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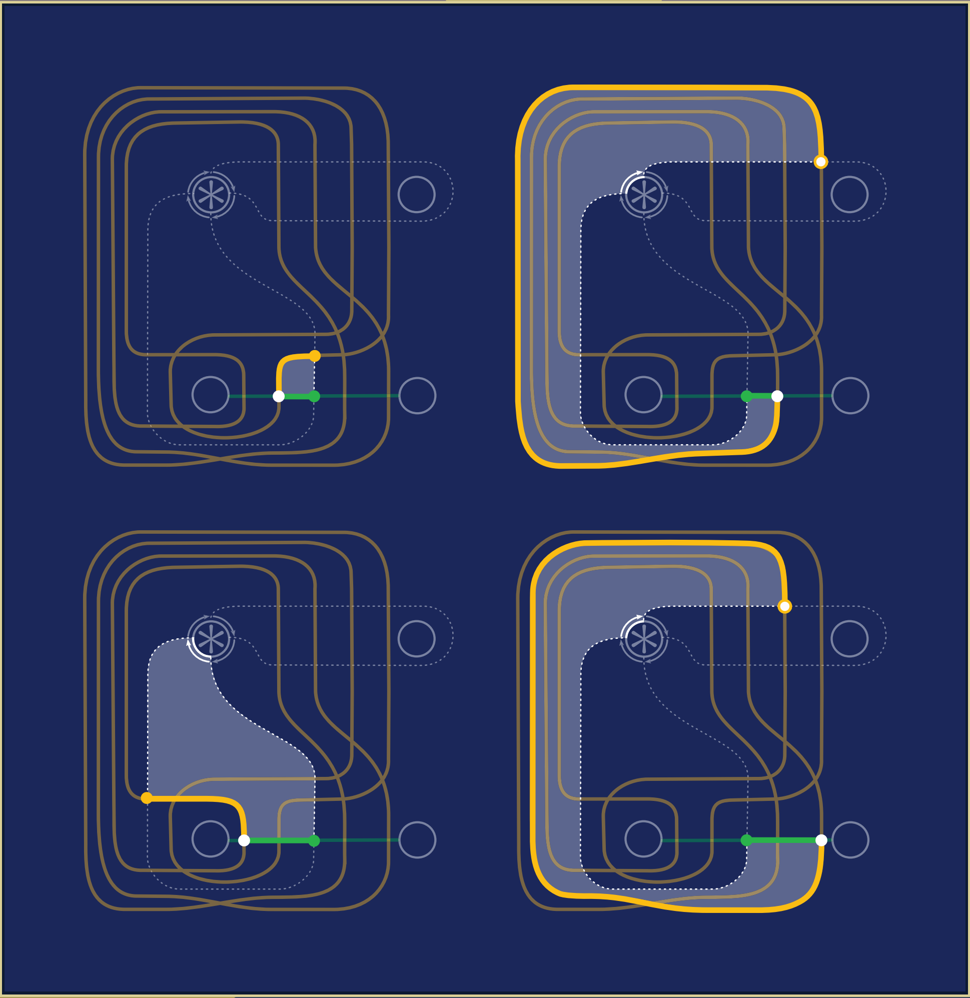
Gauge Theory and Low-Dimensional Topology: Progress and Interaction

Broken Lefschetz fibrations, branched coverings, and braided surfaces

Mark C. Hughes



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We discuss an important class of fibrations on smooth 4-manifolds, called broken Lefschetz fibrations. We outline their connection to symplectic and near-symplectic structures, describe their topology, and discuss several approaches to their construction. We focus on new techniques involving branched coverings and braided surfaces with folds, and provide explicit examples of fibrations constructed using these approaches.

1. Fibrations on 4-manifolds

Fibrations on smooth manifolds have played an important role in the development of low-dimensional topology. These fibrations show up naturally from the viewpoint of algebraic geometry, but have broad generalizations that extend outside of their algebrogeometric origins. They provide very useful topological frameworks to study geometric objects, like contact, symplectic, and Stein manifolds. Furthermore, they can be used to describe 3- and 4-dimensional manifolds in terms of diffeomorphism groups of surfaces, a viewpoint which can be especially fruitful.

In this paper we discuss Lefschetz fibrations and broken Lefschetz fibrations, and survey the main results on these structures. After defining them and describing their connection to symplectic and near-symplectic structures, we will outline several important constructions and provide examples. These examples focus on explicit constructions using branched coverings and braided surfaces. Although these techniques can often be used to construct explicit broken Lefschetz fibrations on 4-manifolds directly from a given handle decomposition, they rely on the construction of a certain branched covering with orientable branch locus and prescribed boundary, which cannot always be achieved.

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This first section contains definitions of the various fibration structures that we will be concerned with on 3- and 4-manifolds, as well as descriptions of their topology.

1A. ***Singular fibrations on 4-manifolds.*** Let X be a smooth, compact, connected, oriented 4-manifold, Σ be a compact surface, and let $f : X \to \Sigma$ be a smooth map. A critical point p of f is called a *positive Lefschetz critical point* if there are orientation-preserving local complex coordinates about p on which $f : \mathbb{C}^2 \to \mathbb{C}$ is modeled as $f(u, v) = u^2 + v^2$. If the coordinates around the critical point are instead orientation-reversing, then it is called a *negative Lefschetz critical point*. We will often omit the adjective *positive*, and refer to a positive Lefschetz critical point simply as a *Lefschetz critical point*.

An embedded circle $C \subset X$ of critical points of f is called an *indefinite fold singularity* if f is modeled near points of C by the map

$$(\theta, x, y, z) \mapsto (\theta, x^2 + y^2 - z^2)$$

from $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}$, where C is given locally by $x = y = z = 0$. Indefinite fold singularities are sometimes referred to as *round* 1*-handle singularities* or *broken singularities* in the literature.

A surjective map $f : X \to \Sigma$ is called a *Lefschetz fibration* if all critical points of f are in the interior of X and are positive Lefschetz critical points. It is called an *achiral Lefschetz fibration* if we also allow negative Lefschetz critical points. Finally, we add the adjective *broken* to either of these names to indicate that we also allow indefinite fold singularities in the set of critical points of f. When discussing these maps we will sometimes use the generic term *fibration* to describe a map which can be any of the types defined above.

1B. ***Boundary behavior of fibrations.*** Let M be a 3-dimensional closed smooth oriented manifold. An *open book decomposition* on M is a smooth map $\lambda : M \to D^2$ such that $\lambda^{-1}(\partial D^2)$ is a compact 3-dimensional submanifold on which λ restricts as a surface bundle over $S^1 = \partial D^2$. Furthermore, we require that the closure of $\lambda^{-1}(\operatorname{int} D^2)$ be the disjoint union of solid tori, on which λ is the projection $D^2 \times S^1 \to D^2$. We say that $\lambda^{-1}(0)$ is the *binding* of the open book on M, and for any $p \in S^1$ the compact surface $\Sigma_p = \lambda^{-1}(\{\alpha p \mid 0 \le \alpha \le 1\})$ is the *page* over p. The surface bundle structure on $\lambda^{-1}(\partial D^2)$ induces a monodromy map on the pages of λ.

By a celebrated theorem of Giroux [19], open book decompositions on a closed 3-manifold M (up to a stabilization operation) are in one-to-one correspondence with contact structures on M (up to isotopy). Thus open book decompositions provide a useful topological setting in which to study contact structures on a given closed 3-manifold.

Now suppose that X is a smooth 4-manifold and Σ is a compact surface, that $\partial X \neq \varnothing$ is connected, and that $f : X \to \Sigma$ is a fibration. Then we say that f is *convex*, if

- $\Sigma = D^2$,
- $f(\partial X) = D^2$, and
- $f|_{\partial X} : \partial X \to D^2$ is an open book decomposition on ∂X.

We say that f is *concave* if there is a disk $D \subset \operatorname{int} \Sigma$ such that

- $f(\partial X) = D$, and
- $f|_{\partial X} : \partial X \to D$ is an open book decomposition on ∂X.

Finally, f is said to be *flat* if

- $f(\partial X) = \partial \Sigma$, and
- $f|_{\partial X} : \partial X \to \partial \Sigma$ is a nonsingular fiber bundle.

The fibers of a flat fibration are all closed surfaces, and the boundary ∂X consists of the fibers above $\partial \Sigma$. The fibers of a convex fibration all have boundary, and ∂X is comprised of the fibers above $\partial \Sigma = \partial D^2$, along with the boundaries of the fibers above $\operatorname{int} D^2$. In contrast, concave fibrations will have both closed fibers and fibers with boundary. Indeed, the fibers above $\operatorname{int} D \subset \Sigma$ will have boundary, while all other fibers will be closed.

Suppose now that $f_1 : X_1 \to \Sigma$ is a concave fibration, $f_2 : X_2 \to D^2$ is a convex fibration, and that there is an orientation-reversing diffeomorphism $\phi : \partial X_1 \to \partial X_2$ which respects the open book decompositions. Then f_1 and f_2 can be glued together, to give a fibration $f : X_1 \cup_\phi X_2 \to \Sigma$. This gives a very useful method for constructing fibrations on closed 4-manifolds. Indeed, one effective strategy is to divide the closed manifold X into simpler pieces X_1 and X_2, on which convex and concave fibrations can be constructed. In general these maps will induce different open book decompositions along their common boundary. If, however, these fibrations can be modified so that they agree along $\partial X_1 = \partial X_2$, then they can be glued to give a fibration on all of X. See [1; 17; 18] for approaches to matching these boundary fibrations which make use of Giroux's theorem and Eliashberg's classification of overtwisted contact structures.

1C. ***Monodromy around Lefschetz critical points.*** The regular fibers of a flat or convex (achiral) Lefschetz fibration $f : X \to \Sigma$ will all be surfaces of the same diffeomorphism type, which we call the *genus of* f. Lefschetz fibrations of genus $g \geq 2$ can be determined entirely by their *monodromy representations*. Let $\Sigma^* \subset \Sigma$ denote the set of regular values of f, and let $p \in \Sigma \setminus \Sigma^*$ be a critical value. If $\gamma \subset \Sigma^*$ is an oriented loop based at $q \in \Sigma^*$ which travels counterclockwise

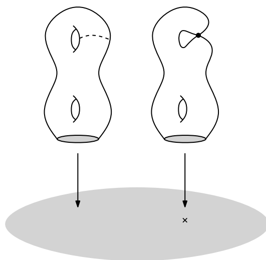

Figure 1. Vanishing cycle of Lefschetz critical point.

around p and no other critical values, then a trivialization of the bundle $f^{-1}(\gamma)$ over γ induces a diffeomorphism of the fiber F_q above q. This diffeomorphism will be a positive (negative) Dehn twist if p corresponds to a positive Lefschetz critical point (respectively, negative Lefschetz critical point). The cycle along which this Dehn twist takes place is called the *vanishing cycle* associated to the critical point. As we approach the critical fiber F_p, the corresponding vanishing cycles in nearby regular fibers shrink down to a single transverse intersection in F_p (see Figure 1 where the vanishing cycle is denoted with a dashed line).

The monodromy of a regular fiber provides a useful way to describe Lefschetz fibrations. More precisely, suppose that $f : X \to \Sigma$ is a Lefschetz fibration that has m critical points and that f is injective on the set of critical points. Suppose further that Σ is either S^2 or D^2, and hence Σ^* is an m-times punctured sphere or disk. Fix a basepoint $q \in \Sigma^*$, and a collection of oriented simple closed curves $\gamma_1, \dots, \gamma_m$ based at q, which are disjoint away from q, where γ_j travels counterclockwise around the j-th puncture of Σ^* and no other punctures. Note that the loops $\gamma_1, \dots, \gamma_m$ then generate $\pi_1(\Sigma^*; q)$, and that we can order them so that the product $\gamma_1 \cdots \gamma_m$ is null-homotopic when $\Sigma = S^2$, and homotopic to $\partial\Sigma$ when $\Sigma = D^2$. Finally, let $F_q = f^{-1}(q)$ be the fiber above $q \in \Sigma^*$, and let $\mathcal{M}(F_q)$ denote the mapping class group of F_q (i.e., the group of orientation-preserving diffeomorphisms of F_q fixing the boundary pointwise, mod isotopy rel boundary).

Then to each loop γ_j we can associate an element $\varphi_j \in \mathcal{M}(F_q)$, which is represented by a positive Dehn twist along the corresponding vanishing cycle. In the case when $\Sigma = S^2$ these elements must additionally satisfy $\varphi_1\varphi_2 \cdots \varphi_m = 1 \in \mathcal{M}(F_q)$, since the product of the loops $\gamma_1, \dots, \gamma_m$ is trivial in $\pi_1(\Sigma^*; q)$. The fibration f then determines a homomorphism $\Lambda : \pi_1(\Sigma^*; q) \to \mathcal{M}(F_q)$, called the *monodromy representation* of f. Note that Λ is only determined by f up to conjugation by a fixed element in $\mathcal{M}(F_q)$ along with changes in the set of the generating loops γ_j. Conversely, given a set of generating loops $\gamma_1, \dots, \gamma_m$ as above, and a collection of positive Dehn twists $\tau_1, \dots, \tau_m \in \mathcal{M}(F_q)$, we can construct a Lefschetz fibration $f : X \to \Sigma$ whose monodromy representation satisfies $\Lambda(\gamma_j) = \tau_j$ for each j (in the case when $\Sigma = S^2$ we must additionally require that $\tau_1 \cdots \tau_m = 1 \in \mathcal{M}(F_q)$).

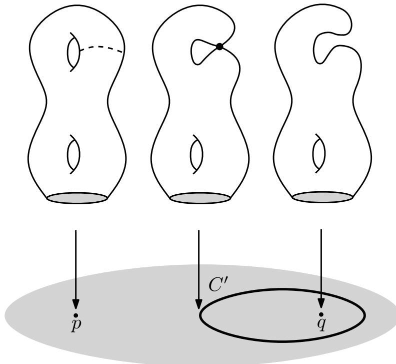

Figure 2. Passing an indefinite fold singularity.

Thus problems involving Lefschetz fibrations can be reformulated and successfully studied in terms of factorizations of mapping class group elements.

This monodromy description of a Lefschetz fibration on X can be adapted further to encode information about embedded surfaces in X. By selecting points $s_1, \ldots, s_k \in F_q$, we can instead consider the group $\mathcal{M}(F_q; \{s_1, \ldots, s_k\})$ of isotopy classes of orientation-preserving diffeomorphisms of F_q which fix the boundary pointwise and preserve $\{s_1, \ldots, s_k\}$ setwise. If we think of our monodromy representation as taking values in $\mathcal{M}(F_q; \{s_1, \ldots, s_k\})$, the trace of the marked points under the monodromy can be completed to an embedded surface in X, called a *multisection* of the fibration. See Baykur and Hayano's work in [7] or [8] for more details.

1D. ***The topology of broken Lefschetz fibrations.*** Suppose now that $f : X \to \Sigma$ is a broken fibration, with indefinite fold singularity along an embedded circle C. Suppose that $C' \subset \Sigma$ is the image of C under f, and that C' is embedded. Let p and q be nearby regular points sitting on opposite sides of C'. Suppose for concreteness that $p = (\theta, -1)$ and $q = (\theta, 1)$ for some $\theta \in S^1$ in the coordinate charts described above. Then the fiber F_q above q can be obtained from F_p by 0-surgery along a pair of points in F_p. Equivalently, F_p can be obtained from F_q by 1-surgery along a simple closed curve (see Figure 2). Indeed, we can think of the coordinate charts describing the indefinite fold singularity as defining an S^1-family of local Morse functions, each with a single index 1 critical point. In particular, for a broken fibration with connected fiber, the genus of the fiber changes by ± 1 each time we cross the image of an indefinite fold singularity in Σ.

Now suppose that $f : X \to D^2$ is a Lefschetz fibration, possibly achiral, possibly broken. Let K be a framed knot in $f^{-1}(\partial D^2) \subset \partial X$, which can be isotoped so that it lies entirely on the interior of a single fiber. Then we can attach a 2-handle along K to yield a new manifold with boundary which we denote X'. If we choose the framing along K so that it is one less than the induced fiber framing, then f will extend to a fibration on X' with a new Lefschetz critical point in the newly added 2-handle. If we instead choose K to have framing one greater than the induced fiber framing, f will instead extend to a fibration on X' with an additional negative Lefschetz critical point (see, e.g., [17]).

Suppose again that $f : X \to D^2$ is a fibration as above, but that we have now chosen two disjoint knots K_1 and K_2 in ∂X, each of which gives a section of f restricted to $f^{-1}(\partial D^2) \subset \partial X$. Then we obtain a new manifold X'' by attaching $S^1 \times D^1 \times D^2$ to ∂X along K_1 and K_2, by identifying $S^1 \times \{-1\} \times D^2$ and $S^1 \times \{1\} \times D^2$ with tubular neighborhoods of K_1 and K_2, respectively. In this case the fibration f will extend to X'', with a single indefinite fold singularity along $S^1 \times \{0\} \times \{0\}$. Indeed, the knots K_1 and K_2 intersect each of the boundary fibers in a pair of points, which specify the locations of the 0-surgeries that take place as we pass the indefinite fold image. We will sometimes refer to this procedure as attaching a *round* 1*-handle* to X, as $S^1 \times D^1 \times D^2$ can be thought of as an S^1-family of 3-dimensional 1-handles $D^1 \times D^2$, which are attached to X fiberwise along the boundary. Alternatively, we can split $S^1 \times D^1 \times D^2$ into a 4-dimensional 1-handle and 2-handle pair, where the 2-handle runs over the 1-handle twice geometrically, but zero times algebraically.

The effect on X of adding a round 1-handle is the same as gluing a fibered cobordism to ∂X, where each fiber over S^1 is the standard Morse theoretic cobordism obtained by adding a 3-dimensional 1-handle to a thickened surface. Broken Lefschetz fibrations and round 1-handle attachments are studied in detail by Baykur in [4], where he also defines generalized n-dimensional round j-handles, for any index j in any dimension n. In what follows we will sometimes find it convenient to refer to 4-dimensional round 2-handles, which are the product of a 3-dimensional 2-handle with S^1 (these are, of course, just upside-down round 1-handles, and will not warrant any further discussion here).

As in the case of Lefschetz critical points, we also obtain monodromy descriptions of the indefinite fold singularities. The monodromy of the fibration outside a new indefinite fold singularity will depend on the framings of the tubular neighborhoods of K_1 and K_2, or alternatively, on the framing k of the 2-handle in the 4-dimensional handle pair description. Indeed, suppose that F is a fiber of the fibration f before attaching the round 1-handle, and that the monodromy around the boundary ∂D^2 is given by a map $\varphi : F \to F$. Then adding the new round 1-handle changes the fibers along the boundary by replacing two disks D_1 and D_2 in F with $S^1 \times [0, 1]$. The new monodromy will be given by the restriction of φ to $F \backslash (D_1 \cup D_2)$, with $-k$ Dehn twists along the cycle $S^1 \times \left\{\frac{1}{2}\right\}$ (i.e., $|k|$ positive Dehn twists if k is negative, and $|k|$ negative Dehn twists if k is positive).

Combining the above monodromy descriptions of indefinite fold singularities with those of Lefschetz critical points gives monodromy representations of broken

Lefschetz fibrations. Returning to the notation from Section 1C, suppose that $f : X \to \Sigma$ has a single indefinite fold singularity along a loop C, that $f(C)$ is embedded in Σ, and that the basepoint q and the images of each Lefschetz critical point are on the side of $f(C)$ with higher genus fibers (or lower Euler characteristic, in the case of disconnected fibers). Suppose that in addition to the loops $\gamma_1, \dots, \gamma_m$ in Σ^*, we have also selected an embedded arc γ in Σ from the basepoint q to the image of the round 1-handle singularity, which is disjoint from the loops γ_i away from q. Then given $\gamma, \gamma_1, \dots, \gamma_m$, the manifold X can be reconstructed from the mapping class group elements $\varphi_1, \dots, \varphi_m \in \mathcal{M}(F_q)$, together with a loop in F_q specifying the location of the 1-surgery that corresponds to crossing the image of the indefinite fold singularity from the high genus side to the low genus side along γ. Such monodromy descriptions of *simplified broken Lefschetz fibrations* are studied in detail by Baykur and Hayano [6].

1E. ***Symplectic and near-symplectic structures.*** Lefschetz fibrations are of great interest in 4-manifold topology, in large part due to theorems of Donaldson [14] and Gompf [20] relating them to symplectic 4-manifolds. A *symplectic form* on a smooth oriented 4-manifold X is a closed, nondegenerate 2-form ω, whose wedge product square $\omega \wedge \omega$ is a volume form inducing the given orientation on X. A *symplectic manifold* is a manifold equipped with a symplectic form.

Donaldson proved that any symplectic 4-manifold admits a *Lefschetz pencil*. That is, there is a finite set of points $B \subset X$ and a smooth map $F : X \backslash B \to \mathbb{CP}^1$ which is a Lefschetz fibration, and around each point of B the map F is locally modeled by the projectivization map $\mathbb{C}^2 \backslash \{0\} \to \mathbb{CP}^1$. Blowing up at the points in B gives an honest Lefschetz fibration; thus Donaldson's result can be restated by saying that any symplectic 4-manifold admits a Lefschetz fibration after blow-ups. Gompf proved the converse to this, by showing that any manifold which admits a Lefschetz pencil also admits a symplectic structure.

A similar relationship exists between broken Lefschetz fibrations and *near-symplectic structures*. Let ω be a smooth closed 2-form with $\omega^2 \geq 0$, and set $Z = \{\omega = 0\}$. Then ω is called a *near-symplectic structure* on X if $\omega^2 > 0$ on the complement of Z, and for each point in Z there is a neighborhood U such that the map $U \to \Lambda^2(T^*U)$ induced by ω has rank 3. This implies that the zero locus Z is a family of embedded circles. Manifolds admitting near-symplectic structures are quite common. Indeed, any closed oriented smooth 4-manifold with $b_2^+(X) > 0$ admits a near-symplectic structure (see [22]).

Analogous to the relationship between Lefschetz pencils and symplectic structures, Auroux, Donaldson, and Katzarkov [2] proved the following: a smooth 4-manifold X admits a near-symplectic structure with zero locus Z if and only if it admits a broken Lefschetz pencil f with indefinite fold singularities along Z, and there is a class $\omega \in H^2(X)$ that evaluates positively on every component of every fiber of f. Here, a broken Lefschetz pencil on X is a finite set of points $B \subset X$, and a smooth map $F : X \setminus B \to \mathbb{CP}^1$ which is a broken Lefschetz fibration, and around each point of B the map F is locally modeled by the projectivization map as above. These structures can be chosen to be compatible, in the sense that if we specify either a near-symplectic structure or broken Lefschetz pencil, then the other object may be chosen so that the regular fibers of the pencil are symplectic away from the singular locus.

Broken Lefschetz fibrations and near-symplectic structures gained a great deal of attention following [2], due in part to constructions of new Floer theoretic invariants and a conjectured relationship to gauge theory and the Seiberg–Witten invariants of 4-manifolds. In [34; 35], Perutz defines and studies the *Lagrangian matching invariant*, which counts pseudoholomorphic multisections of a broken Lefschetz fibration, subject to certain Lagrangian boundary conditions (alternatively, these can be thought of as pseudoholomorphic sections of an associated family of symmetric products of the nonsingular fibers). The Lagrangian matching invariant is a near-symplectic generalization of the Donaldson–Smith invariants defined on symplectic Lefschetz fibrations, which were shown by Usher to be equivalent to the Seiberg–Witten invariants of the underlying 4-manifold for fibrations of high degree [42]. Similarly, the Lagrangian matching invariant can also be compared to the Seiberg–Witten invariants, and Perutz conjectures these invariants are in fact equivalent. The relationship between broken Lefschetz fibrations and Seiberg–Witten invariants is studied further by Baykur in [4], where he discusses vanishing results for Seiberg–Witten invariants under a near-symplectic fiber sum operation and presents numerous examples.

Broken Lefschetz fibrations have also been used to define invariants outside of the Floer and gauge theoretic worlds. In [5], Baykur defines the *broken genera* of an oriented 4-manifold X, which are diffeomorphism invariants constructed using a family of simplified broken Lefschetz fibrations on X. These are defined in terms of the minimal genus of a regular fiber among all simplified broken Lefschetz fibration on X (or an associated blow-up of X), whose fiber realizes a certain homology class in $H_2(X; \mathbb{Z})$. In addition to defining these invariants, Baykur shows that these invariants are able to distinguish infinitely many exotic smooth structures among manifolds of the same homeomorphism type.

1F. ***Existence of fibrations on closed 4-manifolds.*** Besides establishing a relationship between near-symplectic structures and broken Lefschetz fibrations, Auroux, Donaldson, and Katzarkov also constructed a fibration on S^4 with a single indefinite fold singularity, and no other critical points. As S^4 is clearly not near-symplectic, this raised the question of determining which smooth oriented 4-manifolds admit broken Lefschetz fibrations.

This question, and related ones, were answered in stages by several authors. In [17], Etnyre and Fuller proved that after surgery along an embedded circle every smooth closed 4-manifold admits an achiral Lefschetz fibration. Gay and Kirby proved in [18] that every smooth closed 4-manifold admits a broken achiral Lefschetz fibration. Building on the work of Saeki in [41], Baykur used singularity theory to prove that all closed orientable smooth 4-manifolds admit broken Lefschetz fibrations in [3]. Moreover, Akbulut and Karakurt [1], Baykur [30, Appendix B], and Lekili [30] demonstrated that the negative Lefschetz singularities in Gay and Kirby's construction can be eliminated, and hence provided an alternate proof that every closed oriented smooth 4-manifold admits a broken Lefschetz fibration.

The constructions in [1; 17; 18] each involve cutting X up into pieces and constructing the desired fibrations on the pieces separately as described in Section 1B, before regluing. The main differences lie in the modifications that are made to the fibrations to match the boundary open book decompositions. In either approach however, the core argument is the same, relying on machinery from contact topology to ensure that the open book decompositions match along the boundaries before the pieces are reglued. More precisely, the fibrations are first modified to ensure that both boundary open book decompositions support overtwisted contact structures, and then to arrange that both of these contact structures are homotopic. By Eliashberg's classification of overtwisted contact structures [16], the two contact structures must then be isotopic, and hence by Giroux's theorem [19], the boundary open book decompositions will agree after some number of positive stabilizations (which can be realized by further modifications to the fibrations). This process is, of course, nonconstructive due to its reliance on these deep classification results.

Baykur and Lekili's constructions instead focused on studying deformations of generic maps near their singularities. More precisely, they both show that a generic indefinite surjective map $X \to S^2$ can be modified near its critical points to obtain a broken Lefschetz fibration $f : X \to S^2$. These early singularity theory constructions did not (in general) produce broken Lefschetz fibrations with embedded images of their indefinite fold singularities, however. More recent work of Baykur and Saeki [9; 10] improves upon these techniques, by presenting explicit algorithms to convert an arbitrary broken Lefschetz fibration into one with connected fibers and a single indefinite fold singularity with embedded image.

In the case when $b_2^+(X) > 0$, or equivalently when X is near-symplectic, the near-symplectic structure can be used to construct broken Lefschetz fibrations and pencils with additional desired properties. For example, it can be shown that any near-symplectic structure is cohomologous to a near-symplectic form which has connected zero locus, and this can be used to show that in this case X admits a broken Lefschetz pencil with connected fibers and at most one indefinite fold singularity, and that the indefinite fold image is embedded.

Aside from the existence results mentioned above, the uniqueness question for broken Lefschetz fibrations has also been studied. In [43], Williams establishes a set of modifications to broken Lefschetz fibrations which preserve the homotopy class of the fibration map, and proves that they are sufficient to relate any two broken Lefschetz fibrations in the same homotopy class. He also obtains a set of moves relating all broken Lefschetz fibrations on a given 4-manifold (even nonhomotopic ones) by adding in an additional projection move.

Finally, it is worth noting that Lefschetz fibrations have also been extended to certain nonorientable manifolds. In [33], Miller and Ozbagci show that any nonorientable handlebody without 3- and 4-handles admits a Lefschetz fibration over the disk. The fibers of these fibrations are nonorientable surfaces with nonempty boundary.

2. Braided surfaces in $D^2 \times D^2$

In addition to the constructions described in Section 1F, (broken) Lefschetz fibrations can also be obtained by way of branched coverings and braided surfaces. More precisely, fibrations on X can be obtained by constructing and modifying certain coverings $h : X \to D^2 \times D^2$, which are branched along properly embedded surfaces in $D^2 \times D^2$. To obtain a fibration, we will require that these branch loci are braided surfaces with folds in $D^2 \times D^2$. This approach can be carried out directly on a given handle decomposition of the 4-manifold, and yield explicit broken Lefschetz fibrations. In this section we define braided surfaces and a generalization, before outlining this technique and providing explicit examples in later sections.

2A. ***Braided ribbon surfaces.*** Rudolph defined a *braided surface* [37] to be a smooth properly embedded oriented surface $S \subset D^2 \times D^2$ on which the projection to the second factor $\mathrm{pr}_2 : D^2 \times D^2 \to D^2$ restricts as a simple branched covering. Examples of these braided surfaces can be obtained by taking intersections of nonsingular complex plane curves with 4-balls in $\mathbb{C}^2$, and they can be used to study the links that arise as their boundaries in $S^3 = \partial D^4$ (see, e.g., [38; 39; 40]). See Figure 3. The boundary of a braided surface will be a closed braid in the solid torus $D^2 \times S^1 \subset \partial(D^2 \times D^2)$.

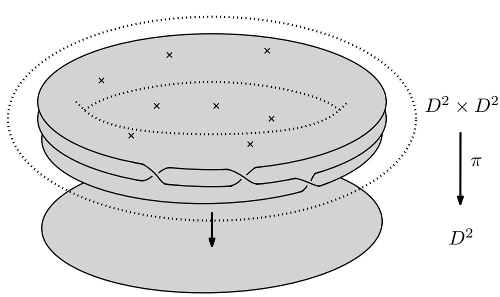

Figure 3. Braided ribbon surface.

Let S be a braided surface. In a neighborhood of any branch point p of the covering $\mathrm{pr}_2|_S$, there are local complex coordinates u and v on D^2 such that S is given by the equation $u^2 = v$ in the coordinates (u, v) on $D^2 \times D^2$. We say that p is a *positive* branch point if these coordinates can be taken to be orientation-preserving, and a *negative* branch point otherwise.

One feature of Rudolph's braided surfaces is that they are all necessarily *ribbon*. A properly embedded surface S in $D^4 = \{(z, w) : |z|^2 + |w|^2 \leq 1\}$ is said to be *ribbon embedded* if the function $|z|^2 + |w|^2$ restricts to S as a Morse function with no local maximal points on $\mathrm{int}\, S$. A properly embedded surface in D^4 is said to be *ribbon* if it is isotopic to a surface which is ribbon embedded. By fixing an identification of $D^2 \times D^2$ with D^4, we can similarly consider ribbon surfaces in $D^2 \times D^2$ (the definition of ribbon embeddings in $D^2 \times D^2$ will depend on our choice of identification, though the resulting class of ribbon surfaces will not).

Rudolph proved that any orientable ribbon surface in $D^2 \times D^2$ is isotopic to a braided surface, though in general this isotopy cannot be chosen to fix ∂S even if ∂S is already a closed braid in $D^2 \times S^1 \subset \partial(D^2 \times D^2)$. Rudolph's braiding algorithm involves manipulating a ribbon *immersed* surface in $\mathbb{R}^3$, and hence can't be applied to nonribbon surfaces in $D^2 \times D^2$.

Viro defined a similar notion which he called a 2-*braid*, by additionally requiring that $\partial S \subset D^2 \times S^1$ be a trivial closed braid (i.e., $\partial S = P \times S^1$ for some finite subset $P \subset D^2$). Viro's 2-braids come equipped with a closure operation yielding closed surfaces in S^4, and in a September 1990 lecture at Osaka City University, Viro proved a 4-dimensional Alexander theorem by showing that every closed oriented surface in S^4 is isotopic to the closure of a 2-braid. These 2-braids were also studied extensively by Kamada [24; 25; 26; 27; 28], who proved a 4-dimensional Markov theorem relating any two 2-braids with isotopic closures.

Braided surfaces admit monodromy representations, similar to the multisections described in Section 1C. Let $S \subset D^2 \times D^2$ be a braided surface, and let D^* be the regular values of the restriction $\mathrm{pr}_2|_S$ (i.e., the complement of the images of the branched points of S under $\mathrm{pr}_2|_S$). Then D^* will be a punctured disk, and after fixing a basepoint $q \in \partial D^*$ we can choose a collection of oriented simple closed curves $\gamma_1, \ldots, \gamma_m$ based at q, which are disjoint away from q, and such that γ_j travels counterclockwise around the j-th puncture of D^* and no other punctures. We can order the loops $\gamma_1, \ldots, \gamma_m$ so that the product $\gamma_1 \cdots \gamma_m$ is homotopic to ∂D^2.

If the restriction $\mathrm{pr}_2|_S$ is an n-sheeted branched covering of D^2, then for each j the set $(\mathrm{pr}_2|_S)^{-1}(\gamma_j)$ will be a closed n-stranded braid in the solid torus $(\mathrm{pr}_2)^{-1}(\gamma_j)$. Each of the $(\mathrm{pr}_2|_S)^{-1}(\gamma_j)$ will be the closure of a braid of the form $\alpha_j = \beta_j^{-1}\sigma_{i_j}^{\pm 1}\beta_j$, where σ_{i_j} is one of the standard Artin generators of the n-strand braid group B_n, and $\beta_j \in B_n$. Equivalently, if D_q denotes the fiber of pr_2 above the point q, and $(\mathrm{pr}_2|_S)^{-1}(q) = \{s_1, \ldots, s_n\} \subset D_q$, then each loop γ_j will induce a monodromy

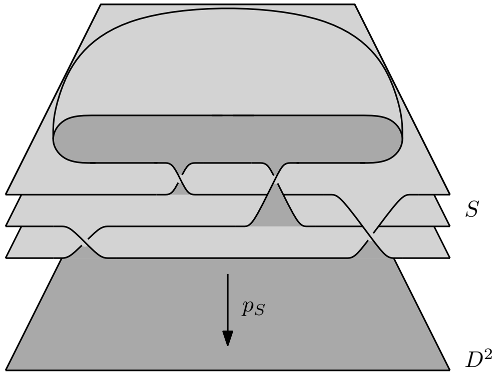

Figure 4. Cross section of a braided surface with folds.

map $\varphi_j \in \mathcal{M}(D_q; \{s_1, \dots, s_n\})$, which swaps precisely two points in $\{s_1, \dots, s_n\}$ along some arc in D_q, and leaves the others points fixed. The family of monodromy maps $\varphi_1, \dots, \varphi_m$ (resp. family of braids $\alpha_1, \dots, \alpha_m$) will be determined by S up to conjugation by a fixed element of $\mathcal{M}(D_q; \{s_1, \dots, s_n\})$ (resp. conjugation by a fixed braid in B_n), as well as changes in the choice of loops $\gamma_1, \dots, \gamma_m$. Conversely, a family of such maps in $\mathcal{M}(D_q; \{s_1, \dots, s_n\})$ or braids in B_n define a braided surface in $D^2 \times D^2$ up to isotopy through braided surfaces.

2B. ***Braided surfaces with folds.*** The surfaces in $D^2 \times D^2$ we use to construct broken Lefschetz fibrations will not in general be ribbon, and hence cannot be braided via Rudolph's algorithm. We thus consider a less restrictive notion of braiding, which we define now.

Let $\phi : F \to \Sigma$ be a smooth map of oriented surfaces. Then a *fold of F with respect to ϕ* is an embedded circle $C \subset F$, so that

(1) ϕ restricts to an embedding on C,

(2) F and Σ both admit coordinate charts of the form $S^1 \times [-1, 1]$ around $C = S^1 \times \{0\}$ and $\phi(C) = S^1 \times \{0\}$, on which ϕ is given by $(\theta, t) \mapsto (\theta, t^2)$,

Now let $S \subset D^2 \times D^2$, and let pr_S denote the restriction of pr_2 to S. We say that S is a *braided surface with folds* if the critical points of pr_S all correspond either to isolated simple branch points or folds of S with respect to pr_S. Moreover, we will often assume that the critical values in D^2 form a set of embedded concentric circles (corresponding to folds), with isolated critical values lying inside the innermost circle. See Figure 4 for a cross sectional diagram of a braided surface with a single fold. We prove the following.

Theorem 2.1. *Let S be a smooth oriented surface properly embedded in $D^2 \times D^2$. Then S is isotopic to a braided surface with folds and only positive branch points. If ∂S is already a closed braid, then the isotopy can be chosen rel ∂S.*

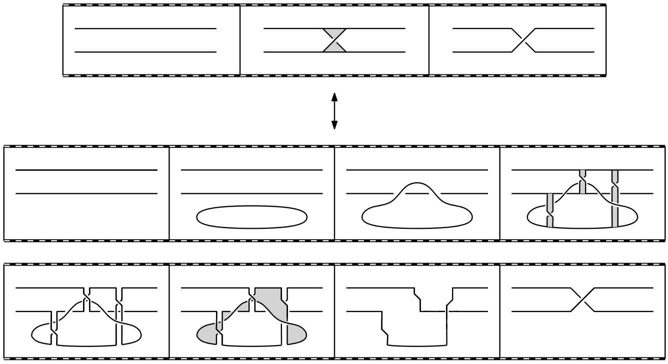

Figure 5. Replacing a negative branch point with three positive branch points and a fold.

Proof. In [23], the author proves that every such surface $S \subset D^2 \times D^2$ is isotopic to a braided surface with caps. Here, a cap is an embedded disk in S on which the projection pr_2 restricts as an embedding, and whose boundary is a fold circle as defined above. Moreover, the isotopy arranging S as a braided surface with caps can be taken rel ∂S is ∂S if already a closed braid. Alternatively, one could start with a bridge trisection of the surface S in the standard genus zero trisection of $D^4 = D^2 \times D^2$, which Meier [32] proved is always possible.

In order to ensure only positive branch points, we replace any negative branch points as shown in Figure 5. More precisely, if $p \in S$ is a negative branch point, then we can choose some parametrized neighborhood V around p so that $S \cap V$ is described locally by the motion picture diagram shown at the top of Figure 5. We can then remove $S \cap V$ from S, and replace it with the surface whose local motion picture description is shown at the bottom of Figure 5. This removes the original negative branch point p, and replaces it with three positive branch points, and a single fold circle. To see that these two surfaces are isotopic rel ∂S, we construct the isotopy shown in Figure 6 using band slides. □

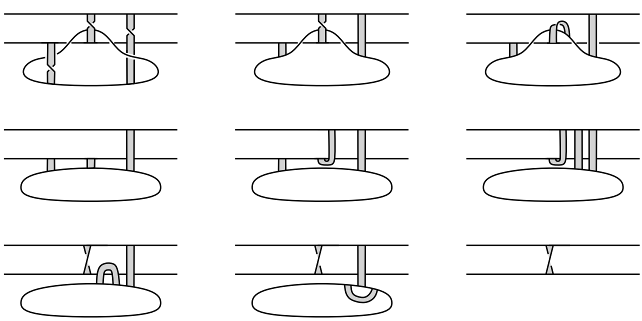

Figure 6. An isotopy which takes the replacement surface to the original surface with a single negative branch point.

3. Broken Lefschetz fibrations from branched coverings

In this section we describe how (broken) Lefschetz fibrations can be constructed via branched coverings and braided surfaces. We then provide examples of this construction in Section 4.

3A. ***Lefschetz fibrations from branched coverings.*** Suppose X is an oriented 4-dimensional 2-handlebody with a fixed handle decomposition that has no 3- or 4-handles. Then we can construct a simple branched covering $H : X \to D^2 \times D^2$ branched along an orientable ribbon surface S. Further, we can assume S is a braided surface by Rudolph's algorithm. Then the composition $X \xrightarrow{H} D^2 \times D^2 \xrightarrow{\mathrm{pr}_2} D^2$ is an achiral Lefschetz fibration, with a positive Lefschetz critical point (resp. negative Lefschetz critical point) for each positive (resp. negative) branch point of $S \to D^2$. Thus if S has only positive branch points, we obtain a true Lefschetz fibration. In fact, Loi and Piergallini [31] show that any sufficiently nice Lefschetz fibration over D^2 necessarily factors in this way.

Using these constructions, Loi and Piergallini also prove that for an oriented connected compact 4-manifold X with boundary, the existence of a Stein structure is equivalent to the existence of a Lefschetz fibration over D^2 with all vanishing cycles nonseparating in the fiber. By considering the associated simple branched covering restricted to ∂X, it follows that a 3-manifold is Stein fillable if and only if it admits a positive open book decomposition.

Now suppose we start instead with a handlebody description of a 4-manifold X which has 3- and 4-handles. As noted above we can construct a branched covering of the 0-,1-, and 2-handles over $D^2 \times D^2$, branched along a ribbon surface. Once we try to extend this covering to the 3- and 4-handles however, the branch locus is no longer ribbon, and may additionally have cusp and node singularities.

3B. ***Broken Lefschetz fibrations from branched coverings.*** Our method for creating broken Lefschetz fibrations on handlebodies with 3- and 4-handles is based on Proposition 3.1, which takes as input a simple branched covering $h : X \to D^2 \times D^2$ with orientable branch locus, and yields a broken Lefschetz fibration $g : X \to D^2$. This approach can then be combined with techniques of Gay and Kirby to produce broken Lefschetz fibrations over S^2 on many closed 4-manifolds.

Proposition 3.1 is a generalization of Proposition 1.2 of [31] to branched coverings with nonribbon branch loci.

Proposition 3.1. *Suppose that X is a smooth 4-manifold with boundary, and that $h : X \to D^2 \times D^2$ is a simple branched covering with branch locus $B_h \subset D^2 \times D^2$ an embedded orientable surface. Then there is an isotopy $\phi_t : D^2 \times D^2 \to D^2 \times D^2$, $\phi_0 = \mathrm{id}_{D^2 \times D^2}$, such that $\mathrm{pr}_2 \circ \phi_1 \circ h : X \to D^2$ is a broken Lefschetz fibration.*

Proof. By Theorem 2.1, B_h is isotopic in $D^2 \times D^2$ to a braided surface with folds and only positive branch points. Let ϕ_t be an isotopy of $D^2 \times D^2$ which takes B_h to such a surface. Let $H = \phi_1 \circ h$ denote the isotoped branched covering, and let B_H denote its branch locus. Away from the preimages of the critical points of $\mathrm{pr}_2|_{B_H}$, the composition $g = \mathrm{pr}_2 \circ H$ is a regular map. By [31] the map g has a Lefschetz critical point for every positive branch point of $\mathrm{pr}_2|_{B_H}$.

To see that the fold lines of B_H give indefinite fold singularities, note that along these fold lines B_H is locally embedded as $\mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$, by $(s, r) \mapsto (0, r, s, r^2)$. Furthermore, near nonsingular points of B_H, H can be written in complex coordinates as $(u, v) \mapsto (u^2, v)$, where B_H is given locally by $u = 0$. Combining these two local models yields a map of the required local form. Furthermore, the folds of B_H can be pushed out so that they lie above a neighborhood of the boundary of D^2, so that their images form a collection of concentric circles in D^2 which enclose the Lefschetz critical values. □

Remark 3.2. Note that Proposition 3.1 holds more generally than stated above. Indeed, by [3; 30] any generic map $X \to D^2$ can be perturbed to become a broken Lefschetz fibration, and by [9; 10] any map $X \to S^2$ can be converted to a broken Lefschetz fibration whose fibers are all connected, and whose indefinite fold singularities are connected with embedded image. The proof of Proposition 3.1 is what will be most useful to us, since the branched covering $h : X \to D^2 \times D^2$ and the isotopy $\phi_t : D^2 \times D^2 \to D^2 \times D^2$ can often be constructed by hand from a given Kirby diagram of X (see Section 4).

3C. ***Broken Lefschetz fibrations on closed 4-manifolds.*** We will now show how Proposition 3.1 can be used in many cases to construct a broken Lefschetz fibration $f : X \to S^2$ on a closed orientable 4-manifold X from a given handle decomposition. Let $F \subset X$ be a closed surface with $F \cdot F = 0$, and consider a tubular neighborhood νF of F. For simplicity, we describe first the construction in the case that $F \cong S^2$, and hence $\nu F \cong S^2 \times D^2$. Such a neighborhood can sometimes be identified in the handle diagram of X as a 2-handle attached along a 0-framed unknot together with the 0-handle of X. If no such $S^2 \times D^2$ can be identified, it can be added to the diagram by adjoining a canceling 2- and 3-handle pair, where the 2-handle is attached along a 0-framed unknot. We will think of the union of this 2-handle with the 0-handle to which it is attached as forming νF.

3D. ***Building the concave piece.*** We describe how to construct a concave broken fibration $f : \nu F \to S^2$ with a single indefinite fold singularity and no Lefschetz critical points. This construction is originally due to Auroux, Donaldson, and Katzarkov [2], as part of their construction of a broken Lefschetz fibration on S^4, though our description follows that in [18].

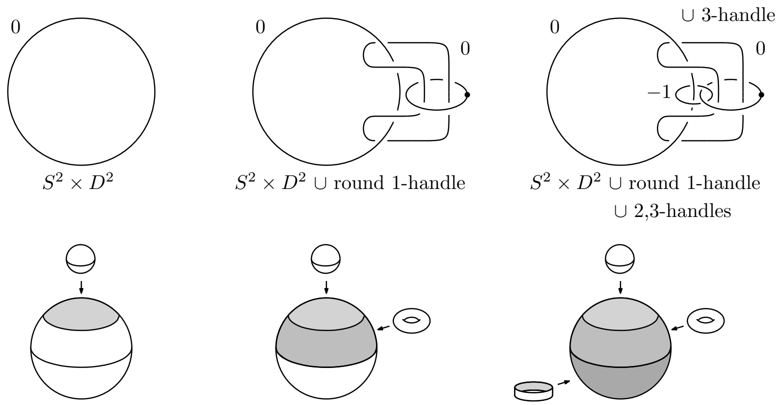

Figure 7. Concave broken fibration on $S^2 \times D^2$.

We begin by identifying the target of the projection $\mathrm{pr}_2 : S^2 \times D^2 \to D^2$ with the northern polar cap in S^2. This defines a fibration of $S^2 \times D^2$ with fiber S^2 over this region (see the bottom-left diagram in Figure 7). Expressing $S^2 \times D^2$ with the usual handlebody diagram (top-left, Figure 7), we can add a 1-handle and 0-framed 2-handle to this diagram, as in the top middle diagram. Taken together, these two handles can be interpreted as a round 1-handle, which is attached to $S^2 \times D^2$ along two sections of the existing fibration restricted to the boundary. We can thus extend this fibration over the round 1-handle, giving a fibration over the northern hemisphere with an indefinite fold singularity over the arctic circle. Fibers between the equator and the arctic circle will be obtained from the polar fibers by 0-surgery, and hence will be tori. Note that the fibration we have constructed so far is flat along its boundary.

Finally, we add an additional 2-handle H_2, and a 3-handle H_3 to our diagram (top-right, Figure 7). The attaching circle of H_2 is a section of the flat fibration restricted to the boundary, and hence the fibration can be extended over H_2, by projecting it to the southern hemisphere (with fiber D^2). In other words, thinking of H_2 as $D^2 \times D^2$ attached along $\partial D^2 \times D^2$, we think of H_2 as sitting as a D^2-bundle over the southern hemisphere, with projection map $D^2 \times D^2 \to D^2$ being given by projection onto the first factor. Note that we choose the attaching circle of H_2 so that it runs over the existing 1-handle from the round 1-handle once, and has framing -1. While these choices are not necessary to ensure the fibration extends, they are made to allow for the handle cancellations described below.

After extending over H_2 the resulting fibration is concave. The page of the boundary open book decomposition is a torus with a single hole (which resulted from attaching the 2-handle H_2), while its binding will be the belt-sphere of H_2.

The attaching sphere of the new 3-handle H_3 is arranged so that it intersects the binding at its north and south poles, and so that it intersects each page in a properly

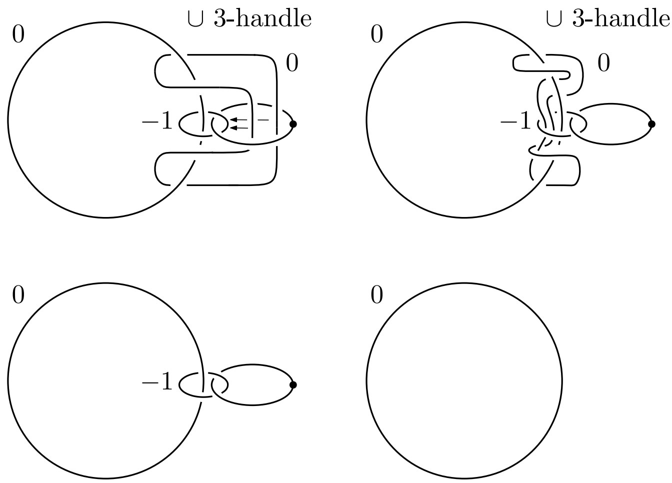

Figure 8. Sliding handles in the concave fibration to identify canceling pairs. After sliding one of the 0-framed 2-handles over the (-1)-framed 2-handle H_2 twice, the slid handle becomes a 0-framed unknot that can be canceled with the 3-handle. The handle H_2 can then be canceled with the 1-handle, leaving behind $S^2 \times D^2$.

embedded arc. The fibration can then be extended across H_3, resulting in no new critical points. This extension changes the D^2 fibers over the southern hemisphere by adding a 2-dimensional 1-handle, yielding annular fibers. On the other hand, the pages of the boundary open book change by the *removal* of a neighborhood of a properly embedded arc from the puncture torus pages (the intersection of the original page with the attaching sphere of H_3), yielding annular pages.

This gives a concave broken fibration as depicted in the bottom-right diagram of Figure 7, with a single indefinite fold singularity, and no positive or negative Lefschetz critical points. Moreover, after sliding the 0-framed 2-handle from the round 1-handle over H_2 twice as shown in Figure 8, we find that the added 1-,2-, and 3-handles all form canceling pairs. Hence the total space of our fibration is diffeomorphic to $S^2 \times D^2 \cong \nu F$. Notice that the induced open book decomposition on $\partial(\nu F)$ will have disconnected binding, which may cause problems when we try to construct a matching convex fibration on $X \setminus \nu F$. We thus instead think of the lone canceling 3-handle as being attached as a 1-handle to $X \setminus \nu F$, and construct a concave fibration f_1 on $X_1 = \nu F \setminus \{3\text{-handle}\}$, whose boundary open book decomposition has punctured torus page and connected binding (see Figure 9).

If instead F has genus $g \geq 1$ we can proceed much as before, either identifying a neighborhood νF in the handle diagram of X, or by adding a standard diagram of $F \times D^2$ with additional 2- and 3-handles to cancel the 1- and 2-handles of νF. More

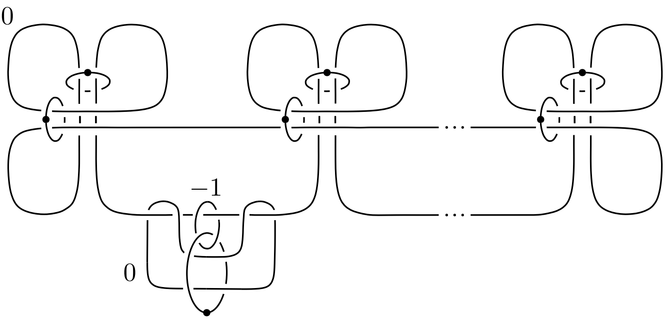

Figure 9. Neighborhood of $F \subset X$ with an extra 2-handle and round 1-handle added.

precisely, ignoring the (-1)-framed 2-handle in Figure 9, and the 1- and 2-handles coming from the round 1-handle, the remaining handles (together with the 0-handle) give a diagram for $F \times D^2$. This diagram can be placed in any handlebody diagram of a 4-manifold X without changing the diffeomorphism type of X, provided we also add a 0-framed 2-handle attached along a meridian for each 1-handle circle, along with a single extra 3-handle. To this diagram we could then add a round 1-handle and pair of 2- and 3-handles (see Figure 9) and continue as above.

3E. ***Building the convex piece.*** Let $Y = X \backslash X_1$. We now discuss how to use a handle structure on Y to build a convex fibration $g : Y \to D^2$, so that it extends the open book decomposition $\lambda : \partial Y \to D^2$ induced by the concave fibration $f_1 : \nu F \to S^2$. We attempt to do this in three steps:

(1) Express the open book decomposition λ as $\lambda = \mathrm{pr}_2 \circ h$, where $h : \partial Y \to \partial(D^2 \times D^2)$ is a simple covering branched along a closed braid in $\partial(D^2 \times D^2)$, and $\mathrm{pr}_2 : D^2 \times D^2 \to D^2$ is the projection.

(2) Extend the branched covering h to a covering $H : Y \to D^2 \times D^2$ branched along an orientable surface.

(3) Use Proposition 3.1 to obtain the desired broken Lefschetz fibration.

Part (1) is always possible. Indeed, let P be the page of λ, with monodromy $\tau : P \to P$. Then by choosing a suitable (degree ≥ 3 and simple) branched covering $\alpha : P \to D^2$, the map τ is the lift of a map $\widehat{\tau} : D^2 \to D^2$ which fixes the branch locus of α setwise [21; 29]. If K is the binding of λ, then this allows us to write $\partial Y \backslash \nu K$ as a branched covering of the solid torus $D^2 \times \partial D^2$ branched over a closed braid. A matching (unbranched) covering $\nu K \to \partial D^2 \times D^2$ can be glued to this covering to give the desired map $h : \partial Y \to \partial(D^2 \times D^2)$.

Problems may arise when we try to carry out part (2) of the above process, however. The covering h can always be extended to a branched covering $H : Y \to D^2 \times D^2$, though the branch locus may not be an orientable embedded surface.

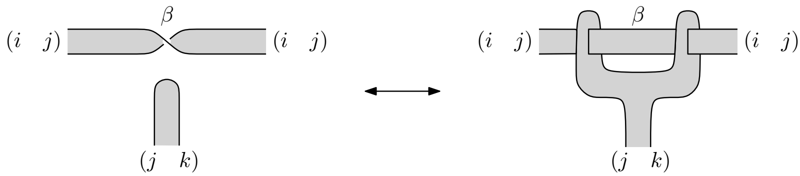

Figure 10. Fixing a nonorientable band in the branch locus of $\widehat{h}$.

To see how this covering is constructed, fix some choice of relative handle decomposition for the pair $(Y, \partial Y)$. The covering $h : \partial Y \to \partial(D^2 \times D^2)$ can be extended to a covering

$$\widehat{h} : \partial Y \times [0, 1] \to \partial(D^2 \times D^2) \times [0, 1]$$

in the usual way. Here, $\partial Y \times [0, 1]$ and $\partial(D^2 \times D^2) \times [0, 1]$ are thought of as collar neighborhoods of ∂Y and $\partial(D^2 \times D^2)$ respectively. Identify ∂Y with $\partial Y \times \{0\}$, and let $\partial_+ Y = \partial Y \times \{1\}$. We now attempt to extend this covering over the handles of Y to construct the desired covering H.

Let $\sigma_1 = D^1 \times D^3$ be a 1-handle, and let $\tau_1 : \sigma_1 \to \sigma_1$ be the involution defined by

$$\tau_1 : (t, x, y, z) \longmapsto (-t, -x, y, z).$$

If σ_1 is a 1-handle in our handle decomposition of $(Y, \partial Y)$, then we can isotope its attaching map $\alpha_1 : S^0 \times D^3 \to \partial_+ Y$ so that it is symmetric with respect to $\widehat{h}$, i.e., so that $\widehat{h} \circ \alpha_1 \circ \tau = \widehat{h} \circ \alpha_1$. Once this is done, by Lemma 6.1 of [11] we can extend $\widehat{h}$ over the 1-handle σ_1, using the quotient induced by τ_1. The result is a branched covering of $(\partial Y \times [0, 1]) \cup \sigma_1$ over $\partial(D^2 \times D^2) \times [0, 1]$, where the new branch locus is obtained by adding a disjoint disk to the branch locus of $\widehat{h}$ in $\partial(D^2 \times D^2) \times [0, 1]$.

Similarly, if $\sigma_2 = D^2 \times D^2$ is instead a 2-handle attached to $\partial_+ Y$ by some attaching map $\alpha_2 : S^1 \times D^2 \to \partial_+ Y$, by [15] we can isotope α_2 so that it becomes symmetric with respect to the involution $\tau_2 : D^2 \times D^2 \to D^2 \times D^2$, defined by

$$\tau_2 : (t, s, x, y) \longmapsto (-t, s, -x, y).$$

Here, the attaching circle of σ_2 will intersect the branching set of $\widehat{h}$ in two points, say p_1 and p_2. Then we can extend the covering $\widehat{h}$ to a branched covering of $(Y \times [0, 1]) \cup \sigma_2$ over $\partial(D^2 \times D^2)$, where the new branch locus is obtained by attaching a single band to the branch locus of $\widehat{h}$ at the points $\widehat{h}(p_1)$ and $\widehat{h}(p_2)$. This band will have n half-twists in it, where n is the framing of σ_2.

When extending $\widehat{h}$ over a 2-handle σ_2, it is possible that the corresponding band β may be attached to the branch locus $B \subset \partial(D^2 \times D^2) \times [0, 1]$ in a nonorientable way. By [12] this can be remedied, by adding (or removing) a half-twist in β as in Figure 10. In this local picture we have pushed B entirely into the 3-dimensional space $\partial(D^2 \times D^2) \times \{1\}$, where it can be depicted as an immersed surface with only ribbon double points. The labels on the components denote the associated monodromy action on the sheets of $\widehat{h}$.

Let Y_2 denote the union of $\partial Y \times [0, 1]$ with the 1- and 2-handles. We can thus extend the branched covering $h : \partial Y \to \partial(D^2 \times D^2)$ to a covering

$$\tilde{h} : Y_2 \to \partial(D^2 \times D^2) \times [0, 1],$$

where the associated branch locus $\widetilde{B} \subset \partial(D^2 \times D^2) \times [0, 1]$ is an embedded orientable surface. If the intersection

$$\widetilde{B}_1 = \widetilde{B} \cap \partial(D^2 \times D^2) \times \{1\}$$

is an unlink, then $\tilde{h}$ can be extended across the 3- and 4-handles to give a branched covering $H : Y \to D^2 \times D^2$ with orientable embedded branch locus. This can be seen by noting that the union of the 3- and 4-handles is a thickened bouquet of circles, which can be expressed as a branched covering of D^4 with branch locus a collection of properly embedded disjoint disks. If $\widetilde{B}_1$ is an unlink, this covering can be glued to $\tilde{h} : Y_2 \to \partial(D^2 \times D^2) \times [0, 1]$ to obtain the desired covering H.

In general however, $\widetilde{B}_1$ will not be an unlink. By [36] we can modify the covering by adding cusp and node singularities on the interior of $\widetilde{B}$ so that $\widetilde{B}_1$ becomes an unlink, though doing so may fail to preserve the required orientability of the branch locus B. When this can be avoided, we can proceed with the rest of the construction to obtain a broken Lefschetz fibration of X over S^2.

3F. ***Broken Lefschetz fibrations on doubles of 4-manifolds.*** We now discuss a situation in which the above construction will always be possible. Let U be a handlebody with single 0-handle and no 4-handles. The *double* of U is the manifold $X = U \cup_{\mathrm{Id}_{\partial U}} \overline{U}$, where $\overline{U}$ denotes the handlebody U with reversed orientation. The handle structure on U induces a handle structure on X in a natural way, by turning the j-handles of $\overline{U}$ upside-down and attaching them as $(4-j)$-handles to U.

Theorem 3.3. *Let X be a smooth, closed, orientable 4-manifold, with handle structure coming from the double of a handlebody U. Then the procedure described in Section 3E will produce a broken Lefschetz fibration $f : X \to S^2$.*

Proof. If $F = S^2$ is a trivially embedded sphere in the 0-handle of X, we can construct a concave fibration of νF over S^2 as in Section 3D. Let $Y = X \setminus \nu F$, and let $\lambda : \partial Y \to D^2$ be the induced open book decomposition. By [18] the monodromy of λ is trivial, and hence it factors through a simple branched covering $h : \partial Y \to \partial(D^2 \times D^2)$ of degree ≥ 3, whose branch locus is a trivial closed braid in $D^2 \times \partial D^2$.

We now proceed to extend the covering h to a covering

$$\tilde{h} : Y_2 \to \partial(D^2 \times D^2) \times [0, 1]$$

with branch locus $\widetilde{B}$. Again we let $\widetilde{B}_1$ be the intersection of $\widetilde{B}$ with $\partial(D^2 \times D^2) \times \{1\}$.

Each 1-handle we extend over contributes an unknot component to $\widetilde{B}_1$ which is unlinked from the other components.

Before extending h across the 2-handles, note that in the induced handle structure on Y, the 2-handles occur pairs. Every 2-handle σ from U is paired with a 2-handle σ' from $\overline{U}$, where σ' is attached along a 0-framed meridian of the attaching circle of σ (see [20]). We can also think of σ' as being attached along the belt sphere of σ.

Extending h over a 2-handle from U changes $\widetilde{B}_1$ by oriented surgery along a band β. On the other hand, since the belt sphere of σ is symmetric with respect to the involution $\tau_2 : \sigma \to \sigma$, extending h across σ' will change $\widetilde{B}_1$ by oriented surgery along a band β' which cancels β. Hence the net effect of extending h across σ and σ' does not change $\widetilde{B}_1$, which thus remains an unlink. □

Any orientable S^2-bundle over a (possibly nonorientable) surface Σ is the double of a D^2-bundle over Σ. See [18] for an alternate construction of broken Lefschetz fibrations on doubles of 2-handlebodies.

3G. ***Connected sums.*** The procedure outlined in Section 3E respects connected sums in the following sense:

Proposition 3.4. *Suppose that X_1 and X_2 are two handlebodies for which the procedure in Section 3E yields broken Lefschetz fibrations $f_1 : X_1 \to S^2$ and $f_2 : X_2 \to S^2$. Then the same procedure can be used to obtain a broken Lefschetz fibration $f : X_1 \# X_2 \to S^2$ which restricts to a concave fibration on $X_1 \backslash D^4 \subset X_1 \# X_2$ and to a convex fibration on $X_2 \backslash D^4 \subset X_1 \# X_2$. Moreover, the ball $D^4 \subset X_1$ can be chosen so that $f|_{X_1 \backslash D^4} = f_1|_{X_1 \backslash D^4}$.*

Proof. The handle structures on X_1 and X_2 yield a handle decomposition of $X_1 \# X_2$ by starting with the 0-handle of X_1 and attaching all 1-,2- and 3-handles of X_1, followed by the 1-, 2-, 3- and 4-handles of X_2.

Cut out a neighborhood of an S^2 from X_1, and construct the concave fibration on νS^2 and the branched covering h as above. The map h can be extended across the 1,2 and 3-handles of X_1 to give a covering

$$h' : X_1 \backslash (\nu S^2 \cup \text{4-handle}) \to \partial(D^2 \times D^2) \times [0, 1].$$

We identify $\partial(D^2 \times D^2) \times [0, 1]$ with $(D^2 \times D^2) \backslash (D' \times D')$, where $D' \subset D^2$ is a small disk containing the origin. Then by Theorem 2.1 the branch locus B' of h' can be braided rel $\partial B'$ so that it is a braided surface with folds in $(D^2 \times D^2) \backslash (D' \times D')$ and only positive branch points. Gluing the map $\mathrm{pr}_2 \circ h'$ to the concave fibration on νS^2 gives a concave fibration $X_1 \backslash$4-handle $\to S^2$. This fibration can either be continued across the 4-handle of X_1 to obtain the fibration $f_1 : X_1 \to S^2$, or across the 1-, 2-, 3- and 4-handles of X_2 to give a fibration $f : X_1 \# X_2 \to S^2$. □

Broken Lefschetz fibrations on connected sums were also built by Baykur in [4] (based on an observation by Perutz [35]), where he also defines a generalization of the symplectic fiber sum operation on near-symplectic broken Lefschetz fibrations.

4. Examples

In this section we compute a few simple examples, to illustrate how the above procedure is carried out.

4A. ***Broken Lefschetz fibration on S^4.*** Consider the diagram of S^4 in Figure 11. As in Figure 7, the union of all 0-, 1-, and 2-handles in this decomposition gives a neighborhood of an unknotted $S^2 \subset S^4$, together with an additional round 1-handle and (ordinary) 2-handle attached. Call the union of these handles X_1, and set $X_2 = S^4 \backslash X_1$. The open book decomposition on $\partial X_1 = \partial X_2$ induced by the concave fibration $f_1 : X_1 \to S^2$ from the above proof will have a punctured torus page with trivial monodromy (see [18]). Hence it can be represented by a 3-fold simple branched covering $h : \partial X_2 \to \partial(D^2 \times D^2)$, and whose branch locus in $\partial(D^2 \times D^2)$ is the closure of the trivial 4-strand braid in $D^2 \times \partial D^2$ (h can be described on each page by the branched covering in Figure 12).

The branched covering h extends to a covering $H : X_2 \to D^4$, which is built by turning the handle decomposition from Figure 11 upside-down, and viewing X_2 as a 0-handle with two 1-handles attached. The 0-handle can be expressed as a 3-fold covering of D^4 branched over two properly embedded unknotted disks. For each 1-handle we extend this covering over, a properly embedded unknotted disk is added to the branch locus. Hence the branch locus B_H of H in $D^4 \cong D^2 \times D^2$ is isotopic to the braided surface $\{p_1, \dots, p_4\} \times D^2$, for some collection of disjoint points $\{p_1, \dots, p_4\} \subset D^2$. The only critical points in the resulting broken Lefschetz fibration $f : S^4 \to S^2$ will thus lie along the indefinite fold singularity in X_1, and we recover Auroux, Donaldson, and Katzarkov's example in [2].

4B. ***S^2-bundles over orientable surfaces.*** Let X be an S^2-bundle over an closed orientable surface of genus g. For simplicity, we consider first the case when $g = 1$.

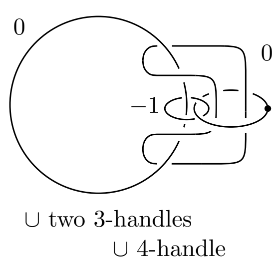

Figure 11. Handlebody structure of a neighborhood of S^2 in S^4.

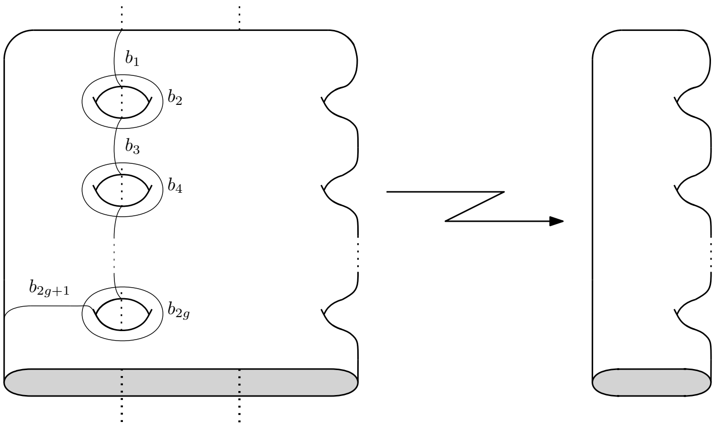

Figure 12. 3-fold branched cover $\Sigma \to D^2$.

Consider the diagram of X in Figure 13, where each 1-handle attaching sphere is paired with the sphere directly across from it. Notice that the diffeomorphism type of X depends only on the parity of n. Assume first that $n = 0$. In this case $X \cong S^2 \times T^2$. While there is an obvious fibration $S^2 \times T^2 \to S^2$, the construction below has the advantage that it can be iterated to construct broken Lefschetz fibrations on connected-sums of S^2-bundles, and generalizes to the twisted bundle $S^2 \tilde{\times} T^2$. Note that broken Lefschetz fibrations on S^2 bundles over T^2 can also be obtained by converting trisection examples given in [10] or [13] to broken Lefschetz fibrations.

We begin by adding a copy of the diagram in Figure 11 (minus the 4-handle) to the diagram of X, which does not change the diffeomorphism type of X. Again, let X_1 denote the union of the 0-handle with the newly added 1-handle and 2-handles, and let $X_2 = X \backslash X_1$. As above, X_1 admits a concave fibration over S^2 and induces an open book decomposition on ∂X_2 with punctured torus page and trivial monodromy. The associated 3-fold branched covering

$$h : \partial X_2 \to \partial(D^2 \times D^2)$$

has branch locus a trivial 4-strand closed braid in $D^2 \times \partial D^2$. We need to extend h

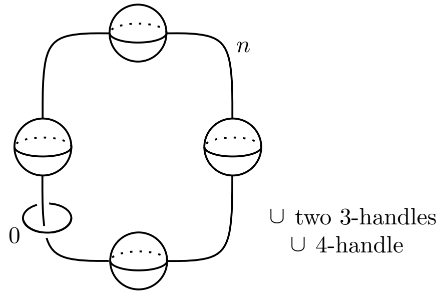

Figure 13. S^2-bundle over torus.

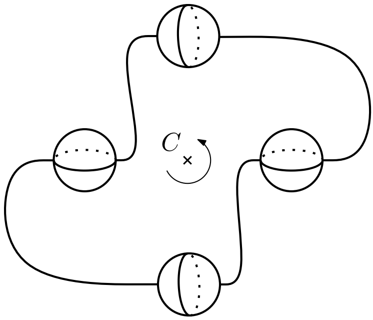

Figure 14. Symmetrizing the handles.

over the handles in Figure 13, as well as the additional 3-handles we introduced when adding the diagram in Figure 11.

In ∂X_2 there are four circles of branch points, corresponding to the four components of the branch locus in $\partial(D^2 \times D^2)$. We can isotope the handle attaching maps so that one of these four circles C skewers the diagram in Figure 13, so that locally the covering looks like rotation of π about the center of the diagram. We first focus on extending the covering over the 1-handles σ_1 and σ_1', and over the 2-handle σ_2 coming from the handle structure on T^2.

Isotope the attaching maps of these handles so that they are symmetric with respect to rotation by π around C, as in Figure 14. We can thus extend the covering $\tilde{h}$ over σ_1, σ_1', and σ_2. Extending over the 1-handles adds a pair of disks to the branch locus, while extending over the 2-handle adds a band. Notice that when the attaching circle of σ_2 runs along the horizontal 1-handle σ_1, it will intersect the branch set in precisely two points. The branch $\widetilde{B}$ locus of

$$\tilde{h} : (\partial X_2 \times [0,1]) \cup \sigma_1 \cup \sigma_1' \cup \sigma_2 \to \partial(D^2 \times D^2) \times [0,1]$$

will be as in Figure 15.

More precisely, let $\widetilde{B}_t = \widetilde{B} \cap (\partial(D^2 \times D^2) \times \{t\})$ for $t \in [0,1]$. The leftmost frame represents $\widetilde{B}_0$, the branch locus of h, where we have suppressed all of the components except for $h(C)$. As t increases, we see two unknotted components appear, corresponding to the 1-handles σ_1 and σ_1', followed by a band surgery corresponding to the 2-handle σ_2. Extending $\tilde{h}$ across the remaining 2-handle in Figure 13 results in an additional band surgery which cancels the first. Note that all of the components in Figure 15 will have the same monodromy as $h(C)$.

The branch locus $\widetilde{B}_1 = \widetilde{B} \cap (\partial(D^2 \times D^2) \times \{1\})$ is thus a six component unlink (three components from Figure 15 and three additional components from $h : \partial X_2 \to \partial(D^2 \times D^2)$ which were suppressed from the diagrams). It only remains to extend this covering over the four 3-handles and unique 4-handle of X_2. It is not hard to see that the union of these higher index handles admits a 3-fold simple

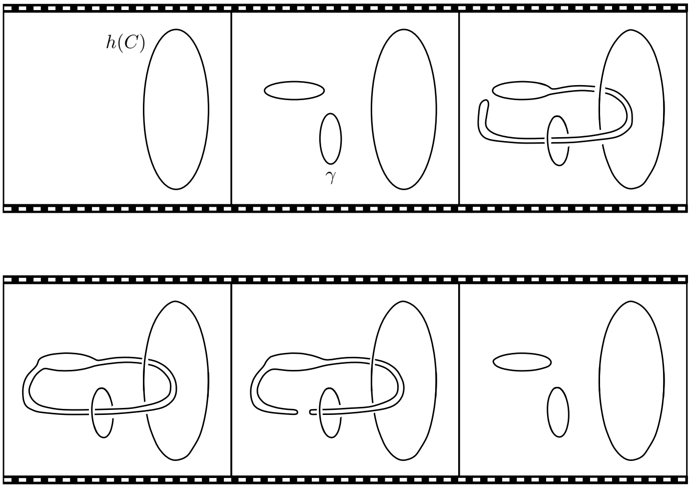

Figure 15. Branch locus $\widetilde{B}$ after extending over 1-handles and σ_2.

branched covering over D^4, with branch locus consisting of six disjoint properly embedded disks in D^4. This covering can thus be glued to $\tilde{h}$ to give a covering $H : X_2 \to D^2 \times D^2$, where these six disks cap off the six component unlink $\widetilde{B}_1$. Let B_H denote the branch locus of H, which consists of $\widetilde{B}$ capped off with these six disks.

Finally, in order to apply Proposition 3.1, we must arrange B_H as a braided surface with folds. By [23] this is equivalent to arranging $B_H \subset D^2 \times D^2$ so that it sits in a collar neighborhood $\partial(D^2 \times D^2) \times [0, 1]$ such that

(1) the restriction to B_H of the projection $\rho : \partial(D^2 \times D^2) \times [0, 1] \to [0, 1]$ is a Morse function, and

(2) $(\rho|_{B_H})^{-1}(t)$ a closed braid in $\partial(D^2 \times D^2) \times \{t\}$ for all regular values t.

Figure 16 shows how this can be done. Again we start with the component $h(C)$ (hiding the three other components), and introduce two new unknots corresponding to extending the branched covering over the 1-handles. The key difference now is that at every regular level the branch locus must be a closed braid. Hence, the band corresponding to σ_2 now shows up first as a maximal point, which is then completed by adding two half-twisted bands via saddle points in the seventh frame. The second band surgery takes place in the ninth frame. Finally the branch locus is simplified to the trivial 3-strand braid, which is capped of by three minimal points (the other unseen three unknot components are similarly capped off).

The resulting broken achiral Lefschetz fibration $\mathrm{pr}_2 \circ H : X_2 \to D^2$ has an indefinite fold singularity for each maximal point of B_H (which shows up along the boundary of the maximal disk), and a positive or negative Lefschetz critical point for each saddle point. Hence $\mathrm{pr}_2 \circ H$ has three indefinite fold singularities, two

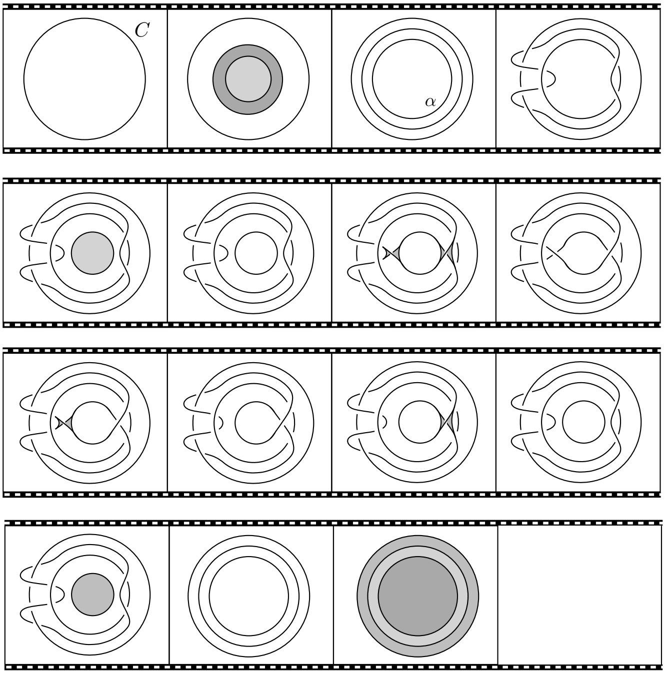

Figure 16. Branch locus B_H as a braided surface with folds.

positive Lefschetz critical points, and two negative Lefschetz critical points. The negative Lefschetz critical points can be replaced by the isotopy in Theorem 2.1, and the monodromy information of the fibration can be read off of Figure 16.

Now suppose that X is the S^2-bundle over T^2 given by Figure 13 with $n = 1$, i.e., $X \cong S^2 \tilde{\times} T^2$. Then the branch locus $\widetilde{B}$ will be as in Figure 15, except that the band corresponding to σ_2 will have a single half-twist, and hence $\widetilde{B}$ will be nonorientable. This can be remedied by involving another component of the branch locus $h : \partial X_2 \to \partial(D^2 \times D^2)$, and performing a move as in Figure 10 (see Figure 17, where the monodromy information must be chosen to agree with the labels in Figure 10).

When braided, this move introduces a new local maximal point, and two new saddle points (one of each sign). Hence the resulting broken achiral Lefschetz fibration has an additional indefinite fold singularity, positive Lefschetz critical point, and negative Lefschetz critical point when compared to the fibration constructed on $S^2 \times T^2$.

If X is a S^2-bundle over a higher surface of genus $g > 1$, we can start instead with the diagram in Figure 18. The associated branch locus will be as in Figure 16, except that the innermost strand α will be replaced by $2g - 1$ parallel strands, and hence the fibration $\mathrm{pr}_2 \circ H : X_2 \to D^2$ will now have $2g + 1$ indefinite fold singularities. The number of saddle points will be the same as in Figure 16, and

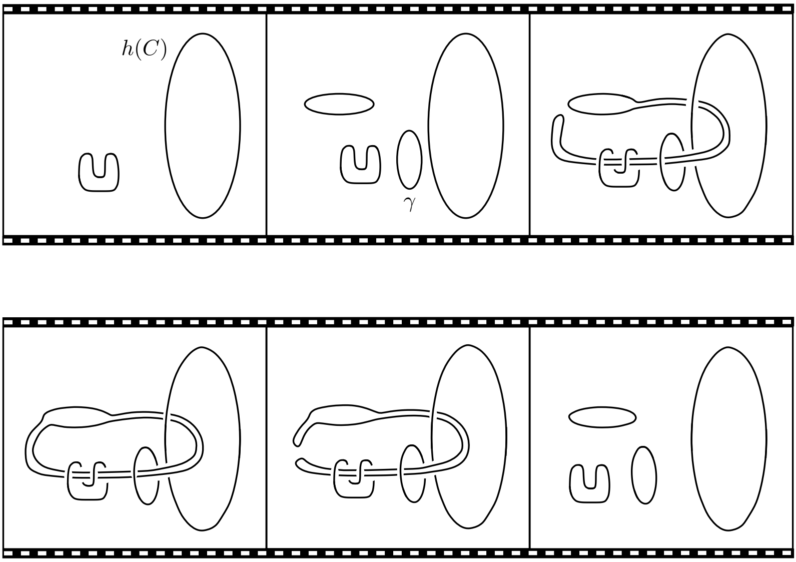

Figure 17. Branch locus $\widetilde{B}$ for $S^2 \tilde{\times} T^2$.

hence the resulting fibration will have two positive Lefschetz critical points and two negative Lefschetz critical points (where the negative critical points can be replaced as described above).

4C. *S^2-bundles over* $\mathbb{RP}^2$. We now consider S^2-bundles over $\mathbb{RP}^2$, which can be described by the diagram in Figure 19. Proceeding as above, we can arrange the component C of the branch set so that it sits vertically in the diagram between the two strands of the attaching circle of the n-framed 2-handle σ_2, and so that the attaching maps of σ_2 and the 1-handle σ_1 are symmetric with respect to rotation about C. For $n = 0$ and $n = 1$ the branch locus $\widetilde{B}$ will be as in Figures 15 and 17, respectively,

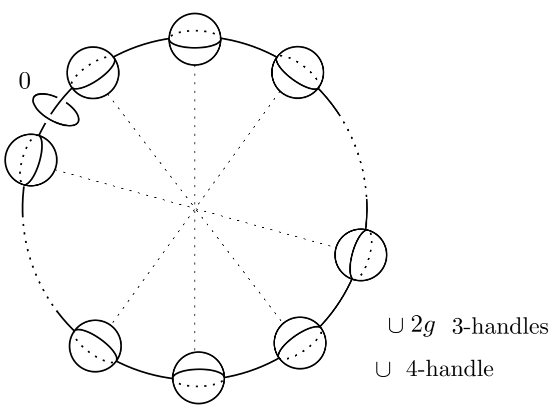

Figure 18. S^2-bundle over genus g surface.

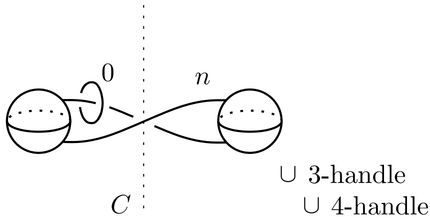

Figure 19. S^2-bundle over $\mathbb{RP}^2$.

except that the second unknot components (labelled by γ and corresponding to the extra 1-handle) will not be present. After filling in the higher index handles and braiding the resulting branch locus B_H, the result will be the same as in Figure 16, except that in the second still only the outermost new component will appear.

4D. ***Connected sums.*** The above constructions can be repeated to give broken Lefschetz fibrations on connected sums. For example, instead of capping off the unknot components in the third to last still of Figure 16, the movie (or another similar braided movie) could be repeated.

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References

[1] S. Akbulut and c. Karakurt, "Every 4-manifold is BLF", *J. Gökova Geom. Topol. GGT* **2** (2008), 83–106. MR Zbl

[2] D. Auroux, S. K. Donaldson, and L. Katzarkov, "Singular Lefschetz pencils", *Geom. Topol.* **9** (2005), 1043–1114. MR Zbl

[3] R. I. Baykur, "Existence of broken Lefschetz fibrations", *Int. Math. Res. Not.* **2008** (2008), art. id. rnn101. MR Zbl

[4] R. I. Baykur, "Topology of broken Lefschetz fibrations and near-symplectic four-manifolds", *Pacific J. Math.* **240**:2 (2009), 201–230. MR Zbl

[5] R. I. Baykur, "Broken Lefschetz fibrations and smooth structures on 4-manifolds", pp. 9–34 in *Proceedings of the Freedman Fest*, Geom. Topol. Monogr. **18**, Geom. Topol. Publ., Coventry, 2012. MR Zbl

[6] R. I. Baykur and K. Hayano, "Broken Lefschetz fibrations and mapping class groups", pp. 269–290 in *Interactions between low-dimensional topology and mapping class groups*, Geom. Topol. Monogr. **19**, Geom. Topol. Publ., Coventry, 2015. MR Zbl

[7] R. I. Baykur and K. Hayano, "Hurwitz equivalence for Lefschetz fibrations and their multisections", pp. 1–24 in *Real and complex singularities*, Contemp. Math. **675**, Amer. Math. Soc., Providence, RI, 2016. MR Zbl

[8] R. I. Baykur and K. Hayano, "Multisections of Lefschetz fibrations and topology of symplectic 4-manifolds", *Geom. Topol.* **20**:4 (2016), 2335–2395. MR Zbl

[9] R. I. Baykur and O. Saeki, "Simplifying indefinite fibrations on 4-manifolds", 2017. arXiv

[10] R. I. Baykur and O. Saeki, "Simplified broken Lefschetz fibrations and trisections of 4-manifolds", *Proc. Natl. Acad. Sci. USA* **115**:43 (2018), 10894–10900. MR Zbl

[11] I. Berstein and A. L. Edmonds, "On the construction of branched coverings of low-dimensional manifolds", *Trans. Amer. Math. Soc.* **247** (1979), 87–124. MR Zbl

[12] I. Bobtcheva and R. Piergallini, "Covering moves and Kirby calculus", 2007. arXiv

[13] N. A. Castro and B. Ozbagci, "Trisections of 4-manifolds via Lefschetz fibrations", *Math. Res. Lett.* **26**:2 (2019), 383–420. MR Zbl

[14] S. K. Donaldson, "Lefschetz fibrations in symplectic geometry", pp. 309–314 in *Proceedings of the International Congress of Mathematicians*, *Vol. II* (Berlin, 1998), 1998. MR Zbl

[15] A. L. Edmonds, "Extending a branched covering over a handle", *Pacific J. Math.* **79**:2 (1978), 363–369. MR Zbl

[16] Y. Eliashberg, "Classification of overtwisted contact structures on 3-manifolds", *Invent. Math.* **98**:3 (1989), 623–637. MR Zbl

[17] J. B. Etnyre and T. Fuller, "Realizing 4-manifolds as achiral Lefschetz fibrations", *Int. Math. Res. Not.* **2006** (2006), art. id. 70272. MR Zbl

[18] D. T. Gay and R. Kirby, "Constructing Lefschetz-type fibrations on four-manifolds", *Geom. Topol.* **11** (2007), 2075–2115. MR Zbl

[19] E. Giroux, "Géométrie de contact: de la dimension trois vers les dimensions supérieures", pp. 405–414 in *Proceedings of the International Congress of Mathematicians*, *Vol. II* (Beijing, 2002), Higher Ed. Press, Beijing, 2002. MR Zbl

[20] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics **20**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[21] H. M. Hilden, "Three-fold branched coverings of S^3", *Amer. J. Math.* **98**:4 (1976), 989–997. MR Zbl

[22] K. Honda, "Transversality theorems for harmonic forms", *Rocky Mountain J. Math.* **34**:2 (2004), 629–664. MR Zbl

[23] M. C. Hughes, "Braiding link cobordisms and non-ribbon surfaces", *Algebr. Geom. Topol.* **15**:6 (2015), 3707–3729. MR Zbl

[24] S. Kamada, "2-dimensional braids and chart descriptions", pp. 277–287 in *Topics in knot theory* (Erzurum, 1992), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **399**, Kluwer Acad. Publ., Dordrecht, 1993. MR Zbl

[25] S. Kamada, "Alexander's and Markov's theorems in dimension four", *Bull. Amer. Math. Soc.* (*N.S.*) **31**:1 (1994), 64–67. MR Zbl

[26] S. Kamada, "On braid monodromies of non-simple braided surfaces", *Math. Proc. Cambridge Philos. Soc.* **120**:2 (1996), 237–245. MR Zbl

[27] S. Kamada, "Arrangement of Markov moves for 2-dimensional braids", pp. 197–213 in *Low-dimensional topology* (Funchal, 1998), Contemp. Math. **233**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[28] S. Kamada, *Braid and knot theory in dimension four*, Mathematical Surveys and Monographs **95**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl

[29] C. Labruère and L. Paris, "Presentations for the punctured mapping class groups in terms of Artin groups", *Algebr. Geom. Topol.* **1** (2001), 73–114. MR Zbl

[30] Y. Lekili, "Wrinkled fibrations on near-symplectic manifolds", *Geom. Topol.* **13**:1 (2009), 277–318. MR Zbl

[31] A. Loi and R. Piergallini, "Compact Stein surfaces with boundary as branched covers of B^4", *Invent. Math.* **143**:2 (2001), 325–348. MR Zbl

[32] J. Meier, "Filling braided links with trisected surfaces", 2020. arXiv

[33] M. Miller and B. Ozbagci, "Lefschetz fibrations on nonorientable 4-manifolds", *Pacific J. Math.* **312**:1 (2021), 177–202. MR Zbl

[34] T. Perutz, "Lagrangian matching invariants for fibred four-manifolds, I", *Geom. Topol.* **11** (2007), 759–828. MR Zbl

[35] T. Perutz, "Lagrangian matching invariants for fibred four-manifolds, II", *Geom. Topol.* **12**:3 (2008), 1461–1542. MR Zbl

[36] R. Piergallini, "Four-manifolds as 4-fold branched covers of S^4", *Topology* **34**:3 (1995), 497–508. MR Zbl

[37] L. Rudolph, "Braided surfaces and Seifert ribbons for closed braids", *Comment. Math. Helv.* **58**:1 (1983), 1–37. MR Zbl

[38] L. Rudolph, "Special positions for surfaces bounded by closed braids", *Rev. Mat. Iberoamericana* **1**:3 (1985), 93–133. MR Zbl

[39] L. Rudolph, "Quasipositivity as an obstruction to sliceness", *Bull. Amer. Math. Soc.* (*N.S.*) **29**:1 (1993), 51–59. MR Zbl

[40] L. Rudolph, "Knot theory of complex plane curves", pp. 349–427 in *Handbook of knot theory*, Elsevier B. V., Amsterdam, 2005. MR Zbl

[41] O. Saeki, "Elimination of definite fold", *Kyushu J. Math.* **60**:2 (2006), 363–382. MR Zbl

[42] M. Usher, "The Gromov invariant and the Donaldson–Smith standard surface count", *Geom. Topol.* **8** (2004), 565–610. MR Zbl

[43] J. Williams, "The h-principle for broken Lefschetz fibrations", *Geom. Topol.* **14**:2 (2010), 1015–1061. MR Zbl

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Gauge Theory and Low-Dimensional Topology: Progress and Interaction

This volume is a proceedings of the 2020 BIRS workshop *Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4*. This was the 6th iteration of a recurring workshop held in Banff. Regrettably, the workshop was not held onsite but was instead an online (Zoom) gathering as a result of the Covid-19 pandemic. However, one benefit of the online format was that the participant list could be expanded beyond the usual strict limit of 42 individuals. It seemed to be also fitting, given the altered circumstances and larger than usual list of participants, to take the opportunity to put together a conference proceedings.

The result is this volume, which features papers showcasing research from participants at the 6th (or earlier) *Interactions* workshops. As the title suggests, the emphasis is on research in gauge theory, contact and symplectic topology, and in low-dimensional topology. The volume contains 16 refereed papers, and it is representative of the many excellent talks and fascinating results presented at the Interactions workshops over the years since its inception in 2007.

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