THE OPEN BOOK SERIES 5

Gauge Theory and Low-Dimensional Topology: Progress and Interaction

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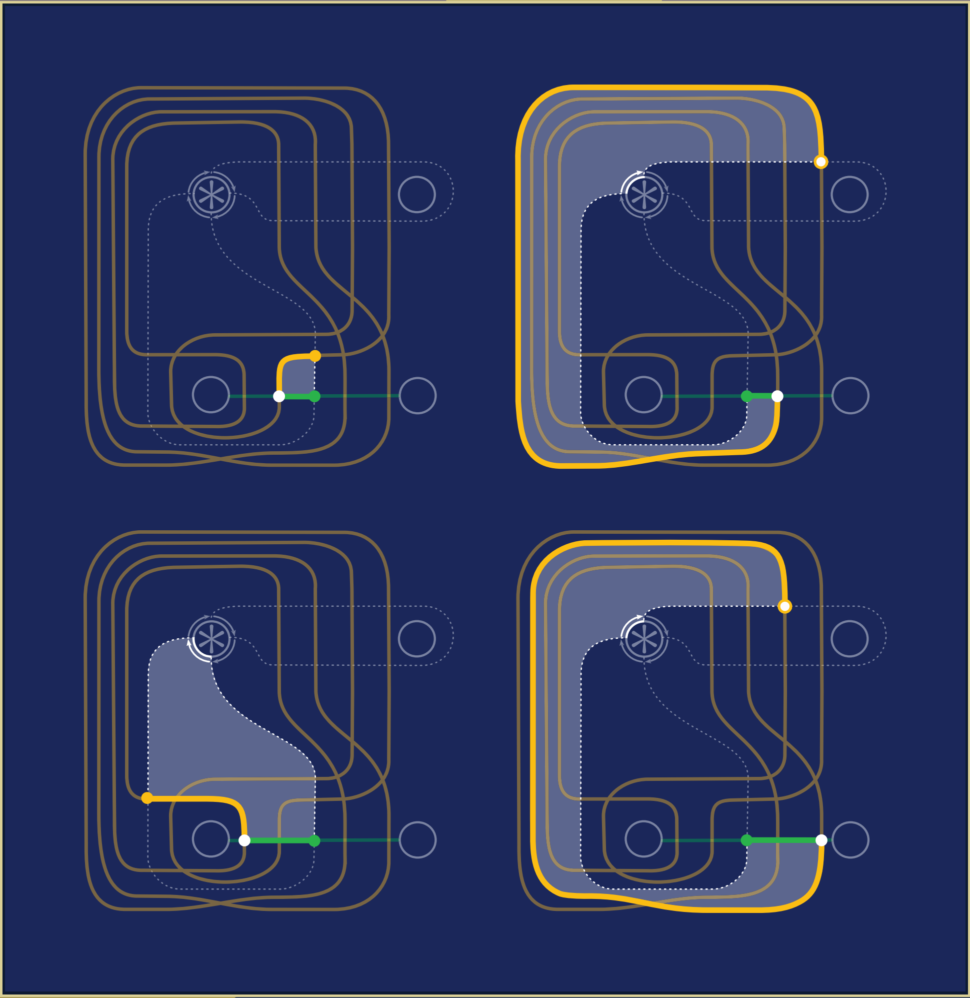
Robert Lipshitz



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A remark on quantum Hochschild homology

Robert Lipshitz

We observe that quantum Hochschild homology is a composition of two familiar operations, and give a short proof that it gives an invariant of annular links, in some generality. Much of this is implicit in Beliakova, Putyra and Wehrli's work.

Beliakova, Putyra and Wehrli studied various kinds of traces, in relation to annular Khovanov homology [2]. In particular, to a graded algebra and a graded bimodule over it, they associate a quantum Hochschild homology of the algebra with coefficients in the bimodule, and use this to obtain a deformation of the annular Khovanov homology of a link. A spectral refinement of the resulting invariant was recently given by Akhmechet, Krushkal and Willis [1].

Before giving our reformulation, we recall Beliakova, Putyra, and Wehrli's definition.

Definition 1 [2, Section 3.8.5]. Let A be a graded ring, M a chain complex of graded A-bimodules (so M is bigraded), and $q \in A$ an invertible central element with grading 0. The *quantum Hochschild complex* of A with coefficients in M and parameter q has chain groups $qCH_n(A; M) = M \otimes_{\mathbb{Z}} A^{\otimes_{\mathbb{Z}} n}$ and differential

$$\begin{aligned}\partial(m\otimes a_1\otimes\cdots\otimes a_n) = ma_1\otimes a_2\otimes\cdots\otimes a_n + \sum_{i=1}^{n-1}(-1)^i m\otimes a_1\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n \\ +(-1)^n q^{-|a_n|} a_n m\otimes a_1\otimes\cdots\otimes a_{n-1}.\end{aligned}$$

The homology of this complex is the *quantum Hochschild homology* $qHH_\bullet(A; M)$ of A with coefficients in M and parameter q.

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The goal of this note is to reformulate this operation and deduce that it often leads to annular link invariants. The data of A and q specifies a ring homomorphism $f_q : A \to A$ defined on homogeneous elements a of A by

$$f_q(a) = q^{-|a|}a,$$

where $|a|$ denotes the grading of a. We can twist the left action on the A-bimodule M by f_q to obtain a new bimodule ${}_{f_q}M$ which is equal to M as a right A-module and has left action given by the composition $A \otimes_{\mathbb{Z}} {}_{f_q}M \xrightarrow{f_q \otimes \mathbb{I}} A \otimes_{\mathbb{Z}} M \xrightarrow{m} M = {}_{f_q}M$. This operation is a special case of tensor product:

$${}_{f_q}M \cong {}_{f_q}A \otimes_A M$$

(compare [2, Section 3.8.3]).

Our first observation is:

Proposition 2. *The quantum Hochschild homology of A with coefficients in M is isomorphic to the ordinary Hochschild homology of A with coefficients in ${}_{f_q}M$.*

Proof. This is immediate from the definitions. □

Call a chain complex of graded A-bimodules M *weakly central* if for any graded A-bimodule N there is a quasi-isomorphism $M \otimes_A^L N \simeq N \otimes_A^L M$.

Lemma 3. *The bimodule ${}_{f_q}A$ is weakly central.*

Proof. The isomorphism $M \otimes_A {}_{f_q}A \to {}_{f_q}A \otimes M$ sends m to $q^{-|m|}m$. □

We turn next to annular link invariants. Consider the category Tan with one object for each even integer and $\mathrm{Hom}(2m, 2n)$ given by the set of isotopy classes of $(2m, 2n)$-tangles (embedded in $D^2 \times [0, 1]$). Given a (graded) algebra A, a *very weak action* of Tan on the category of A-modules is a choice of quasi-isomorphism class of chain complex of (graded) A-bimodules $C(T)$ for each $T \in \mathrm{Hom}(2m, 2n)$ so that $C(T_2 \circ T_1)$ is quasi-isomorphic to $C(T_2) \otimes_A^L C(T_1)$. For example, if we take A to be the direct sum of the Khovanov arc algebras [4] then Khovanov defined a very weak action of Tan on ${}_A\mathsf{Mod}$, and if we define A to be the direct sum of the Chen–Khovanov algebras [3] then Chen and Khovanov defined a very weak action of Tan on ${}_A\mathsf{Mod}$. (In fact, in both cases, they did more; cf. Remark 6.)

Any $(2n, 2n)$-tangle $T \subset D^2 \times [0, 1]$ has an *annular closure* in $D^2 \times S^1$.

Proposition 4. *Fix a very weak action of* Tan *on* ${}_A\mathsf{Mod}$ *and fix a weakly central A-bimodule P. Then, for any $(2n, 2n)$-tangle T, the isomorphism class of $HH_*(A; C(T) \otimes_A^L P)$ is an invariant of the annular closure of T.*

(Compare [2, Corollary 3.23].)

Proof. This is immediate from the definitions and the trace property of Hochschild homology, i.e., that, given A-bimodules M and N,

$$HH_*(A; M \otimes_A^L N) \cong HH_*(A; N \otimes_A^L M). \qquad \square$$

The following is part of Beliakova, Putyra and Wehrli's Theorem B [2]:

Corollary 5. *Up to isomorphism, the quantum Hochschild homology of the Chen–Khovanov bimodule associated to a* $(2n, 2n)$*-tangle* T *is an invariant of the annular closure of* T.

Proof. This is immediate from Lemma 3, Proposition 4, and the fact that the Chen–Khovanov bimodules induce a very weak action of Tan [3]. $\square$

Remark 6. To keep this note short, we have not discussed functoriality of these annular link invariants under annular cobordisms. To do so, one replaces Tan by an appropriate 2-category of tangles and weak centrality by a notion keeping track of the isomorphisms. See Beliakova, Putyra and Wehrli [2] for further discussion.

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Gauge Theory and Low-Dimensional Topology: Progress and Interaction

This volume is a proceedings of the 2020 BIRS workshop *Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4*. This was the 6th iteration of a recurring workshop held in Banff. Regrettably, the workshop was not held onsite but was instead an online (Zoom) gathering as a result of the Covid-19 pandemic. However, one benefit of the online format was that the participant list could be expanded beyond the usual strict limit of 42 individuals. It seemed to be also fitting, given the altered circumstances and larger than usual list of participants, to take the opportunity to put together a conference proceedings.

The result is this volume, which features papers showcasing research from participants at the 6th (or earlier) *Interactions* workshops. As the title suggests, the emphasis is on research in gauge theory, contact and symplectic topology, and in low-dimensional topology. The volume contains 16 refereed papers, and it is representative of the many excellent talks and fascinating results presented at the Interactions workshops over the years since its inception in 2007.

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