THE OPEN BOOK SERIES 5

Gauge Theory and Low-Dimensional Topology: Progress and Interaction

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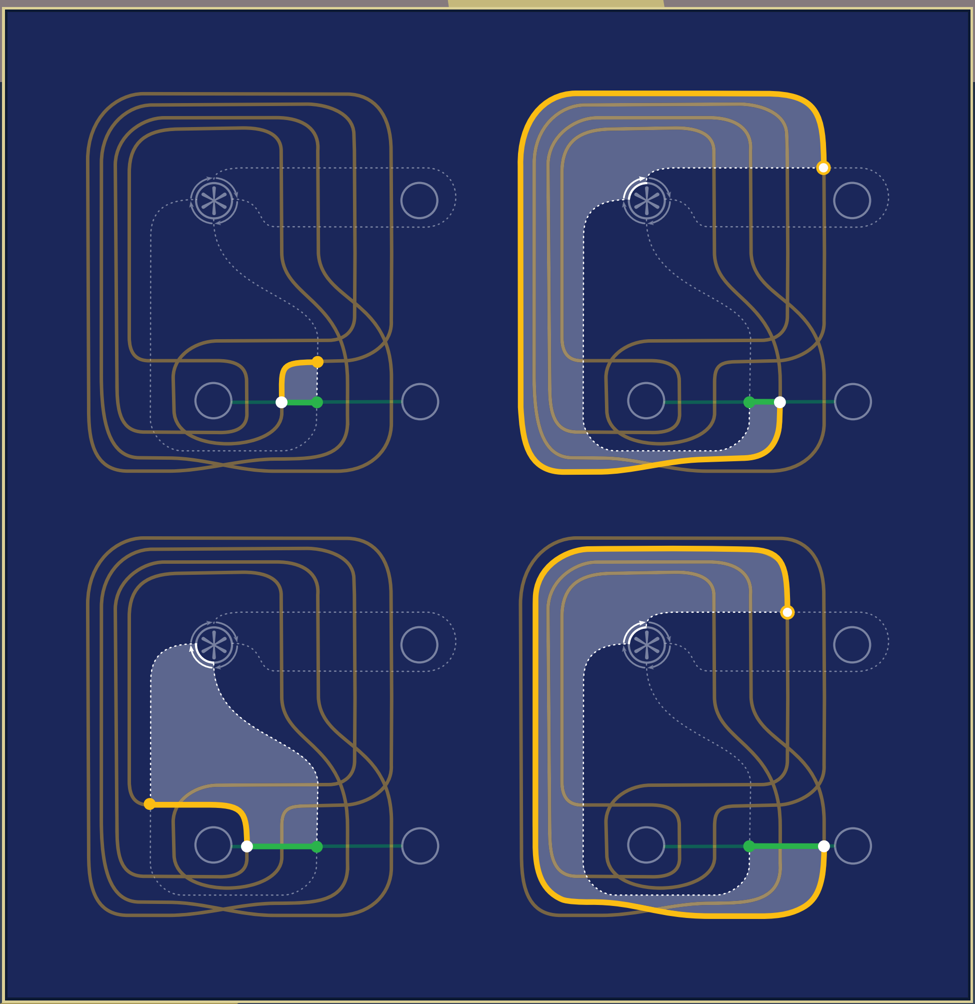
Olga Plamenevskaya



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THE OPEN BOOK SERIES 5 (2022)
Gauge theory and low-dimensional topology: progress and interaction
https://doi.org/10.2140/obs.2022.5.269

On uniqueness of symplectic fillings of links of some surface singularities

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We consider the canonical contact structures on links of rational surface singularities with reduced fundamental cycle. These singularities can be characterized by their resolution graphs: the graph is a tree, and the weight of each vertex is no greater than its negative valency. The contact links are given by the boundaries of the corresponding plumbings. In a joint work with L. Starkston, we have previously shown that if the weight of each vertex in the graph is at most -5, the contact structure has a unique symplectic filling (up to symplectic deformation and blow-up); the proof was based on a symplectic analog of de Jong and van Straten's description of smoothings of these singularities. Here, we give a short self-contained proof of the uniqueness of fillings, via analysis of positive monodromy factorizations for planar open books supporting these contact structures.

1. Introduction

In this note, we consider links of complex surface singularities, equipped with their canonical contact structures. Let $X \subset \mathbb{C}^N$ be a singular complex surface with an isolated singularity at the origin. For small $r > 0$, the intersection $Y = X \cap S_r^{2N-1}$ with the sphere $S_r^{2N-1} = \{|z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 = r\}$ is a smooth 3-manifold called the link of the singularity $(X, 0)$. The induced contact structure ξ on Y is the distribution of complex tangencies to Y, and is referred to as the canonical or Milnor fillable contact structure on the link. The contact manifold (Y, ξ), which we will call the contact link, is independent of the choice of r, up to contactomorphism.

Our main result, Theorem 1, states that for a certain class of singularities, the canonical contact structure on the link has a unique symplectic filling (up to blow-up and symplectic deformation). This theorem was originally proved in [23]; here, we

Partially supported by NSF grant DMS-1906260.
MSC2020: 57K33, 57K43.
Keywords: links of singularities, symplectic fillings.

will give a new proof, from a different perspective. The sufficient condition will be stated in terms of the dual resolution graph of the singularity. Recall that for a normal surface singularity $(X, 0)$, this graph is defined as follows. Consider a resolution of the singularity, i.e., a proper birational morphism $\pi : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth. We can assume that the exceptional divisor $\pi^{-1}(0)$ has normal crossings. This means that $\pi^{-1}(0) = \bigcup_{v \in G} E_v$, where the irreducible components E_v are smooth complex curves that intersect transversally at double points only. The (dual) resolution graph encodes the topology of the resolution: the vertices $E \in G$ correspond to the exceptional curves and are weighted by the self-intersection $E \cdot E$ of the corresponding curve, while the edges of G record intersections of different irreducible components. Up to contactomorphism, the link of the singularity with its canonical contact structure can be reconstructed from the graph G and the data of self-intersections and genera of exceptional curves, as the boundary of the plumbing of symplectic disk bundles over surfaces according to G.

In this paper, we only work with *rational* singularities; then G is always a tree, and each exceptional curve has genus 0. The following assumption plays the key role in this paper: for every exceptional curve E, we require that the self-intersection $E \cdot E$ and the valency $a(E)$ of the corresponding vertex in G satisfy the inequality

$$a(E) \leq -E \cdot E. \tag{1}$$

Plumbing graphs with this property are sometimes referred to as “graphs with no bad vertices” in low-dimensional topology; a bad vertex, by definition, has valency greater than its negative weight. (The boundary of the corresponding plumbing is a Heegaard Floer L-space [21].) If the dual resolution graph is a tree with the above property, $(X, 0)$ is a *rational singularity with reduced fundamental cycle*. In the literature, singularities of this type are also known as *minimal* singularities [13].

We will give a direct new proof of the following theorem, first established in [23]:

Theorem 1 [23]. *Suppose that* $(X, 0)$ *is a rational surface singularity with reduced fundamental cycle, and assume additionally that every exceptional curve in its resolution has self-intersection at most* -5. *Then the contact link* (Y, ξ) *of* $(X, 0)$ *has a unique minimal weak symplectic filling, which is Stein.*

In the special case where the resolution graph is star-shaped with three legs, this fact is proved in [5, Theorem 2.7, Remark 2.8], by a different method.

Symplectic and Stein fillings of links of surface singularities are of interest because of the connection to algebrogeometric questions, namely to the smoothings of the singularity. The Milnor fiber of each smoothing of $(X, 0)$ gives a Stein filling of its link (Y, ξ); another Stein filling can be provided by the minimal resolution of the singularity, after deforming the symplectic form. (Rational surface singularities are always smoothable, with an "Artin smoothing component" whose Milnor fiber gives the same Stein filling as the resolution. In particular, for the singularities in Theorem 1, the filling can be viewed as the resolution or as the Milnor fiber for the Artin smoothing.) An important question is whether *all* Stein fillings of a given surface singularity arise in this way [16]. Although this correspondence breaks down when the singularity is sufficiently complicated [1; 2; 23], the answer is positive for certain simple classes of singularities. Namely, all Stein fillings come from Milnor fibers or the minimal resolution for (S^3, ξ_{std}) [6], for links of simple and simple elliptic singularities [19; 20], for lens spaces (links of cyclic quotient singularities) [15; 17], and in general for quotient singularities [4; 22]. Theorem 1 significantly extends this list.

Our interest in the question of Theorem 1 was motivated by a (very special case of) a conjecture of Kollár on deformations of rational surface singularities [14]. The conjecture asserts that every exceptional curve has self-intersection at most -5 in the resolution of a rational singularity, then the base space of a semiuniversal deformation of this singularity has a unique component. For the case of rational singularities with reduced fundamental cycle, the conjecture was established by de Jong and van Straten [12]; in particular, it follows that under the hypotheses of Theorem 1, the singularity has a unique smoothing component. Our Theorem 1 gives the symplectic analog of this statement.

The proof we gave in [23] comes as a side product of the theory developed in that article, where we describe symplectic fillings of the corresponding class of singularities via a symplectic analog of de Jong and van Straten's construction. Fillings are encoded by certain configurations of symplectic disks in $\mathbb{C}^2$; we were then able to apply a lemma of de Jong and van Straten to establish "combinatorial uniqueness" of the corresponding disk arrangements, and then finish the argument via topological considerations.

In this paper, we will instead give a direct proof of Theorem 1, working with open book factorizations. As a corollary, we get a symplectic proof that all smoothings of the corresponding singularity are diffeomorphic. We will assume that the reader is familiar with the basics of open book decompositions for contact 3-manifolds; see [7] for a survey. Under the hypotheses of the theorem, the canonical contact structures on the links of singularities admit planar open books. (This follows from a construction of Gay and Mark [10]; see Section 2. Planarity was also a key ingredient that allowed us to build an analog of the de Jong–van Straten theory in [23].) In the planar case, symplectic fillings can be studied via theorems of Wendl and Niederkruger [18; 25]: every minimal symplectic filling is symplectic deformation equivalent to a Lefschetz fibration over a disk with the same planar fiber P. The classification of fillings then reduces to enumerating positive factorizations of the monodromy of the given open book.

In general, finding all positive factorizations of the monodromy is a daunting task, even in the planar case. The question is much easier if one only seeks to determine the image of the Dehn twists of the factorization in the *abelianization* of the mapping class group of the page. This is equivalent to finding the *homology classes* of the curves about which the Dehn twists are performed; we also disregard the order of the twists. This easier question can be studied by counting how many times the Dehn twists enclose each hole in the planar page, and how many times they enclose each pair of holes. (The planar page is a disk with holes, and we say that a simple closed curve in a disk with holes encloses a hole if the curve separates the hole from the outer boundary component of the disk.) If P is planar, any two factorizations of the boundary-fixing monodromy $\phi : P \to P$ can be connected by a sequence of lantern relations, and it follows that the number of Dehn twists enclosing a given hole (or a given pair of holes) is *independent* of the factorization of ϕ. Thus, we can introduce the multiplicity $m(v)$ of a hole v with respect to the monodromy ϕ, and similarly the joint multiplicity $m(v_1, v_2)$ of a pair of holes v_1, v_2. Knowing these multiplicities, one can attempt to describe possible other factorizations of ϕ, by examining the combinatorics of how the Dehn twists can enclose the holes. This method was introduced in [24] to classify fillings of certain lens spaces.

Once we understand the Dehn twists in the factorization at the level of homology classes of the curves, additional information is needed to find the isotopy classes of the curves. In the case at hand, this step is possible because the given monodromy admits a positive factorization into Dehn twists about *disjoint* curves.

In [23], the combinatorial part of the proof was based on the description of fillings via a symplectic analog of the de Jong–van Straten construction. We then used the result of [12] asserting uniqueness of a combinatorial solution for a certain curve arrangement problem. For the second part of the proof, we gave a direct mapping class argument. The purpose of the note is to give a direct multiplicity-count argument for the first part; see Lemma 2. For the second part, we essentially repeat the reasoning from [23]; this argument, based on right-veering properties, is given in Lemma 3 for completeness.

It is interesting to note that our direct argument for the combinatorics of Dehn twists follows the strategy of [12, Theorem 6.23]: we translate their proof from the incidence matrices to multiplicities of holes, and provide some extra details where needed.

2. Proof of Theorem 1

To begin, we recall the construction of the open books supporting the canonical contact structures for the class of singularities that satisfy (1) [10, Theorem 1.1]. Starting with the plumbing graph G, the construction given by Gay and Mark

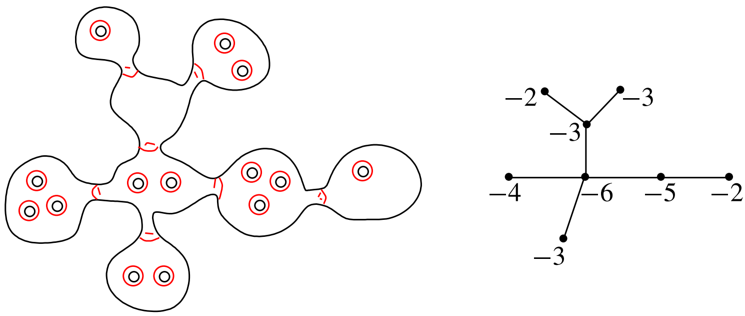

Figure 1. The Gay–Mark open book supporting the canonical contact structure on the link of the singularity with dual resolution graph shown on the right. The page of the open book has genus 0 and is constructed from the spheres with holes corresponding to the vertices of the graph. Each sphere is connected to the other spheres by necks that correspond to the edges; the total number of holes and necks for each sphere equals the negative self-intersection of the vertex. The monodromy is the product of the positive Dehn twists about the boundaries of the holes and the meridians of the necks; these curves are shown in red.

produces a planar Lefschetz fibration compatible with the symplectic resolution of a rational singularity $(X, 0)$ with reduced fundamental cycle. (The symplectic structure on the plumbing can be deformed to the corresponding Stein structure.) We describe the induced planar open book on the link (Y, ξ). To construct the page of the Gay–Mark open book, take a sphere S_E for each vertex $E \in G$ and cut $-a(E) - E \cdot E \geq 0$ disks out of this sphere. (As before, $a(E)$ is the valency of the vertex E; the number of disks is nonnegative by (1).) Next, make a connected sum of these spheres with holes by adding a connected sum neck for each edge of G. For a sphere S_E corresponding to the vertex E, the number of necks equals the number of edges adjacent to E, i.e., its valency $a(E)$. The resulting surface S has genus 0 because G is a tree. See Figure 1 for an example. The open book monodromy is given by the product of positive Dehn twists around each of the holes and around the meridians of the necks. We will call this product the *standard* factorization of the Gay–Mark monodromy.

To examine positive factorizations of this open book, we first we put the resolution graph in the following special form, as in [12]. We choose the vertices $E_1, E_2, \ldots E_k$ and partition the remaining vertices into subsets $R_2, \ldots, R_k$ as shown in Figure 2, so that for any vertex $F \in R_j$, the length $l(E_j, F)$ of the chain from F to E_j satisfies $l(E_j, F) \leq j-1$. Here, the length of chain means the number of edges; for example, the statement means that every vertex in R_2 is directly connected to E_2 by a single edge. This can always be achieved via the following procedure. We choose E_1 to be the endpoint of a longest chain C in the graph; then E_1 is necessarily a leaf

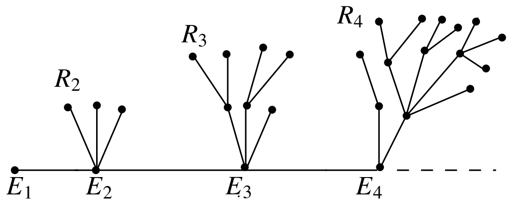

Figure 2. A graph with conveniently arranged vertices: after removing $E_1, E_2, \ldots$, the remaining vertices are partitioned into subsets $R_2, R_3, \ldots$. Every vertex in the set R_j connects to the vertex E_j by a chain with fewer than j edges.

vertex of G. Let E_2 be its adjacent vertex, and let E_3 be the vertex on the chain C that is adjacent to E_2. Removing E_2 from G, we get one connected component consisting of E_1, another that contains E_3, and possibly a number of other vertices in the remaining components. Let R_2 be the set of these remaining vertices. Each vertex $F \in R_2$ must be a leaf vertex connected to E_2 (otherwise we can build a chain longer than C by going to R_2 instead of E_1); thus the condition $l(E_2, F) \leq 1$ is satisfied. If E_3 is a leaf vertex, it can be included in R_2, and the procedure is over. If E_3 is not a leaf vertex, and every other vertex in $G \setminus (E_1 \cup R_2)$ is connected to E_3 by a path of at most 2 edges, then all remaining vertices can be included in E_3. Otherwise we consider the vertex E_4 preceding E_3 on the path C. Removing E_3, we set aside the two components of $G \setminus E_3$ that contain E_1 resp. E_4 and let E_3 be the set of all remaining vertices. Again, since no chain in the graph G can be longer than C, every vertex $F \in R_3$ must be connected to E_3 by a path of no more than 2 edges, satisfying $l(F, E_3) \leq 2$. We continue this process to define E_5, R_4, etc., stopping when we reach k such that all the remaining vertices in G can be connected to E_k by a path no longer than $(k-1)$ edges, and thus placed in R_k. See Figure 2. We will say that vertices of the graph are *conveniently arranged* if they are partitioned into subsets as above.

For the resolution graph G with conveniently arranged vertices, we build the Gay–Mark open book as in Figure 1. We will identify the planar page of this open book with a disk with holes, so that the outer boundary of the disk corresponds to the boundary of one of the holes associated to the vertex E_1. This identification and the choice of the outer boundary component of the disk will be fixed from now on, for the statement and the proof of Lemma 2. In the standard factorization of the Gay–Mark monodromy, there is a sequence of the Dehn twists $D_1, D_2, \ldots, D_k$ around a nested collection of curves $\gamma_1, \ldots, \gamma_k$, such that

(i) D_1 is the twist around the outer boundary component γ_1 of the page, and therefore D_1 encloses all the holes;

(ii) D_j is the twist around the neck γ_j between E_{j-1} and E_j for $j = 2, \ldots, k$, so that D_j encloses all the holes corresponding to $E_j, R_j, E_{j+1}, R_{j+1}, \ldots, E_k, R_k$.

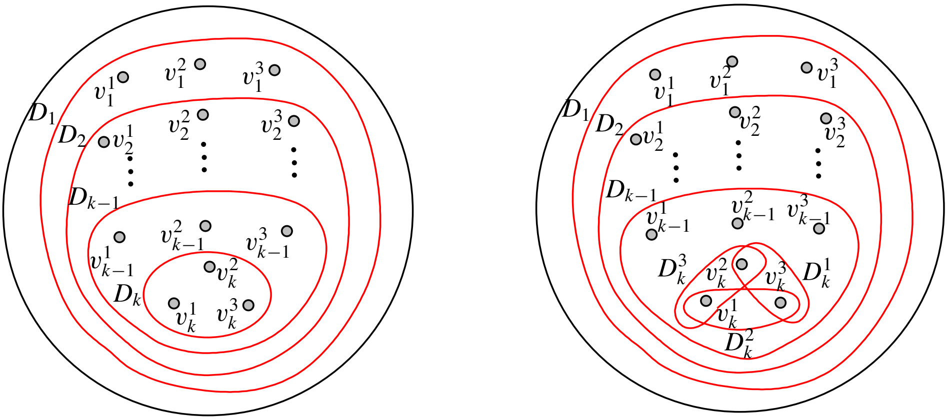

Figure 3. Property F1 (left) and Property F2 (right) for a chosen subset of holes $v_1^1, v_1^2, v_1^3, \dots v_k^1, v_k^2, v_k^3$ and the Dehn twists that enclose them in a factorization. Note that we only require that the *homology classes* of the curves are as schematically shown; isotopy classes may look different from the picture.

The curves $\{\gamma_j\}_{j=1}^{k}$ cut the disk page into annular domains $\{V_j\}_{j=1}^{k-1}$ such that V_j is bounded by γ_j, γ_{j+1} for $j = 1, \dots, k-1$, and a disk V_k bounded by γ_k. It follows that the joint multiplicity of any two holes from V_j is at least j. In the Gay–Mark construction, the holes from V_j are associated to vertices E_j, R_j of the graph G.

We now make a choice of a certain ordered subset of holes in the page. Because the valency of E_1 is one, and the self-intersection $E_1 \cdot E_1$ is at most -5, the corresponding annular domain V_1 contains $-E_1 \cdot E_1 - 2 \geq 3$ holes. We label three of these holes as v_1^1, v_1^2, v_1^3. Next, again because self-intersections of vertices are at most -5, we can pick three holes v_2^1, v_2^2, v_2^3 in the domain V_2. We require that v_2^1, v_2^2, v_2^3 satisfy an additional condition $m(v_2^r, v_2^s) = 2$: if R_2 is nonempty, we make sure that no two holes are in the same branch of R_2 to avoid higher joint multiplicities. For $j = 3, \dots k$, we proceed to pick v_j^1, v_j^2, v_j^3 in the domain V_j, choosing different branches of R_j if R_j is nonempty, so that $m(v_j^r, v_j^s) = j$ for any pair of indices $r, s = 1, 2, 3$. By construction, we have

$$m(v_i^r, v_j^s) = \min(i, j) \tag{2}$$

for any two chosen holes v_i^r, v_j^s.

The choice of the holes $v_1^1, v_1^2, v_1^3, \dots, v_k^1, v_k^2, v_k^3$ will be fixed. By construction, the standard factorization of the Gay–Mark open book satisfies the following:

Property F1. The factorization includes Dehn twists $D_1, \dots, D_k$ such that

- the Dehn twist D_j encloses the holes v_i^1, v_i^2, v_i^3 for all $i \geq j$.

This is illustrated in Figure 3. Note that we have only listed the Dehn twists that correspond to the edges of the chain $E_1, \dots, E_k$. The Dehn twists that correspond to edges the sets R_j are not listed above; for each R_j, the corresponding Dehn twists enclose holes in the domain V_j. While these Dehn twists may be nested, the hypothesis that chains of edges in R_j connecting to E_j have length at most $j-1$ gives a bound on a number of twists enclosing any hole w in V_j in the standard factorization: there are at most j nested Dehn twists inside V_j (including the Dehn twist around the boundary of w), in addition to $D_1, \dots, D_j$. This means that in ϕ, the multiplicity of the hole w is at most $2j$.

For an inductive step in our proof, we will consider graphs G satisfying a weaker hypothesis: when the vertices of G are conveniently arranged, we require that the self-intersection $E_k \cdot E_k$ be less than or equal to -4, whereas $E \cdot E \leq -5$ for every other vertex $E \in G$. Note that in the case where $E_k \cdot E_k = -4$, we can still choose a labeled collection of holes $v_1^1, v_1^2, v_1^3, \dots, v_k^1, v_k^2, v_k^3$ as above. Because $E_k \cdot E_k = -4$, we can use the lantern relation to replace the product of D_k and three other Dehn twists (around holes or necks in the sphere corresponding to E_k) by three Dehn twists D_k^1, D_k^2, D_k^3. For this new factorization, we have:

Property F2. The factorization includes Dehn twists $D_1, \dots, D_{k-1}, D_k^1, D_k^2, D_k^3$ such that:

- D_j encloses the holes v_i^1, v_i^2, v_i^3 for all $i \geq j$, for each for each $j = 1, \dots, k-1$.
- D_k^1 encloses v_k^2, v_k^3 but not v_k^1.
- D_k^2 encloses v_k^1, v_k^3 but not v_k^2.
- D_k^3 encloses v_k^1, v_k^2 but not v_k^3.

Under the hypotheses of the following lemma, we will show that an *arbitrary* factorization of the monodromy of the Gay–Mark open book must have Property F1 or Property F2. This will be a step in the argument showing that any monodromy factorization must be standard if the self-intersection of each vertex of G is at most -5.

Lemma 2. *Suppose that the vertices of the graph G are conveniently arranged, with distinguished vertices $E_1, E_2, \dots, E_k$, and the corresponding sets $R_2, \dots, R_k$. Assume that the self-intersections of all vertices in the graph are at most -5, except possibly E_k, which has self-intersection at most -4. Suppose also that there is a collection of holes $\{v_j^1, v_j^2, v_j^3\}_{j=1}^k$, chosen as above. Then*:

(1) *If $E_k \cdot E_k = -4$, then* ***every*** *monodromy factorization includes*
   - (a) *the Dehn twist around the outer boundary component of the page*,
   - (b) *a collection of Dehn twists* (*containing the twist around the outer boundary*) *that has Property $F1$ or Property $F2$.*

(2) *If $E_k \cdot E_k \leq -5$, then every monodromy factorization is homologically equivalent to the standard one.*

Proof. We will build an inductive argument, with double induction on k and the number of vertices in the graph.

The base of induction is given by $k = 1$ and $k = 2$. The case of $k = 1$ corresponds to lens spaces and was treated in [24]. (This is an easy exercise on computing multiplicities.) The case $k = 2$ is straightforward but more tedious; we check it after explaining the induction step.

For now, assume that $k \geq 2$, and that the statement of the lemma is established for all graphs where the chain of distinguished vertices $E_1, E_2, \ldots$ has length at most k. Consider the graph G with conveniently arranged vertices with a longer chain $E_1, \ldots, E_{k+1}$, and the remaining vertices partitioned into the sets $R_2, \ldots, R_{k+1}$. Take a new graph G', obtained from G by removing all vertices of R_{k+1} and E_{k+1}, increasing by 1 the self-intersection of E_k, and keeping the same self-intersection for all other vertices. The Gay–Mark open books (P, ϕ) and (P', ϕ'), representing, respectively, the contact links of singularities with graphs G and G', are related as follows. To obtain the page P' from P, we cap off all the holes in P associated to E_{k+1} and R_{k+1}; the boundary-fixing diffeomorphism ϕ then induces ϕ'. (Note that $E_k \cdot E_k$ increases by 1 in the graph G' since the corresponding subsurface in the page has fewer boundary components now: removing E_{k+1}, R_{k+1} is the same as pinching off the neck connecting the E_k-sphere to the E_{k+1}-sphere.)

Fix an arbitrary factorization Φ of the Gay–Mark open book for G. When the holes in P are capped off to obtain P', Φ induces the factorization Φ' for the open book (P', ϕ'). Since by assumption $E_k \cdot E_k \leq -5$ in G, the self-intersection of the corresponding vertex is at most -4 in G'. The induction hypothesis applies to the graph G', and therefore, the conclusion of the lemma holds for the factorization Φ' of the monodromy ϕ'. In particular, there is a Dehn twist $T' = T'_1$ around the outer boundary component of P' in the factorization Φ', and moreover, there are Dehn twists $T'_2, T'_3, \ldots, T'_{k-1}$, and T'_k (or $T'_{k,1}, T'_{k,2}, T'_{k,3}$) that have Property F1 (or, respectively, Property F2). These Dehn twists must be induced by the corresponding Dehn twists $T = T_1, T_2, \ldots, T_{k-1}$ and T_k (or $T_{k,1}, T_{k,2}, T_{k,3}$) in the factorization Φ of $\phi : P \to P$.

To show that Φ has a Dehn twist around the outer boundary component of P, we need to check that T encloses all the holes corresponding to E_{k+1} and R_{k+1} that were removed from P to obtain P'; we already know that T encloses all the holes that P inherits from P'. For the sake of contradiction, let $v = v^s_{k+1}$ be a hole associated to E_{k+1} or R_{k+1}, and suppose that it is not enclosed by T. First assume that the factorization Φ' has property F1, so that the factorization Φ includes Dehn twists $T = T_1, T_2, \ldots, T_{k-1}, T_k$ as above. We examine the multiplicities of the selected holes. Because these multiplicities can be computed from the standard factorization of $\phi : P \to P$, by (2) we know that the joint multiplicity $m(v, v^s_j) = j$ for $i = 1, 2, 3$ and $j = 1, \ldots, k$.

If $T = T_1$ does not enclose v, the holes v and v_j^s are enclosed together by at most $j-1$ of the Dehn twists $T_1, T_2, \dots, T_{k-1}, T_k$. Even if v is enclosed by all of $T_2, \dots, T_{k-1}, T_k$, it follows that there must be an additional Dehn twist τ_j^s enclosing both v and v_j^s. Observe that the Dehn twists τ_j^i must be all distinct (that is, $\tau_j^s = \tau_i^r$ only if $i = j, r = s$): the joint multiplicity $m(v_j^s, v_i^r)$ of any two distinct holes v_j^s, v_i^r is already realized by $T_1, T_2, \dots, T_{k-1}, T_k$, so no additional Dehn twist can enclose them both. It follows that the hole v must be enclosed by at least $3k$ distinct Dehn twists $\tau_1^1, \tau_1^2, \tau_1^3, \dots, \tau_k^1, \tau_k^2, \tau_k^3$ in the factorization Φ, in addition to $k-1$ Dehn twists $T_2, \dots, T_{k-1}, T_k$. It is not hard to see that if v is not enclosed by some of the twists among $T_2, \dots, T_{k-1}, T_k$, then each missing twist will need to be replaced by several individual twists to achieve $m(v, v_j^s) = j$. It follows that v is enclosed by at least $3k + k - 1 = 4k - 1$ twists. To obtain a contradiction, we compute the multiplicity $m(v)$ in the monodromy ϕ. The hole v is associated to E_{k+1} or to some vertex E in R_{k+1}; in the standard factorization of ϕ, it is enclosed by the small twist around the hole v, by the outer boundary twist, as well as by the Dehn twists corresponding to the edges in the chain in G from E_1 to E_{k+1} and then the chain from E_{k+1} to E, if the latter chain is present. Since $E \in R_{k+1}$, and by construction the length of the chain from E_{k+1} to any vertex in R_{k+1} is at most k, we see that $m(v) \leq 2k + 2$. This is a contradiction since $2k + 2 < 4k - 1$ for $k \geq 2$.

Similar reasoning leads to the same conclusion in the case where the factorization Φ' has Property F2 instead of Property F1. As above, we see that if v is not enclosed by $T = T_1$, there must be at least $3(k-1)$ distinct Dehn twists $\tau_1^1, \tau_1^2, \tau_1^3, \dots, \tau_{k-1}^1, \tau_{k-1}^2, \tau_{k-1}^3$ in the factorization Φ to achieve $m(v, v_j^s) = j$ for $j = 1, \dots, k-1,\ \ s = 1, 2, 3$. The holes v_k^1, v_k^2, v_k^3 need a bit more attention. Indeed, when v is not enclosed by $T = T_1$, there are at most $k-2$ twists among $T_2, \dots, T_{k-1}$ enclosing v and v_k^s for each $s = 1, 2, 3$. The joint multiplicity $m(v, v_k^s) = k$ can be achieved if v is enclosed by all three twists $T_{k,1}, T_{k,2}, T_{k,3}$, in addition to all of $T_2, \dots, T_{k-1}$ and $\tau_1^1, \tau_1^2, \tau_1^3, \dots, \tau_{k-1}^1, \tau_{k-1}^2, \tau_{k-1}^3$. This would give $m(v) \geq 3k + (k-1)$ as before. Another case is when one of the twists $T_{k,1}, T_{k,2}, T_{k,3}$ (say $T_{k,3}$) does not enclose v. In that case, two additional twists, distinct from all of the above, enclosing, respectively, v and v_k^1 (but not v_k^2 or v_k^3) and v and v_k^2 (but not v_k^1 or v_k^3), are needed (again for the reason of joint multiplicities). This would still yield $m(v) > 3k + (k-1)$. As in the case of Property F1, the multiplicity of v will be even higher if v is not enclosed by some of the twists among $T_2, \dots, T_{k-1}, T_k$. As above, we get a contradiction since $m(v) \leq 2k + 2$, as computed from the standard factorization of Φ.

At this point, we have shown that the Gay–Mark open book for the resolution graph G must have an outer boundary twist T in any factorization Φ, assuming that the smaller graph G' satisfies the conclusion of the lemma. To prove the other statements of the lemma for G, we will now reduce to a different smaller graph $\widetilde{G}$.

In the page P, consider all the holes associated to the vertex $E_1 \in G$. We know that these holes have joint multiplicity 1 with any other hole in P, thus they cannot be enclosed by any twists other than T that involve several holes. Since there is one boundary Dehn twist δ_i around each of these holes in the standard factorization, so that the multiplicity of each hole is 2, these boundary twists must be present in Φ as well. It follows that the factorization Φ has the form $\Phi = T\delta_1\delta_2 \cdots \delta_m \circ \widetilde{\Phi}$, where $\widetilde{\Phi}$ is supported in $\widetilde{P} = P \setminus V_1$, the part of the page P associated to $G \setminus E_1$. Remove the vertex E_1 and its connecting edge from the graph G, and consider the resulting graph $\widetilde{G}$, keeping the same self-intersections of vertices. Clearly, $\widetilde{\Phi}$ gives a factorization of the Gay–Mark open book associated to $\widetilde{G}$. If self-intersections of all vertices of G are at most -5, the same holds for $\widetilde{G}$. The graph $\widetilde{G}$ has fewer vertices, so by the induction hypothesis, the factorization $\widetilde{\Phi}$ must be standard. It follows that the factorization Φ of the Gay–Mark open book for G is standard as well, proving part (2) of the lemma.

To make the induction step work for part (1), assume that in G, the vertex E_{k+1} has self-intersection at most -4, while all the other vertices have self-intersection at most -5. When we remove E_1 to form the graph $\widetilde{G}$, the vertices of $\widetilde{G}$ may no longer be conveniently arranged; after vertices are rearranged, we need to have the (-4) vertex at the appropriate position to apply the induction hypothesis. We must rearrange the vertices of $\widetilde{G}$ to have a chain $\widetilde{E}_1, \widetilde{E}_2, \dots$, with the other vertices partitioned into the sets $\widetilde{R}_1, \widetilde{R}_2, \dots$, so that the length of any chain in E_j, R_j is at most $j-1$. Consider the vertex E_2 in G. If R_2 is not empty in G, we can pick a vertex of R_2 to play the role $\widetilde{E}_1$ in $\widetilde{G}$, let $\widetilde{R}_2$ be the remaining vertices of R_2, and set $\widetilde{E}_2 = E_2$, $\widetilde{E}_3 = E_3, \dots, \widetilde{R}_3 = R_3$, $\widetilde{R}_4 = R_4, \dots$. In this case, the (-4) vertex E_{k+1} in G becomes the vertex $\widetilde{E}_{k+1}$ at the end of the chain $\widetilde{E}_1, \widetilde{E}_2, \dots$ in $\widetilde{G}$, as required. If R_2 is empty in G, we check if R_3 has a chain of (maximum possible) length 2. If so, we rearrange the vertices: let $\widetilde{E}_3 = E_3$, pick $\widetilde{E}_1$ and $\widetilde{E}_2$ forming a length 2 chain in R_3 (with $\widetilde{E}_1$ being the leaf vertex). Let $\widetilde{R}_2$ consist of all vertices other than $\widetilde{E}_1$, $\widetilde{E}_3$, and let $\widetilde{R}_3$ consist of all remaining vertices of R_3, together with the old vertex E_2. For $j \geq 4$, we have $\widetilde{E}_j = E_j$, $\widetilde{R}_j = R_j$, so the (-4) vertex remains in the right place for the graph $\widetilde{G}$, which is now conveniently arranged. See Figure 4. If there are no chains of length 2 in R_3, we similarly examine R_4 to see if there are chains of length 3. If so, we flip the graph to put this length 3 chain into the position of vertices $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$, make the vertices E_2, E_3 and all of R_3 be part of the new set R_4; the graph is now conveniently arranged, and the (-4) vertex does not move. If there are no length 3 chains in R_4, we look at R_5, etc. To summarize, the above procedure means that we can conveniently rearrange the vertices of $\widetilde{G}$ without moving the (-4) vertex whenever for some $j = 2, \dots, k$, the set R_j in G has a chain of the maximum possible length $j-1$. If such a chain does not exist, we check if R_{k+1} has a chain of length k. If so, this chain will become the

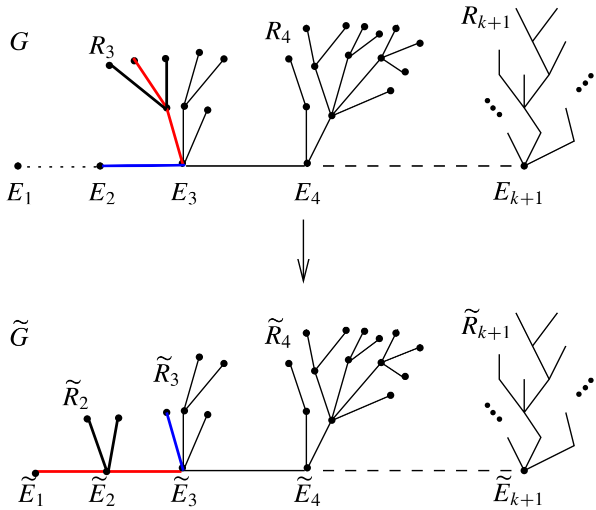

Figure 4. After removing the vertex E_1 from G, we flip a chain in the graph to make the new graph $\widetilde{G}$ conveniently arranged, while keeping in place the vertex E_k. The graph G is shown at the top, the new graph $\widetilde{G}$ at the bottom. The picture illustrates the situation where R_2 is empty, and we flip a length 2 chain in R_3, together with all the edges and vertices attached to this chain in R_3. The vertex E_2 becomes part of the new set $\widetilde{R}_3$.

new chain $\widetilde{E}_1, \widetilde{E}_2, \dots,$ the old vertices $E_2, \dots, E_k$ as well as the sets $R_2, \dots, R_k$ will all be in $\widetilde{R}_{k+1}$, and the graph will be conveniently rearranged without moving the (-4) vertex $E_{k+1} = \widetilde{E}_{k+1}$. Lastly, if each chain R_j, $j = 2, \dots, k+1$ has length at most $j-2$ in G, we can set $\widetilde{E}_1 = E_2, \widetilde{E}_2 = E_3, \dots, \widetilde{E}_k = E_{k+1}$ and $\widetilde{R}_2 = R_3, \dots, \widetilde{R}_k = R_{k+1}$, so that the graph $\widetilde{G}$ will be conveniently arranged, and the vertex $\widetilde{E}_k \in \widetilde{G}$ will have self-intersection -4.

With the rearrangement in place, part (1) follows by induction: if the factorization Φ of the Gay–Mark open book for G has the form $\Phi = T\delta_1\delta_2\cdots\delta_m \circ \widetilde{\Phi}$, where $\widetilde{\Phi}$ is the factorization of the Gay–Mark open book for $\widetilde{G}$, and part (1) of the lemma holds for $\widetilde{\Phi}$, then clearly the same is true for Φ.

We now return to the base of induction and check the case $k = 2$. In this case, the graph is star-shaped with legs of length 1, with E_2 in the center. As above, the page P is identified with the disk whose outer boundary corresponds to one of the holes associated to E_1; there are at least three holes v_1^1, v_1^2, v_1^3 in V_1. First, we claim that any factorization has a Dehn twist enclosing all of these holes. If not, we must have two distinct Dehn twists τ_1 and τ_2, τ_1 enclosing v_1^1, v_3^1 and τ_2 enclosing v_2^1, v_3^1, because $m(v_1^r, v_1^s) = 1$. Since $m(v_3^1) = 2$ and there are at least 4 holes in P having joint multiplicity 2 with v_3^1, one of these Dehn twists, say τ_1, contains an additional hole w. But then there must be two additional Dehn twists in the factorization, enclosing, respectively, v_2^1, v_1^1 and v_2^1, w, which is impossible since $m(v_2^2) = 2$. Thus, there is a Dehn twist τ enclosing v_1^1, v_1^2, v_1^3. We would like to show that τ encloses all the holes in P. Suppose not, and let w be a hole not enclosed by τ. Then there must be distinct Dehn twists τ_1, τ_2, τ_3, enclosing, respectively, w, v_1^1, w, v_1^2, and w, v_1^3. Since $m(v_1^1) = m(v_1^2) = m(v_1^3) = 2$, there cannot be any other Dehn twists enclosing v_1^1, v_1^2, v_1^3, and since $m(v_1^r, v) = 1$ for $r = 1, 2, 3$ and any other hole v, every hole v must be either in τ or in *each* of τ_1, τ_2, τ_3 (but not simultaneously in τ and τ_i, $i = 1, 2, 3$). For the hole w, two cases are possible: (1) w belongs to E_2, in which case $m(w) = 3$, so there are no Dehn twists except τ_1, τ_2, τ_3 enclosing w; or (2) w belongs to R_2, in which case $m(w) = 4$, and there is exactly one additional Dehn twist τ'. In either case, there must exist another hole w' such that $m(w, w') = 2$; however, we can only get $m(w, w') = 3$ (if w' is in all three of τ_1, τ_2, τ_3) or $m(w, w') = 1$ (if w' is in τ and τ'). It follows that the Dehn twist τ must enclose all holes in P.

We conclude that the factorization includes Dehn twists around the curves that are homologous, and therefore isotopic, to the boundaries of all the holes associated to E_1. As above, these can be removed from consideration. The same argument works for any leaf vertex of the graph, reducing the question to the situation of only one vertex, E_2. This is the case $k = 1$ representing an open book for a lens space as in [24]; if $E_2 \cdot E_2 \leq -5$, there is a unique factorization, and if $E_2 \cdot E_2 = -4$, then the only other option for the homology classes of curves comes from the lantern relation. □

By Lemma 2, we now know that under the hypotheses of Theorem 1, the Dehn twists in every positive factorization are performed about the curves in the same *homology* classes as the Dehn twists in the standard factorization. We now show that the curves are in the same *isotopy* classes.

Lemma 3. *Let (P, ϕ) be a planar open book whose monodromy ϕ admits a factorization Φ into a product of positive Dehn twists about* ***disjoint*** *simple closed curves in P. Suppose that Φ' is another positive factorization of ϕ, such that Φ is homologically equivalent to Φ'. Then the factorizations Φ and Φ' are the same, up to the order of Dehn twists.*

Proof. After reordering, we can write $\Phi = D_1 D_2 \cdots D_l \delta_1 \delta_2 \cdots \delta_n$, where the δ_i's are Dehn twists about the boundary-parallel curves, and $D_1, D_2, \dots, D_l$ are the Dehn twists around disjoint curves $\gamma_1, \dots, \gamma_l$ in P that are not boundary parallel.

Then, again after reordering, we have $\Phi' = T_1 T_2 \cdots T_l \delta_1 \delta_2 \cdots \delta_n$, where the Dehn twists D_j and T_j are performed about homologous curves in P: indeed, every boundary-parallel curve γ_j is determined by its homology class, uniquely up to isotopy. We can thus remove the Dehn twists $\delta_1, \delta_2, \dots, \delta_n$ from consideration. We will use the same notation, $\Phi = D_1 D_2 \cdots D_l$ and $\Phi' = T_1 T_2 \cdots T_l$ for the two

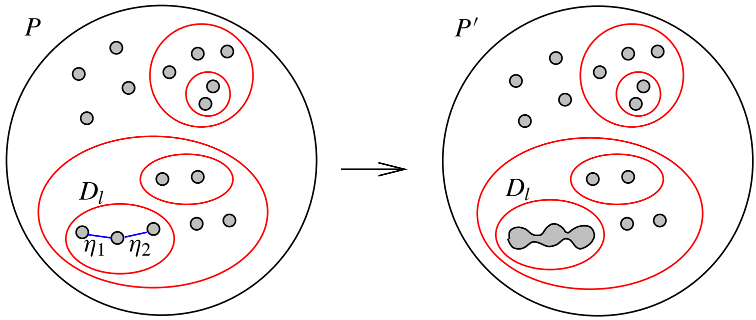

Figure 5. After cutting P along arcs $\eta_1, \eta_2, \ldots$, the Dehn twist D_l becomes boundary-parallel in the new surface P'.

factorizations of the diffeomorphism

$$\phi = D_1 D_2 \cdots D_l = T_1 T_2 \cdots T_l. \tag{3}$$

We will prove the lemma by induction on the number l of the nonboundary parallel Dehn twists. Identifying P with a disk with holes, we can assume that γ_l is an innermost curve in the collection $\gamma_1, \gamma_2, \ldots, \gamma_l$. Suppose that γ_l encloses r holes. Choose a collection of arcs $\eta_1, \eta_2, \ldots, \eta_{r-1}$ connecting these holes and disjoint from γ_l, so that after cutting along these arcs, the holes become a single hole, and the domain enclosed by γ_l becomes an annulus (which deformation retracts to γ_l). See Figure 5. By construction, the arcs $\eta_1, \eta_2, \ldots, \eta_{r-1}$ are disjoint from the support of each of the Dehn twists $D_1, D_2, \ldots, D_l$, thus the diffeomorphism $\phi = D_1 D_2 \cdots D_l$ fixes each of these arcs. As in [3, Proposition 3] and [9, Section 2], we now make the following key observation: after an isotopy removing nonessential intersections, all arcs $\eta_1, \ldots, \eta_{r-1}$ must be also disjoint from the support of each of the Dehn twists $T_1, T_2, \ldots, T_l$. To see this, we recall that each right-handed Dehn twist is a right-veering diffeomorphism of the oriented surface P [11]. If α and β are two arcs with the same endpoint $x \in \partial S$, we say that β lies to the right of α if the pair of tangent vectors $(\dot{\beta}, \dot{\alpha})$ at x gives the orientation of P. The right-veering property of a boundary-fixing map $\tau : P \to P$ means that for every simple arc α with endpoints on ∂P, the image $\tau(\alpha)$ is either isotopic to α or lies to the right of α at both endpoints, once all nonessential intersections between α and $\tau(\alpha)$ are removed. Now, suppose that the support of the Dehn twist T_j essentially intersects one of the arcs, say η_1. Then the curve $T_j(\eta_1)$ is not isotopic to η_1 (see, e.g., [8, Proposition 3.2]), so $T_j(\eta_1)$ lies to the right of η_1. Since the composition of right-veering maps is right-veering, we can only get curves that lie further to the right of η_1 after composing with the other Dehn twists $T_1, \ldots, T_l$. However, the composition $\phi = T_1 T_2 \cdots T_j \cdots T_l$ fixes η_1, a contradiction.

Once we know that the support of all the Dehn twists is disjoint from all of the arcs $\eta_1, \ldots, \eta_{r-1}$, we can cut the page P along these arcs, and consider the image of the relation (3) in the resulting cut-up surface P'. In P', we have that (the induced diffeomorphisms) T_l and D_l are Dehn twists around the curve homologous to the boundary of the same hole, and therefore, $T_l = D_l$ as Dehn twists in P'. It follows that for the Dehn twists (induced by) $D_1, \dots, D_{l-1}$ and $T_1, \dots, T_{l-1}$ in P', we have

$$D_1 D_2 \cdots D_{l-1} = T_1 T_2 \cdots T_{l-1}.$$

By the induction hypothesis, we can conclude that for each $j = 1, \dots, l-1$, the Dehn twists D_j and T_j are performed about isotopic curves in P'. It follows that each pair D_j, T_j gives the same Dehn twists in P, for each $j = 1, \dots, l$. □

Proof of Theorem 1. Under the hypotheses of Theorem 1, the contact 3-manifold (Y, ξ) is supported by an open book with planar page P. Theorems of Wendl and Niederkruger then imply that up to blow-up and deformation of the symplectic form, every weak symplectic filling has a Lefschetz fibration whose fiber is given by P; the monodromy of the fibration is the monodromy of the open book. The Lefschetz fibration is described by its vanishing cycles, or, equivalently, by a positive factorization of the monodromy. Lemmas 2 and 3 show that the positive monodromy factorization is unique. □

Acknowledgements

Originally, Theorem 1 was proved in joint work with Laura Starkston [23]. The alternative proof given here resulted from the author's attempts to understand the combinatorial constructions of [12, Theorem 6.23] to open book factorizations.

This article was written for Proceedings of the 2020 BIRS workshop on Interactions of Gauge Theory and Contact and Symplectic Topology. The author benefited greatly from the series of the BIRS workshops on these topics and would like to thank the organizers of all the past workshops of the series.

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Received 17 Feb 2021.

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ISSN: 2329-9061 (print), 2329-907X (electronic)

ISBN: 978-1-935107-11-8 (print), 978-1-935107-10-1 (electronic)

First published 2022.

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THE OPEN BOOK SERIES 5

Gauge Theory and Low-Dimensional Topology: Progress and Interaction

This volume is a proceedings of the 2020 BIRS workshop *Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4*. This was the 6th iteration of a recurring workshop held in Banff. Regrettably, the workshop was not held onsite but was instead an online (Zoom) gathering as a result of the Covid-19 pandemic. However, one benefit of the online format was that the participant list could be expanded beyond the usual strict limit of 42 individuals. It seemed to be also fitting, given the altered circumstances and larger than usual list of participants, to take the opportunity to put together a conference proceedings.

The result is this volume, which features papers showcasing research from participants at the 6th (or earlier) *Interactions* workshops. As the title suggests, the emphasis is on research in gauge theory, contact and symplectic topology, and in low-dimensional topology. The volume contains 16 refereed papers, and it is representative of the many excellent talks and fascinating results presented at the Interactions workshops over the years since its inception in 2007.

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