THE OPEN BOOK SERIES 5

Gauge Theory and Low-Dimensional Topology: Progress and Interaction

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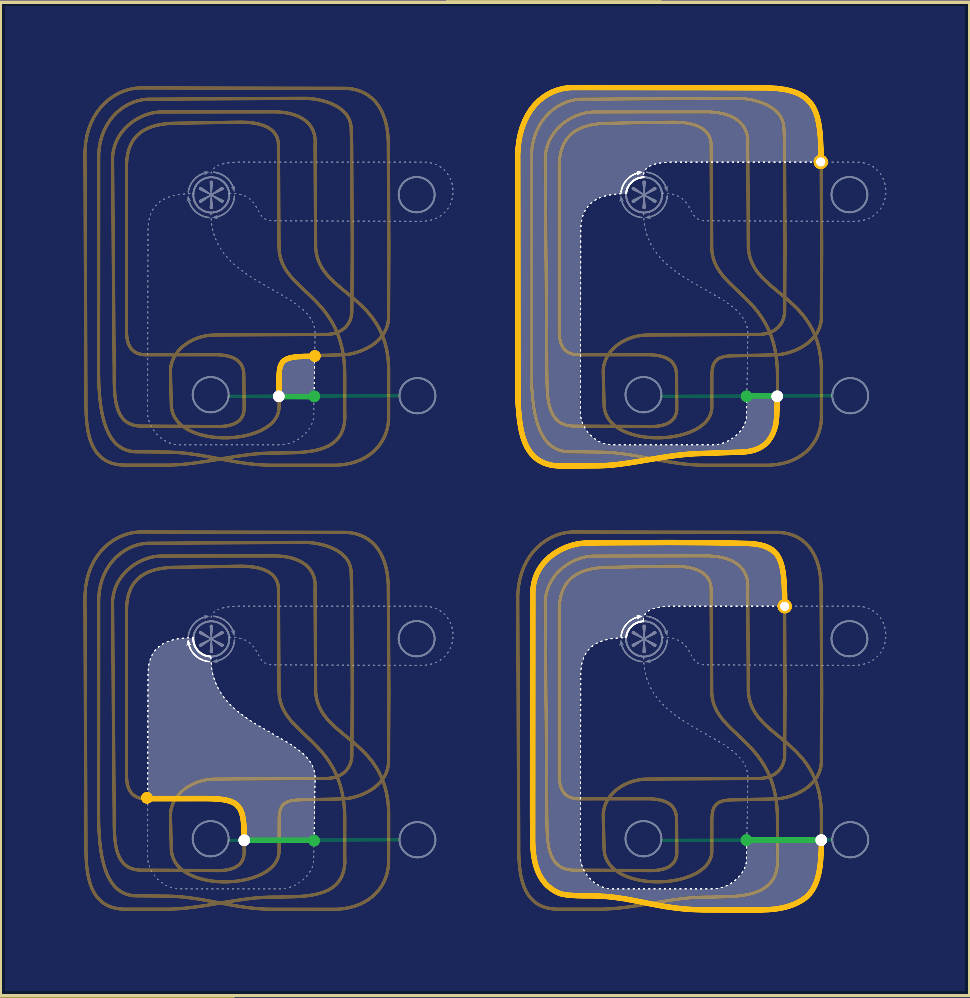
Peter Lambert-Cole



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Lecture notes on trisections and cohomology

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We describe several geometric interpretations of $H_2(X)$ when X is a trisected 4-manifold. The main insight is that, by analogy with Hodge theory and sheaf cohomology in algebraic geometry, classes in $H_2(X)$ can be usefully interpreted as "(1,1)"-classes. First, we reinterpret work of Feller, Klug, Schirmer and Zemke and of Florens and Moussard on the (co)homology of trisected 4-manifolds in terms of the Čech cohomology of presheaves on X, in both the case of singular and de Rham cohomology. We then discuss complex line bundles, almost-complex structures, spin structures and $\text{Spin}^{\mathbb{C}}$-structures on trisected 4-manifolds.

1. Introduction

A motivating question in 4-manifold topology is the following:

Question 1.1. To what extent are general 4-manifolds similar to projective complex surfaces?

Donaldson showed that, like projective surfaces, every closed symplectic manifold admits a Lefschetz pencil [4]. Later, Auroux, Donaldson and Katzarkov showed that near-symplectic manifolds admit so-called broken Lefschetz pencils[1] [1]. Baykur then proved that every closed, oriented, smooth 4-manifold admits a broken Lefschetz fibration over S^2 [2]. This gives one sense in which all such 4-manifolds are similar to projective surfaces.

It is a classical fact, known as Theorem B, that over a Stein domain, coherent sheaves have no higher cohomology. That is, if Z is Stein and $\mathcal{F}$ is a coherent sheaf, then $H^i(Z;\mathcal{F}) = 0$ for $i > 0$. A consequence is that if X is a complex manifold, $\mathcal{F}$ is a coherent sheaf, and $\mathcal{Z} = \{Z_i\}$ is an open cover of X by Stein domains, then

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[1]The term "singular Lefschetz pencil" was used in [1].

the sheaf cohomology of $\mathcal{F}$ can be computed by the Čech complex with respect to the open cover $\mathcal{Z}$:

$$H^*(X;\mathcal{F}) \cong \check{H}^*(\mathcal{Z};\mathcal{F}).$$

On a projective surface, Hodge theory implies that Dolbeault cohomology refines de Rham cohomology. Specifically, there is an isomorphism

$$H^k(X;\mathbb{C}) \cong \bigoplus_{i+j=k} H^{i,j}_{\bar\partial}(X;\mathbb{C}).$$

In addition, Dolbeault's theorem states that Dolbeault cohomology is isomorphic to the cohomology of the sheaf of holomorphic differential forms:

$$H^{i,j}_{\bar\partial}(X;\mathbb{C}) \cong H^i(X;\Omega^j).$$

Moreover, applying Serre duality to the constant sheaf $\underline{\mathbb{C}}$ shows that there is an isomorphism

$$H^{i,j}_{\bar\partial}(X;\mathbb{C}) \cong H^{n-i,n-j}_{\bar\partial}(X;\mathbb{C}),$$

where n is the complex dimension of X.

Interestingly, trisections of 4-manifolds reveal similar results for singular and de Rham cohomology. The four-dimensional handlebody $\natural_k S^1 \times B^3$ admits a Stein structure. Thus, since every closed 4-manifold admits a trisection, it can be covered by three domains that admit Stein structures. In addition, by slightly enlarging the sectors of trisection, we get an open cover $\mathcal{T} = \{U_1, U_2, U_3\}$, where

(1) U_i is diffeomorphic to $\natural_{k_i} S^1 \times B^3$,

(2) $U_i \cap U_j$ is diffeomorphic to $\natural_g S^1 \times B^3$, and

(3) $U_1 \cap U_2 \cap U_3$ is diffeomorphic to $\Sigma_g \times D^2$.

Let $\mathcal{C}^i$ denote the presheaf on X defined as

$$\mathcal{C}^i(U) := H^i(U;\mathbb{Z}).$$

It is clear that $\mathcal{C}^i$ is a presheaf. However, in general it is not a sheaf as it satisfies the gluing axiom but not the locality axiom. In particular, it is not separated. Nonetheless, we can compute the Čech cohomology $\check{H}^*(\mathcal{T},\mathcal{C}^i)$ of the presheaf $\mathcal{C}^i$ with respect to the open cover $\mathcal{T}$.

Methods to compute the homology of 4-manifolds from a trisection have been given by Feller, Klug, Schirmer and Zemke [5] and by Florens and Moussard [6]. Reinterpreting their results, we get the following theorems:

Theorem 1.2 (Hodge/Dolbeault theorem). *There is an isomorphism*

$$H^k(X;\mathbb{Z}) \cong \bigoplus_{i+j=k} \check{H}^i(\mathcal{T},\mathcal{C}^j).$$

Moreover, we have the following "Hodge diamond" for the cohomology of a trisected 4*-manifold*:

$$\begin{array}{ccccc} & & H^4(X;\mathbb{Z}) & & \\ & 0 & & H^3(X;\mathbb{Z}) & \\ 0 & & H^2(X;\mathbb{Z}) & & 0 \\ & H^1(X;\mathbb{Z}) & & 0 & \\ & & H^0(X;\mathbb{Z}) & & \end{array}$$

In particular, the Čech complex $\check{C}^*(\mathcal{T}, \mathcal{C}^1)$ — representing the middle diagonal of the Hodge diamond — is essentially given in [6, Section 2.1] but not described as such.

We can also interpret the symmetry of the Hodge diamond as Serre duality.

Theorem 1.3 (Serre duality). *There is an isomorphism*

$$\check{H}^i(\mathcal{T}, \mathcal{C}^j) \otimes \mathbb{R} \cong \check{H}^{2-i}(\mathcal{T}, \mathcal{C}^{2-j}) \otimes \mathbb{R}.$$

1A. ***Second cohomology as* (1, 1)*-classes.*** By analogy with complex geometry, we refer to any class in $\check{H}^1(\mathcal{T}, \mathcal{C}^1) \cong H^2(X;\mathbb{Z})$ as a (1, 1)*-class*. On a projective surface, the Lefschetz theorem states that the integral (1,1)-classes are precisely those that can be represented by a divisor. The proof of Theorem 1.2 further implies that every class of is a (1,1)-class.

Theorem 1.4. *Every class in $H^2(X;\mathbb{Z})$ is a* (1,1)*-class with respect to the trisection $\mathcal{T}$. Specifically*,

$$H^2(X;\mathbb{Z}) \cong \check{H}^1(\mathcal{T}, \mathcal{C}^1).$$

Unpacking the definition of Čech cohomology, this means that every element of $H^2(X)$ is represented by a triple $(\beta_1, \beta_2, \beta_3)$ where β_λ is a 1-dimensional cohomology class on the handlebody H_λ of the trisection. We will describe several geometric interpretations of this.

(1) **De Rham cohomology**: Every class $\omega \in H^2_{DR}(X)$ can be represented by a triple $(\beta_1, \beta_2, \beta_3)$ where β_λ is a closed 1-form on H_λ.

(2) **$\mathbb{C}$-line bundles**. Recall that isomorphism classes of $\mathbb{C}$-line bundles over X are classified by $H^2(X;\mathbb{Z})$ and homotopy classes of maps from H_λ to S^1 are classified by $H^1(H_\lambda;\mathbb{Z})$. Take a $\mathbb{C}$-line bundle E with first Chern class $c_1(E)$. Then E can be trivialized over each sector Z_λ of the trisection and the triple $(\beta_1, \beta_2, \beta_3)$ corresponding to $c_1(E)$ determines the transition maps (up to homotopy).

(3) Spin$^{\mathbb{C}}$**-structures**. The set of Spin$^{\mathbb{C}}$-structures on X is an affine copy of $H^2(X;\mathbb{Z})$. Following Gompf, we show how to interpret a Spin$^{\mathbb{C}}$-structure as an almost-complex structure on the spine of the trisection. Then, the action of $H^2(X;\mathbb{Z})$ can be described in terms of "Lutz twists" along a collection of curves representing homology classes in $H_1(H_\lambda)$ that are hom-dual to $(\beta_1,\beta_2,\beta_3)$.

2. Singular cohomology

2A. ***Sheaves.*** We first review the basics of sheaves and Čech cohomology. Let X be a topological space and let R be a commutative ring.

Definition 2.1. A *presheaf of R-modules* $\mathcal{F}$ on X consists of

(1) an R-module $\mathcal{F}(U)$ for each open set U, and
(2) a restriction map $\rho_{U,V}:\mathcal{F}(U)\to\mathcal{F}(V)$ if V is contained in U.

Furthermore, the restriction maps satisfy the relations

(1) $\rho_{U,U}:\mathcal{F}(U)\to\mathcal{F}(U)$ is the identity homomorphism, and
(2) $\rho_{U,W}=\rho_{V,W}\circ\rho_{U,V}$ if $W\subset V\subset U$.

If $s\in\mathcal{F}(U)$ and $V\subset U$, then we will denote $\rho_{U,V}(s)$ by $s|_V$.

Exercise 2.2. Suppose that X is a smooth manifold. Check that the following are presheaves:

(1) $\mathcal{R}$ is the constant presheaf, where $\mathcal{R}(U)=R$ and the restriction map is the identity.
(2) $\mathcal{C}^i$ is a presheaf of $\mathbb{Z}$-modules, where $\mathcal{C}^i(U)=H^i(U;\mathbb{Z})$ and the restriction maps are given by the inclusion map.
(3) $\mathcal{DR}^i$ is a presheaf of $\mathbb{R}$-modules, where $\mathcal{DR}^i(U)=H^i_{DR}(U,\mathbb{R})$ and the restriction maps are given by the inclusion map.
(4) Ω^p is a sheaf of $\mathbb{R}$-modules, where $\Omega^p(U)$ is the set of p-forms on U and the restriction maps are given by the inclusion map.

Definition 2.3. A *sheaf of R-modules* is a presheaf of R-modules that satisfy the further conditions:

(1) (*locality*) If $\{U_i\}_{i\in I}$ is an open covering of U and if $s,t\in\mathcal{F}(U)$ satisfy $s|_{U_i}=t|_{U_i}$ for all $i\in I$, then $s=t$.
(2) (*gluing*) Let $\{U_i\}_{i\in I}$ be an open covering of U and let $\{s_i\in\mathcal{F}(U_i)\}_{i\in I}$ be a collection of local sections such that

$$s_i|_{U_i\cap U_j}=s_j|_{U_i\cap U_j}$$

for all $i,j\in I$. Then there is a section $s\in\mathcal{F}(U)$ such that $s|_{U_i}=s_i$ for all $i\in I$.

Exercise 2.4. Show that $\mathcal{R}$ and Ω^p are sheaves, but $\mathcal{C}^i$ and $\mathcal{DR}^i$ are not sheaves in general.

2B. ***Čech cohomology.*** We refer the reader to [3] for a discussion of Čech cohomology in general. To simplify the exposition, we restrict to open covers consisting of at most three open sets.

Let X be a smooth manifold, let $\mathcal{F}$ be a presheaf of R-modules, and let $\mathcal{U} = \{U_1, U_2, U_3\}$ be an open cover of X. The *Čech cochain groups* are defined to be

$$\begin{aligned}
\check{C}^0(\mathcal{U}, \mathcal{F}) &= \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \oplus \mathcal{F}(U_3),\\
\check{C}^1(\mathcal{U}, \mathcal{F}) &= \mathcal{F}(U_1 \cap U_2) \oplus \mathcal{F}(U_1 \cap U_3) \oplus \mathcal{F}(U_2 \cap U_3),\\
\check{C}^2(\mathcal{U}, \mathcal{F}) &= \mathcal{F}(U_1 \cap U_2 \cap U_3).
\end{aligned}$$

For $1 \le i < j \le 3$, denote the restriction maps by

$$\begin{aligned}
\rho_{i,ij} &: \mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j),\\
\rho_{ij,123} &: \mathcal{F}(U_i \cap U_j) \to \mathcal{F}(U_1 \cap U_2 \cap U_3).
\end{aligned}$$

The *Čech coboundary map* is defined to be

$$\begin{aligned}
\delta^{-1} &= 0,\\
\delta^0 &= (\rho_{1,12} - \rho_{2,12}) \oplus (\rho_{1,13} - \rho_{3,13}) \oplus (\rho_{2,23} - \rho_{3,23}),\\
\delta^1 &= \rho_{12,123} - \rho_{13,123} + \rho_{23,123},\\
\delta^2 &= 0.
\end{aligned}$$

The *Čech cohomology* $\check{H}^*(\mathcal{U}, \mathcal{F})$ of the presheaf $\mathcal{F}$ with respect to the open cover $\mathcal{U}$ is defined to be

$$\check{H}^i(\mathcal{U}, F) = \frac{\ker(\delta^i)}{\operatorname{Im}(\delta^{i-1})}.$$

Exercise 2.5. Find open covers $\mathcal{U} = \{U_1, U_2, U_3\}$ and compute the Čech cohomology of the sheaf $\mathcal{R}$ for the following topological spaces:

(1) S^1.

(2) $S^1 \vee S^1$.

(3) S^2.

2C. ***Notational setup.*** Let $X = Z_1 \cup Z_2 \cup Z_3$ be a trisection of X, let $Y_\lambda = \partial Z_\lambda$ and let $H_\lambda = \mathbb{Z}_{\lambda-1} \cap Z_\lambda$. Let Σ be the central surface. The inclusion

$$\iota_\lambda : \Sigma \to H_\lambda$$

induces two maps

$$(\iota_\lambda)_* : H_1(\Sigma) \to H_1(H_\lambda), \qquad (\iota_\lambda)^* : H^1(H_\lambda) \to H^1(\Sigma).$$

Define subspaces

$$L_\lambda := \ker((\iota_\lambda)_*) \subset H_1(\Sigma), \qquad M_\lambda := \mathrm{Im}((\iota_\lambda)^*) \subset H^1(\Sigma).$$

We can use the intersection pairing $\langle -, - \rangle_\Sigma$ on $H_1(\Sigma)$ to define an isomorphism $\pi : H_1(\Sigma) \to H^1(\Sigma)$ by setting

$$\pi(x) = \langle -, x \rangle_\Sigma.$$

Furthermore, we have inclusion maps $\kappa_{i,j} : H_j \hookrightarrow Y_i$ and $\rho_i : Y_i \to Z_i$ for $i = 1, 2, 3$ and $j = i-1, i$. These induce maps

$$(\kappa_{i,j})_* : H_1(H_j) \to H_1(Y_i), \qquad (\rho_i)_* : H_1(Y_i) \to H_1(Z_i),$$
$$(\kappa_{i,j})^* : H^1(Y_i) \to H^1(H_j), \qquad (\rho_i)^* : H^1(Z_i) \to H^1(Y_i).$$

2D. *Hodge diamond.* The results in [5; 6] compute homology. In particular, we have the following expression for $H_*(X)$.

Theorem 2.6 [6]. *The homology of X with $\mathbb{Z}$-coefficients is the homology of the complex*

$$0 \to \mathbb{Z} \xrightarrow{0} (L_1 \cap L_2) \oplus (L_2 \cap L_3) \oplus (L_3 \cap L_1) \xrightarrow{\zeta} L_1 \oplus L_2 \oplus L_3 \xrightarrow{\iota} H_1(\Sigma) \xrightarrow{0} \mathbb{Z} \to 0,$$

where $\zeta(a, b, c) = (c - a, a - b, b - c)$ and $\iota(a, b, c) = a + b + c$.

The middle terms of this complex are essentially the Čech complex.

Proposition 2.7. *There is a chain complex isomorphism*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (L_1 \cap L_2) \oplus (L_2 \cap L_3) \oplus (L_3 \cap L_\alpha) & \xrightarrow{\zeta} & L_\alpha \oplus L_2 \oplus L_3 & \xrightarrow{\iota} & H_1(\Sigma) & \longrightarrow & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle 0} \\
0 & \longrightarrow & \bigoplus_\lambda H^1(Z_\lambda) & \xrightarrow{\delta_1} & \bigoplus_\lambda H^1(H_\lambda) & \xrightarrow{\delta_2} & H^1(\Sigma) & \longrightarrow & 0
\end{array}$$

The second complex of this proposition is exactly the Čech complex of $\mathcal{C}^1$ with respect to $\mathcal{T}$, thus by applying Poincaré duality we obtain the following corollary.

Corollary 2.8. *For $i = 1, 2, 3$, there are isomorphisms*

$$H_{4-i}(X; \mathbb{Z}) \cong H^i(X; \mathbb{Z}) \cong \check{H}^{i-1}(\mathcal{T}, \mathcal{C}^1).$$

Proof of Proposition 2.7. By definition, $Z_\lambda = \natural_{k_\lambda} S^1 \times B^3$ and $Y_\lambda = \partial Z_\lambda = \#_{k_\lambda} S^1 \times S^2$. In particular,

$$H_1(Z_\lambda) \cong H_1(Y_\lambda) \cong \mathbb{Z}^{k_\lambda}.$$

We can apply the Mayer–Vietoris sequence to the Heegaard splitting $Y_\lambda = H_\lambda \cup H_{\lambda+1}$ to get the sequence

$$\to H_2(H_\lambda) \oplus H_2(H_{\lambda+1}) \to H_2(Y_\lambda) \to H_1(\Sigma) \\ \to H_1(H_\lambda) \oplus H_1(H_{\lambda+1}) \to H_1(Y_\lambda) \to H_0(\Sigma).$$

Since $H_2(H_\lambda) = H_2(H_{\lambda+1}) = 0$, we see that

$$H^1(Y_\lambda) \cong H_2(Y_\lambda) \cong \ker(H_1(\Sigma) \to H_1(H_\lambda) \oplus H_1(H_{\lambda+1})) \cong L_\lambda \cap L_{\lambda+1},$$

where the first isomorphism follows by Poincaré duality. This defines ϕ_1.

Using the long exact sequence of the pair (H_λ, Σ) we obtain

$$H_2(H_\lambda) \to H_2(H_\lambda, \Sigma) \to H_1(\Sigma) \to H_1(H_\lambda) \to .$$

Since $H_2(H_\lambda) = 0$, we see that

$$H^1(H_\lambda) \cong H_2(H_\lambda, \Sigma) \cong \ker(H_1(\Sigma) \to H_1(H_\lambda)) = L_\lambda.$$

This defines ϕ_2. □

The remaining cohomology groups are straightforward to calculate.

Proposition 2.9. *The cohomology groups of $\mathcal{C}^0$ are*

$$\check{H}^0(\mathcal{T}, \mathcal{C}^0) \cong \mathbb{Z}, \qquad \check{H}^1(\mathcal{T}, \mathcal{C}^0) \cong 0, \qquad \check{H}^2(\mathcal{T}, \mathcal{C}^0) \cong 0.$$

Proof. Each open set U_i and each double and triple intersection is connected and so

$$H^0(U_i; \mathbb{Z}) \cong H^0(U_i \cap U_j; \mathbb{Z}) \cong H^0(U_1 \cap U_2 \cap U_3) \cong \mathbb{Z}.$$

The Čech complex is therefore

$$0 \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0.$$

If $\{a, b, c\}$ is a chain in $\check{C}^0(\mathcal{T}, \mathcal{C}^0)$ then

$$\delta^0\{a, b, c\} = \{a - b, b - c, c - a\}.$$

Thus, this chain is coclosed if and only if $a = b = c$. Thus, $\check{H}^0(\mathcal{T}, \mathcal{C}^0) \cong \mathbb{Z}\langle\{a, a, a\}\rangle \cong \mathbb{Z}$. If $\{a, b, c\}$ is a chain in $\check{C}^1(\mathcal{T}, \mathcal{C}^0)$, then

$$\delta^1\{a, b, c\} = \{a + b + c\}.$$

The chain is coclosed if and only if it has the form

$$\{a, b, -a - b\} = a\{1, 0, -1\} + b\{0, 1, -1\}.$$

Both elements $\{1, 0, -1\}$ and $\{0, 1, -1\}$ are in the image of δ^0, so $\check{H}^1(\mathcal{T}, \mathcal{C}^0) \cong 0$. Finally, the differential δ^1 is surjective so $\check{H}^1(\mathcal{T}, \mathcal{C}^0) \cong 0$ as well. □

Proposition 2.10. *The cohomology groups of $\mathcal{C}^2$ are*

$$\check{H}^0(\mathcal{T}, \mathcal{C}^2) \cong 0, \qquad \check{H}^1(\mathcal{T}, \mathcal{C}^2) \cong 0, \qquad \check{H}^2(\mathcal{T}, \mathcal{C}^2) \cong \mathbb{Z}.$$

Exercise 2.11. Prove the proposition. [Hint: what is the rank of $\check{C}^i(\mathcal{T}, \mathcal{C}^2)$?]

3. De Rham

Let $\mathcal{DR}^i$ denote the presheaf on X defined as

$$\mathcal{DR}^i(U) := H^i_{DR}(U; \mathbb{R})$$

3A. *De Rham to Čech isomorphism.*

Theorem 3.1. *There are isomorphisms*

$$\begin{aligned} &H^1_{DR}(X; \mathbb{R}) \cong \check{H}^0(\mathcal{T}, \mathcal{DR}^1), && H^0_{DR}(X; \mathbb{R}) \cong \check{H}^0(\mathcal{T}, \mathcal{DR}^0), \\ &H^2_{DR}(X; \mathbb{R}) \cong \check{H}^1(\mathcal{T}, \mathcal{DR}^1), && H^4_{DR}(X; \mathbb{R}) \cong \check{H}^2(\mathcal{T}, \mathcal{DR}^2), \\ &H^3_{DR}(X; \mathbb{R}) \cong \check{H}^2(\mathcal{T}, \mathcal{DR}^1). \end{aligned}$$

We break up the proof by the degree of the cohomology group:

Degree 0: The cohomology group $H^0_{DR}(X; \mathbb{R})$ consists of constant functions. Given a constant function $C : X \to \mathbb{R}$, its restriction to U_λ is also a constant function $C : U_\lambda \to \mathbb{R}$ and therefore an element of $H^0_{DR}(U_\lambda; \mathbb{R})$. The isomorphism from de Rham to Čech is given by $C \mapsto (C, C, C)$.

Conversely, an element of $\check{H}^0(\mathcal{T}, \mathcal{DR}^0)$ is a triple (C_1, C_2, C_3) of constant functions whose restrictions to the pairwise intersections agree. In other words, $C_1 = C_2 = C_3 = C$. The inverse isomorphism is therefore $(C, C, C) \mapsto C$.

Degree 1: The map from de Rham to Čech is identical to the degree 0 case above. Given some closed 1-form β, the corresponding element in Čech cohomology is given by restricting β to each U_λ.

The inverse isomorphism is more complicated. In particular, an element of $\check{H}^0(\mathcal{T}, \mathcal{DR}^1)$ is a triple $([\beta_1], [\beta_2], [\beta_3])$ of cohomology classes, not specific closed forms. Choose representative closed 1-forms β_1, β_2, β_3. By assumption, the restrictions satisfy

$$[\beta_{\lambda-1}] = [\beta_\lambda] \in H^1_{DR}(U_{\lambda-1} \cap U_\lambda; \mathbb{R}).$$

Therefore, $\beta_\lambda - \beta_{\lambda-1} = dg_\lambda$ for some function $g_\lambda : U_{\lambda-1} \cap U_\lambda \to \mathbb{R}$.

Exercise 3.2. Show that there exist functions $f_\lambda : U_\lambda \to \mathbb{R}$ such that, on $U_{\lambda-1} \cap U_\lambda$,

$$\beta_{\lambda-1} + df_{\lambda-1} = \beta_\lambda + df_\lambda.$$

Consequently, we can represent our original Čech class by the triple

$$(\beta_1 + df_1, \beta_2 + df_2, \beta_3 + df_3)$$

and these 1-forms glue into a global 1-form β.

Degree 2: In this case, the maps in both directions are more complicated and we need to check that they are in fact isomorphisms. First, choose a class $[\omega] \in H^2_{DR}(X; \mathbb{R})$ and represent it by a closed 2-form ω. The restriction $\omega|_{U_\lambda}$ is exact since $H^2_{DR}(U_\lambda; \mathbb{R}) = 0$, thus we can choose a primitive α_λ for $\omega|_{U_\lambda}$. Over the double intersection $U_{\lambda-1} \cap U_\lambda$, the restrictions $\alpha_{\lambda-1}$ and α_λ are both primitives for ω, therefore their difference $\alpha_\lambda - \alpha_{\lambda-1}$ is closed. Consequently, the map from de Rham to Čech is given by

$$\omega \mapsto (\alpha_1 - \alpha_3, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2).$$

There were three sources of indeterminancy:

(1) We could replace α_λ by $\alpha_\lambda + df_\lambda$ for some function $f_\lambda : U_\lambda \to \mathbb{R}$.

(2) We could replace ω by $\omega + d\mu$ for some global 1-form μ.

(3) We could replace the primitive α_λ with $\alpha_\lambda + \rho_\lambda$, where ρ is a closed 1-form on U_λ.

Exercise 3.3. (1) Show that modifying the primitives $\{\alpha_\lambda\}$ by exact 1-forms results in the same Čech cochain.

(2) Show that we can choose primitives for $\omega + d\mu$ that result in the same Čech cochain.

(3) Show that modifying the primitives $\{\alpha_\lambda\}$ by closed 1-forms $\{\rho_\lambda\}$ changes the Čech cochain by a Čech coboundary.

Conversely, given a class in $\check{H}^0(\mathcal{T}, \mathcal{DR}^1)$, choose a fixed cochain $([\beta_1], [\beta_2], [\beta_3])$ and fixed closed 1-forms $\{\beta_1, \beta_2, \beta_3\}$ to represent this class.

Exercise 3.4. (1) There exists a triple of 1-forms $\{\alpha_\lambda\}$ on the open sets $\{U_\lambda\}$ such that $\alpha_\lambda - \alpha_{\lambda-1} = \beta_\lambda$.

(2) The 2-forms $\{d\alpha_1, d\alpha_2, d\alpha_3\}$ glue together to give a global 2-form ω.

(3) Modifying the choices — modifying the Čech cochain by a coboundary, modifying the closed 1-forms $\{\beta_\lambda\}$ by exact 1-forms, and modifying the choices of $\{\alpha_\lambda\}$ — results in a cohomologous 2-form ω'.

Degree 3: Given a class $[\mu] \in H^3_{DR}(X; \mathbb{R})$, represent it by a closed 3-form μ. Since $H^3_{DR}(U_\lambda; \mathbb{R}) = 0$, we can choose a primitive ω_λ for μ over each U_λ. The differences $\omega_\lambda - \omega_{\lambda-1}$ are closed and represent elements of $H^3_{DR}(U_\lambda \cap U_{\lambda-1}; \mathbb{R}) = 0$. In particular, these forms are also exact and we can choose further primitive 1-forms $\{\beta_\lambda\}$.

Restricting to the triple intersection $U_1 \cap U_2 \cap U_3$ we get a 1-form $\beta = \beta_1 + \beta_2 + \beta_3$ that is closed since

$$d\beta = d\beta_1 + d\beta_2 + d\beta_3 = (\omega_1 - \omega_3) + (\omega_2 - \omega_1) + (\omega_3 - \omega_2) = 0.$$

Thus, $[\mu]$ is sent to an element $[\beta] \in H^1_{DR}(U_1 \cap U_2 \cap U_3; \mathbb{R})$ and therefore represents a Čech 2-cocycle.

Exercise 3.5. (1) Show that changing ω_λ by a closed 2-form results in the same Čech 2-cocycle

(2) Show that changing β_λ by a closed 1-form modifies the resulting Čech 2-cocycle by a Čech 2-coboundary.

The inverse map can be constructed by an argument similar to the degree 2 case; we leave it as an exercise.

Exercise 3.6. Construct the inverse map $\check{H}^2(\mathcal{T}, \mathcal{C}^1) \to H^3_{DR}(X; \mathbb{R})$ and show that it is well defined.

Degree 4: The isomorphism is constructed in a analogous method to the degree 3 case and we leave it as an exercise to the reader.

Exercise 3.7. Construct the isomorphism $H^4_{DR}(X; \mathbb{R}) \cong \check{H}^2(\mathcal{T}, \mathcal{C}^2)$.

3B. ***Intersection pairing.*** The intersection pairing on de Rham cohomology can also be expressed in terms of the Čech cohomology of the de Rham presheafs. In particular, we can describe the pairings

$$H^2_{DR}(X) \times H^2_{DR}(X) \to \mathbb{R},$$
$$H^3_{DR}(X) \times H^1_{DR}(X) \to \mathbb{R}.$$

Moreover, we can describe the pairing obtained by integrating a closed p-form over a closed p-dimensional submanifold:

$$H^2_{DR}(X) \times H_2(X; \mathbb{Z}) \to \mathbb{R},$$
$$H^3_{DR}(X) \times H_3(X; \mathbb{Z}) \to \mathbb{R},$$
$$H^4_{DR}(X) \times H_4(X; \mathbb{Z}) \to \mathbb{R}.$$

Theorem 3.8 (intersection pairing). *Let X be a trisected 4-manifold.*

(1) *Let ω_1, ω_2 be a pair of closed 2-forms. Suppose that under the de Rham–Čech isomorphism we have*

$$[\omega_1] \mapsto (\alpha_1, \alpha_2, \alpha_3), \qquad [\omega_2] \mapsto (\beta_1, \beta_2, \beta_3).$$

Then

$$\int_X \omega_1 \wedge \omega_2 = \int_\Sigma \alpha_1 \wedge \beta_2 = \int_\Sigma \alpha_2 \wedge \beta_3 = \int_\Sigma \alpha_3 \wedge \beta_1.$$

(2) *Let μ be a closed 3-form and α be a closed 1-form. Suppose that under the de Rham–Čech isomorphism, we have that $[\mu] \mapsto [\beta]$. Then*

$$\int_X \mu \wedge \alpha = \int_\Sigma \beta \wedge \alpha|_\Sigma .$$

Exercise 3.9. Prove these statements. [Hint: use Stokes' theorem combined with the arguments in the previous subsection.]

To describe the integration pairing, we first fix some notation.

(1) Let $\mathcal{K}$ be an embedded, oriented closed surface in general position with respect to the trisection. Let $\tau_\lambda^{\mathcal{K}}$ denote the tangle $\mathcal{K} \pitchfork H_\lambda$. We orient τ_λ as follows: since $\mathcal{K}$ is oriented, the intersection $F_\lambda = \mathcal{K} \cap Z_\lambda$ is oriented. The boundary ∂F_λ inherits an orientation from F_λ; the tangle $\tau_\lambda^{\mathcal{K}}$ is a subset of this boundary and inherits an orientation.

(2) Let $\mathcal{M}$ be an embedded, oriented, closed hypersurface in general position with respect to the trisection. In particular, the intersection $\mathcal{M} \pitchfork \Sigma$ is a simple closed curve $\gamma_{\mathcal{M}}$.

Theorem 3.10 (integration pairing). *Let X be a trisected 4-manifold.*

(1) *Let ω be a closed 2-form on X that maps to $(\beta_1, \beta_2, \beta_3)$ under the de Rham–Čech isomorphism and let $\mathcal{K}$ be an embedded, oriented closed surface. Then,*

$$\int_{\mathcal{K}} \omega = \sum_{\lambda=1,2,3} \int_{\tau_\lambda^{\mathcal{K}}} \beta_\lambda .$$

(2) *Let μ be a closed 3-form on X that maps to $\beta \in H^1_{DR}(\Sigma)$ under the de Rham–Čech isomorphism and let $\mathcal{M}$ be an embedded, oriented, closed hypersurface. Then,*

$$\int_{\mathcal{M}} \mu = \int_{\gamma_{\mathcal{M}}} \beta .$$

(3) *Let Ω be a closed 4-form on X that maps to $\omega \in H^2_{DR}(\Sigma)$ under the de Rham–Čech isomorphism. Then,*

$$\int_X \Omega = \int_\Sigma \omega .$$

Exercise 3.11. Prove these statements [Hint: again, use Stokes' theorem].

4. Complex line bundles

4A. ***Algebraic topology.*** First, we recall some facts from algebraic topology.

(1) The circle S^1 is a $K(\mathbb{Z}, 1)$. In particular, there is a one-to-one correspondence between classes in $H^1(X; \mathbb{Z})$ and homotopy classes of maps $f : X \to S^1$.

(2) The space $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$. In particular, there is a one-to-one correspondence between classes in $H^2(X;\mathbb{Z})$ and homotopy classes of maps $f : X \to \mathbb{CP}^\infty$. The cohomology ring of $\mathbb{CP}^\infty$ is $\mathbb{Z}[\alpha]$, where α has degree 2, and the identification between maps and cohomology classes is given by

$$f \leftrightarrow f^*(\alpha).$$

(3) The space $\mathbb{CP}^\infty$ is the classifying space for $U(1)$ (equivalently $\mathbb{C}$-line) bundles. In particular, there is a one-to-one correspondence between $\mathbb{C}$-line bundles on X, up to isomorphism, and homotopy classes of maps $f : X \to \mathbb{CP}^\infty$. There is a *tautological line bundle* $E \to \mathbb{CP}^\infty$ and the correspondence between maps and $\mathbb{C}$-bundles is given by

$$f \leftrightarrow f^*(E).$$

(4) The *first Chern class* is a complete invariant of $\mathbb{C}$-line bundles and connects (2) and (3) above. In particular, for the tautological bundle E on $\mathbb{CP}^\infty$ we have

$$c_1(E) = \alpha.$$

Moreover, since Chern classes are characteristic, they are natural with respect to pullbacks and therefore

$$c_1(f^*(E)) = f^*(c_1(E)) = f^*(\alpha).$$

4B. ***Chern classes of $\mathbb{C}$-line bundles.*** Using a trisection $\mathcal{T}$ of X, we can explicitly see the equivalence

$$\{\mathbb{C}\text{-line bundles on X}\}/\{\text{equivalence}\} = \check{H}^1(\mathcal{T}, \mathcal{C}^1) \cong H^2(X;\mathbb{Z}),$$

where an element of $\check{H}^1(\mathcal{T}, \mathcal{C}^1)$ is a "(1, 1)-class".

Line bundles to (1, 1)*-classes.* Take a line bundle E on X. Since each sector Z_λ of a trisection is a 1-handlebody, we can choose a trivialization s_λ of E over Z_λ. Up to homotopy, the potential choices of trivializations are in one-to-one correspondence with elements of $H^1(Z_\lambda;\mathbb{Z}) \cong \mathbb{Z}^{k_\lambda}$. Over the double intersection H_λ, we have two trivializations $s_{\lambda-1}, s_\lambda$. Taking their quotient, we obtain a map

$$g_\lambda := \frac{s_\lambda}{s_{\lambda-1}} \to \mathbb{C}^*.$$

Composing this with the homotopy equivalence $\mathbb{C}^* \simeq S^1$, the map g_λ determines a homotopy class of maps from H_λ to S^1. In other words, the transition function g_λ determines a unique element β_λ of $H^1(H_\lambda;\mathbb{Z})$. Moreover, since

$$g_1 g_2 g_3 = \frac{s_1}{s_3}\frac{s_2}{s_1}\frac{s_3}{s_2} = 1,$$

the resulting triple $(\beta_1, \beta_2, \beta_3)$ is a Čech 1-cocycle in $\check{C}^*(\mathcal{T}, \mathcal{C}^1)$. Modifying the trivialization s_λ by some element of $H^1(Z_\lambda;\mathbb{Z})$ changes the resulting cocycle by a Čech coboundary. In particular, we obtain a well-defined element $c_1(E) \in \check{H}^1(\mathcal{T}, \mathcal{C}^1)$.

$(1, 1)$-*classes to line bundles.* Given a (1,1)-class $(\beta_1, \beta_2, \beta_3) \in \check{H}^1(\mathcal{T}, \mathcal{C}^1)$, we can represent $\beta_\lambda \in H^1(H_\lambda; \mathbb{Z})$ by a map $g_\lambda : H_\lambda \to S^1$. Moreover, given the cocycle condition $\beta_1 + \beta_2 + \beta_3 = 0$ we can assume that $g_1 g_2 g_3 = 1$. In particular, the triple $\{g_1, g_2, g_3\}$ determines a triple of transition functions that allow us to construct a $\mathbb{C}$-bundle over X.

5. Almost-complex structures

An *almost-complex structure* J on X is a fiberwise homomorphism $J : TX \to TX$ such that $J^2 = -I$. This turns every fiber $T_x X$ into a complex vector space, where J is multiplication by i. Consequently, the almost-complex structure determines Chern classes $c_i(TX, J) \in H^{2i}(X; \mathbb{Z})$. The goal of this section is to describe almost-complex structures on the spine of a trisection.

5A. *Field of complex tangencies.* Let $Y^3 \subset X^4$ be a smooth hypersurface and let J be an almost-complex structure. The *field of* J*-complex tangencies* is defined to be

$$\xi := J(TY) \cap TY$$

Exercise 5.1. Show that ξ has rank 2 at every point. [Hint: ξ_x is a J-complex line in $T_x X$.] In particular, ξ is an *oriented* plane field.

Exercise 5.2. Let $\phi : X \to \mathbb{R}$ be a function such that $Y = \phi^{-1}(0)$. Show that the field of J-tangencies is the kernel of the 1-form $d^{\mathbb{C}}\phi = d\phi(J-)$, restricted to Y.

Proposition 5.3. *Let Y be a* 3*-manifold. Homotopy classes of almost-complex structures on $Y \times [0, 1]$ are in one-to-one correspondence with homotopy classes of* (*coorientable*) 2*-plane fields on Y.*

Proof. Let J be an almost-complex structure on $Y \times [0, 1]$ and let ξ_t denote the field of J-tangencies along $Y \times \{t\}$. It is immediately clear that $\{\xi_t\}$ is a homotopy of 2-plane fields. Furthermore, let J_s be a family of almost-complex structures and let $\xi_{s,t}$ denote the field of J_s-tangencies along $Y \times \{t\}$. Again, this clearly gives a 2-parameter homotopy of plane fields on Y.

Now let ξ be an oriented, coorientable 2-plane field and choose a fiberwise metric g on ξ. We can define an almost-complex structure $J : \xi \to \xi$ using the metric as follows. Locally, we can choose an oriented, orthonormal frame $\{e_1, e_2\}$ and define

$$J(e_1) = e_2 \quad \text{and} \quad J(e_2) = -e_1$$

and extend linearly.

Exercise 5.4. Show that, up to homotopy, this J does not depend on the metric g or the local orthonormal frame.

Next, let Λ be an oriented line field that coorients ξ. After choosing a metric h on Λ, we get a unit-length section σ of Λ and can extend J from ξ to TX by defining

$$J(\partial_t) = \sigma \quad \text{and} \quad J(\sigma) = -\partial_t.$$

Exercise 5.5. Show that, up to homotopy, this J does not depend on the homotopy class of $J|_\xi$, the homotopy class of Λ, or the metric h.

Finally, we have to check that every J on $Y \times [0, 1]$ can be constructed in this way. Choose some J and define $E = \langle \partial_t, J(\partial_t) \rangle$ and $\Lambda = TY \cap E$. Choose a nonvanishing section σ of Λ. Then,

$$J(\partial_t) = f\partial_t + g\sigma$$

for some functions f, g. By assumption, $\{\partial_t, J\partial_t\}$ is an oriented basis for E and therefore $g > 0$. Since J preserves ξ, we can define a family J_s of almost-complex structures for $s \in [0, 1]$ by defining

$$J_s|_\xi = J \quad \text{and} \quad J_s(\partial_t) = sf\partial_t + g\sigma.$$

After scaling the metric so that $|g\sigma| = 1$, we have that J_0 is an almost-complex structure of the form constructed above and J_1 is our original J. □

Exercise 5.6. Show that $\Sigma \times D^2$ admits an almost-complex structure J with $c_1(J) = 0$. [Hint: embed Σ in $\mathbb{C}^2$.]

Lemma 5.7. *The spine of a trisection admits an almost-complex structure J.*

Proof. By the previous exercise, we can choose some J on a tubular neighborhood of the central surface Σ. The remaining task is to extend it across each handlebody H_λ. The almost-complex structure J determines a hyperplane field ξ_λ in a neighborhood of $\partial H_\lambda = \Sigma$.

Exercise 5.8. Show $\langle e(\xi_\lambda), [\Sigma] \rangle = \langle c_1(J), [\Sigma] \rangle = 0$. [Hint: Choose a section σ of ξ_λ and a normal vector field ν to H_λ. Then $\det(\nu, \sigma) = 0$ precisely where $\sigma = 0$.]

Consequently, it is possible to extend ξ_λ across H_λ and by Proposition 5.3, this determines a homotopy class of J in a neighborhood of H_λ. □

5B. ***First Chern class of J.*** Given some J on the spine of a trisection, we can construct a 1-complex C_J in the spine that represents the Poincaré dual to $c_1(TX, J)$.

The central surface Σ is canonically framed. In particular, we can choose coordinates (s, t) on D^2 such that pulling back the coordinates by the projection

$$\pi : \nu(\Sigma) \cong \Sigma \times D^2 \to D^2$$

we have that

$$\Sigma = \pi^{-1}(0), \qquad H_2 = \pi^{-1}(0, t) \text{ for } t \geq 0,$$
$$H_1 = \pi^{-1}(s, 0) \text{ for } s \leq 0, H_3 = \pi^{-1}(-x, x) \text{ for } x \geq 0.$$

Consider the *conormal sequence* for the central surface Σ:

$$0 \to N^*\Sigma \to T^*X \to T^*\Sigma \to 0.$$

A *coframing* of Σ is a trivialization of its conormal bundle. Since $N^*\Sigma$ is an $\mathbb{R}^2$-bundle, a coframing is determined by a single, nowhere-vanishing section. Moreover, it is clear from the conormal sequence that such a section is given by a nowhere-vanishing 1-form whose restriction to Σ is identically 0. An almost-complex structure J determines a dual almost-complex structure $J^t : T^*X \to T^*X$. Inserting this, we get a (nonexact) sequence

$$N^*\Sigma \to T^*X \xrightarrow{J^t} T^*X \to T^*\Sigma.$$

Given a section α of $N^*\Sigma$, we can push it through this sequence to get a 1-form $\widetilde{\alpha}$ on Σ, defined to be

$$\widetilde{\alpha} = \alpha(J-)|_{\Sigma}.$$

Exercise 5.9. A *complex point* of Σ is a point $x \in \Sigma$ such that $J(T_x\Sigma) = T_x\Sigma$. Show that $\widetilde{\alpha}$ vanishes at precisely the complex points of Σ.

Exercise 5.10. By a C^∞-small perturbation of Σ, we can assume that Σ has finitely many complex points [Hint: what are the dimensions of the Grassmannians $\mathrm{Gr}_{\mathbb{R}}(2,4)$ and $\mathrm{Gr}_{\mathbb{C}}(1,2)$?]

Recall the normal coordinates (s,t) on $\Sigma \times D^2$. The pair ds, dt of 1-forms gives a coframing of Σ. Define

$$\beta_1 := \widetilde{ds}, \qquad \beta_2 := \widetilde{dt}, \qquad \beta_3 = -\widetilde{ds} - \widetilde{dt}.$$

Exercise 5.11. Show that $\beta_1 \wedge \beta_2 \neq 0$, except at the complex points of Σ. In particular, β_1 vanishes at $x \in \Sigma$ if and only if β_2 vanishes at x.

Exercise 5.12. Suppose that β_1, β_2, viewed as sections of $T^*\Sigma$, are transverse to the 0-section. Show that at each complex point $x \in \Sigma$, the indices of the vanishing of β_1 and β_2 at x agree.

Exercise 5.13. Show that β_λ extends to a 1-form on the handlebody H_λ of the trisection such that $\ker(\beta_\lambda)$ is the field of J-complex tangencies along H_λ.

Choose vector fields $\{v_1, v_2\}$ on Σ such that

$$\beta_1(v_1) = 0, \quad \beta_2(v_1) = \beta_1(v_2) \geq 0, \quad \beta_2(v_2) = 0$$

and set $v_3 = -v_1 - v_2 \in \ker(\beta_3)$. Since $v_\lambda \in \ker(\beta_\lambda)$, we can extend v_λ to a section of ξ_λ over H_λ.

For notational purposes, let ν_λ be a normal vector fields to H_λ such that near Σ, we have

$$u_1 = \partial_s, \quad u_2 = \partial_t, \quad u_3 = -\partial_s - \partial_t.$$

Exercise 5.14. Show that the pairs

$$\{u_1, v_1\}, \quad \{u_2, v_2\}, \quad \{u_3, v_3\}$$

determine the same section of $\det(TX, J)$ over Σ.

Proposition 5.15. *Let J be an almost-complex structure on the spine of a trisection $\mathcal{T}$ of X. Choose vector fields $\{v_\lambda \subset \xi_\lambda\}$ as above and let $\tau_\lambda = v_\lambda^{-1}(0)$. The 1-complex*

$$C_J = \tau_1 \cup \tau_2 \cup \tau_3$$

is the intersection of $PD(c_1(J))$ with the spine of the trisection $\mathcal{T}$.

Proof. The bivector $u_\lambda \wedge v_\lambda$ determines a section of the determinant line bundle over H_λ. The vector u_λ is everywhere normal to H_λ and nonvanishing, while v_λ is tangent and vanishes along τ_λ. By the previous exercise, we obtain a section of the determinant bundle on the entire spine that vanishes precisely along the 1-complex C_J. □

6. Spin$^{\mathbb{C}}$-structures

A standard interpretation of a spin structure on a manifold X is a trivialization of TX over the 1-skeleton that extends across the 2-skeleton. A similar interpretation of Spin$^{\mathbb{C}}$-structures, due to Gompf, is an almost-complex structure over the 2-skeleton that extends across the 3-skeleton.

6A. *Handle decompositions.* Every trisection $\mathcal{T}$ of X determines an *inside-out* handle decomposition as follows.

(1) Start with a neighborhood $\nu(\Sigma)$ of the central surface. This is diffeomorphic to $\Sigma \times D^2$ and can be built in the standard way using a 0-handle, $2g$ 1-handles, and a 2-handle. The boundary of this neighborhood is $\Sigma \times S^1$.

(2) Next, attach a neighborhood $\nu(H_\lambda)$ of each 3-dimensional piece of the trisection. The solid handlebody H_λ is built from a single 3-dimensional 0-handle and g 3-dimensional 1-handles. Upside-down, this becomes g 2-handles and a single 3-handle. Fix some distinct angular points $\theta_1, \theta_2, \theta_3 \in S^1$ in positive cyclic order. Then attaching $\nu(H_\lambda)$ is equivalent to the following. Attach g 2-handles along a cut system of curves on $\Sigma \times \{\theta_\lambda\}$ with surface framing. After this surgery, the surface $\Sigma \times \{\theta_\lambda\}$ is now an essential 2-sphere and the 4-dimensional 3-handle is attached along this 2-sphere. The resulting boundary of the 4-manifold has three components Y_1, Y_2, Y_3 with $Y_3 \cong \#_{k_i} S^1 \times S^2$.

(3) Last, attach the 4-dimensional sectors. These are 4-dimensional 1-handlebodies; upside-down they consist of k_i 3-handles and a single 4-handle. The 3-handles are attached along the essential spheres in $\#_{k_i} S^1 \times S^2$. The resulting boundary is three copies of S^3, which is where the 4-handles are attached.

The *outside-in* handle decomposition determined by $\mathcal{T}$ is the handle decomposition obtained by turning the inside-out handle decomposition upside-down.

6B. *Spin structures.* A standard interpretation of a spin structure on a manifold X is a trivialization of TX over the 1-skeleton that extends across the 2-skeleton. Now, consider the inside-out handle decomposition of X determined by a trisection $\mathcal{T}$. The 1-skeleton of X is contained in the 1-skeleton of $\nu(\Sigma)$. Thus, every spin structure of X restricts to a spin structure on $\nu(\Sigma)$; moreover, since spin structures are stable, every spin structure of X restricts to a spin structure on the central surface Σ.

Recall that there exist two spin structures on S^1 and exactly one extends across D^2. The spin structures on a closed, oriented surface Σ are classified by maps

$$q : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2,$$

where $q(\gamma) = 0$ if the spin structure, restricted to a curve representing γ, is the spin structure that extends across the disk. This map is a quadratic enhancement of the intersection form on $H_1(\Sigma)$; in particular, it satisfies the relation

$$q(x+y) = q(x) + q(y) + \langle x, y\rangle \mod 2. \tag{1}$$

Let $\boldsymbol{\alpha} = \{\alpha_i\}$ be a cut system of curves on Σ. We say that $q(\boldsymbol{\alpha}) = 0$ if $q(\alpha_i) = 0$ for every $\alpha_i \in \boldsymbol{\alpha}$. Note that by the relation in (1), if $q(\boldsymbol{\alpha}) = 0$, then for every cut system $\boldsymbol{\alpha}'$ obtained by handlesliding some curves in $\boldsymbol{\alpha}$, we also have $q(\boldsymbol{\alpha}') = 0$.

Proposition 6.1. *Let $\mathcal{T}$ be a trisection of X with trisection diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Then X admits a spin structure if and only if there exists a quadratic enhancement $q : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ such that*

$$q(\boldsymbol{\alpha}) = q(\boldsymbol{\beta}) = q(\boldsymbol{\gamma}) = 0.$$

Moreover, the set of spin structures is in one-to-one correspondence with such quadratic enhancements.

Proof. Each q corresponds to a spin structure on Σ and therefore a trivialization of TX over its 1-skeleton. In the inside-out handle decomposition, we have $3g+1$ 2-handles. One 2-handle corresponds to the 2-handle of Σ; by assumption the trivialization extends over this handle. The remaining 2-handles are attached along the curves of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ with surface framing. Consequently, the trivialization of TX extends across such a handle if and only if the spin structure, restricted to the attaching circle, is the spin structure on S^1 that extends across the disk. □

6C. *Lutz twists.* A Lutz twist is a method for modifying a 2-plane field ξ along an embedded curve γ.

Fix a metric and orthonormal framing of TH_λ. Let ξ be a 2-plane field on H_λ. Then ξ determines a map $\psi : H_\lambda \to S^2$, by sending the unit normal vector to ξ to its direction in $\mathbb{R}^3$ using the framing of TH_λ. Now let γ be an embedded curve in H_λ. The image $\psi(\gamma)$ is a closed loop S^2, which is contractible and therefore this path is homotopic to a constant path at the north pole. Consequently, we can homotope ξ and assume that $\psi(\gamma)$ is the constant map to the north pole. Geometrically, this means that tangent vector γ' is perpendicular to ξ at every point along γ.

Definition 6.2. A *Lutz twist* of ξ consists of the following operation. Choose a framed neighborhood of γ, with coordinates (r, θ, t). Assume that $\xi = \ker(dt)$. Now, choose smooth functions f, g such that

(1) $f : [0, 2\epsilon] \to \mathbb{R}$ is identically 0 near the endpoints and nonnegative,

(2) $g : [0, 2\epsilon] \to \mathbb{R}$ is increasing, identically -1 near 0, identically 0 near ϵ, and identically 1 near 2ϵ.

Replace ξ with

$$\widehat{\xi} = \ker(g\,dt + f\,d\theta).$$

Exercise 6.3. (1) Show that applying two Lutz twists along γ is homotopic to the identity.

(2) We have described a *left-handed* Lutz twist; i.e., the planes make a single left-handed turn along every diameter of the normal disk to γ. We could alternatively do a *right-handed* Lutz twist by choosing f to be nonpositive. Show that left-handed and right-handed Lutz twists result in homotopic plane fields.

A Lutz twist changes the relative Euler class of the plane field ξ_λ. Let τ denote a fixed trivialization of ξ_λ along Σ and define the relative Euler class $e(\xi_\lambda, \tau) \in H^2(H_\lambda, \Sigma) \cong H_1(H_\lambda)$.

Lemma 6.4. *For a Lutz twist along γ, the relative Euler classes satisfy*

$$e(\xi, \tau) - e(\widehat{\xi}, \tau) = 2[\gamma] \in H_1(H_\lambda).$$

Proof. We can extend τ to a framing that is $\{\partial_r, \partial_\theta\}$ in a tubular neighborhood of γ. This framing must vanish along γ and so $e(\xi_\lambda, \tau) = A + [\gamma]$ for some $A \in H_1(H_\lambda)$. However, after the Lutz twist, we can use the same framing, which still vanishes along γ, except with opposite sign. Thus $e(\widehat{\xi_\lambda}, \tau) = A - [\gamma]$. □

6D. *Action of $H^2(X; \mathbb{Z})$.* The set of Spin$^{\mathbb{C}}$-structures on X is an affine copy of $H^2(X; \mathbb{Z})$. This means that $H^2(X; \mathbb{Z})$ acts freely and transitively on the set of Spin$^{\mathbb{C}}$-structures. That is, given a Spin$^{\mathbb{C}}$-structure $\mathfrak{s}$ and some nonzero $A \in H^2(X; \mathbb{Z})$, there is a distinct Spin$^{\mathbb{C}}$-structure $\mathfrak{s}' = \mathfrak{s} + A$. Furthermore, the first Chern classes satisfy

$$c_1(\mathfrak{s} + A) = c_1(\mathfrak{s}) + 2A.$$

To describe the action of $H^2(X;\mathbb{Z})$ on the set of Spin$^{\mathbb{C}}$-structures, we interpret $H^2(X)$ by dualizing the complex in Proposition 2.7. This is a complex

$$0 \to H_1(\Sigma) \to \bigoplus_\lambda H_1(H_\lambda) \to \bigoplus_\lambda H_1(Z_\lambda) \to 0$$

whose middle homology group is isomorphic to $H^2(X;\mathbb{Z})$. In particular, it consists of triples $(a, b, c) \in \bigoplus_\lambda H_1(H_\lambda)$ such that

$$a - b = 0 \in H_1(Z_1), \quad b - c = 0 \in H_1(Z_2), \quad c - a = 0 \in H_1(Z_3)$$

modulo the image of $H_1(\Sigma)$.

In order to move from almost-complex structures to Spin$^{\mathbb{C}}$-structures, we need the following facts.

Lemma 6.5. *Let X be a closed 4-manifold with a handle decomposition. Let J be an almost-complex structure on the 2-skeleton X_2 and let ξ be the field of J-tangencies along the boundary $Y_2 := \partial X_2$. In particular, ξ is the 2-plane field $TY_2 \cap J(TY_2)$. Then J extends across a 3-handle attached along a 2-sphere $S \subset Y_2$ if and only if $\langle e(\xi), [S]\rangle = 0$.*

Proof. One direction is obvious: if a 3-handle is attached along S then $[S] = 0$ in $H_2(X;\mathbb{Z})$. Thus $\langle e(\xi), [S]\rangle = \langle c_1(J), [S]\rangle = 0$.

Conversely, suppose that $\langle e(\xi), [S]\rangle = 0$. There is a homotopy $\{\xi_t\}$ of 2-plane fields from $\xi = \xi_0$ to ξ_1 such that ξ_1 is the standard, *negative* tight contact structure in a neighborhood of S. There is an almost-complex structure J on $Y \times [0, 1]$ whose restriction to $Y \times \{t\}$ is precisely ξ_t.

Finally, we can attach a 3-handle by turning a Stein 1-handle upside-down. A 1-handle addition is cobordism from $S^0 \times B^3$ to $B^1 \times S^2$; when $S^0 \times B^3$ has the standard tight contact structure, the almost-complex structure can be extended across the cobordism and induces the standard tight (positive) contact structure on $B^1 \times S^2$. Turning this upside-down, the induced contact structure is *negative* on the neighborhood of S^2. Therefore, the almost-complex structure extends across the 3-handle. □

Choose a thickening of the spine and let $\{\widehat{Y}_\lambda\}$ denote its boundary components. If J is an almost-complex structure on the spine, let $\{\widehat{\xi}_\lambda\}$ denote the fields of J-complex tangencies.

Corollary 6.6. *An almost-complex structure J on the spine of the trisection $\mathcal{T}$ of X is a* Spin$^{\mathbb{C}}$*-structure if and only if the plane field $\widehat{\xi}_\lambda$ satisfies $e(\widehat{\xi}_\lambda) = 0$ for all $\lambda = 1, 2, 3$.*

We can now define the action of $H^2(X;\mathbb{Z})$ on a Spin$^{\mathbb{C}}$-structure $\mathfrak{s}$.

(1) We can view $\mathfrak{s}$ as an almost-complex structure on the spine such that the Euler classes $e(\widehat{\xi_\lambda})$ all vanish.

(2) Given $A \in H^2(X;\mathbb{Z})$, represent it by a triple (a,b,c) in $\bigoplus_\lambda H_1(H_\lambda)$. We can represent each element a, b, c, by an embedded collection of curves $\{\gamma_\lambda \subset H_\lambda\}$.

(3) Modify J by a Lutz twist on every component of γ_λ for $\lambda = 1, 2, 3$.

Exercise 6.7. Show that after the Lutz twists, we still have that $e(\widehat{\xi_\lambda}) = 0$ for each $\lambda = 1, 2, 3$.

Consequently, the resulting almost-complex structure also extends across the 3-handles and determines a $\mathrm{Spin}^{\mathbb{C}}$-structure.

Exercise 6.8. Show that modifying a $\mathrm{Spin}^{\mathbb{C}}$-structure $\mathfrak{s}$ by the Lutz twist along $A \sim (a,b,c)$, the first Chern classes satisfy

$$c_1(\mathfrak{s}+A) = c_1(\mathfrak{s}) + 2A.$$

[Hint: how does the Lutz twist affect the 1-complex C_J from Proposition 5.15?]

References

[1] D. Auroux, S. K. Donaldson, and L. Katzarkov, "Singular Lefschetz pencils", *Geom. Topol.* **9** (2005), 1043–1114. MR Zbl

[2] R. I. Baykur, "Existence of broken Lefschetz fibrations", *Int. Math. Res. Not.* **2008** (2008), art. id. rnn101. MR Zbl

[3] G. E. Bredon, *Sheaf theory*, vol. 170, 2nd ed., Graduate Texts in Mathematics, Springer-Verlag, New York, 1997. MR

[4] S. K. Donaldson, "Lefschetz pencils on symplectic manifolds", *J. Differential Geom.* **53**:2 (1999), 205–236. MR Zbl

[5] P. Feller, M. Klug, T. Schirmer, and D. Zemke, "Calculating the homology and intersection form of a 4-manifold from a trisection diagram", *Proc. Natl. Acad. Sci. USA* **115**:43 (2018), 10869–10874. MR Zbl

[6] V. Florens and D. Moussard, "Torsions and intersection forms of 4-manifolds from trisection diagrams", *Canad. J. Math.* **74**:2 (2022), 527–549. MR

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Gauge Theory and Low-Dimensional Topology: Progress and Interaction

This volume is a proceedings of the 2020 BIRS workshop *Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4*. This was the 6th iteration of a recurring workshop held in Banff. Regrettably, the workshop was not held onsite but was instead an online (Zoom) gathering as a result of the Covid-19 pandemic. However, one benefit of the online format was that the participant list could be expanded beyond the usual strict limit of 42 individuals. It seemed to be also fitting, given the altered circumstances and larger than usual list of participants, to take the opportunity to put together a conference proceedings.

The result is this volume, which features papers showcasing research from participants at the 6th (or earlier) *Interactions* workshops. As the title suggests, the emphasis is on research in gauge theory, contact and symplectic topology, and in low-dimensional topology. The volume contains 16 refereed papers, and it is representative of the many excellent talks and fascinating results presented at the Interactions workshops over the years since its inception in 2007.

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