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Gauge Theory and Low-Dimensional Topology: Progress and Interaction

A note on thickness of knots

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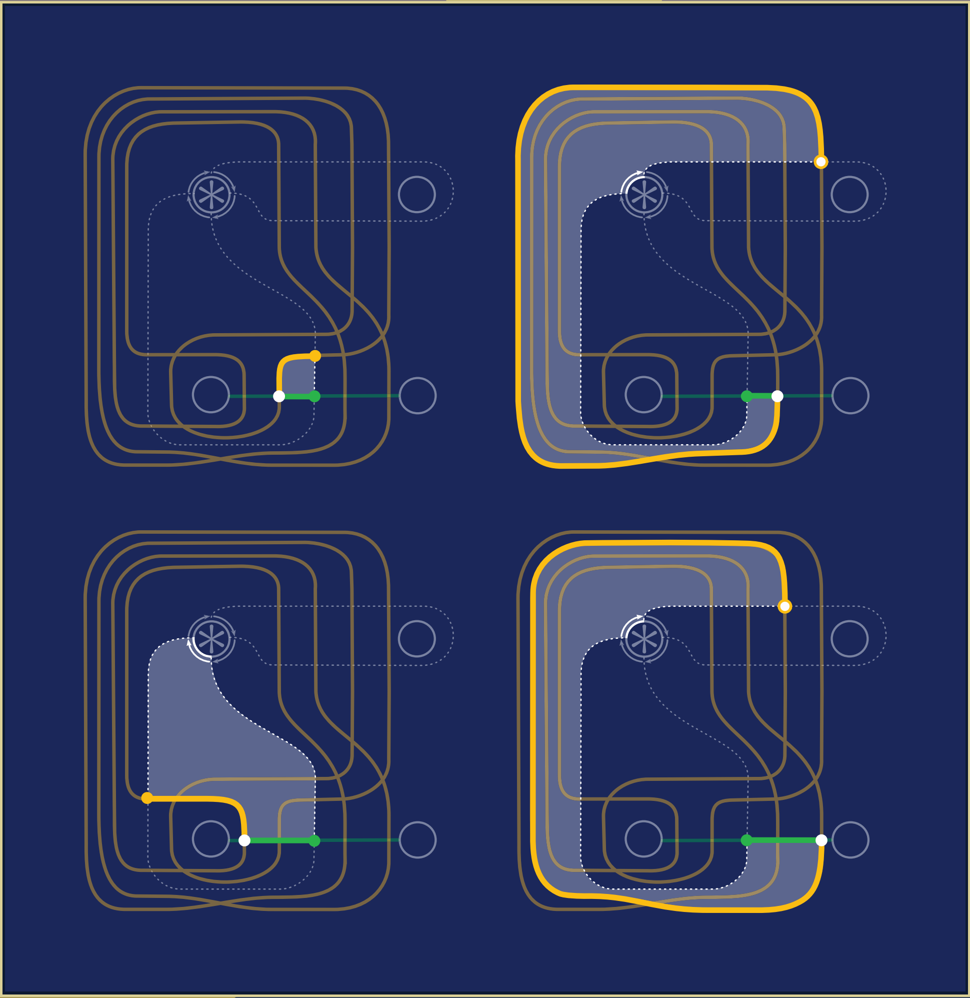

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A note on thickness of knots

András I. Stipsicz and Zoltán Szabó

We introduce a numerical invariant $\beta(K) \in \mathbb{N} \cup \{0\}$ of a knot $K \subset S^3$ which measures how nonalternating K is. We prove an inequality between $\beta(K)$ and the (knot Floer) thickness $th(K)$ of a knot K. As an application we show that all Montesinos knots have thickness at most one.

1. Introduction

A knot $K \subset S^3$ is *alternating* if it admits a diagram with the property that when traversing through the diagram, we alternate between over- and under-crossings. (An intrinsic definition of alternating knots has been recently found by Greene and Howie [5; 6].) A diagram of K partitions the plane into domains (the connected components of the complement of the projection), and the alternating property can be rephrased by saying that on the boundary of each domain each edge connects an under-crossing with an over-crossing. Indeed, this observation provides a way to measure how far a knot is from being alternating. We introduce the following definition:

Definition 1.1. Suppose that D is the diagram of a given knot $K \subset S^3$. A domain d of D is *good* if any edge on the boundary of d connects an over- and an under-crossing. The domain d is *bad* if it is not good. The number of bad domains of the diagram D is denoted by $B(D)$.

Clearly, the diagram D is alternating if and only if $B(D) = 0$. Indeed, by taking

$$\beta(K) = \min\{B(D) \mid D \text{ is a diagram for } K\},$$

we get a knot invariant, which satisfies $\beta(K) = 0$ if and only if K is an alternating knot. As it is typical for knot invariants given by minima of quantities over all diagrams, it is easy to find an upper bound on $\beta(K)$ (by determining $B(D)$ for a diagram of K), but it is harder to actually compute its value.

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Keywords: knot Floer homology, thickness, alternating knots.

As it turns out, knot Floer homology provides a lower bound for $\beta(K)$ through the *thickness* of K. Recall that $\widehat{\mathrm{HFK}}(K)$, the hat-version of knot Floer homology of K, is a finite-dimensional bigraded vector space over the field $\mathbb{F}$ of two elements. By collapsing the Maslov and Alexander gradings M and A on $\widehat{\mathrm{HFK}}(K)$ to $\delta = A - M$, we get a graded vector space $\widehat{\mathrm{HFK}}^{\delta}(K)$. The thickness $th(K)$ of K is the largest possible difference of δ-gradings of two homogeneous (nonzero) elements of this vector space. It is known that for an alternating knot K the δ-graded Floer homology is in a single δ-grading (determined by the signature of the knot); hence if K is alternating, then $th(K) = 0$. (Knots satisfying $th(K) = 0$ are called *thin* knots, hence alternating knots are thin.)

With this definition in place, the main result of this paper is as follows:

Theorem 1.2. *Suppose that $K \subset S^3$ is a nonalternating knot. Then*

$$th(K) \leq \tfrac{1}{2}\beta(K) - 1. \tag{1-1}$$

While the thickness of K can be used to estimate how nonalternating K is, (1-1) can also be used to estimate $th(K)$ by finding appropriate diagrams of K. In particular, the formula can be applied to show the following:

Corollary 1.3 (Lowrance [7]). *Suppose K is a Montesinos knot. Then*, $th(K) \leq 1$.

Remark 1.4. • A quantity similar to $\beta(K)$ has been introduced by Turaev [10], now called the *Turaev genus* $g_T(K)$. An inequality similar to (1-1) for the Turaev genus and the (knot Floer) thickness $th(K)$ was shown by Lowrence in [7]. As the Turaev genus of nonalternating Montesinos knots is known to be equal to 1 [1; 2], our Corollary 1.3 also follows from [7].

• Indeed, a simple argument (due to Adam Lowrence (personal communication, 2020)) shows that

$$g_T(K) \leq \tfrac{1}{2}\beta(K) - 1,$$

since by [7, Theorem 4.1] for a diagram D of K we have $g_T(D) = th(C^{\delta}_{D,p})$ (with the notation of Section 2).

• Similar observations regarding the relation between the Turaev genus g_T and β have been communicated to us by Homayun Karimi and Seungwon Kim (2020).

The formula (1-1) can be used in a further way: by a recent result of Zibrowius [11], mutation does not change $\widehat{\mathrm{HFK}}^{\delta}(K)$, and hence leaves $th(K)$ unchanged. Consequently, besides isotopies, we can change a diagram by mutations to get better estimates for $th(K)$ through $B(D)$ for a diagram D of a mutant.

The paper is organized as follows. In Section 2, we recall basics of knot Floer homology and prove the theorem stated above. In Section 3, we give the details of the proof of Corollary 1.3, and finally in Section 4, we list some further properties and questions regarding β.

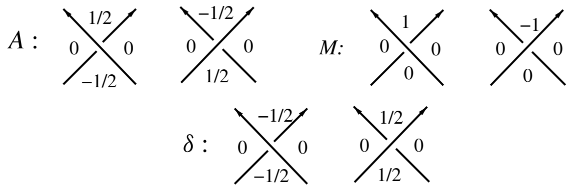


Figure 1. The local contributions for A, M and $\delta = A - M$ at a crossing. The Kauffman state distinguishes a corner at the crossing, and we take the value in that corner as a contribution of the crossing to A, M or δ of the Kauffman state at hand.

2. The knot Floer homology thickness of knots

Suppose that $V = \sum_a V_a$ is a finite-dimensional graded vector space, where $V_a \subset V$ is the subspace of homogeneous elements of grading $a \in \mathbb{R}$. The *thickness* $th(V)$ of V is by definition the largest possible difference between gradings of (nonzero) homogeneous elements:

$$th(V) = \max\{a \in \mathbb{R} \mid V_a \neq 0\} - \min\{a \in \mathbb{R} \mid V_a \neq 0\}.$$

Suppose now that the graded vector space V is endowed with a boundary operator ∂ of degree 1; then the homology $H(V, \partial)$ also admits a natural grading from the grading of V. As $H(V, \partial)$ is the quotient of a subspace of V, it is easy to see that

$$th(H(V, \partial)) \leq th(V).$$

The hat version of knot Floer homology (over the field $\mathbb{F}$ of two elements) of a knot $K \subset S^3$ is a finite-dimensional bigraded vector space $\widehat{\mathrm{HFK}}(K) = \sum_{M,A} \widehat{\mathrm{HFK}}_M(K, A)$. By collapsing the two gradings to $\delta = A - M$, we get the δ-graded invariant $\widehat{\mathrm{HFK}}^{\delta}(K)$. The thickness of $\widehat{\mathrm{HFK}}^{\delta}(K)$ is by definition the thickness $th(K)$ of K.

Knot Floer homology is defined as the homology of a chain complex, which we can associate to a diagram of the knot (and some further choices). Indeed, for a given diagram D of a knot K, fix a marking, that is, a point of D which is not a crossing. Consider the bigraded vector space $C_{D,p}$ (graded by the Alexander and the Maslov gradings A and M) associated to the marked diagram (D, p), which is generated over $\mathbb{F}$ by the Kauffman states of the marked diagram, a concept which we recall below.

Suppose that for the marked diagram (D, p) of the knot K, the set of crossings is denoted by $Cr(D)$, the set of domains by $Dom(D)$, and $Dom_p(D)$ denotes the set of those domains which do not contain p on their boundary. A *Kauffman state* κ is a bijection $\kappa : Cr(D) \to Dom_p(D)$ with the property that for a crossing $c \in Cr(D)$ the value $\kappa(c)$ is one of the (at most four) domains meeting at c. The Alexander, Maslov and δ-gradings of a Kauffman state are computed by summing the local contributions at each crossing, as given by the diagrams of Figure 1.

According to [8] there is a boundary map $\partial : C_{D,p} \to C_{D,p}$ of bidegree $(-1, 0)$ (in the bigrading (M, A)) with the property that $H(C_{D,p}, \partial)$ is isomorphic to the knot Floer homology $\widehat{\mathrm{HFK}}(K)$ of K (as a bigraded vector space). By collapsing the two gradings A and M to $\delta = A - M$, we get the graded vector spaces $(C^{\delta}_{D,p}, \partial)$ and its homology $\widehat{\mathrm{HFK}}^{\delta}(K)$. As $\widehat{\mathrm{HFK}}^{\delta}(K)$ is the quotient of a subspace of $C^{\delta}_{D,P}$, we have that

$$th(\widehat{\mathrm{HFK}}^{\delta}(K)) \leq th(C^{\delta}_{D,p}, \partial).$$

Proposition 2.1. *Suppose that D is a diagram of the knot K. If D is not an alternating diagram, then*

$$th(C^{\delta}_{D,p}) \leq \tfrac{1}{2}B(D) - 1.$$

Proof. Fix a marked point p on D, and consider the δ-graded chain complex $(C^{\delta}_{D,p}, \partial)$ generated by the Kauffman states of (D, p).

The δ-grading at a positive crossing is either 0 or $\frac{1}{2}$, and at a negative crossing it is either 0 or $-\frac{1}{2}$. So we can express the δ-grading of a Kauffman state κ as the sum

$$\tfrac{1}{4}\mathrm{wr}(D) + \sum_{c \in Cr} f(\kappa(c)),$$

where wr is the writhe of the diagram, and f is a function on the Kauffman corners, which is either $\frac{1}{4}$ or $-\frac{1}{4}$ (depending on the chosen quadrant at the crossing c).

Simple computation shows that for a good domain each corner in the domain gives the same f-value; hence for different Kauffman states the contributions from this particular domain are the same. This is no longer true for a bad domain, but the difference of two contributions is at most $\frac{1}{2}$. When determining the possible maximum of $\delta(x) - \delta(x')$ for two homogeneous elements $x, x' \in C^{\delta}_{D,p}$, the contributions from the writhe cancel, and so do the contributions from good domains, while bad domains contribute at most $\frac{1}{2}$. This shows that $th(C^{\delta}_{D,p}) \leq \frac{1}{2}B(D)$.

By assumption, D is not alternating; hence there is a bad domain, with an edge showing that it is bad. Choose the marking p on such an edge. Since this edge guarantees that the two domains having it on their boundary are both bad, while these two bad domains do not get Kauffman corners, we get that $th(C_{D,p})$ is bounded by

$$\tfrac{1}{2}(B(D) - 2) = \tfrac{1}{2}B(D) - 1,$$

concluding the proof. □

Proof of Theorem 1.2. Suppose that K is not alternating. Then any diagram D of K is nonalternating; hence we have that

$$th(K) \leq th(C^{\delta}_{D,p}) \leq \tfrac{1}{2}B(D) - 1.$$

Since $\beta(K)$ is computed from the minimum of the right-hand side of this inequality, the proof follows at once. □

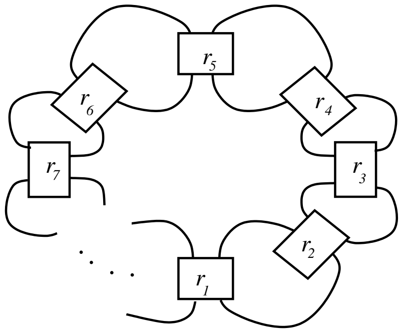


Figure 2. The Montesinos knot $M(r_1, \dots, r_n)$. The box containing r_i denotes the algebraic tangle determined by the rational number $r_i = \beta_i/\alpha_i$ (cf. Figure 3). In order to have a knot, at most one of the α_i can be even.

3. Montesinos knots

Montesinos knots are straightforward generalizations of pretzel knots; a diagram involving rational tangles defining the Montesinos knot $M(r_1, \dots, r_n)$ is shown by Figure 2. (A box with a rational number r_i in it symbolizes the tangle shown by Figure 3.) We allow any of the r_i to be equal to ± 1. Notice that the order of $(r_1, \dots, r_n)$ is important; those r_i which are equal to ± 1 can be commuted with any other parameter through a simple isotopy of the diagram.

Lemma 3.1. *Consider the diagram of the Montesinos knot $M(r_1, \dots, r_n)$ given by Figure 2. It can be isotoped to a diagram with at most four bad domains.*

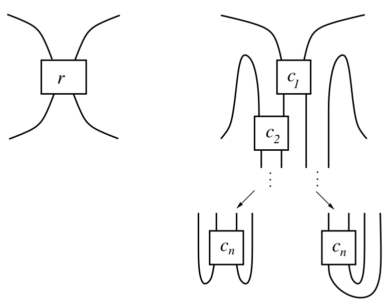


Figure 3. The rational tangle corresponding to $r \in \mathbb{Q}$. The rational number r determines the coefficients c_i through its continued fraction expansion. The boxes with $c_i \in \mathbb{Z}$ on the right denote $|c_i|$ half twists (right-handed for positive, left-handed for negative c_i). Depending on the parity of n (the number of c_i's) we have two different finishing forms. The tangle is alternating (as part of a knot or link) if the c_i alternate in sign.

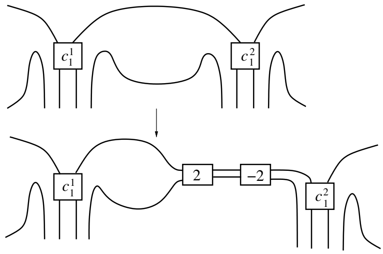


Figure 4. The introduction of cancelling twists to turn domains between tangles to be good.

Proof. Recall that a rational tangle has the form given by Figure 3. Adapting the isotopies described in [4], we can achieve that all tangles are alternating; hence the potentially bad domains are the ones between the tangles, together with the central and the unbounded domains. The number of bad domains between the tangles can be reduced by the following observation. The domain between two tangles is bad if the first coefficients c_1^1 and c_1^2 of the two rational numbers determining the tangles have opposite signs, say $c_1^1 > 0$ and $c_1^2 < 0$. Then by Reidemeister-II moves we can introduce canceling twistings, as shown by Figure 4, and then commute the first twisting (in the figure given by the box with 2 in it) between the first and second tangles of the Montesinos knot. All domains between the boxes will become good, except the ones connecting the first tangle with the newly introduced twists and the second tangle also connecting it with the newly introduced twists. After these alterations, make sure that (by the adaptation of [4]) all tangles are isotoped to be alternating. In total the new diagram then has four bad domains, concluding the proof. □

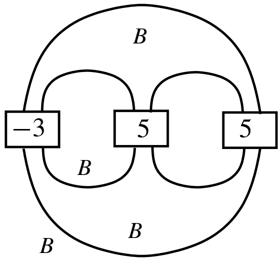
Proof of Corollary 1.3. For a Montesinos knot $M(r_1, \dots, r_n)$, an appropriate isotopy of the diagram of Figure 2 (as given by Lemma 3.1) gives a diagram with at most four bad domains. The application of Theorem 1.2 concludes the argument. □

Remark 3.2. Using the mutation invariance of $th(K)$, Lemma 3.1 can be avoided: by mutations, any Montesinos knot $M(r_1, \dots, r_n)$ can be moved to $M(q_1, \dots, q_n)$ with the same rational parameters in a different order so that q_i and q_{i+1} have the same sign with at most one exception. Isotoping the diagram so that the tangles are alternating, the mutated diagram then has at most 4 bad domains. Using the result of [11, Theorem 0.1] then the corollary follows as before.

4. Further properties

It is a standard fact that the knot Floer homology of the connected sum of two knots is the tensor product of the knot Floer homologies:

$$\widehat{\mathrm{HFK}}(K_1 \# K_2) \cong \widehat{\mathrm{HFK}}(K_1) \otimes \widehat{\mathrm{HFK}}(K_2).$$



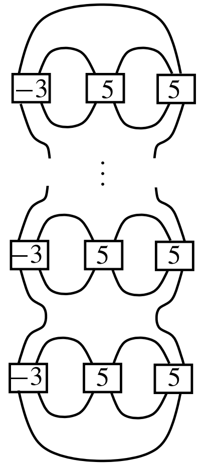


Figure 5. The knot K_n. In (a) the pretzel knot $P(-3, 5, 5)$ is shown. The B symbols signify the bad domains. (A box containing the integer n denotes $|n|$ half twists, right-handed for $n > 0$ and left-handed for $n < 0$.) In (b) we provide a diagram of K_n, where the connected sum is taken at bad domains.

From this (bigraded) isomorphism it follows that

$$th(K_1\#K_2) = th(K_1) + th(K_2).$$

The behaviour of $\beta(K)$ is less clear under connected summing. Suppose that K_1, K_2 are both nonalternating knots. By taking the connected sum of two diagrams D_1, D_2 for these knots at bad edges (i.e., arcs on the boundary of bad domains verifying that the domains are bad), we get that

$$B(D_1\#D_2) = B(D_1) + B(D_2) - 2,$$

immediately implying that

$$\beta(K_1\#K_2) \leq \beta(K_1) + \beta(K_2) - 2.$$

Motivated by the equality for the thickness th, we arrive at the following conjecture:

Conjecture 4.1. If K_1, K_2 are two nonalternating knots, then

$$\beta(K_1\#K_2) = \beta(K_1) + \beta(K_2) - 2.$$

Sharpness. It is not hard to find knot diagrams for which (1-1) is sharp. Indeed, the standard diagram of the pretzel knot $P(-3, 5, 5)$ admits four bad domains (see Figure 5(a)), while an explicit calculation of $\widehat{\mathrm{HFK}}(P(-3, 5, 5))$ shows

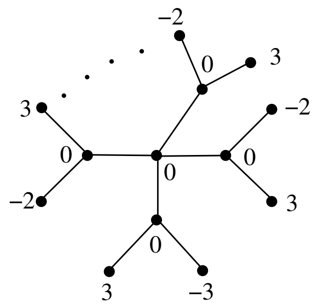


Figure 6. The planar weighted tree defining the knot C_n. The central vertex is of framing 0, and has $n+1$ neighbours, all with framing 0. The first of these vertices is connected to two leaves of framings 3 and -3, while the further vertices are connected to two leaves with framings 3 and -2.

that $th(P(-3,5,5)) = 1$. Consider the n-fold connected sum $K_n = \#_n P(-3,5,5)$; connect summing the diagrams at bad edges (in the above sense) we get a sequence of knots K_n and diagrams D_n for them with the properties that $th(K_n) = n$ and $B(D_n) = 2n+2$; see Figure 5(b). The nonalternating knots K_n then satisfy $n = th(K_n) = \frac{1}{2}\beta(K_n) - 1$.

Arborescent examples. A family of knots (and links) can be specified by combinatorial means as follows. Consider a planar tree (a graph with no circles), with an integer attached to each vertex. An embedded surface can be constructed from the tree by the following algorithm: for each vertex consider a twisted band, with the integer attached to the vertex prescribing the number of half-twists introduced. (The boundary of such a band is the $T_{2,n}$ torus knot or link, where $n \in \mathbb{Z}$ is the decoration of the vertex.) If two vertices are connected in the tree by an edge, plumb the two surfaces together. The boundary of the resulting surface is an arborescent knot (or link). To make the definition precise (i.e., to get a well-defined knot or link) further information is needed, prescribing the location of the plumbing on each band, relative to the twisting; see [3]. We will not make this distinction here for two reasons: (a) the different choices one can make for a given graph result in mutation equivalent knots, and since the thickness is mutation invariant, different choices make no effect on our calculations, and (b) in the example we will show below, the nodes (i.e., vertices of degree more than 2) have framing 0, hence the above mentioned choice makes no difference.

It is easy to see that pretzel knots (and more generally Montesinos knots) are all arborescent; these knots correspond to graphs with a single node. (Such graphs are called star-shaped.) Consider the family of knots C_n defined by the graph of Figure 6. (For diagrams of the knots C_n, see Figure 7.) Computer calculations [9]

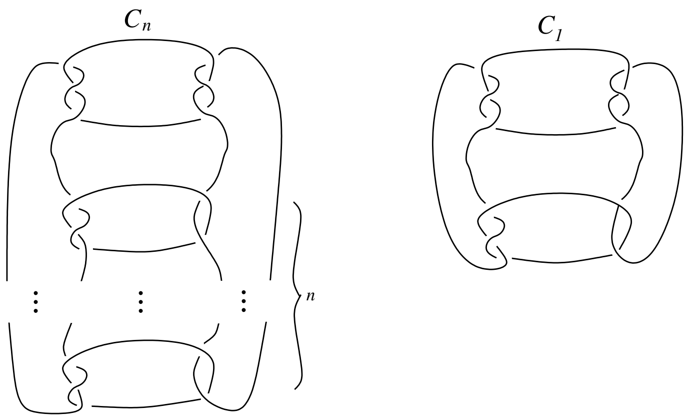


Figure 7. The knot C_n in general and C_1 in particular.

show that C_n has thickness n once $n \leq 4$. (For $n = 1$ the knot C_1 is a pretzel knot having thickness equal to 1, while for $n = 0$ the knot C_0 is the connected sum of a right-handed and a left-handed trefoil; hence it is thin.) These cases lead us to expect that $th(C_n) = n$ holds in general. Indeed, it is not hard to find a diagram for C_n with $2n + 2$ bad domains; hence $th(C_n) \leq n$ follows from our main result, and the above mentioned calculations suggest that we have equality here. More generally, it would be interesting to see if there is a simple relation between the number of nodes of a (weighted) tree and the thickness of a corresponding knot; maybe the thickness is at most the number of nodes.

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THE OPEN BOOK SERIES 5

Gauge Theory and Low-Dimensional Topology: Progress and Interaction

This volume is a proceedings of the 2020 BIRS workshop *Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4*. This was the 6th iteration of a recurring workshop held in Banff. Regrettably, the workshop was not held onsite but was instead an online (Zoom) gathering as a result of the Covid-19 pandemic. However, one benefit of the online format was that the participant list could be expanded beyond the usual strict limit of 42 individuals. It seemed to be also fitting, given the altered circumstances and larger than usual list of participants, to take the opportunity to put together a conference proceedings.

The result is this volume, which features papers showcasing research from participants at the 6th (or earlier) *Interactions* workshops. As the title suggests, the emphasis is on research in gauge theory, contact and symplectic topology, and in low-dimensional topology. The volume contains 16 refereed papers, and it is representative of the many excellent talks and fascinating results presented at the Interactions workshops over the years since its inception in 2007.

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