nonzero constant c, the map*

$$\hat{\rho} : (\mathbb{P}^*, c \, dx \wedge dy) \to (\chi(S^2, 4)^*, \widehat{\omega}_{(S^2, 4)})$$

is a PSL$(2, \mathbb{Z})$ *equivariant symplectomorphism.*[1]

The proof, as well as an exposition of the simpler case of SU(2) character variety of the torus, is contained in Section 5.

The outline of the proof of the flat case of Theorem A is the following. Holonomy identifies the flat moduli spaces with character varieties. When $\mathcal{L}$ is empty, Theorem A follows from Weil's identification of the Zariski tangent spaces of character varieties with cohomology and Poincaré-Lefschetz duality. Symplectic reduction is used to extend to the case when $\mathcal{L}$ is nonempty. We use only Poincaré

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[1]The constant c equals $\frac{1}{2}$.

duality with local coefficients, which is briefly reviewed, to highlight the fact that the proof that the image of the differential is *maximal* isotropic (the subtlest part of any proof) does not depend on the deeper result that the nondegenerate 2-form is closed, i.e., symplectic.

To put Theorem A in context, notice that the Lagrangian immersion r_U depends on the perturbation data π. Building on ideas from [1; 10; 32] and many others, Wehrheim and Woodward developed quilted Lagrangian–Floer homology and Floer field theory [29; 30]; a framework that aims to produce a $2+1$ or, more generally, $(2, 0)+1$ TQFT which factors as the composition of the (perturbed) flat moduli space functor, followed by passing to the Lagrangian Floer theory of the flat moduli spaces of surfaces equipped with the Atiyah–Bott symplectic form. This can be considered as an approach to realizing a *bordered* Lagrangian Floer theory of character varieties of surfaces and 3-manifolds, as was done for Heegaard–Floer theory in [23].

Even in the lucky case that one finds perturbation data π for which $\mathcal{M}_\pi(X, \mathcal{L})$ is smooth of the correct dimension and Lagrangian immerses into the smooth stratum $\mathcal{M}(\partial(X, \mathcal{L}))^*$, an understanding of how this immersion depends on π is necessary in order to extract topological information. If each perturbation curve is parallel to an embedded curve in the boundary surface, for example, then varying the perturbation parameters changes r by a Hamiltonian isotopy, and in particular has no effect on the topology of $\mathcal{M}_\pi(X, \mathcal{L})$. This is discussed in Section 6B; see also [17].

However, for general choices of perturbation data, perturbations typically change the topology of $\mathcal{M}_\pi(X, \mathcal{L})$; indeed, the primary purpose of using perturbation curves is precisely to smooth $\mathcal{M}_\pi(X, \mathcal{L})$. As we discuss in Section 6C, Floer field theory posits that, nevertheless, the resulting immersions should be independent of π in a Floer-theoretic sense (i.e., isomorphic in some Fukaya category).

2. Review of Poincaré duality and symplectic forms

2A. *Preliminaries and notation.*

2A1. *Symplectic linear algebra.* Let A denote a finite-dimensional $\mathbb{R}$-vector space. A skew symmetric bilinear form $\omega : A \times A \to \mathbb{R}$ is called a *symplectic form* if it is nondegenerate, that is, the map $A \to \mathrm{Hom}(A, \mathbb{R})$ given by $a \mapsto \omega(a, -)$ is an isomorphism. If A admits a symplectic form, then its dimension is even; denote the dimension by $2n$.

A *coisotropic subspace* $C \subset A$ is a subspace satisfying

$$\mathrm{Annihilator}(C) := \{a \in A \mid \omega(c, a) = 0 \text{ for all } c \in C\} \subset C,$$

and an *isotropic subspace* $C \subset A$ is a subspace satisfying

$$C \subset \mathrm{Annihilator}(C) := \{a \in A \mid \omega(c, a) = 0 \text{ for all } c \in C\}.$$

A *Lagrangian subspace* is a coisotropic and isotropic subspace, that is, a subspace $L \subset A$ satisfying $\mathrm{Annihilator}(L) = L$. Equivalently, a Lagrangian subspace $L \subset A$ is a coisotropic (or isotropic) subspace satisfying $\dim L = \frac{1}{2} \dim A$.

Symplectic reduction refers to the following process. Given any coisotropic subspace $C \subset A$, the restriction

$$\omega|_C : C \times C \to \mathbb{R}$$

may be degenerate, but $\omega|_C$ descends to a symplectic form $\widehat{\omega}$ on

$$\widehat{C} := C/\mathrm{Annihilator}(C).$$

Furthermore if $L \subset A$ is any Lagrangian subspace of (A, ω), then

$$\hat{L} := (L \cap C)/\big(L \cap \mathrm{Annihilator}(C)\big) \subset \widehat{C}$$

is a Lagrangian subspace of $(\widehat{C}, \widehat{\omega})$. The subquotient $(\widehat{C}, \widehat{\omega})$ is called the *symplectic reduction of (A, ω) with respect to C* and the Lagrangian subspace $\hat{L} \subset \widehat{C}$ the *symplectic reduction of L with respect to C*.

2A2. *The Lie group G.* Let G be connected compact Lie group. Its Lie algebra $\mathfrak{g}$ admits a positive definite symmetric Ad-invariant bilinear form, so we fix one and denote it by $\{\,,\,\} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Fix a conjugacy class $J \subset G$. It is well known that G embeds in some $\mathbb{R}^N$ as an algebraic variety. Thus, if F_g denotes the free group on g generators, $\mathrm{Hom}(F_g, G) = G^g$ is an affine real-set variety, with tangent space $\mathfrak{g}^g$ at the trivial homomorphism.

Our main focus, and the context for Theorem A, concerns the case when $G = \mathrm{SU}(2)$, $\{v, w\} = -\frac{1}{2} \mathrm{Tr}(vw)$, and $J \subset \mathrm{SU}(2)$ is the conjugacy class of traceless matrices. To keep notation compact, we identify $\mathrm{SU}(2)$ with the group of unit quaternions

$$\{a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \mid a^2 + b^2 + c^2 + d^2 = 1\}$$

and its Lie algebra su(2) with the purely imaginary quaternions $\{b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}\}$. With this identification, $\mathrm{Re} : \mathrm{SU}(2) \to [-1, 1]$ corresponds to $\frac{1}{2} \mathrm{Tr}$.

2A3. *Notation used for 2- and 3-manifolds.* Throughout this article, the notation Y, S, C, V, and F is fixed as follows.

First, S denotes a possibly disconnected compact oriented surface without boundary. Denote by S_+ the path connected based space obtained by first adding a disjoint base point to S, then attaching a 1-cell from this new base point to each path component of S.

Next, C denotes a finite disjoint union of m circles in the surface S. Number its components C_i, $i = 1, \ldots m$. We assume that either each C_i is oriented, or the chosen conjugacy class $J \subset G$ is invariant under inversion in G, so that the condition that a homomorphism from $\pi_1(S_+) \to G$ takes each loop C_i into J makes sense.

The pair (S, C) determines a decomposition of S into two surfaces as follows.

Denote by V a tubular neighborhood of C. Then $V \subset S$ is a disjoint union of oriented annuli, one around each C_i.

Denote by F the complementary surface, determining the decomposition:

$$S = F \cup_{\partial F = \partial V} V. \tag{2-1}$$

Orient V and F as subsurfaces of S.

Finally, Y denotes a compact, connected, and oriented 3-manifold with boundary $\partial Y = S$; that is, we assume that an orientation preserving identification of ∂Y with S is given. Fix a base point in the interior of Y, and extend the inclusion $S = \partial Y \subset Y$ to a based embedding of S_+ into Y.

2A4. *Tangles.* A tangle $(X, \mathcal{L})$ consists of a connected compact oriented 3-manifold X, and $\mathcal{L} \subset X$ a properly embedded compact 1-submanifold with boundary. Thus $\mathcal{L}$ consists of a disjoint union of n intervals and k interior circles.

A tangle $(X, \mathcal{L})$ gives rise to a triple (Y, S, C) as above by taking

$$Y = X \setminus N(\mathcal{L}),$$

where $N(\mathcal{L})$ denotes a tubular neighborhood of $\mathcal{L}$, and letting $S = \partial Y$. Then let $C \subset S$ denote a union of $m = n + k$ meridians of $\mathcal{L}$, one for each component, viewed as a curve in S. As before, Y, S, C determine V and F. Orientations of the tangle components are equivalent to orientations of the components of C.

The process $(X, \mathcal{L}) \Rightarrow (Y, S, C)$ is nearly reversible, by attaching 2-handles to Y along C and setting $\mathcal{L}$ to be union of the co-cores of the 2-handles. The resulting tangle is obtained from $(X, \mathcal{L})$ by removing k disjoint small balls from the interior of X, each meeting a different closed component of $\mathcal{L}$ in a trivial arc. This results in a tangle with no closed components, but which has the same character variety as the starting tangle. We use the notation (Y, S, C) in our arguments as it leads to simpler expressions, but state consequences in terms of the tangle $(X, \mathcal{L})$, as they are clearly seen as morphisms in a (more familiar) $(2, 0) + 1$ cobordism category.

2B. *Poincaré duality and intersection forms.* We begin by recalling the statement of Poincaré-Lefschetz duality with local coefficients for an oriented compact connected n-manifold M with boundary ∂M equipped with a finite cell decomposition (see [8] for a careful exposition and proofs, and [24] for an elementary derivation using dual regular cell decompositions). Denote by $\xi \in C_n(M, \partial M; \mathbb{Z})$ a cellular chain representing the fundamental class.

Fix a base point in M and some homomorphism $\pi_1 M \to \Gamma$ to some group Γ, and denote by $M_\Gamma \to M$ the corresponding Γ cover, equipped with the lifted cell structure and its cellular left Γ covering action. The cellular $\mathbb{Z}$ chain complexes of M_Γ and $(M_\Gamma, \partial M_\Gamma)$ are denoted by $C_*^\Gamma(M)$ and $C_*^\Gamma(M, \partial M)$. These are free finitely

generated left $\mathbb{Z}\Gamma$ modules: a choice of $\mathbb{Z}\Gamma$ basis is given by arbitrarily choosing lifts of cells of M (respectively of cells which intersect the interior of M). A choice of cellular approximation of the diagonal determines a chain level cap product:

$$\cap \xi : \mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*^\Gamma(M, \partial M), \mathbb{Z}\Gamma) \to C_*^\Gamma(M). \tag{2-2}$$

The abelian group $\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*^\Gamma(M, \partial M, \mathbb{Z}\Gamma)$ admits a left $\mathbb{Z}\Gamma$ structure by $(\gamma \cdot x)(\sigma) := x(\sigma) \cdot \gamma^{-1}$. The Poincaré duality theorem asserts that the map of (2-2) is a chain homotopy equivalence of free left $\mathbb{Z}\Gamma$ chain complexes.

Given any left $\mathbb{Z}\Gamma$ module W, we have that

$$\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*^\Gamma(M, \partial M), W) = \mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*^\Gamma(M, \partial M), \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} W$$

and hence capping with ξ induces a chain homotopy equivalence

$$\cap \xi : \mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*^\Gamma(M, \partial M), W) \to C_*^\Gamma(M) \otimes_{\mathbb{Z}\Gamma} W.$$

Now suppose that W is a finite-dimensional $\mathbb{R}$ vector space equipped with a positive definite inner product $\{\,,\,\} : W \times W \to \mathbb{R}$ and $\Gamma \to O(W)$ a representation, determining the left $\mathbb{Z}\Gamma$ module structure on W. There is an algebraic chain isomorphism of $\mathbb{R}$ chain complexes:

$$C_*^\Gamma(M) \otimes_{\mathbb{Z}\Gamma} W \cong \mathrm{Hom}_{\mathbb{R}}\big(\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*^\Gamma(M), W), \mathbb{R}\big), \ c \otimes w \mapsto \big(h \mapsto \{h(c), w\}\big).$$

Composing with the chain homotopy equivalence $\cap \xi$, and using the universal coefficient theorem for $\mathbb{R}$, we obtain an isomorphism

$$H^*(M, \partial M; W) \cong H^*(\mathrm{Hom}_{\mathbb{R}}(\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*^\Gamma(M), W), \mathbb{R})) = \mathrm{Hom}_{\mathbb{R}}(H^*(M; W), \mathbb{R})$$

whose adjoint is the (by construction nondegenerate) *cohomology intersection pairing over W*

$$H^*(M, \partial M; W) \times H^*(M; W) \to \mathbb{R}. \tag{2-3}$$

The cohomology intersection pairing can also be expressed in terms of cup products:

$$(x, y) \mapsto \{x \cup y\} \cap \xi, \tag{2-4}$$

where $\{\,\} : H^n(M, \partial M; W \otimes_{\mathbb{Z}\Gamma} W) \to H^n(M, \partial M; \mathbb{R})$ is induced by the coefficient homomorphism $W \otimes_{\mathbb{Z}\Gamma} W \to \mathbb{R}$ determined by the bilinear form $\{\,,\,\}$.

When the boundary of M is empty, this pairing, which we denote by

$$\omega_M : H^*(M; W) \times H^*(M; W) \to \mathbb{R}, \tag{2-5}$$

is therefore a nondegenerate inner product on the $\mathbb{R}$ vector space $H^*(M; W)$.

If ∂M is nonempty, precomposing the injective adjoint

$$H^*(M; W) \to \mathrm{Hom}_{\mathbb{R}}(H^*(M, \partial M; W); \mathbb{R})$$

with the restriction map $H^*(M, \partial M; W) \to H^*(M; W)$ yields a map

$$H^*(M, \partial M; W) \to \mathrm{Hom}_{\mathbb{R}}(H^*(M, \partial M; W); \mathbb{R})$$

with kernel equal to the kernel of $H^*(M, \partial M; W) \to H^*(M; W)$. An equivalent statement is that the pairing

$$\omega_{(M, \partial M)} : H^*(M, \partial M; W) \times H^*(M, \partial M; W) \to \mathbb{R} \qquad (2\text{-}6)$$

has radical equal to $\ker H^*(M, \partial M; W) \to H^*(M; W)$.

Taking gradings into account, when $\dim M = 2n$ restriction defines a pairing

$$\omega_{(M, \partial M)} : H^n(M, \partial M; W) \times H^n(M, \partial M; W) \to \mathbb{R} \qquad (2\text{-}7)$$

with radical equal to $\ker H^n(M, \partial M; W) \to H^n(M; W)$.

When $\dim M = 4\ell + 2$, for example when M is a surface, the pairings ω_M and $\omega_{(M, \partial M)}$ are skew-symmetric:

$$\omega_M(x, y) = -\omega_M(y, x) \quad \text{and} \quad \omega_{(M, \partial M)}(x, y) = -\omega_{(M, \partial M)}(y, x).$$

3. Two- and three-dimensional manifolds and symplectic linear algebra

3A. *A symplectic form on the first cohomology of surface.* Recall that S_+ denotes the path connected CW complex obtained by adding a disjoint base point to the oriented surface S and a 1-cell connecting each path component of S to this base point. Let $\Gamma = \pi_1(S_+)$. Its universal cover $\widetilde{S}_+ \to S_+$ is a regular Γ cover, and hence so is its restriction over S,

$$\widetilde{S} \to S.$$

A representation $\rho : \Gamma \to G$ is fixed.

Then ρ determines, via the adjoint action of G, the representation $\mathrm{Ad}\,\rho : \Gamma \to O(\mathfrak{g})$, and hence cohomology groups $H^*(S; \mathfrak{g})$, $H^*(S, \partial S; \mathfrak{g})$, and $H^*(\partial S; \mathfrak{g})$, with, for example,

$$H^*(S; \mathfrak{g}) := H^*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*^{\Gamma}(S), \mathfrak{g})).$$

If we wish to emphasize ρ, we write $H^*(S; \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

Equation (2-5) shows that, since the boundary of S is empty,

$$\omega_S : H^1(S; \mathfrak{g}) \times H^1(S; \mathfrak{g}) \to \mathbb{R} \qquad (3\text{-}1)$$

is a nondegenerate skew-symmetric form.

On the other hand, if C is nonempty, so that the boundary of F is nonempty, equation (2-7) shows that

$$\omega_{(F, \partial F)} : H^1(F, \partial F; \mathfrak{g}) \times H^1(F, \partial F; \mathfrak{g}) \to \mathbb{R}$$

is in general a degenerate skew symmetric form with radical equal to

$$\ker H^1(F, \partial F; \mathfrak{g}) \to H^1(F; \mathfrak{g}).$$

A degenerate skew symmetric form induces a nondegenerate form on the quotient by its radical. The exact sequence of the pair $(F, \partial F)$ shows that $\omega_{(F, \partial F)}$ descends to a *nondegenerate* skew symmetric form

$$\widehat{\omega}_F : \widehat{H}^1(F; \mathfrak{g}) \times \widehat{H}^1(F; \mathfrak{g}) \to \mathbb{R}, \tag{3-2}$$

where

$$\widehat{H}^1(F; \mathfrak{g}) = \text{Image } H^1(F, \partial F; \mathfrak{g}) \to H^1(F; \mathfrak{g}) = \ker H^1(F; \mathfrak{g}) \to H^1(\partial F; \mathfrak{g}).$$

Summarizing:

Proposition 3.1. *Let S be a compact oriented surface without boundary and $\rho : \pi_1 S_+ \to G$ a representation. Then $(H^1(S; \mathfrak{g}_{\text{Ad}\,\rho}), \omega_S)$ is a symplectic vector space. If $F \subset S$ is the complement of a nonempty disjoint union of annuli V, then $(\widehat{H}^1(F; \mathfrak{g}_{\text{Ad}\,\rho}), \widehat{\omega}_F)$ is a symplectic vector space.*

In the following diagram of inclusions, the two vertical and two horizontal rows are exact, and the isomorphisms are excisions ($\mathfrak{g}_{\text{Ad}\,\rho}$-coefficients omitted):

$$
\begin{array}{ccc}
H^1(S, F) \xrightarrow{\cong} H^1(V, \partial V) \\
\downarrow{q_1} \qquad\qquad \downarrow{q_2} \\
\longrightarrow H^1(S, V) \xrightarrow{\alpha} H^1(S) \xrightarrow{\beta} H^1(V) \xrightarrow{\gamma} H^2(S, V) \quad (3\text{-}3) \\
\cong \downarrow{p_0} \qquad \downarrow{p_1} \qquad \downarrow{p_2} \qquad \cong \downarrow \\
\longrightarrow H^1(F, \partial F) \xrightarrow{a} H^1(F) \xrightarrow{b} H^1(\partial F) \xrightarrow{c} H^2(F, \partial F)
\end{array}
$$

Proposition 3.2. *The kernel of β is a coisotropic subspace of $H^1(S; \mathfrak{g})$ with annihilator $\ker p_1$, and hence p_1 induces a symplectomorphism*

$$\ker \beta / \ker p_1 \xrightarrow{\cong} \widehat{H}^1(F; \mathfrak{g}).$$

Proof. The surface V is a disjoint union of annuli. The composition $\{0\} \times S^1 \subset \partial(I \times S^1) \subset I \times S^1$ a homotopy equivalence, and hence the restriction $H^1(I \times S^1) \to H^1(\partial(I \times S^1))$ is injective with any coefficients. Hence p_2 is injective, and $q_2 = 0$.

If $s \in H^1(S)$ satisfies $\omega_S(s, \alpha(y)) = 0$ for all $y \in H^1(S, V)$, we then have that $\omega_{F, \partial F}(p_1(s), a \circ p_0(y)) = 0$ for all $y \in H^1(S, V)$. Hence $\omega_{F, \partial F}(p_1(s), a(z)) = 0$ for all $z \in H^1(F, \partial F)$. Since the pairing $H^1(F, \partial F) \times H^1(F) \to \mathbb{R}$ is nondegenerate (equation (2-3)), this implies that $p_1(s) = 0$. In other words, the annihilator of $\text{im}\,\alpha = \ker \beta$ is contained in $\ker p_1$.

Since $q_2 = 0$, $\ker p_1 \subset \ker \beta$. Therefore $\ker \beta$ contains its annihilator and hence is coisotropic.

It remains to show that $\ker p_1 = \text{image } q_1$ is contained in the annihilator of $\ker \beta$. Given $x \in H^1(S, V)$ and $y \in H^1(S, F)$, we have that $\omega_S(\alpha(x), q_1(y)) = 0$ since the cup product

$$H^1(S, V) \times H^1(S, F) \to H^2(S)$$

factors through $H^2(S, F \cup V) = 0$ (see (2-4)). $\qquad\square$

Corollary 3.3. *If $L \subset H^1(S; \mathfrak{g}_{\text{Ad}\,\rho})$ is any Lagrangian subspace, then the image*

$$p_1(L \cap \ker \beta) \subset \widehat{H}^1(F; \mathfrak{g}_{\text{Ad}\,\rho})$$

is a Lagrangian subspace.

3B. *Restriction from a 3-manifold with boundary.* Recall that Y is a compact, connected, oriented 3-manifold with boundary $S = \partial Y$, extended to an embedding $S_+ \subset Y$.

Assume that the representation $\rho : \pi_1(S_+) \to G$ is a restriction of a representation (of the same name) $\rho : \pi_1(Y) \to G$.

Lemma 3.4. *The image of the restriction map,*

$$L_Y := \text{Image } r : H^1(Y; \mathfrak{g}_{\text{Ad}\,\rho}) \to H^1(S; \mathfrak{g}_{\text{Ad}\,\rho}),$$

is a Lagrangian subspace of $(H^1(S; \mathfrak{g}_{\text{Ad}\,\rho}), \omega_S)$.

Proof. In the following diagram, the middle row is part of the exact sequence of the pair (Y, S). The vertical arrows all isomorphisms, with the downward pointing isomorphisms Poincaré-Lefschetz duality. The diagram commutes up to sign [26]. We suppress the $\mathfrak{g}_{\text{Ad}\,\rho}$ coefficients.

$$
\begin{array}{ccccc}
\text{Hom}(H_1(Y); \mathbb{R}) & \longrightarrow & \text{Hom}(H_1(S); \mathbb{R}) & \xrightarrow{\delta^*} & \text{Hom}(H_2(Y, S); \mathbb{R}) \\
\uparrow & & \uparrow & & \uparrow \\
H^1(Y) & \xrightarrow{\ \ r\ \ } & H^1(S) & \xrightarrow{\ \ \nu\ \ } & H^2(Y, S) \\
\downarrow & & \downarrow & & \downarrow \\
H_2(Y, S) & \xrightarrow{\ \ \delta\ \ } & H_1(S) & \longrightarrow & H_1(Y)
\end{array}
$$

The image of r equals the kernel of ν, which is isomorphic to the kernel of δ^*. The kernel of δ^* is isomorphic to the cokernel of its dual δ, which in turn is isomorphic to the cokernel of r. Hence the image and cokernel of r are isomorphic, and so $\dim(\text{image}(r)) = \frac{1}{2} \dim H^1(S)$.

Commutativity of the following diagram is a consequence of naturality of cup product and Poincaré duality:

$$
\begin{array}{ccccc}
H^1(Y; \mathfrak{g}) \times H^1(Y; \mathfrak{g}) & \xrightarrow{\cup_{\{,\}}} & H^2(Y; \mathbb{R}) & \xrightarrow{\cap[Y,S]} & H_1(Y, S; \mathbb{R}) \\
\downarrow{r \times r} & & \downarrow & & \downarrow \\
H^1(S; \mathfrak{g}) \times H^1(S; \mathfrak{g}) & \xrightarrow{\cup_{\{,\}}} & H^2(S; \mathbb{R}) & \xrightarrow{\cap[S]} & H_0(S; \mathbb{R}) \xrightarrow{\epsilon} \mathbb{R} \\
& & & & \downarrow \\
& & & & H_0(Y; \mathbb{R})
\end{array}
$$

Exactness of the vertical sequence shows that the dotted arrow is zero, which implies that the image of r is isotropic, and therefore Lagrangian. □

Recall that the boundary $S = \partial Y$ is equipped with a embedded collection $C \subset S$ of circles, with tubular neighborhood V and complementary subsurface F, producing the decomposition $S = F \cup V$ as in (2-1). Consider the ladder of exact sequences, with all maps induced by inclusions. The $\mathfrak{g}_{\mathrm{Ad}\,\rho}$ coefficients are suppressed. The bottom two rows coincide with those of (3-3).

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & H^1(Y, V) & \xrightarrow{A} & H^1(Y) & \xrightarrow{B} & H^1(V) & \longrightarrow \cdots \\
& \downarrow{r_0} & & \downarrow{r} & & \| & \\
\cdots \longrightarrow & H^1(S, V) & \xrightarrow{\alpha} & H^1(S) & \xrightarrow{\beta} & H^1(V) & \longrightarrow \cdots \\
& \cong\downarrow{p_0} & & \downarrow{p_1} & & \downarrow{p_2} & \\
\cdots \longrightarrow & H^1(F, \partial F) & \xrightarrow{a} & H^1(F) & \xrightarrow{b} & H^1(\partial F) & \longrightarrow \cdots
\end{array} \quad (3\text{-}4)
$$

A diagram chase shows that Image $r \cap \ker \beta = $ Image $\alpha \circ r_0$. Hence

$$p_1(\text{Image } r \cap \ker \beta) = \text{Image } a \circ p_0 \circ r_0 : H^1(Y, V) \to H^1(F) \subset \ker b = \widehat{H}^1(F)$$

Lemma 3.4 and Corollary 3.3 imply the following.

Corollary 3.5. *Suppose that $\partial Y = S = V \cup F$ with V a disjoint union of annuli. Then*

$$L_{Y,V} := \text{Image } H^1(Y, V; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(F; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \subset \widehat{H}^1(F; \mathfrak{g}_{\mathrm{Ad}\,\rho})$$

is a Lagrangian subspace. Moreover, $L_{Y,V}$ is the symplectic reduction of L_Y with respect to $\ker \beta : H^1(S; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(V; \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

4. Character varieties, relative character varieties, and their tangent spaces

4A. *Character varieties.*

Definition 4.1. Given a finitely presented group Γ, its *G character variety* $\chi_G(\Gamma)$ is the real semialgebraic set defined to be the orbit space of the *G*-conjugation

action on the affine $\mathbb{R}$-algebraic set $\mathrm{Hom}(\Gamma, G)$. A set of g generators of Γ embeds $\mathrm{Hom}(\Gamma, G)$ in G^g equivariantly, and, since G is a compact Lie group, $\mathrm{Hom}(\Gamma, G)$ is an affine $\mathbb{R}$-algebraic set, with orbit space $\chi_G(\Gamma)$ a semialgebraic set. We call $\chi_G(\Gamma)$ the *G-character variety of* Γ. If Z is a path connected space, write $\chi_G(Z)$ instead of $\chi_G(\pi_1(Z))$. The character variety of a nonpath connected space, *by definition*, is the cartesian product of the character varieties of its path components.

As observed by Weil [31], the *formal tangent space at* the conjugacy class of *any* representation $\rho : \Gamma \to G$ to the character variety $\chi_G(\Gamma)$ is naturally identified with first cohomology:

$$T_\rho \chi_G(\Gamma) = H^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\,\rho}). \tag{4-1}$$

We take (4-1) as the definition of the formal tangent space at ρ for any $\rho \in \chi_G(\Gamma)$. Recall that $H^1(\Gamma; \mathfrak{g}_{\mathrm{Ad}\,\rho})$ and $H^1(X; \mathfrak{g}_{\mathrm{Ad}\,\rho})$ are canonically isomorphic for any space X with fundamental group Γ.

Weil's argument is based on the calculation that if a path of representations is expressed in the form $\rho_s = \exp(\alpha_s)\rho_0$ for some path $\alpha_s : \Gamma \to \mathfrak{g}$, then $\frac{d}{ds}|_{s=0} \alpha_s : \Gamma \to \mathfrak{g}$ is a 1-cocycle [6].

4B. *Relative character varieties.* As above, assume that Y is a compact connected 3-manifold with boundary $S = \partial Y$, $C \subset S$ is a union of m circles C_i, V is the tubular neighborhood of C in S, with complementary surface F.

Assume further that either J is invariant via the inversion map of G (as is the case for the conjugacy class of traceless matrices in $\mathrm{SU}(2)$) or else assume that every circle C_i is equipped with an orientation.

Then define the *relative character variety*

$$\chi_{G,J}(Y, C) \subset \chi_G(Y) \tag{4-2}$$

to be the subvariety consisting of conjugacy classes of representations $\pi_1(Y) \to G$ which send (any based representative of the homotopy class of) each circle in C into J. Define the *formal tangent space* of $\chi_{G,J}(Y, C)$ at ρ to be

$$T_\rho \chi_{G,J}(Y, C) = \ker H^1(Y; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(C; \mathfrak{g}_{\mathrm{Ad}\,\rho}). \tag{4-3}$$

Given an oriented surface F, define

$$\chi_{G,J}(F, \partial F) = \chi_{G,J}\big(F \times [0, 1], \partial F \times \{\tfrac{1}{2}\}\big).$$

Its formal tangent space is

$$T_\rho \chi_{G,J}(F, \partial F) = \ker H^1(F; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(\partial F; \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \widehat{H}^1(F; \mathfrak{g}_{\mathrm{Ad}\,\rho}). \tag{4-4}$$

4C. *Regular points.* The term "formal tangent space" may be replaced by its usual elementary differential topology notion in neighborhoods of *regular points* of $\chi_{G,J}(Y, C)$ and $\chi_{G,J}(F, \partial F)$, as we next explain. In brief, as elsewhere in gauge theory, a regular point is one which has a neighborhood diffeomorphic to Euclidean space of the correct index-theoretic dimension. We provide a stripped-down explanation, suitable for our purposes, of what this means, for the benefit of the reader.

First, given a connected compact surface F (with possibly empty boundary) we call $\rho \in \chi_{G,J}(F, \partial F)$ a regular point provided ρ has a neighborhood $U \subset \chi_{G,J}(F, \partial F)$ so that $\dim \widehat{H}^1(F; \mathfrak{g}_{\mathrm{Ad}\,\rho'})$ is independent of $\rho' \in U$.

Next, given a disjoint union of connected compact surfaces $F = \bigsqcup_i F_i$, call

$$\rho \in \chi_{G,J}(F, \partial F) = \prod_i \chi_{G,J}(F_i, \partial F_i)$$

a regular point provided each of its components is a regular point.

Finally, for a pair (Y, C) (with C possibly empty) $\rho \in \chi_{G,J}(Y, C)$ is called a regular point provided ρ admits a neighborhood $U \subset \chi_{G,J}(Y, C)$ so that for all $\rho' \in U$:

- The restriction map $\chi_{G,J}(Y, C) \to \chi_{G,J}(F, \partial F)$ takes ρ' to a regular point, and

- $\dim T_{\rho'} \chi_{G,J}(Y, C) = \frac{1}{2} \dim T_\rho \chi_{G,J}(F, \partial F) = \frac{1}{2} \sum_i \dim T_\rho \chi_{G,J}(F_i, \partial F_i)$.

Hence, if $\rho \in \chi_{G,J}(Y, C)$ is a regular point, the map $\chi_{G,J}(Y, C) \to \chi_{G,J}(F, \partial F)$, which takes a representation of a 3-manifold group to its restriction to the boundary surface, is, near ρ, a smooth map of a smooth n-disk into $\mathbb{R}^{2n}$ for some n.

Notice that $\chi_{G,J}(Y, C)$ is the preimage of the point $(J, \ldots, J)$ under the restriction map

$$\chi_G(Y) \to \chi_G(C) = \prod_{i=1}^m \chi_G(C_i) = (G/_{\mathrm{conjugation}})^m,$$

and that $\chi_{G,J}(F, \partial F)$ is the preimage of the point $(J, \ldots, J)$ under the restriction map

$$\chi_G(F) \to \chi_G(\partial F).$$

Since $C \subset V$ is a deformation retract, the exact sequence of the pair shows that

$$T_\rho \chi_{G,J}(Y, C) \cong \mathrm{Image}\ H^1(Y, V; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(Y; \mathfrak{g}_{\mathrm{Ad}\,\rho}).$$

The image of the differential of the restriction map $\chi_{G,J}(Y, C) \to \chi_{G,J}(F, \partial F)$ at $\rho \in \chi_{G,J}(Y, C)$ is therefore identified with $L_{Y,V}$, the image of the composition

$$H^1(Y, V; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^1(F, \partial F; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to \widehat{H}^1(F; \mathfrak{g}_{\mathrm{Ad}\,\rho}),$$

which by Corollary 3.5 is a Lagrangian subspace of $(H^1(F; \mathfrak{g}_{\mathrm{Ad}\,\rho}), \widehat{\omega}_F)$. In sum:

Corollary 4.2. *If $\rho \in \chi_{G,J}(Y, C)$ is a regular point, then there exists a neighborhood of ρ so that the differential of the restriction $\chi_{G,J}(Y, C) \to \chi_{G,J}(F, \partial F)$ at any point ρ' in this neighborhood has image a Lagrangian subspace of $(\widehat{H}^1(F; \mathfrak{g}_{\mathrm{Ad}\, \rho'}), \widehat{\omega}_F)$.*

It is known that for surfaces, with the exception of a few low genus cases, the regular points coincide with the irreducible representations.

4D. *Symplectic structure.* The proof of Corollary 4.2 does not rely of the following fundamental result of Atiyah and Bott [3] and its extensions due to Goldman [12], Karshon [19], Biswas and Guruprasad [4], King and Sengupta [20], Guruprasad, Huebschmann, Jeffrey and Weinstein [14].

Theorem 4.3. *On the top stratum of regular points of $\chi_G(S)$ and $\chi_{G,J}(F, \partial F)$, the 2-forms ω_S and $\widehat{\omega}_F$ are closed, that is, are symplectic forms.*

In light of this result, Corollary 4.2 can be restated as follows.

Theorem 4.4. *Suppose that $\rho \in \chi_{G,J}(Y, C)$ is a regular point. Then there exists a neighborhood U of ρ in $\chi_{G,J}(Y, C)$ so that the restriction of r to U,*

$$r|_U : U \to \chi_{G,J}(F, \partial F),$$

is a Lagrangian embedding.

In particular, if $\chi_{G,J}(Y, C)$ contains only regular points, then the restriction map is a Lagrangian immersion.

Given a tangle $(X, \mathcal{L})$, we write $\chi_{G,J}(X, \mathcal{L})$ rather than $\chi_{G,J}(Y, C)$, where Y, S, C, F and V are determined by $(X, \mathcal{L})$ as in Section 2A4. Also write $\chi_{G,J}(\partial(X, \mathcal{L}))$ rather than $\chi_{G,J}(F, \partial F)$. Theorem 4.4 can be restated in the new notation as follows.

Corollary 4.5. *Suppose that $\rho \in \chi_{G,J}(X, \mathcal{L})$ is a regular point. Then there exists a neighborhood U of ρ in $\chi_{G,J}(X, \mathcal{L})$ so that the restriction of r to U,*

$$r|_U : U \to \chi_{G,J}(\partial(X, \mathcal{L})),$$

is a Lagrangian embedding.

In particular, if $\chi_{G,J}(X, \mathcal{L})$ contains only regular points, then the restriction map is a Lagrangian immersion.

In what follows, we simply write $\chi(A)$ for $\chi_G(A)$ and $\chi(A, B)$ for $\chi_{G,J}(A, B)$. In addition, we denote by $\chi(S)^* \subset \chi(S)$ and $\chi(F, \partial F)^* \subset \chi(F, \partial F)$ the *smooth top strata*, as real algebraic varieties, equipped with the symplectic forms ω_S, $\widehat{\omega}_F$.

5. Three pillowcases

In this section, take $G = \mathrm{SU}(2)$ viewed as the unit quaternions, with Lie algebra su(2) the purely imaginary quaternions. Let $J \subset \mathrm{SU}(2)$ denote the conjugacy class of unit quaternions with zero real part, so $J = \mathrm{su}(2) \cap \mathrm{SU}(2)$, a 2-sphere.

5A. *The quotient of the torus by the elliptic involution.* Let $\mathbb{T}$ denote $U(1) \times U(1)$ with its symplectic form $dx \wedge dy$ and symplectic $\mathrm{SL}(2, \mathbb{Z})$ action given in (1-2). Multiplication by $-1 \in \mathrm{SL}(2, \mathbb{Z})$ is central and hence the quotient

$$\mathbb{P} = \mathbb{T}/\{\pm 1\}$$

inherits a $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\{\pm 1\}$ action. The quotient map $\mathbb{T} \to \mathbb{P}$ is the 2-fold branched cover of the 2-sphere with branch points the four points $\{(\pm 1, \pm 1)\}$. The complement, $\mathbb{P}^*$ of the four branch points is a smooth surface, with symplectic form $dx \wedge dy$ and symplectic $\mathrm{PSL}(2, \mathbb{Z})$ action, and the restriction

$$\mathbb{T}^* \to \mathbb{P}^*$$

is a smooth symplectic 2-fold covering map.

5B. *The* $\mathrm{SU}(2)$ *character variety of the genus 1 surface.* Consider the genus one closed oriented surface T, equipped with generators $\mu, \lambda \in \pi_1 T$ represented by a pair of oriented loops intersecting geometrically and algebraically once. The Dehn twists D_μ, D_λ about μ and λ induce the automorphisms

$$D_\mu : (\mu \mapsto \mu, \lambda \mapsto \mu\lambda) \quad \text{and} \quad D_\lambda : (\mu \mapsto \mu\lambda^{-1}, \lambda \mapsto \lambda).$$

of $\pi_1(T, t_0) \cong \mathbb{Z}\mu \oplus \mathbb{Z}\lambda$ (where t_0 lies outside the support of these 2 Dehn twists). These automorphisms induce, by precomposition, homeomorphisms

$$D_\mu^*, D_\lambda^* : \chi(T) \to \chi(T).$$

Let $\mathcal{H}(T) = \mathrm{Hom}(\pi_1(T), \mathrm{SU}(2))$. Denote by

$$p : \mathcal{H}(T) \to \chi(T)$$

the (surjective) orbit map of the conjugation action.

Define

$$\rho : \mathbb{T} \to \mathcal{H}(T), \ \rho(e^{x\boldsymbol{i}}, e^{y\boldsymbol{i}}) = (\mu \mapsto e^{x\boldsymbol{i}}, \ \lambda \mapsto e^{y\boldsymbol{i}}). \tag{5-1}$$

Proposition 5.1. *The Dehn twists D_μ and D_λ define a symplectic $\mathrm{PSL}(2, \mathbb{Z})$ action on $(\chi(T), \omega_T)$. The map ρ of (5-1) descends to a $\mathrm{PSL}(2, \mathbb{Z})$ equivariant homeomorphism*

$$\rho : \mathbb{P} \to \chi(T)$$

which restricts to a $\mathrm{PSL}(2, \mathbb{Z})$ equivariant symplectomorphism

$$(\mathbb{P}^*, c \, dx \wedge dy) \to (\chi(T)^*, \omega_T)$$

on the top strata of the $\mathrm{SU}(2)$ character varieties.

Proof. It is elementary to show that $\rho : \mathbb{P} \to \chi(T)$ is a well defined homeomorphism, as well as an analytic diffeomorphism of the smooth strata $\mathbb{P}^* \to \chi(T)^*$.

We show that $\rho : (\mathbb{P}^*, dx \wedge dy) \to (\chi(T)^*, \omega_T)$ is a symplectomorphism. Since this is a local statement, we work in $\mathbb{R}^2$ for simplicity. Define $m = \rho \circ e : \mathbb{R}^2 \to \chi(T)$, where $e(x, y) = (e^{x\boldsymbol{i}}, e^{y\boldsymbol{i}})$.

The differential of m at $(x, y) \in \mathbb{R}^2$, $dm : T_{(x,y)}\mathbb{R}^2 \to T_{m(x,y)}(\mathrm{SU}(2) \times \mathrm{SU}(2))$ is given by

$$dm\left(\tfrac{\partial}{\partial x}\right) = \left(\mu \mapsto e^{x\boldsymbol{i}}\boldsymbol{i}, \lambda \mapsto 0\right), \ dm\left(\tfrac{\partial}{\partial y}\right) = \left(\mu \mapsto 0, \lambda \mapsto e^{y\boldsymbol{i}}\boldsymbol{i}\right).$$

Following Weil [31], left translation in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ identifies these with the su(2)-valued 1-cochains

$$z_x = (\mu \mapsto \boldsymbol{i}, \lambda \mapsto 0), \ z_y = (\mu \mapsto 0, \lambda \mapsto \boldsymbol{i}) \in C^1(T; \mathrm{su}(2)_{\mathrm{Ad}\, m(x,y)}).$$

The subspaces $L = \boldsymbol{i}\mathbb{R}$ and $V = \boldsymbol{j}\mathbb{R} + \boldsymbol{k}\mathbb{R}$ are invariant and complementary with respect to $\mathrm{Ad}\, m(x, y)$, and therefore

$$H^1(T; \mathrm{su}(2)_{\mathrm{Ad}\, m(x,y)}) = H^1(T; L_{\mathrm{Ad}\, m(x,y)}) \oplus H^1(T; V_{\mathrm{Ad}\, m(x,y)}).$$

Note that the action $\mathrm{Ad}\, m(x, y)$ on L is trivial, since $L = \boldsymbol{i}\mathbb{R}$ and $m(x, y)$ has values in the abelian subgroup $\{e^{\boldsymbol{i}u}\}$. Hence

$$H^1(T; L_{\mathrm{Ad}\, m(x,y)}) \cong H^1(T; \mathbb{R}) \cong \mathbb{R}^2.$$

The branched cover $m : \mathbb{R}^2 \to \chi(T)$ is a local diffeomorphism near any

$$(x, y) \in (\mathbb{R}^2)^* = \mathbb{R}^2 \setminus (\pi\mathbb{Z})^2,$$

and hence it follows that for such (x, y),

$$H^1(T; \mathbb{R}) \otimes \mathbb{R}\boldsymbol{i} = H^1(T; L_{\mathrm{Ad}\, m(x,y)}) = \mathrm{Span}\{z_x, z_y\} \quad \text{and} \quad H^1(T; V_{\mathrm{Ad}\, m(x,y)}) = 0$$

(these calculations can also be easily checked directly), so that

$$T_{m(x,y)}(\chi(T)) = H^1(T; \mathbb{R}) \otimes \mathbb{R}\boldsymbol{i}.$$

The cup product

$$H^1(T; L_{\mathrm{Ad}\, m(x,y)}) \times H^1(T; L_{\mathrm{Ad}\, m(x,y)}) \to H^2(T; \mathbb{R})$$

is thereby identified with

$$H^1(T; \mathbb{R}) \otimes \mathbb{R}\boldsymbol{i} \times H^1(T; \mathbb{R}) \otimes \mathbb{R}\boldsymbol{i} \to H^2(T; \mathbb{R}),$$
$$(z_1 \otimes \boldsymbol{i}, z_2 \otimes \boldsymbol{i}) = (z_1 \cup z_2) \cdot (-\mathrm{Re}(\boldsymbol{ii})) = z_1 \cup z_2.$$

In particular, $z_x = \mu^* \otimes \boldsymbol{i}$ and $z_y = \lambda^* \otimes \boldsymbol{i}$, where

$$\mu^*, \lambda^* \in H^1(T; \mathbb{Z}) = \mathrm{Hom}(H_1(T; \mathbb{Z}), \mathbb{Z})$$

is the basis dual to μ, λ. This basis is symplectic with respect to the (usual, untwisted) intersection form. Hence

$$\left((p \circ m)^*(\omega_T)\right)|_{(x,y)}\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right) = \omega_T(z_x, z_y) = (\mu^* \cup \lambda^*) \cap [T] = 1$$
$$= dx \wedge dy\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right) = 1.$$

Naturality of cup products shows that D_μ^*, D_λ^* preserve the symplectic form ω_T. Next,

$$D_\mu^*(m(x, y)) = m(x, y) \circ D_\mu$$
$$= \left(\mu \mapsto m(x, y)(\mu) = e^{x i}, \lambda \mapsto m(x, y)(\mu\lambda) = e^{(x+y)i}\right)$$

and similarly

$$D_\lambda^*(m(x, y)) = m(x, y) \circ D_\lambda = \left(\mu \mapsto e^{(x-y)i}, \lambda \mapsto e^{yi}\right).$$

It follows that the subgroup of $\mathrm{Homeo}(\chi(T))$ generated by D_μ^* and D_λ^* pulls back, via the homeomorphism $\rho : \mathbb{P} \to \chi(T)$, to the subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ generated by

$$\rho^*(D_\mu^*) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho^*(D_\lambda^*) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

These two matrices generate $\mathrm{PSL}(2, \mathbb{Z})$ (see [25]), finishing the proof. $\qquad \square$

5B1. *The solid torus and the restriction to its boundary.* Let X denote the solid torus with boundary T. Equip T with based loops μ, λ generating $\pi_1(T)$, so that μ is trivial in $\pi_1(X)$ and λ generates $\pi_1(X)$. Then

$$\chi(X) = \chi(\mathbb{Z}\lambda) = \mathrm{SU}(2)/_{\mathrm{conjugation}}.$$

An explicit slice of the conjugation action $\mathrm{Hom}(\mathbb{Z}\lambda, \mathrm{SU}(2)) \to \chi(\mathbb{Z}\lambda)$ is given by the map

$$[0, \pi] \to \mathrm{Hom}(\mathbb{Z}\lambda, \mathrm{SU}(2)), \quad s \mapsto (\lambda \mapsto e^{is}) \qquad (5\text{-}2)$$

with composition $[0, \pi] \to \chi(X) \xrightarrow{\mathrm{Re}_\lambda} [-1, 1]$ equal to the analytic isomorphism $\cos(s)$. Simple cohomology calculations show $\dim H^1(\mathbb{Z}; \mathfrak{su}(2))$ equals 1 when $0 < s < \pi$ and equals 3 when $s = 0$ or π. This shows that the interior of the interval forms the smooth top stratum of $\chi(\mathbb{Z})$, and the endpoints are singular.

Since $\mu = 1 \in \pi_1(X)$, the restriction-to-the boundary map

$$\chi(X) \to \chi(T) \qquad (5\text{-}3)$$

is easily computed, in $\mathbb{P}$, to be the smooth (necessarily Lagrangian) embedded arc given by

$$[0, \pi] \ni s \mapsto [e^{si}, 1] \in \mathbb{P} \qquad (5\text{-}4)$$

with endpoints at $[-1, 1]$ and $[1, 1]$.

5C. *The traceless* $\mathrm{SU}(2)$ *character variety of the 4-punctured 2-sphere.* Let $4D^2 \subset S^2$ be four disjoint open disks, and set

$$F = S^2 \setminus 4D^2, \quad \partial F = 4S^1.$$

Then $\pi_1(F)$ has presentation

$$\pi_1(F) = \langle a, b, c, d \mid abcd = 1 \rangle$$

and is free on a, b, c. Set

$$\mathcal{H}(S^2, 4) = \left\{ \rho \in \mathrm{Hom}(\pi_1(F), \mathrm{SU}(2)) \mid \rho(a), \rho(b), \rho(c), \rho(d) \in J \right\}.$$

Let $p : \mathcal{H}(S^2, 4) \to \chi(S^2, 4)$ denote the orbit map of the conjugation action.
   Define

$$\hat{\rho} : \mathbb{T} \to \mathcal{H}(S^2, 4), \quad (e^{x\boldsymbol{i}}, e^{y\boldsymbol{i}}) \mapsto \left(a \mapsto \boldsymbol{j}, \; b \mapsto e^{x\boldsymbol{i}}\boldsymbol{j}, \; c \mapsto e^{y\boldsymbol{i}}\boldsymbol{j} \right). \qquad (5\text{-}5)$$

A half Dehn twist of (D^2, p, q), where p, q are a fixed pair of interior points, is a homeomorphism of the disk which fixes the boundary and permutes p and q, and which generates the infinite cyclic mapping class group rel boundary of a disk with two marked interior points. The generator which veers to the right is called a positive half Dehn twist.

Proof of Theorem B. That $\hat{\rho} : \mathbb{P} \to \chi(S^2, 4)$ is a well defined homeomorphism, as well as an analytic diffeomorphism of the smooth strata $\mathbb{P}^* \to \chi(S^2, 4)^*$, is simple; its proof can be found in [22; 18].
   We show that $\hat{\rho} : (\mathbb{P}^*, c \, dx \wedge dy) \to (\chi(S^2, 4)^*, \widehat{\omega}_{(S^2, 4)})$ is a symplectomorphism, for some constant c. Since this is a local statement, we work in $\mathbb{R}^2$ for simplicity. Define $\widehat{m} = \hat{\rho} \circ e : \mathbb{R}^2 \to \chi(S^2, 4)$, where $e(x, y) = (e^{x\boldsymbol{i}}, e^{y\boldsymbol{i}})$.
   The differential of $\widehat{m}$ at $(x, y) \in \mathbb{R}^2$ is given by

$$d\widehat{m}\left(\tfrac{\partial}{\partial x}\right) = \left(a \mapsto 0, b \mapsto e^{x\boldsymbol{i}}\boldsymbol{i}\boldsymbol{j}, c \mapsto 0 \right),$$
$$d\widehat{m}\left(\tfrac{\partial}{\partial y}\right) = \left(a \mapsto 0, b \mapsto 0, c \mapsto e^{y\boldsymbol{i}}\boldsymbol{i}\boldsymbol{j} \right).$$

Using left translation $L_{g^{-1}} : \mathrm{SU}(2) \to \mathrm{SU}(2)$ to identify $T_g \, \mathrm{SU}(2)$ with its Lie algebra $\mathrm{su}(2) = T_1 \, \mathrm{SU}(2)$, transforms $d\widehat{m}\left(\tfrac{\partial}{\partial x}\right)$ and $d\widehat{m}\left(\tfrac{\partial}{\partial y}\right)$ to the $\mathrm{su}(2)$-valued 1-cochains

$$z_x = \left(a \mapsto 0, b \mapsto -\boldsymbol{i}, c \mapsto 0 \right), \quad z_y = \left(a \mapsto 0, b \mapsto 0, c \mapsto -\boldsymbol{i} \right).$$

Since $\widehat{m}$ is a local diffeomorphism away from $(\pi\mathbb{Z})^2$ [15], and $\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}$ span $T_{(x,y)}\mathbb{R}^2$, the cohomology classes $[z_x], [z_y]$ span

$$\widehat{H}^1(S^2 - 4D^2; \mathrm{su}(2)_{\mathrm{Ad}\,\widehat{m}(x,y)}) = T_{[\widehat{m}(x,y)]}(\chi(S^2, 4)).$$

For any $(x, y) \in \mathbb{R}^2 \setminus (\pi \mathbb{Z})^2$, the adjoint action

$$\operatorname{Ad} \widehat{m}(x, y) : \pi_1(S^2 - 4D^2) \to \operatorname{GL}(\operatorname{su}(2))$$

reduces as the direct sum

$$\operatorname{Ad} \widehat{m}(x, y) = \operatorname{Ad} \widehat{m}(x, y)_1 \oplus \operatorname{Ad} \widehat{m}(x, y)_2 : \pi_1(S^2 - 4D^2) \to \operatorname{GL}(\mathbb{R}\boldsymbol{i}) \times \operatorname{GL}(\mathbb{C}\boldsymbol{j})$$

and hence

$$\widehat{H}^1(S^2 - 4D^2; \operatorname{su}(2)_{\operatorname{Ad} \widehat{m}(x,y)})$$
$$= \widehat{H}^1(S^2 - 4D^2; \mathbb{R}\boldsymbol{i}_{\operatorname{Ad} \widehat{m}(x,y)_1}) \oplus \widehat{H}^1(S^2 - 4D^2; \mathbb{C}\boldsymbol{j}_{\operatorname{Ad} \widehat{m}(x,y)_2}).$$

Since $\mathbb{R}\boldsymbol{i}$ and $\mathbb{C}\boldsymbol{j}$ are orthogonal, the symplectic form $\widehat{\omega}_{S^2,4}$ splits orthogonally

$$\widehat{\omega}_{S^2,4} = \widehat{\omega}^1_{S^2,4} \oplus \widehat{\omega}^2_{S^2,4}.$$

The cocyles z_x and z_y lie in the first summand, and hence they span the first summand and the second summand is zero (these two facts can also be easily calculated directly). Hence $\widehat{\omega}_{S^2,4} = \widehat{\omega}^1_{S^2,4}$.

The representation on the first summand is independent of (x, y): indeed a, b, and c (and hence also d) act by -1 for all x, y. The cocycles z_x, z_y are independent of x, y, and hence

$$\widehat{m}^*(\widehat{\omega}_{S^2,4})|_{(x,y)}\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right) = \widehat{\omega}^1_{S^2,4}(z_x, z_y) = c \, dx \wedge dy\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right)$$

for a nonzero constant c (since z_x, z_y span).

The half-Dehn twists along the disks H_1 and H_2 illustrated in Figure 1 induce automorphisms of $\pi_1(S^2 \setminus 4, s_0)$, for a base point chosen outside the supports of H_1 and H_2, as indicated in the figure. These automorphisms are given by

$$H_1 : a \mapsto a, \ b \mapsto b, \ c \mapsto d = \bar{c}\bar{b}\bar{a}$$
$$H_2 : a \mapsto a, \ b \mapsto cd\bar{c} = \bar{b}\bar{a}\bar{c}, \ c \mapsto c$$

and induce homeomorphisms $H_1^*, H_2^* : \chi(S^2, 4) \to \chi(S^2, 4)$. Naturality of cup products shows that H_1^*, H_2^* preserve the symplectic form $\widehat{\omega}_{(S^2,4)}$.

Next,

$$H_1^*(m(x, y))$$
$$= \big(a \mapsto m(x, y)(a) = \boldsymbol{j}, b \mapsto m(x, y)(b) = e^{x\boldsymbol{i}}, c \mapsto m(x, y)(\bar{c}\bar{b}\bar{a}) = e^{(y-x)\boldsymbol{i}}\boldsymbol{j}\big)$$
$$= m(x, y - x)$$

and similarly

$$H_2^*(m(x, y)) = \big(a \mapsto \boldsymbol{j}, b \mapsto m(x, y)(\bar{b}\bar{a}\bar{c}) = e^{(x+y)\boldsymbol{i}}\boldsymbol{j}, c \mapsto e^{y\boldsymbol{i}}\big) = m(x + y, y).$$

It follows that the subgroup of $\mathrm{Homeo}(\chi(S^2, 4))$ generated by H_1^* and H_2^* pulls back, via the homeomorphism $\hat{\rho} : \mathbb{P} \to \chi(T)$, to the subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ generated by

$$\hat{\rho}^*(H_1^*) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \hat{\rho}^*(H_2^*) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

These two matrices generate $\mathrm{PSL}(2, \mathbb{Z})$ (see [25]), finishing the proof. $\qquad \square$

Mapping classes of $(S^2, 4)$ permute the four punctures. The subgroup of the mapping class group of $(S^2, 4)$ which fixes the point labeled a in Figure 1 can be shown to act on $\chi(S^2, 4)$; this subgroup is isomorphic to $\mathrm{PSL}(2, \mathbb{Z})$ (see, e.g., [9]), generated by these half twists.

6. Perturbations

The holonomy perturbation process is easily understood, as well as motivated, in the language of Weinstein composition of Lagrangian immersions.

6A. Composition. Given any two (set) maps

$$\alpha : A \to M \quad \text{and} \quad \beta = \beta_M \times \beta_N : B \to M \times N,$$

define the *composition* $(A \times_M B, \beta_N^\alpha)$ by

$$A \times_M B := \{(a, b) \in A \times B \mid \alpha(a) = \beta_M(b)\} = (\alpha \times \beta_M)^{-1}(\Delta_M) \qquad (6\text{-}1)$$

and

$$\beta_N^\alpha : A \times_M B \to N, \quad \beta_N^\alpha(a, b) := \beta_N(b). \qquad (6\text{-}2)$$

6A1. Composition in character varieties. Recall that if $\rho : \Gamma \to G$ is a homomorphism,

$$\mathrm{Stab}(\rho) = \{g \in G \mid g\rho(\gamma)g^{-1} = \rho(\gamma) \text{ for all } \gamma \in \Gamma\}.$$

The proof of the following lemma can be found in [15, Lemma 4.2].

Lemma 6.1. *Suppose that Γ_0, Γ_1, H are groups, $h_0 : H \to \Gamma_0$, $h_1 : H \to \Gamma_1$ homomorphisms. Set $\Gamma = \Gamma_0 *_H \Gamma_1$, the pushout along h_0, h_1. There is a surjection*

$$\chi(\Gamma) \to \chi(\Gamma_0) \times_{\chi(H)} \chi(\Gamma_1)$$

with fiber over $([\rho_0], [\rho_1])$ the double coset

$$\mathrm{Stab}(\rho_0) \backslash \mathrm{Stab}(\rho_0|_H) / \mathrm{Stab}(\rho_1).$$

The fibers $\mathrm{Stab}(\rho_0) \backslash \mathrm{Stab}(\rho_0|_H) / \mathrm{Stab}(\rho_1)$ are called *gluing parameters*.

Proposition 6.2. *If $Z = Z_0 \cup_\Sigma Z_1$ is a decomposition of a compact 3-manifold along a closed separating surface Σ, with $\pi_1(\Sigma) \to \pi_1(Y_0)$ surjective, then $\chi(Z) \to \chi(Z_0) \times_{\chi(\Sigma)} \chi(Z_1)$ is a homeomorphism, in fact, an algebraic isomorphism.*

Proof. Choose $([\rho_0], [\rho_1]) \in \chi(Z_0) \times_{\chi(\Sigma)} \chi(Z_1)$. Since $\pi_1(\Sigma) \to \pi_1(Y_0)$ is surjective, ρ_0 and $\rho_0|_{\pi_1(\Sigma)}$ have the same image, and hence equal stabilizers. Thus $\text{Stab}(\rho_0) \backslash \text{Stab}(\rho_0|_\Sigma) / \text{Stab}(\rho_1)$ is a single point, and the proof follows from Lemma 6.1. $\qquad\square$

Corollary 6.3. *If $Z = Z_0 \cup_\Sigma Z_1$ with Z_0 a handlebody, then*

$$\chi(Z) = \chi(Z_0) \times_{\chi(\Sigma)} \chi(Z_1).$$

6A2. *Composition in the Weinstein category.* The Weinstein category [32] is, roughly speaking, a category with objects symplectic manifolds and morphisms Lagrangian immersions. Composition is not always defined, however. The following criterion ensures that a composition of Lagrangian immersions is defined.

Lemma 6.4 [13, §4.1; 5, Lemma 2.0.5]. *Suppose M, N are symplectic manifolds,*

$$\alpha : A \to M \quad and \quad \beta = \beta_M \times \beta_N : B \to M^- \times N$$

are Lagrangian immersions (with M^- obtained from M by reversing the sign of the symplectic form). If $\alpha \times \beta_M$ is transverse to the diagonal $\Delta_M \in M \times M$, then $A \times_M B$ is a smooth manifold and $\beta_N^\alpha : A \times_M B \to N$ is a Lagrangian immersion.

When the transversality assumption in Lemma 6.4 holds, one says *the composition*

$$(A \times_M B, \beta_N^\alpha)$$

of (A, α) and (B, β) is defined and immersed.

In cases where the transversality assumption in Lemma 6.4 does not hold, a differential topological approach to remedying the situation would be to deform either or both of the immersions α, β. In order to retain the symplectic properties, one would typically deform them by Hamiltonian flows. In the context in this article, where character varieties correspond to flat moduli spaces in the gauge theoretic framework, we seek deformations in Lemma 6.4 that correspond to *holonomy perturbations* in the gauge theory context; we describe these in the next section.

6B. SU(2) *and holonomy perturbations.* We return to the pair (Y, C), determining $S = V \cup F$ as above, so $S = \partial Y$ and $V = nbd(C)$.

Suppose that $e : D^2 \times S^1 \hookrightarrow \text{Int}(Y)$ is an embedding of a solid torus. Denote its image by Y_0, the closure of the complement by Y_1, and the separating torus by T, so that $Y = Y_0 \cup_T Y_1$. Corollary 6.3 shows that

$$\chi(Y, C) = \chi(Y_0) \times_{\chi(T)} \chi(Y_1, C),$$

so that $\chi(Y, C) \to \chi(F, \partial F)$ is exhibited as the composition of

$$\alpha : \chi(Y_0) \to \chi(T) \quad and \quad \beta : \chi(Y_1, C) \to \chi(T) \times \chi(F, \partial F).$$

For a deformation $\alpha_\pi : \chi(Y_0) \to \chi(T)$ of α, or more generally a family of functions α_π, $\pi \in \mathcal{U}$ with $\mathcal{U}$ a manifold (for example, a small open interval), we can view the deformed composition

$$\chi_\pi(Y, C) := \chi(Y_0) \circ_\pi \chi(Y_1, C), \quad \beta^{\alpha_\pi} : \chi(Y_0) \circ_\pi \chi(Y_1, C) \to \chi(F, \partial F) \quad \text{(6-3)}$$

as a *perturbed character variety with perturbation data* π.

Restrict to the case when $G = \mathrm{SU}(2)$ and J is the conjugacy class of imaginary unit quaternions. Then $\chi(Y_0)$ is simply an arc $[e^{is}]$, $0 \le s \le \pi$, and the immersion to $\chi(\partial Y_0)$ is $\alpha : [e^{is}] \mapsto [e^{is}, 1]$ in the pillowcase, described in (5-3) and (5-4). This map descends from the smooth, $\mathbb{Z}_2$ equivariant map $e^{is} \to (e^{is}, 1)$ from S^1 to T.

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, odd, 2π periodic function. Then f determines a $\mathbb{Z}_2$ equivariant Hamiltonian deformation $e^{is} \mapsto (e^{is}, e^{if(s)})$, inducing the deformation of α given by

$$\alpha_f : [0, \pi] \to \chi(S^1 \times S^1), \quad \alpha_f(s) = [e^{is}, e^{if(s)}]. \quad \text{(6-4)}$$

The definition extends easily to the setting of a finite disjoint collection of embeddings of solid tori $e = \{e_i\}_{i=1}^k$ of disjointly embedded solid tori in Y and a corresponding collection of smooth, odd, periodic functions $f_1, \ldots, f_k$ as above. Denote by π this set of perturbation data $\{(e_i, f_i)\}_{i=1}^k$. Letting Y_0 denote the union of the solid tori, (6-3) defines a way to deform the character variety.

Notice that it suffices to think of e as framed link in Y, since isotopic embeddings yield equal perturbed character varieties.

Definition 6.5. Let $\mathcal{V}$ be the vector space of smooth, odd, 2π periodic functions $\mathbb{R} \to \mathbb{R}$. Fix (Y, C) as above, where C may or may not be nonempty). Given perturbation data

$$\pi = (e, f) = \left(\bigsqcup_{i=1}^k e_i : D^2 \times S^1 \subset Y, f = (f_i) \in \mathcal{V}^k \right),$$

define $\chi_\pi(Y, C)$ to be the resulting perturbed (traceless) character variety.

In light of (6-4), $\chi_\pi(Y, C)$ has the following explicit description.

Proposition 6.6. *Let (Y, C) be a compact oriented 3-manifold with a collection of curves C in its boundary. Let $\pi = (e, f)$ be a choice of perturbation data, and define $Y_0 \subset Y$ to be the disjoint union of solid tori $\bigsqcup_i e_i(D^2 \times S^1)$. Finally define $\lambda_i, \mu_i \in \pi_1(Y \setminus Y_0)$ to be the loops $e_i(S^1 \times \{1\})$ and $e_i(\{1\} \times \partial D^2)$, connected to the base point in some way. Then*

$$\chi_\pi(Y, C) = \{\rho \in \chi(Y \setminus Y_0, C) \mid \text{ if } \rho(\lambda_i) = e^{is}, \text{ then } \rho(\mu_i) = e^{if(s)}\}.$$

Theorem 6.7 [16; 27]. *Given a set of perturbation data $\pi = (e, f)$ as in Definition 6.5, there is a holonomy perturbation $h_{(e, f)}$ of the flatness equation on $\mathrm{SU}(2)$ connections for which the perturbed flat moduli space is identified with the perturbed character variety as described above.*

Remark 6.8. More flexible holonomy perturbations can be defined using a solid handlebody, rather than disjoint solid tori (see, for example, [10]). In light of Corollary 6.3, there is a similar composition interpretation of the perturbed character variety, with Y_0 the handlebody. But in this setting, an explicit description of the perturbed character variety and the counterpart to the restriction map in (6-4), are not known to the authors.

We can now prove the following theorem, which is equivalent to Theorem A.

Theorem 6.9. *Suppose $A \in \chi_\pi(Y, C)$ is a regular point. Then A admits a neighborhood U so that $r|_U : U \to \chi(F, \partial F)^*$ is a Lagrangian embedding.*

Proof. Apply Lemma 6.4.   □

6C. *Dependence on perturbations.* In [16], the second author showed that (when $\mathcal{L}$ is empty) different holonomy perturbations in general yield Legendrian cobordant immersions. Unfortunately this usually does not guarantee that they have isomorphic Floer homology. We now outline how one can address this point using Wehrheim and Woodward's quilt theory [28] (a rigorous formulation of the Weinstein category [33]), as well as its extension to the immersed case as developed by Bottman and Wehrheim [5].

One can find a discussion of holonomy perturbations in cylinders $(S, p) \times [0, 1]$, with p a finite set of points in a surface S, in [17]. In particular, Theorem 6.3 of that article states the following. Take perturbation data π with framed perturbation curve obtained by pushing a simple closed curve in $S \setminus p$ into the interior of $S \times I$. Then define the 1-parameter family of holonomy perturbations $s\pi$, $s \in [0, \epsilon)$.

The restriction

$$\chi_{s\pi}((S, p) \times I) \to \chi(S_0, p) \times \chi(S_1, p)$$

can be identified with the family of graphs of a Hamiltonian isotopy of $\chi(S)$ known as the *Goldman twist flow* associated to the simple closed curve [12]. This implies that the holonomy perturbation process can be viewed as a combination of a decomposition induced by cutting a 3-manifold along a separating torus T, followed by a perturbation as in (6-3), with $\alpha_{s\pi} : \chi(S^1 \times D^2) \to \chi(T)$ the composition of the unperturbed inclusion α_0 followed by a small time flow of the Hamiltonian Goldman twist flow associated to a curve in this torus.

We formalize this in the following way. Call two Lagrangian immersions

$$\iota_0 : L_0 \looparrowright M, \quad \iota_1 : L_1 \looparrowright M$$

secretly Hamiltonian isotopic if they can be expressed as compositions with some $\beta : \Lambda \looparrowright X \times M$:

$$i_0 : L_0 \looparrowright M = \beta_M^{j_0} : L_0' \times_X \Lambda \looparrowright M,$$
$$i_1 : L_1 \looparrowright M = \beta_M^{j_1} : L_1' \times_X \Lambda \looparrowright M,$$

in such a way that $j_0, j_1 : L'_0, L'_1 \hookrightarrow X$ are Hamiltonian isotopic in X. It follows from the discussion in Section 6B that different choices of holonomy perturbations induce secretly Hamiltonian isotopic immersed Lagrangians.

Being secretly Hamiltonian isotopic does not necessarily imply being Hamiltonian isotopic. Indeed, L_0 and L_1 need not even be diffeomorphic. Moreover, in the absence of extra hypotheses (embedded composition, monotonicity, exactness, etc.), the conclusion of the Wehrheim–Woodward composition theorem [28] need not hold. In particular, given a third Lagrangian L_2, $HF(L_0, L_2)$ and $HF(L_1, L_2)$ need not be isomorphic, even if they are both well-defined.

However, provided that

- all Lagrangian immersions and symplectic manifolds satisfy suitable assumptions so to be able to define Lagrangian (quilted) Floer homology,

- all Lagrangian immersions come equipped with suitable bounding cochains, in a way consistent with composition,

then it would follow from the Bottman–Wehrheim conjecture [5, §4.4] (see also a similar statement in [11]) that the secretly Hamiltonian isotopic Lagrangian immersions L_0 and L_1, when paired with any test Lagrangian $L_2 \subset M$, produce isomorphic Floer homology groups $HF(L_0, L_2) \simeq HF(L_1, L_2)$ (since these would respectively correspond to the quilted Floer homology groups $HF(L'_0, \Lambda, L_2)$ and $HF(L'_1, \Lambda, L_2)$, which are isomorphic).

In that sense secretly Hamiltonian isotopic Lagrangian immersions L_0 and L_1 can be thought as being equivalent. In particular, the problem of dependence of Floer theory on the choice of holonomy perturbation is seen as a special case of the general problem of dependence of quilted Floer homology on composition.

Acknowledgements

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The cover image is based on an illustration from the article "Khovanov homology and strong inversions", by Artem Kotelskiy, Liam Watson and Claudius Zibrowius (see p. 232).

Gauge Theory and Low-Dimensional Topology: Progress and Interaction

This volume is a proceedings of the 2020 BIRS workshop *Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4*. This was the 6th iteration of a recurring workshop held in Banff. Regrettably, the workshop was not held onsite but was instead an online (Zoom) gathering as a result of the Covid-19 pandemic. However, one benefit of the online format was that the participant list could be expanded beyond the usual strict limit of 42 individuals. It seemed to be also fitting, given the altered circumstances and larger than usual list of participants, to take the opportunity to put together a conference proceedings.

The result is this volume, which features papers showcasing research from participants at the 6th (or earlier) *Interactions* workshops. As the title suggests, the emphasis is on research in gauge theory, contact and symplectic topology, and in low-dimensional topology. The volume contains 16 refereed papers, and it is representative of the many excellent talks and fascinating results presented at the Interactions workshops over the years since its inception in 2007.

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