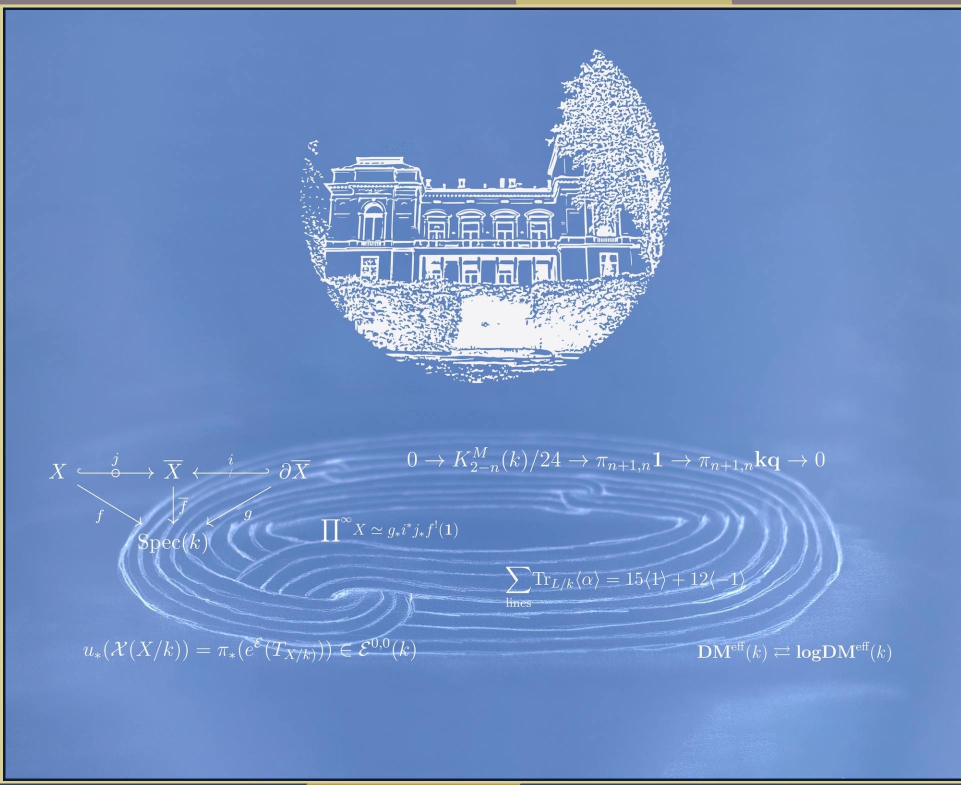
THE OPEN BOOK SERIES 6

Motivic Geometry

Oslo, 2020/21

edited by
Paul Arne Østvær



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Based on lectures given at the Centre for Advanced Study (CAS) of the Norwegian Academy of Science and Letters, this book provides a panorama of developments in motivic homotopy theory and related fields.

A common goal of the research program underlying this volume is the understanding of the geometric nature of spaces, revealed through algebraic and homotopical invariants. The articles in this volume, contributed by leading experts, together touch on an extensive network of related topics in algebraic geometry, homotopy theory, K-theory and related areas.

The volume has a significant expository component, making it accessible to students, while also containing information and in-depth discussion of interest to all practitioners including specialists.

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Preface

In the academic year 2020/21, the Centre for Advanced Study (CAS) at the Norwegian Academy of Science and Letters organized a program focused on Motivic Geometry. This program facilitated rich interactions among researchers in motivic homotopy theory and related fields, united by a shared goal of understanding the geometric nature of spaces revealed through algebraic and homotopical invariants. We are currently at a pivotal moment, aiming to deepen this creative interplay, with dramatic and unexpected connections coming into focus.

One of the program's objectives was to showcase the diversity of motivic homotopy theory and its most intriguing recent developments. It addressed a wide array of topics, including algebraic vector bundles, affine algebraic geometry, classification of varieties, enriched enumerative geometry, equivariant Witt cohomology, framed correspondences, Hodge theory of p-adic varieties, isotropic motivic homotopy theory, K-theory, logarithmic motives, Milnor–Witt homotopy sheaves, motivic nearby cycles, motivic stable homotopy groups, p-adic motivic cohomology, representations of the motivic Galois group, reciprocity sheaves, slice spectral sequences, stable $\mathbb{A}^1$-homotopy at infinity, and strict $\mathbb{A}^1$-invariance.

The program's lecture series introduced current advancements in this field. It was tailored for experts but presented in an accessible way for those with a general background.

This volume reflects the spirit of the lecture series and the accompanying conferences. Consequently, this volume contains a significant expository and didactic component for the younger audience for whom the lecture series was primarily intended. It provides an overview of the field's current state and is intended for both beginners and specialists.

Most of the six chapters expand on the speakers' lectures and, as mentioned above, cover a wide range of topics. Namely:

- *Notes on motivic infinite loop space theory*
  by Tom Bachmann and Elden Elmanto
- *An introduction to six-functor formalisms*
  by Martin Gallauer

- *Introduction to framed correspondences*
  by Marc Hoyois and Nikolai Opdan
- *Lectures on the cohomology of reciprocity sheaves*
  by Nikolai Opdan and Kay Rülling
- *The Grothendieck ring of varieties and algebraic K-theory of spaces*
  by Oliver Röndigs
- *Stable homotopy groups of motivic spheres*
  by Oliver Röndigs and Markus Spitzweck

. . . .

We would like to express our gratitude to the authors for their valuable contributions. We are also thankful to Silvio Levy and Alex Scorpan for their invaluable support in preparing this volume for The Open Book Series published by MSP (Mathematical Sciences Publishers). Special thanks are due to CAS for supporting the idea of these proceedings.

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Notes on motivic infinite loop space theory

Tom Bachmann and Elden Elmanto

In fall of 2019, the Thursday seminar at Harvard University studied motivic infinite loop space theory. As part of this, the authors gave a series of talks outlining the main theorems of the theory, together with their proofs, in the case of infinite perfect fields. In winter of 2021/2, Bachmann taught a topics course at LMU Munich on strict $\mathbb{A}^1$-invariance of framed presheaves (which is one of the main theorems, but was not covered in detail during the Thursday seminar). These are our extended notes on these topics.

1. Introduction

We shall assume knowledge of the basic notions of unstable motivic homotopy theory; see, for example, [6, §2.2] for a review and [2] for an introduction. We shall also use freely the language of ∞-categories as set out in [27; 28].

Given a base scheme S, we have the presentably symmetric monoidal ∞-category $\mathcal{S}\mathrm{pc}(S)$ of *motivic spaces*, and a functor $\mathrm{Sm}_S \to \mathcal{S}\mathrm{pc}(S)$ which preserves finite products (and finite coproducts). We write $\mathcal{S}\mathrm{pc}(S)_* = \mathcal{S}\mathrm{pc}(S)_{*/}$ for the presentably symmetric monoidal ∞-category of pointed motivic spaces; we use the smash product symmetric monoidal structure. Let $\mathbb{P}^1_S$ be pointed at 1; this defines an object of $\mathcal{S}\mathrm{pc}(S)_*$. We write $\Sigma^\infty : \mathcal{S}\mathrm{pc}(S)_* \to \mathcal{SH}(S)$ for the universal presentably symmetric monoidal ∞-category under $\mathcal{S}\mathrm{pc}(S)_*$ in which $\mathbb{P}^1$ becomes $\otimes$-invertible.

MSC2020: 14F42.
Keywords: motivic homotopy, infinite loop spaces, framed correspondences.

Denote by $\mathcal{SH}(S)^{\mathrm{veff}} \subset \mathcal{SH}(S)$ the closure under colimits of the essential image of the functor Σ^∞.

The aim of motivic infinite loop space theory is to describe the category $\mathcal{SH}(S)^{\mathrm{veff}}$. It turns out that there is a good answer to this problem if $S = \mathrm{Spec}(k)$, where k is a perfect field. This uses the notion of *framed transfers*, first discovered by Voevodsky [36]. The theory was taken up, and many important results proved, by Garkusha–Panin [18] and their numerous collaborators; see, for example, [1; 10; 12; 17; 20]. Unfortunately their results rely on a process called σ-stabilization which interacts poorly with symmetric monoidal structures. (This is somewhat similar to the problem of constructing a good smash product of spectra in the early days of stable homotopy theory.) This problem was overcome by Elmanto–Hoyois–Khan–Sosnilo–Yakerson [15]; their main contribution is the invention of the notion of *tangentially framed correspondences* and an accompanying symmetric monoidal ∞-category $\mathrm{Corr}^{\mathrm{fr}}(k)$.

Using this category, motivic infinite loop space theory can be stated as follows.

Theorem 1.1. *For a perfect field k, there exists a canonical, symmetric monoidal equivalence of ∞-categories*

$$\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}} \simeq \mathcal{SH}(k)^{\mathrm{veff}}.$$

Here $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$ is a category obtained from $\mathrm{Corr}^{\mathrm{fr}}(k)$ by the usual procedure (consisting of sifted-free cocompletion and motivic localization); it is semiadditive and $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ denotes its subcategory of grouplike objects.

The principal aim of these notes is to explain how to prove this theorem, assuming that k is infinite. Our secondary aim is to reformulate some of the technical results of [1; 10; 12; 17; 20] (those that we need in order to prove Theorem 1.1) in the language of ∞-categories. As it turns out, this simplifies many of the statements and also many of the proofs. Given this focus, we do not treat here the construction of the category $\mathrm{Corr}^{\mathrm{fr}}(k)$ and we refer freely to [15] for this and many basic results about framed motivic spaces. We also do not discuss alternative (1-categorical) approaches to the theory [16; 19].

Organization. The proof of Theorem 1.1 consists mainly in two steps. Firstly we show that there is an equivalence $\mathcal{SH}^{\mathrm{fr}}(k) \simeq \mathcal{SH}(k)$; here $\mathcal{SH}^{\mathrm{fr}}(k)$ is obtained from $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)$ by inverting the framed motivic space corresponding to $\mathbb{P}^1$. This is known as the *reconstruction theorem*. Then we show that the canonical functor $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k) \to \mathcal{SH}^{\mathrm{fr}}(k) \simeq \mathcal{SH}(k)$ is fully faithful. This is called the *cancellation theorem*.

In Section 2 we prove the reconstruction theorem modulo a technical result, known as the *cone theorem*. We then spend all of Section 3 on proving the cone theorem. In Section 4 we prove the cancellation theorem, modulo strict

$\mathbb{A}^1$-invariance of framed presheaves. In Section 5 we prove strict $\mathbb{A}^1$-invariance (of $\mathbb{A}^1$-invariant framed presheaves over infinite perfect fields).

Notation and scope. It is not our intention to faithfully reproduce the results, arguments or notation from any of the many references. Instead our aim is to outline the theory of motivic infinite loop spaces (over a perfect field) from the perspective of ∞-categories. Consequently we follow notation and terminology from [15], and not other references. In particular by *framed correspondences* we mean the ∞-categorical version as devised in [15]; sometimes we may add the word "tangentially" for clarity but often we may not. When referring to Voevodsky's original notion, we speak about *equationally framed correspondences*.

2. The reconstruction theorem

Primary sources: [15; 18].

2A. ***Setup.*** Let S be a scheme. Recall from [15, §4] that there is a symmetric monoidal, semiadditive ∞-category $\mathrm{Corr}^{\mathrm{fr}}(S)$ and a symmetric monoidal functor $\gamma : \mathrm{Sm}_{S+} \to \mathrm{Corr}^{\mathrm{fr}}(S)$.[1] It preserves finite coproducts [15, Lemma 3.2.6] and is essentially surjective (by construction); we refer the reader to [15, 3.2.2] for the most important properties. We denote by $\gamma^* : \mathcal{P}_\Sigma(\mathrm{Sm}_{S+}) \to \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))$ its sifted cocontinuous extension.[2] Write $\mathcal{S}\mathrm{pc}(S)_*$ for the localization of $\mathcal{P}_\Sigma(\mathrm{Sm}_{S+})$ at the generating motivic equivalences, that is, (generating) Nisnevich equivalences and $\mathbb{A}^1$-homotopy equivalences, and $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ for the localization of $\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))$ at the images of the generating motivic equivalences under γ^*. Let $\mathbb{P}^1 \in \mathcal{S}\mathrm{pc}(S)_*$ be pointed at 1. Recall that for any presentably symmetric monoidal ∞-category $\mathcal{C}$ and any object $P \in \mathcal{C}$ there is a universal presentably symmetric monoidal ∞-category under $\mathcal{C}$ in which P becomes $\otimes$-invertible [33, §2.1]; we denote it by $\mathcal{C}[P^{-1}]$.

The following is the main result of this section, which we will give a proof of assuming the comparison results explained in 2E.

Theorem 2.1 (reconstruction). *The induced functor*

$$\gamma^* : \mathcal{S}\mathrm{pc}(S)_*[(\mathbb{P}^1)^{-1}] \to \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)[\gamma^*(\mathbb{P}^1)^{-1}]$$

is an equivalence.

We write $\mathcal{SH}(S) = \mathcal{S}\mathrm{pc}(S)_*[(\mathbb{P}^1)^{-1}]$ and $\mathcal{SH}^{\mathrm{fr}}(S) = \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)[\gamma^*(\mathbb{P}^1)^{-1}]$. We shall prove the result when $S = \mathrm{Spec}(k)$ is the spectrum of an infinite field. The result for general S is reduced to this case in [25] (using [15, §B]).

[1] Recall that for a category with finite coproducts and a final object $*$, $\mathcal{C}_+ \subset \mathcal{C}_{*/}$ denotes the subcategory on objects of the form $c \coprod *$. We mainly use this in conjunction with the equivalence $\mathcal{P}_\Sigma(\mathcal{C}_+) \simeq \mathcal{P}_\Sigma(\mathcal{C})_*$ [6, Lemma 2.1].

[2] We denote by $\mathcal{P}_\Sigma(\mathcal{C}) = \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{pc})$ the nonabelian derived category of $\mathcal{C}$.

Remark 2.2. While $\mathcal{SH}^{\mathrm{fr}}(S)$ appears to be a more complicated ∞-category than $\mathcal{SH}(S)$ (the ∞-category of motivic spectra), the point of motivic infinite loop space theory (and the rest of this note) is to give explicit formulas for mapping spaces in $\mathcal{SH}^{\mathrm{fr}}(S)$, at least when S is the spectrum of a perfect field. More precisely: the functor $\mathcal{S}\mathrm{pc}(S)_* \to \mathcal{SH}(S)$ is far from being fully faithful, while the cancellation theorem Theorem 4.3 asserts that the functor $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)_* \to \mathcal{SH}^{\mathrm{fr}}(S)$ is fully faithful on grouplike objects.

2B. *Preliminary reductions.* The functor γ^* preserves colimits by construction, so has a right adjoint γ_*. The stable presentable ∞-category $\mathcal{SH}(S)$ is compactly generated by objects of the form $\Sigma^\infty_+ X \wedge (\mathbb{P}^1)^{\wedge n}$, for $X \in \mathrm{Sm}_S$ and $n \in \mathbb{Z}$. Similarly $\mathcal{SH}^{\mathrm{fr}}(S)$ is compactly generated by $\gamma^*(\Sigma^\infty_+ X \wedge (\mathbb{P}^1)^{\wedge n})$. It follows that $\gamma_* : \mathcal{SH}^{\mathrm{fr}}(S) \to \mathcal{SH}(S)$ is conservative and preserves colimits.

Conservativity of γ_* implies that in order to prove that γ^* is an equivalence, it suffices to show that it is fully faithful, or equivalently that the unit of adjunction $u : \mathrm{id} \to \gamma_*\gamma^*$ is an equivalence. Indeed the composite

$$\gamma_* \xrightarrow{u\gamma_*} \gamma_*\gamma^*\gamma_* \xrightarrow{\gamma_* c} \gamma_*$$

is the identity (γ^* and γ_* being adjoints), the first transformation is an equivalence by assumption, hence so is the second one, and finally so is the counit c since γ_* is conservative.

Since γ_* preserves colimits, the class of objects on which u is an equivalence is closed under colimits. Hence it suffices to show that u is an equivalence on the generators.

Given any adjunction $\gamma^* : \mathcal{C} \leftrightarrows \mathcal{D} : \gamma_*$ with γ^* symmetric monoidal, the right adjoint γ_* satisfies a projection formula for strongly dualizable objects: if $P \in \mathcal{C}$ is strongly dualizable, then there is an equivalence of functors $\gamma_*(- \otimes \gamma^* P) \simeq \gamma_*(-) \otimes P$. Indeed we have a sequence of binatural equivalences

$$\begin{aligned}
\mathrm{Map}(-, \gamma_*(- \otimes \gamma^* P)) &\simeq \mathrm{Map}(\gamma^*(-), - \otimes \gamma^* P) \\
&\simeq \mathrm{Map}(\gamma^*(- \otimes P^\vee), -) \\
&\simeq \mathrm{Map}(- \otimes P^\vee, \gamma_*(-)) \\
&\simeq \mathrm{Map}(-, \gamma_*(-) \otimes P),
\end{aligned}$$

and hence the result follows by the Yoneda lemma.

Since $\Sigma^\infty \mathbb{P}^1 \in \mathcal{SH}(S)$ is invertible and hence strongly dualizable, in order to prove Theorem 2.1 it is thus enough to show that for every $X \in \mathrm{Sm}_S$, the unit map

$$\Sigma^\infty_+ X \to \gamma_*\gamma^*\Sigma^\infty_+ X \in \mathcal{SH}(S)$$

is an equivalence. Using Zariski descent, we may further assume that X is affine.

2C. *Recollections on prespectra.* Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category, and $P \in \mathcal{C}$. We denote by $\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ the ∞-category whose objects are sequences $(X_1, X_2, \dots)$ with $X_i \in \mathcal{C}$, together with "bonding maps" $P \otimes X_i \to X_{i+1}$. The objects are called *prespectra*. The morphisms are the evident commutative diagrams. We call $X = (X_n)_n \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ an Ω-*spectrum* if the adjoints of the bonding maps, $X_i \to \Omega_P X_{i+1}$, are all equivalences. Here $\Omega_P : \mathcal{C} \to \mathcal{C}$ denotes the right adjoint of the functor $\Sigma_P := P \otimes (-)$. We denote by $L_{\mathrm{st}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P) \subset \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ the subcategory of Ω-spectra. The inclusion has a left adjoint which we denote by $L_{\mathrm{st}} : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P) \to L_{\mathrm{st}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$; the maps inverted by L_{st} are called stable equivalences.

Remark 2.3. If P is a symmetric object, i.e., for some $n \geq 2$ the cyclic permutation on $P^{\otimes n}$ is homotopic to the identity, then $L_{\mathrm{st}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P) \simeq \mathcal{C}[P^{-1}]$. This is proved in [33, Corollary 2.22]. See also [22, Theorems 10.1 and 10.3].

2C1. *Spectrification.* There is a natural transformation

$$\Sigma_P \Omega_P \xrightarrow{c} \mathrm{id} \xrightarrow{u} \Omega_P \Sigma_P.$$

Using this we can build a functor $Q_1 : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ with the property that for $X = (X_n)_n \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$ we have $Q_1(X)_n = \Omega_P X_{n+1}$. There is a natural transformation $\mathrm{id} \to Q_1$. Iterating this construction and taking the colimit we obtain

$$\mathrm{id} \to Q := \operatorname*{colim}_n Q_1^{\circ n}.$$

The following is well known (see, for example, [22]).

Lemma 2.4. *Let $X \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{C}, P)$.*

(1) *The map $X \to QX$ is a stable equivalence.*

(2) *If Ω_P preserves filtered colimits (i.e., $P \in \mathcal{C}$ is compact), then QX is an Ω-spectrum.*

2C2. *Prolongation.* Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. Following Hovey [22, Lemma 5.2], we call F *prolongable* if we are provided with a natural transformation $\tau : \Sigma_P F \to F \Sigma_P$; we denote the data of a prolongable functor as a pair (F, τ). Equivalently, we should provide a natural transformation $F \to \Omega_P F \Sigma_P$. In any case, there is an obvious category of prolongable endofunctors (having objects the pairs (F, τ) as above). Any prolongable functor (F, τ) induces an endofunctor

$$F : Sp^{\mathbb{N}}(\mathcal{C}, P) \to Sp^{\mathbb{N}}(\mathcal{C}, P), (X_n)_n \mapsto (F X_n)_n.$$

The bonding maps of FX are given by

$$\Sigma_P F(X_n) \xrightarrow{\tau_{X_n}} F(\Sigma_P X_n) \xrightarrow{F b_n} F(X_{n+1}),$$

where $b_n : \Sigma_P X_n \to X_{n+1}$ is the original bonding map.

Example 2.5. The functor $F_n = \Omega_P^n \Sigma_P^n$ is prolongable by $\Omega_P^n u \Sigma_P^n : F_n \to \Omega_P F_n \Sigma_P$, where $u : \mathrm{id} \to \Omega_P \Sigma_P$ is the unit transformation. One checks easily that

$$F_n \Sigma^\infty X \simeq Q_1^{\circ n} \Sigma^\infty X.$$

The transformation $\Omega_P^n u \Sigma_P^n$ defines a morphism $F_n \to F_{n+1}$ of prolongable functors; let F_∞ be its colimit. Then one checks that

$$F_\infty \Sigma^\infty X \simeq Q \Sigma^\infty X.$$

Example 2.6. The functor $F = \Sigma_P$ can be prolonged a priori in (at least) two ways: via the canonical isomorphism $\tau_1 : \Sigma_P F = \Sigma_P \Sigma_P = F \Sigma_P$ and via the switch map $\tau_2 : \Sigma_{P \otimes P} \to \Sigma_{P \otimes P}$. Then $F_1 \simeq F_2$ as prolongable functors if and only if the switch map on $P \otimes P$ is the identity. (Sometimes F_1 is called the *fake suspension* functor.)

Example 2.7. Let $F : \mathcal{C} \to \mathcal{C}$ be a lax $\mathcal{C}$-module functor, so that in particular for each $A \in \mathcal{C}$ we are given a transformation $\Sigma_A F \to F \Sigma_A$. Specializing to $A = P$ we obtain a prolongable functor $\tilde{F}$, natural in the lax $\mathcal{C}$-module functor F. The functor F_n (from Example 2.5) is a lax $\mathcal{C}$-module functor, via

$$A \otimes \underline{\mathrm{Hom}}(P^{\otimes n}, P^{\otimes n} \otimes X) \to \underline{\mathrm{Hom}}(P^{\otimes n}, P^{\otimes n} \otimes A \otimes X), \quad \text{“}(a \otimes f) \mapsto c_a \otimes f\text{”},$$

where c_a denotes the “constant map at a”.

Suppose that P is strongly dualizable with dual $P^\vee$. Then F_n and $\tilde{F}_n$ have equivalent underlying functors. However, their prolongation are described in different ways. The functor F_n can be written as

$$P^{\vee \otimes n} \otimes P^{\otimes n} \xrightarrow{u} P^{\vee \otimes n} \otimes P^{\otimes n} \otimes P^\vee \otimes P \xrightarrow{\sigma_{324}} P^{\vee \otimes n} \otimes P^\vee \otimes P \otimes P^{\otimes n} \simeq P^{\vee \otimes n+1} \otimes P^{\otimes n+1}.$$

On the other hand the prolongation of $\tilde{F}_n$ can be written as

$$P^{\vee \otimes n} \otimes P^{\otimes n} \xrightarrow{u} P^{\vee \otimes n} \otimes P^{\otimes n} \otimes P^\vee \otimes P \xrightarrow{\sigma_{123}} P^\vee \otimes P^{\vee \otimes n} \otimes P^{\otimes n} \otimes P \simeq P^{\vee \otimes n+1} \otimes P^{\otimes n+1}.$$

They are isomorphic if and only if the $(n+1)$-fold cyclic permutation acts trivially on $P^{\otimes n+1}$.

Example 2.8. Let $\mathrm{id} \xrightarrow{u} F_1 \xrightarrow{\rho} \mathrm{id}$ be a retraction of prolongable functors. Since $\rho : F_1 \to \mathrm{id}$ is a morphism of prolongable functors, the following square commutes:

$$\begin{array}{ccc} \Omega_P \Sigma_P = F_1 & \xrightarrow{\Omega_P u \Sigma_P} & \Omega_P F_1 \Sigma_P = \Omega_P^2 \Sigma_P^2 \\ \rho \downarrow & & \downarrow \Omega_P \rho \Sigma_P \\ \mathrm{id} & \xrightarrow{u} & F_1 = \Omega_P \Sigma_P. \end{array}$$

Hence $u\rho \simeq \Omega_P \rho u \Sigma_P \simeq \mathrm{id}_{\Omega_P \Sigma_P}$ and so u and ρ are inverse equivalences.

Remark 2.9. The prolongations using lax module structures interacts more reasonably with categorical constructions than the one via units of adjunction. For this reason it is more natural to have a retraction $\mathrm{id} \xrightarrow{u'} \tilde{F}_1 \xrightarrow{\rho} \mathrm{id}$. If P is 2-symmetric (i.e., the switch on $P^{\otimes 2}$ is the identity), then $F_1 \simeq \tilde{F}_1$ and $u' \simeq u$, under this equivalence. Hence u' and ρ are inverse equivalences. This holds more generally if P is n-symmetric for any $n \geq 2$; this is the content of Voevodsky's cancellation theorem. See Theorem 4.7 in Section 4.

2D. *Notions of framed correspondences.*

2D1. *(Tangentially) framed correspondences.* We have the lax Sm_{S+}-module functor

$$h^{\mathrm{fr}} : \mathrm{Sm}_{S+} \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*, \quad X_+ \mapsto \gamma_*\gamma^* X_+.$$

(Sections of $h^{\mathrm{fr}}(X)$ over Y are called *(tangentially) framed correspondences from Y to X*, but we will not use this terminology much. We will usually drop the adjective "tangentially".) We extend this to a sifted cocontinuous functor

$$h^{\mathrm{fr}} : \mathcal{P}_\Sigma(\mathrm{Sm}_S)_* \simeq \mathcal{P}_\Sigma(\mathrm{Sm}_{S+}) \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*.$$

Of course $\gamma_*\gamma^*$ is already sifted cocontinuous, so $h^{\mathrm{fr}} \simeq \gamma_*\gamma^*$ and this is just a notational change.

2D2. *Equationally framed correspondences.* There are explicitly defined lax Sm_{S+}-module functors [15, §2.1]

$$h^{\mathrm{efr},n} : \mathrm{Sm}_{S+} \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*$$

and natural transformations $\sigma : h^{\mathrm{efr},n} \to h^{\mathrm{efr},n+1}$. (Sections of $h^{\mathrm{efr},n}(X)$ over Y are called *equationally framed correspondences from Y to X of level n*.) We denote by

$$h^{\mathrm{efr},n} : \mathcal{P}_\Sigma(\mathrm{Sm}_S)_* \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*$$

the sifted cocontinuous extensions, and by

$$h^{\mathrm{efr}} : \mathcal{P}_\Sigma(\mathrm{Sm}_S)_* \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_*$$

the colimit along σ. We will elaborate on this in Section 3C.

2D3. *Relative equationally framed correspondences.* Let $U \subset X \in \mathrm{Sm}_S$ be an open immersion. There are explicitly defined presheaves

$$h^{\mathrm{efr},n}(X, U) \in \mathcal{P}_\Sigma(\mathrm{Sm}_{S+});$$

they depend functorially on the pair (X, U) and are lax modules, in a way which we will not elaborate on. (Sections of $h^{\mathrm{efr},n}(X, U)$ over Y are called *relative* *equationally framed correspondences from Y to (X, U) of level n.*) For us the most important case is where $X = X' \times \mathbb{A}^m$ and $U = X' \times \mathbb{A}^m \setminus X' \times \{0\}$; we put

$$h^{\mathrm{efr},n}(X', \mathcal{O}^n) = h^{\mathrm{efr},n}(X' \times \mathbb{A}^m, X' \times \mathbb{A}^m \setminus X' \times \{0\}).$$

These assemble into lax Sm_{S+}-module functors $\mathrm{Sm}_{S+} \to \mathcal{P}_\Sigma(\mathrm{Sm}_{S+})$. We will elaborate on this in Section 3B.

2E. ***Comparison results.*** We now explain the comparison results which go into the proof of the reconstruction theorem.

2E1. *Equationally framed versus tangentially framed.* There is a canonical transformation

$$h^{\mathrm{efr}} \to h^{\mathrm{fr}} \in \mathrm{Fun}(\mathrm{Sm}_{S+}, \mathcal{P}_\Sigma(\mathrm{Sm}_{S+})),$$

which is a motivic equivalence (objectwise) [15, Corollaries 2.2.20 and 2.3.25]. Since motivic equivalences are stable under (sifted) colimits, the sifted cocontinuous extension of the natural transformation is still a motivic equivalence objectwise. The transformations are compatible with the lax module structures.

2E2. *The cone theorem.* There is a canonical transformation

$$h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}(X, U);$$

here the left-hand side is obtained by sifted cocontinuous extension. This is a motivic equivalence for X affine, provided the base is an infinite field. This is known as the cone theorem, and will be treated in Section 3.

The natural transformation

$$h^{\mathrm{efr},n}(X \times \mathbb{A}^m / X \times \mathbb{A}^m \setminus X \times 0) \to h^{\mathrm{efr},n}(X, \mathcal{O}^m)$$

can be promoted to a lax module transformation.

2E3. *Voevodsky's lemma.* We denote by $T \in \mathcal{P}_\Sigma(\mathrm{Sm}_{S+})$ the presheaf quotient $\mathbb{A}^1/\mathbb{G}_m$. It comes equipped with a canonical map of presheaves $a : \mathbb{P}^1 \to L_{\mathrm{Zar}}T$ by presenting the domain as a (Zariski-local) pushout $\mathbb{A}^1 \cup_{\mathbb{G}_m} \mathbb{A}^1 \simeq \mathbb{P}^1$.

There is a canonical equivalence of lax module functors

$$h^{\mathrm{efr},n}(X, \mathcal{O}^m) \to \Omega^n_{\mathbb{P}^1} L_{\mathrm{Nis}} \Sigma^{n+m}_T X_+.$$

Following [18, §3], this is known as Voevodsky's lemma; see [15, Appendix A] for a proof. The equivalence is compatible with the natural stabilization maps (increasing n) on both sides.

2F. *Proof of reconstruction.* Write $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S) = L_{\mathrm{Nis}}\mathcal{P}_\Sigma(\mathrm{Sm}_S)$ and $\mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S) = L_{\mathrm{Nis}}\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))$.

Lemma 2.10. *The forgetful functor $\mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S) \to \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)$ preserves and detects motivic equivalences.*

Proof. Immediate from [15, Proposition 3.2.14]. □

Since $\gamma^* : \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_* \to \mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S)$ is symmetric monoidal, it induces a functor $\gamma^*_{\mathbb{N}}$ upon passage to prespectra. We obtain an adjunction

$$\gamma^*_{\mathbb{N}} : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1) \leftrightarrows \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S), \gamma^*\mathbb{P}^1) : \gamma^{\mathbb{N}}_*;$$

the right adjoint $\gamma^{\mathbb{N}}_*$ is given by the formula $\gamma^{\mathbb{N}}_*(X)_n \simeq \gamma_*(X_n)$. We call a map $X \to Y \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1)$ a *level motivic equivalence* if each map $X_n \to Y_n$ is a motivic equivalence, and similarly for framed prespectra. The saturated class generated by level motivic equivalences and stable equivalences is called stable motivic equivalences. Local objects for this class of maps are called motivic Ω-spectra; these are the prespectra $X = (X_n)_n$ such that X is an Ω-spectrum and each X_n is motivically local.

Corollary 2.11. *The functor $\gamma^{\mathbb{N}}_*$ preserves and detects stable motivic equivalences.*

Proof. Since $\gamma^{\mathbb{N}}_*$ preserves motivic Ω-spectra (from its formula above) it is enough to show that it commutes with spectrification. Let $X = (X_n)_n$ be a prespectrum. By Lemma 2.4(2), its spectrification is given by

$$(QL_{\mathrm{mot}}X)_n = \operatorname*{colim}_i \Omega^i_{\mathbb{P}^1} L_{\mathrm{mot}} X_{n+i}.$$

Since $\gamma_* : \mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S) \to \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*$ preserves motivic equivalences, filtered colimits (both by Lemma 2.10), and $\mathbb{P}^1$-loops, the result follows. □

We also note the following.

Lemma 2.12. *There are canonical equivalences*

$$L_{\mathrm{st,mot}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1) \simeq \mathcal{SH}(S)$$

and

$$L_{\mathrm{st,mot}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}^{\mathrm{fr}}_{\mathrm{Nis}}(S), \gamma^*\mathbb{P}^1) \simeq \mathcal{SH}^{\mathrm{fr}}(S).$$

Proof. We prove the result for unframed spectra; the other case is similar. It is easy to see that $L_{\mathrm{mot}}\mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1) \simeq \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{pc}(S)_*, \mathbb{P}^1)$ (see, for example, [3, Lemma 26]). But $\mathbb{P}^1$ is symmetric in $\mathcal{S}\mathrm{pc}(S)_*$ [24, Lemma 6.3], and hence the result follows from Remark 2.3. □

Let $G : \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_* \to \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*$ be an endofunctor. We say that G is *mixed prolongable* if we are given a natural transformation $\Sigma_{\mathbb{P}^1} G \to G\Sigma_T$. Then G naturally induces a functor

$$G : \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, T) \to \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1).$$

Let $G_n = \Omega^n_{\mathbb{P}^1}\Sigma^n_T$. This is mixed prolongable via

$$\Omega^n_{\mathbb{P}^1}\Sigma^n_T \xrightarrow{\Omega^n_{\mathbb{P}^1} u \Sigma^n_T} \Omega^n_{\mathbb{P}^1}\Omega_T\Sigma^{n+1}_T \xrightarrow{a^*} \Omega^{n+1}_{\mathbb{P}^1}\Sigma^{n+1}_T;$$

here $a : \mathbb{P}^1 \to T$ is the canonical map and $u : \mathrm{id} \to \Omega_T\Sigma_T$ is the unit of adjunction. For $X \in \mathrm{Sm}_S$, let $\Sigma^\infty_T X$ denote the associated T-suspension prespectrum. Then

$$G_0\Sigma^\infty_T X = (X, T \wedge X, T^2 \wedge X, \dots) \in \mathcal{S}\mathrm{p}^{\mathbb{N}}(\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(S)_*, \mathbb{P}^1)$$

is a spectrum motivically equivalent to $\Sigma^\infty_{\mathbb{P}^1} X$. By Corollary 2.11 and Lemma 2.12 it is hence enough to show that

$$G_0\Sigma^\infty_T X \to \gamma^{\mathbb{N}}_* \gamma^*_{\mathbb{N}} G_0\Sigma^\infty_T X$$

is a stable motivic equivalence. There are canonical maps of mixed prolongable functors $G_0 \to G_1 \to \cdots$, and one checks that

$$QG_0\Sigma^\infty_T X \simeq \operatorname*{colim}_i G_i\Sigma^\infty_T X.$$

In particular the map

$$G_0\Sigma^\infty_T X \to \operatorname*{colim}_i G_{2i+1}\Sigma^\infty_T X$$

is a stable equivalence.

The functor $\Omega^n_{\mathbb{P}^1}\Sigma^n_T$ is mixed prolongable in another way, using the lax module structure. Denote the mixed prolongable functor obtained in this way by $\tilde{G}_n$. Arguing as in Example 2.7, G_n and $\tilde{G}_n$ differ by cyclic permutations of $\mathbb{P}^1, T$ of order $n+1$. Note that the functor $\underline{\mathrm{Hom}}(-,-)$ preserves $\mathbb{A}^1$-homotopy equivalences in both variables. Since the cyclic permutation on $(\mathbb{P}^1)^{\wedge 2n+1}$ is $\mathbb{A}^1$-homotopic to the identity,[3] and the same holds for T, we deduce $G_{2i+1} \overset{\mathbb{A}^1}{\simeq} \tilde{G}_{2i+1}$ as prolongable functors. We learn that the canonical map

$$G_0\Sigma^\infty_T X \to \operatorname*{colim}_i G_{2i+1}\Sigma^\infty_T X \overset{\mathbb{A}^1}{\simeq} \operatorname*{colim}_i \tilde{G}_{2i+1}\Sigma^\infty_T X \simeq \operatorname*{colim}_i \tilde{G}_i\Sigma^\infty_T X$$

is a stable $\mathbb{A}^1$-equivalence.

Let E_i denote the sifted cocontinuous approximation[4] of $\tilde{G}_i$, so that there is a map $E_i \to \tilde{G}_i$ of mixed prolongable functors. We can view h^{efr} (and h^{fr}) as mixed

[3]Observe the equality $(1, 2, 3)(3, 4, 5)\cdots(2n-1, 2n, 2n+1) = (1, 2, \dots, 2n, 2n+1)$.

[4]In the sense that we restrict G_i to Sm_{S+} and then take the sifted cocontinuous extension.

prolongable functors (note that they preserve Nisnevich equivalences in $\mathcal{P}_\Sigma(\mathrm{Sm}_{S+})$ by [15, Propositions 2.3.7(ii) and 2.1.5(iii)] and so descend to Nisnevich sheaves) by using their lax module structures. By Voevodsky's lemma and the fact that both functors are sifted cocontinuous extensions from smooth schemes, $E_i \simeq h^{\mathrm{efr},i}$ as lax modules and hence as mixed prolongable functors. Thus by the cone theorem (see Theorem 3.1 and Remark 3.4 in Section 3 for more details), the map

$$E_i \Sigma^\infty_T X \to \tilde{G}_i \Sigma^\infty_T X$$

is a level motivic equivalence (here we use that the base is an infinite field). We obtain the commutative diagram

$$\begin{array}{ccccccc} G_0\Sigma^\infty_T X & \xrightarrow{L_{\mathrm{st}}} & G_\infty\Sigma^\infty_T X & \xleftrightarrow{L_{\mathbb{A}^1}} & \tilde{G}_\infty\Sigma^\infty_T X & & \\ & \searrow & & L_{\mathrm{mot}} \nearrow & & & \\ & & E_\infty\Sigma^\infty_T X & \xrightarrow{\simeq} & h^{\mathrm{efr}}\Sigma^\infty_T X & \xrightarrow{L_{\mathrm{mot}}} & h^{\mathrm{fr}}\Sigma^\infty_T X. \end{array}$$

All maps are the canonical ones; labels on the arrows denote the type of equivalence. The composite $G_0\Sigma^\infty_T X \to h^{\mathrm{fr}}\Sigma^\infty_T X \simeq \gamma^{\mathbb{N}}_* \gamma^*_{\mathbb{N}} G_0 \Sigma^\infty_T X$ is the unit of adjunction. The diagram proves this unit is a stable motivic equivalence. This concludes the proof.

3. The cone theorem

Primary sources: [10; 20].

3A. ***Introduction.*** The cone theorem is the determination of the motivic homotopy type of $h^{\mathrm{efr}}(X/U)$, i.e., the "framed cone" of an open immersion $U \hookrightarrow X$ where X is smooth. In the proof of the reconstruction theorem, coupled with Voevodsky's lemma (Lemma 3.2), it relates the endofunctor on pointed Nisnevich sheaves given by $\Omega_{\mathbb{P}^1}\Sigma_T$ and the sifted cocontinuous extension of a framed model of this functor.

Theorem 3.1. *Let k be an infinite field, X a smooth affine k-scheme, and $U \subset X$ open. Then there is a canonical motivic equivalence*

$$h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}(X, U).$$

For now we work over an arbitrary base scheme S. We have already discussed Voevodsky's lemma that describes $h^{\mathrm{efr},n}(X)$ in terms of maps of pointed sheaves (see Section 2E3). In general we can describe the sections of the (pointed) sheaf

$$L_{\mathrm{Nis}}(X/U),$$

as follows. Define

$$Q(X, U)(T) = \{(Z, \phi) \mid Z \subset T \text{ closed}, \phi : T^h_Z \to X,\ \phi^{-1}(X \setminus U) = Z\},$$

which is pointed at $(\varnothing, \mathrm{can})$. Here T_Z^h denotes the henselization of T in Z. There is canonical map

$$Q(X, U) \to L_{\mathrm{Nis}}(X/U),$$

which sends a section (Z, ϕ) over T to the map

$$T \simeq L_{\mathrm{Nis}}(T_Z^h \amalg_{T_Z^h \setminus Z} T \setminus Z) \xrightarrow{\phi} X/U.$$

Lemma 3.2 [15, Proposition A.1.4]. *The map $Q(X, U) \to L_{\mathrm{Nis}}(X/U)$ is an isomorphism.*

The presheaf of equationally framed correspondences of level n can be phrased in these terms. Let us elaborate on how this is done. Recall that we have n closed immersions $(\mathbb{P}^1)^{n-1} \hookrightarrow (\mathbb{P}^1)^n$ as the components of the "divisor at ∞" — so that $\bigcup (\mathbb{P}^1)^{\times n-1}$ is the divisor $\partial\mathbb{P}$. We then have the fiber sequence (in sets)

$$\begin{aligned} h^{\mathrm{efr},n}(X)(T) &\to Q(\mathbb{A}^n \times X, \mathbb{A}^n_X \setminus 0_X)((\mathbb{P}^1)^{\times n} \times T) \\ &\to \prod_{1 \le i \le n} Q(\mathbb{A}^n \times X, \mathbb{A}^n_X \setminus 0_X)((\mathbb{P}^1)^{\times n-1} \times T). \end{aligned}$$

Via Lemma 3.2, $h^{\mathrm{efr},n}(X)$ is isomorphic to

$$\underline{\mathrm{Hom}}_{\mathcal{P}_\Sigma(\mathrm{Sm}_{S+})}((\mathbb{P}^1)^{\wedge n} \wedge (-)_+, L_{\mathrm{Nis}}(T^{\wedge n} \wedge X_+)).$$

3B. *Relative equationally framed correspondences.* We elaborate on the discussion in Section 2D3. Throughout X is a smooth affine S-scheme and we have a cospan of S-schemes

$$Y \xhookrightarrow{i} X \xhookleftarrow{j} X \setminus Y (=: U),$$

where i is a closed immersion and j is its open complement. The presheaf of *relative equationally framed correspondences* $h^{\mathrm{efr},n}(X, U)$ is then defined via a similar formula:

$$\begin{aligned} h^{\mathrm{efr},n}(X, U)(T) &\to Q(\mathbb{A}^n \times X, \mathbb{A}^n_X \setminus (0 \times Y))((\mathbb{P}^1)^{\times n} \times T) \\ &\to \prod_{1 \le i \le n} Q(\mathbb{A}^n \times X, \mathbb{A}^n_X \setminus (0 \times Y))((\mathbb{P}^1)^{\times n-1} \times T). \end{aligned}$$

The next lemma follows from the above discussion.

Lemma 3.3. *There is a canonical isomorphism of sheaves of sets*

$$h^{\mathrm{efr},n}(X, U) \simeq \underline{\mathrm{Hom}}_{\mathcal{P}_\Sigma(\mathrm{Sm}_{S+})}((\mathbb{P}^1)^{\wedge n} \wedge (-)_+, L_{\mathrm{Nis}}(T^{\wedge n} \wedge (X/U))).$$

Remark 3.4. Consider the functor $G : \mathcal{P}_\Sigma(\mathrm{Sm}_{S+}) \to \mathcal{P}_\Sigma(\mathrm{Sm}_{S+})$ given by

$$G(P) = \underline{\mathrm{Hom}}_{\mathcal{P}_\Sigma(\mathrm{Sm}_{S+})}((\mathbb{P}^1)^{\wedge n} \wedge (-)_+, L_{\mathrm{Nis}}(T^{\wedge n} \wedge P)).$$

Write $c : E \to G$ for the sifted-cocontinuous approximation of G (that is, the left-Kan extension of $E|_{\mathrm{Sm}_{S+}}$). Then by Voevodsky's lemma we have $E \simeq h^{\mathrm{efr},n}$. Consequently we obtain a natural map

$$c_{X/U} : h^{\mathrm{efr},n}(X/U) \simeq E(X/U) \to G(X/U) \simeq h^{\mathrm{efr},n}(X, U).$$

This is the map of Theorem 3.1.

Explicitly, elements of $h^{\mathrm{efr},n}(X, U)(T)$ are described as (equivalence classes of) tuples

$$(Z, (\phi, g), W),$$

where

(1) $Z \hookrightarrow \mathbb{A}^n_T$ is a closed subscheme, finite over T,
(2) W is an étale neighborhood of Z in $\mathbb{A}^n_T$,
(3) $(\phi, g) : W \to \mathbb{A}^n \times X$ is a morphism such that

$$Z = (\phi, g)^{-1}(0 \times Y) = \phi^{-1}(0) \cap g^{-1}(Y).$$

For example, suppose $X = \mathbb{A}^1$ and $U = \mathbb{G}_m$. Then $h^{\mathrm{efr}}_n(\mathbb{A}^1, \mathbb{G}_m)$ is isomorphic to

$$\underline{\mathrm{Hom}}_{\mathcal{P}_\Sigma(\mathrm{Sm}_{S+})}((\mathbb{P}^1)^{\wedge n} \wedge (-)_+, L_{\mathrm{Nis}} T^{\wedge n+1}).$$

Remark 3.5. The subscheme Z in the definition of $Q(X, U)((\mathbb{P}^1)^{\wedge n})$ is not required to be finite. However, in the definition of $h^{\mathrm{efr},n}(X, U)$, the Z appearing is a closed subset of both $(\mathbb{P}^1)^{\times n}$ and $\mathbb{A}^n$, so both proper and affine, hence finite.

We will also need the next presheaf.

Definition 3.6. Let $h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U) \subset h^{\mathrm{efr}}(X, U)$ be the subpresheaf consisting of those $(Z, (\phi, g), W)$ where $\phi^{-1}(0) \to T$ is quasifinite.

Remark 3.7. Recall that the scheme W in an equationally framed correspondence is well defined only up to refinement. If $p : W' \to W$ is such a refinement and $\phi^{-1}(0)$ is quasifinite, then so is $(\phi \circ p)^{-1}(0)$, p being quasifinite. The converse need not hold.

Example 3.8. In $h^{\mathrm{efr},1}(\mathbb{A}^1, \mathbb{G}_m)(k)$, we have the cycle $c = (Z = 0_k, (0, x), \mathbb{A}^1)$, where 0 indicates the constant function at zero, so we are considering the zero locus of the map

$$(0, x) : \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1.$$

In this situation, $0^{-1}(0) = \mathbb{A}^1$ and hence is *not* quasifinite over the base field, so $c \notin h^{\mathrm{efr},1}_{\mathrm{qf}}(\mathbb{A}^1, \mathbb{G}_m)(k)$. On the other hand $0^{-1}(0) \cap x^{-1}(0) = 0$, which restores the finiteness of Z, as needed. Generically, we should expect quasifiniteness of $\phi^{-1}(0)$ — the only function we need to avoid in the above example is literally the constant function at zero.

The relevance of the quasifinite version is the following.

Construction 3.9. We have a map

$$h^{\mathrm{efr},n}(X) \to h^{\mathrm{efr},n}(X, U), \quad (W, (\phi, g), Z) \mapsto (W, (\phi, g),\ Z_Y = \phi^{-1}(0) \cap g^{-1}(Y)),$$

which factors as

$$h^{\mathrm{efr},n}(X) \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U) \subset h^{\mathrm{efr},n}(X, U), \tag{1}$$

since $\phi^{-1}(0)$ is, in fact, finite. Now, consider the diagram

$$h^{\mathrm{efr},n}(X \amalg U) \rightrightarrows h^{\mathrm{efr},n}(X),$$
$$(Z, (\phi, g), W) \mapsto ((Z, (\phi, \nabla \circ g), W), (Z_X, (\phi_X, g_X), W_X)),$$

where $(Z_X, (\phi_X, g_X), W_X)$ is the component of $(Z, (\phi, g), W)$ over X, and $\nabla : X \amalg U \to X$ is the fold map. Denote the set-theoretic coequalizer of this diagram (taken sectionwise) by $\tau_{\leq 0} h^{\mathrm{efr},n}(X/U)$. (This notation is justified in Section 3C.) The map (1) then further factors as

$$\begin{array}{ccc} & h^{\mathrm{efr}}(X) & \\ \swarrow & & \searrow \\ \tau_{\leq 0} h^{\mathrm{efr},n}(X/U) & \dashrightarrow & h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U). \end{array}$$

We can explicitly describe the sections of the presheaf $\tau_{\leq 0} h^{\mathrm{efr},n}(X/U)$: if $T \in \mathrm{Sm}_S$, then $\tau_{\leq 0} h^{\mathrm{efr},n}(X/U)(T)$ is the quotient of $h^{\mathrm{efr},n}(X)$ modulo the equivalence relation generated by

$$(W, (\phi, g), Z) \sim (W', (\phi', g'), Z'),$$

whenever there exists $(W'', (\phi'', g''), Z'')$ such that $g'' : W'' \to U \subset X$, $W = W' \amalg W''$ up to refining the étale neighborhoods, and $(g, \phi) = (g', \phi') \amalg (g'', \phi'')$.

Remark 3.10. We warn the reader that the canonical map $h^{\mathrm{efr},n}(X \amalg Y) \to h^{\mathrm{efr},n}(X) \times h^{\mathrm{efr}}(Y)$ is not an equivalence (unless $X = \varnothing$ or $Y = \varnothing$). It becomes so after applying $L_{\mathbb{A}^1}$ and letting $n \to \infty$ [15, Remark 2.19; 18, Theorem 6.4].

Lemma 3.11. *Let S be any scheme. The map $\tau_{\leq 0} h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)$ is an L_{Nis}-equivalence.*

Proof. Let T be the henselization of a smooth S-scheme in a point. It suffices to show that the map on sections over T is both surjective and injective.

Surjectivity: Take $(Z, (\phi, g), W) \in h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)(T)$ and put $V = \phi^{-1}(0)$, so that $Z = V \cap g^{-1}(Y)$. We may assume that W is affine (see [15, Lemma A.1.2(ii)]), and hence so is V. Since V is quasifinite, we may write

$$V = V_1 \amalg \cdots \amalg V_{n+1},$$

where V_i is local and finite over T for $i \le n$, and $V_{n+1} \to T$ misses the closed point [34, Tag 04GJ]. Similarly $Z = Z_1 \amalg \cdots \amalg Z_d$. We may assume that $Z_i \subset V_i$ and $d \le n$ (note that $Z_i \to T$ hits the closed point by finiteness, and hence $Z_i \not\subset V_{n+1}$). Removing $V_{d+1} \cup \cdots \cup V_{n+1}$ from W, we may also assume that $n = d$ and $Z_{n+1} = \varnothing$. In particular V is finite over T. It remains to prove that $V \to \mathbb{A}^n_T$ is a closed immersion. Denote by $\bar{V} \subset \mathbb{A}^n_T$ the image of V, which is a closed subscheme finite over T. We can write $W_{\bar{V}} = W_1 \amalg W_2$, where W_1 is finite over $\bar{V}$ and W_2 misses the closed points. Then $W_2 \subset W$ is closed and misses all closed points of V, so $V \subset W \setminus W_2 =: W'$. Now $W'_{\bar{V}} = W_1$ and so $W'_V \to V$ is finite étale; also $W'_Z \to Z$ is an isomorphism, whence so is $W'_V \to V$ [34, Tag 04GK]. It follows that $W'_V \simeq V \to W'$ is a closed immersion, and hence $V \to \mathbb{A}^n_T$ is a locally closed immersion (using fpqc descent [34, Tag 02L6]). Since V is finite, this is a closed immersion.

Injectivity: Consider two cycles

$$c = (Z, (\phi, g), W), \quad c' = (Z', (\phi', g'), W'),$$

with the same image in $h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)$. Put $Z_1 = Z \cap g^{-1}(Y)$ and $Z'_1 = Z' \cap g^{-1}(Y)$. In other words $Z_1 = Z'_1$ and there exists an étale neighborhood W'' refining W and W' such that $(\phi, g)|_{W''} = (\phi', g')|_{W''}$. We may write $Z = C \amalg D$, where $D \cap Z_1 = \varnothing$ and every component of C meets Z_1 (using again [34, Tag 04GJ]). Shrinking W to remove D replaces c by a cycle with the same image in $\tau_{\le 0} h^{\mathrm{efr},n}(X/U)$; we may thus assume that $D = \varnothing$. Now $\sigma : W''_Z \to Z$ is open and its image contains all closed points, so σ is surjective. Since every closed point of Z lifts along σ and σ is étale, it follows that σ admits a section [34, Tags 04GJ and 04GK]. Thus, shrinking W'' if necessary, we may assume that it is an étale neighborhood of Z. Arguing the same way for Z' concludes the proof. □

3C. ***Quotients versus homotopy quotients.*** The quotient X/U is given by the geometric realization of the following diagram in presheaves (also called a "bar construction"):

$$X_+ \leftleftarrows (X \amalg U)_+ \Lleftarrow (X \amalg U \amalg U)_+ \cdots . \tag{2}$$

By definition (as sifted-colimit preserving extensions) we get that $h^{\mathrm{efr}}(X/U)$ is the colimit of the simplicial diagram

$$h^{\mathrm{efr}}(X) \leftleftarrows h^{\mathrm{efr}}(X \amalg U) \Lleftarrow h^{\mathrm{efr}}(X \amalg U \amalg U) \cdots . \tag{3}$$

We remark that the first two maps coincide with those from Construction 3.9. There is thus a canonical map

$$h^{\mathrm{efr},n}(X/U) \to \tau_{\leq 0} h^{\mathrm{efr},n}(X/U),$$

which witnesses 0-truncation of the resulting geometric realization.

Construction 3.12. Composing with the map from Construction 3.9, we get maps

$$h^{\mathrm{efr},n}(X/U) \to \tau_{\leq 0} h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U) \hookrightarrow h^{\mathrm{efr},n}(X, U).$$

The composite is the map in question in the cone theorem.

We now claim that the first map is an equivalence, i.e., $h^{\mathrm{efr},n}(X/U)$ is 0-truncated.

Construction 3.13. Let $\mathbf{efr}(X, U)(T)$ denote the following (1-)category (in fact, a poset):

- The objects are elements of $h^{\mathrm{efr}}(X)(T)$.
- There is a morphism

$$(Z, (\phi, g), W) \to (Z', (\phi', g'), W')$$

  if and only if there exists a decomposition $Z \amalg Z'' = Z'$, $g'|_{Z''}$ factors through $U \subset X$, and $(\phi', g')|_{W'^h_Z} = (\phi, g)|_{W^h_Z}$.

Lemma 3.14. *There is canonical equivalence*

$$|N_\bullet \mathbf{efr}(X, U)(T)| \simeq h^{\mathrm{efr}}(X/U).$$

Proof. For this proof we will abbreviate $(W, (\phi, g), Z)$ as (Z, Φ); as we manipulate these cycles what happens on the data of the étale neighborhood and defining functions will be clear. For each n, we have a map

$$N_n \mathbf{efr}(X, U)(T) \to h^{\mathrm{efr}}(X \amalg U^{\amalg n})(T),$$

given by

$$(Z_0, \Phi_0) \to \cdots \to (Z_n, \Phi_n) \mapsto (Z_0 \amalg (Z_1 \setminus Z_0) \amalg (Z_2 \setminus Z_1) \amalg (Z_n \setminus Z_{n-1}), \Phi_n).$$

On the other hand, if $(Z, \Phi) \in h^{\mathrm{efr}}(X \amalg U^{\amalg n})(T)$ we get cycles $\{Z'_i, \Phi'_i\}_{i \geq 1}$ by pulling back along the various inclusions $\{\iota_i : U \hookrightarrow X \amalg U^{\amalg n}\}$ and also a cycle (Z_0, Φ_0) by pulling back along $X \hookrightarrow X \amalg U^{\amalg n}$. This defines an element $N_n \mathbf{efr}(X, U)(T)$ by setting $(Z_i, \Phi) = (Z_0 \amalg Z'_1 \amalg \cdots Z'_i, \Phi_i)$, with the maps determined. These maps induce mutual inverses of simplicial sets. □

Lemma 3.15. *The space* $|N_\bullet \mathbf{efr}(X, U)(T)|$ *is* 0*-truncated.*

Proof. Consider the subcategory

$$\mathbf{efr}(X, U)(T)^0 \subset \mathbf{efr}(X, U)(T),$$

consisting of those cycles (Z, Φ) such that no (nonempty) connected component of Z factors through U. Then $\mathbf{efr}(X, U)(T)^0$ is a category with no nonidentity arrows, whence $|N_\bullet \mathbf{efr}(X, U)(T)^0|$ is 0-truncated. The inclusion $\mathbf{efr}(X, U)(T)^0 \to \mathbf{efr}(X, U)(T)$ admits a right adjoint (given by discarding all components of Z that factor through U), and hence induces an equivalence on classifying spaces. The result follows. □

It follows that the canonical map

$$h^{\mathrm{efr},n}(X/U) \to \tau_{\leq 0} h^{\mathrm{efr},n}(X/U)$$

is a sectionwise equivalence of spaces. Combining Lemmas 3.11, 3.14 and 3.15, we have proved the following result.

Theorem 3.16. *Let S be a scheme. The map*

$$h^{\mathrm{efr},n}(X/U) \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)$$

is an L_{Nis}-equivalence.

3D. ***Moving into quasifinite correspondences.*** In order to complete the proof of the cone theorem, we will need the following result.

Theorem 3.17. *Let $S = Spec(k)$, where k is an infinite field. The inclusion of presheaves*

$$h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U) \hookrightarrow h^{\mathrm{efr},n}(X, U),$$

is an $L_{\mathbb{A}^1}$-equivalence.

This is a moving lemma in motivic homotopy theory.

Remark 3.18. In [20], this moving lemma was discovered for $X = \mathbb{A}^n$ and $U = \mathbb{A}^n \backslash 0$ which suffices for the purposes of computing the framed motives of algebraic varieties. We will follow the treatment [10] which performs the moving lemma for more general pairs.

For the rest of this section, we work over an infinite field. We fix the smooth affine scheme X, its open subscheme U and closed complement Y. Write $\overline{X}$ for a projective closure of X, and $\overline{Y} := \overline{X} \setminus U$. By considering the Segre embedding, we find a very ample line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n \times \overline{X}$ with a section x_0 such that $x_0|_{\mathbb{A}^n \times X}$ is nonvanishing. We also have sections $x_1, \ldots, x_n \in H^0(\mathbb{P}^n \times \overline{X}, \mathcal{O}(1))$ such that $x_i/x_0|_{\mathbb{A}^n \times X}$ are the usual coordinates on $\mathbb{A}^n$. Denote by $\mathcal{N} \subset \mathbb{P}^n \times \overline{X}$ the closed subscheme which is the first-order thickening of $0 \times \overline{Y}$. Pick $d > 0$. Set

$$\vec{x} = (x_1 x_0^{d-1}, x_2 x_0^{d-1}, \ldots, x_n x_0^{d-1}) \in H^0(\mathbb{P}^n \times \overline{X}, \mathcal{O}(d)^{\oplus n})$$

and

$$H^0(\mathbb{P}^n \times \overline{X}, \mathcal{O}(d)^{\oplus n}) \supset \Gamma_d := \{\vec{s} \mid \vec{s}|_{\mathcal{N}} = \vec{x}|_{\mathcal{N}}\}.$$

Note that Γ_d is a finite-dimensional[5] affine k-space, which we will view as an affine scheme.

Suppose that $\vec{s} = (s_1, \ldots, s_n) \in \Gamma_d(k)$. Then $\vec{s}|_{\mathbb{A}^n \times X}/x_0^d$ defines a regular map $\mathbb{A}^n \times X \to \mathbb{A}^n$. Combining with the projection $\mathbb{A}^n \times X \to X$ we obtain

$$f_{\vec{s}} : \mathbb{A}^n \times X \to \mathbb{A}^n \times X.$$

By construction, $f_{\vec{s}}$ is the identity in the first-order neighborhood of $0 \times Y \subset \mathbb{A}^n \times X$. This has the following significance.

Lemma 3.19. *Let $\varphi : W \to \mathbb{A}^n_X$ be arbitrary. Set $Z = \varphi^{-1}(0 \times Y) \subset W$. For $\vec{s} \in \Gamma(k)$ we have*

$$(f_{\vec{s}} \circ \varphi)^{-1}(0 \times Y) = Z \amalg Z'$$

(*for some Z' depending on $\vec{s}$*).

Proof. Let $Z_1 = f_{\vec{s}}^{-1}(0 \times Y)$. It suffices to prove that $0 \times Y \to Z_1$ is an open (whence clopen) immersion. Since $f_{\vec{s}}|_{\mathcal{N}} = \mathrm{id}$, we get $Z_1 \cap \mathcal{N} = Z$. In other words, if I is the sheaf of ideals defining $0 \times Y$, then $I|_{Z_1} = I^2|_{Z_1}$. The result follows by [34, Tag 00EH]. □

Construction 3.20. If $(Z, (\phi, g), W) \in h^{\mathrm{efr}}(X, U)(T)$, then we define

$$\vec{s} \cdot (Z, (\phi, g), W) = (Z, (f_{\vec{s}} \circ (g, \phi)), W \setminus Z').$$

This makes sense by Lemma 3.19 and yields, in fact, an action

$$\Gamma_d \times h^{\mathrm{efr},n}(X, U) \to h^{\mathrm{efr},n}(X, U),\ (\vec{s}, \Phi) \mapsto \vec{s} \cdot \Phi.$$

Multiplication by x_0 induces an injection $\Gamma_d \to \Gamma_{d+1}$. Write $\Gamma_\infty = \bigcup_d \Gamma_d$. Note that the action of Γ_d on $h^{\mathrm{efr},n}(X, U)$ factors through multiplication by d and hence induces an action by Γ_∞.

We need to be able to draw paths in Γ_d with controlled properties. This is made precise by the next result, whose proof will be discussed in Section 3F.

Lemma 3.21. *Let $T_1, \ldots, T_n \in \mathrm{Sm}_k$, $c_i \in h^{\mathrm{efr},n}(X, U)(T_i)$, and $V_i \subset \Gamma_\infty$ finite dimensional.*

Then there exists $\vec{\gamma} \in \Gamma_\infty \setminus \bigcup_i V_i$ such that, for all i, if $V_i' \subset \Gamma_\infty$ is the cone on V_i with tip $\vec{\gamma}$, then for all $\vec{v} \in V_i' \setminus V_i$ we have $\vec{v} \cdot c_i \in h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)(T_i)$. We can arrange that if $\vec{x} \in V_i'$ then already $\vec{x} \in V_i$.

[5]This is the reason for compactifying X.

Remark 3.22. Taking $V_i = \{\vec{x}\}$, the lemma in particular asserts that we can use paths in Γ_∞ to make correspondences quasifinite. The more general case $V_i \neq \{\vec{x}\}$ is used to show that these paths are essentially unique.

3E. *Filtration and finishing the proof.* Granting ourselves the above lemma, we finish the proof of the cone theorem.

We begin with some preparations. Let A be a category and $D : A \to \mathcal{P}(\mathrm{Sm}_k)$ be an A-indexed diagram. We construct a simplicial object

$$\mathrm{Tel}_A(D)_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{P}(\mathrm{Sm}_k))$$

by setting

$$\mathrm{Tel}_A(D)_n = \coprod_{i_0 \to i_1 \to \cdots \to i_n \in \mathcal{C}} D(i_0).$$

The simplicial structure maps involve the cosimplicial structure maps in the standard cosimplicial category $[\bullet]$ and the functoriality of D. This is a standard construction; see, for example, [13, §4]. The standard cosimplicial affine scheme $\mathbb{A}^\bullet$ yields a functor $S^\bullet_{\mathbb{A}} : \mathcal{P}(\mathrm{Sm}_k) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{P}(\mathrm{Sm}_k))$ which has a left adjoint $|-|_{\mathbb{A}^1}$.

Lemma 3.23. *The geometric realization* $|\mathrm{Tel}_A(D)_\bullet|_{\mathbb{A}^1}$ *is* $\mathbb{A}^1$*-equivalent to* $\mathrm{colim}_A\, D$.

Proof. For $F \in \mathcal{P}(\mathrm{Sm}_k)$ $\mathbb{A}^1$-invariant we have

$$\mathrm{Map}(|X_\bullet|_{\mathbb{A}^1}, F) \simeq \mathrm{Map}(X_\bullet, S^\bullet_{\mathbb{A}} F) \simeq \mathrm{Map}(X_\bullet, cF) \simeq \mathrm{Map}(|X_\bullet|, F),$$

where cF denotes the constant simplicial presheaf. The result follows since the usual geometric realization of $\mathrm{Tel}_A(D)_\bullet$ is a standard model for the sectionwise homotopy colimit of D [13, §4]. □

Proof of Theorem 3.17. We shall supply a filtered poset A as well as systems of subpresheaves

$$\{h^{\mathrm{efr},n}(X,U)^\alpha\}_{\alpha\in A} \subset h^{\mathrm{efr},n}(X,U), \quad \{h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U)^\alpha\}_{\alpha\in A} \subset h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U)$$

such that

$$h^{\mathrm{efr},n}(X,U) = \bigcup_{\alpha\in A} h^{\mathrm{efr},n}(X,U)^\alpha \quad \text{and} \quad h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U) = \bigcup_{\alpha\in A} h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U)^\alpha.$$

Next we construct for $\alpha = (\alpha_0 \leq \cdots \leq \alpha_n)$, $\alpha_i \in A$, maps

$$r_\alpha : h^{\mathrm{efr},n}(X,U)^{\alpha_0} \times \mathbb{A}^n \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U)$$

and

$$H_\alpha : \mathbb{A}^1 \times h^{\mathrm{efr},n}(X,U)^{\alpha_0} \times \mathbb{A}^n \to h^{\mathrm{efr},n}(X,U),$$
$$K_\alpha : \mathbb{A}^1 \times h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U)^{\alpha_0} \times \mathbb{A}^n \to h^{\mathrm{efr},n}_{\mathrm{qf}}(X,U),$$

all compatible with the (co)simplicial structure maps. Applying $|-|_{\mathbb{A}^1}$, we obtain via Lemma 3.23 a map

$$|r_\bullet|_{\mathbb{A}^1} : |\mathrm{Tel}_A(h^{\mathrm{efr},n}(X,U)^{(-)}|_{\mathbb{A}^1} \simeq \operatorname*{colim}_A h^{\mathrm{efr},n}(X,U)^{(-)} \simeq h^{\mathrm{efr},n}(X,U) \to h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U).$$

The construction is arranged in such a way that $|H_\bullet|_{\mathbb{A}^1}$ and $|K_\bullet|_{\mathbb{A}^1}$ exhibit homotopies making the following triangles commute:

$$\begin{array}{ccc} h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U) & \xrightarrow{\mathrm{id}} & h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U) \\ \downarrow & \nearrow_{|r_\bullet|_{\mathbb{A}^1}} & \\ h^{\mathrm{efr},n}(X,U), & & \end{array} \qquad \begin{array}{ccc} h^{\mathrm{efr},n}(X,U) & \xrightarrow{\mathrm{id}} & h^{\mathrm{efr},n}(X,U) \\ \downarrow{\scriptstyle |r_\bullet|_{\mathbb{A}^1}} & \nearrow & \\ h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U). & & \end{array}$$

Set

$$\tilde{A} = \big\{\vec{s}, \{(T_1, c_1, V_1), \dots, (T_n, c_n, V_n)\} \mid \\ \vec{s} \in \Gamma_\infty,\ V_i \subset \Gamma_\infty,\ T_i \in \mathrm{Sm}_k,\ c_i \in h^{\mathrm{efr},n}(X,U)(T_i)\big\}.$$

Here V_i is a finite-dimensional, affine subspace. Let $A \subset \tilde{A}$ be the subset of elements having the following properties:

- $\vec{s} \in V_i$, $\vec{x} \notin V_i$.
- For all i and all $\vec{s'} \in V_i^{\vec{x}} \setminus \{\vec{x}\}$ we have $\vec{s'} \cdot c_i \in h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)(T_i)$.

Here $V_i^{\vec{x}}$ denotes the affine subspace generated by V_i and $\vec{x}$.

For $\alpha = (\vec{s}, M) \in A$ we denote by $h^{\mathrm{efr},n}(X,U)^\alpha \subset h^{\mathrm{efr},n}(X,U)$ the subpresheaf generated by the sections c for $(T, c, V) \in M$ (ignoring the V component), and similarly $h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)^\alpha$ is the subpresheaf generated by those c which happen to be quasifinite. We put an ordering on A by declaring that $(\vec{s}, M) \le (\vec{t}, N)$ if for all $(T, c, V) \in M$ we have $(T, c, V^{\vec{t}}) \in N$. It is immediate from Lemma 3.21 that this makes A into a filtered poset and that the filtrations of $h^{\mathrm{efr},n}(X,U)$ and $h_{\mathrm{qf}}^{\mathrm{efr},n}(X,U)$ are exhaustive.

With this preparation out of the way, let $\alpha = (\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n) \in A$, with $\alpha_i = (\vec{s}_i, M_i)$. We set

$$\begin{aligned} r_\alpha(c, \lambda) &= \vec{s}(\lambda) \cdot c, \\ H_\alpha(t, c, \lambda) &= (t\vec{s}(\lambda) + (1-t)\vec{x}) \cdot c, \\ K_\alpha(t, c, \lambda) &= (t\vec{s}(\lambda) + (1-t)\vec{x}) \cdot c. \end{aligned}$$

Here $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{A}^n$ and

$$\vec{s}(\lambda) = \Big(1 - \sum_i \lambda_i\Big)\vec{s}_0 + \sum_i \lambda_i \vec{s}_i.$$

The cosimplicial structure on $\mathbb{A}^\bullet$ comes from viewing $\mathbb{A}^n$ as the subspace of $\mathbb{A}^{n+1}$ where the sum of the coordinates is 1. With this interpretation, it is clear that this construction is compatible with the simplicial structure. It remains to show that the maps r_α and K_α land in $h_{\mathrm{qf}}^{\mathrm{efr},n}(X, U)$. Let $(T, c, V) \in M_0$. Let V' be the affine subspace generated by V and all the s_i. One checks by induction that $(T, c, V') \in M_n$. The required quasifiniteness follows (recall that by assumption, $\vec{v} \cdot c$ is quasifinite for $\vec{v} \in (V')^{\vec{x}} \setminus \vec{x} \supset V'$). □

3F. *Proof of Lemma 3.21.* We now prove the key moving lemma, following arguments of Druzhinin. We will in fact establish the following stronger result.

Theorem 3.24. *Let $T \in \mathrm{Sm}_k$ $c \in h^{\mathrm{efr},n}(X, U)(T)$, $V \subset \Gamma_{d'}$. There exists $d'' > d'$ such that for all $d \geq d''$, there is an open, nonempty subset $U_d \subset \Gamma_d$ of "allowable cone points". (That is, any $\vec{\gamma} \in U_d$ has the required properties for the single correspondence c.)*

Lemma 3.21 follows from this by applying the theorem to each (T_i, c_i, V_i) and picking a rational point in the intersection of the sets U_d obtained (which is possible because this intersection is a nonempty, open subset of an affine space and k is infinite).

We spend the rest of the section proving this result. Fix

$$c = (W, (\phi, g), Z) \in h^{\mathrm{efr}}(X, U)(T).$$

The canonical map (induced by $W \to T$ and $(\phi, g) : W \to \mathbb{A}^n \times X$)

$$\psi : W \to T \times \mathbb{A}^n \times X$$

is finite over $T \times 0 \times Y$. Since the quasifinite locus is open [34, Tag 01TI], there exists an open neighborhood $W' \subset W$ of Z such that $\psi|_{W'}$ is quasifinite. Replacing W by W', we may assume that ψ is quasifinite. Let $m > 0$ and consider the map

$$\psi^m : W^{\times_T m} \to T \times (\mathbb{A}^n \times X)^m.$$

It is still quasifinite. Define

$$T \times (\mathbb{A}^n \times X)^m \supset \mathcal{E}_m := \{(t, p_1, \dots, p_m) \mid t \in T,\ p_i \in \mathbb{A}^n \times X,\ p_i \neq p_j,\ p_i \notin 0 \times Y\}.$$

Consider further

$$W^{\times_T m} \times \Gamma_d \supset B_{m,d} := \big\{(w_1, \dots, w_m, \vec{s}) \mid \psi(w_1, \dots, w_m) \in \mathcal{E}_m,\ (f_{\vec{s}} \circ (\phi, g))(w_i) \in 0 \times (X \setminus Y), \\ (f_{\vec{s}} \circ (\phi, g))^{-1}(0 \times (X \setminus Y)) \text{ not quasifinite at } w_i\big\}$$

and

$$T \times \Gamma_d \supset B_d := \{(t, \vec{s}) \mid (f_{\vec{s}} \circ (\phi, g))^{-1}(0 \times (X \setminus Y)) \text{ not quasifinite over } t\}.$$

There is an evident map $B_{m,d} \to B_d$. We shall prove the following:

(1) For any m, d, the map $B_{m,d} \to B_d$ is surjective with fibers of dimension $\geq m$.
(2) For fixed m, and $d = d(m)$ sufficiently large, we have

$$\dim B_{m,d} \leq \dim T + \dim \Gamma_d.$$

We deduce that

$$\dim B_d \overset{(1)}{\leq} \dim B_{m,d} - m \overset{(2)}{\leq} \dim \Gamma_d + \dim T - m.$$

Choosing $m \geq \dim T + \dim V + 2$, we can ensure that

$$\dim B_d \leq \dim \Gamma_d - \dim V - 2.$$

Write $p : B_d \amalg \{\vec{x}\} \to \Gamma_d$ for the projection and inclusion. Write

$$q : (B_d \amalg \{\vec{x}\}) \times V \times \mathbb{A}^1 \to \Gamma_d, (b, v, t) \mapsto tp(b) + (1-t)v.$$

Then the image of q has dimension $< \dim \Gamma_d$, and so the complement of the closure of the image of q is a nonempty open $U_d \subset \Gamma_d$.

Proof of Theorem 3.24. Let $\vec{s} \in \Gamma_d \setminus p(B_d)$. We claim that $\vec{s} \cdot c \in h^{\mathrm{efr},n}_{\mathrm{qf}}(X, U)$. Indeed if $\varphi = f_{\vec{s}} \circ (\phi, g)$ then we know that $\varphi^{-1}(0 \times Y) = Z \amalg Z'$. We also know that $\varphi^{-1}(0 \times (X \setminus Y))$ is quasifinite over T. Replacing W by $W \setminus Z'$ we arrange that $\varphi^{-1}(0 \times Y) = Z$ is finite over T. The claim follows.

Now let $\vec{\gamma} \in U_d$, $v \in V$, and $t \in \mathbb{A}^1$. If $t\vec{\gamma} + (1-t)v = b \in p(B_d) \cup \{\vec{x}\}$ with $t \neq 0$ then

$$\vec{\gamma} = \frac{1}{t}b + \frac{t-1}{t}v,$$

which contradicts the construction of U_d. In other words, if V' is the cone on V with tip $\vec{\gamma}$ and $b \in V' \setminus V$, then $b \notin B$ and so $b \cdot c$ is quasifinite, as needed. Similarly $\vec{x} \notin V'$ unless $\vec{x} \in V$. □

The main idea for proving (1) is that if a morphism (of finite type, say) is not quasifinite over some point, then the fiber must have dimension ≥ 1. Taking m-fold products, we obtain something of dimension $\geq m$.

Proof of (1). We may base change to an algebraically closed field, and it suffices to treat fibers over closed (hence rational) points. Thus let $t \in T$, $\vec{s} \in \Gamma_d$ be closed points with $(t, \vec{s}) \in B_d$. Set $\varphi = f_{\vec{s}} \circ (\phi, g) : W \to \mathbb{A}^n \times X$, so that $A := \varphi^{-1}(0 \times (X \setminus Y))$ is not quasifinite over t. Let $A_1 \subset A$ be a positive-dimensional component of the fiber over t (which exists because A is not quasifinite over t). Since ψ is quasifinite,

$$B := \psi(A_1) \subset \{t\} \times (\mathbb{A}^n_X \setminus 0_Y) \subset T \times \mathbb{A}^n \times X$$

is infinite. By Chevalley's theorem [34, Tag 054K], B is a finite disjoint union of locally closed subsets, and hence contains an infinite subset $B_0 \subset B$ which is a scheme. Being of finite type over a field, B_0 has positive dimension. Let $C \subset B_0^m$ be the subscheme of distinct points. Since $\dim B_0 \geq 1$ we have $\dim C \geq m$. By construction the image of $(B_{m,d})_{t,\vec{s}} \to \mathcal{E}_m$ contains C. It follows that

$$\dim(B_{m,d})_{t,\vec{s}} \geq \dim C \geq m,$$

as needed. □

For proving (2), we may (and will) ignore the quasifiniteness condition in the definition of $B_{m,d}$. The main idea is that the condition $f_{\vec{s}}((\phi, g)(w)) \in 0 \times X$ is equivalent to the vanishing of n sections at w, and hence m such conditions should have codimension $mn = \dim W^{\times_T m} - \dim T$.

Proof of (2). We may base change to an algebraically closed field. Let

$$W^{\times_T m} \supset \mathcal{W}_m := \psi^{-1}(\mathcal{E}_m)$$

so that we have a map $q : B_{m,d} \to \mathcal{W}_m$. Since $\dim \mathcal{W}_m \leq \dim T + mn$, it will suffice to show that the fibers of q (over closed points) have dimension $\leq \dim \Gamma - mn$. Let $(w_1, \dots, w_m) \in \mathcal{W}_m$ have image $(p_1, \dots, p_m) \in \mathcal{E}_m$. Put

$$\Gamma_{d,(p_1,\dots,p_m)} = \{\vec{s} \in \Gamma_d \mid f_{\vec{s}}(p_i) \in 0 \times X\}.$$

Then $q^{-1}(w_1, \dots, w_m) \subset \Gamma_{d,(p_1,\dots,p_m)}$; hence it suffices to show $\dim \Gamma_{d,(p_1,\dots,p_m)} \leq \dim \Gamma_d - mn$. We have an exact sequence

$$0 \to \Gamma_{d,(p_1,\dots,p_m)} \to \Gamma_d \xrightarrow{\text{ev}} \bigoplus_i H^0(p_i, \mathcal{O}(d)^{\oplus n}).$$

Since the right-hand term has dimension mn, it is sufficient to prove that the evaluation map ev is surjective. Set $\mathcal{K} = \ker(\mathcal{O}(d)^{\oplus n} \to \mathcal{O}(d)^{\oplus n}|_{\mathcal{N}})$, so that $\Gamma_d = \{\vec{x}\} + H^0(\mathbb{P}^n \times \overline{X}, \mathcal{K})$. By construction $p_i \notin \mathcal{N}$, and hence $\mathcal{K}|_{p_i} = \mathcal{O}(d)^{\oplus n}|_{p_i}$. The result thus follows from Lemma 3.25 below (applied with $\mathcal{F} = \mathcal{K}$). □

We used the following well-known result.

Lemma 3.25. *Let X be a projective scheme over a field, $\mathcal{F}$ a coherent sheaf on X and $m \geq 0$. There exists N such that for all $d \geq N$ and distinct rational points $p_1, \dots, p_m \in X$, the map*

$$H^0(X, \mathcal{F}(d)) \to \bigoplus_i H^0(p_i, \mathcal{F}(d))$$

is surjective.

Proof. Replacing $\mathcal{F}$ by the pushforward along an embedding of X into projective space, we may assume that $X = \mathbb{P}^n$. Given a surjection $\mathcal{F}' \to \mathcal{F}$, the result for $\mathcal{F}'$ implies the one for $\mathcal{F}$. The result for $\mathcal{F}_1, \mathcal{F}_2$ implies it for $\mathcal{F}_1 \oplus \mathcal{F}_2$. Hence it suffices to prove the result for $\mathcal{F} = \mathcal{O}$ (use [34, Tag 01YS]). We can find $L_{ij} \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ such that $L_{ij}(p_i) = 0$ but $L_{ij}(p_j) \neq 0$. Then for fixed j, the section

$$s_j = \prod_{i \neq j} L_{ij} \in H^0(\mathbb{P}^n, \mathcal{O}(m-1))$$

has $s_j(p_j) \neq 0$ but $s_j(p_i) = 0$ for all $i \neq j$. This shows that $N = m - 1$ works (in this case). □

4. The cancellation theorem

Primary sources: [1; 15; 37].

After these lecture notes were written, some of the ideas from this section were used in [5]; that work may also serve as a somewhat more formal exposition of some of the ideas presented here.

4A. *Group-complete framed spaces.*

Lemma 4.1 [15, Proposition 3.2.10(iii)]. *The category $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ is semiadditive.*

It follows that, for every $\mathcal{X} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ and $X \in \mathrm{Sm}_S$, $\pi_0\mathcal{X}(X)$ is an abelian monoid.

Definition 4.2. We call $\mathcal{X}$ group-complete (or grouplike) if $\pi_0\mathcal{X}(X)$ is, for every $X \in \mathrm{Sm}_S$. We denote by $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}} \subset \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ the subcategory of group-complete spaces.

The group-complete spaces are closed under limits and filtered colimits (in fact all colimits), and hence the inclusion $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}} \subset \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ admits a left adjoint $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{gp}}$ which is easily seen to be symmetric monoidal. The functor $\Omega^\infty : \mathcal{SH}(S) \simeq \mathcal{SH}^{\mathrm{fr}}(S) \to \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)$ has image contained in $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}}$. It follows that $\Sigma^\infty : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S) \to \mathcal{SH}^{\mathrm{fr}}(S) \simeq \mathcal{SH}(S)$ inverts group completions and so factors through a symmetric monoidal, cocontinuous functor

$$\Sigma^\infty : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}} \to \mathcal{SH}(S).$$

The following is the main result.

Theorem 4.3 ($\mathbb{P}^1$-cancellation). *If k is a perfect field, then*

$$\Sigma^\infty : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathcal{SH}(k)$$

is fully faithful.

Remark 4.4. The essential image of Σ^∞ is closed under colimits and known as the subcategory of *very effective spectra*.

Remark 4.5. The theorem is equivalent to showing that for $\mathcal{X}, \mathcal{Y} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ we have $\mathrm{Map}(\mathcal{X}, \mathcal{Y}) \simeq \mathrm{Map}(\Sigma_{\mathbb{P}^1}\mathcal{X}, \Sigma_{\mathbb{P}^1}\mathcal{Y})$, and this is further equivalent to showing that

$$\mathcal{Y} \to \Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1}\mathcal{Y}$$

is an equivalence. Here $\Sigma^1_{\mathbb{P}} : \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ is the functor of tensor product with the image of $\mathbb{P}^1$ in $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$.

Since $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$, it suffices to prove separate statements for these two suspensions. This is how we shall establish Theorem 4.3.

4B. S^1-*cancellation.*

Proposition 4.6. *For $\mathcal{X} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$, the canonical map*

$$\mathcal{X} \to \Omega_{S^1}\Sigma_{S^1}\mathcal{X}$$

is an equivalence.

Proof. Let $\mathcal{Y} \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))^{\mathrm{gp}}$. We shall first determine $\Sigma_{S^1}\mathcal{Y}$. Let $X \in \mathrm{Sm}_S$. There is a finite coproduct preserving functor $c_X : \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Corr}^{\mathrm{fr}}(S)$ sending $*$ to X. Its sifted cocontinuous extension admits, by Proposition C.1 in [6], a right adjoint $c_{X*} : \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S)) \to \mathcal{P}_\Sigma(\mathrm{Span}(\mathrm{Fin})) \simeq \mathrm{CMon}(\mathcal{S}\mathrm{pc})$, which preserves limits and sifted colimits, and hence all colimits by semiadditivity and [6, Lemma 2.8]. We deduce that

$$(\Sigma_{S^1}\mathcal{Y})(X) \simeq \Sigma_{S^1}(\mathcal{Y}(X)) \in \mathrm{CMon}(\mathcal{S}\mathrm{pc}). \tag{4}$$

This implies both that $\Sigma_{S^1}\mathcal{Y}$ is group-complete and, using that $\mathrm{CMon}(\mathcal{S}\mathrm{pc})^{\mathrm{gp}} \simeq \mathcal{SH}_{\geq 0}$ [28, Remark 5.2.6.26], that

$$\mathcal{Y} \to \Omega_{S^1}\Sigma_{S^1}\mathcal{Y} \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))^{\mathrm{gp}}$$

is an equivalence. To promote this to the same statement for $\mathcal{X} \in \mathcal{S}\mathrm{pc}^{\mathrm{fr}}(S)^{\mathrm{gp}}$, it is enough to show that whenever $\mathcal{Y}$ is motivically local, the same holds for $L_{\mathrm{Nis}}\Sigma_{S^1}\mathcal{Y}$; indeed Ω_{S^1} is computed sectionwise and hence preserves Nisnevich equivalences. Equation (4) shows that $\Sigma_{S^1}\mathcal{Y}$ is $\mathbb{A}^1$-invariant; the result thus follows from Corollary 5.4 in Section 5. □

4C. *Abstract cancellation.*

The following is extracted from [37, §4].

Theorem 4.7. *Let $\mathcal{C}$ be a symmetric monoidal 1-category and $G \in \mathcal{C}$ a symmetric object. Suppose that the functor $\Sigma_G := G \otimes -$ admits a right adjoint Ω_G. Note that Ω_G is canonically a lax $\mathcal{C}$-module functor. Suppose that the unit transformation*

$$u : \mathrm{id}_{\mathcal{C}} \to \Omega_G\Sigma_G$$

admits a retraction ρ in the category of lax $\mathcal{C}$-module functors. Then u, ρ are inverse isomorphisms.

Remark 4.8. If $\mathcal{C}$ is an ∞-category and ρ is a lax $\mathcal{C}$-module retraction of the unit transformation $u : \mathrm{id}_{\mathcal{C}} \to \Omega_G \Sigma_G$, then the same conclusion holds (apply the theorem to $h\mathcal{C}$).

Remark 4.9. Since $\mathcal{C}$ is a 1-category, a lax $\mathcal{C}$-module structure on an endofunctor $F : \mathcal{C} \to \mathcal{C}$ just consists of compatible morphisms $X \otimes F(Y) \to F(X \otimes Y)$ for all $X, Y \in \mathcal{C}$. A transformation $\alpha : F \to G$ being a lax $\mathcal{C}$-module transformation is a property: it is the requirement that for $X, Y \in \mathcal{C}$, the following square commutes:

$$\begin{array}{ccc} X \otimes F(Y) & \xrightarrow{\mathrm{id}_X \otimes \alpha_Y} & X \otimes G(Y) \\ \downarrow & & \downarrow \\ F(X \otimes Y) & \xrightarrow{\alpha_{X \otimes Y}} & G(X \otimes Y). \end{array}$$

Example 4.10. A lax $\mathcal{C}$-module transformation $\alpha : \mathrm{id} \to \mathrm{id}$ (of $\mathrm{id}_{\mathcal{C}}$ with its canonical $\mathcal{C}$-module structure) is completely determined by $\alpha_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}$. In particular ρ being a retraction of u is equivalent to the composite

$$\mathbb{1} \xrightarrow{u_{\mathbb{1}}} \Omega_G \Sigma_G \xrightarrow{\rho_{\mathbb{1}}} \mathbb{1}$$

being the identity.

To simplify notation, from now on we will write $\underline{\mathrm{Hom}}(G, -)$ for Ω_G, and also use suggestive notation like $\otimes \mathrm{id}_Y : \underline{\mathrm{Hom}}(A, B) \to \underline{\mathrm{Hom}}(A \otimes Y, B \otimes Y)$, when convenient.

Lemma 4.11. *For $X, Y \in \mathcal{C}$, the following diagram commutes:*

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(G, G \otimes X) & \xrightarrow{\rho_X} & \underline{\mathrm{Hom}}(\mathbb{1}, X) \\ {\scriptstyle \otimes \mathrm{id}_Y}\downarrow & & {\scriptstyle \otimes \mathrm{id}_Y}\downarrow \\ \underline{\mathrm{Hom}}(G \otimes Y, G \otimes X \otimes Y) & \xrightarrow{\Omega_Y \rho_{X \otimes Y}} & \underline{\mathrm{Hom}}(Y, X \otimes Y). \end{array}$$

Proof. Decompose the diagram as

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(G, G \otimes X) & \xrightarrow{\rho_X} & \underline{\mathrm{Hom}}(\mathbb{1}, X) \\ {\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow \\ \underline{\mathrm{Hom}}(Y, Y \otimes \underline{\mathrm{Hom}}(G, G \otimes X)) & \xrightarrow{\underline{\mathrm{Hom}}(Y, Y \otimes \rho_X)} & \underline{\mathrm{Hom}}(Y, Y \otimes \underline{\mathrm{Hom}}(\mathbb{1}, X)) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(Y, \underline{\mathrm{Hom}}(G, G \otimes X \otimes Y)) & \xrightarrow{\underline{\mathrm{Hom}}(Y, \rho_{X \otimes Y})} & \underline{\mathrm{Hom}}(Y, \underline{\mathrm{Hom}}(\mathbb{1}, X \otimes Y)) \\ {\scriptstyle \simeq}\downarrow & & {\scriptstyle \simeq}\downarrow \\ \underline{\mathrm{Hom}}(G \otimes Y, G \otimes X \otimes Y) & \xrightarrow{\Omega_Y \rho_{X \otimes Y}} & \underline{\mathrm{Hom}}(Y, X \otimes Y). \end{array}$$

Here the middle vertical transformations are the lax module structure maps, and the bottom vertical isomorphisms hold in any symmetric monoidal category. The upper and lower squares commute by naturality, and the middle one by assumption of ρ being a lax module transformation. The vertical composites are given by $\otimes\,\mathrm{id}_Y$. This concludes the proof. □

Proof of Theorem 4.7. Let $X \in \mathcal{C}$. It suffices to show that the composite $\Omega_G \Sigma_G X \xrightarrow{\rho_X} X \xrightarrow{u_X} \Omega_G \Sigma_G X$ is the identity. Let $n \geq 2$ and $\alpha : G^{\otimes n} \to G^{\otimes n}$ be an automorphism. Consider the composite

$$\begin{aligned} p(\alpha) : \underline{\mathrm{Hom}}(G, G \otimes X) &\xrightarrow{\mathrm{id}_{G^{\otimes n-1}} \otimes} \underline{\mathrm{Hom}}(G^{\otimes n}, G^{\otimes n} \otimes X) \\ &\xrightarrow{c_\alpha} \underline{\mathrm{Hom}}(G^{\otimes n}, G^{\otimes n} \otimes X) \\ &\xrightarrow{\rho^{n-1}} \underline{\mathrm{Hom}}(G, G \otimes X), \end{aligned}$$

where c_α denotes the conjugation by α.

Note that the map "$\mathrm{id}_{G^{\otimes n-1}} \otimes$" is a composite of units u, and hence by assumption of ρ being a retraction, we get $p(\mathrm{id}) = \mathrm{id}$.

On the other hand let $\alpha = \sigma$ be the cyclic permutation of $G^{\otimes n}$. Then the first $n-2$ applications of ρ are again "canceling out identities", so that $p(\sigma)$ is the same as the composite

$$\underline{\mathrm{Hom}}(G, G \otimes X) \xrightarrow{f} \underline{\mathrm{Hom}}(G^{\otimes 2}, G^{\otimes 2} \otimes X) \xrightarrow{f_2} \underline{\mathrm{Hom}}(G, G \otimes X),$$

where f_1 "inserts id_G in the middle", and "f_2 applies ρ at the front". Lemma 4.11 implies that this is the same as $u_X \rho_X$.

Hence if G is n-symmetric, then since $\sigma = \mathrm{id}$ we find that

$$u_X \rho_X = p(\sigma) = p(\mathrm{id}) = \mathrm{id}\,.$$

This concludes the proof. □

4D. *Twisted framed correspondences.* Using [14, §B] it is possible to construct a symmetric monoidal ∞-category $\mathrm{Corr}^{\mathrm{fr}}_L(S)$ with the following properties:

- Its objects are pairs (X, ξ) with $X \in \mathrm{Sm}_S$ and $\xi \in K(X)$.
- The morphisms from (X, ξ) to (Y, ζ) are given by spans

  $$X \xleftarrow{f} Z \xrightarrow{g} Y,$$

  where Z is a *derived* scheme and f is a quasismooth morphism, together with a trivialization

  $$f^*(\xi) + L_f \simeq g^*(\eta) \in K(Z).$$
- There is a symmetric monoidal functor $\delta : \mathrm{Corr}^{\mathrm{fr}}(S) \to \mathrm{Corr}^{\mathrm{fr}}_L(S)$ which sends X to $(X, 0)$ and induces the evident maps on mapping spaces.

It follows that the tensor product in $\mathrm{Corr}_L^{\mathrm{fr}}(S)$ is given by the product of schemes, and the functor δ is faithful (induces monomorphisms on mapping spaces).

The following will be helpful.

Lemma 4.12. *A span*

$$X \xleftarrow{f} Z \to Y \in \mathrm{Map}_{\mathrm{Corr}_L^{\mathrm{fr}}(S)}((X, 0), (Y, 0))$$

is in the image of δ if and only if f is finite.

Proof. The only concern is that Z might be a derived scheme instead of a classical one; by [14, Lemma 2.2.1] this cannot happen. □

We mainly introduce the category $\mathrm{Corr}_L^{\mathrm{fr}}(S)$ for a technically convenient reason: all of its objects are strongly dualizable.

Proposition 4.13. *Let $X \in \mathrm{Sm}_S$. The spans*

$$* \leftarrow X \xrightarrow{\Delta} X \times X$$

and

$$X \times X \xleftarrow{\Delta} X \to *$$

admit evident framings, and exhibit (X, L_X) as the dual of $(X, 0)$ in $\mathrm{Corr}_L^{\mathrm{fr}}(S)$.

Proof. This kind of duality happens in all span categories; we just need to verify that the spans are frameable and that the induced framings of the compositions are trivial. All of this is easy to verify. For example $X \times X$ really means $(X, 0) \otimes (X, L_X) = (X \times X, p_2^* L_X)$, and hence to frame the first span we need to exhibit a path

$$0 + L_X \simeq \Delta^* p_2^* L_X,$$

but this holds on the nose since $\Delta^* p_2^* \simeq \mathrm{id}$; to frame the second span we need to exhibit a path

$$\Delta^* p_2^* L_X + L_\Delta \simeq 0$$

which is possible in K-theory since the composite $X \xrightarrow{\Delta} X \times X \xrightarrow{p_1} X$ is the identity, so $0 = L_{\mathrm{id}} \simeq L_\Delta + \Delta^* L_{p_1}$ and finally $L_{p_1} \simeq p_2^* L_X$ by base change. □

The following will be helpful later to exhibit spans.

Construction 4.14. Suppose we are given $X, G \in \mathrm{Sm}_S$, a map $f : X \times G \to \mathbb{A}^1$ and a path $L_G \simeq 1 \in K(G)$. Then there is a span

$$D(f) : (X \xleftarrow{p_1} Z(f) \xrightarrow{p_2} G) \in \mathrm{Map}_{\mathrm{Corr}_L^{\mathrm{fr}}(S)}((X, 0), (G, 0));$$

the framing is given by

$$L_{p_1} \simeq L_{Z(f)/X\times G} + L_{X\times G/X} \simeq -1 + L_G \simeq 0 \in K(Z(f)),$$

where we have used that $L_{Z(f)/X\times G} \simeq -1$ via f and $L_G \simeq 1$ by assumption.

We will always apply this construction with $G = \mathbb{A}^1 \setminus 0$, so that there is a canonical trivialization of L_G.

4E. *A general construction.* Given $X, Y \in \mathrm{Sm}_S$, for notational convenience we will write $f : X \rightsquigarrow Y$ for $f \in \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}_L(S)}((X, 0), (Y, 0))$.

Construction 4.15. Let $A, G \in \mathrm{Sm}_S$ and $\alpha : A \times G \rightsquigarrow G$. We obtain a $\mathrm{Corr}^{\mathrm{fr}}_L(S)$-module transformation

$$\rho_\alpha : \Omega_G \Sigma_G \to \Omega_A \in \mathrm{End}(\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}_L(S)))$$

as follows: via strong dualizability (Proposition 4.13), we can rewrite the source and target and consider the transformation

$$G^\vee \otimes G \otimes - \xrightarrow{\alpha^\vee \otimes \mathrm{id}_-} A^\vee \otimes -,$$

where $\alpha^\vee : G^\vee \otimes G \to A^\vee$ is obtained from α in the evident manner.

We will eventually apply this with $G = \mathbb{A}^1 \setminus 0$ and $A = \mathbb{A}^1$ or $A = *$.

Remark 4.16. Let $X, Y \in \mathrm{Sm}_S$. Given a span

$$G \times Y \leftarrow Z \to G \times X,$$

the transformation ρ_α produces a span

$$A \times Y \leftarrow \rho_\alpha(Z) \to X.$$

Write α as

$$A \times G \leftarrow C \to G.$$

Tracing through the definitions, one finds that

$$\rho_\alpha(Z) = Z \times_{G \times G} C,$$

with an evident induced framing.

Lemma 4.17. *The transformation ρ_α satisfies the following properties.*

(1) *Given $Z' : X \rightsquigarrow X'$ and $Z : G \times Y \rightsquigarrow G \times X$ we have*

$$\rho_\alpha((\mathrm{id}_G \otimes Z') \circ Z) \simeq (\mathrm{id}_A \otimes Z') \circ \rho_\alpha(Z).$$

(2) *Given $Z' : Y \rightsquigarrow Y$ and $Z : G \times Y \rightsquigarrow G \times X$ we have*

$$\rho_\alpha(Z \circ (\mathrm{id}_G \otimes Z')) \simeq \rho_\alpha(Z) \circ (\mathrm{id}_G \otimes Z').$$

(3) *Given $i : A' \rightsquigarrow A$, we have*

$$\rho_{i^*\alpha} \simeq i^* \rho_\alpha.$$

Proof. Evident from the naturality of the construction. □

Now define

$$M_\alpha(Y,X) \subset \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}(S)}(G \times Y, G \times X)$$

to consist of the disjoint union of those path components corresponding to spans $G \times Y \leftarrow Z \rightarrow G \times X$ such that $\rho_\alpha(Z)$ is finite. Then (1) and (2) of Lemma 4.17 translate (using Lemma 4.12) into

(1) $(\mathrm{id}_G \otimes Z') \circ M_\alpha(Y,X) \subset M_\alpha(Y,X')$, and
(2) $M_\alpha(Y,X) \circ (\mathrm{id}_G \otimes Z') \subset M_\alpha(Y',X)$.

Construction 4.18. Define a subfunctor

$$F_\alpha \Omega_G \Sigma_G \hookrightarrow \Omega_G \Sigma_G \in \mathrm{End}(\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S)))$$

via

$$(F_\alpha \Omega_G \Sigma_G X)(Y) = M_\alpha(X,Y).$$

The lax monoidal natural transformation

$$\Omega_G \delta_* \delta^* \Sigma_G \simeq \delta_* \Omega_G \Sigma_G \delta^* \xrightarrow{\rho_\alpha} \delta_* \Omega_A \delta^* \simeq \Omega_A \delta_* \delta^*$$

restricts by construction to a natural transformation

$$\rho_\alpha : F_\alpha \Omega_G \Sigma_G \rightarrow \Omega_A,$$

which we will think of as

$$\rho_\alpha : A \otimes F_\alpha \Omega_G \Sigma_G \rightarrow \mathrm{id}\,.$$

Take $A = \mathbb{A}^1$, $G = \mathbb{A}^1 \setminus 0$ and suppose that $\rho_\alpha(\mathrm{id}_G)$ is finite. Then the unit transformation

$$\mathrm{id} \rightarrow \Omega_G \Sigma_G$$

factors through $F_\alpha \Omega_G \Sigma_G$. We obtain two $\mathbb{A}^1$-homotopic transformations

$$\rho_{i_0^*\alpha}, \rho_{i_1^*\alpha} : F_\alpha \Omega_G \Sigma_G \rightarrow \mathrm{id} \in \mathrm{End}(\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(S))).$$

4F. $\mathbb{G}_m$*-cancellation.* Let $G = \mathbb{A}^1 \setminus 0$.

Definition 4.19. We define maps $G \times G \rightarrow \mathbb{A}^1$ via

$$g_n^+(t_1,t_2) = t_1^n + 1 \quad \text{and} \quad g_n^-(t_1,t_2) = t_1^n + t_2.$$

We further define maps $\mathbb{A}^1 \times G \times G \rightarrow \mathbb{A}^1$ via

$$h_n^\pm(t,t_1,t_2) = t g_n^\pm(t_1,t_2) + (1-t) g_m^\pm(t_1,t_2).$$

Recall the associated spans from Construction 4.14. Put

$$F_i = \bigcap_{m,n \geq i} [F_{D(h_{m,n}^+)} \cap F_{D(h_{m,n}^-)}] \subset \Omega_G \Sigma_G.$$

Lemma 4.20. *We have*

$$\operatorname*{colim}_i F_i \simeq \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}(S)}(X, Y).$$

Proof. We follow [37, Lemma 4.1 and Remark 4.2]. Suppose $Y \leftarrow Z \rightarrow X \in \mathrm{Map}_{\mathrm{Corr}^{\mathrm{fr}}(S)}(Y, X)$. We shall exhibit an integer N such that for all $m, n > N$ the projection $Z' = \rho_{D(h^{\pm}_{m,n})}(Z) \rightarrow Y \times \mathbb{A}^1$ is finite; this will prove what we want. Write $f_1, f_2 : Z \rightarrow G$ for the two projections. Using Zariski's main theorem, we can form a commutative diagram

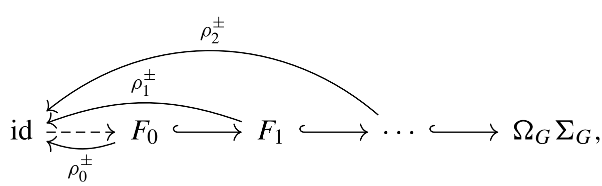
$$\begin{array}{ccc} Z & \longrightarrow & \bar{C} \\ {\scriptstyle f_1 \times p_Y}\big\downarrow & & {\scriptstyle \bar{f}_1 \times p_Y}\big\downarrow \\ G \times Y & \longrightarrow & \mathbb{P}^1 \times Y, \end{array}$$

where $\bar{f}_1 \times p_Y$ is finite. There exists N such that the rational function $\bar{f}_1^N / f_2$ is regular in a neighborhood U_0 of $\bar{f}_1^{-1}(0)$ and $f_2 / \bar{f}_1^N$ is regular in a neighborhood U_∞ of $\bar{f}_1^{-1}(\infty)$. We have the function h equals $h^{\pm}_{m,n}(t, f_1, f_2)$ on $Z \times \mathbb{A}^1$, and Remark 4.16 implies $Z' = Z(h) \subset Z \times \mathbb{A}^1$. The composite $\bar{C} \times \mathbb{A}^1 \rightarrow \mathbb{P}^1 \times Y \times \mathbb{A}^1 \rightarrow Y \times \mathbb{A}^1$ is projective, and $Z(h) \rightarrow Y \times \mathbb{A}^1$ is affine. We will finish the proof by showing that $i^{\pm} : Z(h) \rightarrow \bar{C} \times \mathbb{A}^1$ is a closed immersion for $n, m > N$; indeed then $Z(h) \rightarrow Y \times \mathbb{A}^1$ will be both proper and affine, and hence finite as desired.

Note that h^+ extends to the regular map $t\bar{f}_1^m + (1-t)\bar{f}_1^n + 1 : \bar{C} \rightarrow \mathbb{P}^1$, which does not vanish if $\bar{f}_1 \in \{0, \infty\}$. Thus i^+ is always a closed immersion.

Now we deal with i^-. Let $U_1 = \bar{f}_1^{-1}(G)$. A morphism being a closed immersion is local on the target [34, Tag 01QO], so it is enough to show that i is a closed immersion over U_0, U_∞ and U_1. This is clear for U_1. Consider the function $h_0 = t\bar{f}_1^n / f_2 + (1-t)\bar{f}_1^m / f_2 + 1$. By construction, this is regular on h_0, so $Z(h_0) \subset U_0$ is closed. Also by construction, $h_0 = 1$ if $\bar{f}_1 = 0$, and $h^- = f_2 h_0$ on $U_0 \setminus 0$, where f_2 is a unit. It follows that $Z(h_0) = U_0 \cap Z(h)$. A similar argument works for U_∞. □

Using Construction 4.18, we thus obtain a sequence of lax module transformations



where the arrows to the right form a colimit diagram. The dashed arrow might not exist, but the lemma above implies that its composite sufficiently far to the right does, and this is all we need.[6] For $m \geq n$, the $h^{\pm}_{m,n}$ induce $\mathbb{A}^1$-homotopies making

[6] One may verify that the arrow actually does exist.

the following diagram commute:

$$\begin{array}{ccc} F_n & \hookrightarrow & F_m \\ {\scriptstyle \rho_n^\pm}\downarrow & \swarrow {\scriptstyle \rho_m^\pm} & \\ \mathrm{id} & & \end{array}$$

Applying $L_{\mathbb{A}^1}$, there are thus induced transformations on the colimit

$$L_{\mathbb{A}^1} \underset{\rho^\pm}{\overset{u}{\rightleftarrows}} L_{\mathbb{A}^1}\Omega_G\Sigma_G.$$

After group completion, we may take the difference, and hence obtain

$$\rho = \rho^+ - \rho^- : L_{\mathbb{A}^1}^{\mathrm{gp}}\Omega_G\Sigma_G \to L_{\mathbb{A}^1}^{\mathrm{gp}}.$$

We are now ready to prove our main result.

Theorem 4.21. *Let k be an infinite perfect field. Then the unit transformation*

$$u : \mathrm{id} \to \Omega_{\mathbb{G}_m}\Sigma_{\mathbb{G}_m} \in \mathrm{End}(\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}})$$

is an equivalence.

Proof. We seek to apply the abstract cancellation Theorem 4.7 (in the guise of Remark 4.8). Note that $\mathbb{G}_m$ is symmetric in $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$: $T \simeq S^1 \wedge \mathbb{G}_m$ is symmetric by the usual argument, and S^1 is (symmetric and) semi-invertible (by S^1-cancellation, i.e., Proposition 4.6). We have already constructed a lax module transformation

$$\rho : L_{\mathrm{mot}}^{\mathrm{gp}}\Omega_G\Sigma_G \to L_{\mathrm{mot}}^{\mathrm{gp}}.$$

Corollary 5.5 in Section 5 shows that $L_{\mathrm{mot}}^{\mathrm{gp}}\Omega_G\Sigma_G \simeq \Omega_G\Sigma_G L_{\mathrm{mot}}^{\mathrm{gp}}$, and hence we obtain a lax module transformation

$$\rho : \Omega_G\Sigma_G \to \mathrm{id} \in \mathrm{End}(\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}).$$

In $\mathcal{S}\mathrm{pc}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ there is a splitting $G \simeq \mathbb{1} \oplus \mathbb{G}_m$, and hence a retraction $\mathbb{G}_m \to G \to \mathbb{G}_m$. This induces a retraction of lax module functors

$$\Omega_{\mathbb{G}_m}\Sigma_{\mathbb{G}_m} \to \Omega_G\Sigma_G \to \Omega_{\mathbb{G}_m}\Sigma_{\mathbb{G}_m},$$

which in particular allows us to build the lax module transformation

$$\rho' : \Omega_{\mathbb{G}_m}\Sigma_{\mathbb{G}_m} \to \Omega_G\Sigma_G \to \mathrm{id}.$$

In order to apply the abstract cancellation theorem, it remains to verify that $\rho' u \simeq \mathrm{id}$. Via Example 4.10, for this it suffices to compute the effect of $\rho' u$ on $\mathrm{id}_{\mathbb{1}}$. Now $u(\mathrm{id}_{\mathbb{1}}) = \mathrm{id}_{\mathbb{G}_m}$, which corresponds to $\mathrm{id}_G - p \in \mathrm{Hom}(G, G)$, where $p : G \to * \to G$, and so $\rho' u(\mathrm{id}_{\mathbb{1}}) = \rho(\mathrm{id}_G) - \rho(p)$. The result thus follows from Lemma 4.22. $\square$

Lemma 4.22. *For each $n > 0$ we have*

(1) $\rho_n^+(p) = \rho_n^-(p)$, *and*
(2) $\rho_n^+(\mathrm{id}_G) \overset{\mathbb{A}^1}{\simeq} \rho_n^-(\mathrm{id}_G) + \mathrm{id}_{\mathbb{1}}$.

Proof. This is essentially [37, Lemma 4.3].

Note that p is represented by the correspondence $G \xleftarrow{\simeq} G \xrightarrow{1} G$, so that by Remark 4.16, $\rho_n^{\pm}(p)$ is represented by $Z(g_n^{\pm}(t, 1)) \to *$. But $g_n^+(t, 1) = g_n^-(t, 1)$, whence (1).

Similarly $\rho_n^{\pm}(\mathrm{id}_G)$ is represented by $Z_{\pm} := Z(g_n^{\pm}(t, t))$, so $Z_+ = Z(t^n + 1)$ and $Z_- = Z(t^n + t)$, where both $t^n + 1, t^n + t$ are viewed as functions on $\mathbb{A}^1 \setminus 0$. Consider $H = D(t^n + ts + 1 - s) : \mathbb{A}^1 \rightsquigarrow *$, where we view h as a function $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. Then H provides an $\mathbb{A}^1$-homotopy between $D(t^n + 1)$ and $D(t^n + t)$, where this time we view $t^n + 1, t^n + t$ as functions on $\mathbb{A}^1$. Now

$$Z(t^n + 1 \mid \mathbb{A}^1) = Z(t^n + 1 \mid \mathbb{A}^1 \setminus 0) = Z^+,$$

whereas

$$Z(t^n + t \mid \mathbb{A}^1) = Z(t^n + t \mid \mathbb{A}^1 \setminus 0) \amalg \{0\} = Z^- \amalg \{0\}.$$

Since $0 \subset \mathbb{A}^1 \to *$ defines the identity correspondence, H provides the desired homotopy. □

5. Strict $\mathbb{A}^1$-invariance

Primary sources: [12; 17].

5A. ***Introduction.*** The title of this section derives from the following. Write $\mathcal{P}_\Sigma(\mathrm{Sm}_k, \mathrm{Ab})$ for the category of additive presheaves of abelian groups on Sm_k.

Definition 5.1. Let $F \in \mathcal{P}_\Sigma(\mathrm{Sm}_k, \mathrm{Ab})$. Then F is called $\mathbb{A}^1$*-invariant* (or sometimes ($\mathbb{A}^1$-)*homotopy invariant*) if for all $X \in \mathrm{Sm}_k$, the canonical map $F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism.

Also, F is called *strictly* $\mathbb{A}^1$*-invariant* if for all $n \geq 0$ and all $X \in \mathrm{Sm}_k$ the canonical map $H^n_{\mathrm{Nis}}(X, F) \to H^n_{\mathrm{Nis}}(X \times \mathbb{A}^1, F)$ is an isomorphism.

Remark 5.2. Observe that if F is an abelian presheaf, then F is $\mathbb{A}^1$-invariant if and only if the map $F(X \times \mathbb{A}^1) \to F(X)$ induced by the zero section $X \to X \times \mathbb{A}^1$ is injective. We will use this without further comment throughout the sequel.

There are two important observations regarding this:

(1) If F is an $\mathbb{A}^1$-invariant presheaf, it need not be the case that $a_{\mathrm{Nis}}F$ is $\mathbb{A}^1$-invariant (let alone strictly $\mathbb{A}^1$-invariant).
(2) If F is an $\mathbb{A}^1$-invariant sheaf, it need not be strictly $\mathbb{A}^1$-invariant.

However, it turns out that in the presence of transfers, neither of these problems occurs. The first general results in this direction were obtained by Voevodsky in [35]. Here is a version for framed presheaves.

Theorem 5.3. *Let k be a field, and $F \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Ab})$. Suppose that F is $\mathbb{A}^1$-invariant.*

(1) *For $U \subset \mathbb{A}^1$ open we have $H^0_{\mathrm{Nis}}(U, F) \simeq F(U)$.*
(2) *The sheafification $a_{\mathrm{Nis}}F$ is $\mathbb{A}^1$-invariant.*
(3) *If k is perfect, then $a_{\mathrm{Nis}}F$ is strictly $\mathbb{A}^1$-invariant.*

We can escalate the above result as follows.

Corollary 5.4. *Let k be a perfect field, and $F \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k))^{\mathrm{gp}}$ be $\mathbb{A}^1$-invariant. Then $L_{\mathrm{Nis}}F$ is $\mathbb{A}^1$-invariant, and hence motivically local.*

Proof. By an induction on the Postnikov tower, or equivalently using the (strongly convergent) descent spectral sequence, this is immediate from Theorem 5.3. □

We can also deduce the following fact, which is very important for the cancellation theorem.

Corollary 5.5. *Let k be a perfect field. On the category $\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k))^{\mathrm{gp}}$, the canonical transformation $L_{\mathrm{mot}}\Omega_{\mathbb{G}_m} \to \Omega_{\mathbb{G}_m}L_{\mathrm{mot}}$ is an equivalence.*

Proof. Using [30, Lemma 6.1.3], it suffices to prove that the map induces an equivalence on sections over fields. Thus let K/k be a field extension. By Corollary 5.4, $L_{\mathrm{mot}} = L_{\mathrm{Nis}}L_{\mathbb{A}^1}$. Note that $\Omega_{\mathbb{G}_m}$ commutes with $L_{\mathbb{A}^1}$ (see, e.g., [4, Lemma 4]) and fields are stalks for the Nisnevich topology; hence it is enough to show that

$$(\Omega_{\mathbb{G}_m}L_{\mathbb{A}^1}\mathcal{X})(K) \to (\Omega_{\mathbb{G}_m}L_{\mathrm{Nis}}L^1_{\mathbb{A}}\mathcal{X})(K)$$

is an equivalence. By another induction on the Postnikov tower / descent spectral sequence, we reduce to showing that for $F \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Ab})$ which is $\mathbb{A}^1$-invariant, one has

$$H^n_{\mathrm{Nis}}(\mathbb{G}_{m\,K}, F) = \begin{cases} F(\mathbb{G}_{m\,K}), & n = 0, \\ 0, & \text{else.} \end{cases}$$

The first case is immediate from Theorem 5.3(1). Assume that $n \geq 1$. Theorem 5.3(3) asserts that $a_{\mathrm{Nis}}F$ is strictly $\mathbb{A}^1$-invariant. Since $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m \in \mathcal{S}\mathrm{pc}(k)$ we find that $H^n_{\mathrm{Nis}}(\mathbb{G}_{m\,K}, F) = H^{n+1}_{\mathrm{Nis}}(\mathbb{P}^1_K, F)$. The result thus follows from the fact that $\mathbb{P}^1$ has Nisnevich cohomological dimension one [32, Proposition 3.1.8] and thus its Nisnevich cohomology vanishes whenever $n + 1 \geq 2$. □

The remainder of this section is devoted to proving Theorem 5.3.

Notation and conventions. From now on, all cohomology will be Nisnevich cohomology, i.e., $H^* := H^*_{\mathrm{Nis}}$.

Given a scheme X and a point $x \in X$, we write X_x for the local scheme $\mathrm{Spec}(\mathcal{O}_{X,x})$ and X_x^h for the henselian local scheme $\mathrm{Spec}(\mathcal{O}^h_{X,x})$. If $S \subset X$ is a finite set of points, we write X_S for the semilocalization of X in S (see Section 5E2 for more about semilocalizations).

Recall that a scheme is called *essentially smooth affine over k* if it can be written as a cofiltered limit of smooth affine k-schemes, with étale transition maps. Observe that essentially smooth affine schemes are affine of finite dimension and have local rings which are integral domains, but need not be Noetherian (though we will not use non-Noetherian schemes in any relevant way).

5B. ***A formalism for strict $\mathbb{A}^1$-invariance.*** We shall prove Theorem 5.3 following the strategy developed in [17] and explained further in [9; 11].

Throughout we fix a field k. Consider an abelian presheaf $F \in \mathcal{P}_\Sigma(\mathrm{SmAff}_k, \mathrm{Ab})$. As usual, we extend F to essentially smooth affine schemes by continuity: if $X = \lim_i X_i$, where each X_i is smooth affine (and the transition maps are étale, so that X is essentially smooth), then $F(X) := \mathrm{colim}_i\, F(X_i)$. We isolate the following four properties which F may satisfy.

Definition 5.6 (IA). We say that F satisfies *injectivity on the affine line* (*IA*) if the following holds. For any finitely generated, separable field extension K/k (automatically essentially smooth) and open subschemes $\varnothing \neq V_1 \subset V_2 \subset \mathbb{A}^1_K$ (automatically affine), the restriction $F(V_2) \to F(V_1)$ is injective.

Definition 5.7 (EA). We say that F satisfies *excision on the relative affine line* (*EA*) if the following holds. For any essentially smooth affine scheme U and affine open subscheme $V \subset \mathbb{A}^1_U$ containing 0_U, restriction induces an isomorphism

$$F(\mathbb{A}^1_U \setminus 0_U)/F(\mathbb{A}^1_U) \simeq F(V \setminus 0_U)/F(V).$$

(Note that $\mathbb{A}^1_U \setminus 0_U$ and $V \setminus 0_U$ are indeed affine.)

We require that if K/k is a finitely generated, separable field extension, $z \in \mathbb{A}^1_K$ a closed point, $V \subset \mathbb{A}^1_K$ an open neighborhood of z, then

$$F(\mathbb{A}^1_K \setminus z)/F(\mathbb{A}^1_K) \simeq F(V \setminus z)/F(V).$$

Definition 5.8 (IL). We say that F satisfies *injectivity for henselian local schemes* (*IL*) if the following holds. For any essentially smooth, henselian local scheme U with generic point η, the restriction $F(U) \to F(\eta)$ is injective.

Definition 5.9 (EE). We say that F satisfies *étale excision* (*EE*) if the following holds. Let $\pi : X' \to X$ be a local morphism of local schemes which can be obtained as a cofiltered limit of étale morphisms of smooth k-schemes (with étale transition maps). Let $Z \subset X$ be a principal closed subscheme such that $\pi^{-1}(Z) \to Z$ is an isomorphism. Then the canonical map

$$F(X \setminus Z)/F(X) \to F(X' \setminus \pi^{-1}(Z))/F(X')$$

is an isomorphism.

Remark 5.10. Observe that if X is affine and $Z \subset X$ is a principal closed subscheme, then $X \setminus Z$ is (a principal open) affine. The above axioms are often stated in a more general form without affineness or principality assumptions. As we will see in this section, our weak form of the axioms is enough to deduce strict $\mathbb{A}^1$-invariance.

We shall also use the notion of contraction.

Definition 5.11. Let F be a presheaf. We denote by F_{-1} the presheaf $X \mapsto F(X \times \mathbb{G}_m)/F(X)$, and by F_{-n} the n-fold iterate of this construction. We also write F'_{-1} for the presheaf $X \mapsto F(X \times \mathbb{G}_m)/F(X \times \mathbb{A}^1)$.

Remark 5.12. F'_{-1} is the definition of contraction in [29, §23]. This yields the most natural result in Lemma 5.15. For $\mathbb{A}^1$-invariant presheaves, the two notions coincide. For non-$\mathbb{A}^1$-invariant presheaves, the definition of F_{-1} we gave seems more standard.

The main result of this section is as follows.

Theorem 5.13. *Let k be a perfect field. Let $\mathcal{C}$ be a collection of abelian presheaves on* SmAff_k *which is closed under $F \mapsto F_{-1}$ and $F \mapsto H^i(-, F)$. Assume that whenever $F \in \mathcal{C}$ is an $\mathbb{A}^1$-invariant presheaf, it satisfies IA, EA, IL and EE.*

Let $F \in \mathcal{C}$ be $\mathbb{A}^1$-invariant. Then for every essentially smooth (*not necessarily affine*) *k-scheme X we have*

$$H^i(X \times \mathbb{A}^1, F) \simeq H^i(X, F).$$

Note that since the category of Nisnevich sheaves on Sm_k is the same as the category of Nisnevich sheaves on SmAff_k, $H^i(X, F)$ makes sense even if X is not affine. The next lemma states that Nisnevich sheafification of a presheaf with IA, EA, IL and EE does not change its values on opens of (relative) $\mathbb{A}^1$'s.

Lemma 5.14. *Let K/k be a finitely generated, separable field extension and F a presheaf satisfying IA, EA, IL and EE. Let $U \subset \mathbb{A}^1_K$ be open. Then $F(U) \simeq (a_{\mathrm{Nis}}F)(U)$ and $H^i(U, F) = 0$ for $i > 0$.*

Proof. Let $X = \mathbb{A}^1_K$.

We first establish the following claim: $(*)$ if $U \subset \mathbb{A}^1_K$ is open, $z_1, \ldots, z_n \in U$ are distinct closed points, then

$$F(U \setminus \{z_1, \ldots, z_n\})/F(U) \simeq \bigoplus_{i=1}^{n} F(U^h_{z_i} \setminus z_i)/F(U^h_{z_i}).$$

If $n = 1$, this follows by combining EA and EE. Now let $n > 1$, and assume the claim proved for $n - 1$. Combining IA and the case $n = 1$, we have a short exact sequence

$$0 \to F(U \backslash \{z_1, \dots, z_{n-1}\})/F(U) \to F(U \backslash \{z_1, \dots, z_n\})/F(U) \to F(U^h_{z_n} \backslash z_n)/F(U^h_{z_n}) \to 0.$$

By induction, the first term is isomorphic to $\bigoplus_{i=1}^{n-1} F(U^h_{z_i} \setminus z_n)/F(U^h_{z_i})$, and we can thus split the sequence. This proves the claim.

Consider, on X_{Nis}, the sequence of sheaves

$$0 \to a_{\mathrm{Nis}} F \to \bigoplus_{\eta \in U^{(0)}} F(\eta) \to \bigoplus_{z \in U^{(1)}} F(U^h_z \setminus z)/F(U^h_z) \to 0,$$

where $U \to X$ is an arbitrary affine étale scheme. Observe that the second and third terms are skyscraper sheaves, and so are acyclic (see, e.g., [31, proof of Lemma 5.42]). We argue that this sequence is exact. For this we need only consider the case where $U = \eta$ (a generic point of some étale X-scheme), and the case where U is henselian local of dimension 1, so in particular has only two points. Both sequences are exact; the only nontrivial point is injectivity of $F(U) \to F(\eta)$ which is IL.

It follows that we may compute $H^i(U, F)$ using the above resolution; in particular $H^i = 0$ for $i > 1$. Let $U \subset X$. We first compute $H^0(U, F)$: It consists of those elements $a \in F(\eta)$ (where η is the generic point of U) such that for every closed point $z \in U$, a is in the image of $F(X^h_z) \to F(X^h_z \setminus z)$. Let a be such an element. Then there exists $\varnothing \neq V \subset U$ and $a' \in F(V)$ such that $a = a'|_\eta$. Let $z \in U \setminus V$ and put $V' = V \cup \{z\}$. Note that $V' \subset U$ is open since its complement consists of finitely many (closed) points. By $(*)$ with $n = 1$ we have $F(V)/F(V') \simeq F((V'_z)^h \setminus z)/F((V'_z)^h)$. The image of a' in the right-hand group vanishes by assumption, hence it vanishes in the left-hand group. In other words there exists $a'' \in F(V')$ extending a'. Repeating this argument finitely many times we conclude that $F(U) \to H^0(U, F)$ is surjective. The map is injective by IA, and hence an isomorphism.

It remains to prove that $H^1(U, F) = 0$. In other words, given distinct closed points $z_1, \dots, z_n \in U$ we must prove that $F(\eta) \to \bigoplus_{i=1}^n F(U^h_{z_i} \setminus z_i)/F(U^h_{z_i})$ is surjective. This follows from $(*)$, since it identifies the right-hand side with a quotient of $F(U \setminus \{z_1, \dots, z_n\})$. □

The following is essentially [29, Theorem 23.12].

Lemma 5.15. *Let X be essentially smooth and affine, $i : Z \hookrightarrow X$ a principal, essentially smooth closed subscheme, and F satisfy the properties EA, EE. Write $j : U = X \setminus Z \to X$ for the complementary open immersion. Suppose that we are given étale neighborhoods $(X, Z) \leftarrow (\Omega, Z) \to (\mathbb{A}^1_Z, Z)$.*

There is a short exact sequence of Nisnevich sheaves on X

$$a_{\mathrm{Nis}} F \to a_{\mathrm{Nis}} j_* j^* F \to a_{\mathrm{Nis}} i_* F'_{-1} \to 0.$$

If F satisfies IL, then $a_{\mathrm{Nis}} F \to a_{\mathrm{Nis}} j_* j^* F$ is injective.

Proof. We shall use without further comment the fact that Z_{Nis} has a conservative family of stalk functors of the form $F \mapsto F(U^h_x \times_X Z)$, where $U \to X$ is étale and $x \in U$.

Denote by $F_{(X,Z)}$ the sheaf $a_{\mathrm{Nis}} i^*(j_* j^* F/F)$ on Z_{Nis}. By adjunction we obtain a map $a_{\mathrm{Nis}}(j_* j^* F/F) \to i_* F_{(X,Z)}$. Checking on stalks, we see that this is an equivalence. Since i_* commutes with a_{Nis}, our task is to prove that $F_{(X,Z)} \simeq a_{\mathrm{Nis}} F'_{-1}$. If $(X', Z) \to (X, Z)$ is an étale neighborhood, there is an induced map $F_{(X,Z)} \to F_{(X',Z)}$; again checking on stalks we see that this is an equivalence. Using the étale neighborhoods provided by the hypothesis, we may thus assume that $X = \mathbb{A}^1_Z$. For $U \to Z$ étale, $\mathbb{A}^1_U \to \mathbb{A}^1_Z$ is étale, and $F'_{-1}(U) = (j_* j^* F/F)(\mathbb{A}^1_U)$; this induces a map $F'_{-1} \to i^* j_* j^* F/F$. We shall prove that this is an equivalence.

We check this on stalks. Let $Z' \to Z$ be étale and $x \in Z'$. We shall consider the stalk at x; to simplify notation replace Z by Z'. Consider the commutative diagram

$$\begin{array}{ccccccc}
(\mathbb{A}^1_Z)^h_x & \longrightarrow & (\mathbb{A}^1_{Z^h_x})_x & \longrightarrow & \mathbb{A}^1_{Z^h_x} & \longrightarrow & \mathbb{A}^1_Z \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
B & \longrightarrow & A & \longrightarrow & Z^h_x & \longrightarrow & Z.
\end{array}$$

The right-most vertical map is the canonical inclusion, and all squares are defined to be cartesian. Note that A is local and a localization of Z^h_x containing x; thus $A \simeq Z^h_x$. Similarly B is henselian local, pro-étale over A and contains x; thus $B \simeq Z^h_x$ also. Now we get

$$\begin{aligned}
[i^*(j_* j^* F/F)](Z^h_x) &\simeq (j_* j^* F/F)((\mathbb{A}^1_Z)^h_x) \\
&\simeq F((\mathbb{A}^1_Z)^h_x \setminus Z^h_x)/F((\mathbb{A}^1_Z)^h_x) \\
&\overset{EE}{\simeq} F((\mathbb{A}^1_{Z^h_x})_x \setminus Z^h_x)/F((\mathbb{A}^1_{Z^h_x})_x) \\
&\overset{EA}{\simeq} F(\mathbb{A}^1_{Z^h_x} \setminus Z^h_x)/F(\mathbb{A}^1_{Z^h_x}) \\
&\simeq F'_{-1}(Z^h_x).
\end{aligned}$$

For the last part, it suffices to observe that if X is henselian local with generic point η and $U \subset X$ is nonempty, then

$$F(X) \simeq (a_{\mathrm{Nis}} F)(X) \to (a_{\mathrm{Nis}} F)(U) \to (a_{\mathrm{Nis}} F)(\eta) \simeq F(\eta)$$

is injective by IL, and hence so is $(a_{\mathrm{Nis}} F)(X) \to (a_{\mathrm{Nis}} F)(U)$. □

Remark 5.16. Let $X \in \mathrm{Sm}_k$, $Z \subset X$ a smooth, principal closed subscheme. Then locally on X, étale neighborhoods as required in Lemma 5.15 exist. See, for example, [8, §5.9].

Lemma 5.17. *Suppose that F satisfies IA and F_{-1} satisfies IL. Then the canonical map $a_{\mathrm{Nis}} F_{-1} \to (a_{\mathrm{Nis}} F)_{-1}$ is an injection.*

Proof. Using IL, it suffices to prove that $F_{-1}(K) \to (a_{\mathrm{Nis}} F)_{-1}(K)$ is injective. Hence we need to prove that $F(\mathbb{A}^1_K \setminus 0) \to (a_{\mathrm{Nis}} F)(\mathbb{A}^1_K \setminus 0)$ is injective. This follows from IA. □

Proof of Theorem 5.13. To begin with, note that if F is $\mathbb{A}^1$-invariant then so is F_{-n}. We shall use this freely in the sequel.

As a first step, we shall prove that if $F \in \mathcal{C}$ is $\mathbb{A}^1$-invariant then so is $a_{\mathrm{Nis}} F$. Since F is $\mathbb{A}^1$-invariant it satisfies IA, EA, IL and EE (by assumption on $\mathcal{C}$) and so Lemma 5.14 applies. Let $X \in \mathrm{SmAff}_k$. We must prove that $H^0(X \times \mathbb{A}^1, F) \to H^0(X, F)$ is injective. Consider the diagram

$$\begin{array}{ccc} H^0(X, F) & \longrightarrow & \prod_{\eta \in X^{(0)}} H^0(\eta, F) \\ \uparrow & & \uparrow \\ H^0(\mathbb{A}^1_X, F) & \longrightarrow & \prod_{\eta \in X^{(0)}} H^0(\mathbb{A}^1_\eta, F) \\ \downarrow & & \downarrow \\ \prod_{x \in U} H^0(U^h_x, F) & \longrightarrow & \prod_{x \in U} H^0(\eta_x, F), \end{array}$$

where the product is over points of $(\mathbb{A}^1_X)_{\mathrm{Nis}}$ and η_x is the generic point of U^h_x. This lies over a point of $\mathbb{A}^1_\eta$, so the bottom right-hand map is defined and the diagram commutes. The bottom left-hand map is injective (since $H^0(-, F)$ is Nisnevich-separated) and the bottom horizontal map is injective by IL; hence the middle horizontal map is injective. Consequently the top left-hand map is injective as soon as the top right hand map is. This reduces the claim to the case $X = \eta$, which holds by Lemma 5.14.

Next we will prove by induction on n that if $F \in \mathcal{C}$ is $\mathbb{A}^1$-invariant, then $H^n(-, F)$ is also $\mathbb{A}^1$-invariant. The case $n = 0$ has been dealt with. In particular we may assume that F is a sheaf. Let $n > 0$ and suppose that all smaller n have been established.

Step (1). If $j : U \to X$ is a principal open immersion of smooth k-schemes, with smooth closed complement $Z \hookrightarrow X$, then we claim

$$R^i j_* F = 0 \quad \forall\, 0 < i < n.$$

For this consider the presheaf $G = H^i(-, F)$, which we know is $\mathbb{A}^1$-invariant by induction. The problem is local on X, so by Remark 5.16 we may apply Lemma 5.15

to G and obtain an exact sequence (using that $G_{-1} \simeq G'_{-1}$ by Remark 5.12)

$$0 \to a_{\mathrm{Nis}} G \to a_{\mathrm{Nis}} j_* j^* G \to i_* a_{\mathrm{Nis}} G_{-1} \to 0.$$

We have $a_{\mathrm{Nis}} G = 0$, $a_{\mathrm{Nis}} j_* j^* G = R^i j_* F$. By Lemma 5.17 we have

$$a_{\mathrm{Nis}} G_{-1} \hookrightarrow (a_{\mathrm{Nis}} G)_{-1} = 0,$$

which proves the claim.

Step (2). If X is an essentially smooth scheme, $U \subset X$ a principal open subscheme with essentially smooth closed complement, then we claim that $H^n(\mathbb{A}^1_X, j_* j^* F) \to H^n(\mathbb{A}^1_U, F)$ is injective. To prove this, we may assume X smooth. Consider the cofiber sequence $j_* j^* F \to R j_* j^* F \to C$. By step (1), C has cohomology sheaves concentrated in degree $\geq n$. Hence in the long exact sequence

$$H^{n-1}(\mathbb{A}^1_X, C) \to H^n(\mathbb{A}^1_X, j_* j^* F) \to H^n(\mathbb{A}^1_X, R j_* j^* F)$$

the first term vanishes, so the second map is injective. The result follows since the last term identifies with $H^n(\mathbb{A}^1_U, F)$ by the previous step.

Step (3). If X is an essentially smooth scheme, $U \subset X$ a principal open subscheme with essentially smooth closed complement Z, and $z \in Z$, then $H^n(\mathbb{A}^1_{X^h_z}, F) \to H^n(\mathbb{A}^1_{X^h_z}, j_* j^* F)$ is injective. Again we may assume that X is smooth. The problem being local around z, via Remark 5.16, gives us étale neighborhoods $(X, Z) \leftarrow (\Omega, Z) \to (\mathbb{A}^1_Z, Z)$. Taking the product with $\mathbb{A}^1$, we may thus apply Lemma 5.15 to $\mathbb{A}^1_Z \hookrightarrow \mathbb{A}^1_X$ and get an exact sequence $0 \to F \to j_* j^* F \to i_* F_{-1} \to 0$ on $(\mathbb{A}^1_X)_{\mathrm{Nis}}$. Taking $H^1(\mathbb{A}^1_{X^h_z}, -)$ yields a long exact sequence, part of which reads

$$H^{n-1}(\mathbb{A}^1_{X^h_z}, j_* j^* F) \xrightarrow{a} H^{n-1}(\mathbb{A}^1_{Z^h_z}, F_{-1}) \to H^n(\mathbb{A}^1_{X^h_z}, F) \to H^n(\mathbb{A}^1_{X^h_z}, j_* j^* F).$$

Thus, it suffices to prove a is surjective. If $n > 1$ then $H^{n-1}(\mathbb{A}^1_{Z^h_z}, F_{-1}) \simeq H^{n-1}(Z^h_z, F_{-1}) = 0$, by induction on n. It remains to prove that $H^0(\mathbb{A}^1_{X^h_z}, j_* j^* F) \to H^0(\mathbb{A}^1_{Z^h_z}, F_{-1})$ is surjective; in other words, that the map $F(\mathbb{A}^1_{X^h_z \setminus Z^h_z}) \to F_{-1}(\mathbb{A}^1_{Z^h_z})$ is surjective. Since F is $\mathbb{A}^1$-invariant, this is just $F(X^h_z \setminus Z^h_z) \to F_{-1}(Z^h_z)$, which is the evaluation of the surjective map of sheaves $j_* j^* F \to F_{-1}$ on X^h_z, and hence surjective.

Conclusion. Let X be an essentially smooth scheme. Write $f : X \to \mathbb{A}^1_X$ for the inclusion at 0. We seek to prove that $F \to R f_* F$ induces an equivalence on H^n, and we already know this for H^i, $i < n$. We shall prove this by induction on $d = \dim X$. Let C be the cofiber of $F \to R f_* F$. Then $R f_* F \simeq F \oplus C$ (via $p : \mathbb{A}^1_X \to X$) and so we must prove that C has cohomology concentrated in degrees $> n$; this may be checked stalkwise. In other words we must prove that if X is an essentially smooth, henselian local scheme of dimension d, then $H^n(\mathbb{A}^1_X, F) \to H^n(X, F)$ is an isomorphism; equivalently we may prove that it is injective. If $d = 0$, X is the spectrum of a field, and we are reduced to Lemma 5.14. Thus $d > 0$ and we can find a principal open $U \subset X$ with essentially smooth closed complement Z (here we use that k is perfect). Consider the commutative diagram

$$\begin{array}{ccccc} H^n(\mathbb{A}^1_X, F) & \xrightarrow{(2)} & H^n(\mathbb{A}^1_X, j_*j^*F) & \xrightarrow{(1)} & H^n(\mathbb{A}^1_U, F) \\ \downarrow & & \downarrow & & \downarrow \\ H^n(X, F) & \longrightarrow & H^n(X, j_*j^*F) & \longrightarrow & H^n(U, F). \end{array}$$

The maps in the top composite are injective, by the steps indicated above them. The right-hand vertical map is injective by induction on d. It follows that the left-hand vertical map is injective, as desired. □

Remark 5.18. The proof shows that the perfectness assumption on k is only needed to ensure $\mathbb{A}^1$-invariance of $H^i(-, F)$ for $i > 0$.

5C. *Framed pretheories.*

As in the last section, we consider an abelian presheaf $F \in \mathcal{P}_\Sigma(\mathrm{SmAff}_k, \mathrm{Ab})$, extended by continuity to essentially smooth affine schemes. The next definition is the framed analog of the notion of pretheories introduced by Voevodsky; see [35].

Definition 5.19. By a structure of *framed pretheory* on F we mean the following data: for every $X \in \mathrm{SmAff}_k$, $C \to X$ a smooth relative curve, $\mu : \Omega^1_{C/X} \simeq \mathcal{O}_C$, $f \in \mathcal{O}(C)$ and decomposition $Z(f) = Z \amalg Z'$ with Z finite over X, we are given

$$\mathrm{tr}(f)^\mu_Z : F(C) \to F(X).$$

The transfers must satisfy the following properties:

(1) If $Z = Z_1 \amalg Z_2$, then $\mathrm{tr}(f)^\mu_Z = \mathrm{tr}(f)^\mu_{Z_1} + \mathrm{tr}(f)^\mu_{Z_2}$.

(2) If $p : (C', Z') \to (C, Z)$ is an étale neighborhood, then $\mathrm{tr}(f)^\mu_Z = \mathrm{tr}(f \circ p)^{p^*\mu}_{Z'} \circ p^*$.

(3) Given $\alpha : X' \to X$ let $\alpha' : X' \times_X C \to C$ be the induced map. Then

$$\alpha^* \circ \mathrm{tr}(f)^\mu_Z = \mathrm{tr}(f \circ \alpha')^{\alpha'^*\mu}_{\alpha'^{-1}(Z)} \circ \alpha'^*.$$

(4) Suppose that $Z \to X$ is an isomorphism with inverse i. Then

$$\mathrm{tw}(f)^\mu_Z := \mathrm{tr}(f)^\mu_Z \circ (C \to X)^* : F(X) \to F(X)$$

is an isomorphism and $\mathrm{tr}(f)^\mu_Z = \mathrm{tw}(f)^\mu_Z \circ i^*$.

(5) Fix a section $i : X \to C$ with image Z and a trivialization μ. Assume that X is semilocal. Then, there exists $\lambda \in H^0(Z, C_{Z/C})$ such that for any $f \in I_Z(C)$ with $df = \lambda$ we have $\mathrm{tw}(f)^\mu_Z = \mathrm{id}$.

Note that condition (3) implies that the transfers on F extend to essentially smooth schemes, so (5) makes sense.

Example 5.20. Let $F \in \mathcal{P}_\Sigma(\mathrm{Corr}_k^{\mathrm{fr}}, \mathrm{Ab})$. Then F admits a structure of framed pretheory as follows. Given $(C \to X, \mu, f, Z)$, $Z \to X$ is syntomic by [15, Proposition 2.1.16]. We define the K-theoretic trivialization of the cotangent complex $L_{Z/X}$ as

$$\tau : L_{Z/X} \simeq L_{Z/C} + L_{C/X}|_Z \overset{f,\mu}{\simeq} -\mathcal{O} + \mathcal{O} \simeq 0 \in K(Z).^7$$

Consequently we obtain a framed correspondence $X \overset{\tau}{\longleftarrow} Z \to C$, pullback along which defines $\mathrm{tr}(f)_Z^\mu$. All axioms are easily verified. (For the last axiom, one may argue as follows. Since Z is semilocal, $C_{Z/C}$ admits a nonvanishing section λ'. Together λ', μ determine a trivialization of $L_{Z/X} \simeq L_{X/X} = 0$, whence a class in $K_1(Z)$. Since Z is semilocal, $K_1(Z) \simeq \mathcal{O}^\times(Z)$ [38, Lemma III.1.4]. Hence replacing λ' by $\lambda := u\lambda'$ for well-chosen u ensures that the trivialization is the canonical one, and thus $\mathrm{tw}(f)$ is the identity morphism.)

Suppose we are given $(C \to X, f, \mu, Z)$ as in Definition 5.19, a map $g : C \to Y \in \mathrm{SmAff}_k$, $U \subset X$ and $U' \subset Y$ open. Assume that $g^{-1}(Y \backslash U') \cap Z$ lies over $X \backslash U$. Write $C_U \subset C$, $Z_U \subset Z$ for the canonical open subschemes. Then $C_U \cap g^{-1}(U') \to C_U$ is an étale (in fact open) neighborhood of Z_U. We may form the diagram

$$\begin{tikzcd} F(Y) \arrow[rr, "g^*"] \arrow[d] & & F(C) \arrow[r, "\mathrm{tr}(f)_Z^\mu"] \arrow[d] & F(X) \arrow[d] \\ F(U') \arrow[r, "g^*"] & F(C_U \cap g^{-1}(U')) \arrow[rr, bend right, "\mathrm{tr}"'] & F(C_U) \arrow[l] \arrow[r, "\mathrm{tr}"] & F(U). \end{tikzcd}$$

The maps labeled tr are the evident transfers, and the unlabeled maps are pullbacks along evident inclusions. The diagram commutes by properties (2) and (3).

Construction 5.21. Taking vertical cokernels in the outer rectangle of the above diagram, we obtain a map

$$F(U')/F(Y) \to F(U)/F(X).$$

Definition 5.22. Let X, Y be essentially smooth over k, $X \leftarrow C \xrightarrow{g} Y$ a span. We call data

$$\Phi = (X \leftarrow C \to Y, f, \mu, Z) : X \rightsquigarrow Y,$$

such that $(C \to X, f, \mu, Z)$ satisfies the assumptions of Definition 5.19, a *curve correspondence* from X to Y and put

$$\Phi^* = \mathrm{tr}(f)_Z^\mu \circ g^* : F(Y) \to F(X).$$

[7] Note that $L_{Z/C} \simeq I/I^2[1]$ where I is the ideal defining Z. By hypothesis, I/I^2 is an invertible $\mathcal{O}_Z$-module and the class of f in I/I^2 provides a trivialization of this line bundle

If we are further given $U \subset X$, $U' \subset Y$ such that $g^{-1}(Y \setminus U') \cap Z$ lies over $X \setminus U$, we call the data a *curve correspondences of pairs*, denote it by

$$\Phi = (X \leftarrow C \to Y, f, \mu, Z) : (X, U) \rightsquigarrow (Y, U')$$

and write

$$\Phi^* : F(U')/F(Y) \to F(U)/F(X)$$

for the map of Construction 5.21.

Lemma 5.23. *Let $\Phi = (X \leftarrow C \xrightarrow{g} Y, f, \mu, Z) : (X, U) \rightsquigarrow (Y, U')$ be a curve correspondence of pairs. Suppose that $Z \subset g^{-1}(U')$. Then $\Phi^* = 0$.*

Proof. By axiom (2) we can replace C by $g^{-1}(U')$. Since now Φ^* factors through $F(U')/F(U') = 0$, the result follows. □

5D. *Injectivity on the relative affine line.* In this section we establish IA for $\mathbb{A}^1$-invariant framed pretheories.

Lemma 5.24. *Let $U \in \mathrm{SmAff}_k$, $V_1 \subset V_2 \subset \mathbb{A}^1_U$ affine and open. Assume that $\mathbb{A}^1_U \setminus V_2$ and $V_2 \setminus V_1$ are finite over U. Then there exist curve correspondences $\Phi, \Phi^- : V_2 \rightsquigarrow V_1$ and $\Theta : V_2 \times \mathbb{A}^1 \rightsquigarrow V_2$ such that in any framed pretheory, $i_0^*\Theta^* = \Phi^* i^*$ and $i_1^*\Theta^* - (\Phi^-)^* i^*$ is invertible. Here i denotes the inclusion $V_1 \to V_2$ and $i_s : V_2 \to V_2 \times \mathbb{A}^1$ is the inclusion at s.*

Proof. We begin by constructing certain functions $f, g \in k[\mathbb{A}^1_U \times_U V_2]$. We shall denote the first coordinate by y and the second by x. We will arrange that f, g are, respectively, monic in y of degrees n and $n-1$ (for some n sufficiently large). We shall ensure that

$$f|_{(\mathbb{A}^1_U \setminus V_1)\times_U V_2} = 1,\ \ g|_{(\mathbb{A}^1_U \setminus V_2)\times_U V_2} = (y-x)^{-1},\ \ g|_{(V_2 \setminus V_1)\times_U V_2} = 1,\ \ g|_{Z(y-x)} = 1.$$

To do this, note that each of the subschemes we are restricting to is finite over V_2, and apply Lemma 5.25 below.

Let $h \in k[\mathbb{A}^1 \times V_2 \times \mathbb{A}^1]$ be given by

$$h = (1-t)f + t(y-x)g;$$

here t denotes the third coordinate. Note that h is monic in y. Define

$$\Phi = (V_2 \leftarrow V_1 \times_U V_2 \to V_1, f, dy, Z(f)) : V_2 \rightsquigarrow V_1,$$
$$\Theta = (V_2 \times \mathbb{A}^1 \xleftarrow{pr_2} V_2 \times_U V_2 \times \mathbb{A}^1 \xrightarrow{pr_1} V_2, h, dy, Z(h)) : V_2 \times \mathbb{A}^1 \rightsquigarrow V_2.$$

Note that here by f we implicitly denote its restriction to $V_1 \times_U V_2$, and similarly for h. Since h is monic in y, $Z(h) \subset \mathbb{A}^1 \times V_2 \times \mathbb{A}^1$ is finite over $V_2 \times \mathbb{A}^1$. By construction, h is constantly equal to 1 on $(\mathbb{A}^1_U \setminus V_2) \times_U V_2 \times \mathbb{A}^1$. Thus $Z(h)$ is completely contained in $V_2 \times_U V_2 \times \mathbb{A}^1$, and so Θ is well defined. A similar argument applies to Φ.

Since $h|_{t=0} = f$ we find (using Definition 5.19(3)) that

$$i_0^*\Theta^* = \Phi^* i^*.$$

Since $h|_{t=1} = (y-x)g$ has vanishing locus splitting into two disjoint pieces, we find (using Definition 5.19(1)) that

$$i_1^*\Theta^* = (V_2 \xleftarrow{pr_2} V_2 \times_U V_2 \xrightarrow{pr_1} V_2, (y-x)g, dy, Z(y-x))^* \\ + (V_2 \times_U V_2, (y-x)g, dy, Z(g))^*.$$

The first term is invertible by Definition 5.19(4). Note that $Z(g) \subset V_1 \times_U V_2$. Thus we can define

$$\Phi^- = (V_2 \leftarrow V_1 \times_U V_2 \to V_1, (y-x)g, dy, Z(g)) : V_2 \rightsquigarrow V_1,$$

concluding the proof. □

Lemma 5.25. *Let U be an affine scheme and $Z \subset \mathbb{A}^1_U$ a closed subscheme which is finite over U. Let $\bar{f} \in \mathcal{O}(Z)$. Then for n sufficiently large there exists a monic $f \in \mathcal{O}(\mathbb{A}^1_U)$ of degree n with $f|_Z = \bar{f}$.*

Proof. Let $U = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(A[T]/I)$. Since Z is finite, there exist $g_1, \dots, g_r \in A[T]$ whose images generate $A[T]/I$ as an A-module. Let n be larger than the maximum of the degrees of the g_i. We claim that f as desired can be found for such n. Indeed note that *any* $\bar{h} \in A[T]/I$ admits a lift $h \in A[T]$ of degree $< n$; in fact we can choose the lift to be an A-linear combination of the g_i. Now let f_1 be an arbitrary lift of $\bar{f} - T^n$ of degree $< n$, and put $f = T^n + f_1$. □

Theorem 5.26. *Let U be essentially smooth over k, $V_1 \subset V_2 \subset \mathbb{A}^1_U$ affine and open. Assume that $\mathbb{A}^1_U \setminus V_2$ and $V_2 \setminus V_1$ are finite over U. Let F be an $\mathbb{A}^1$-invariant framed pretheory. Then $F(V_2) \to F(V_1)$ is injective. In particular F satisfies IA.*

Proof. All our open immersions are affine, hence quasicompact [34, Tag 01K4] and so of finite presentation [34, Tag 01TU]. It follows that when writing $U = \lim_i U_i$ as a cofiltered limit of smooth affine schemes, we may assume we are given $V_1' \subset V_2' \subset \mathbb{A}^1_{U_0}$ affine open with base change V_i, such that $\mathbb{A}^1_{U_0} \setminus V_2'$ and $V_2' \setminus V_1'$ are finite over U_0 [34, Tags 01ZM, 01ZO and 0EUU]. By continuity of F, we may thus assume $U \in \mathrm{SmAff}_k$. Now let $x \in F(V_2)$ with $i^*(x) = 0$. Using $\mathbb{A}^1$-invariance we find that, in the notation of Lemma 5.24,

$$0 = \Phi^* i^*(x) - (\Phi^-)^* i^*(x) = i_0^*\Theta^*(x) - (\Phi^-)^* i^*(x) = i_1^*\Theta^*(x) - (\Phi^-)^* i^*(x).$$

But $i_1^*\Theta^* - (\Phi^-)^* i^*$ is invertible, so $x = 0$. □

5E. ***Geometric preliminaries.*** The proofs of the other axioms are similar to the one for IA, but significantly more elaborate. We collect here some results from algebraic geometry that we shall use.

5E1. *Serre's theorem.* Let A be a Noetherian ring and $X \to \mathrm{Spec}(A)$ a projective A-scheme with ample line bundle $\mathcal{O}(1)$. Then for any coherent sheaf $\mathcal{F}$ on A, $i > 0$ and n sufficiently large, $H^i(X, \mathcal{F}(n)) = 0$ [34, Tag 0B5T(4)]. An immediate consequence, which we shall often use, is if $\mathcal{F} \to \mathcal{G}$ is a surjection of coherent sheaves, then for n sufficiently large, $H^0(X, \mathcal{F}(n)) \to H^0(X, \mathcal{G}(n))$ is surjective. (Indeed this follows from vanishing of $H^1(X, \ker(\mathcal{F} \to \mathcal{G})(n))$.) In particular, if $Z \subset X$ is closed, then $H^0(X, \mathcal{F}(n)) \to H^0(Z, \mathcal{F}(n))$ is surjective. This is deduced by taking $\mathcal{G} = i_* i^* \mathcal{F}$, where $i : Z \to X$ is the closed immersion.

5E2. *Semilocal schemes.* We call a scheme *semilocal* if it has only finitely many closed points *and is affine*. Note that if X is an affine scheme (or more generally an AF-scheme)[8] and $x_1, \dots, x_n \in X$, then

$$X_{x_1,\dots,x_n} := \lim_{U \supset \{x_1,\dots,x_n\}} U$$

is a semilocal scheme, where the limit is over all open neighborhoods of the x_i. Indeed every such neighborhood is quasiaffine, and hence contains a smaller *affine* neighborhood of the finitely many points [21, Corollaire 4.5.4]. (But note that, for example, if X is the affine line with the origin doubled and $x_1, x_2 \in X$ are the origins, then X_{x_1,x_2} is not separated, and hence not semilocal.)

We shall frequently use the following properties of semilocal schemes.

(1) If X is semilocal and $X' \to X$ is finite then X' is semilocal.
(2) If X is semilocal and L is a line bundle (or more generally a vector bundle of constant rank) on X, then L is trivial [7, Lemma 1.4.4].[9]
(3) If X is semilocal and $Y \to X$ is a closed immersion, then $\mathcal{O}^\times(X) \to \mathcal{O}^\times(Y)$ is surjective.[10]

5E3. *Some general position arguments.* The following is essentially Lemma 4.1 in [9].

Lemma 5.27. *Let U be a local Noetherian scheme with infinite residue field. Let $\overline{C'} \to \overline{C}$ be a finite morphism of projective curves over U; $\mathcal{Z}', D' \subset \overline{C'}$ closed subschemes finite over U with $\mathcal{Z}' \cap D' = \varnothing$; $\Delta'_Z \subset \mathcal{Z}'$ a principal closed subscheme; and $\overline{C'} \setminus D'$ smooth affine over U.*

[8] This means that every finite set of points is contained in an open affine; for example, a scheme quasiprojective over an affine base.

[9] Here is a proof. Let $X_0 \subset X$ denote the closed subscheme which is the disjoint union of the closed points of X. Then $L|_{X_0}$ admits a nonvanishing section, which can (X being affine) be lifted to a section of L on X. Its vanishing locus avoids X_0 and is closed, and hence empty.

[10] Here is a proof. Let $X'_0 \subset X$ denote the closed subscheme which is the disjoint union of the closed points of $X \setminus Y$. Then $Y \amalg X'_0 \to X$ is a closed immersion. Now given $a \in \mathcal{O}^\times(Y)$, as before we can lift the nonvanishing section $(a, 1) \in \mathcal{O}(Y \amalg X'_0)$ to a section $\tilde{a} \in \mathcal{O}(X)$, which is nonvanishing.

Assume that the composite $\Delta'_Z \to \overline{C}' \to \overline{C}$ is a closed immersion. Then for n sufficiently large there exists a section $\xi \in H^0(\overline{C}', \mathcal{O}(n))$ such that $Z(\xi) \cap \mathcal{Z}' = \Delta'_Z$, $Z(\xi) \cap D' = \varnothing$, and $Z(\xi) \to \overline{C}' \to \overline{C}$ is a closed immersion.

Proof. Let us begin with the following preparatory remarks. Let $X \subset \overline{C}'$ be a closed subscheme which is finite over $\overline{C}$. Let $T \subset \overline{C}$ be the set of points t such that the geometric fiber $X_{\bar{t}} \to \bar{t}$ is not a closed immersion. Then T is the support of the coherent sheaf $\mathrm{cok}(\mathcal{O}_{\overline{C}} \to \pi_* \mathcal{O}_X)$, and hence closed in $\overline{C}$. Since a proper morphism is a closed immersion if and only if it is unramified and radicial [34, Tags 01S2 and 04XV], we see that $X \to \overline{C}$ is a closed immersion if and only if $T = \varnothing$. In particular $X \to \overline{C}$ is a closed immersion if and only if its restriction to the closed fiber over U is.

By assumption $\Delta'_Z \subset \mathcal{Z}'$ is principal, say cut out by a section $t \in H^0(\mathcal{Z}', \mathcal{O})$. Since $\mathcal{Z}'$ is semilocal, $\mathcal{O}(1)|_{\mathcal{Z}'}$ admits a nonvanishing section d. Let $\xi \in H^0(\overline{C}', \mathcal{O}(n))$ be a section such that $\xi|_{\mathcal{Z}'} = td^n$. Let $x \in U$ be the closed point. For any scheme $S \to U$, denote by S_x the fiber over x. Assume that $Z(\xi) \cap D'_x = \varnothing$ and $Z(\xi)_x \to \overline{C}_x$ is a closed immersion. Then $Z(\xi) \cap D' = \varnothing$ (being proper over U with empty closed fiber), so $Z(\xi) \to U$ is finite (being proper and affine [34, Tag 01WN]). Hence by the preparatory remarks, $Z(\xi) \to \overline{C}$ is a closed immersion. That is, such a ξ satisfies the required properties. Let $M = \mathcal{O}_{\overline{C}'_x} \times_{\mathcal{O}_{\mathcal{Z}'_x}} \mathcal{O}_{\mathcal{Z}'}$. Then $\mathcal{O}_{\overline{C}'} \to M$ is surjective by [34, Tag 0C4J] and so is surjective on H^0 after twisting up sufficiently. Thus it suffices to construct ξ on the closed fiber (satisfying the additional condition that $\xi|_{\mathcal{Z}'_x} = td^n$, so that $Z(\xi) \cap \mathcal{Z}' = \Delta'_Z$).

We may thus assume that U is the spectrum of an infinite field k. For each point $x \in D'$, pick a trivialization s_x of $\mathcal{O}(1)|_x$. Let $n > 0$ and $\Gamma \subset H^0(\overline{C}', \mathcal{O}(n))$ consist of those sections s such that $s|_x = s_x^n$ for all $x \in D'$ and $s|_{\mathcal{Z}'} = td^n$. We must show that there exists (for n sufficiently large) $s \in \Gamma$ such that $Z(s) \to \overline{C}$ is a closed immersion. Let $T \subset \overline{C} \times \Gamma$ denote the subset of pairs (s, c) such that $Z(s) \to \overline{C}$ is not a closed immersion over the geometric point $\bar{c}$. We claim that $\dim T < \dim \Gamma$. This implies that the complement of the closure of the image of $T \to \Gamma$ is nonempty, and hence has a rational point (k being infinite and Γ an affine space). The preparatory remarks show that any such rational point corresponds to a closed immersion, as desired.

To prove the claim, we may base change to an algebraic closure of k, and hence assume k is algebraically closed. Note that for n sufficiently large, for any $x_1, x_2 \in \overline{C}'$ the map

$$H^0(\overline{C}', \mathcal{O}(n)) \to H^0(Z_{x_1,x_2} \cup D' \cup \mathcal{Z}', \mathcal{O}(n))$$

is surjective, where $Z_{x_1,x_2} = Z(I(x_1)I(x_2))$. (Indeed there is a closed subscheme of $\mathbb{P}(H^0(\overline{C}', \mathcal{O}(n))) \times \overline{C}' \times \overline{C}'$ witnessing the failure of this condition, so the set of points (x_1, x_2) satisfying the condition is open, but for every (x_1, x_2) and n sufficiently large the condition holds, so we conclude by quasicompactness of $\overline{C'} \times \overline{C'}$.) Now let $x_1, x_2 \in \overline{C'} \setminus (D' \cup \mathcal{Z}')$. Then $H^0(Z_{x_1,x_2} \cup D' \cup \mathcal{Z}', \mathcal{O}(n)) \twoheadrightarrow H^0(Z_{x_1,x_2}, \mathcal{O}(n)) \simeq k^2$. Now let $c \in \overline{C}$ and $s \in \Gamma$. Then $Z(s) \to \overline{C}$ can only fail to be a closed immersion over c if either there exist $x_1 \neq x_2 \in \overline{C'_c}$ with $s(x_1) = 0 = s(x_2)$, or there exists $x \in \overline{C'_c}$ such that s vanishes to order ≥ 2 at x. By the above remark, either condition is of codimension 2 on Γ, provided $\mathcal{Z}'_c = \varnothing$. For the finitely many other points c, the only condition is that $Z(s)_c$ contains other points, which is of codimension 1 on Γ by similar arguments. It follows that all but finitely many fibers of $T \to \overline{C}$ have dimension $\leq \dim \Gamma - 2$, and the remaining ones have dimension $\leq \dim \Gamma - 1$. Since $\dim \overline{C} = 1$, this concludes the proof. □

In the rest of this section we will establish a moving lemma. The core argument uses the method of general projections, which we encapsulate in the following.

Theorem 5.28. *Let k be a field, $X \subset \mathbb{A}^N_k$ a closed subscheme of dimension d, $Z \subset \mathbb{A}^N$ of dimension $\leq d-1$, $S \subset \mathbb{A}^N$ a finite set of closed points (i.e., a subscheme of dimension 0). Then, for a general linear projection $\pi : \mathbb{A}^N \to \mathbb{A}^d$,*

(1) *$\pi|_X : X \to \mathbb{A}^d$ is finite,*

(2) *if X is smooth, then $\pi|_X$ is étale at all points of $S \cap X$,*

(3) *$\pi^{-1}(\pi(S)) \cap Z \subset S$.*

Proof. This is proved, for example, in [26, §3.2]. Specifically (1) is proved just before the beginning of §3.2.1 and (3) is proved in the case $S = \{s\}$, $s \notin Z$, in §3.2.1. The same proof works for $s \in Z$ (our statement is slightly different than the one in the reference, to allow this situation). The case of general S follows. In §3.2.2, (2) is proved. □

Remark 5.29. Note that a dense open subset of an affine space over an *infinite* field contains a rational point. It follows that in the case of an infinite field, there is an actual linear projection $\pi : \mathbb{A}^N_k \to \mathbb{A}^d_k$ satisfying all the properties.

The following is our moving lemma. It is essentially the same as [9, Lemma 3.7].

Proposition 5.30. *Let k be an infinite field, $X \in \mathrm{SmAff}_k$, $Z \subset X$ a nowhere dense closed subscheme and $\pi : X' \to X \in \mathrm{SmAff}_k$ an étale neighborhood of Z. Write $Z' \subset X'$ for the lift of Z. Let $T' \subset Z'$ be a finite set of closed points and put $T = \pi(T') \subset Z$. Let U (resp. U') be the semilocalization of X in T (resp. X' in T'). There exist commutative diagrams of essentially smooth, affine k-schemes:*

$$\begin{array}{ccccc} U' & \xrightarrow{s'} & C' & \xrightarrow{v'} & X' \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\varpi}\downarrow & & {\scriptstyle\pi}\downarrow \\ U & \xrightarrow{s} & C & \xrightarrow{v} & X \end{array}$$

and

$$\begin{array}{ccccc} C' & \xrightarrow{j'} & \overline{C'} & & \\ {\scriptstyle \varpi}\downarrow & & {\scriptstyle \overline{\varpi}}\downarrow & & \\ C & \xrightarrow{j} & \overline{C} & \longrightarrow & S, \end{array}$$

such that the following hold:

(1) *The composites* $U \to C \to X$ *and* $U' \to C' \to X'$ *are the canonical inclusions.*

(2) j, j' *are open immersions*; $\overline{C'}, \overline{C}$ *are projective curves over* S; *and* C, C' *are smooth affine over* S.

(3) $\overline{\varpi} : \overline{C'} \to \overline{C}$ *is finite and* $\varpi : C' \to C$ *is étale.*

(4) $\mathcal{Z} := v^{-1}(Z)$ *and* $\mathcal{Z}' := v'^{-1}(Z')$ *are finite over* S, *and in fact* $\mathcal{Z}' \xrightarrow{\simeq} \mathcal{Z}$.

(5) $D' := \overline{C'} \setminus \overline{C}$ *and* $D := \overline{C'} \setminus \overline{C}$ *are finite over* S, *and* $\overline{\varpi}(D') \supset D$.

(6) *We have* $D = Z(d)$ *for some section* d *of an ample line bundle* $\mathcal{O}(1)$ *on* $\overline{C}$.

(7) $\Omega^1_{C/S}$ *is trivial* (*and hence so is* $\Omega^1_{C'/S}$).

Remark 5.31. We can base change $\overline{C}$, $\overline{C'}$ and so on along $U \to C \to S$. Utilizing also the diagonal maps $U \to U \times_S U$ and $U' \to U' \times_S U$, we find that in Proposition 5.30 we may set $S = U$, which is the case of interest. In this case, we have extra properties:

(1) The map $U \xrightarrow{s} C$ is a section of the separated morphism $C \to U$, whence its image is a closed subscheme (in fact an effective Cartier divisor) $\Delta \subset C$, which is isomorphic to U.

(2) The map $U \simeq \Delta \subset C \xrightarrow{v} X$ is the canonical inclusion.

(3) We have $\Delta \cap \mathcal{Z} \simeq Z_T$ via the projection to U.

(4) The composite $Z'_{T'} \hookrightarrow U' \xrightarrow{s'} C' \to U$ is a closed immersion, whence ($C' \to U$ being separated) we obtain a closed subscheme $\Delta'_Z \subset C'$ mapping isomorphically to $Z'_{T'} \simeq Z_T \subset U$ under the projection. Note that $\Delta'_Z \subset \mathcal{Z}'$.

Remark 5.32. In the situation of Remark 5.31, let $C'' = C' \times_U U'$ and $\overline{C''} = \overline{C'} \times_U U'$. We obtain the commutative diagram

$$\begin{array}{ccccc} C'' & \longrightarrow & C' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \\ \overline{C''} & \longrightarrow & \overline{C'} & & \\ \downarrow & & \downarrow & & \\ U' & \longrightarrow & U. & & \end{array}$$

Denote the composite $C'' \to C' \to X'$ by v'' and put $\mathcal{Z}'' = v''^{-1}(Z')$. Let D'' be the preimage of D' in $\overline{C''}$. Note:

(1) $\mathcal{Z}''$, D'' are finite over U'.

(2) The pullback of $\mathcal{O}(1)$ to $\overline{C''}$ exhibits C'' as a projective curve over U'.

(3) The map $s : U' \to C''$ induces a closed immersion $U' \to C''$; denote its image by Δ''. Then $\Delta'' \cap \mathcal{Z}''$ maps isomorphically to $Z'_{T'}$ (and to Δ'_Z).

(4) The composite $U' \simeq \Delta'' \to X'$ is the canonical inclusion.

Proof of Proposition 5.30. Shrinking X, X' if necessary, and arguing on connected components, we may assume that X', X are pure of dimension d and $\pi^{-1}(Z) = Z'$. Using Zariski's main theorem [34, Tag 05K0], we obtain a dense open immersion $X' \hookrightarrow \overline{X'}$ over X with $\overline{X'} \to X$ finite. Choose an embedding $X \hookrightarrow \mathbb{A}^N$. Using general projections (Theorem 5.28) we find a linear map $p_1 : \mathbb{A}^N \to \mathbb{A}^d$ such that $X \to \mathbb{A}^d$ is finite, $X \to \mathbb{A}^d$ is étale at T, and $p_1^{-1}(p_1(T)) \cap Z \subset T$. Let $X_0 \subset X$ be an affine open neighborhood of T such that $X_0 \to \mathbb{A}^d$ is étale (recall that any open neighborhood of T contains an affine open neighborhood of T, as explained in Section 5E2) and put

$$X_0' = \pi^{-1}(X_0), \quad Z_0 = Z \cap X_0.$$

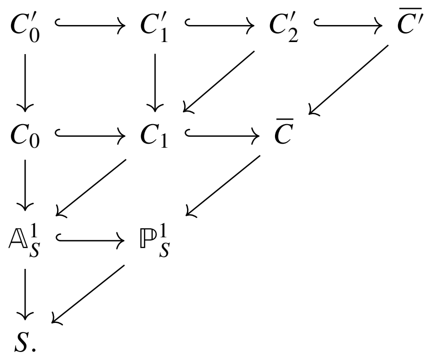
Using general projections again, we find a linear map $p_2 : \mathbb{A}^d \to \mathbb{A}^{d-1}$ such that $p_1(Z) \to \mathbb{A}^{d-1}$ is finite, $p_1(X \setminus X_0) \to \mathbb{A}^d$ is finite, and

$$p_2^{-1}(p_2(p_1(T))) \cap p_1(Z \setminus Z_0) = \varnothing$$

(the latter is possible since $p_1(Z \setminus Z_0)$ does not contain $p_1(T)$, by construction). Let S be the semilocalization of $\mathbb{A}^{d-1}$ at the closed subscheme $p_2(p_1(T))$. At this point we have the diagram

$$\begin{array}{ccccc} X_0' & \hookrightarrow & X' & \longrightarrow & \overline{X}' \\ \downarrow & & \downarrow{\scriptstyle \pi} & \swarrow & \\ X_0 & \hookrightarrow & X & & \\ \downarrow & \swarrow & & & \\ \mathbb{A}^d & & & & \\ \downarrow & & & & \\ \mathbb{A}^{d-1}. & & & & \end{array}$$

Base changing the above diagram along $S \to \mathbb{A}^{d-1}$, we obtain the schemes C_0', C_1', C_2', C_0, C_1, $\mathbb{A}^1_S$ of the commutative diagram



By construction $C_1 \to \mathbb{A}^1_S$ and $C_2' \to C_1$ are finite. We obtain $\overline{C}$ by compactifying $C_1 \to \mathbb{A}^1_S \to \mathbb{P}^1_S$ and $\overline{C'}$ by compactifying $C_2' \to C_1 \to \overline{C}$ (using Zariski's main theorem again). Write $v_1 : C_1 \to X$ for the canonical map. By construction $\mathcal{Z} := v_1^{-1}(Z)$ is finite over S. Also, $v_1^{-1}(Z \setminus Z_0)$ is finite over S but its image misses the closed points; hence it must be empty. In other words $v_1^{-1}(Z) \subset C_0$. Let $v_1' : C_1' \to X'$ be the canonical map. Then $\mathcal{Z}' := (v_1')^{-1}(Z') \to \mathcal{Z}$ is an isomorphism as needed. Since the square part of the diagram is cartesian, we find that also $\mathcal{Z}' \subset C_0'$. We shall find at the end a section $d \in H^0(\overline{C}, \mathcal{O}(n))$ such that $D := Z(d)$ is finite over U, $\overline{C} \setminus C_0 \subset D$ and $D \cap \mathcal{Z} = \varnothing$. We put $C = \overline{C} \setminus D$ and let C' be the preimage of C in $\overline{C'}$. Since C_0 is affine and $C \subset C_0$ is a principal open subset (note $\mathcal{O}(1)$ is trivial over $\mathbb{A}^1$ and hence over C_0), C_0 is affine. The same argument applies to C'. Since $C' \to C \to \mathbb{A}^1_S$ are étale, the canonical modules vanish as needed. Since $\overline{X'} \setminus X_0' \to \mathbb{A}^{d-1}$ is finite so is $C_2' \setminus C_0' \to S$; from this one deduces that D' is finite over S.[11] The natural maps $U \to C_0$ and $U' \to C_0'$ factor through C and C', respectively, since the images of the closed points T do. It follows that the theorem is proved, up to constructing d.

First note that $D_0 := \overline{C} \setminus C_0$ is finite over S. Indeed it is proper, so we need only establish quasifiniteness; but $C_1 \setminus C_0 \to S$ is finite by construction and so is $\overline{C} \setminus C_1 \to (\mathbb{P}^1_S \setminus \mathbb{A}^1_S) \simeq S$, as needed.[12] For a closed point $s \in S$, let R be a connected component of dimension 1 of $\overline{C}_s$. Since $(D_0)_s \to s$ is finite, it cannot contain all of R; let $x_R \in R \setminus D$. Now pick d such that $d|_{D_0} = 0$, $d|_{\mathcal{Z}} \neq 0$ and $d(x_R) \neq 0$ for all such (R, s) (of which there are only finitely many). This satisfies the required properties. (The only nontrivial claim is that $Z(d) \to S$ is finite. But

[11] Note that if $A \to B$ is finite, $B \subset \overline{B}$ is a dense open immersion, and $A \to \overline{A} \to \overline{B}$ is a compactification, then $A \to \overline{A}|_B$ is both closed ($A \to B$ being finite) and also a dense open immersion, whence an isomorphism. That is, the $\overline{A} \setminus A$ lies completely over $\overline{B} \setminus B$.

[12] See the previous footnote.

using properness and semicontinuity of fiber dimension [34, Tag 0D4I], it suffices to prove quasifiniteness over the closed points, which we have ensured.) □

5F. *Injectivity for semilocal schemes.* We now verify that any framed pretheory satisfy IL (in fact we prove a somewhat stronger property).

Lemma 5.33. *Let k be an infinite field, $X \in \mathrm{SmAff}_k$, $Z \subset X$ closed and nowhere dense, $x_1, \dots, x_n \in Z$, U the semilocalization of X in the x_i. Then there are curve correspondences $\Phi, \Phi^- : U \rightsquigarrow X \setminus Z$, $\Theta : U \times \mathbb{A}^1 \rightsquigarrow X$ such that, for any framed pretheory,*

$$i_0^* \Theta^* = \Phi^* \circ (X \setminus Z \to X)^*$$

and

$$i_1^* \Theta^* = \mathrm{tw} \circ (U \to X)^* + (\Phi^-)^* \circ (X \setminus Z \to X)^*,$$

where tw *is some automorphism of $F(U)$.*

Proof. We first show that we may assume $x_1, \dots, x_n \in X$ are closed. Indeed if not, pick closed specializations $y_1, \dots, y_n$. Note that $y_i \in Z$ (Z being closed) and $x_i \in X_{y_1,\dots,y_n}$ ($X_{y_1,\dots,y_n}$ being an intersection of open subsets containing y_i). Applying the claim with the y_i in place of the x_i yields curve correspondences Φ', Θ' over $X_{y_1,\dots,y_n}$. Pulling them back along $X_{x_1,\dots,x_n} \to X_{y_1,\dots,y_n}$ yields the desired result.

Hence from now on we assume that the x_i are closed. Apply Proposition 5.30 and Remark 5.31 to the identity map $(X, Z) \to (X, Z)$, with $T = \{x_1, \dots, x_n\}$. We hence obtain a diagram

$$X \xleftarrow{v} C \xrightarrow{j} \overline{C} \xrightarrow{p} U,$$

where $\overline{C} \to U$ is a projective curve with ample line bundle $\mathcal{O}(1)$, $D = \overline{C} \setminus Z$ is given by $Z(d)$ for some $d \in \mathcal{O}(1)$ and is finite over U, $\Omega^1_{C/U}$ is trivial, $Z' := v^{-1}(Z)$ is finite over U, j is an open immersion and C is smooth over U.

Note that $\Delta : U \to C$ is a regular immersion of codimension 1, and hence $\Delta \subset \overline{C}$ is a divisor. In particular $\mathcal{O}(-\Delta)$ (the ideal sheaf defining Δ) is a line bundle on $\overline{C}$, with inverse $\mathcal{O}(\Delta)$. We shall show at the end, for n sufficiently large, we can find sections $s \in H^0(\overline{C}, \mathcal{O}(n))$, $s' \in H^0(\overline{C}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta))$ such that

- $s|_{Z'}$ and $s'|_{Z' \cup D \cup \Delta}$ are nonvanishing,
- $s|_D = s' \otimes \delta$, where $\delta \in H^0(\overline{C}, \mathcal{O}(\Delta))$ defines Δ.

Set $\tilde{s} = (1-t)s + ts' \otimes \delta$. Since $\tilde{s}$ is constantly nonzero on $D \times \mathbb{A}^1$, $Z(\tilde{s}) \subset C \times \mathbb{A}^1$ and so this is affine and proper, whence finite, over $U \times \mathbb{A}^1$. Similarly $Z(s)$, $Z(s')$ are finite over U. Note that $Z(s)$, $Z(s') \subset C \setminus Z'$, and $Z(s' \otimes \delta) = Z(s') \amalg Z(\delta)$ (since

$Z(\delta) = \Delta$ and so $Z(s') \cap Z(\delta) = \varnothing$). Pick an isomorphism $\mu : \Omega^1_{C/U} \simeq \mathcal{O}_C$. Put

$$\begin{aligned} \Theta &= (U \times \mathbb{A}^1 \leftarrow C \times \mathbb{A}^1 \xrightarrow{v} X, \tilde{s}/d^n, \mu, Z(\tilde{s})), \\ \Phi &= (U \leftarrow C \setminus Z' \xrightarrow{v} X \setminus Z, s/d^n, \mu, Z(s)), \\ \Phi^- &= (U \leftarrow C \setminus Z' \xrightarrow{v} X \setminus Z, s' \otimes \delta/d^n, \mu, Z(s')). \end{aligned}$$

Then, by construction, $i_0^*\Theta^* = \Phi^* \circ (X \setminus Z \to X)^*$ and

$$\begin{aligned} i_1^*\Theta^* &= (U \leftarrow C \xrightarrow{v} X, s' \otimes \delta/d^n, \mu, Z(s' \otimes \delta))^* \\ &= (U \leftarrow C \xrightarrow{v} X, s' \otimes \delta/d^n, \mu, Z(\delta))^* + (\Phi^-)^* \circ (X \setminus Z \to X)^*. \end{aligned}$$

Since $Z(\delta) \to U$ is an isomorphism, the first term is $\mathrm{tw}(s' \otimes \delta/d^n)^{\mu}_{Z(\delta)} \circ (U \simeq \Delta \to X)^*$ by Definition 5.19(4). We conclude since $U \simeq \Delta \to X$ is the canonical map, by construction.

It remains to construct s, s'. For n large enough, both

$$H^0(\overline{C}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta)) \to H^0(Z' \cup D \cup \Delta, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta))$$

and

$$H^0(\overline{C}, \mathcal{O}(n)) \to H^0(Z' \amalg D, \mathcal{O}(n))$$

are surjective (see Section 5E1). Since $Z' \cup D \cup \Delta$ is semilocal (being proper and quasifinite, hence finite, over U), $\mathcal{O}(n) \otimes \mathcal{O}(-\Delta)$ admits a nonvanishing section on it (see Section 5E2); let s' be any lift thereof. Note that

$$H^0(Z' \amalg D, \mathcal{O}(n)) \simeq H^0(Z', \mathcal{O}(n)) \times H^0(D, \mathcal{O}(n));$$

let s be any lift of $(s' \otimes \delta|_{Z'}, 1)$, where $1 \in H^0(Z', \mathcal{O}(n))$ is a nonvanishing section. The required properties hold by construction. □

Theorem 5.34. *Let U be a semilocal scheme, essentially smooth over an infinite field k. Let $Z \subset U$ be a closed subscheme not containing any connected component of U. Then for any $\mathbb{A}^1$-invariant framed pretheory F, the restriction $F(U) \to F(U \setminus Z)$ is injective.*

Proof. Since U is semilocal, it has only finitely many connected components. Since $F(A \amalg B) \simeq F(A) \times F(B)$, we may argue separately for each connected component of U; hence we may assume that U is connected. If U has only one point, the result is trivial. We may thus assume that the subset $Z_0 \subset U$ of closed points is a proper closed subscheme. Replacing Z by $Z \cup Z_0$, we may assume that Z contains all closed points. Replacing Z by a larger proper closed subscheme, we may also assume that Z is finitely presented (e.g., principal). Write $U = \lim_i V_i$, where $V_i \in \mathrm{SmAff}_k$. Replacing V_i by its single connected component containing the image of U, we may assume each V_i is connected. Since Z is a finitely presented closed subscheme, without loss of generality we may assume that $Z = Z_0 \times_{V_0} U$, where $Z_0 \subset V_0$ is closed. If $Z_i := Z_0 \times_{V_0} V_i$ contains all of V_i, then Z contains all of X, which is not the case. It follows that Z_i is nowhere dense in V_i, for each i. Let $x_1^{(i)}, \ldots, x_n^{(i)} \in Z_i$ denote the images of the closed points of X. Let $U_i = (V_i)_{x_1^{(i)}, \ldots, x_n^{(i)}}$ be the semilocalization (see Section 5E2). By continuity, it will suffice to prove that $F(U_i) \to F(U_i \setminus Z_i)$ is injective for each i.

In other words we may assume that $U = V_{x_1, \ldots, x_n}$ where V is a smooth affine scheme, $Z \subset V$ nowhere dense, $x_i \in Z$. Let $X \subset V$ be an open affine neighborhood of the x_i. Lemma 5.33 shows that

$$\ker(F(X) \to F(X \setminus Z)) \subset \ker(F(X) \to F(U)).$$

Indeed if $x \in F(X)$ with $x|_{X \setminus Z} = 0$, then

$$\begin{aligned} 0 = \Phi^*(x|_{X\setminus Z}) - (\Phi^-)^*(x|_{X\setminus Z}) &= \Phi^*(x|_{X\setminus Z}) - i_0^*\Theta^*(x) \\ &= \Phi^*(x|_{X\setminus Z}) - i_1^*\Theta^*(x) = -\mathrm{tw}(x|_U) \end{aligned}$$

and so $x|_U = 0$, tw being invertible. Now taking the (filtered, whence exact) colimit over all such X we obtain the desired result. ☐

Corollary 5.35. *Let U be a semilocal connected scheme, essentially smooth over an infinite field k. Write $\eta \in U$ for the generic point. Then for any $\mathbb{A}^1$-invariant framed pretheory F, the map $F(X) \to F(\eta)$ is injective. In particular F satisfies IL.*

Proof. By Theorem 5.34, $F(U) \to F(V)$ is injective for every nonempty open affine subscheme $V \subset U$. The result follows by taking the filtered colimit over all such V. ☐

5G. *Excision on the relative affine line.*

We now proceed with EA.

Lemma 5.36. *Let $U \in \mathrm{SmAff}_k$, $V \subset \mathbb{A}^1_U$ open, and $0_U \subset V$. Write, for the open immersion of pairs, $i : (V, V \setminus 0) \to (\mathbb{A}^1_U, \mathbb{A}^1_U \setminus 0)$. There exist curve correspondences of pairs*

$$\begin{aligned} \Phi, \Psi &: (\mathbb{A}^1_U, \mathbb{A}^1_U \setminus 0) \rightsquigarrow (V, V \setminus 0), \\ \Theta_1 &: (\mathbb{A}^1_U, \mathbb{A}^1_U \setminus 0) \times \mathbb{A}^1 \rightsquigarrow (\mathbb{A}^1_U, \mathbb{A}^1_U \setminus 0), \\ \Theta_2 &: (V, V \setminus 0) \times \mathbb{A}^1 \rightsquigarrow (V, V \setminus 0) \end{aligned}$$

such that for any framed pretheory

(1) $i_0^*\Theta_1^* = \Phi^* i^*$, $i_1^*\Theta_1^*$ *is invertible*,

(2) $i_0^*\Theta_2^* = i^*\Psi^*$, $i_1^*\Theta_2^*$ *is invertible.*

Note that here we are using the pullback along a curve correspondence of pairs from Construction 5.21; thus, for example, Φ^* is a map

$$\Phi^* : F(V \setminus 0)/F(V) \to F(\mathbb{A}^1_U \setminus 0)/F(\mathbb{A}^1_U).$$

Proof. (1) We shall construct sections

$$s \in H^0(\mathbb{P}^1 \times \mathbb{A}^1_U, \mathcal{O}(n)) \quad \text{and} \quad s' \in H^0(\mathbb{P}^1 \times \mathbb{A}^1_U, \mathcal{O}(n-1)),$$

for some $n > 0$, satisfying the following properties. Denote the coordinate on $\mathbb{A}^1$ by x and on $\mathbb{P}^1$ by $y = (Y_0 : Y_1)$. Let $\delta = Y_1 - xY_0 \in H^0(\mathbb{P}^1 \times \mathbb{A}^1_U, \mathcal{O}(1))$; observe that $Z(\delta)$ defines the diagonal $\{x = y\} \hookrightarrow \mathbb{P}^1 \times \mathbb{A}^1_U$. Let $D = \mathbb{P}^1_U \setminus V$. We shall ensure that

- $s|_{D\times\mathbb{A}^1}$, $s'|_{0\times\mathbb{A}^1_U}$, $s'|_{Z(\delta)}$ and $s'|_{\infty\times\mathbb{A}^1_U}$ are all nonvanishing,
- $s|_{0\times\mathbb{A}^1_U} = \delta s'$, and
- $s|_{\infty\times\mathbb{A}^1_U} = \delta s'$.

Put $\tilde{s} = (1-t)s + t\delta s' \in H^0(\mathbb{P}^1 \times \mathbb{A}^1_U \times \mathbb{A}^1, \mathcal{O}(n))$. Now consider the function $f = s/Y_0^n$ on $V \times \mathbb{A}^1 \subset \mathbb{P}^1 \times \mathbb{A}^1_U$ and the function $\tilde{f} = \tilde{s}/Y_0^n$ on $\mathbb{A}^3_U \subset \mathbb{P}^1 \times \mathbb{A}^1_U \times \mathbb{A}^1$. We claim that

$$\Phi = (\mathbb{A}^1_U \xleftarrow{pr_x} V \times \mathbb{A}^1 \xrightarrow{pr_y} V, f, dy, Z(f)),$$
$$\Theta_1 = (\mathbb{A}^1_U \times \mathbb{A}^1 \xleftarrow{pr_x} \mathbb{A}^3_U \xrightarrow{pr_y} \mathbb{A}^1_U, \tilde{f}, dy, Z(\tilde{f}))$$

are curve correspondences of pairs satisfying the required properties.

To begin with, since $\tilde{s}$ is constantly nonzero over ∞, we find that $Z(\tilde{s}) = Z(\tilde{f})$. In particular this is both proper and affine, whence finite, over $\mathbb{A}^1_U \times \mathbb{A}^1$. Similarly $Z(f) = Z(s)$ is finite over $\mathbb{A}^1_U$. Also, $\tilde{s}|_{y=0}$ is constantly equal to $\delta s'$, which vanishes there only if $y = x$, that is, $x = 0$. It follows that Θ_1 and Φ are well-defined curve correspondences of pairs as displayed in the proposition. By construction we have $i_0^*\Theta_1^* = \Phi^* i^*$. On the other hand, $Z(\delta) \cap Z(s') = \varnothing$ by assumption and thus $Z(\delta s') = Z(\delta) \amalg Z(s')$. So we get

$$i_1^*\Theta_1^* = (\mathbb{A}^1_U \xleftarrow{pr_x} \mathbb{A}^2_U \xrightarrow{pr_y} \mathbb{A}^1_U, \delta s'/Y_0^n, dy, Z(\delta))^* + (\mathbb{A}^1_U \xleftarrow{pr_x} \mathbb{A}^2_U \xrightarrow{pr_y} \mathbb{A}^1_U, \delta s'/Y_0^n, dy, Z(s'))^*.$$

The first term is invertible by Definition 5.19(4), since $Z(\delta)$ maps isomorphically to $\mathbb{A}^1_U$ via both x and y. The second term vanishes by Lemma 5.23, since $Z(s') \subset (\mathbb{A}^1_U \setminus 0) \times \mathbb{A}^1$ by construction. We have thus proved (1) up to constructing s and s'.

We now construct s and s'. By Serre's theorem (see Section 5E1) we may ensure

$$s|_{D\times\mathbb{A}^1} = Y_1^n, \quad s|_{0\times\mathbb{A}^1_U} = Y_0^{n-1}\delta,$$
$$s'|_{\infty\times\mathbb{A}^1_U} = Y_1^{n-1}, \quad s'|_{0\times\mathbb{A}^1_U} = Y_0^{n-1}, \quad s'|_{Z(\delta)} = Y_0^{n-1}.$$

Since Y_1 only vanishes at $0 \notin D$, $s|_{D\times\mathbb{A}^1}$ is nonvanishing. The other nonvanishing conditions hold for similar reasons. Since $\delta = Y_1$ at ∞ (i.e., $Y_0 = 0$), $s = s'\delta$ there. The agreement at 0 holds by construction.

(2) The argument is very similar. One constructs sections $s \in H^0(\mathbb{P}^1 \times \mathbb{A}^1_U, \mathcal{O}(n))$ and $s' \in H^0(\mathbb{P}^1 \times V, \mathcal{O}(n-1))$ such that

- $s|_{D\times V}$, $s'|_{0\times V}$, $s'|_{Z(\delta)}$ and $s'|_{D\times V}$ are all nonvanishing,
- $s|_{0\times V} = \delta s'$, and
- $s|_{D\times V} = \delta s'$.

This is done by using Serre's theorem to ensure that

$$s|_{D\times V} = Y_1^n, \quad s|_{0\times V} = Y_0^{n-1}\delta,$$
$$s'|_{D\times V} = Y_1^n \delta^{-1}, \quad s'|_{0\times V} = Y_0^{n-1}, \quad s'|_{Z(\delta)} = Y_0^{n-1},$$

and arguing as before. Put $\tilde{s} = (1-t)s + t\delta s' \in H^0(\mathbb{P}^1 \times V \times \mathbb{A}^1)$. Arguing as before that this is well defined, we obtain curve correspondences of pairs

$$\Psi = (\mathbb{A}^1_U \xleftarrow{pr_x} V \times \mathbb{A}^1 \xrightarrow{pr_y} V, s/Y_0^n, dy, Z(s))$$

and

$$\Theta_2 = (V \times \mathbb{A}^1 \xleftarrow{pr_x} V \times_U V \times \mathbb{A}^1 \xrightarrow{pr_y} V, \tilde{s}/Y_0^n, dy, Z(\tilde{s})).$$

One checks as before that these satisfy the required properties. □

We also have the following variant.

Lemma 5.37. *Let K be a field, $z \in \mathbb{A}^1_K$ closed and $V \subset \mathbb{A}^1_K$ an open neighborhood of z. Write $i : (V, V \setminus z) \to (\mathbb{A}^1_K, \mathbb{A}^1_K \setminus z)$ for the open immersion of pairs. There exist curve correspondences of pairs*

$$\Phi, \Psi : (\mathbb{A}^1_K, \mathbb{A}^1_K \setminus z) \rightsquigarrow (V, V \setminus z),$$
$$\Theta_1 : (\mathbb{A}^1_K, \mathbb{A}^1_K \setminus z) \times \mathbb{A}^1 \rightsquigarrow (\mathbb{A}^1_K, \mathbb{A}^1_K \setminus z),$$
$$\Theta_2 : (V, V \setminus z) \times \mathbb{A}^1 \rightsquigarrow (V, V \setminus z)$$

such that for any framed pretheory

(1) $i_0^*\Theta_1^* = \Phi^* i^*$, $i_1^*\Theta_1^*$ *is invertible,*
(2) $i_0^*\Theta_2^* = i^*\Psi^*$, $i_1^*\Theta_2^*$ *is invertible.*

Proof. The proof is almost the same as for Lemma 5.36. Let d be the degree of z; then there exists a section $\nu \in H^0(\mathbb{P}^1 \times \mathbb{A}^1_K, \mathcal{O}(d))$ such that $Z(\nu) = z \times \mathbb{A}^1$. Now replace $\mathcal{O}(1)$ by $\mathcal{O}(d)$, t_1 by ν and t_0 by t_0^d in the previous argument. □

We can use this to prove EA.

Theorem 5.38. *Let U be essentially smooth, affine over a field k, and $V \subset \mathbb{A}^1_U$ an open subscheme containing 0_U. Let F be an $\mathbb{A}^1$-invariant framed pretheory. Then restriction induces*

$$F(\mathbb{A}^1_U \setminus 0_U)/F(\mathbb{A}^1_U) \simeq F(V \setminus 0_U)/F(V).$$

Similarly if K is a field, $z \in \mathbb{A}^1_K$ is closed and V is an open neighborhood, then

$$F(\mathbb{A}^1_K \setminus z)/F(\mathbb{A}^1_K) \simeq F(V \setminus z)/F(V).$$

Proof. Since $V \to \mathbb{A}^1_U$ is affine it is quasicompact [34, Tag 01K4], and hence of finite presentation [34, Tag 01TU]. It follows that when writing $U = \lim_i U_i$ as a cofiltered limit of smooth affine schemes, we may assume the existence of $V_0 \subset \mathbb{A}^1_{U_0}$ affine open with base change V [34, Tags 01ZM and 0EUU]. By continuity of F, we may thus assume that U is smooth over k. The first statement now follows from Lemma 5.36. (Recall that if $i : A \to B$, $\Phi, \Psi : B \to A$ are maps of sets such that $\Phi \circ i : A \to A$ and $i \circ \Psi : B \to B$ are invertible, then $A \xrightarrow{i} B \xleftarrow{\Phi} A$ is injective so i is injective, and $B \xrightarrow{\Psi} A \xrightarrow{i} B$ is surjective so i is surjective, that is, i is invertible.)

The second statement is immediate from Lemma 5.37. □

5H. *Étale excision.*

Finally we treat EE.

Lemma 5.39. *Let $X \in \mathrm{SmAff}_k$, $Z \subset X$ a closed subscheme and*

$$\pi : (X', Z') \to (X, Z) \in \mathrm{SmAff}_k$$

an étale neighborhood. Let $z' \in Z'$ and put $z = \pi(z') \in Z$. Let $U = X_z$, $U' = X'_{z'}$.

Write $i : (U, U \setminus Z) \to (X, X \setminus Z)$ and $i' : (U', U' \setminus Z') \to (X', X' \setminus Z')$ for the open immersions of pairs. There exist curve correspondences of pairs

$$\begin{aligned} \Phi, \Psi &: (U, U \setminus Z) \rightsquigarrow (X', X' \setminus Z'), \\ \Theta_1 &: (U, U \setminus Z) \times \mathbb{A}^1 \to (X, X \setminus Z), \\ \Theta_2 &: (U', U' \setminus Z') \times \mathbb{A}^1 \to (X', X' \setminus Z') \end{aligned}$$

such that for any framed pretheory

(1) $i_0^* \Theta_1^* = \Phi^* \pi^*$, $i_1^* \Theta_1^* = \mathrm{tw} \circ i^*$, *where* tw *is some automorphism of $F(U)$,*

(2) $i_0^* \Theta_2^* = \pi^* \Psi^*$, $i_1^* \Theta_2^* = i'^*$.

Proof. Shrinking X' if necessary, we may assume that $Z' = \pi^{-1}(Z)$.

(1) We apply Proposition 5.30 and Remark 5.31 and hence obtain a diagram in notation as stated there (with $S = U$). We shall (at the end) find sections $s \in H^0(\overline{C}, \mathcal{O}(n))$ and $s' \in H^0(\overline{C}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta))$ such that

- $s'|_{D \cup \mathcal{Z} \cup \Delta}$ is nonvanishing,
- $s|_{D \cup \mathcal{Z}} = s' \otimes \delta$, where $\delta \in H^0(\overline{C}, \mathcal{O}(\Delta))$ defines Δ,
- $Z(s) = Z_0 \amalg Z_1$, where ϖ is an étale neighborhood of Z_0 and $Z_1 \cap \mathcal{Z} = \varnothing$.

Put

$$\tilde{s} = (1-t)s + t\delta \otimes s' \in H^0(\overline{C} \times \mathbb{A}^1, \mathcal{O}(n)).$$

Then $\tilde{s}$ is constantly nonvanishing on $D \times \mathbb{A}^1$. In particular $Z(\tilde{s}) \subset C \times \mathbb{A}^1$ is finite (since it is proper and affine) over $U \times \mathbb{A}^1$. Also, $\tilde{s}$ is constantly equal to $\delta \otimes s'$ on $\mathcal{Z} \times \mathbb{A}^1$. It follows that $Z(\tilde{s}) \cap \mathcal{Z} = \Delta \cap \mathcal{Z}$ lies over $Z_z \subset U$. Choose a trivialization $\mu : \Omega^1_{C/U} \simeq \mathcal{O}_C$. Consider

$$\Theta_1 = (U \times \mathbb{A}^1 \leftarrow C \times \mathbb{A}^1 \xrightarrow{v} X, \tilde{s}/d^n, \mu, Z(\tilde{s})) : (U, U \setminus Z) \times \mathbb{A}^1 \rightsquigarrow (X, X \setminus Z).$$

What we have said so far shows that this is a well-defined curve correspondence of pairs. We get

$$i_1^* \Theta_1^* = (U \leftarrow C \to X, \delta \otimes s'/d^n, \mu, Z(\delta))^* + (U \leftarrow C \to X, \delta \otimes s'/d^n, \mu, Z(s'))^*.$$

Since $Z(s') \cap \mathcal{Z} = \varnothing$, the second term vanishes by Lemma 5.23. Since $U \simeq Z(\delta) \to X$ is the canonical map, the first term is $\mathrm{tw}(\delta \otimes s'/d^n, \mu, Z(s')) \circ i^*$, as needed. Similarly

$$i_0^* \Theta_1^* = (U \leftarrow C \to X, s/d^n, \mu, Z_0)^* + (U \leftarrow C \to X, s/d^n, \mu, Z_1)^*.$$

Since $Z_1 \cap \mathcal{Z} = \varnothing$, the second term vanishes. On the other hand, by construction, $C' \to C$ is an étale neighborhood of Z_0. Let

$$\Phi = (U \leftarrow C' \xrightarrow{v'} X', s/d^n, \mu, Z_0).$$

This has the required property, by Definition 5.19(2).

It remains to construct s, s'. Since $\Delta \subset C$ is an effective Cartier divisor, and $\mathcal{Z}$ is semilocal, $\Delta \cap \mathcal{Z} \to \mathcal{Z}$ principal, and hence so is (the isomorphic map) $\Delta'_{\underline{Z}} \to \mathcal{Z}'$. Applying Lemma 5.27, we obtain an effective divisor $Z(\xi) =: Z_0 \subset C \subset \overline{C}$ (finite over U) such that ϖ is an étale neighborhood and $Z_0 \cap \mathcal{Z} = \Delta \cap \mathcal{Z}$ (as schemes). Let $\zeta \in \mathcal{O}(-Z_0)$ be the section defining Z_0. Pick n large enough such that both of the maps

$$H^0(\overline{C}, \mathcal{O}(n) \otimes \mathcal{O}(-Z_0)) \to H^0(\mathcal{Z} \cup D, \mathcal{O}(n) \otimes \mathcal{O}(-Z_0))$$

and

$$H^0(\overline{C}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta)) \to H^0(\mathcal{Z} \cup D \cup \Delta, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta))$$

are surjective. Since $\mathcal{Z} \cup D$ is semilocal, $\mathcal{O}(n) \otimes \mathcal{O}(-Z_0)$ admits a nonvanishing section on it. Let ζ' be a lift of such a nonvanishing section to $\overline{C}$ and put $s = \zeta \otimes \zeta'$. By construction $s|_{\mathcal{Z} \amalg D} = s_0 \otimes \delta$, for some nonvanishing section

$$s_0 \in H^0(\mathcal{Z} \cup D, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta)).$$

We may find a nonvanishing section $s_1 \in H^0(\mathcal{Z} \cup D \cup \Delta, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta))$ extending s_0 (see Section 5E2); finally let $s' \in H^0(\overline{C}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta))$ be any lift of s_1. The required properties hold by construction.

(2) We apply Proposition 5.30 and Remark 5.32 and hence obtain a diagram in notation as stated there. Let $\lambda \in H^0(\Delta'', \mathcal{O}(-\Delta''))$ be a generator as in Definition 5.19(5) (with $Z = \Delta'' \subset C''$). We shall at the end find sections $s \in H^0(\overline{C'}, \mathcal{O}(n))$ and $s' \in H^0(\overline{C''}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta''))$ such that

- $s'|_{\mathcal{Z}''}$ and $s|_{D'}$ are nonvanishing,
- $s'|_{\Delta''} = d^n \lambda$,
- $s'|_{D''} = \delta^{-1} \overline{\varpi''}^*(s)$, where $\delta \in H^0(\overline{C''}, \mathcal{O}(\Delta''))$ defines Δ'',
- $\overline{\varpi''}^*(s)|_{\mathcal{Z}''} = s' \otimes \delta$.

Put $\tilde{s} = (1-t)\overline{\varpi''}^*(s) + t\delta \otimes s'$. As before $\tilde{s}$ is constantly nonvanishing over $D'' \times \mathbb{A}^1$, and hence $Z(\tilde{s}) \subset C''$ is finite over $U' \times \mathbb{A}^1$. Also $\tilde{s}$ is constantly equal to $s' \otimes \delta$ on $\mathcal{Z}'' \times \mathbb{A}^1$, and so $Z(\tilde{s}) \cap \mathcal{Z}''$ lies over $Z'_{z'} \subset U'$. We thus obtain a curve correspondence of pairs

$$\Theta_2 = (U' \times \mathbb{A}^1 \leftarrow C'' \times \mathbb{A}^1 \xrightarrow{v''} X', \tilde{s}/d^n, \mu, Z(\tilde{s})) : (U', U' \backslash Z') \times \mathbb{A}^1 \rightsquigarrow (X', X' \backslash Z').$$

Similarly we obtain

$$\Psi = (U \leftarrow C \xrightarrow{v'} X', s/d^n, \mu, Z(s)) : (U, U \setminus Z) \to (X', X' \setminus Z').$$

Arguing as before we see that $i_0^* \Theta_2^* = \pi^* \Psi^*$ and

$$i_1^* \Theta_2^* = \mathrm{tw}(\delta \otimes s'/d^n)^{\mu}_{Z(\delta)} \circ i^*.$$

Since $Z(\delta) = \Delta''$ and $d(\delta \otimes s'/d^n) = \lambda$, by construction (see Definition 5.19(5)) we have $\mathrm{tw}(\delta \otimes s'/d^n)^{\mu}_{Z(\delta)} = \mathrm{id}$, as needed.

It thus remains to construct s, s'. Write $Z_1 = \mathcal{Z}'' \cap \Delta''$ and Z_2 for its image in C'. Then $Z_1 \simeq Z_2 \simeq Z_z$, and $Z_1 \xrightarrow{\simeq} Z_2 \times_U U'$. The closed immersion $Z_2 \to \mathcal{Z}'$ is isomorphic to $\Delta \cap \mathcal{Z} \to \mathcal{Z}$, whence locally principal, and so principal since $\mathcal{Z}'$ is semilocal. Let $\rho \in \mathcal{O}(\mathcal{Z}')$ cut out Z_2, so that $\overline{\varpi}^*(\rho)$ cuts out Z_1. We may thus write $\overline{\varpi}^*(\rho) = \rho' \otimes \delta$, with $\rho' \in H^0(\mathcal{Z}'', \mathcal{O}(-\Delta))$ a generator. Now $\lambda|_{Z_1}$ and $\rho'|_{Z_1}$ both generate $\mathcal{O}(-\Delta)|_{Z_1}$ and hence differ by a unit. Since $\mathcal{O}^\times(\mathcal{Z}') \to \mathcal{O}^\times(Z_2) \simeq \mathcal{O}^\times(Z_1)$ is surjective ($\mathcal{Z}'$ being semilocal), we may multiply ρ by a unit and so assume that $\lambda|_{Z_1} = \rho'|_{Z_1}$. Since $\mathcal{Z}'' \cup \Delta''$ is the pushout in schemes of $\mathcal{Z}'' \leftarrow \mathcal{Z}'' \cap \Delta \to \Delta''$ [34, Tag 0C4J], there exists $\tilde{\lambda} \in H^0(\mathcal{Z}'' \cup \Delta'', \mathcal{O}(-\Delta''))$ such that $\tilde{\lambda}|_{\mathcal{Z}''} = \rho'$ and $\tilde{\lambda}|_{\Delta''} = \lambda$. Choose n large enough such that

$$H^0(\overline{C'}, \mathcal{O}(n)) \to H^0(D' \amalg \mathcal{Z}', \mathcal{O}(n))$$

and

$$H^0(\overline{C''}, \mathcal{O}(n) \otimes \mathcal{O}(-\Delta'')) \to H^0(D'' \amalg (\Delta'' \cup \mathcal{Z}''), \mathcal{O}(n) \otimes \mathcal{O}(-\Delta''))$$

are surjective. Let s' be a lift of $(1, \rho d^n)$, where $1 \in H^0(D', \mathcal{O}(n))$ is a nonvanishing section. Let s be a lift of $(\delta^{-1} \cdot \overline{\varpi}^*(1), \tilde{\lambda} d^n)$. The required properties hold by construction. □

Theorem 5.40. *Let k be an infinite field and $\pi : U' \to U$ a cofiltered limit of étale morphisms of smooth k-schemes. Assume that U', U are local schemes and π is a local morphism. Let $Z' \subset U'$, $Z \subset U$ be finitely presented closed subschemes such that π induces an isomorphism of Z' onto Z. Let F be an $\mathbb{A}^1$-invariant framed pretheory. Then π^* induces*

$$F(U \setminus Z)/F(U) \simeq F(U' \setminus Z')/F(U').$$

Proof. Since Z, Z' are finitely presented we may without loss of generality assume that $\pi = \lim_\alpha \pi_\alpha$, where $\pi_\alpha : (X'_\alpha, Z'_\alpha) \to (X_\alpha, Z_\alpha)$ is an étale neighborhood of smooth affine k-schemes, and $Z = \lim_\alpha Z_\alpha$, $Z' = \lim_\alpha Z'_\alpha$. Write z_α, z'_α for the images in X_α, X'_α of the closed point. Set $U_\alpha = (X_\alpha)_{z_\alpha}$, $U'_\alpha = (X'_\alpha)_{z'_\alpha}$ and write $\bar{\pi}_\alpha : U'_\alpha \to U_\alpha$ for the restriction of π_α. Consider the commutative diagram

$$\begin{array}{ccc} F(U_\alpha \setminus Z_\alpha)/F(U_\alpha) & \xrightarrow{\bar{\pi}^*_\alpha} & F(U'_\alpha \setminus Z'_\alpha)/F(U'_\alpha) \\ {\scriptstyle i^*_\alpha}\uparrow & & \uparrow{\scriptstyle i'^*_\alpha} \\ F(X_\alpha \setminus Z_\alpha)/F(X_\alpha) & \xrightarrow{\pi^*_\alpha} & F(X'_\alpha \setminus Z'_\alpha)/F(X'_\alpha). \end{array}$$

Lemma 5.39 yields equations

$$\Phi^*_\alpha \pi^*_\alpha = \mathrm{tw} \circ i^*_\alpha \quad \text{and} \quad \bar{\pi}^*_\alpha \Psi^*_\alpha = i'^*_\alpha.$$

These show that

$$\ker(\pi^*_\alpha) \subset \ker(i^*_\alpha) \quad \text{and} \quad \mathrm{cok}(i'^*_\alpha) \twoheadrightarrow \mathrm{cok}(\bar{\pi}^*_\alpha).$$

Taking the colimit over all α concludes the proof (since $\lim_\alpha i_\alpha : \lim_\alpha U_\alpha \to \lim_\alpha X_\alpha$ is an isomorphism, and similarly for i'). □

5I. *Conclusion.*

We have proved the following.

Theorem 5.41. *Any $\mathbb{A}^1$-invariant framed pretheory* (*see Definition 5.19*) *over an infinite field satisfies the axioms IA, EA, IL, and EE of Section 5B.*

Proof. Combine Theorems 5.26, 5.38 and 5.40, and Corollary 5.35. □

We can now prove the main theorem.

Proof of Theorem 5.3. We first consider the case where k is infinite. We know that the forgetful functor $\mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k)) \to \mathcal{P}_\Sigma(\mathrm{Sm}_k)$ commutes with $\Omega_{\mathbb{G}_m}$ (for trivial reasons) and L_{Nis} (by [15, Proposition 3.2.14]). Thus if $F \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k), \mathrm{Ab})$, then so are F_{-1} and $H^i(-; F)$. Suppose that F is $\mathbb{A}^1$-invariant. Then $H^i(-, F)$ is $\mathbb{A}^1$-invariant by Theorem 5.13 (which applies because of Theorem 5.41 and Example 5.20), provided k is perfect. For H^0 we do not need perfectness; see Remark 5.18. The fact that F coincides with its sheafification on open subsets of $\mathbb{A}^1$ is Lemma 5.14.

Now let k be finite and $F \in \mathcal{P}_\Sigma(\mathrm{Corr}^{\mathrm{fr}}(k), \mathcal{SH})$ be $\mathbb{A}^1$-invariant. It suffices to prove that $F \to L_{\mathrm{mot}} F$ is a Nisnevich local equivalence (indeed then $L_{\mathrm{Nis}} F \simeq L_{\mathrm{mot}} F$ is $\mathbb{A}^1$-invariant). That is, we must prove that $F(X) \simeq (L_{\mathrm{mot}} F)(X)$ for any X which is essentially smooth, henselian local over k. Arguing as in [15, Corollary B.2.5], for this it suffices to prove that if k'/k is an infinite perfect extension of k, then $F(X_{k'}) \simeq (L_{\mathrm{mot}} F)(X_{k'})$. Since $X_{k'}$ is a finite disjoint union of henselian local schemes, this follows (using that L_{mot} commutes with essentially smooth base change [23, Lemma A.4]) from $L_{\mathrm{Nis}}(F|_{\mathrm{Sm}_{k'}}) \simeq L_{\mathrm{mot}}(F|_{\mathrm{Sm}_{k'}})$, which we have already established. □

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An introduction to six-functor formalisms

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These are notes for a mini-course given at the summer school and conference The six-functor formalism and motivic homotopy theory in Milan 9/2021. They provide an introduction to the formalism of Grothendieck's six operations in algebraic geometry and end with an excursion to rigid-analytic motives.

The notes do not correspond precisely to the lectures delivered but provide a more self-contained account for the benefit of the audience and others. No originality is claimed.

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0. Introduction

The main goal of these lectures is to touch upon the following questions regarding six-functor formalisms:

(1) *Why care about them?*

(2) *What are they?*

(3) *How to construct them?*

Needless to say, our answers are far from complete. (We will try to give references for the reader who wants to venture further but they will certainly not be exhaustive either.) In short, they are:

(1) *Why?* If one cares about cohomology then one should care about the six operations because the latter *enhance* the former. Grothendieck's relative point of view is baked into the formalism, connecting it well with modern algebraic geometry. And, finally, the formalism has proven highly successful in the last decades. (This is particularly apparent in motivic homotopy theory, the other main topic of the summer school. Unfortunately, we will treat this last point only briefly and leave much to the other talks.)

(2) *What?* First the confession: there will be no definition of six-functor formalisms in these lectures. Just as 'cohomology' is arguably not a precisely defined term and varies from context to context, we cannot expect its enhancement to admit a definition pleasing everyone. Instead we will try to give a glimpse of the six functors in action, and we will describe a convenient and precise framework to think about them. This framework consists of *coefficient systems* which encode a minimal set of structure and axioms one would like a six-functor formalism to enjoy.

(3) *How?* Given the power of the formalism it is unsurprising that all known examples required major efforts, often by many mathematicians, until they were established. (And in several areas, a 'complete' formalism is still very much a work in progress.) We will focus on arguably the most common and serious stumbling block, the construction of the *exceptional functoriality*. In a slightly different direction, such difficulties can be circumvented altogether by constructing six-functor formalisms out of already-established ones. And finally, we will discuss by way of illustration, an example (from rigid-analytic geometry) of a recent new addition to the list of six-functor formalisms.

We assume that the reader is familiar with basic scheme theory and has seen derived categories before. Section 1 is written in the language of triangulated categories although the axioms are barely used. In Sections 2 and 3 we use the language of stable ∞-categories but some help is provided and much of it can also be understood just at the level of underlying triangulated categories. We imagine that the more exposition to the various cohomology theories for schemes (or for other geometric objects) the reader has had, the easier it will be to follow the text.

1. Why?

Here we will try to motivate the study and development of six-functor formalisms. The point of view we will try to convey is that

six-functor formalisms enhance cohomology.

Interspersedly, we will also make comments about the related question why six-functor formalisms arose historically in the first place although this is not our focus. There is little rigorous mathematics to be found here — for that we ask the reader to wait until Sections 2 and 3.

Remark 1.1. Another natural way to answer the question in the title would be to list applications of the theory and thereby argue for its importance. We will not do that here and, in any case, compiling an even approximately complete list would seem a daunting task. Indeed, the language and theory of six-functor formalisms permeates much of modern algebraic geometry and beyond, and has spawned entire fields of research. The development of, for example, étale cohomology, perverse sheaves, or motivic homotopy theory is quite unthinkable in the absence of the six operations.

1A. *A hierarchy of invariants.*

Example 1.2. If you are studying a topological space X, a useful invariant to know about is the sequence of Betti numbers $b_n(X)$, the latter measuring the number of n-dimensional holes in X. Famously, Noether explained how these numbers are just shadows of the homology of X, these being a sequence of abelian groups $H_n(X)$ measuring the difference between cycles and boundaries on X. Thus homology is a richer invariant than the Betti numbers since there is a way to go from the former to the latter but no way (in general) to reverse this process:

$$\begin{array}{c} H_n(X) \\ {\scriptstyle \text{rank}}\big\downarrow \quad \not\big\uparrow \\ b_n(X) \end{array}$$

Example 1.3. Now imagine instead a variety X over a finite field $k = \mathbb{F}_q$ (of cardinality $q = p^r$, say). If you are an arithmetic geometer, chances are you would like to know the number of rational points, that is, solutions to the polynomial equations defining X, possibly over finite extensions of k. The ζ-function of X conveniently packages this information:

$$\zeta_X(T) = \exp\left(\sum_{n\geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} T^n\right) \in \mathbb{Q}[\![T]\!].$$

If X is smooth and proper, Weil [49] predicted that this function is very nicely behaved: it should be a rational function, satisfy a certain functional equation, and one should have tight control over the zeroes and poles. (Weil also proved this for curves.) He suggested that these desirable properties would follow from a well-behaved cohomology theory for varieties over finite fields, a suggestion which was eventually realized by the concerted effort of many mathematicians, including Grothendieck, Serre and Deligne: the theory of ℓ-adic cohomology was at least partly developed to settle the Weil conjectures.

Here then we find a similar situation as in topology in that cohomology groups are richer invariants than individual numbers and that a certain behavior of the former implies a certain behavior of the latter:

$$\begin{array}{c} H^\bullet(X_{\bar{k}}; \mathbb{Q}_\ell) \\ \text{traces of (iterated) Frobenii} \rightsquigarrow \\ \zeta_X(T) \end{array}$$

What is of interest to us in this historical example is that the 'good behavior' of these cohomology groups $H^n(X_{\bar{k}}; \mathbb{Q}_\ell)$ was in turn deduced from properties of an even richer invariant, the ℓ-adic constructible derived category:

$$\begin{array}{ccc} \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X_{\bar{k}}; \mathbb{Q}_\ell) & & \text{category-level invariant} \\ \text{hom-groups} \rightsquigarrow & & \\ H^\bullet(X_{\bar{k}}; \mathbb{Q}_\ell) & & \text{set-level invariant} \\ \text{trace of Frobenius} \rightsquigarrow & & \\ \zeta_X(T) & & \text{element-level invariant} \end{array}$$

Summarizing, in order to prove certain things about element-level invariants mathematicians in this case have found themselves proving things about category-level invariants two levels up and deducing the former from the latter.

Remark 1.4. Jumping ahead of ourselves for a moment, we can say that

six-functor formalisms govern the behavior of certain category-level invariants.

Therefore we can expect that this formalism will be useful in proving things about certain set-level invariants, namely the cohomology of 'spaces' (these could be topological spaces or spaces appearing in algebraic geometry and even further beyond). To say something more precise we have to appreciate one important feature which arises in set-level and category-level invariants but is absent at the lower end of the hierarchy. This we will try to do in Section 1B.

Remark 1.5. The Weil conjectures (Example 1.3) are discussed from this point of view in an unpublished note of Voevodsky [47]. In the same note (from the year 2000) he expresses the view that the development of the six-functor formalism would become one of the main technical tools in advancing motivic homotopy theory: a view which over the last 20 years has certainly materialized!

1B. ***Relative point of view.*** Grothendieck famously stressed the 'relative point of view', replacing, for example, schemes by morphisms of schemes as the fundamental object of study. This shift is also apparent in the development of ℓ-adic cohomology and the proof of the Weil conjectures (Example 1.3).

Remark 1.6. Even if one is ultimately interested in the cohomology of a single variety X it is often necessary to invoke other, related varieties and their cohomologies in the process, for example, in arguments that proceed by induction on the dimension, or when covering X by simpler pieces. It then becomes important to study the cohomology groups not in isolation but together with the maps

$$f^* : H^\bullet(Y) \to H^\bullet(Y') \tag{1.7}$$

for all morphisms $f : Y' \to Y$.

And even if not passing through other varieties, the action of a morphism on cohomology provides additional, often very interesting information about the varieties involved. In the discussion of the Weil conjectures (Example 1.3) we already saw an example of this phenomenon: the action of the Frobenius endomorphism is used to express the number of rational points in terms of cohomology.

Remark 1.8. For the same reason, even if one is interested in proper varieties it is sometimes necessary to invoke nonproper varieties and their cohomologies in the process. The latter are typically less well behaved and to make up for that, the notion of *cohomology with compact support* was developed. Thus in addition to cohomology groups, we also want to study the groups $H^\bullet_c(X)$ and their dependence on X.

1C. ***The six functors in topology.*** In Section 1A we saw that moving up along the hierarchy of invariants, cohomology is replaced by sheaves, and in Section 1B we stressed the need to adopt a relative point of view. Putting the two together one arrives at the study of assignments

$$\text{spaces} \to \text{categories},$$

which send a space to some category of sheaves on that space, and where morphisms of spaces induce functors between the corresponding categories of ‘sheaves’. (As we saw in Example 1.3, these are not necessarily literally sheaves but something related, such as derived categories of sheaves.) The latter are examples of the functors giving ‘six-functor formalisms’ their name. Sometimes they are also called *operations* since they operate on sheaves.

For Weil and Grothendieck, a good cohomology theory for varieties over finite fields was to behave similarly to the cohomology of topological spaces. It is therefore prudent to look at the topological situation first. This we do briefly here. References that include much more detail include [16; 27; 28].

Example 1.9. Let us go back to the topological Example 1.2. For a nice enough[1] space X the cohomology $H^\bullet(X)$ coincides with the sheaf cohomology of the constant sheaf on X. The most familiar operations on (abelian) sheaves associated with a continuous map $f : X \to Y$ are the *inverse image* (or *pull-back*) and *direct image* (or *push-forward*), respectively:

$$\mathrm{Sh}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Sh}(X)$$

Recall that $f_*\mathcal{F}$ is the sheaf whose sections on an open subset $U \subseteq Y$ are given by $\Gamma(f^{-1}(U), \mathcal{F})$. The functor f^* is left adjoint to f_*, and takes $\mathcal{G} \in \mathrm{Sh}(Y)$ to a sheaf satisfying $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)}$ for all $x \in X$. More explicitly, $f^*\mathcal{G}$ is the sheaf associated to the presheaf $V \mapsto \mathrm{colim}_{f(V)\subseteq U}\, \mathcal{G}(U)$, where U runs over the open neighborhoods of $f(V)$.

Note that if $Y = *$ is just a point, the functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(*) \simeq \mathrm{Mod}(\mathbb{Z})$ coincides with the global sections functor. We deduce that the right-derived functor coincides with sheaf cohomology:

$$\mathsf{R}^n f_*(\mathcal{F}) \cong H^n(X; \mathcal{F}). \tag{1.10}$$

Thus we may view the derived push-forward as a relative and enhanced version of cohomology.

Example 1.11. Continuing with Example 1.9, another familiar operation is the *direct image with compact support* (or *compactly supported push-forward*). It is defined as a subfunctor of the direct image functor:

$$\Gamma(U, f_!\mathcal{F}) := \{s \in \Gamma(U, f_*\mathcal{F}) = \Gamma(f^{-1}(U), \mathcal{F}) \mid s \text{ has compact support}\}.$$

Note that again, for $Y = *$ a point, the direct image with compact support recovers cohomology with compact support:

$$\mathsf{R}^n f_!(\mathcal{F}) \cong H^n_{\mathrm{c}}(X; \mathcal{F}). \tag{1.12}$$

[1] For example, cohomologically locally connected in the sense of [40].

Remark 1.13. The functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ does not admit an adjoint in general. This together with the fact that we are ultimately interested in derived functors leads us to consider derived categories of sheaves instead. It turns out that at least for locally compact Hausdorff spaces, the functor acquires a right adjoint:

$$\mathsf{D}(\mathrm{Sh}(X)) \underset{f^!}{\overset{\mathsf{R}f_!}{\rightleftarrows}} \mathsf{D}(\mathrm{Sh}(Y))$$

The functor $f^!$ is called the *exceptional inverse image* (or *pull-back*). Accordingly, $f_!$ is sometimes also called the *exceptional direct image* (or *push-forward*).[2]

Remark 1.14. Together with the tensor product and internal hom of sheaves we have collected all six functors:

$$\begin{array}{c} (\otimes^{\mathsf{L}}, \mathsf{R}\underline{\mathrm{Hom}}) \\[1ex] \mathsf{D}(\mathrm{Sh}(X)) \\ \mathsf{L}f^* \Big\uparrow \Big\downarrow \mathsf{R}f_* \qquad \mathsf{R}f_! \Big\downarrow \Big\uparrow f^! \\ \mathsf{D}(\mathrm{Sh}(Y)) \end{array}$$

It is customary to drop the symbols R and L for derived functors as the context usually makes it clear when derived functors are intended.

1D. *Enhancing cohomology: structure.*

Convention 1.15. Let us now abstract from the specific topological situation and instead assume that with each 'space' (topological space, scheme, stack, ...) X we are given a closed tensor triangulated category $(C(X), \otimes, \underline{\mathrm{Hom}})$ and with each morphism of spaces $f : X \to Y$ two adjunctions $f^* \dashv f_*$, $f_! \dashv f^!$ of exact functors:

$$\begin{array}{c} (\otimes, \underline{\mathrm{Hom}}) \\[1ex] C(X) \\ f^* \Big\uparrow \Big\downarrow f_* \Leftarrow f_! \Big\downarrow \Big\uparrow f^! \\ C(Y) \end{array}$$

The arrow $\Leftarrow$ indicates a natural transformation $f_! \Rightarrow f_*$ (which, in topology, is induced by the inclusion of sections with compact support). We also assume that

[2] Another common name is *f-upper-shriek* for $f^!$ and *f-lower-shriek* for $f_!$.

f^* is endowed with a symmetric monoidal structure. As a first approximation, the category $C(X)$ may be thought of as a 'derived category of sheaves on X' although we don't want to assume that this is literally the case. To have more neutral language we will refer to objects in $C(X)$ as *coefficients*, just as one speaks of cohomology with coefficients.

While we won't stress this aspect, it is important that the dependence on X and f is 'pseudofunctorial'. For example, we should in addition be given natural isomorphisms $g^*f^* \cong (fg)^*$ satisfying familiar cocycle conditions, and $\mathrm{id}^* \cong \mathrm{id}$. For a precise formulation we refer to [15, Definition 2.2].

Remark 1.16. We will now discuss how this basic setup allows us to recover structure present in cohomology. In Section 1E we will see some properties of the six functors and how these properties govern the behavior of cohomology.

Remark 1.17. The identifications in (1.10) and (1.12) show how the sheaf operations allow us to recover cohomology of spaces. In the basic setup of Convention 1.15 we may take this as our *definition*. Let $p : X \to B$ be a morphism of spaces, where we think of B as a 'base space', fixed by the context. For any coefficient $\mathscr{F} \in C(X)$, the *cohomology* (resp. *with compact support*) is

$$H^\bullet(X; \mathscr{F}) := p_*\mathscr{F} \in C(B) \quad (\text{resp. } H^\bullet_{\mathrm{c}}(X; \mathscr{F}) := p_!\mathscr{F} \in C(B)).$$

When $\mathscr{F} = \mathbb{1}$ is the tensor unit we denote these coefficients simply by $H^\bullet(X)$ and $H^\bullet_{\mathrm{c}}(X)$, respectively.

In order to obtain actual cohomology groups one may take appropriate homomorphism groups:

$$H^n(X; \mathscr{F}) := \hom_{C(B)}(\mathbb{1}, p_*\mathscr{F}[n]) \quad (\text{resp. } H^n_{\mathrm{c}}(X; \mathscr{F}) := \hom_{C(B)}(\mathbb{1}, p_!\mathscr{F}[n]))$$

Remark 1.18. One can also define *homology* and *Borel–Moore homology*, generalizing these theories from topology, like so:

cohomology	$p_*p^*\mathbb{1}$	$H^\bullet$
cohomology with compact support	$p_!p^*\mathbb{1}$	$H^\bullet_{\mathrm{c}}$
homology	$p_!p^!\mathbb{1}$	$H_\bullet$
Borel–Moore homology	$p_*p^!\mathbb{1}$	$H^{\mathrm{BM}}_\bullet$

Example 1.19. Let k be a field in which the prime ℓ is invertible and such that $\mathrm{cd}_\ell(k) < \infty$. Then one has a structure as described in Convention 1.15 which sends each finite-type k-scheme (or even algebraic stack) X to the ℓ-adic constructible derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X; \mathbb{Q}_\ell)$ (see, for example, [35], although much of it goes back to SGA, particularly [44; 45]). In this case the cohomology (resp. with compact support) as defined in Remark 1.17 recovers ℓ-adic cohomology (resp. with compact support).

Here are some more examples:[3]

coefficients	cohomology groups
$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X;\mathbb{Q}_\ell)$ constructible ℓ-adic sheaves	ℓ-adic cohomology
$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X(\mathbb{C});\mathbb{Z})$ constructible analytic sheaves	Betti cohomology
$\mathsf{D}^{\mathrm{b}}_{\mathrm{h}}(\mathscr{D}_X)$ holonomic $\mathscr{D}$-modules	de Rham cohomology
$\mathsf{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ coherent sheaves	coherent cohomology
$\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$ mixed Hodge modules	absolute Hodge cohomology
$\mathsf{DM}(X)$ Voevodsky motivic sheaves	(weight-0) motivic cohomology
$\mathsf{SH}(X)$ stable motivic homotopy sheaves	stable motivic (weight-0) cohomotopy groups

Remark 1.20. Consider now a relative situation

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & * & \end{array}$$

The unit of the adjunction $f^* \dashv f_*$ induces a morphism

$$\eta : q_* \to q_* f_* f^* \cong p_* f^*$$

and thus a morphism in cohomology

$$H^\bullet(Y, \mathscr{F}) \to H^\bullet(X, f^*\mathscr{F}).$$

If $\mathscr{F} = \mathbb{1}_X$ one recovers the action of f on the cohomology of X as in (1.7).

Remark 1.21. With compactly supported cohomology the situation is more subtle. In the topological context, a natural map

$$\Gamma_{\mathrm{c}}(Y, F) \to \Gamma_{\mathrm{c}}(X, f^*F)$$

is defined when $f : X \to Y$ is proper. Namely, in that case pulling back sections restricts to those with compact support. This map is in turn induced by the same unit of the adjunction,

$$\eta : q_! \to q_! f_* f^* \cong p_! f^*,$$

using that $f_! = f_*$ as f is proper:

$$H^n_{\mathrm{c}}(Y, \mathscr{F}) \to H^n_{\mathrm{c}}(X, f^*\mathscr{F}).$$

Similar functoriality exists for proper morphisms of schemes and other ‘spaces’.

Exercise 1.22. Describe the functoriality of homology.

[3] Some of these are only partial examples, in others certain technical assumptions are required.

Remark 1.23. Let $p : X \to B$ be a morphism and recall that the inverse image $p^* : C(B) \to C(X)$ is symmetric monoidal. It follows formally that its right adjoint $p_* : C(X) \to C(B)$ sends commutative algebras to commutative algebras. In particular, the cohomology $H^\bullet(X) = p_*\mathbb{1} \in C(B)$ has the structure of a commutative algebra. We may view this as an enhancement of the *cup product* in cohomology. Indeed, evaluating the multiplication through appropriate hom-groups $\hom_{C(B)}(\mathbb{1}, -[n])$ one obtains a cup product at the level of cohomology groups:

$$\cup : H^a(X) \times H^b(X) \to H^{a+b}(X)$$

Remark 1.24. Needless to say, this section does not exhaust all structures of interest in cohomology. For example, *vanishing* and *nearby cycles* are additional concepts of interest. Another example will play a more important role in Section 2. The classical theorem of de Rham and its algebraic geometry version of Grothendieck identifies cohomology groups associated with different theories (de Rham and singular cohomology). Relatedly, *Chern classes* and *regulator maps* may be seen as morphisms from certain cohomology groups of one theory to those of another. We want to think of these as underlying '(iso)morphisms of six-functor formalisms'. For example, the Beilinson regulator maps algebraic K-theory classes to absolute Hodge cohomology, and this should arise from a family of *Hodge realization functors*

$$\rho^*_{\mathrm{H}}(X) : \mathsf{DM}^{\mathrm{c}}(X) \to \mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$$

from categories of (constructible) motivic sheaves that 'realize' the underlying Hodge cohomology of motives. Ideally we would like these functors to be suitably compatible with the six operations.[4]

1E. *Enhancing cohomology: properties.*

We now turn to properties of the six functors and related properties in cohomology. We will discuss here only some of the many possibilities. Our selection is geared towards the approach to six-functor formalisms described in Section 2.

Remark 1.25. To avoid a possible confusion let us stress: the point is not (at least, not always) that results about a given cohomology theory come for free using six-functor formalisms. But the difficulty can sometimes be shifted from establishing them directly to establishing that the cohomology theory underlies a six-functor formalism. We will return to this in Sections 2 and 3.

1E1. *Proper push-forward.* We already mentioned in the topological context that $f_! = f_*$ whenever f is proper. The same is true for six-functor formalisms in general: whenever f is 'proper' (for example, a proper morphism of schemes), the transformation $f_! \to f_*$ is an isomorphism.

[4]This particular example is taken up again in Example 3.19.

1E2. *Duality.* An important impetus for developing the six-functor formalism was what is now sometimes called *Grothendieck duality*, as, for example, in [23; 24]. In a very limited sense, in our setup this can be viewed as the computation of $f^!\mathbb{1}$ for $f : X \to *$ smooth.

Example 1.26 (topology). Let X be a smooth manifold of dimension d and let $f : X \to *$ be the unique map. For a ring Λ, one finds that $f^!\Lambda = \omega_{X,\Lambda}[d]$ is the (shifted) Λ-orientation sheaf. Thus X is orientable if and only if $\omega_{X,\mathbb{Z}}$ is the constant sheaf with value $\mathbb{Z}$. In that case, $\omega_{X,\Lambda}$ is constant for every ring Λ.

Example 1.27 (coherent). Let X be a smooth k-variety of dimension d. If the map $f : X \to \operatorname{Spec}(k)$ denotes the structure morphism then $f^!k \cong \omega_X[d]$ is the (shifted) canonical sheaf on X.

Example 1.28 (ℓ-adic). Let X be a smooth k-variety of dimension d and ℓ a prime invertible in k. Then $f^!\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(d)[2d]$ where (d) denotes the d-th Tate twist.

Corollary 1.29. *With the assumptions of Examples 1.26–1.28, respectively, one has*:

(a) *Poincaré duality (topology): If X is orientable, $H^n_c(X;\mathbb{Q})^* \cong H^{d-n}(X;\mathbb{Q})$.*

(b) *Poincaré duality (ℓ-adic): $H^n_c(X;\mathbb{Q}_\ell)^* \cong H^{2d-n}(X;\mathbb{Q}_\ell(d))$.*

(c) *Serre duality: If X is proper, $H^n(X;\mathcal{O}_X)^* \cong H^{d-n}(X;\omega_X)$.*

Proof. This follows from the adjunction isomorphisms

$$H^n_c(X)^* \cong \operatorname{hom}(f_!f^*\mathbb{1}[n], \mathbb{1}) \cong \operatorname{hom}(\mathbb{1}, f_*f^!\mathbb{1}[-n])$$

together with the computations reported in Examples 1.26–1.28. □

Remark 1.30. The coefficient $f^!\mathbb{1}$ tries to be a dualizing object. Verdier duality is concerned with the functors $\mathbb{D} = \underline{\operatorname{Hom}}(-, f^!\mathbb{1})$ and asks under which conditions one has isomorphisms such as

$$\mathrm{id} \xrightarrow{\sim} \mathbb{D}\circ\mathbb{D}, \quad \mathbb{D}g_! \xrightarrow{\sim} g_*\mathbb{D}, \quad g^*\mathbb{D} \xrightarrow{\sim} \mathbb{D}g^!.$$

It provides a relative version and generalization of duality phenomena such as the ones of Corollary 1.29.

Exercise 1.31 (Atiyah duality). Let $f : X \to B$ be a smooth and proper morphism. Show that the coefficient $H^\bullet(X) = H^\bullet_c(X)$ is *rigid*, with $\otimes$-dual given by $H_\bullet(X) = H^{\mathrm{BM}}_\bullet(X)$.[5]

You will want to use the following two fundamental properties:

[5]We use 'rigid' instead of 'strongly dualizable'. Recall that an object a in a symmetric monoidal category is rigid if there is an object a^* (called its $\otimes$-*dual*) and morphisms $\mathbb{1} \to a\otimes a^*$ and $a^*\otimes a \to \mathbb{1}$ satisfying the identities familiar from adjunctions; see [17].

(a) **(Proper projection formula)** For arbitrary f,

$$f_!(f^*F \otimes G) \xrightarrow{\sim} F \otimes f_!G.$$

(b) If f is smooth then

$$f^*F \otimes f^!H \xrightarrow{\sim} f^!(F \otimes H).$$

Note that the two morphisms are related by adjunction. The second one is a form of relative purity, to which we now turn.

1E3. *Relative purity.*

Remark 1.32. Previously we 'computed' $f^!\mathbb{1}$ in the case where $f : X \to *$ is smooth. It is natural to want to generalize this to arbitrary *smooth* morphisms $f : X \to Y$,[6] and this is provided by *relative purity*. It implies:

(1) The difference between $f^!$ and f^* is measured yet again by $f^!\mathbb{1}$:

$$f^!\mathbb{1} \otimes f^*F \xrightarrow{\sim} f^!F.$$

(This is equivalent to Exercise 1.31(b).)

(2) The coefficient $f^!\mathbb{1}$ is $\otimes$-invertible.

The coefficient $f^!\mathbb{1}$ arises from the *Thom construction* (see below) applied to the relative tangent bundle T_f, and this information is very useful in computations. We interpret the equivalence $f^!\mathbb{1} \otimes (-) =: \{T_f\}$ as a 'twist' by the relative tangent bundle and may therefore rewrite

$$\{T_f\}f^* \simeq f^!. \tag{1.33}$$

Example 1.34. In the ℓ-adic setting[7] the Thom construction depends only on the rank of the vector bundle and the relative purity isomorphism reads as (see Example 1.44 below)

$$f^*(d)[2d] \simeq f^!. \tag{1.35}$$

Note how (1.35) generalizes Poincaré duality in ℓ-adic cohomology discussed in Section 1E2.

One often abbreviates the operation $(d)[2d]$ by $\{d\}$ and this is our inspiration for the notation in (1.33).

---

[6]In the topological context, this should be interpreted as a topological submersion [28, Definition 3.3.1].

[7]This is more generally true for *orientable* theories (that is, those with a good notion of Chern classes). Implicitly, we also used the canonical isomorphism $f^*\{d\} \simeq \{d\}f^*$ which, in the notation introduced in Remark 1.38, is a particular instance of $f^*\{V\} \simeq \{f^{-1}V\}f^*$ for any vector bundle V on Y.

Before discussing the Thom construction, let us note some important consequences of relative purity.

Remark 1.36. As a consequence we note that for f smooth, the inverse image functor f^* admits a left adjoint

$$f_\sharp = f_!\{T_f\} \quad \dashv \quad f^*.$$

It satisfies the (Smooth projection formula):

$$f_\sharp(f^*\mathcal{F} \otimes \mathcal{G}) \xrightarrow{\sim} \mathcal{F} \otimes f_\sharp\mathcal{G},$$

which arises, by adjunction, from the composite

$$f^*\mathcal{F} \otimes \mathcal{G} \to f^*\mathcal{F} \otimes f^* f_\sharp\mathcal{G} \xrightarrow{\sim} f^*(\mathcal{F} \otimes f_\sharp\mathcal{G})$$

of the unit of the adjunction $f_\sharp \dashv f^*$ and the monoidality of f^*.

Example 1.37. Note that for $f : X \to B$ smooth, the homology of X may be expressed alternatively as

$$H_\bullet(X) = f_! f^! \mathbb{1} \simeq f_\sharp f^* \mathbb{1}.$$

Remark 1.38. In terms of this left adjoint we can describe the Thom construction as follows. Let $p : V \to X$ be a vector bundle with zero section $s : X \hookrightarrow V$. The V-twist is defined as

$$\{V\} := p_\sharp s_* : C(X) \to C(X).$$

Then the *Thom construction* applied to V is defined as the evaluation of this functor at the unit, that is,

$$\mathrm{Th}(V) = \mathbb{1}\{V\} = p_\sharp s_* \mathbb{1}.$$

Example 1.39. As mentioned in Remark 1.32, for f smooth we have

$$f^! \mathbb{1} \simeq \mathrm{Th}(T_f).$$

If f is étale then the relative tangent bundle $T_f = X$ is of rank zero hence $\mathrm{Th}(T_f) \simeq \mathbb{1}$, and one deduces that $f^! \simeq f^*$. (Whatever 'étale' means in contexts other than schemes, we would expect this last property to hold.)

Exercise 1.40. Explain as a consequence that Borel–Moore homology is contravariantly functorial with respect to étale morphisms.

Remark 1.41. The Thom construction yields a morphism from the monoid (with respect to direct sums) of isomorphism classes of vector bundles on X to the Picard group of $C(X)$, which can be extended to virtual vector bundles [14; 41], perfect complexes and that passes to the level of K-theory. In particular, there is a group homomorphism

$$K_0(X) \xrightarrow{\mathrm{Th}(-)} \mathrm{Pic}(C(X)).$$

For a modern approach to this construction, see, for example, [7, §16], and for a more general discussion of purity we refer to [12; 13].

Remark 1.42. Continue with the setup of Remark 1.38 and denote by $j : V\backslash X \hookrightarrow V$ the inclusion of the complement of the zero section. From the localization property discussed just below in Section 1E4 we deduce an exact triangle

$$p_\sharp j_\sharp j^* p^* \mathbb{1} \to p_\sharp p^* \mathbb{1} \to p_\sharp s_* s^* p^* \mathbb{1} \tag{1.43}$$

in $C(X)$ which exhibits $\mathrm{Th}(V)$ as the cone of the canonical morphism between homologies relative to X,

$$H_\bullet(V\backslash X) \to H_\bullet(V).$$

These are the analogues of the sphere and disk bundle associated with V in topology, respectively, and this justifies labeling the construction ‘Thom construction’.

Locally every vector bundle is trivial so we better understand these first. By Remark 1.41, it is enough to understand the rank-1 case.

Example 1.44. Let $V = \mathbb{A}^1_X$ be the trivial vector bundle of rank 1 on X. A fundamental property of the six-functor formalisms in algebraic geometry that we are interested in here is the contractibility of the affine line:

($\mathbb{A}^1$-homotopy) $p_\sharp p^* \xrightarrow{\sim} \mathrm{id}$.

This implies that $H_\bullet(\mathbb{A}^1_X) = H_\bullet(X)$. By Exercise 1.45 below, the exact triangle (1.43) in $C(X)$ becomes

$$\mathbb{1} \oplus \mathbb{1}\{1\}[-1] \to \mathbb{1} \to \mathrm{Th}(\mathbb{A}^1_X)$$

so that $\mathrm{Th}(\mathbb{A}^1_X) = \mathbb{1}\{1\}$.

Exercise 1.45. Show that the homology of $\mathbb{G}_m = \mathbb{A}^1\backslash 0$ splits:

$$H_\bullet(\mathbb{G}_m) = \mathbb{1} \oplus \tilde{H}_\bullet(\mathbb{G}_m)$$

for some coefficient $\tilde{H}_\bullet(\mathbb{G}_m)$ (which we think of as the *reduced* homology of $\mathbb{G}_m$). We define the *Tate twists* (and shifts thereof)

$$\mathbb{1}(1) := \tilde{H}_\bullet(\mathbb{G}_m)[-1], \quad \mathbb{1}\{1\} := \tilde{H}_\bullet(\mathbb{G}_m)[1].$$ [8]

Using that open covers give rise to Mayer–Vietoris exact triangles,[9] show also that

$$H_\bullet(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{1}\{1\}.$$

Remark 1.46. We deduce from the preceding discussion the property

[8] There is a monoid morphism $\mathbb{N} \to K_0(X)$ that takes n to the class of the trivial vector bundle $\mathbb{A}^n_X$. Then this notation becomes compatible with the previous one; cf. Remark 1.41.

[9] This is a particular instance of Exercise 1.64 below.

(Tate stability) $\mathbb{1}\{1\} = \mathrm{Th}(\mathbb{A}^1)$ is $\otimes$-invertible.

We may then define, for any coefficient F and $n \in \mathbb{Z}$, $F\{n\} = F \otimes \mathbb{1}\{n\}$.

Remark 1.47. As soon as we have discussed smooth base change (Remark 1.61) we can establish that the Thom construction is Zariski local in a suitable sense. Together with the localization property we are about to discuss (Remark 1.51), (Tate stability) therefore is seen to imply the $\otimes$-invertibility of all Thom coefficients. In the same vein, ($\mathbb{A}^1$-homotopy) is enough to imply the contractibility of any vector bundle.

1E4. *Localization.* Let $i : Z \hookrightarrow X$ be a closed immersion with open complement $j : X \backslash Z \hookrightarrow X$.

Convention 1.48. One defines the *cohomology of X with support in Z* to be

$$H^\bullet_Z(X) := i_! i^! \mathbb{1} \in C(X).$$

(In other words, it is the homology of Z relative to X.)

Example 1.49. Applying $\hom_{C(X)}(\mathbb{1}, -[n])$ this recovers the corresponding notion in topology and in ℓ-adic cohomology. In the coherent context, these groups may be better known under the name of *local cohomology* (*with respect to* Z).

Remark 1.50. There is a so-called *localization triangle* of functors $C(X) \to C(X)$:

$$i_! i^! \to \mathrm{id} \to j_* j^*,$$

which we may apply to the tensor unit $\mathbb{1}$ to obtain (with base X)

$$H^\bullet_Z(X) \to H^\bullet(X) \to H^\bullet(X \backslash Z).$$

The associated long exact sequence is a well-known cohomological tool:

$$\cdots \to H^n_Z(X) \to H^n(X) \to H^n(X \backslash Z) \to H^{n+1}_Z(X) \to \cdots .$$

Remark 1.51. The localization triangle is in turn a consequence of the localization property of six-functor formalisms:

(Localization) The sequence of triangulated categories

$$C(Z) \xrightarrow{i_*} C(X) \xrightarrow{j^*} C(U)$$

is a localization sequence.

This means that one is in the situation of a recollement [8, § 1.4], and in particular that

(1) $i_* \simeq i_!$, j_*, $j_!$ are fully faithful,

(2) the composites $j^* i_*$, $i^! j_*$ and $i^* j_!$ all vanish,

(3) the pairs (i^*, j^*) and $(i^!, j^!)$ are each conservative,

(4) one has another localization sequence $j_! j^! \to \mathrm{id} \to i_* i^* \to^+$.

Remark 1.52. The last triangle may also be written as (with base space X)

$$H_c^\bullet(X \backslash Z) \to H_c^\bullet(X) \to H_c^\bullet(Z)$$

and gives rise to the usual long exact sequence of pairs in compactly supported cohomology. This follows from the identifications $j^* = j^!$ (Example 1.39) and $i_* = i_!$ (Section 1E1).

Example 1.53. (1) By (Localization), $C(\varnothing) \simeq 0$ (take $i = \mathrm{id} : \varnothing \to \varnothing$).

(2) Now let X_{red} be X with the reduced scheme structure, and $i : X_{\mathrm{red}} \hookrightarrow X$ the obvious closed immersion. It follows from (Localization) together with part (1) that $i_* : C(X_{\mathrm{red}}) \xrightarrow{\sim} C(X)$ is an equivalence. In other words, six-functor formalisms are insensitive to nilpotent thickenings.

Remark 1.54. In the ℓ-adic setting, localization is an easy property. In other contexts, however, it can be a substantial theorem. For example, for $\mathcal{D}$-modules fully faithfulness of i_* is known as *Kashiwara's lemma*. In motivic homotopy theory, the localization sequence is a fundamental result of Morel and Voevodsky which they call *the glueing theorem*.

1E5. *Blow-up.* The relation between the cohomology of a variety X and its blow-up $\tilde{X} = \mathrm{Bl}_Z(X)$ is as simple as one might hope but it encodes a fundamental property of six-functor formalisms, namely proper base change.

Convention 1.55. We place ourselves in a more general situation, with a commutative diagram of the following shape:

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{\tilde{i}} & \tilde{X} & \xleftarrow{\tilde{j}} & \tilde{X}\backslash\tilde{Z} \\ {\scriptstyle p'}\downarrow & \searrow^{w} & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle =} \\ Z & \xrightarrow[i]{} & X & \xleftarrow[j]{} & X\backslash Z \end{array} \tag{1.56}$$

We assume that p is proper and i a closed immersion, both squares are Cartesian and the right vertical arrow is an isomorphism. In this situation, the left part of the diagram is called an *abstract blow-up square*.

We now want to explain why there is an exact triangle (the *blow-up exact triangle*)

$$H^\bullet(X) \to H^\bullet(Z) \oplus H^\bullet(\tilde{X}) \to H^\bullet(\tilde{Z}). \tag{1.57}$$

"Proof". The functoriality of cohomology easily gives the two morphisms in (1.57) so that the composite is zero (this involves introducing a sign, as usual). We will

cheat a little bit and assume that cones are functorial so we get a canonical morphism from the cone of the first map to $H^\bullet(\tilde{Z})$ and it suffices to show this map is invertible. By (Localization), this in turn can be checked after applying each of i^* and j^*. The upshot of this little game is that we may prove $i^*(1.57)$ and $j^*(1.57)$ are exact triangles.

Let us write the two candidate triangles in terms of the six operations:

$$\begin{aligned} i^*\mathbb{1} &\to i^*i_*i^*\mathbb{1} \oplus i^*p_*p^*\mathbb{1} \to i^*w_*w^*\mathbb{1}, \\ j^*\mathbb{1} &\to j^*i_*i^*\mathbb{1} \oplus j^*p_*p^*\mathbb{1} \to j^*w_*w^*\mathbb{1}. \end{aligned}$$

By (Localization), i_* is fully faithful, $j^*i_* = 0$, and $j^*w_* = j^*i_*p'_* = 0$. Taking this into account the candidate triangles look as follows:

$$\begin{aligned} &\mathbb{1} \to \mathbb{1} \oplus i^*p_*\mathbb{1} \to p'_*\tilde{i}^*\mathbb{1}, \\ &\mathbb{1} \to j^*p_*\mathbb{1} \to 0. \end{aligned}$$

It is clear that what remains to do is to 'commute' p_* with i^* and j^*, respectively. This is precisely the content of proper base change (Remark 1.58). □

Remark 1.58. Let

$$\begin{array}{ccc} V & \xrightarrow{h} & X \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{g} & Y \end{array} \tag{1.59}$$

be a Cartesian square. Using the unit and counit of the adjunctions between inverse and direct image functors we deduce a canonical *Beck–Chevalley* (or, *push-pull*) transformation:

$$g^*f_* \to k_*k^*g^*f_* \simeq k_*h^*f^*f_* \to k_*h^*. \tag{1.60}$$

We note:

(Proper base change) If f is proper then (1.60) is invertible: $g^*f_* \simeq k_*h^*$.

Remark 1.61. Another instance in which the Beck-Chevalley transformation is invertible is:

(Smooth base change) If g is smooth then (1.60) is invertible: $g^*f_* \simeq k_*h^*$.

Exercise 1.62. Recall that by relative purity, the inverse image along a smooth morphism admits a left adjoint $(-)_\sharp$. Construct analogously a transformation

$$h_\sharp k^* \to f^*g_\sharp \tag{1.63}$$

and show that it is invertible if and only if (1.60) is. (This is a general phenomenon in 2-category theory: The transformations (1.60) and (1.63) are "mates".)

Exercise 1.64. Consider again the diagram (1.56). Assume now instead that p is étale, that i is an open immersion, and that the right vertical arrow is an isomorphism on the associated reduced schemes. The left part of the diagram in this case is called a *distinguished Nisnevich square*. Prove in a similar way that there is an associated exact triangle (1.57). (We might call this a *Nisnevich–Mayer–Vietoris triangle*.)

Exercise 1.65. Sometimes, proper and smooth base change instead refer to the following isomorphisms:

(a) Proper base change: $g^* f_! \simeq k_! h^*$.
(b) Smooth base change: $g^! f_* \simeq k_* h^!$.

At least morally, these are nothing but reformulations of (Proper base change) and (Smooth base change), respectively. More exactly, using properties discussed previously (for example, localization and relative purity):

(a) Assume f factors as an open immersion followed by a proper morphism (for example, f is separated and of finite type). Construct a zig-zag of push-pull transformations between $g^* f_!$ and $k_! h^*$. Show that it is an isomorphism if (Proper base change) holds.
(b) Assume g factors as a closed immersion followed by a smooth morphism (for example, g is quasiprojective). Construct a zig-zag of push-pull transformations between $g^! f_*$ and $k_* h^!$. Show that it is an isomorphism if (Smooth base change) holds.

What can you say about the converse statements?

2. What?

What is a six-functor formalism? As mentioned in the introduction, we will not try to give a definition. However, our main goal in this section is to describe an axiomatization of a convenient ‘stand-in’. It encodes a minimal set of structure and properties a six-functor formalism is commonly expected to enjoy. And we show how powerful this notion yet is. For example, most properties discussed in Section 1E are consequences, and the few remaining ones (related to duality) can still be studied within this framework.

This section’s results rely on the work of many mathematicians; see Remark 2.18.

2A. ***A convenient framework.*** From now on we officially restrict to schemes as our ‘spaces’. (But see Section 3C.)

Convention 2.1. Throughout we fix

- B: a base scheme, assumed Noetherian and finite dimensional,
- Sch_B: B-schemes, assumed separated and of finite type over B.

If B is clear from the context or doesn't play a role, we will refer to B-schemes as just schemes and write Sch instead of Sch_B. Note that all schemes considered are Noetherian and finite dimensional, and all morphisms are separated and of finite type. This will come in handy although more general setups are certainly possible.

Remark 2.2. For our framework to be flexible enough it is better to replace triangulated categories by a suitable enhancement. We saw a hint of this at a very basic level in the proof of the blow-up triangle (1.57). More serious uses of an enhancement will be made throughout Sections 2 and 3. We will work with *stable ∞-categories* as developed extensively in [38]. Nevertheless, a reader who is not familiar with this theory may replace them by triangulated categories (or another suitable enhancement) and still get the gist of the text. Most statements would still make sense and might even be true.

Convention 2.3. The ∞-category of stable ∞-categories and exact functors is denoted by $\mathrm{Cat}_\infty^{\mathrm{st}}$. This has a symmetric monoidal structure for which the tensor product $\mathscr{C} \otimes \mathscr{D}$ is the universal recipient of biexact functors from the Cartesian product $\mathscr{C} \times \mathscr{D}$. We identify commutative algebra objects therein with symmetric monoidal stable ∞-categories, and we write $\mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ for the ∞-category of these. (Note that by our convention, the tensor product is exact in both variables.) They are an enhancement of tensor-triangulated categories, where our coefficients lived in Section 1.

Here is the main definition. While a coefficient lives on a single B-scheme we are now interested in the system of all coefficients on all B-schemes. It seems natural to call this data a *coefficient system*. This terminology was introduced in [18]. Others have used different terms; see Section 2C.

Definition 2.4. A *coefficient system* (*over B*) is a functor $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ satisfying the following axioms (where we write $f^* = C(f)$ for f a morphism of B-schemes).

(1) **(Left)** For each smooth morphism $p : Y \to X$ contained in Sch_B, the functor $p^* : C(X) \to C(Y)$ admits a left adjoint $p_\sharp$, and:

(Smooth base change) For each Cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{p'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{p} & X \end{array}$$

in Sch_B, the Beck–Chevalley transformation $p'_\sharp(f')^* \to f^* p_\sharp$ is an equivalence.

(Smooth projection formula) The canonical transformation

$$p_\sharp(p^*(-) \otimes -) \to - \otimes p_\sharp(-)$$

is an equivalence.

(2) **(Right)** For every $X \in \mathrm{Sch}_B$ and every $f : Y \to X$:

(Internal hom) The symmetric monoidal structure on $C(X)$ is closed.
(Push-forward) The pull-back functor f^* admits a right adjoint $f_* : C(Y) \to C(X)$.

(3) **(Localization)** The ∞-category $C(\varnothing) = 0$ is trivial. And for each closed immersion $i : Z \hookrightarrow X$ in Sch_B with complementary open immersion $j : U \hookrightarrow X$, the square (see Remark 2.6 below)

$$\begin{array}{ccc} C(Z) & \xrightarrow{i_*} & C(X) \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ 0 & \longrightarrow & C(U) \end{array} \tag{2.5}$$

is Cartesian in $\mathrm{Cat}_\infty^{\mathrm{st}}$.

(4) For each $X \in \mathrm{Sch}_B$, if $p : \mathbb{A}^1_X \to X$ denotes the canonical projection with zero section $s : X \to \mathbb{A}^1_X$, then:

($\mathbb{A}^1$-homotopy) The functor $p^* : C(X) \to C(\mathbb{A}^1_X)$ is fully faithful.
(Tate stability) The composite $p_\sharp s_* : C(X) \to C(X)$ is an equivalence.

Remark 2.6. Let us comment on these axioms and relate them to what we've seen in Section 1.

(1) The existence of left adjoints $f_\sharp$ to inverse images along smooth morphisms is a consequence of relative purity, and we also discussed (Smooth projection formula) in this context. The (Smooth base change) is another fundamental property although typically formulated as base change of inverse images (along smooth morphisms) against *direct images*. It was shown in Exercise 1.62 that these two formulations are equivalent.

(2) The structure of a coefficient system encodes only inverse images and tensor products. The axiom (Right) ensures that direct images and internal homs exist as well.

(3) By applying (Smooth base change) to the Cartesian square (for i, j as in (Localization))

$$\begin{array}{ccc} \varnothing & \xrightarrow{j'} & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{j} & X \end{array}$$

we obtain the equivalence $j'_\sharp (i')^* \xrightarrow{\sim} i^* j_\sharp$ and the former composite is null-homotopic since $C(\varnothing) = 0$. Taking right adjoints provides the homotopy $j^* i_* \simeq 0$ that is used in (2.5).[10] In the presence of (Push-forward), the square (2.5) is

[10] I'm grateful to Ryomei Iwasa for pointing out that an explanation of the axiom was required.

Cartesian if and only if the sequence of underlying triangulated categories

$$\mathrm{Ho}(C(Z)) \xrightarrow{i_*} \mathrm{Ho}(C(X)) \xrightarrow{j^*} \mathrm{Ho}(C(U))$$

is a localization sequence so we recover the condition discussed in Section 1E4.

(4) The functor p^* in ($\mathbb{A}^1$-homotopy) is fully faithful if and only if the counit $p_\sharp p^* \to \mathrm{id}$ is an equivalence. As observed in Example 1.37, $p_\sharp p^* = p_! p^!$, and

$$p_\sharp p^* F = p_\sharp(\mathbb{1} \otimes p^* F) \xrightarrow{\sim} p_\sharp \mathbb{1} \otimes F = p_\sharp p^* \mathbb{1} \otimes F$$

by (Smooth projection formula). From which one deduces that ($\mathbb{A}^1$-homotopy) is equivalent to the $\mathbb{A}^1$-homotopy property considered in Section 1 (namely that the homology of the affine line is trivial).

(5) Recall that the functor $p_\sharp s_*$ in (Tate stability) was denoted by $\{1\}$ in Section 1.

Example 2.7. All theories mentioned in Section 1 'should' be examples of coefficient systems. This has been established for some of them, partially for others. The only exception to that statement is the bounded derived category of coherent sheaves as usually conceived. It is not invariant with respect to the affine line, and it does not admit $\sharp$-functoriality. Nevertheless, there is work in this direction too; see [43].

In fact, all these examples fall into two important special cases:

Convention 2.8. A coefficient system C is *small* (resp. *presentable*) if the functor takes values in symmetric monoidal small (resp. presentable) ∞-categories (resp. and symmetric monoidal left-adjoint functors).

Remark 2.9. Stable presentable ∞-categories can be viewed as the ∞-categorical version of well-generated triangulated categories. They satisfy a convenient adjoint functor theorem: a functor between presentable ∞-categories is a left adjoint if and only if it preserves colimits. The ∞-category of presentable ∞-categories and left adjoint functors is denoted by Pr^{L}. It is antiequivalent to the ∞-category of presentable ∞-categories and *right* adjoint functors, denoted Pr^{R}.

It follows that a functor $\mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\otimes}$ automatically satisfies (Right).[11]

Definition 2.10. (1) Let $C, D : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ be two coefficient systems. A natural transformation $\phi : C \to D$ is a *morphism of coefficient systems* if, for each smooth morphism f of B-schemes, the induced transformation $f_\sharp \phi \to \phi f_\sharp$ is an equivalence.

[11] By convention, symmetric monoidal presentable ∞-categories are *presentably symmetric monoidal*, that is, the tensor product commutes with colimits in each variable separately. (A better way of saying this is $\mathrm{Pr}^{\mathrm{L},\otimes} = \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ for a suitable symmetric monoidal structure on Pr^{L}, namely the Lurie tensor product.)

(2) We can then define the ∞-category of coefficient systems (over B) as a sub-∞-category of the functor category:

$$\mathrm{CoSy}_B \subseteq \mathrm{Fun}(\mathrm{Sch}_B^{\mathrm{op}}, \mathrm{Cat}_\infty^{\mathrm{st},\otimes}).$$

One has obvious variants for small and presentable coefficient systems, denoted $\mathrm{CoSy}_{\mathrm{B}}^{\mathrm{sm}}$ and $\mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}}$, respectively.

Exercise 2.11. Let $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ satisfying both (Smooth base change) and (Push-forward). Show that the following are equivalent:

(i) C satisfies (Localization).

(ii) C satisfies the following three conditions:

  (1) $C(\varnothing) = 0$.
  (2) For each closed immersion i, the functor i_* is fully faithful.
  (3) If j denotes the open immersion complementary to a closed immersion i then the pair (i^*, j^*) is conservative.

2B. ***Main result.*** The main result comes in two parts: The first wants to say that

coefficient systems underlie six-functor formalisms,

and the second wants to say that

morphisms of coefficient systems underlie morphisms of six-functor formalisms.

We refer to Remark 2.17 for the fine print.

Remark 2.12. If T is a small stable ∞-category then there is an associated presentable stable ∞-category $\mathrm{Ind}(T)$, its *Ind-completion*. As we will discuss in more detail below (Proposition 2.26), this process turns a small coefficient system into a presentable one. So while the main results in this section are stated in the presentable context, they are equally true for small, and therefore for ‘all’, coefficient systems (see Corollary 2.31).

Theorem 2.13. *Let C be a presentable coefficient system over B. Then there are functors* (*which are equal on objects*)

$$C = C^* : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}, \quad C_! : \mathrm{Sch}_B \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}},$$

with global right adjoints

$$C_* : \mathrm{Sch}_B \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{R}}, \quad C^! : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{R}},$$

and for each morphism f of B-schemes, *a transformation $f_! := C_!(f) \to C_*(f) =: f_*$ which is invertible when f is proper*, *satisfying the projection formulae*, *smooth and proper base change*, *relative purity and ‘the rest’.*

Remark 2.14. We refer to [1, Scholie 1.4.2] or [11, Theorem 2.4.50] for more extensive (but still incomplete) lists of properties. Notably not included in this list is everything on duality, for which see Remark 2.17 and Section 2D3. Some aspects of the proof of Theorem 2.13 will be discussed in Section 3.

Theorem 2.15. *Let $\phi : C \to D$ be a morphism of presentable coefficient systems. Then there are natural transformations*

$$f^*\phi \xrightarrow{\sim} \phi f^*, \quad \phi(-)\otimes\phi(-) \xrightarrow{\sim} \phi((-)\otimes(-)), \qquad f_!\phi \xrightarrow{\sim} \phi f_!,$$
$$\phi f_* \to f_*\phi, \quad \phi\underline{\mathrm{Hom}}(-,-) \to \underline{\mathrm{Hom}}(\phi(-),\phi(-)), \quad \phi f^! \to f^!\phi,$$

the first three of which are always equivalences, and the last three of which are so 'in good cases'.

Remark 2.16. For example, ϕ commutes with direct image along *proper* morphisms, and with exceptional inverse image along *smooth* morphisms. Further 'good cases' will be discussed in Section 2D3.

Remark 2.17. Theorems 2.13 and 2.15 are our main justification for viewing coefficient systems as a stand-in for six-functor formalisms. Let us repeat the caveats already alluded to:

(1) The results are on the face of it about presentable coefficient systems only. But analogous statements can be deduced for small coefficient systems (Corollary 2.31). And all known examples are either small or presentable.

(2) Theorem 2.13 does not say anything about duality, an important topic in the context of six-functor formalisms (as briefly discussed in Section 1). This is a consequence of our goal to be as encompassing as possible. For general coefficient systems, duality cannot be expected unless one restricts to coefficients that are 'small' in a certain sense. This will be taken up again in our short discussion of constructibility (Section 2D3).

(3) The last caveat is related. Namely, Theorem 2.15 does not quite say that a morphism of coefficient systems 'commutes' with the six operations. However, in good cases it does so when restricted to 'constructible coefficients'; see Section 2D3.

Remark 2.18. It is clear that in Theorem 2.13 the extension of a coefficient system $C = C^*$ to $C_!$ is essentially unique (see, for details, Section 3A). The importance of the axioms (Localization) and ($\mathbb{A}^1$-homotopy) in *constructing* the exceptional functoriality was first observed by Voevodsky and was formalized in his notion of *cross-functors* [15]. A version of Theorem 2.13 was first proved by Ayoub [1]. He worked at the level of triangulated categories and restricted to quasiprojective morphisms. The latter restriction was removed by Cisinski and Déglise [11] albeit with an additional axiom. The homotopy-theoretic difficulties in lifting these results to ∞-categories were addressed by another host of mathematicians, including Liu and Zheng [36; 37], Robalo [42] and Khan [31].

Many consequences can be deduced from Theorems 2.13 and 2.15. We refer to [1; 2; 11] for comprehensive treatments. As an example, we mention the following result. It can be viewed as a distillation of the properties discussed in Section 1E5 and in the language of ∞-categories it becomes arguably even more powerful.

Corollary 2.19. *Let C be a presentable coefficient system. The underlying functor*

$$C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

is a cdh-sheaf.

Remark 2.20. For the precise meaning of this statement for general topologies we refer to [18, §2] or [6, Definition 2.3.1]. In the particular case of the cdh-topology, there is a very convenient criterion, however, for which see the proof below.

In practice, this means that C can be studied locally for the cdh-topology. For example, if $B = \mathrm{Spec}(k)$ with k a characteristic-zero field, then C is uniquely determined by its restriction to Sm_k, the category of smooth k-varieties.

Proof. By (Localization), the ∞-category $C(\varnothing)$ is final. It then remains to check that C takes distinguished Nisnevich (resp. abstract blow-up) squares to Cartesian squares in Cat_∞. This amounts essentially to the existence of Nisnevich–Mayer–Vietoris triangles and (abstract) blow-up triangles, which we deduced in Section 1E5 from localization and smooth and proper base change. All of these hold in C, by Theorem 2.13.

For similar proofs, see [11, §3.3.a–b; 26, §6.3]. □

2C. ***Other approaches.*** The framework of coefficient systems is closely related to others in the literature. Let us summarize some of these relations, without trying to be exhaustive.

Remark 2.21. (1) A functor $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ is a coefficient system if and only if passing to homotopy categories gives a *closed symmetric monoidal stable homotopy 2-functor* $\mathrm{Ho}(C) : \mathrm{Sch}_B^{\mathrm{op}} \to \mathsf{TrCat}^\otimes$ in the sense of [1]. Similarly, a natural transformation $\phi : C \to D$ between coefficient systems is a morphism of coefficient systems if and only if passing to homotopy categories produces a morphism of symmetric monoidal stable homotopy 2-functors.

(2) A functor $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ is a coefficient system if and only if passing to homotopy categories gives a *motivic triangulated category* $\mathrm{Ho}(C) : \mathrm{Sch}_B^{\mathrm{op}} \to \mathsf{TrCat}^\otimes$ in the sense of [11]. This is not completely obvious since in loc. cit. an additional axiom (Adj) is assumed. It follows from Theorem 2.13 that this axiom is automatic.

(3) It follows from this last observation that presentable coefficient systems also have been considered before, under the name of *motivic categories of coefficients* [31].

(4) Ultimately, a more complete and thus satisfying framework for six-functor formalisms might be provided by the technology of [21], using the $(\infty, 2)$-*category of correspondences*. However, some of this technology rests on assumptions that are — as far as we are aware — not yet verified in the literature.[12]

2D. ***Internal structure of framework.*** From a bird's-eye view, the framework of coefficient systems consists of cohomology theories and their manifold relations. For example, Grothendieck's comparison isomorphism between algebraic de Rham cohomology and Betti cohomology should be reflected in an isomorphism of coefficient systems over $\mathrm{Spec}(\mathbb{C})$,

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}} \otimes \mathbb{C} \simeq \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}},$$

that is, by an enhanced version of the Riemann–Hilbert correspondence between (derived) constructible sheaves and regular holonomic $\mathscr{D}$-modules (cf. Remark 1.24). In particular, extending scalars at the level of cohomology groups is thus reflected by an operation at the level of coefficient systems.

This and many more phenomena should, in other words, be reflected in a rich internal structure of the ∞-category CoSy. We will be able to provide just a glimpse of this structure if only because mathematicians have barely started to investigate it systematically.

2D1. *Initial object.* Let's say we wanted to construct the 'universal' coefficient system, that is, the initial object of CoSy. We would probably start with the initial required structure and then try to freely enforce the axioms of coefficient systems one by one. As we will see, this can in fact be done, more or less, and the resulting coefficient system turns out to be SH, (stable) motivic homotopy theory!

Remark 2.22. One might find this result remarkable. Without mentioning SH in the definition, the ∞-category of coefficient systems knows about it in a strong sense. It is probably less remarkable once one remembers that the approach to six-functor formalisms axiomatized in the notion of coefficient systems goes back to Voevodsky's study of the functoriality of $\mathsf{SH}(X)$ in X.

We now put this into practice, trying to construct the universal coefficient system. For more details and generalizations, see [19].

Construction 2.23. The construction proceeds in several steps.

[12] In any case, presentable coefficient systems should extend uniquely to this framework; see [31, §4.2].

(1) Coefficient systems encode the $()^*$- and $\otimes$-structure. The initial (as well as final) functor doing so is

$$* : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$$

that sends every scheme to the final category with the only possible symmetric monoidal structure.[13]

(2) This is not the initial coefficient system because morphisms of coefficient systems are required to commute with $\sharp$-push-forwards. For example, given a coefficient system C and smooth morphism $p : P \to X$, we have an object

$$H_\bullet(P) := p_\sharp \mathbb{1} \in C(X)$$

and by (Smooth base change) and (Smooth projection formula) we have canonical equivalences

$$H_\bullet(P) \otimes H_\bullet(P') \cong H_\bullet(P \times_X P')$$

in $C(X)$. With some work one can show that

$$H_\bullet : \mathrm{Sm}_X \to C(X)$$

defines a functor, symmetric monoidal with respect to the Cartesian structure on smooth X-schemes, and this suggests that the functor

$$X \mapsto (\mathrm{Sm}_X)^\times \in \mathrm{Cat}_\infty^\otimes$$

is worth a closer look. In fact, with more work one can show that it is the initial functor satisfying (Left).

(3) Passing to the next axiom we see that (Right) is not satisfied by this functor. The only way we know of producing right adjoints in this context is to pass to presentable ∞-categories in order to invoke adjoint functor theorems. Thus,

$$X \mapsto \mathcal{P}(\mathrm{Sm}_X),$$

the category of presheaves on Sm_X with the pointwise symmetric monoidal structure (which is the Day convolution in this case). This forces us to work in the context of presentable ∞-categories from now on though. (Or at least ∞-categories admitting small colimits.)

(4) It is unclear how one would go about freely enforcing (Localization). On the other hand, the axiom seems to be saying that many questions about a coefficient system can be studied locally for the Zariski topology. And indeed, we saw that it plays an integral role in proving cdh-descent (Corollary 2.19). This suggests that we could get some way towards the axiom by restricting to sheaves for the

[13] Note that stability is a *condition* in the ∞-categorical world so we will not restrict to stable ∞-categories initially but rather enforce it eventually. (In fact, it will come for free.)

cdh-topology. And this 'works' except for the fact that the cdh topology isn't a very natural topology on *smooth* schemes.[14] It turns out that the Nisnevich topology, lying between the Zariski and the cdh-topology, works even better and eventually gives the 'same' result:

$$X \mapsto \mathsf{L}_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_X).$$

Here we write $\mathsf{L}_{\mathrm{Nis}}$ for the (accessible) localization of presentable ∞-categories with respect to Nisnevich–Čech covers.

(5) Enforcing the next two axioms, ($\mathbb{A}^1$-homotopy) and (Tate stability), seems comparatively straightforward: we formally invert the canonical projection $\mathbb{A}^1_P \to P$ for each smooth $P \to X$, and we formally $\otimes$-invert the cofiber of $P \xrightarrow{\infty} \mathbb{P}^1_P$. To make sense of the cofiber it is necessary to pass freely to pointed ∞-categories.[15] This doesn't bother us in the least since in the end we want to end up in stable ∞-categories anyway. Thus we set

$$X \mapsto (\mathsf{L}_{\mathbb{A}^1\cup\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_X)_\bullet)[(\mathbb{P}^1, \infty)^{\otimes -1}].$$

The ∞-category on the right is $\mathsf{SH}(X)$, the *stable motivic* (*or* $\mathbb{A}^1$-)*homotopy category* on X. It is a presentable symmetric monoidal ∞-category. Note that since $(\mathbb{P}^1, \infty) = (\mathbb{G}_\mathrm{m}, 1) \otimes S^1$ (see Exercise 1.45), the ∞-category is automatically stable.

Remark 2.24. It follows that the resulting functor $\mathsf{SH} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L},\otimes}_{\mathrm{st}}$ has the required shape and it remains to verify the axioms of a coefficient system. All of them are formal except for (Localization). The latter is proved by Morel and Voevodsky [39] under the name of *the glueing theorem*.

Remark 2.25. Summarizing, there are at least three ways of thinking about stable motivic homotopy theory:

(a) Explicitly, the ∞-category $\mathsf{SH}(X)$ can be constructed as

$$(\mathsf{L}_{\mathbb{A}^1\cup\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_X)_\bullet)[(\mathbb{P}^1, \infty)^{\otimes -1}].$$

(b) Robalo [42] shows that this symmetric monoidal presentable ∞-category also admits a characterization: any $\otimes$-functor $\mathrm{Sm}_X \to D$ into a stable presentable ∞-category factors uniquely through $\mathsf{SH}(X)$ as soon as it satisfies Nisnevich-excision, $\mathbb{A}^1$-invariance, and $\otimes$-inverts $(\mathbb{P}^1, \infty)$.

(c) By the discussion above [19], the coefficient system SH is the initial object of $\mathrm{CoSy}^{\mathrm{Pr}}$.

This ties back to Section 1A: While the second point concerns cohomology theories (at the set-level), the third point is the exact analogue at the category-level and concerns six-functor formalisms.

[14] See [32] for how to circumvent this problem.

[15] There are also subtle technical difficulties related to the $\otimes$-structure for which we refer to [42].

2D2. *Ind-completion.* Since Theorems 2.13 and 2.15 apply to presentable coefficient systems only, but many of the coefficient systems considered in Section 1 are small, it is very useful to have a process that takes a small coefficient system and outputs a presentable one. This process is simply Ind-completion (see Remark 2.12).

Proposition 2.26. *There is a functor*

$$\mathrm{Ind} : \mathrm{CoSy}_{\mathrm{B}}^{\mathrm{sm}} \to \mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}},$$

which takes a small coefficient system C to the functor $X \mapsto \mathrm{Ind}(C(X))$, the target being endowed with the Day convolution product. □

Exercise 2.27. Prove this result. (A useful fact is that if $f \dashv g$ is an adjunction between small stable ∞-categories, then their unique colimit-preserving extensions $\mathrm{Ind}(f) \dashv \mathrm{Ind}(g)$ again form an adjunction between their Ind-completions.)

Remark 2.28. So, given a small coefficient system C, we apply Theorem 2.13 to $\mathrm{Ind}(C)$ and obtain the full six operations on the system of ∞-categories $\mathrm{Ind}(C(X))$. However, at this point we do not know whether the exceptional functoriality restricts to the subsystem $C(X) \subset \mathrm{Ind}(C(X))$. It turns out that it does and the proof is not difficult.

Lemma 2.29. *Let $f : X \to Y$ be a morphism of B-schemes, let $M \in C(X)$ and $N \in C(Y)$. Then*

$$f_!M \in C(Y), \quad f^!N \in C(X).$$

Proof sketch. For the first statement we factor f as an open immersion followed by a proper morphism and reduce to proving each case separately. In the latter case we have $f_!M = f_*M \in C(Y)$ and we win. In the former we have $f_!M = f_\sharp M \in C(Y)$ and we win again.

For the second statement we use (Localization) to show that an object $L \in \mathrm{Ind}(C(X))$ belongs to $C(X)$ if (and only if) $L|_{U_i} \in C(U_i)$ for some open cover (U_i) of X. In other words, the question is local on X. In particular, we can assume that f is quasiprojective, and factor it as a closed immersion followed by a smooth morphism. The first case then follows from (Localization) and the second case follows from relative purity. □

Exercise 2.30. Fill in the details of this proof sketch.

Corollary 2.31. *Theorems 2.13 and 2.15 admit analogues for small coefficient systems.*[16]

[16] Of course, in this case the four functors $()^*, ()_*, ()_!, ()^!$ take values in *small* stable ∞-categories.

2D3. *Constructibility.* Let C be a coefficient system.

Convention 2.32. Denote by $C^{\mathrm{gm}}(X) \subset C(X)$ the smallest full sub-∞-category that

(1) contains $f_\sharp \mathbb{1}\{n\}$ for $f : Y \to X$ smooth, $n \in \mathbb{Z}$, and

(2) is stable and closed under direct factors (we call such subcategories *thick*).

This defines the subfunctor $C^{\mathrm{gm}} \subseteq C$ *of geometric origin* (see Lemma 2.33).

Lemma 2.33. *$C^{\mathrm{gm}} \subseteq C : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ is a subfunctor and the inclusion $C^{\mathrm{gm}} \to C$ commutes with $f_\sharp$ for f smooth.*

Proof. This follows immediately from the axioms (Smooth base change) and (Smooth projection formula). □

Example 2.34. (1) For $C = \mathsf{SH}$, the geometric part coincides with the compact part: the objects of $\mathsf{SH}^{\mathrm{gm}}(X)$ are precisely the compact objects in $\mathsf{SH}(X)$.[17] Also, SH is compactly generated so that $\mathrm{Ind}(\mathsf{SH}^{\mathrm{gm}}) = \mathsf{SH}$.

(2) The same is true for $C = \mathsf{DM}_{\mathbb{B}}$, Beilinson motives in the sense of [11] (or rather, their ∞-categorical enhancement). That is, $\mathsf{DM}_{\mathbb{B}}^{\mathrm{gm}}$ is the compact part and $\mathrm{Ind}(\mathsf{DM}_{\mathbb{B}}^{\mathrm{gm}}) = \mathsf{DM}_{\mathbb{B}}$. If k is a field there is a canonical equivalence

$$\mathsf{DM}_{\mathbb{B}}^{\mathrm{gm}}(\mathrm{Spec}(k)) = \mathsf{DM}^{\mathrm{gm}}(k; \mathbb{Q})$$

with (the ∞-categorical enhancement of) Voevodsky's category of geometric motives with rational coefficients [48].

(3) In the ℓ-adic setting, 'of geometric origin' is close to 'bounded-constructible'; see [10].

Remark 2.35. Let C be a presentable (or just cocomplete) coefficient system. By Section 2D1, there is a unique morphism of coefficient systems $\mathsf{SH} \to C$, and $C^{\mathrm{gm}} \subseteq C$ is exactly the thick subfunctor generated by the image of $\mathsf{SH}^{\mathrm{gm}}$.

We should now address the question whether C^{gm} is a coefficient system as well. We will not state sufficient conditions here and refer to the literature instead:

Theorem 2.36 [3, §3; 11, §4.2]. *In 'good cases', C^{gm} is a coefficient system.*

Example 2.37. An example to which the theorem applies is Beilinson motives (Example 2.34). In particular one obtains, in this case, a very satisfying picture

[17] Recall that in an ∞-category $\mathscr{C}$ with filtered colimits, an object M is compact if $\mathrm{Map}_{\mathscr{C}}(M, -) : \mathscr{C} \to \mathscr{S}$ to the ∞-category of spaces preserves filtered colimits. If $\mathscr{C}$ is a stable ∞-category, this can be tested at the level of homotopy categories and is equivalent to M being compact in the sense of triangulated categories: $\hom_{\mathrm{Ho}(\mathscr{C})}(M, -)$ preserves direct sums.

translating between small and compactly generated coefficient systems:

$$\mathrm{CoSy}^{\mathrm{sm}} \ni \mathsf{DM}_{\text{Б}}^{\mathrm{gm}} \underset{(-)^{\mathrm{gm}}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathsf{DM}_{\text{Б}} \in \mathrm{CoSy}^{\mathrm{Pr}}$$

Corollary 2.38. *In the same 'good cases' assume $\phi : C \to D$ is a morphism of coefficient systems. Then*

$$\phi|_{C^{\mathrm{gm}}} : C^{\mathrm{gm}} \to D$$

commutes with all six functors.

Remark 2.39. This improves on Theorem 2.15 in 'good cases'.

Remark 2.40. There is a more general notion of *constructibility* for coefficient systems over B. Instead of the generating set $\{f_\sharp \mathbb{1}\{n\}\}$ one may consider the set $\{f_\sharp p^* F\}$ where F runs through a specified set of coefficients on the base B, $f : Y \to X$ is smooth, and $p : Y \to B$ is the structure morphism. We recover the geometric part by allowing only Tate twists as coefficients on B.

The more general notion is useful in the study of duality phenomena, one of the topics of Section 1 which wasn't addressed by the notion of coefficient systems alone. We refer again to [3, §3; 11, §4.2] for in-depth discussions.

2D4. *Miscellanea.* Many other topics could be discussed in the framework of coefficient systems, for example:

(1) We saw in Corollary 2.19 that coefficient systems satisfy cdh-descent. Some of them satisfy *descent* with respect to stronger topologies, however, such as étale descent (and therefore eh-descent) or h-descent [11, §3]. This can be useful in extending coefficient systems from schemes to algebraic stacks via an atlas, say.

(2) It makes sense to consider *linear* coefficient systems and scalar extension. For example, in some cases being $\mathbb{Q}$-linear implies h-descent [11, §3.3.d]. A general discussion of scalar extension can be found in [18, §8], and we will discuss one application of this technique in Section 3B.

(3) *Orientable* coefficient systems are somewhat simpler to work with in the sense that 'all Thom twists are Tate twists' (see Example 1.34 and [11, §2.4.c]).

In these and many other cases there should be corresponding initial objects (similarly to Section 2D1).

Let us mention just two instances of possibly more surprising phenomena. As remarked at the beginning of this section, clearly, a lot remains to be explored!

Example 2.41. There is a functor

$$\exp : \mathrm{CoSy}_B \to \mathrm{CoSy}_B$$

which ‘exponentiates’ a coefficient system, and whose study we initiated in [22]. When applied to $\mathsf{DM}_{\mathbb{B}}$ it produces a new coefficient system $\mathsf{DM}_{\mathbb{B}}^{\exp}$ that should enhance Fresán and Jossen’s theory of exponential motives [20].[18] And when applied to mixed Hodge modules, it should produce an enhancement of Kontsevich and Soibelman’s exponential mixed Hodge structures [34]. An interesting aspect of this construction is that every exponentiated coefficient system comes with an additional ‘seventh’ operation, the Fourier transform familiar from the ℓ-adic theory as well as $\mathscr{D}$-modules (see, for example, [29; 30]).

Example 2.42. Let $\mathbb{F}_q$ be a finite field and choose an algebraic closure $\mathbb{F}$. If C is a coefficient system on $\mathbb{F}$-schemes, one can define a functor

$$C^{\mathrm{W}} : \mathrm{Sch}_{\mathbb{F}_q}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$$

by the formula, for any $\mathbb{F}_q$-scheme X,

$$C^{\mathrm{W}}(X) = \lim\Big(C(X \times_{\mathbb{F}_q} \mathbb{F}) \underset{\mathrm{id}}{\overset{\mathrm{Fr}}{\rightrightarrows}} C(X \times_{\mathbb{F}_q} \mathbb{F})\Big),$$

where Fr denotes the q-Frobenius on X and the limit is taken in $\mathrm{Cat}_\infty^{\mathrm{st},\otimes}$. The superscript is in honor of Weil since in the case of ℓ-adic cohomology, the ∞-category $C^{\mathrm{W}}(X)$ can be seen as a derived category of Weil sheaves [25]. With some work (see Exercise 2.43 below) one shows that this underlies a functor

$$(-)^{\mathrm{W}} : \mathrm{CoSy}_{\mathrm{Spec}(\mathbb{F})} \to \mathrm{CoSy}_{\mathrm{Spec}(\mathbb{F}_q)}.$$ [19]

Exercise 2.43. The goal of this extended exercise is to prove C^{W} of Example 2.42 is a coefficient system. This can be done as follows:

(1) Let $\omega : C \to D$ be a natural transformation of functors $\mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ and assume that

(1.1) D is a coefficient system,

(1.2) C admits left adjoints $p_\sharp$ for smooth morphisms p, and ω commutes with them,

(1.3) $\omega_X : C(X) \to D(X)$ is conservative for each $X \in \mathrm{Sch}_B^{\mathrm{op}}$.

Show the functor C satisfies (Smooth base change), (Smooth projection formula), ($\mathbb{A}^1$-homotopy) as well as items (1) and (3) of Exercise 2.11.

[18] More precisely, $\mathsf{DM}_{\mathbb{B}}^{\exp}(k)$ bears to their theory the same relation as $\mathsf{DM}_{\mathbb{B}}(k)$ to Nori motives, for $k \subseteq \mathbb{C}$ a field.

[19] This example was brought to my attention by Joshua Lieber.

(2) Assume in addition that C satisfies (Right) and that ω commutes with f_* for all immersions f. Then C also satisfies (Localization) and (Tate stability), hence is a coefficient system.

(3) Use the previous point to show that C^{W} of Example 2.42 is a coefficient system. *Hint*: for any diagram $F : I \to \mathrm{Cat}_\infty^{\mathrm{st}}$ the canonical functor $\lim_I F \to \prod_{i \in I_0} F(i)$ is conservative.

3. How?

The question alluded to in the title can be understood in at least two ways:

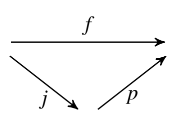
(A) How to construct six-functor formalisms in general?

(B) How to obtain six-functor formalisms from coefficient systems? That is, how to prove Theorem 2.13?

The two are related. Often the $\otimes$-structure and $*$-functoriality is produced without much effort and it is the !-functoriality that poses the most serious difficulties. Below we will focus on the problem of constructing exceptional direct and inverse images, and we will refer to the literature for the problem of proving the expected properties.

3A. ***Exceptional functoriality for coefficient systems.*** We start with question (B) and for this we want to follow the strategy employed by Deligne to produce exceptional functoriality in ℓ-adic cohomology [45, §XVII.3, 5.1]. As we will see, working in the generality we do, additional difficulties arise that need to be addressed.

Remark 3.1. Let f be a morphism of B-schemes. Since f is separated and of finite type (Convention 2.1), we may use Nagata compactification to find a factorization



(3.2)

with j an open immersion and p a proper morphism. We would then like to set

$$f_! := p_* j_\sharp$$

but this definition poses several difficulties:

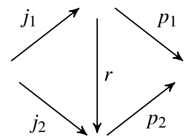
(1) *Well-definedness*: Is it 'independent' of the factorization?

(2) *Right-adjoint*: Why is there a right adjoint $f^!$?

(3) *Functoriality*: In what sense is it functorial in f?

We will address each of these difficulties in turn.

3A1. *Well-definedness.* Consider the category $\mathrm{Comp}(f)$ of compactifications of f: its objects are factorizations as in (3.2), with morphisms (necessarily proper) making the obvious diagram commute:

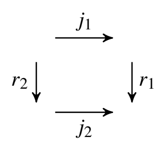


The category $\mathrm{Comp}(f)$ is easily seen to be cofiltered. In comparing $(p_1)_*(j_1)_\sharp$ with $(p_2)_*(j_2)_\sharp$ we may therefore assume a morphism r as above. We then find

$$(p_1)_*(j_1)_\sharp \simeq (p_2)_* r_*(j_1)_\sharp \overset{!}{\simeq} (p_2)_*(j_2)_\sharp,$$

where the last identification would follow if we could 'commute' $j_\sharp$'s with p_*'s. This is known as the *support property*:

(Support) Given a Cartesian square



with r_1 proper and j_2 an open immersion, the induced transformation $(j_2)_\sharp(r_2)_* \to (r_1)_*(j_1)_\sharp$ is an equivalence.

Exercise 3.3. Show that (Proper base change) $\Rightarrow$ (Support).

Remark 3.4. As a consequence of Exercise 3.3 it is natural to try to establish (Proper base change). We sketch the main ideas that go into deducing it from the axioms of a coefficient system. The same strategy will be employed in Section 3A2.

(1) Recall that we are given (1.59) with f proper, and would like to show the transformation $g^* f_* \to k_* h^*$ from (1.60) to be an equivalence. By (Localization) and Chow's lemma we reduce to f projective, $f = pi$ where i is a closed immersion and $p : \mathbb{P}^d_Y \to Y$ is the canonical projection. This reduction step is written out in detail in [6, 4.1.1.(1)].

(2) The case of i_* ('closed base change') follows easily from (Localization) so we further reduce to the case of p_*.

(3) Hence, in addition to being projective, p is also smooth of relative dimension d so we expect to observe Atiyah duality (Exercise 1.31). In other words, one ought to be able to show a canonical equivalence

$$p_* \simeq p_\sharp\{-T_p\}. \tag{3.5}$$

Although this is a rather explicit problem, the proof is long and involved. Moreover, constructing a candidate for the equivalence also involves proving some form of purity. We refer to [1, Théoréme 1.7.9] for details.

(4) Having established the equivalence (3.5), we are reduced to show that both $p_\sharp$ and $\{T_p\}$ 'commute with inverse images'. This is exactly (Smooth base change) and closed base change (recall Remark 1.38).

3A2. *Right-adjoint.* It is clear that the functor $j_\sharp$ admits a right adjoint, namely j^*. To show that p_* does as well (for p proper) we will use the adjoint functor theorem for presentable ∞-categories. In other words we will show that p_* preserves colimits. The advantage of this formulation of the problem is that it becomes amenable to the same attack as the one employed in proving (Proper base change) above: one reduces to projective and further to smooth projective morphisms and then obtains the identification (3.5). Both of the functors on the right are left adjoints and we conclude.

3A3. *Functoriality.* Well-definedness discussed in Section 3A1 is only one aspect of the problem that is posed by functoriality. Recall that we want to construct a functor $C_! : \mathrm{Sch}_B \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Deligne achieved this at the level of triangulated categories by setting, for $f : X \to Y$,

$$f_! := \varinjlim_{(p,j)\in\mathrm{Comp}(f)^{\mathrm{op}}} p_* j_\sharp, \tag{3.6}$$

using that $\mathrm{Comp}(f)$ is cofiltered and the functor

$$* \circ \sharp : \mathrm{Comp}(f)^{\mathrm{op}} \to \mathrm{Hom}(C(X), C(Y))$$

sends morphisms to isomorphisms, by (Support). But even constructing such a functor $* \circ \sharp$ is a daunting task in the context of ∞-categories as it would involve providing, in addition to the homotopies of (Support), homotopies between these and so on ad infinitum.

Remark 3.7. One solution to this homotopy theoretic problem was developed in [36], based on multisimplicial sets. It is very general but unfortunately rather complicated. We would like to describe a more elementary solution specific to the given problem. It is based on our recent collaboration with Ayoub and Vezzani [6].

Remark 3.8. The basic idea is very simple. Let $f : X \to Y$ be a morphism of B-schemes which admits a compactification $\bar{f}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \end{array} \tag{3.9}$$

Here, the square commutes, j and k are open immersions and $\bar{X}$ and $\bar{Y}$ are proper B-schemes. We obtain a diagram of solid arrows

$$\begin{array}{ccc} C(X) & \overset{f_!}{\dashrightarrow} & C(Y) \\ {\scriptstyle j_\sharp}\downarrow & & \downarrow{\scriptstyle k_\sharp} \\ C(\bar{X}) & \xrightarrow{\bar{f}_*} & C(\bar{Y}) \end{array}$$

and as $\bar{f}$ is proper we would like to define $f_!$ so that the square 'commutes'. Again, this is not a tenable strategy in the context of ∞-categories. Being commutative is not a property but a structure and we are back to the exact same issue as before.

However, we can avoid this issue with the following trick. Define a full sub-∞-category $C(X, \bar{X})_!$ of $C(\bar{X})$ as the essential image of $j_\sharp$, and similarly for k. (In particular, we have an equivalence $C(X) \simeq C(X, \bar{X})_!$.) (Support) implies that $\bar{f}_*$ restricts to a morphism $C(X, \bar{X})_! \to C(Y, \bar{Y})_!$. The gain is that the functor $\bar{f}_*$ is already part of a functor $C_* : \mathrm{Sch}_B \to \mathrm{Pr}^{\mathrm{R}}$ which encodes these higher homotopies.

After outlining the basic idea we can now summarize the construction of the functor $C_!$.

Construction 3.10. We will use the diagram

$$\begin{array}{ccccc} & & \mathrm{Comp}_B & & \\ & {\scriptstyle\omega}\swarrow & & \searrow{\scriptstyle\pi} & \\ \mathrm{Sch}_B & & & & \mathrm{Sch}_B^{\mathrm{prop}} \end{array}$$

in which Comp_B denotes the category whose objects are pairs $(X, \bar{X})$ as above and whose morphisms are pairs $(f, \bar{f})$ as in (3.9). Forgetting $\bar{X}$ (resp. X) defines the functor ω (resp. π). (Here, $\mathrm{Sch}_B^{\mathrm{prop}}$ denotes the category of B-schemes and proper morphisms.)

Starting with C and passing to C_* as above we obtain the functor $C_* \circ \pi : \mathrm{Comp}_B \to \mathrm{Pr}^{\mathrm{R}}$ that informally can be described as

$$(X, \bar{X}) \mapsto C(\bar{X}), \quad (f, \bar{f}) \mapsto \bar{f}_*.$$

In fact, this functor takes values in Pr^{L} as well, by Section 3A2.

If $C(-,-)_! : \mathrm{Comp}_B \to \mathrm{Pr}^{\mathrm{L}}$ denotes the full subfunctor of $C_* \circ \pi$ considered in Remark 3.8 then we define

$$C_! := \mathrm{LKE}_\omega C(-,-)_! : \mathrm{Sch}_B \to \mathrm{Pr}^{\mathrm{L}},$$

the left-Kan extension along ω. This last step thus removes the dependency of the factorization in a similar way as in (3.6).

Remark 3.11. It is not difficult to prove that $C_!(f)$ recovers $p_* j_\sharp$ up to homotopy for any factorization as in (3.2). For details, we refer to [6, §4.3]. From there, one can go on and prove the expected properties of this six-functor formalism; see [1] or [11].

3B. *Motivic coefficient systems.* Once one has Theorem 2.13 at one's disposal, of course, question (A) becomes: how to construct coefficient systems? In this brief section we will describe an elegant and powerful procedure that has been employed in the literature to produce 'motivic' coefficient systems. This topic would have just as well fit in with Section 2D.

Remark 3.12. Let $C \in \mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}}$ be a coefficient system, say presentable to fix our ideas. As we saw in Section 2D1, there is an essentially unique morphism $\rho_C^* : \mathsf{SH} \to C$ from the initial object which can be viewed as the (homological) C-realization: For any B-scheme X, the functor $\rho_C^*(X) : \mathsf{SH}(X) \to C(X)$ sends a smooth X-scheme Y to its homology coefficient $H_\bullet(Y)$ in $C(X)$. This functor admits a right adjoint $\rho_*^C(X) : C(X) \to \mathsf{SH}(X)$ that has a canonical lax symmetric monoidal structure (since $\rho_C^*(X)$ underlies a symmetric monoidal functor). In particular we see that $\rho_*^C(B)\mathbb{1} \in \mathrm{CAlg}(\mathsf{SH}(B))$ is a motivic ring spectrum which we denote by $\mathscr{C}$.

This object represents C-cohomology in the sense that for any smooth B-scheme X, we have by adjunction

$$\pi_0 \mathrm{Map}_{\mathsf{SH}(B)}(X, \mathscr{C}(m)[n]) \simeq \pi_0 \mathrm{Map}_{C(B)}(H_\bullet(X), \mathbb{1}(m)[n]) \simeq H^n(X; \mathbb{1}(m)).$$

The observation we want to make now is that *every* motivic ring spectrum represents some cohomology theory.

Convention 3.13. Let $\mathscr{A} \in \mathrm{CAlg}(\mathsf{SH}(B))$ be a motivic ring spectrum and denote (abusively) by $\mathscr{A}_X := f^*\mathscr{A} \in \mathrm{CAlg}(\mathsf{SH}(X))$ its pull-back to any B-scheme $f : X \to B$. The association $X \mapsto \mathrm{Mod}_{\mathscr{A}_X}(\mathsf{SH}(X)) =: \mathsf{SH}(X; \mathscr{A})$ underlies a functor

$$\mathsf{SH}(-; \mathscr{A}) : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\otimes} \tag{3.14}$$

that — in anticipation of the next theorem — we call the *motivic coefficient system represented by* $\mathscr{A}$.

Theorem 3.15. *The functor* $\mathsf{SH}(-; \mathscr{A})$ *of* (3.14) *is a presentable coefficient system and the canonical 'free functor'* $\rho_{\mathscr{A}}^* : \mathsf{SH} \to \mathsf{SH}(-; \mathscr{A})$ *is a morphism of presentable coefficient systems.*

A proof of this result can be found in [18, Theorem 8.10]; see also [11, §7.2, 17.1].

Remark 3.16. We may now combine the constructions of Remark 3.12 and Convention 3.13. That is, in the situation of Remark 3.12 we obtain a factorization of $\rho_C^* : \mathsf{SH} \to C$ through the *motivic coefficient system associated with C*:

$$\mathsf{SH} \xrightarrow{\rho_{\mathscr{C}}^*} \mathsf{SH}(-;\mathscr{C}) \xrightarrow{\tilde{\rho}_C^*} C.$$

The induced functor $\tilde{\rho}_C^*$ factors further through the localizing subfunctor $\tilde{C}$ of C generated by the part of geometric origin (Section 2D3). By a tilting argument, the resulting morphism

$$\tilde{\rho}_C^* : \mathsf{SH}(-;\mathscr{C}) \to \tilde{C}$$

is in fact an equivalence in some cases of interest; see [11, Theorem 17.1.5].

Remark 3.17. In summary, we have procedures which can be upgraded to functors:

$$\mathrm{CAlg}(\mathsf{SH}(B)) \underset{\rho_*(B)\mathbb{1}}{\overset{\mathsf{SH}(-;-)}{\rightleftarrows}} \mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}}$$

Example 3.18 [11, 17.1.7]. Consider the Betti realization functor

$$\rho_{\mathrm{B}}^* : \mathsf{SH}(\mathrm{Spec}(\mathbb{C})) \to \mathsf{D}(\mathbb{Q})$$

that sends a smooth complex scheme X to the rational singular chain complex $\mathrm{Sing}(X^{\mathrm{an}}) \otimes \mathbb{Q}$ on the underlying complex analytic space. It is naturally symmetric monoidal — in fact, it is part of a morphism of coefficient systems on complex schemes [3]:

$$\mathsf{SH} \to \mathsf{D}((-)^{\mathrm{an}};\mathbb{Q}).$$

The associated motivic ring spectrum $\mathscr{B} := \rho_*^B \mathbb{Q} \in \mathrm{CAlg}(\mathsf{SH}(\mathbb{C}))$ is the (*rational*) *Betti spectrum* that represents Betti cohomology. We will now describe the resulting coefficient system $\mathsf{SH}(-;\mathscr{B})$ more explicitly, following [5, §1.6].

First, observe that for a general complex scheme X, the functor

$$\tilde{\rho}_{\mathrm{B}}^*(X) : \mathsf{SH}(X;\mathscr{B}) \to \mathsf{D}(X^{\mathrm{an}};\mathbb{Q})$$

is far from an equivalence. Instead, it factors through

$$\mathsf{SH}(X;\mathscr{B}) \to \mathrm{Ind}(\mathsf{D}_{\mathrm{c}}^{\mathrm{b}}(X;\mathbb{Q})) \to \mathsf{D}(X^{\mathrm{an}};\mathbb{Q}),$$

where the second arrow is the colimit-preserving functor extending the identity on $\mathsf{D}_{\mathrm{c}}^{\mathrm{b}}(X;\mathbb{Q})$. The first functor in this factorization is in fact fully faithful, and the image is generated under colimits, desuspensions and truncations (with respect to the canonical t-structure) by sheaves of the form $f_*\mathbb{Q}$, where $f : Y \to X$ is proper; see [5, Theorem 1.93].

Example 3.19 [18]. Saito's derived categories of mixed Hodge modules do not, in an obvious way, admit an enhancement to a coefficient system. (As a result, the Hodge realization functors are not known to commute with the six functors on compact objects.) On the other hand, there is a Hodge realization functor

$$\rho^*_{\mathrm{H}} : \mathsf{SH}(\mathrm{Spec}(\mathbb{C})) \to \mathsf{D}(\mathrm{Ind}(\mathrm{MHS}^{\mathrm{p}}_{\mathbb{Q}}))$$

with values in the derived ∞-category of Ind-completed polarizable mixed Hodge structures over $\mathbb{Q}$. The associated motivic ring spectrum $\mathcal{H} := \rho^{\mathrm{H}}_* \mathbb{Q}(0)$ is the *absolute Hodge spectrum* that represents absolute Hodge cohomology. Drew calls the resulting coefficient system $\mathsf{SH}(-; \mathcal{H})$ *motivic Hodge modules*, and they satisfy many of the properties expected of a coefficient system that should capture mixed Hodge modules of geometric origin. In line with this, he conjectures that for each complex scheme X, the triangulated category of compact objects in $\mathrm{Ho}(\mathsf{SH}(X; \mathcal{H}))$ embeds fully faithfully into Saito's $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$.

Example 3.20 [46]. As in Example 3.19, until recently there was no known enhancement of Voevodsky's category of motives over a field, $\mathsf{DM}^{\mathrm{gm}}(\mathrm{Spec}(k); \mathbb{Z})$ to a coefficient system in mixed characteristic. (The situation was better understood with rational coefficients and/or in equal characteristic.) Spitzweck constructs a motivic ring spectrum $\mathcal{M} \in \mathsf{SH}(\mathrm{Spec}(\mathbb{Z}))$ that represents Bloch–Levine motivic cohomology and then defines

$$\mathsf{SH}(-; \mathcal{M}) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L},\otimes}_{\mathrm{st}}$$

that can be seen as a coefficient system of integral motivic sheaves. Over a field k the compact part of $\mathsf{SH}(\mathrm{Spec}(k); \mathcal{M})$ is equivalent to $\mathsf{DM}^{\mathrm{gm}}(\mathrm{Spec}(k); \mathbb{Z})$, while with rational coefficients and for any scheme X one recovers Beilinson motives:

$$\mathsf{SH}(X; \mathcal{M} \otimes \mathbb{Q}) \simeq \mathsf{DM}_{\mathrm{B}}(X).$$

3C. ***Exceptional functoriality for* RigSH.** We have two goals for this last section. First, we want to say something regarding question (A) at the beginning of Section 3. And secondly, we want to give an example of a six-functor formalism outside the world of schemes (and topological spaces) that have dominated the discussion so far.

Remark 3.21. In the context of schemes, Theorem 2.13 provides a very useful criterion for recognizing six-functor formalisms. In contexts that are not too different from schemes one can hope to establish a similar criterion; see, for example, [33] for (certain) algebraic stacks. However, in general one shouldn't expect the axioms of coefficient systems — even if interpreted appropriately — to be sufficient to guarantee the existence of !-functoriality.

Rigid (or 'nonarchimedean') analytic geometry is arguably an example of a theory that is too different for a successful transfer. In the following pages we want to describe how a different kind of transfer allows one to construct !-functors (and prove the expected properties) on the 'universal' rigid-analytic theory, namely rigid-analytic stable motivic homotopy theory RigSH. This is a report on the work with Ayoub and Vezzani [6] already mentioned in Section 3A3.

Remark 3.22. Rigid-analytic geometry is the analogue of complex-analytic geometry over nonarchimedean fields, e.g., p-adic fields. The theory retains both algebraic and analytic aspects, and it has found many applications in arithmetic algebraic geometry, particularly in the wake of Scholze's work on perfectoid spaces and p-adic geometry.

Rigid-analytic spaces in the sense this term is used in [6] form a category RigSpc that encompasses both Tate's rigid-analytic varieties (and Berkovich spaces) as well as a large class of adic spaces (e.g., all 'stably uniform' ones [9]) in the sense of Huber. While ridding the treatment of unnecessary Noetherianity assumptions was a goal of [6], these technical details will not concern us in this short outline.

Remark 3.23. The construction of RigSH is originally due to Ayoub [4] and modeled on Morel and Voevodsky's construction of SH (see Section 2D1). In fact, the two are entirely parallel according to the 'dictionary'

$$\begin{aligned} \mathrm{Sch} &\leftrightsquigarrow \mathrm{RigSpc}, \\ \mathbb{A}^1 &\leftrightsquigarrow \mathbb{B}^1, \\ \mathbb{G}_\mathrm{m} &\leftrightsquigarrow \mathbb{T}. \end{aligned}$$

Here $\mathbb{B}^1$ is the closed unit ball and $\mathbb{T} \subset \mathbb{B}^1$ the annulus.

Unsurprisingly and in a completely parallel fashion, RigSH comes with a closed symmetric monoidal structure and $*$-functoriality. However, there is no analogue of Theorem 2.13 available, and Ayoub was able to construct the !-functoriality only for morphisms that arise as the analytification of algebraic morphisms (that is, those coming from Sch).[20] The original goal of [6] was to remedy this.

Remark 3.24. Let us explain why an analogue of Theorem 2.13 is not available and in fact might not be expected. Indeed, in following the strategy of Section 3A one encounters the following problems in the rigid-analytic world:

(a) The analogue of Exercise 3.3 does not hold (a priori), that is, proper base change does not imply the support property. The underlying reason is that while (Localization) holds for RigSH, it is of limited use since the complement of an open immersion is not, typically, a rigid-analytic space.

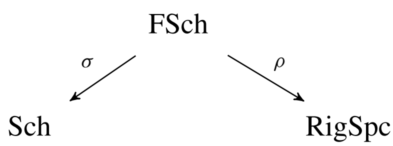
(b) At several places in Section 3A we used Chow's lemma to reduce questions about proper morphisms to projective ones. However, an analogue of Chow's lemma is not available in rigid-analytic geometry, thus making this strategy infeasible.

[20] In fact, he obtained this as an application of a version of Theorem 2.13.

Remark 3.25. On the other hand, morphisms locally of finite type between rigid-analytic spaces are still *weakly* compactifiable, at least locally. More precisely, every $f : X \to Y$ locally of finite type is, locally on X, the composition of a locally closed immersion followed by a proper morphism. Therefore, once one knows (Support) and the existence of right adjoints to proper push-forwards one can then follow essentially the same strategy in constructing the exceptional functoriality as in Section 3A3. The existence of the required right adjoints follows easily from the fact that the ∞-categories $\mathsf{RigSH}(X)$ are compactly generated and that the inverse image functors along proper morphisms preserve compact objects.

In the remainder of this section we will sketch how to prove (Support).

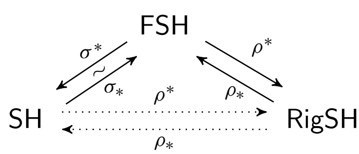
Remark 3.26. The proof still employs a transfer from algebraic to rigid-analytic geometry albeit in a very different way. It is based on Raynaud's approach to rigid-analytic geometry that can be roughly described by the picture



(3.27)

Here, formal schemes sit at the top and admit two functors: the 'special fiber' that associates to $\mathscr{X}$ its underlying topological space with the reduced scheme structure $\sigma(X)$, and the 'generic fiber' ρ that is a categorical localization. More precisely, the category RigSpc is, as a first approximation, the localization of FSch with respect to so-called 'admissible blow-ups': blow-ups with center 'contained in the special fiber'. This approximation becomes correct if one imposes finiteness conditions on the formal schemes involved (adic with finitely generated ideals of definition) and if one allows rigid-analytic spaces to be glued along open immersions.

Remark 3.28. Passing to stable motivic homotopy theory in the three contexts in parallel gives rise to a roof like so,



where the components of the natural transformations $(-)^*$ are symmetric monoidal functors with right adjoints $(-)_*$, and where $\sigma^* \dashv \sigma_*$ is an adjoint equivalence, by localization for FSH. We continue to denote by ρ^* (resp. ρ_*) the functors at the bottom that make the triangle commute.

The basic idea in proving (Support) for RigSH is to apply ρ_* to the morphism $(j_2)_\sharp(r_2)_* \to (r_1)_*(j_1)_\sharp$ and use (Support) for SH to show it is an equivalence. This requires two inputs:

(a) The functor ρ_* needs to be sufficiently conservative. While it isn't on the nose, it is still true (and easy to prove) that the family (for fixed S)

$$\left(\mathsf{RigSH}(S) \xrightarrow{f^*} \mathsf{RigSH}(X) \xrightarrow{\rho_*^{\mathscr{X}}} \mathsf{SH}(\sigma(\mathscr{X}))\right)_{f,\mathscr{X}}$$

is jointly conservative, where $f : X \to S$ runs through smooth morphisms of rigid-analytic spaces and $\mathscr{X}$ is a chosen formal model of X.

(b) It is clear that f^* (resp. ρ_*) commutes with the $(j_i)_\sharp$ and the $(p_i)_*$ (resp. with the $(p_i)_*$) so it remains to prove that ρ_* commutes with the $(j_i)_\sharp$.

This last point turns out to be quite involved and required a systematic study of RigSH. We will not go into the details here and refer to [6, Theorem 4.1.3] instead. On the other hand, this systematic study leads to other results of independent interest which we do want to mention.

Theorem 3.29 [6, Theorem 3.3.3]. (1) *The components of the natural transformation* $\mathsf{SH}(\sigma(-), \rho_*\mathbb{1}) \to \mathsf{RigSH}(\rho(-))$ *are fully faithful.*

(2) *The natural transformation* $\mathsf{SH}^{\text{ét}}(\sigma(-), \rho_*\mathbb{Q}) \to \mathsf{RigSH}^{\text{ét}}(\rho(-), \mathbb{Q})$ *exhibits the latter as the rig-étale sheafification of the former.*

Here, the natural transformations in the statement are between $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$-valued functors on $\mathrm{RigSpc}^{\mathrm{op}}$ (viewed as having the same objects as FSch; see Remark 3.26). The notation $\mathsf{SH}(X, A)$ already employed in Section 3B is a shorthand for the ∞-category of A-modules, A being a commutative algebra object in $\mathsf{SH}(X)$. The first part of Theorem 3.29 can be read as saying that a whole chunk of RigSH admits a completely algebraic description. We call this chunk the part of good reduction and denote it by $\mathsf{RigSH}^{\mathrm{gr}}$. In fact, in good cases the commutative algebra $\rho_*\mathbb{1}$ can be computed. For example, over the p-adic integers $\rho_*^{\mathbb{Z}_p}\mathbb{1} \simeq H^\bullet(\mathbb{G}_{\mathrm{m}})$ and we deduce that

$$\mathsf{SH}^{\mathrm{uni}}(\mathbb{F}_p) \xrightarrow{\sim} \mathsf{RigSH}^{\mathrm{gr}}(\mathbb{Q}_p),$$

where the domain denotes the *unipotent motivic spectra*, that is, the localizing sub-∞-category of $\mathsf{SH}(\mathbb{G}_{\mathrm{m},\mathbb{F}_p})$ generated by the constant motivic spectra.

Finally, the second part of Theorem 3.29 gives a precise measure of the failure of all rigid-analytic motives to be of good reduction. In comparison to the first part, some additional hypotheses are necessary, for example étale-(hyper)sheafification and $\mathbb{Q}$-linearity are enough. All in all, Theorem 3.29 is a vast generalization of [4, Scholie 1.3.26.(1)] which inspired the strategy in the first place.

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Introduction to framed correspondences

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We give an overview of the theory of framed correspondences in motivic homotopy theory. Motivic spaces with framed transfers are the analogue in motivic homotopy theory of E_∞-spaces in classical homotopy theory, and in particular they provide an algebraic description of infinite $\mathbb{P}^1$-loop spaces. We will discuss the foundations of the theory (following Voevodsky, Garkusha, Panin, Ananyevskiy, and Neshitov), some applications such as the computations of the infinite loop spaces of the motivic sphere and of algebraic cobordism (following Elmanto, Hoyois, Khan, Sosnilo, and Yakerson), and some open problems.

1. Introduction

Voevodsky introduced in an unpublished note [13] framed correspondences to get a more computation-friendly model for motivic homotopy theory. In Section 5 we will see how this can be used in practice to get results for algebraic cobordism. First, we introduce framed correspondences as a tool to answer two fundamental questions:

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Question 1.1. *What kind of transfers do cohomology theories represented by motivic spectra have*?

Question 1.2. *Is every cohomology theory with these transfers represented by a motivic spectrum*?

These questions will be answered in Sections 3 and 4, respectively.

We begin with an analysis of these problems in classical topology.

Definition 1.3. Let **Man** denote the category of smooth manifolds and **Spc** the ∞-category of spaces. A *cohomology theory* on **Man** is a functor

$$F : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Spc}$$

satisfying

(1) descent with respect to arbitrary open coverings,
(2) homotopy invariance: $F(M) \xrightarrow{\simeq} F(M \times \mathbb{R})$ for all $M \in \mathbf{Man}$.

We have the following classification.

Theorem 1.4. *The evaluation functor*

$$\{\textit{cohomology theories on } \mathbf{Man}\} \xrightarrow{\simeq} \mathbf{Spc}, \quad F \longmapsto F(*),$$

is an equivalence.

Proof. The constant sheaf $\underline{X}$ on **Man** with fiber a space X is given by

$$M \mapsto \mathrm{Map}(M, X).$$

(This is a nontrivial but well-known computation, which uses the fact that manifolds are sufficiently nice topological spaces, in particular locally contractible.) This computation shows that $\underline{X}$ is homotopy invariant. In particular, $X \mapsto \underline{X}$ is the left adjoint to the given evaluation functor, and since $\underline{X}(*) = X$ it is fully faithful. To show that it is an equivalence of categories it remains to show that evaluation on the point is conservative.

Let f be a morphism between cohomology theories and suppose that it is an isomorphism on a point. We need to show that it is an isomorphism in general. By homotopy invariance, we see that f is an isomorphism on the Euclidean spaces $\mathbb{R}^n$ for all n, and by using the descent property on good covers (covers are copies of $\mathbb{R}^n$ such that all intersections are either empty or isomorphic to $\mathbb{R}^m$) we can deduce that it is indeed an isomorphism for all manifolds M. □

In light of Theorem 1.4, cohomology theories can be described by giving a space X. If X happens to be the infinite loop space of a spectrum E, then the associated cohomology theory

$$M \mapsto \mathrm{Map}(M, \Omega^\infty E)$$

acquires some extra structure.

Example 1.5 (Atiyah duality). If M is compact, then $\Sigma^\infty_+ M$ has as dual the Thom spectrum M^{-T_M}, where T_M is the tangent bundle of M. This implies that for every morphism $f : M \to N$ between compact manifolds there is an induced morphism of Thom spectra $M^{-T_M} \leftarrow N^{-T_N}$. Mapping these spectra into another spectra E we get a covariant pushforward map

$$\mathrm{Map}(M^{-T_M}, E) \to \mathrm{Map}(N^{-T_N}, E).$$

If T_M and T_N are oriented with respect to E (e.g., if E is complex oriented and M and N are complex manifolds), then the pushforward involves ordinary shifts.

One can more generally define a cohomological pushforward along any proper morphism $f : M \to N$ between (not necessarily compact) smooth manifolds. This is a consequence of the "formalism of six operations", which is a vast generalization of Poincaré/Atiyah duality introduced by Grothendieck. For any morphism $f : M \to N$ in **Man**, there is the usual pullback/pushforward adjunction

$$\mathrm{Shv}(M, \mathbf{Spt}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Shv}(N, \mathbf{Spt}),$$

as well as the "exceptional" adjunction

$$\mathrm{Shv}(M, \mathbf{Spt}) \underset{f^!}{\overset{f_!}{\rightleftarrows}} \mathrm{Shv}(N, \mathbf{Spt}).$$

Here, $f_!$ is the pushforward with compact support, which is essentially determined by the following two properties:

- If f is proper, then $f_! = f_*$.
- If f is étale (i.e., a local homeomorphism), then $f^! = f^*$.

Defining the virtual tangent bundle of f by

$$T_f := T_M - f^* T_N,$$

there exists then a canonical natural transformation

$$\mathfrak{p}_f : \Sigma^{T_f} f^* \to f^!,$$

where Σ^{T_f} denotes suspension by the virtual tangent bundle, i.e., smashing with its Thom spectrum. This functor is an automorphism of $\mathrm{Shv}(M, \mathbf{Spt})$, and locally on M, it is the ordinary suspension by the rank of T_f. We do not explain here the construction of $\mathfrak{p}_f$. Suffice it to say, because any morphism factors as the composition of a submersion and a closed immersion and because of the tubular neighborhood theorem, the transformation $\mathfrak{p}_f$ is essentially determined by the cases where f is a submersion (in which case it is an *equivalence* $\Sigma^{T_f} f^* \simeq f^!$), and where f is the zero section of a vector bundle. We will explain later the algebro-geometric analogue of $\mathfrak{p}_f$ in more details.

If E_M denotes the constant sheaf on M with value E, we obtain morphisms

$$\Gamma(M, \Sigma^{T_f} E_M) \simeq \Gamma(M, \Sigma^{T_f} f^* E_N) \xrightarrow{\mathfrak{p}_f} \Gamma(M, f^! E_N) \simeq \mathrm{Map}(\mathbb{S}_M, f^! E_N).$$

If f is proper we furthermore have morphisms

$$\mathrm{Map}(\mathbb{S}_M, f^! E_N) \simeq \mathrm{Map}(f_* f^*(\mathbb{S}_N), E_N) \to \mathrm{Map}(\mathbb{S}_N, E_N) = \Gamma(N, E_N).$$

Composing these morphisms gives us a pushforward map

$$\Gamma(M, \Sigma^{T_f} E_M) \to \Gamma(N, E_N)$$

also known as the *transfer* along f. This provides an answer to Question 1.1 in the classical topological context.

For Question 1.2, the question becomes whether one can recover the spectrum structure on E from the infinite delooping $\Omega^\infty E$ by using these transfers. In particular, if $f : M \to N$ is a finite étale map (i.e., a finite covering map), then $T_f = 0$, in which case there is a canonical transfer

$$\mathrm{Map}(M, \Omega^\infty E) \to \mathrm{Map}(N, \Omega^\infty E).$$

By taking N to be a point and M to be two distinct points, the transfer provides an addition map

$$\Omega^\infty E \times \Omega^\infty E \xrightarrow{+} \Omega^\infty E.$$

We need to encode these transfers in a suitably coherent manner to recover the commutativity of $+$ up to coherent homotopy. Historically, this has been difficult because one lacked the tools to precisely express the required coherence. To achieve this we introduce the 2-category of correspondences:

Definition 1.6. The *category of finite étale correspondences* in **Man**, denoted by $\mathbf{Cor}^{\text{fét}}(\mathbf{Man})$, is the 2-category where objects are smooth manifolds, and morphisms from M to P are spans

$$\begin{array}{ccc} & N & \\ \swarrow^{\text{f. ét.}} & & \searrow \\ M & & P, \end{array}$$

such that the map $N \to M$ is finite étale. These spans are known as *correspondences*. Composition of two correspondences $(M \leftarrow N \to P)$ and $(P \leftarrow Q \to R)$ is defined as the pullback in the diagram

$$\begin{array}{ccccccc} & & & \text{pullback} & & & \\ & & \swarrow & & \searrow & & \\ & N & & & & Q & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ M & & & P & & & R. \end{array}$$

The pullback exists in **Man** because the map $Q \to P$ is finite étale.

The following theorem then provides an answer to Question 1.2.

Theorem 1.7. *There exists a functor*

$$\begin{aligned} \mathbf{Spt} &\to \{\textit{cohomology theories on } \mathbf{Cor}^{\text{fét}}(\mathbf{Man})\}, \\ E &\mapsto \mathrm{Map}(-, \Omega^\infty E), \end{aligned}$$

which restricts to an equivalence between connective spectra $\mathbf{Spt}_{\geq 0}$ *and grouplike cohomology theories on* $\mathbf{Cor}^{\text{fét}}(\mathbf{Man})$.

Such cohomology theories are automatically evaluated in the ∞-category of E_∞-spaces, hence the "grouplike" condition means that the addition structure

$$\Omega^\infty E \times \Omega^\infty E \to \Omega^\infty E$$

should be a group, that is, elements should have inverses up to homotopy. We refer to [2, Appendix C] for a proof of Theorem 1.7 given Theorem 1.4.

2. Transfers for motivic spectra

We will now focus on Question 1.1, i.e., transfers for motivic spectra.

Let $\mathcal{SH}(S)$ denote the ∞-category of motivic spectra over S. There is a six-functor formalism on the system of categories $S \mapsto \mathcal{SH}(S)$, that is, functors

$$f^*, f_*, f^!, f_!, \underline{\mathrm{Hom}}, \otimes,$$

with adjunctions $f^* \dashv f_*$ and $f_! \dashv f^!$, and equivalences $f^! \simeq f^*$ if f is étale and $f_! \simeq f_*$ if f is proper.

For the case of smooth manifolds, we saw in the previous section that there are transfers for any proper morphism. A key difference in the algebraic-geometric context is that we are now also willing to consider schemes which are not smooth. This changes the picture since morphisms of smooth manifolds are very far from being arbitrary morphisms, and in particular they are always of local complete intersections. The condition of local complete intersection turns out to be the most general condition for transfers to exist in this setting.

Definition 2.1. A morphism of schemes $f : X \to S$ is a *local complete intersection* (abbreviated "lci") if locally on X it factors as

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & V \\ & \overset{f}{\searrow} & \downarrow{\scriptstyle p} \\ & & S, \end{array}$$

where i is a closed immersion which locally is cut out by a regular sequence (i.e., a *regular* closed immersion) and p is a smooth morphism.

For any local complete intersection morphism $f : X \to S$ with a global factorization as above, we can define the *virtual tangent bundle* of f by

$$T_f := i^* T_p - N_i \in K(X).$$

Here, T_p is the relative tangent bundle of p, N_i is the normal bundle of i, and $K(X)$ denotes the K-theory space of the scheme X. One can show that T_f does not depend on the factorization of f (it is in fact the image in $K(X)$ of the tangent complex of f, which is defined even if no factorization exists).

We will now sketch the fundamental construction of Déglise, Jin, and Khan [3], which is the source of the transfers in stable motivic homotopy theory. We recall that any element $\xi \in K(X)$ has an associated Thom spectrum in $\mathcal{SH}(X)$, and smashing with it defines a self-equivalence

$$\Sigma^{\xi} : \mathcal{SH}(X) \xrightarrow{\simeq} \mathcal{SH}(X).$$

Construction 2.2 [3]. *For an lci morphism $f : X \to S$ with a global factorization, we construct a canonical transformation*

$$\mathfrak{p}_f : \Sigma^{T_f} f^* \to f^!.$$

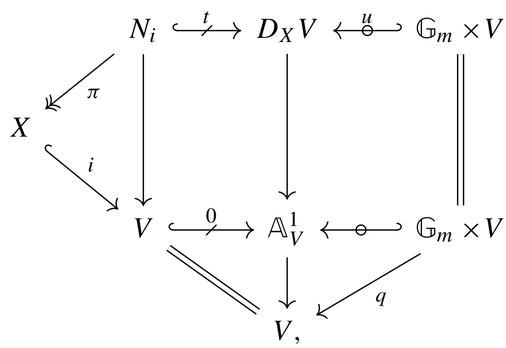
Sketch of proof. Because of the global lci factorization, it suffices to consider the case of smooth morphisms and regular closed immersions.

If f is smooth we let $\mathfrak{p}_f$ be the well-known purity equivalence

$$\Sigma^{T_f} f^* \simeq f^!$$

due to Voevodsky and Ayoub.

If $f = i : X \hookrightarrow V$ is a regular closed immersion we consider the deformation of V to the normal cone, which is a scheme $D_X V$ that fits in a diagram



where $0 : V \hookrightarrow \mathbb{A}^1_V$ is the zero section, $\pi : N_i \to X$ is the normal bundle of i, $\mathbb{G}_m \times V \hookrightarrow D_X V$ and $\mathbb{G}_m \times V \hookrightarrow \mathbb{A}^1_V$ are the open complements, and both squares are Cartesian.[1] Let p denote the composition of the maps $D_X V \to \mathbb{A}^1_V \to V$.

The top row gives rise to a localization triangle

$$t_* t^! \to \mathrm{id}_{\mathcal{SH}(D_X V)} \to u_* u^! \xrightarrow{\partial} t_* t^! [1],$$

and in $\mathcal{SH}(\mathbb{G}_m)$ there is a canonical (universal unit) map $S^0 \to \mathbb{G}_m$. Suspending this map with S^1 we get a new map

$$S^1 \xrightarrow{(*)} S^1 \wedge \mathbb{G}_m = T,$$

where $T = S^1 \wedge \mathbb{G}_m$ is the Thom space of the trivial line bundle. We want to define a natural transformation

$$\mathfrak{p}_i : i^* \to \Sigma^{N_i} i^!,$$

or equivalently, by adjunction, a map

$$\mathrm{id} \to i_* \Sigma^{N_i} i^!.$$

This we obtain from the composition

$$\mathrm{id}_{\mathcal{SH}(V)}[1] \xrightarrow{(*)} q_* \Sigma_T q^* \simeq q_* q^! \simeq p_* u_* u^! p^! \xrightarrow{\partial} p_* t_* t^! p^! [1] \simeq i_* \pi_* \pi^! i^! [1],$$

and using the purity equivalence $\mathfrak{p}_\pi : \pi^* \Sigma^{N_i} \simeq \pi^!$ and the fact that π^* is a fully faithful functor, we further get

$$i_* \pi_* \pi^! i^! \simeq i_* \Sigma^{N_i} i^!.$$

Composing these maps gives the desired natural transformation

$$\mathrm{id}_{\mathcal{SH}(V)} \to i_* \Sigma^{N_i} i^!.$$

[1] For an arbitrary closed immersion i, the fiber of $D_X V$ over 0 is the normal cone of i, which agrees with the normal bundle in case of a regular immersion.

It remains to prove that this procedure is independent of the factorization, composing various f's, etc. We refer the curious reader to [3] for a thorough treatment of these facts. □

Definition 2.3. Let $E \in \mathcal{SH}(S)$ be a motivic spectrum over S. For a morphism $f : X \to S$ and an element $\xi \in K(X)$ we define

$$E(X,\xi) := \mathrm{Map}_{\mathcal{SH}(X)}(\mathbb{1}_X, \Sigma^{\xi} f^* E) \in \mathbf{Spc}\,.$$

Example 2.4. If $\xi = n$, then $\Sigma^{\xi} \cong \Sigma^{2n,n}$. Thus

$$\pi_i E(X,\xi) = E^{2n-i,n}(X).$$

Example 2.5. If $E = \mathrm{KGL}$, then $E(X,\xi) \cong \mathrm{KH}(X)$, where KH denotes the homotopy K-theory of X. Note that this does not depend on ξ.

Example 2.6. If $E = H\mathbb{Z}$ and X is smooth over a field or a discrete valuation ring, then $E(X,\xi) \cong z^r(X,\bullet)$ where $r = \mathrm{rank}(\xi)$. Here $z^r(X,\bullet)$ denotes Bloch's cycle complex.

Theorem 2.7. *If $f : X \to S$ is proper and lci, then the natural transformation $\mathfrak{p}_f : \Sigma^{T_f} f^* \to f^!$ induces a transfer map*

$$E(X, T_f + f^*\xi) \to E(S,\xi),$$

where $\xi \in K(S)$.

Remark 2.8. More generally, if $f : X \to S$ is proper and X^{der} is a derived structure on X which is lci/quasismooth over S, a theorem by Adeel A. Khan [10] gives a transfer map

$$E(X, T_f^{\mathrm{der}}) \to E(S).$$

3. The ∞-category of framed correspondences

Following the approach to Question 1.1 in classical topology, we are interested in those transfers that do not shift the degree, i.e., that induce a covariant functoriality on the "unshifted" cohomology theory. For this to work we saw in the previous section that we need the virtual tangent bundle of a map f to vanish. We are thus led to the following definition:

Definition 3.1. Suppose $f : X \to S$ is a lci morphism. A (stable 0-dimensional) *framing* of f is a trivialization of the virtual tangent bundle T_f, that is, a path in $K(X)$ between T_f and 0.

If f is a framed proper lci morphism, we see by Theorem 2.7 that it induces a transfer map in cohomology that does not shift the degree. This provides an answer to Question 1.1.

Definition 3.2. We define the ∞-category $\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S)$ with objects smooth S-schemes and morphisms given as spans $(X \xleftarrow{f} Z \to Y)$, where f is finite lci and framed (in the sense that there is given a path $\alpha : T_f \xrightarrow{\simeq} 0$ in $K(Z)$).

Composition of morphisms $(X \leftarrow Z \to Y)$ and $(Y \leftarrow W \to V)$ is defined by pullback

$$\begin{array}{ccccccc} & & & \text{pullback} & & & \\ & & \swarrow^{h} & & \searrow^{b} & & \\ & Z & & & & W & \\ \swarrow^{f} & & \searrow^{a} & & \swarrow^{g} & & \searrow \\ X & & & Y & & & V, \end{array}$$

where $T_{f\circ h} \simeq T_h + h^*T_f$, $T_h \simeq b^*(T_g) \simeq 0$ and $h^*T_f \simeq 0$.

Remark 3.3. The condition for a morphism to be finite lci and framed implies that it is flat. This is relevant because it forces the condition of being lci to be stable under base change. In general the lci condition is only stable under tor-independent base change.

Remark 3.4. The reason why we only consider finite maps is that we want the framed transfers to be compatible with the Nisnevich topology. That is, we want the Nisnevich sheafication of a presheaf with framed transfers to have framed transfers, and this requires finite morphisms. It is possible to relax the finiteness condition by instead considering proper morphisms; see Question 5.12. However, we must then instead consider derived schemes since such morphisms are not flat.

Remark 3.5. The category $\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S)$ is semiadditive (i.e., finite sums and finite products coincide) and has a symmetric monoidal structure given by the Cartesian products $(X \otimes Y = X \times_S Y)$. There are canonical functors

$$\mathbf{Cor}^{\mathrm{f\acute{e}t}}(\mathrm{Sm}_S) \to \mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S),$$

because finite étale morphisms have a canonical framing. By forgetting the framing we have a functor

$$\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S) \to \mathbf{Cor}^{\mathrm{fsyn}}(\mathrm{Sm}_S)$$

to the category of finite syntomic (flat and lci) correspondences. A further forgetful functor takes us to Voevodsky's category of correspondences $\mathbf{Cor}(\mathrm{Sm}_S)$.

4. The recognition principle

We here mention three fundamental theorems, due to work by Garkusha, Panin, Ananyevskiy, and Neshitov, and Elmanto, Hoyois, Khan, Sosnilo, and Yakerson. These will allow us to prove a recognition principle for framed correspondences that answers Question 1.2.

Definition 4.1. Let $\mathcal{H}(S)$ denote the ∞-category of $\mathbb{A}^1$-invariant Nisnevich sheaves on Sm_S, and let $\mathcal{H}^{\mathrm{fr}}(S)$ be the ∞-category of $\mathbb{A}^1$-invariant Nisnevich sheaves on $\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S)$ (i.e., presheaves on $\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S)$ whose restriction to Sm_S are $\mathbb{A}^1$-invariant Nisnevich sheaves). We define $\mathcal{SH}(S)$ and $\mathcal{SH}^{\mathrm{fr}}(S)$ as the ∞-categories of T-spectra in $\mathcal{H}(S)$ and $\mathcal{H}^{\mathrm{fr}}(S)$, respectively.

The functor $\mathrm{Sm}_S \to \mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S)$ induces adjoint functor pairs

$$\begin{array}{ccc} \mathcal{H}(S) & \underset{\text{forget}}{\overset{\text{free}}{\rightleftarrows}} & \mathcal{H}^{\mathrm{fr}}(S) \\ \Sigma^\infty_+ \downarrow \uparrow \Omega^\infty & & \Sigma^\infty_{\mathrm{fr}} \downarrow \uparrow \Omega^\infty_{\mathrm{fr}} \\ \mathcal{SH}(S) & \underset{\text{forget}}{\overset{\text{free}}{\rightleftarrows}} & \mathcal{SH}^{\mathrm{fr}}(S). \end{array}$$

Theorem 4.2 (reconstruction theorem, [7, Theorem 16]). *For every scheme S, the canonical functor*

$$\mathcal{SH}(S) \xrightarrow{\sim} \mathcal{SH}^{\mathrm{fr}}(S)$$

is an equivalence of ∞-categories.

Thus every motivic spectrum has a unique structure of framed transfers.

This result requires new ideas since the analogue statement is false for Voevodsky's correspondences because it would yield an equivalence $\mathcal{SH}^{\mathrm{fr}}(S) \simeq \mathbf{DM}(S)$.

We provide a brief sketch of the main ideas involved in the proof.

Sketch of proof. The key idea is to apply Voevodsky's lemma: that is, if X is a scheme, and $U \subset X$ is an open subscheme, we can identify $L_{\mathrm{Nis}}(X/U)(Y)$ with the set of pairs (Z, ϕ) such that $Z \subset Y$ is a closed subset and $\phi : Y^h_Z \to X$ is a map such that $\phi^{-1}(X \setminus U) = Z$.

Then consider, for $X, Y \in \mathrm{Sm}_S$, the mapping space

$$\mathrm{Map}((\mathbb{P}^1)^{\wedge n} \wedge X_+, L_{\mathrm{Nis}}(\mathbb{A}^n / \mathbb{A}^n - 0) \wedge Y_+) \cong (\Omega^n_{\mathbb{P}^1} L_{\mathrm{Nis}} \Sigma^n_T Y_+)(X).$$

Applying Voevodsky's lemma we find that this equals the set of pairs (Z, ϕ) such that

(1) $Z \subset (\mathbb{P}^1_X)^n$ is a closed subscheme,
(2) $Z \cap \partial(\mathbb{P}^1_X)^n = \varnothing$,
(3) $\phi : ((\mathbb{P}^1_X)^n)^h_Z \to \mathbb{A}^1_Y$ such that $\phi^{-1}(0) = Z$.

Thus Z is cut out by n equations with codimension n, and, in particular, it is framed. This shows that there is a forgetful map to $\mathbf{Cor}^{\mathrm{fr}}_S(X, Y)$. We conclude by [5, Corollary 2.3.27] which says that the forgetful map

$$\operatorname*{colim}_{n\to\infty} \Omega^n_{\mathbb{P}^1} L_{\mathrm{Nis}} \Sigma^n_T Y_+ \to \mathbf{Cor}^{\mathrm{fr}}_S(-, Y)$$

is a motivic equivalence if $Y \in \mathrm{Sm}_S$. □

Theorem 4.3 (cancellation theorem, [5, Proposition 5.2.7]). *Let k be a perfect field. The functors*

$$S^1 \wedge (-),\ \mathbb{G}_m \wedge (-) : \mathcal{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathcal{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}$$

are fully faithful, where $\mathcal{H}^{\mathrm{fr}}(k)^{\mathrm{gp}}$ is the full subcategory of $\mathcal{H}^{\mathrm{fr}}(k)$ consisting of grouplike objects.

Let L_{Zar} denote the Zariski sheafication functor and $L_{\mathbb{A}^1}$ the functor defined by

$$(L_{\mathbb{A}^1} F)(U) = |F(U \times \mathbb{A}^\bullet)|.$$

Theorem 4.4 (strict $\mathbb{A}^1$-invariance theorem, [5, Theorem 3.4.11]). *Let k be a perfect field. If $F \in \mathcal{P}_\Sigma(\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_k))^{\mathrm{gp}}$, then $L_{\mathrm{Zar}} L_{\mathbb{A}^1} F$ is an $\mathbb{A}^1$-invariant Nisnevich sheaf.*

Theorems 4.3 and 4.4 are analogues of Voevodsky's results for finite correspondences ([14] and [11, §3.2]) and the proofs are similar. For Theorem 4.3, Bachmann has a paper [1] where he proves a cancellation theorem for finite flat correspondences, which applies to these framed correspondences as well.

The next result is an analogue of Theorem 1.7 for framed motivic spectra and thus answers Question 1.2.

Corollary 4.5 (motivic recognition principle, [5, Theorem 3.5.16]). *Let k be a perfect field. There is an equivalence*

$$\mathcal{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \simeq \mathcal{SH}(k)^{\mathrm{veff}},$$

where $\mathcal{SH}(k)^{\mathrm{veff}} \subset \mathcal{SH}(k)$ denotes the full subcategory generated under colimits by suspension spectra $\Sigma^\infty_+ X$.

Proof. Theorem 4.3 implies that the infinite suspension functor

$$\Sigma^\infty_{\mathrm{fr}} : \mathcal{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathcal{SH}^{\mathrm{fr}}(k)$$

is fully faithful. Using Theorem 4.2 we identify its essential image as $\mathcal{SH}(k)^{\mathrm{veff}}$. □

The main application of Theorem 4.2 is that the framed suspension functor $\Sigma^\infty_{\mathrm{fr}}$ becomes a "machine" for producing motivic spectra: Let $F \in \mathcal{P}_\Sigma(\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S))$ be a presheaf with framed transfers. Then

$$\Sigma^\infty_{\mathrm{fr}} F \in \mathcal{SH}^{\mathrm{fr}}(S),$$

which by Theorem 4.2 can by identified with an object of $\mathcal{SH}(S)$. This allows one to build motivic spectra from presheaves with framed transfers.

In the special case of $S = \operatorname{Spec} k$, where k is a perfect field, we can use Corollary 4.5 to compute the infinite loop space of such a spectrum by using the equivalence

$$\Omega^\infty_{\mathrm{fr}} \Sigma^\infty_{\mathrm{fr}} F \simeq L_{\mathrm{Zar}} L_{\mathbb{A}^1} F^{\mathrm{gp}}.$$

We have the following consequences of this in action:

Example 4.6. Applying the infinite framed suspension functor $\Sigma^\infty_{\mathrm{fr}}$ to the right side of this diagram one obtains the left side:

$$
\begin{array}{ccccccc}
\mathbb{1} & & & \qquad & \mathrm{FSyn}^{\mathrm{fr}} & & \\
\downarrow & & & & \downarrow & & \\
\mathrm{MSL} & \longrightarrow & \mathrm{MGL} & & \mathrm{FSyn}^{\mathrm{or}} & \longrightarrow & \mathrm{FSyn} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{ko} & \longrightarrow & \mathrm{kgl} & & \mathrm{FGor}^{\mathrm{or}} & \longrightarrow & \mathrm{FFlat} \\
\| & & \| & & \downarrow & & \downarrow \\
\mathrm{ko} & \longrightarrow & \mathrm{kgl} & & \mathrm{Vect}^{\mathrm{sym}} & \longrightarrow & \mathrm{Vect} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H\widetilde{\mathbb{Z}} & \longrightarrow & H\mathbb{Z} & & \underline{\mathrm{GW}} & \longrightarrow & \underline{\mathbb{Z}},
\end{array}
$$

In more details:

- $\mathrm{FSyn}^{\mathrm{fr}}$ is the stack of framed finite syntomic schemes, which is the unit object in $\mathcal{P}_\Sigma(\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_k))$. The claim that $\Sigma^\infty_{\mathrm{fr}}\mathrm{FSyn}^{\mathrm{fr}} \simeq \mathbb{1}$ is therefore tautological.
- FSyn is the stack of finite syntomic schemes and $\mathrm{FSyn}^{\mathrm{or}}$ is the stack of finite syntomic schemes with an orientation (i.e., a trivialization of the dualizing sheaf). That these give MGL and MSL over a general base is proved in [4, Theorems 3.4.1, 3.4.3].
- Vect is the stack of vector bundles and FFlat is the stack of finite flat schemes. That both give the effective K-theory spectrum kgl over a field is proved in [8, Corollary 5.2, Theorem 5.4].
- $\mathrm{Vect}^{\mathrm{sym}}$ is the stack of vector bundles with a nondegenerate symmetric bilinear form, and $\mathrm{FGor}^{\mathrm{or}}$ is the stack of oriented finite Gorenstein (dualizing sheaf is a line bundle) schemes. That both give the effective hermitian K-theory spectrum ko over a field of characteristic not 2 is proved in [9, Proposition 7.7, Theorem 7.12].
- $\underline{\mathbb{Z}}$ is the constant sheaf with fiber $\mathbb{Z}$. It is proved in [7, Theorem 21] that $\Sigma^\infty_{\mathrm{fr}}\underline{\mathbb{Z}} \simeq H\mathbb{Z}$ over any Dedekind domain, where $H\mathbb{Z}$ is the motivic spectrum representing Bloch–Levine motivic cohomology.
- Finally, $\underline{\mathrm{GW}}$ is the sheaf of unramified Grothendieck–Witt rings, that is, the Zariski sheafification of the usual Grothendieck–Witt groups. It is proved in [9, Theorem 8.3] that $\Sigma^\infty_{\mathrm{fr}}\underline{\mathrm{GW}} \simeq H\widetilde{\mathbb{Z}}$ over any Dedekind domain in which 2 is invertible.

Here is one nice application of this picture: all morphisms on the right-hand side are easily seen to be E_∞-ring maps, hence so are all morphisms on the left-hand side.

5. Algebraic cobordism

The topic for this section is algebraic cobordism, following [4]. Specifically, we will apply the above theory to provide a framed model for algebraic cobordism which allows for more friendly computations.

> *In particular the question about the convergence of the divisibility spectral sequence for algebraic cobordism would benefit greatly from knowing that elements of algebraic cobordism can be represented by some kind of geometric objects.*
>
> -V. Voevodsky, 2000, My view of the current state of motivic homotopy theory [12, p. 11].

We begin by considering the moduli stack PQSm of proper quasismooth derived schemes. It is natural to study this stack in connection with algebraic cobordism, because these are precisely the schemes that give rise to algebraic cobordism classes. Namely, a proper quasismooth derived scheme of virtual dimension $-n$ has a cobordism class in $\mathrm{MGL}^{2n,n}$ by, for instance, using the transfers from Section 2. The $\mathbb{A}^1$-localization

$$L_{\mathbb{A}^1}\mathrm{PQSm} := |\mathrm{PQSm}(- \times \mathbb{A}^\bullet)|$$

can be described informally as follows. Points in $L_{\mathbb{A}^1}\mathrm{PQSm}$ are schemes, and a path between two points is a scheme over $\mathbb{A}^1$, which may be viewed as a cobordism from the fiber over 0 to the fiber over 1 (see Figure 1). A path between two paths is then a cobordism of cobordisms, etc. The analogous construction in topology, with $\mathbb{R}^1$ in place of $\mathbb{A}^1$, is one way to define the cobordism spaces that appear, for example, in the cobordism hypothesis. One may thus interpret the functor $L_{\mathbb{A}^1}$ as forming the cobordism spaces of moduli stacks of schemes.

Recall the forgetful functor

$$\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S) \to \mathbf{Cor}^{\mathrm{fsyn}}(\mathrm{Sm}_S)$$

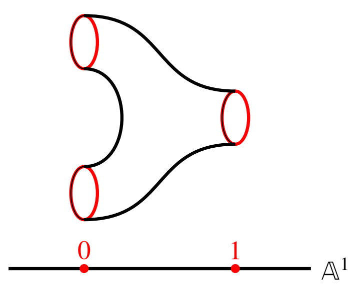


Figure 1. A cobordism from two circles to one.

from the ∞-category of framed correspondences of smooth schemes to the 2-category of finite syntomic correspondences of smooth schemes, given by forgetting the framing and using that the morphism in the left span is finite syntomic.

Let FQSm^n denote the moduli stack of finite quasismooth derived S-schemes of relative dimension $-n$. This is obviously a presheaf on $\mathbf{Cor}^{\mathrm{fsyn}}(\mathrm{Sm}_S)$, and hence also on $\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S)$, using the above forgetful functor.

Theorem 5.1. *For every scheme S and every integer $n \geq 0$ there is an equivalence*

$$\Sigma^n_T \mathrm{MGL} \simeq \Sigma^\infty_{\mathrm{fr}} \mathrm{FQSm}^n$$

in $\mathcal{SH}(S) \simeq \mathcal{SH}^{\mathrm{fr}}(S)$.

The main ingredient of the proof is to understand the framed models for Thom spectra of virtual vector bundles coming from Theorem 5.10. We therefore suspend the proof of this theorem until we have established a proof of the result.

Remark 5.2. There is an equivalence

$$\mathrm{FQSm}^0 \cong \mathrm{FSyn}.$$

The next theorem is a specialization of Theorem 5.1 to the case of perfect fields.

Theorem 5.3. *If k is a perfect field and $n \geq 0$, then*

$$\Omega^\infty_T \Sigma^n_T \mathrm{MGL} \simeq L_{\mathrm{Zar}} L_{\mathbb{A}^1} (\mathrm{FQSm}^n)^{\mathrm{gp}}.$$

In particular,

$$\Omega^\infty_T \mathrm{MGL} \simeq L_{\mathrm{Zar}} L_{\mathbb{A}^1} \mathrm{FSyn}^{\mathrm{gp}}.$$

If $n \geq 1$, then

$$\Omega^\infty \Sigma^n_T \mathrm{MGL} \simeq L_{\mathrm{Nis}} L_{\mathbb{A}^1} \mathrm{FQSm}^n.$$

Remark 5.4. The last point of Theorem 5.3 means that one can trade the group completion by instead taking Nisnevich sheafification. More precisely, the Nisnevich sheaf $L_{\mathrm{Nis}} L_{\mathbb{A}^1} \mathrm{FQSm}^n$ is connected, hence is already grouplike, so we do not need to take its group completion.

A posteriori, we know by Morel's connectivity theorem that $\Omega^\infty \Sigma^n_T \mathrm{MGL}$ is $(n-1)$-connected.

Remark 5.5. We state Theorems 5.1 and 5.3 for MGL, but there are analogous statements for all Thom spectra that one would like to think about, for example, MSL, MSp, $\mathbb{S}$, etc.

Theorem 5.6. *For every scheme S there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Mod}_{\mathrm{MGL}}(\mathcal{SH}(S)) \simeq \mathcal{SH}^{\mathrm{fsyn}}(S),$$

where $\mathcal{SH}^{\mathrm{fsyn}}(S)$ is analogous to $\mathcal{SH}^{\mathrm{fr}}(S)$ using $\mathbf{Cor}^{\mathrm{fsyn}}(\mathrm{Sm}_S)$ in lieu of $\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S)$.

Theorem 5.7. *If k is a perfect field, then*

$$\mathrm{Mod}_{\mathrm{MGL}}(\mathcal{SH}(S)^{\mathrm{veff}}) \simeq \mathcal{H}^{\mathrm{fsyn}}(k)^{\mathrm{gp}}.$$

This theorem is a combination of Theorem 5.6 with a cancellation theorem in $\mathcal{H}^{\mathrm{fsyn}}(k)^{\mathrm{gp}}$.

Framed models for effective Thom spectra.

Definition 5.8. Consider a smooth S-scheme X and $\xi \in K(X)_{\geq 0}$. We define a framed presheaf

$$(X, \xi)^{\mathrm{fr}} \in \mathcal{P}_{\Sigma}(\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S))$$

by sending a scheme U to the space of spans $(U \xleftarrow{f} Z \xrightarrow{g} X)$ where f is finite and quasismooth together with $\alpha : T_f \simeq -g^*(\xi) \in K(Z)$.

This definition is not enough to define a presheaf on $\mathbf{Cor}^{\mathrm{fr}}(\mathrm{Sm}_S)$; there is some work involved in making this precise which can be found in [4, Appendix B].

Remark 5.9. If $\mathrm{rank}(\xi) = 0$, then Z is underived. For example,

$$(X, 0)^{\mathrm{fr}} = \mathbf{Cor}^{\mathrm{fr}}_S(-, X).$$

These presheaves turn out to be framed models for effective Thom spectra.

Theorem 5.10 [4, Theorem 3.2.1]. *There is an equivalence*

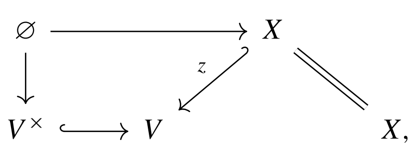
$$\mathrm{Th}_X(\xi) \simeq \Sigma^{\infty}_{\mathrm{fr}}(X, \xi)^{\mathrm{fr}}$$

in $\mathcal{SH}(S) \simeq \mathcal{SH}^{\mathrm{fr}}(S)$.

The proof consists of two parts: first constructing a map between these two objects, and then showing that it is an equivalence.

Proof. Suppose that ξ is a vector bundle V on X, and let $V^{\times}$ denote the complement of the zero section inside V. We define a map as follows:

Consider the diagram



where z is the zero section. The canonical equivalence $T_z \simeq -V$ defines a map

$$V \to (X, V)^{\mathrm{fr}},$$

and we get a diagram

$$\begin{array}{ccc} (V^{\times})^{\mathrm{fr}} & & \\ \big\downarrow & \overset{0}{\searrow} & \\ V^{\mathrm{fr}} & \longrightarrow & (X, V)^{\mathrm{fr}} \\ \big\downarrow & \nearrow & \\ \mathrm{Th}_X(V)^{\mathrm{fr}}, & & \end{array} \tag{5-1}$$

where the dotted arrow is the comparison map we were looking for.

One can show that the map is a motivic equivalence by reducing to the case where the base is a field and using a moving lemma from [6].

The general case for virtual tangent bundles is now mostly formal. The main point is to use the identification $K(-) \simeq L_{\mathrm{Zar}}\mathrm{Vect}(-)^{\mathrm{gp}}$. The Thom spectrum functor $\mathrm{Th}_X : \mathrm{Vect}(X) \to \mathcal{SH}(X)$ takes the direct sum of vector bundles to the tensor product of spectra and lands in the Picard ∞-groupoid $\mathrm{Pic}(\mathcal{SH}(X))$, and thus factors uniquely through the group completion of $\mathrm{Vect}(X)$.

We can then formally extend (5-1) to a map

$$\mathrm{Th}_X(\xi)^{\mathrm{fr}} \to (X, \xi)^{\mathrm{fr}},$$

which is natural in the virtual vector bundle ξ over X.

To see that this map is an equivalence we use that virtual vector bundles of rank ≥ 0 are actual vector bundles Zariski locally. □

We state a result about K-theory that is needed in the proof of Theorem 5.1.

Proposition 5.11 (Bhatt–Lurie, [4, A.0.6]). *Let R be a commutative ring. Then the K-theory functor*

$$K : \mathrm{CAlg}_R^{\mathrm{der}} \to \mathbf{Spc}$$

is left-Kan extended from $\mathrm{CAlg}_R^{\mathrm{Sm}}$.

Proof. The formula for left-Kan extension is

$$\mathrm{LKE}(K \mid \mathrm{CAlg}_R^{\mathrm{Sm}})(T) = \operatorname*{colim}_{S \to T} K(S),$$

where $S \in \mathrm{CAlg}_R^{\mathrm{Sm}}$. The index category $\{S \to T\}$ has finite coproducts given by tensor products, which make it sifted.

By the identity

$$K = \mathrm{Vect}^{\mathrm{gp}}$$

and the fact that the group completion commutes with sifted colimits, we are reduced to show that $\mathrm{Vect}(-)$ is left-Kan extended from CAlg_R^{Sm}. Using that

$\mathrm{Vect} = \mathrm{colim}_n \mathrm{Vect}_{\leq n}$ we reduce to showing that each $\mathrm{Vect}_{\leq n}$ is left-Kan extended. One way of showing this is by taking the Jouanolou device

$$U_n \to \mathrm{Gr}_n,$$

which is an affine bundle whose total space is affine. Then there is a map

$$\coprod_{k \leq n} U_k \xrightarrow{f} \mathrm{Vect}_{\leq n}.$$

The map f is a sectionwise surjection because we are dealing with affine schemes, so that vector bundles are generated by their global sections. Hence it may be lifted to a point in the Grassmannian, which further can be lifted to the Jouanolou device since it is an affine bundle over an affine scheme.

We therefore have

$$\mathrm{Vect}_{\leq n} \simeq \mathrm{colim}\, \check{C}_\bullet(f),$$

where $\check{C}_\bullet(f) = \Delta^{\mathrm{op}} \to \mathrm{Ind}(\text{smooth affine schemes})$. The same remains true of all the self-intersections of $\coprod_{k \leq n} U_k$, since $\mathrm{Vect}_{\leq n}$ is a stack with smooth affine diagonals. Thus the iterated fiber product of $\coprod_{k \leq n} U_k$ is still a colimit of smooth and affine schemes. Combining the colimits we thereby find that the presheaf Vect is a colimit of smooth affine R-schemes, which is precisely what it means to be left-Kan extended from $\mathrm{CAlg}_R^{\mathrm{Sm}}$. □

Now for the proof of Theorem 5.1.

Proof of Theorem 5.1. For $n = 0$ we can write

$$\mathrm{MGL} \simeq \operatorname*{colim}_{X,\xi} \mathrm{Th}_X(\xi),$$

by [2, Section 16], where the colimit is taken over $X \in \mathrm{Sm}_S$ and $\xi \in K(X)$ with $\mathrm{rank}(\xi) = 0$. Hence

$$\mathrm{MGL} \simeq \Sigma^\infty_{\mathrm{fr}} \operatorname*{colim}_{X,\xi} (X, \xi)^{\mathrm{fr}}$$

by Theorem 4.2. Further, there is a forgetful map

$$\begin{array}{c} \operatorname*{colim}_{X,\xi} (X, \xi)^{\mathrm{fr}} \\ \downarrow \\ \mathrm{FSyn} \end{array}$$

by using that an object in $\mathrm{colim}_{X,\xi} (X, \xi)^{\mathrm{fr}}$ consists of a span $U \xleftarrow{f} Z \xrightarrow{g} X$ which we forgetfully map to $U \xleftarrow{f} Z$. We claim that this map in an equivalence.

By taking the fiber one obtains a Cartesian square

$$\begin{array}{ccc} \operatorname*{colim}_{X,\xi}(X,\xi)^{\mathrm{fr}} & \longleftarrow & \operatorname*{colim}_{X,\xi}\{Z \xrightarrow{g} X,\, T_f \simeq -g^*(\xi)\} \\ \big\downarrow & & \big\downarrow \\ \mathrm{FSyn} & \longleftarrow & \{U \xleftarrow{f} Z\}. \end{array}$$

Formally rewriting we find that

$$\begin{aligned} \operatorname*{colim}_{X,\xi}\{Z \xrightarrow{g} X,\, T_f \simeq -g^*(\xi)\} &\simeq \operatorname*{colim}_{X,\xi,g}\{T_f \simeq -g^*(\xi)\} \\ &\simeq \operatorname*{colim}_{X,g}\mathrm{fib}_{T_f}(K(X) \xrightarrow{g^*} K(Z)), \end{aligned}$$

and exchanging the fiber and colimit we get

$$\operatorname*{colim}_{X,g}\mathrm{fib}_{T_f}(K(X) \xrightarrow{g^*} K(Z)) \simeq \mathrm{fib}_{T_f}(\operatorname*{colim}_{X,g} K(X) \to K(Z)).$$

This we recognize as the formula for the left-Kan extension, that is,

$$\operatorname*{colim}_{X,g} K(X) \simeq \mathrm{LKE}(K \mid \mathrm{Sm}_S)(Z).$$

It remains to show that

$$\operatorname*{colim}_{X,\xi}\{Z \xrightarrow{g} X,\, T_f \simeq -g^*(\xi)\}$$

is contractible, which we have reduced to showing that

$$\mathrm{fib}_{T_f}(\mathrm{LKE}(K \mid \mathrm{Sm}_S)(Z) \to K(Z))$$

is contractible. However, it follows from Proposition 5.11 that the map

$$\mathrm{LKE}(K \mid \mathrm{Sm}_S)(Z) \to K(Z)$$

is an equivalence, hence the fiber must be trivial.

The proof for $\Sigma^n_T\mathrm{MGL}$ for $n > 0$ is similar, but with $\xi \in K(X)$ such that $\mathrm{rank}(\xi) = n$. □

Question 5.12. *Is Theorem 5.1 still true if one relaxes the "finiteness" condition to a "properness" condition?*

Note that the finiteness condition was not necessary to get a cobordism class. If one drops this condition, then FSyn becomes equal to PQSm^0, which is a much larger stack. To see this one can start with any smooth proper scheme of any dimension n and find a derived scheme structure on it which is quasismooth and of dimension 0; we may for instance take the vanishing locus of the zero function n times.

It turns out that the naive cobordisms space $L_{\mathbb{A}^1}\mathrm{PQSm}^0$ is a group under disjoint union, which is not true for $L_{\mathbb{A}^1}\mathrm{FSyn}$. There should then be maps

$$\mathrm{FSyn} \hookrightarrow \mathrm{PQSm}^0 \xrightarrow{\text{conjecturally}} \Omega^\infty_T \mathrm{MGL}, \tag{5-2}$$

where the last map uses the transfers in MGL-cohomology. This would imply that $\Sigma^\infty_{\mathrm{fr}}\mathrm{PQSm}^0$ has MGL as a retract. Over perfect fields one can therefore conclude that $\Omega^\infty\mathrm{MGL}$ is a direct factor of $L_{\mathrm{Zar}}L_{\mathbb{A}^1}\mathrm{PQSm}^0$. It is an open problem whether they in fact are equivalent.

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Lectures on the cohomology of reciprocity sheaves

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These are the notes accompanying three lectures given by K. Rülling at the Motivic Geometry program at CAS, which aim to give an introduction and an overview of some recent developments in the field of reciprocity sheaves. We begin by introducing the theory of reciprocity sheaves and the necessary background of modulus sheaves with transfers as developed by B. Kahn, H. Miyazaki, S. Saito, and T. Yamazaki. We then explain some basic examples of reciprocity sheaves with a special emphasis on Kähler differentials and the de Rham–Witt complex. After an overview of some fundamental results, we survey the recent work of F. Binda, K. Rülling, and S. Saito on the cohomology of reciprocity sheaves. In particular, we discuss a projective bundle formula, a blow-up formula, and a Gysin sequence, which generalizes work of Voevodsky on homotopy invariant sheaves with transfers. From this, pushforwards along projective morphisms can be constructed, which give rise to an action of projective Chow correspondences on the cohomology of reciprocity sheaves. This generalizes several constructions which originally relied on Grothendieck duality for coherent sheaves and gives a motivic view towards these results.

We then survey some applications which include the birational invariance of the cohomology of certain classes of reciprocity sheaves, many of which were not considered before. Finally, we outline some recent results which were not part of the lecture series.

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Foreword. These are the notes accompanying three lectures[1] given by K. Rülling in October 2020 at the Motivic Geometry program at CAS, which aim to give an introduction and an overview of some recent developments in the field of reciprocity sheaves. We stress that the focus of this lecture series, and the present notes, is on the properties of reciprocity sheaves, on their cohomology, and on applications of the theory, with a particular emphasis on de Rham–Witt sheaves. We do not stress categorical constructions such as the triangulated category of motives with modulus and we do not intend to give a complete overview of the whole theory, which was first and foremost developed by B. Kahn, H. Miyazaki, S. Saito, and T. Yamazaki. We try to keep the informal style of the lectures also in these notes and we do not claim any originality.

1. Reciprocity sheaves

By work of Voevodsky and many others, the general theory of the cohomology of $\mathbb{A}^1$-invariant sheaves with transfers is fully developed. Among the most fundamental properties are the projective bundle formula, the blow-up formula, the Gysin sequence, Gersten resolution, action of proper Chow correspondences, and representability of cohomology theories.

However, the theory has a drawback: Many interesting non-$\mathbb{A}^1$-invariant sheaves share the same properties as above, such as Kähler differentials, smooth commutative unipotent group schemes, étale motivic cohomology with $\mathbb{Z}/p^n$-coefficients (in char $p > 0$), etc. Despite this, they are not representable in the classical motivic theory and hence cannot be studied by motivic methods. This is in part because the $\mathbb{A}^1$-invariant theory only detects log poles, regular singularities and tame ramifications. In order to study more general theories, we need a more general theory than the classical theory provided by Voevodsky.

One approach was recently introduced by Binda, Park, and Østvær in [5]. The basic idea is to generalize the classical motivic homotopy theory by replacing $\mathbb{A}^1$ with the "cube"

$$\overline{\square} := (\mathbb{P}^1, \infty).$$

It is the log scheme whose underlying scheme is $\mathbb{P}^1$ and whose log structure is induced by the inclusion of the divisor $\infty \hookrightarrow \mathbb{P}^1$. Working with log smooth log schemes and a suitable topology, the authors construct in loc. cit. the triangulated category $\mathrm{logDM}^{\mathrm{eff}}(k)$ of effective logarithmic motives. A cohomology theory representable in $\mathrm{logDM}^{\mathrm{eff}}(k)$ has the nice properties listed above (at least under the assumption of the existence of resolutions of singularities). An example of such a theory is the sheaf of log-Kähler differentials which becomes representable in

[1]The lecture series can be found on Youtube at http://tinyurl.com/reciprocity-sheaves-lectures.

this new triangulated category. Embedding the classical triangulated category of motives $\mathrm{DM}^{\mathrm{eff}}(k)$ fully faithfully in $\log\mathrm{DM}^{\mathrm{eff}}(k)$, they construct an enlargement of the classical $\mathbb{A}^1$-invariant theory. So far there is however no pole order or ramification filtration on the sheaves in this category.

The theory of reciprocity sheaves [25; 28] provides another solution: The basic idea, which goes back to Kahn in the 1990s, is to consider only those sheaves whose sections behave in a controlled way at infinity, i.e., replace $\mathbb{A}^1$-invariance by a *modulus condition*. This is a similar condition to the one considered by Rosenlicht and Serre to define the generalized Jacobian for curves.

Modulus á la Rosenlicht and Serre. The definition of a modulus condition goes back to Rosenlicht [39] and Serre [49, III]. They considered the modulus of a rational map from a curve to a commutative algebraic group.

Definition 1.1. Let k be a perfect field, C a smooth projective curve over k with an effective divisor D, $U := C \setminus D$ the complement of D, and G a smooth commutative k-group. Then a k-morphism $a : U \to G$ has *modulus* D if

$$\sum_{x \in U} v_x(f) \cdot \mathrm{Tr}_{x/k}(a(x)) = 0,$$

for all $f \in k(C)^\times$ with $f \equiv 1 \bmod D$, i.e., $f \in \bigcap_{x \in D} \mathrm{Ker}(\mathcal{O}_{X,x}^\times \to \mathcal{O}_{D,x}^\times)$, where v_x denotes the discrete valuation defined by the point x and $\mathrm{Tr}_{x/k} : G(x) \to G(k)$ is the trace;[2] see, for example, [51, Exp XVII, Appendice 2].

The choice of a rational point $x \in U(k)$ gives a universal morphism called the Albanese map

$$\mathrm{alb}_{(C,D)} : U \to \mathrm{Alb}(C, D),$$

with the property that any map $a : U \to G$ to a smooth commutative group scheme G, which satisfies the modulus condition, factors via $\mathrm{alb}_{(C,D)}$.

Let Sm denote the category of smooth separated k-schemes of finite type, in the following simply called smooth k-schemes. For $X, Y \in \mathrm{Sm}$ denote by $\mathbf{Cor}(X, Y)$ the group of finite correspondences from X to Y as introduced by Suslin and Voevodsky, i.e., it is the free abelian group generated by integral closed subschemes of $X \times Y$, which are finite and surjective over a connected component of X. There is a category of finite correspondences **Cor** whose objects are the objects of Sm and with morphisms the finite correspondences. A *presheaf with transfers* is an additive contravariant functor from **Cor** to the category of abelian groups. The category of presheaves with transfers is denoted by **PST**.

[2] If $G = \mathbb{G}_a$ is the additive group, then this is the usual trace. If $G = \mathbb{G}_m$ is the multiplicative group, then this is the norm.

Using the trace construction alluded to above, any smooth commutative k-group admits a structure of a presheaf with transfers by means of which the modulus condition may be reformulated as follows:

Definition 1.2. An element $a \in G(U)$ has *modulus* D if

$$\gamma^* a = 0,$$

for all prime correspondences $\Gamma \in \mathbf{Cor}(\mathbb{P}^1 \setminus \{1\}, U)$ such that

$$\{1\}|_{\Gamma^N} \geq D|_{\Gamma^N},$$

where $\Gamma^N \to \mathbb{P}^1 \times C$ is the normalization of the closure of Γ, and

$$\gamma := i_0^* \Gamma - i_\infty^* \Gamma \in \mathbf{Cor}(\operatorname{Spec} k, U).$$

Modulus pairs. The framework of modulus presheaves with transfers was introduced by Kahn, Miyazaki, S. Saito, and Yamazaki in [26; 27], as a fundamental tool in the construction of their triangulated category of motives with modulus.

Definition 1.3 [26]. Fix a perfect field k. A *modulus pair* $\mathcal{X}$ is a pair (X, D), where X is a separated scheme of finite type over k and D is an effective Cartier divisor (or the empty scheme) on X such that the complement of the support of D in X is smooth. A modulus pair (X, D) is *proper* if X is proper over $\operatorname{Spec} k$.

The group of *modulus correspondences* from (X, D) to (Y, E), denoted by $\underline{\mathbf{M}}\mathbf{Cor}((X, D), (Y, E))$, is the subgroup of $\mathbf{Cor}(X \setminus D, Y \setminus E)$ generated by finite prime correspondences

$$V \subset (X \setminus D) \times (Y \setminus E)$$

such that

(i) the projection $V^N \to X$ is proper,

(ii) $D|_{V^N} \geq E|_{V^N}$,

where $V^N \to X \times Y$ is the normalization of the closure of V in $X \times Y$. An element of

$$\underline{\mathbf{M}}\mathbf{Cor}((X, D), (Y, E))$$

is called a *modulus correspondence* from (X, D) to (Y, E).

The composition of finite correspondences restricts to a composition of modulus correspondences.[3] Hence we can define the category $\underline{\mathbf{M}}\mathbf{Cor}$ as the *category of modulus pairs*, i.e., objects are modulus pairs and morphisms are modulus correspondences.

[3] We remark that condition (i) is essential for this; see [26, Proposition 1.2.3].

The category $\underline{\mathbf{M}}\mathbf{Cor}$ has a monoidal structure given by

$$(X, D) \otimes (Y, E) = (X \times Y, p_X^* D + p_Y^* E),$$

where p_X (resp. p_Y) denotes the projection onto the first (resp. second) factor.

Let $\underline{\mathbf{M}}\mathbf{PST}$ denote the category of presheaves of abelian groups on $\underline{\mathbf{M}}\mathbf{Cor}$. It comes with a monoidal structure $\otimes_{\underline{\mathbf{M}}\mathbf{PST}}$ which via the Yoneda embedding extends the one on $\underline{\mathbf{M}}\mathbf{Cor}$. There is an adjoint functor pair

$$\underline{\omega}_! : \underline{\mathbf{M}}\mathbf{PST} \rightleftarrows \mathbf{PST} : \underline{\omega}^*$$

such that $\underline{\omega}_! G(X) = G(X, \varnothing)$ and $\underline{\omega}^* F(X, D) = F(X \setminus D)$.

The modulus presheaf represented by a modulus pair (X, D) in $\underline{\mathbf{M}}\mathbf{PST}$ is denoted by

$$\mathbb{Z}_{\mathrm{tr}}(X, D) := \underline{\mathbf{M}}\mathbf{Cor}(-, (X, D)).$$

Let $\overline{\square}$ denote the modulus pair $(\mathbb{P}^1, \infty)$, and set

$$h_0^{\overline{\square}}(X, D) := \mathrm{Coker}(\mathbb{Z}_{\mathrm{tr}}(X, D)(- \otimes \overline{\square}) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}_{\mathrm{tr}}(X, D)),$$

which we can consider as the *cubical modulus* version of h_0 of the Suslin complex.

Remark 1.4. Note that we have surjections

$$\mathbb{Z}_{\mathrm{tr}}(X \setminus D) \twoheadrightarrow \underline{\omega}_! h_0^{\overline{\square}}(X, D) \twoheadrightarrow h_0^{\mathbb{A}^1}(X \setminus D),$$

where $\mathbb{Z}_{\mathrm{tr}}(X \setminus D) = \mathbf{Cor}(-, X \setminus D)$ and

$$h_0^{\mathbb{A}^1}(X \setminus D) = \mathrm{Coker}(\mathbb{Z}_{\mathrm{tr}}(X \setminus D)(- \otimes \mathbb{A}^1) \xrightarrow{i_0^* - i_1^*} Z_{\mathrm{tr}}(X \setminus D))$$

is its maximal $\mathbb{A}^1$-invariant quotient.

Indeed, the surjectivity of the first map follows from the right exactness of $\underline{\omega}_!$ and the equality

$$\underline{\omega}_! \mathbb{Z}_{\mathrm{tr}}(X, D) = \mathbb{Z}_{\mathrm{tr}}(X \setminus D).$$

To see this, observe if $V \in \mathbf{Cor}(S, X \setminus D)$ is a finite prime correspondence, then V is already closed in $S \times X$ since it is finite over S. Hence $V \in \underline{\mathbf{M}}\mathbf{Cor}((S, \varnothing), (X, D))$. The second surjection follows directly from the fact that $\mathbb{Z}_{\mathrm{tr}}(X, D)(- \otimes \overline{\square})$ is a subpresheaf of $\mathbb{Z}_{\mathrm{tr}}(X \setminus D)(- \otimes \mathbb{A}^1)$.

We remark that by the above we have, for $S \in \mathrm{Sm}$,

$$\underline{\omega}_! h_0^{\overline{\square}}(X, D)(S) = \mathrm{Coker}(\underline{\mathbf{M}}\mathbf{Cor}((S, \varnothing) \otimes \overline{\square}, (X, D)) \xrightarrow{i_0^* - i_1^*} \mathbf{Cor}(S, X \setminus D)).$$

Definition 1.5 [28, Definition 2.2.4]. Let (X, D) be a proper modulus pair with $U := X \setminus D$, F a presheaf with transfers, and $a \in F(U)$ a section. We say that a has *modulus* (X, D) if the Yoneda map defined by a factors through $\underline{\omega}_! h_0^{\overline{\square}}(X, D)$, that is, there exists a map that makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{Z}_{\mathrm{tr}}(U) & \xrightarrow[\text{Yoneda}]{a} & F \\ & \searrow\!\!\!\!\searrow \qquad \quad \nearrow & \\ & \underline{\omega}_! h_0^{\overline{\square}}(X, D). & \end{array}$$

Remark 1.6. (1) In [28] the pair (X, D) above is called an SC-modulus of a in order to distinguish it from a slightly different notion of modulus which was introduced before in [25]. In [28, Theorem 3.2.1], it is proven that the two notions of modulus coincide as long as $X \setminus D$ is quasiaffine. In the following we will only work with the above definition of modulus and therefore simply say *modulus* instead of *SC-modulus*.

(2) If $F = G$ is a smooth commutative k-group and X is a smooth projective curve, then evaluating the diagram above at k gives back the definition of modulus introduced by Rosenlicht and Serre which was reformulated in Definition 1.2.

Reciprocity sheaves. We are now in the position to define reciprocity sheaves.

Definition 1.7 [28, Definition 2.2.4]. We say that a presheaf with transfers F is a *reciprocity presheaf* if for any smooth k-scheme U, and for all $a \in F(U)$, there exists a proper modulus pair (X, D) such that $U = X \setminus D$ and a has modulus (X, D).

Let **RSC** denote the full subcategory of **PST** consisting of reciprocity presheaves. The category of *reciprocity sheaves* is $\mathbf{RSC}_{\mathrm{Nis}} := \mathbf{RSC} \cap \mathbf{NST}$, where **NST** denotes the subcategory of **PST** consisting of presheaves with transfers which are Nisnevich sheaves on Sm.

Remark 1.8. (1) In loc. cit. the term *presheaves with transfers with SC-reciprocity* is used for what above is called reciprocity presheaf in order to distinguish them from the reciprocity presheaves introduced in [25]. This difference is however not relevant for us since we mostly work with Nisnevich sheaves with transfers, for which the two notions coincide; see [28, Corollary 3.2.3].

(2) A precursor of reciprocity sheaves are the reciprocity functors defined in [24]. These are defined only on function fields and regular curves over such. It follows from the injectivity theorem [25, Theorem 6] that the restriction of any reciprocity sheaf in the sense of Definition 1.7 to fields and regular curves defines a reciprocity functor; see [45, Theorem 5.7].

Example 1.9. Smooth commutative k-groups and homotopy invariant Nisnevich sheaves provide important examples of reciprocity sheaves. For the homotopy invariant sheaves this follows directly from Remark 1.4; for the smooth commutative groups the argument is essentially given by Rosenlicht and Serre;[4] see [49, Chapter III, Theorem 1].

[4] The argument of Rosenlicht and Serre works for curves over an algebraically closed field; see [25, Theorem 4.1.1] for an extension of the argument to the general case.

Proposition 1.10. *The* (*absolute*) *Kähler differentials are reciprocity sheaves*, *i.e.*,

$$\Omega^j \in \mathbf{RSC}_{\mathrm{Nis}} \quad \textit{for all } \ j \geq 0.$$

Proof. As a corollary of [11, Theorem 3.1.8] Kähler differentials have an action of finite transfers. It remains to show that they satisfy the modulus condition in Definition 1.5.

For any form $a \in \Omega^j(U)$ there is a proper modulus pair (X, D) such that $a \in H^0(X, \Omega^j_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D))$. We claim that $(X, 2D)$ is a modulus for a in the sense of Definition 1.5. For an integral smooth k-scheme S the restriction $\Omega^j_S \to \Omega^j_{k(S)}$ is injective, hence we reduce to show the following: If C is a regular projective curve C over a k-function field K which comes with a map to X, such that its image is not contained in D, and $f \in K(C)$ satisfies the condition $f \equiv 1 \bmod 2D_C$, where D_C denotes the pullback of D to C, then we have to show

$$\mathrm{div}_C(f)^* a = 0. \tag{1-1}$$

To this end, observe that the modulus condition for f and the choice of (X, D) imply

$$\mathrm{Res}_x(a_{|C}\, \mathrm{dlog}(f)) = 0,$$

for all $x \in D_C$, where $\mathrm{Res}_x : \Omega^{j+1}_{K(C)} \to \Omega^j_K$ denotes the residue symbol. Since the pullback of a to $C \setminus D_C$ is regular, we have

$$\mathrm{Res}_x(a_{|C}\, \mathrm{dlog}(f)) = v_x(f)\, \mathrm{tr}_{K(x)/K}(a(x)) \quad \text{for all } x \in C \setminus D_C,$$

with $\mathrm{tr}_{K(x)/K} : \Omega^j_{K(x)} \to \Omega^j_K$ the trace. Thus the reciprocity law yields

$$0 = \sum_{x \in C} \mathrm{Res}_x(a\, \mathrm{dlog}(f)) = \sum_{x \in C \setminus D_C} v_x(f)\, \mathrm{tr}_{K(x)/K}(a(x)),$$

which is a reformulation of (1-1). □

Definition 1.11. For a reciprocity presheaf $F \in \mathbf{RSC}$ we form the *modulus presheaf* $\widetilde{F}$ by defining

$$\widetilde{F}(X, D) := \{a \in F(X \setminus D) : a \text{ has modulus } (\overline{X}, \overline{D} + N \cdot B), \text{ for some } N \gg 0\},$$

where $(\overline{X}, \overline{D} + B)$ is a compactification of (X, D), in the sense of [26, Definition 1.8.1], i.e., it is a proper modulus pair with $X = \overline{X} \setminus B$ and $D = \overline{D}_{|X}$. This definition is independent of the choice of the compactification, by, e.g., [47, Remark 1.5].

We have $\widetilde{F} \in \underline{\mathbf{M}}\mathbf{PST}$ and it satisfies:

- $\overline{\square}$-*invariance*: $\widetilde{F}(\mathcal{X} \otimes \overline{\square}) \cong \widetilde{F}(\mathcal{X})$.
- M-*reciprocity*: $\widetilde{F}(X, D) = \varinjlim_N \widetilde{F}(\overline{X}, \overline{D} + N \cdot B)$.
- *Semipurity*: $\widetilde{F}(X, D) \subset \widetilde{F}(X \setminus D, \varnothing)$.

In fact, $\widetilde{F}$ is a $\overline{\square}$-invariant modulus presheaf by [28, Propositions 2.3.7 and 2.4.1], and the other two properties follow directly from the definition.

Definition 1.12. We define $\mathbf{CI}^{\tau}$ as the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}$ consisting of presheaves satisfying cube-invariance and M-reciprocity. We let $\mathbf{CI}^{\tau,\mathrm{sp}}$ denote the full subcategory of $\mathbf{CI}^{\tau}$ consisting of semipure presheaves.

This gives an adjoint functor pair (see [28, Proposition 2.3.7])

$$\underline{\omega}_! : \mathbf{CI}^{\tau,\mathrm{sp}} \rightleftarrows \mathbf{RSC} : \underline{\omega}^{\mathbf{CI}},$$

where $\underline{\omega}^{\mathbf{CI}}(F) = \widetilde{F} \in \mathbf{CI}^{\tau,\mathrm{sp}}$. For a proper modulus pair $\mathcal{X}$ we obtain $\underline{\omega}_! h_0^{\overline{\square}}(\mathcal{X}) \in \mathbf{RSC}$; see [28, Corollary 2.3.5].

Modulus sheaves.

Definition 1.13. A presheaf $G \in \underline{\mathbf{M}}\mathbf{PST}$ is a *modulus sheaf* if for all modulus pairs $\mathcal{X} = (X, D)$ the presheaf on the category of étale X-schemes

$$(U \xrightarrow{\text{ét}} X) \mapsto G(U, D_{|U}) =: G_{\mathcal{X}}(U)$$

is a Nisnevich sheaf.

We let $\underline{\mathbf{M}}\mathbf{NST}$ denote the category of Nisnevich modulus sheaves. Note that $\underline{\omega}_!$ restricts to a functor $\underline{\mathbf{M}}\mathbf{NST} \to \mathbf{NST}$ which we also denote by $\underline{\omega}_!$.

Remark 1.14. Note that $\underline{\mathbf{M}}\mathbf{NST}$ is not the category of sheaves on a site whose underlying category is $\underline{\mathbf{M}}\mathbf{Cor}$. However, there is a subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ which underlies a site associated to a regular and complete cd-structure, such that $G \in \underline{\mathbf{M}}\mathbf{PST}$ is a sheaf in the above sense if and only if the restriction of G is a sheaf on this site; see [26, Proposition 3.2.3].

Theorem 1.15 [26; 27]. *The natural inclusion $\underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{PST}$ admits an exact left adjoint, the so-called sheafification,*

$$\underline{a}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{PST} \to \underline{\mathbf{M}}\mathbf{NST},$$

which sends presheaves with M-reciprocity to sheaves with M-reciprocity.

The existence of the sheafification functor is proven in [26]. It follows from [27, Theorem 2] that it is compatible with M-reciprocity. By [26, Theorem 2] the sheafifcation functor is determined by the formula

$$\underline{a}_{\mathrm{Nis}}(G)_{(X,D)} = \varinjlim f_*(G_{(Y,f^*D),\mathrm{Nis}}),$$

where the colimit is taken over a directed set of proper morphisms $f : Y \to X$ which induce an isomorphism on the complement $Y \setminus f^*D \cong X \setminus D$ and the index Nis on the right denotes the Nisnevich sheafification on the category of étale Y-schemes. Thus,

$$\underline{\omega}_!(\underline{a}_{\mathrm{Nis}}(G)) = (\underline{\omega}_! G)_{\mathrm{Nis}}, \tag{1-2}$$

where the index Nis on the right denotes the Nisnevich sheafification on Sm, which by a result of Voevodsky restricts to a functor **PST** → **NST**. Also,

$$\mathrm{Ext}^i_{\underline{\mathbf{M}}\mathbf{NST}}(\mathbb{Z}_{\mathrm{tr}}(X,D),G)=\varinjlim H^i(Y_{\mathrm{Nis}},G_{(Y,f^*D)}). \tag{1-3}$$

This leads to the following question [26, Question 1]:

Question 1.16. Does (1-3) stabilize for $G\in\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}:=\mathbf{CI}^{\tau,\mathrm{sp}}\cap\underline{\mathbf{M}}\mathbf{NST}$?[5]

A fundamental result by S. Saito generalizes Voevodsky's strict homotopy invariance theorem by proving that Nisnevich sheafification preserves $\overline{\square}$-invariance. The proof requires in particular a delicate extension of Voevodsky's theory of standard triples (see [55, Chapters 3 and 4]) to the setup of modulus pairs.

Theorem 1.17 [47, Theorem 10.1]. *We have*

$$\underline{a}_{\mathrm{Nis}}(\mathbf{CI}^{\tau,\mathrm{sp}})\subset\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}.$$

Corollary 1.18. *For every* $F\in\mathbf{RSC}$, *the Nisnevich sheafification* F_{Nis} *belongs to* $\mathbf{RSC}_{\mathrm{Nis}}$. *In particular*, $\mathbf{RSC}_{\mathrm{Nis}}\subset\mathbf{NST}$ *is a full abelian subcategory.*

Proof. By Theorem 1.17 we have

$$G:=\underline{a}_{\mathrm{Nis}}(\widetilde{F})\in\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}=\mathbf{CI}^{\tau,sp}\cap\underline{\mathbf{M}}\mathbf{NST}.$$

Together with (1-2) we get

$$F_{\mathrm{Nis}}=\underline{\omega}_!(G)\in\mathbf{RSC}\cap\mathbf{NST}=\mathbf{RSC}_{\mathrm{Nis}}. \qquad\square$$

The following purity theorem by S. Saito generalizes the $\mathbb{A}^1$-invariant purity theorem by Voevodsky and will be essential in Section 5.

Theorem 1.19 [47, Theorem 0.2]. *For* $F\in\mathbf{RSC}_{\mathrm{Nis}}$ *and* $x\in X^{(c)}$ *a* c*-codimensional point in* X *we have*

$$H^i_x(X,F)=0\quad\textit{for } i\neq c,$$

and

$$H^c_x(X,F)\simeq F_{-c}(X):=\frac{F((\mathbb{A}^1\setminus\{0\})^c\times x)}{\sum_{i=1}^n F\big((\mathbb{A}^1\setminus\{0\})^{i-1}\times\mathbb{A}^1\times(\mathbb{A}^1\setminus\{0\})^{c-i}\times x\big)}. \tag{1-4}$$

Remark 1.20. The isomorphism in (1-4) depends on the choice of a k-isomorphism

$$k(x)\{t_1,\dots,t_c\}\xrightarrow{\simeq}\mathcal{O}^h_{X,x},$$

[5] It is shown in [40, no. 6.9] that this question has a negative answer if the base field has positive characteristic p, the divisor of the modulus pair (X,D) has a component of multiplicity divisible by p, and $G=\widetilde{\Omega^q}$ with $q\neq 0$ and $q\neq\dim X$. But it remains an interesting question for which G and (X,D) one has a positive answer. Assuming resolutions of singularity this is, for example, the case if $q=\dim X$; see [40, Corollary 7.5].

where $\mathcal{O}^h_{X,x}$ denotes the henselization of $\mathcal{O}_{X,x}$. In particular, it is not functorial (as one directly sees by considering the case $F = \mathbb{G}_a$ and $c = 1$). Note that if F is $\mathbb{A}^1$-invariant the isomorphism is independent of this choice; see [55, Chapter 3, Lemma 4.36].

Using Theorem 1.19 we find that the E_1-complex of the coniveau spectral sequence has vanishing cohomology except in degree zero, hence we obtain the *Cousin resolution*

$$0 \to F_X \to \bigoplus_{x\in X^{(0)}} i_{x*}H^0_x(F) \to \cdots \to \bigoplus_{x\in X^{(c)}} i_{x*}H^c_x(F) \to \cdots \tag{1-5}$$

on X_{Nis} as a generalization of the Gersten resolution in the $\mathbb{A}^1$-invariant case. The injectivity of the first morphism was already proven in [25].

Relation with the logarithmic theory. Further results by S. Saito give the relation with the logarithmic theory of Binda, Park, and Østvær.

Definition 1.21. We say that $\mathcal{X} = (X, D)$ is an *ls modulus pair* if $X \in \mathrm{Sm}$ and D_{red} is a strict normal crossing divisor on X. (Note that D is allowed to be nonreduced.) Let $\underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ denote the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ of ls modulus pairs.

A morphism $f : Y \to X$ of smooth schemes is *transversal* to D if

$$f^{-1}(D_1 \cap \cdots \cap D_r) \hookrightarrow Y$$

is a regular closed embedding of codimension equal to r, for any irreducible components $D_1, \ldots, D_r$ of D_{red}.

Definition 1.22. Let $(X, \mathcal{M})$ be a smooth log smooth scheme, where $\mathcal{M}$ is a monoid sheaf with a multiplicative map $\mathcal{M} \to \mathcal{O}_X$, which is an isomorphism over $\mathcal{O}^\times_X$, defining the log structure. By definition $\operatorname{supp}\mathcal{M}$ denotes the support of the monoid sheaf $\mathcal{M}/\mathcal{O}^\times_X$. We have $(X, \operatorname{supp}\mathcal{M}) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ (see, for example, [5, Lemma A.5.10]) and define, for $F \in \mathbf{RSC}_{\mathrm{Nis}}$,

$$F^{\log}(X, \mathcal{M}) := \widetilde{F}(X, \operatorname{supp}\mathcal{M}).$$

Theorem 1.23 [48, Theorems 6.1 and 6.3]. *Let $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$ denote the category of dividing Nisnevich sheaves with log-transfers on log smooth fs log schemes, in the sense of* [5, Definition 4.2.1]. *Then there exists a functor*

$$\mathbf{RSC}_{\mathrm{Nis}} \longrightarrow \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}$$

sending $F \mapsto F^{\log}$, which is exact and fully faithful. Also, $F^{\log}$ is strictly $\overline{\square}$-invariant, that is, for smooth log smooth schemes $(X, \mathcal{M})$ we have

$$H^i_{\mathrm{dNis}}((X, \mathcal{M}), F^{\log}) = H^i_{\mathrm{dNis}}((X, \mathcal{M}) \times \overline{\square}, F^{\log})$$

and the Nisnevich cohomology of $F_{(X,\mathrm{supp}\,\mathcal{M})}$ (*with the notation from Definition 1.13*) *is representable in the triangulated category of logarithmic motives* $\mathrm{logDM}^{\mathrm{eff}}(k)$ *constructed in* [5]:

$$H^i(X, F_{(X,\mathrm{supp}\,\mathcal{M})}) \cong \mathrm{Hom}_{\mathrm{logDM}^{\mathrm{eff}}(k)}(M(X,\mathcal{M}), F^{\log}[i]).$$

2. De Rham–Witt sheaves as reciprocity sheaves

We give a short introduction to the de Rham–Witt sheaves introduced by Bloch [7] and Illusie [22], describe some basic properties, define transfers, and show that they are reciprocity sheaves. From this one obtains many interesting reciprocity sheaves, which are useful, for example, in the study of crystalline cohomology, the Brauer group, and étale motivic cohomology with p-primary torsion coefficients.

Motivation. Let X be a smooth projective scheme over $\mathbb{F}_{p^n}$ and set

$$H^i := H^i_{\mathrm{crys}}(X/W(\mathbb{F}_{p^n}))\Big[\frac{1}{p}\Big],$$

the i-th crystalline cohomology group, where $W(\mathbb{F}_{p^n})$ denotes the Witt vectors of $\mathbb{F}_{p^n}$. Then H^i is a finite-dimensional vector space over the field $W(\mathbb{F}_{p^n})[1/p]$, and considering the action of Frobenius on it, it becomes a $F := (F_X^n)^*$-crystal. Such crystals have a slope decomposition

$$H^i \cong \bigoplus H^i_\lambda,$$

where λ ranges through the nonnegative rational numbers. Here H^i_λ is a subvector space on which the Frobenius acts with eigenvalues having p-adic valuation equal to λ.

One of the main motivations behind the construction of the de Rham–Witt complex is the wish to understand this slope decomposition from a cohomological point of view. And indeed, the de Rham–Witt complex $W\Omega^*$ computes crystalline cohomology by

$$H^i = H^i(X_{\mathrm{Zar}}, W\Omega^*)\Big[\frac{1}{p}\Big].$$

The Hodge-to-de Rham spectral sequence yields the *slope spectral sequence* given by

$$E_1^{j,i} = H^i(X, W\Omega^j)\Big[\frac{1}{p}\Big] \implies H^*,$$

which degenerates to give

$$\bigoplus_{j\le\lambda<j+1} H^i_\lambda = H^{i-j}(X, W\Omega^j)\Big[\frac{1}{p}\Big].$$

Witt vectors. Let A be an $\mathbb{F}_p$-algebra. We recall that, for $n \geq 1$, the *Witt vectors* of length n of A form a ring whose underlying set is equal to

$$W_n(A) = \{(a_0, \ldots, a_{n-1}) : a_i \in A\},$$

and whose ring structure is defined in such a way that the following properties hold: The map

$$R : W_{n+1}(A) \to W_n(A), \quad (a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_{n-1}),$$

is a ring-map, called the *restriction*. The map

$$F : W_{n+1}(A) \to W_n(A), \quad (a_0, \ldots, a_n) \mapsto (a_0^p, \ldots, a_{n-1}^p),$$

is a ring-map, called the *Frobenius*. The map

$$V : W_n(A) \to W_{n+1}(A), \quad (a_0, \ldots, a_{n-1}) \mapsto (0, a_0, \ldots, a_{n-1}),$$

is a group-map, called the *Verschiebung* (or shift). The map

$$[-] : A \to W_n(A), \quad a \mapsto [a] := (a, 0, \ldots, 0),$$

is multiplicative and is called the Teichmüller lift. Also,

- $W_1(A) = A$ as a ring,
- $(a_0, \ldots, a_{n-1}) = \sum_{i=0}^{n-1} V^i([a_i])$, where $V^i = \underbrace{V \circ \ldots \circ V}_{i-\text{times}}$,
- $FV = VF$ is multiplication by p,
- $V(a) \cdot b = V(a \cdot F(b))$.

Passing to the limit

$$W(A) := \varprojlim_n W_n(A),$$

where the transition maps are given by the restriction, we get a ring which is p-torsion free if A is reduced. For details, see, e.g., [50, Chapter II, §6].

Example 2.1. The Witt vectors have the following properties:

- $W(\mathbb{F}_p) = \varprojlim_n W_n(\mathbb{F}_p) = \mathbb{Z}_p$.
- If A is perfect, that is, the Frobenius is an isomorphism, then $W_n(A)$ is the unique flat $\mathbb{Z}/p^n\mathbb{Z}$-lift of $A/\mathbb{F}_p$.
- The contravariant functor

$$(\text{Schemes}/\mathbb{F}_p)^o \to (\text{Ab-groups}), \quad X \mapsto H^0(X, W_n\mathcal{O}_X),$$

is represented by a ring scheme W_n. Thus the ring $W_n(A)$ is equal to the A-rational points of W_n.

• Any commutative unipotent $\mathbb{F}_p$-group scheme can be embedded into $\bigoplus_{n_i} W_{n_i}$, viewed as a group scheme.

The de Rham–Witt complex. The *de Rham–Witt complex* of an $\mathbb{F}_p$-scheme X, as defined by Bloch [7], Illusie [22], or Kato [29] is a pro-differential graded algebra

$$((W_n\Omega^*, d)_{n\geq 1}, R),$$

where d is the differential and R is the restriction map, such that

$$W_n\Omega^0 = W_n\mathcal{O}_X.$$

It is equipped with an extension of the Frobenius map

$$F : W_{\bullet+1}\Omega^*_X \to W_\bullet\Omega^*_X,$$

and an extension of the Verschiebung map on $W_\bullet\mathcal{O}_X$

$$V : W_\bullet\Omega^*_X \to W_{\bullet+1}\Omega^*_X,$$

which satisfy the following conditions:

- F is a map of graded rings, and V is a map of graded groups.
- The composition of maps FV is given by multiplication by p.
- F-linearity: $V(\alpha)\cdot\beta = V(\alpha\cdot F(\beta))$.
- $FdV = d$.
- $Fd[a] = [a]^{p-1}d[a]$, for $a \in W_\bullet\mathcal{O}_X$.

In fact, in [21] Hesselholt and Madsen show that $W_\bullet\Omega^*_X$ is the initial object in the category of pro-differentially graded algebras with the above properties. They also extend the definition to all $\mathbb{Z}_{(p)}$-algebras.

We have $W_1\Omega^*_X = \Omega^*_{X/\mathbb{F}_p}$, and there is a commutative square

$$\begin{array}{ccc} W_{n+1}\Omega^j_X & \xrightarrow{F} & W_n\Omega^j_X \\ \downarrow & & \downarrow \\ \Omega^j_X & \xrightarrow{C^{-1}} & \Omega^j_X/d\Omega^{j-1}_X, \end{array}$$

i.e., F lifts the inverse Cartier operator C^{-1}, which is determined by the formula $C^{-1}(ad\log b) = a^p d\log b$.

Remark 2.2. (1) For an $\mathbb{F}_p$-algebra A, Bloch constructed the de Rham–Witt complex in [7] as the pro-object

$$W_\bullet\Omega^q_A \cong TS\ker\big(K_{q+1}(A[T]/T^\bullet) \xrightarrow{T\mapsto 0} K_{q+1}(A)\big),$$

where T denotes the p-typical part and S the symbolic part of Quillen K-theory. Bloch's construction was originally limited to the case $\dim A < p$ and $p \neq 2$. This restriction was removed in [29].

(2) Following an idea of Deligne, Illusie constructed the de Rham–Witt complex in [22] as a quotient of $\Omega^q_{W_n\mathcal{O}_X/W_n(\mathbb{F}_p)}$, such that it is the universal example of a pro-dga with a Verschiebung V satisfying certain properties. Then he proves that on this complex an F as above exists.

(3) If X is a smooth scheme over k with a smooth lift over $X_n/W_n(k)$, then

$$W_n\Omega^q_X \cong \mathcal{H}^q(\Omega^*_{X_n/W_n(k)}),$$

and one can show that this isomorphism is independent of the lift; see [23, Chapter III, (1.5)]. As Illusie and Raynaud explain in loc. cit., it was observed by N. Katz that one can take the right-hand side of the above isomorphism as the definition of the de Rham–Witt sheaves (using local lifts of X over $W_n(k)$ and glue) and that it is possible to construct the structure of a pro-dga with maps F and V using this description.

(4) We mention that there are other constructions of the de Rham–Witt complex by Bhatt, Lurie, and Mathew [4], Cuntz and Deninger [16], Hesselholt [20], Hesselholt and Madsen [21], and Langer and Zink [36], each of which works in a different generality, but they all agree for smooth schemes over a perfect field of positive characteristic.

Theorem 2.3 [22, Chapter II, Theorem 1.4]. *Let X be a smooth scheme over a perfect field k of characteristic $p > 0$ and let $u : (X/W_n(k))_{\mathrm{crys}} \to X_{\mathrm{Zar}}$ be the change of sites map. Then there is an isomorphism*

$$Ru_*\mathcal{O}_{X/W_n(k),\mathrm{crys}} \cong W_n\Omega^\bullet_X.$$

We remark that Bloch proved in [7, Chapter III, Theorem (2.1)] such an isomorphism in the limit over n using his K-theoretic construction of the de Rham–Witt complex (as a pro-object) under the additional assumption that $\dim X < p$ and $p \neq 2$. These assumptions were later removed by Kato; see [29, p. 635, Remark 2].

Theorem 2.4 [17, Chapter I, Theorem 4.1 and Chapter II, Theorem 2.2]. *Let X be a smooth scheme over k and let*

$$\pi : W_nX = (|X|, W_n\mathcal{O}_X) \to \operatorname{Spec} W_n(k)$$

be the finite-type morphism of schemes induced by the structure map of X. Then

$$\pi^! W_n(k) \cong W_n\Omega^{\dim X}_X[\dim X],$$

where $\pi^!$ denotes the exceptional inverse image in the derived category of $\mathcal{O}$-modules from Grothendieck duality. There is a canonical isomorphism

$$W_n\Omega^j_X \xrightarrow{\simeq} R\mathcal{H}\mathrm{om}_{W_n\mathcal{O}_X}(W_n\Omega^{\dim X-j}_X, W_n\Omega^{\dim X}_X).$$

Using the above theorem, Gros constructed in [19] a pushforward

$$f_* : Rf_*W_n\Omega^j_Y \to W_n\Omega^{j-r}_X[-r],$$

for a proper morphism $f : Y \to X$ of relative dimension r between smooth schemes.

Proposition 2.5. *The de Rham–Witt sheaves are reciprocity sheaves*, *i.e.*,

$$W_n\Omega^j \in \mathbf{RSC}_{\mathrm{Nis}}$$

for all $j \geq 0$. The maps d, R, F, V are compatible with the transfer structure and hence are morphisms of reciprocity sheaves.

Proof. The finite transfers structure on $W_n\Omega^j$ and its compatibility with d, R, V, F is a consequence of [12, Theorem 3.4.6]. We recall the definition: for $Z \in \mathbf{Cor}(X, Y)$ the correspondence action is given by the composition

$$\begin{aligned} Z^* : W_n\Omega^j(Y) &\xrightarrow{p_Y^*} W_n\Omega^j(X \times Y) \\ &\xrightarrow{\cup \mathrm{cl}_Z} H^{\dim Y}_Z(X \times Y, W_n\Omega^{j+\dim Y}) \\ &\xrightarrow{p_{X*}} W_n\Omega^j(X), \end{aligned}$$

where cl_Z denotes the cycle class and p_{X*} denotes the pushforward with supports from [12, 2.3]. Note that since Y does not need to be proper, this pushforward only exists with support in the finite X-scheme Z. We also want to point out the compatibility of the pushforward, and hence the correspondence action with d, R, F, V, is not obvious and requires Ekedahl's careful analysis of the behavior of these maps under duality; see [17, Chapter III]. We find $W_n\Omega^j \in \mathbf{NST}$. It remains to show that any form $a \in W_n\Omega^j(X)$ has a modulus. This is similar to the case of Kähler differentials in the proof of Proposition 1.10; see [25, Theorem B.2.2] for details. □

Since $\mathbf{RSC}_{\mathrm{Nis}}$ is an abelian category (see Corollary 1.18), Proposition 2.5 gives us many more examples of reciprocity sheaves by taking kernels and quotients of the maps d, R, F, V. In particular we obtain:

(1) $W_n\Omega^* \in \mathrm{Comp}^+(\mathbf{RSC}_{\mathrm{Nis}})$ represents the complex of sheaves sending a smooth scheme X to $Ru_*\mathcal{O}_{X/W_n,\mathrm{crys}}$.

(2) We have

$$B_\infty W_n\Omega^j := \bigcup_{r\geq 0} F^r dW_{n+r}\Omega^{j-1} \in \mathbf{RSC}_{\mathrm{Nis}}\,. \tag{2-1}$$

(3) The *generalized* Artin–Schreier–Witt sequence is the exact sequence

$$0 \to W_n\Omega^j_{X,\log} \to W_n\Omega^j_X/B_\infty \xrightarrow{\bar{F}-1} W_n\Omega^j_X/B_\infty \to 0$$

on $X_{\text{ét}}$, where $W_n\Omega^j_{X,\log}$ is the subsheaf of $W_n\Omega^j_X$ which locally is generated by dlog-forms [22, Chapter I, 5.7], $B_\infty = B_\infty W_n\Omega^j$, and $\bar{F} : W_n\Omega^j_X/B_\infty \to W_n\Omega^j_X/B_\infty$ is induced by "lifting to level $n+1$ and applying $F : W_{n+1}\Omega^j_X \to W_n\Omega^j_X$" (this operation is well defined on the quotient modulo B_∞). The exactness of this sequence can be deduced from [15, §1, Lemma 2; 22, I, (3.21.1.1), (3.21.1.3) and Proposition 3.26 and its proof]. By a famous theorem of Geisser and Levine [18] we have

$$W_n\Omega^j_{X,\log}[-j] \cong \mathbb{Z}/p^n(j)_{X_{\text{ét}}},$$

where the right-hand side denotes the étale motivic complex of weight j with $\mathbb{Z}/p^n$-coefficients. Let $\epsilon : \mathrm{Sm}_{\text{ét}} \to \mathrm{Sm}_{\mathrm{Nis}}$ be the change of sites map. Since $W_n\Omega^j_X/B_\infty$ is a direct limit of sheaves which are successive extensions of coherent $\mathcal{O}$-modules, it is acyclic for $R\epsilon_*$. Thus we obtain

$$R\epsilon_*\mathbb{Z}/p^n(j) \cong (W_n\Omega^j/B_\infty \xrightarrow{\bar{F}-1} W_n\Omega^j/B_\infty)[-j] \in D^b(\mathbf{RSC}_{\mathrm{Nis}}). \tag{2-2}$$

(4) By a result of Voevodsky the prime-to-p part of $R^i\epsilon_*(\mathbb{Q}/\mathbb{Z}(j))$ is homotopy invariant, and combining this with the above yields

$$R^i\epsilon_*(\mathbb{Q}/\mathbb{Z}(j)) \in \mathbf{RSC}_{\mathrm{Nis}} \quad \text{for all } i, j.$$

By the above this is not homotopy invariant only for $i = j+1$. In particular, the Brauer group defines a reciprocity sheaf:

$$X \mapsto \mathrm{Br}(X) = H^0(X, R^2\epsilon_*(\mathbb{Q}/\mathbb{Z}(1))) \in \mathbf{RSC}_{\mathrm{Nis}}.$$

3. Computation of the modulus in examples

We give some computations of the modulus in certain examples. We will see that the modulus detects higher poles and ramifications, which is not captured by the classical $\mathbb{A}^1$-invariant theory.

We let L be a henselian discrete valuation field of geometric type over the perfect base field k, i.e.,

$$L = \mathrm{Frac}\, \mathcal{O}^h_{U,x},$$

where U is a smooth k-scheme and $x \in U^{(1)}$.

For $F \in \mathbf{RSC}_{\mathrm{Nis}}$ we let $F(L) := F(\mathrm{Spec}\, L)$ and

$$\widetilde{F}(\mathcal{O}_L, \mathfrak{m}^{-n}) := \widetilde{F}(\mathrm{Spec}\, \mathcal{O}_L, n \cdot \{\text{closed point}\}).$$

By [44, Theorem 4.15(4)] we obtain for any proper modulus pair (X, D) that $\widetilde{F}(X, D)$ equals

$$\{a \in F(X \setminus D) \mid \rho^* a \in \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-v_L(\rho^* D)}) \quad \forall L \ \forall \rho \in (X \setminus D)(L)\}. \tag{3-1}$$

In order to understand the modulus sheaf $\widetilde{F}$ we have to study the filtration

$$F(\mathcal{O}_L) \subset \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) \subset \cdots \subset \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) \subset \cdots \subset F(L)$$

for all L. For $\mathbb{A}^1$-invariant Nisnevich sheaves we have $\widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) = F(L)$. For a non-$\mathbb{A}^1$-invariant reciprocity sheaf this is an exhaustive increasing filtration, which for varying L is infinite, in the sense that there exists no natural number $n \geq 0$ such that $F(L)$ is equal to $\widetilde{F}(\mathcal{O}_L, \mathfrak{m}^{-n})$ for all L; see [45, Lemma 5.2].

Definition 3.1. The reciprocity sheaf F has *level* $n \geq 0$, if for any smooth k-scheme X and any $a \in F(\mathbb{A}^1 \times X)$ the following implication holds:

$$a_{\mathbb{A}^1_z} \in F(z) \subset F(\mathbb{A}^1_z) \text{ for all } z \in X_{(\leq n-1)} \implies a \in F(X) \subset F(\mathbb{A}^1 \times X),$$

where $a_{\mathbb{A}^1_z}$ denotes the restriction of a to $\mathbb{A}^1_z$ and $X_{(\leq n-1)}$ denotes the set of points in X whose closure has dimension $\leq n-1$.[6]

Clearly, $\mathbb{A}^1$-invariant sheaves have level 0. Any commutative algebraic group G over k has level 1 by [44, Theorem 5.2]. If F has level n it suffices to consider in (3-1) those L which have transcendence degree $\leq n$ over k. For example, if the level is $n = 1$, this can be interpreted as a *cut-by-curves criterion* for determining the modulus of an element $a \in F(U)$. If the level is $n = 2$ we have a *cut-by-surfaces criterion*, etc.

Differential forms and rank-1 connections.

Theorem 3.2 [40, Chapter 6; 44, Chapter 6]. *Let* $\operatorname{char} k = p \geq 0$ *and* $j \geq 1$. *Then the modulus sheaf* $\widetilde{\Omega^j_{/\mathbb{Z}}}$ *has level* $j+1$ *and*

$$\widetilde{\Omega^j_{/\mathbb{Z}}}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) = \begin{cases} \frac{1}{t^{n-1}} \cdot \Omega^j_{\mathcal{O}_L/\mathbb{Z}}(\log t) & \text{if } p = 0 \text{ or } (n, p) = 1, \\ \frac{1}{t^n} \cdot \Omega^j_{\mathcal{O}_L/\mathbb{Z}} & \text{if } p > 0 \text{ and } p \mid n, \end{cases}$$

where $t \in \mathfrak{m}_L$ *is a local parameter.*

If $p = 0$ *and* $\mathrm{Conn}^1(X)$ (*resp.* $\mathrm{Conn}^1_{\mathrm{int}}(X)$) *denotes the group of isomorphism classes of* (*resp. integrable*) *rank*-1 *connections on* X/k, *we have*:

- $\mathrm{Conn}^1 \in \mathbf{RSC}_{\mathrm{Nis}}$ *has level* 2 *and* $\mathrm{Conn}^1_{\mathrm{int}}(X) \in \mathbf{RSC}_{\mathrm{Nis}}$ *has level* 1.
- $\widetilde{\mathrm{Conn}^1_{\mathrm{int}}}(X, D)$ *is the group of isomorphism classes of integrable rank*-1 *connections on* $U = X \setminus D$ *whose nonlog irregularity*[7] *is bounded by* D.

[6] This is equivalent to the motivic conductor of F having level n in the language of [44].

[7] The nonlog irregularity of a rank-1 connection E on $\operatorname{Spec} L$ is zero if this connection extends to $\operatorname{Spec} \mathcal{O}_L$ and else is equal to $\operatorname{irr}(E) + 1$, where $\operatorname{irr}(E)$ denotes the usual irregularity.

- $h^0_{\mathbb{A}^1}(\mathrm{Conn}^1_{\mathrm{int}})(U)$ *is the group of isomorphism classes of regular singular rank-1 connections on* U, *where we denote by* $h^0_{\mathbb{A}^1}(F)$ *the maximal* $\mathbb{A}^1$*-invariant subsheaf of* F.

Using the above formulas and the birational invariance of $\widetilde{\Omega^j_{/\mathbb{Z}}}$ (see (1-3) with $i=0$) it is shown in [40, Corollary 7.3] (see also [42, Theorem 7.1]) that an integral normal Cohen–Macaulay scheme Y of dimension d and of finite type over k has *pseudorational singularities* if for each effective Cartier divisor R, such that $Y \setminus R$ is smooth, the sheaf $\widetilde{\Omega^d}_{(Y,R)}$ is $S2$, i.e., is completely determined by its stalks at the zero- and one-codimensional points of Y. Note that in view of (3-1), this condition can be rephrased as a condition on the local filtrations $\widetilde{\Omega^d}(\mathcal{O}_L, \mathfrak{m}^{-n})$ for various L mapping to Y.

Witt vectors and characters of the abelianized fundamental group. Let $\operatorname{char} k = p > 0$. In order to define the Albanese with modulus in higher dimension, Kato and Russell defined in [31] the filtration

$$\mathrm{fil}^F_r\, W_n(L) := \sum_{j\geq 0} F^j(\mathrm{fil}^{\log}_{r-1}\, W_n(L) + V^{n-s}(\mathrm{fil}^{\log}_r\, W_s(L))), \quad r \geq 0,$$

where $\mathrm{fil}^{\log}_r\, W_n(L) = \{(a_0, \dots, a_{n-1}) \mid p^{n-1-i} v_L(a_i) \geq -r \text{ for all } i\}$, and $s = \min\{n, \mathrm{ord}_p(r)\}$.

Theorem 3.3 [44, Theorem 7.20]. *The Witt sheaf* W_n *has level* 1, *and*

$$\widetilde{W_n}(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \mathrm{fil}^F_r\, W_n(L).$$

In particular,

$$\widetilde{\mathbb{G}_a}(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \begin{cases} \mathcal{O}_L, & r \leq 1, \\ \sum_j F^j\big(\frac{1}{t^{r-1}}\mathcal{O}_L\big), & (p,r)=1, \\ \sum_j F^j\big(\frac{1}{t^r}\mathcal{O}_L\big), & p \mid r. \end{cases}$$

Let $H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cts}}(G_L, \mathbb{Q}/\mathbb{Z})$, where G_L denotes the absolute Galois group of L. As a variant of the Brylinski–Kato filtration [10; 30] Matsuda introduced in [37], on $H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z})$, the filtration

$$\mathrm{fil}_r\, H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z}) := \bigoplus_{l\neq p} H^i_{\text{ét}}(L, \mathbb{Q}_l/\mathbb{Z}_l) \oplus \bigcup_n \mathrm{im}(\mathrm{fil}^F_r\, W_n(L) \to H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z})),$$

where the maps $\mathrm{fil}^F_r\, W_n(L) \to H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z})$ are induced by the isomorphism $H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z}) = \varinjlim_V W_n(L)/(F-1)W_n(L)$ stemming from Artin–Schreier–Witt sequence.[8] This filtration was originally introduced to generalize the Artin-conductor to the case of imperfect residue fields.

[8] Matsuda does not consider the F-saturated filtration, but note that the images of fil^F and fil in the quotient $W_n(L)/(F-1)$ coincide.

Theorem 3.4 [44, Theorem 8.10]. *Let* $\epsilon : \mathrm{Sm}_{\text{ét}} \to \mathrm{Sm}_{\mathrm{Nis}}$ *denote the change of sites. Then* $R^1\epsilon_*\mathbb{Q}/\mathbb{Z} \in \mathbf{RSC}_{\mathrm{Nis}}$ *has level* 1, *and*

$$\widetilde{R^1\epsilon_*(\mathbb{Q}/\mathbb{Z})}(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \mathrm{fil}_r\, H^1_{\text{ét}}(L, \mathbb{Q}/\mathbb{Z}).$$

Remark 3.5. (1) By work of A. Abbes and T. Saito [1] and Y. Yatagawa [57],

$$\mathrm{fil}_r\, H^1_{\text{ét}}(L, \mathbb{Q}/Z) = \mathrm{Hom}_{\mathrm{cts}}(G_L/G_L^{r+}, \mathbb{Q}/\mathbb{Z}),$$

where $\{G_L^j\}_{j\in\mathbb{Q}_{\geq 0}}$ is the Abbes–Saito ramification filtration of G_L and $G_L^{r+} = \bigcup_{s>r} G_L^s$.

(2) Similarly as above one can use (2-2) to determine $\widetilde{R^{j+1}\epsilon_*(\mathbb{Q}/\mathbb{Z}(j))}$ for $j \geq 1$. This is work in progress.

Torsors under finite group schemes in positive characteristic. Let $\mathrm{char}\, k = p > 0$, and let G be a finite commutative k-group scheme. We can write

$$G = G_{\mathrm{em}} \times G_{\mathrm{eu}} \times G_{\mathrm{im}} \times G_{\mathrm{iu}},$$

where G_{em} is an étale multiplicative group (e.g., $\mathbb{Z}/l$), G_{eu} is an étale unipotent group (e.g., $\mathbb{Z}/p$), G_{im} is an infinitesimal multiplicative group (e.g., μ_p), and G_{iu} is an infinitesimal unipotent group (e.g., α_p). Consider the presheaf on Sm

$$X \mapsto H^1(G)(X) := H^1_{\mathrm{fppf}}(X, G),$$

which classifies isomorphism classes of fppf-G-torsors over X.

Theorem 3.6 [44, Theorem 9.12]. • *The presheaf* $H^1(G)$ *belongs to* $\mathbf{RSC}_{\mathrm{Nis}}$ *and has level* 2, *except for the case when* $G_{\mathrm{iu}} = 0$, *in which case it has level* 1.

• $H^1(G_{\mathrm{em}} \times G_{\mathrm{im}}) \in \mathbf{HI}_{\mathrm{Nis}}$.

• *The map* $L \to H^1(\alpha_p)(L)$ *induced by the exact sequence of fppf-sheaves* $0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0$ *restricts to surjections*

$$\widetilde{\mathbb{G}_a}(\mathcal{O}_L, \mathfrak{m}_L^{-r}) \twoheadrightarrow \widetilde{H^1(\alpha_p)}(\mathcal{O}_L, \mathfrak{m}_L^{-r}), \quad r \geq 0.$$

4. Tensor products and twists

The Lax monoidal structure on $\mathbf{RSC}_{\mathbf{Nis}}$.

Definition 4.1. For two reciprocity sheaves F and G we define a new reciprocity sheaf by

$$(F, G)_{\mathbf{RSC}_{\mathrm{Nis}}} := \underline{\omega}_!(h_0^{\overline{\square}}(\widetilde{F} \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \widetilde{G}))_{\mathrm{Nis}} \in \mathbf{RSC}_{\mathrm{Nis}}\,.$$

It is not clear that this induces a monoidal structure since it is not clear that this construction is associative. Also, it is not clear that this is right exact since $\underline{\omega}^{\mathbf{CI}}$ is not right exact. However, it induces a lax monoidal structure; see [45, Corollary 4.18].

Theorem 4.2 [45]. (1) *Let $\mathbf{HI}_{\mathrm{Nis}}$ denote the category of $\mathbb{A}^1$-invariant Nisnevich sheaves with transfers. By Voevodsky it has a symmetric monoidal structure denoted by $\otimes_{\mathbf{HI}_{\mathrm{Nis}}}$. For $F, G \in \mathbf{HI}_{\mathrm{Nis}}$ we have*

$$(F, G)_{\mathbf{RSC}_{\mathrm{Nis}}} = F \otimes_{\mathbf{HI}_{\mathrm{Nis}}} G.$$

(2) *If* $\operatorname{char} k = 0$, *then*

$$(G, A)_{\mathbf{RSC}_{\mathrm{Nis}}} = 0,$$

for any commutative unipotent group G and abelian variety A.

(3) *If* $\operatorname{char} k = 0$, *then there are isomorphisms*

$$(\mathbb{G}_a, \mathbb{G}_m)_{\mathbf{RSC}_{\mathrm{Nis}}} \xrightarrow{\simeq} \Omega^1_{/\mathbb{Z}}, \quad \gamma \otimes a \otimes u \mapsto \gamma^*(p^*a \wedge d\log q^*u),$$

and if we denote by $I_{\Delta_X} \subset \mathcal{O}_{X\times_{\mathbb{Z}}X}$ the ideal sheaf of the diagonal, then

$$(\mathbb{G}_a, \mathbb{G}_a)_{\mathbf{RSC}_{\mathrm{Nis}}}(X) \xrightarrow{\simeq} H^0(X, \mathcal{O}_{X\times_{\mathbb{Z}}X}/I^2_{\Delta_X}), \quad \gamma \otimes a \otimes b \mapsto \gamma^*(p^*a \otimes q^*b),$$ [9]

where $\gamma \in \mathbf{Cor}(X, Y \times Z)$, $a \in \mathbb{G}_a(Y)$, $b \in \mathbb{G}_a(Z)$, $u \in \mathbb{G}_m(Z)$, p and q denote the projections from $Y \times Z$ to Y and $Y \times Z$ to Z, respectively, and the element $\gamma \otimes a \otimes u$ denotes the image of the corresponding element in $(\mathbb{G}_a \otimes_{\mathbf{PST}} \mathbb{G}_m)(X)$ under the natural map $\mathbb{G}_a \otimes_{\mathbf{PST}} \mathbb{G}_m \to (\mathbb{G}_a, \mathbb{G}_m)_{\mathbf{RSC}_{\mathrm{Nis}}}$, similarly with $\gamma \otimes a \otimes b$.

Definition 4.3. For G in $\mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ we define

$$G(n) := h_0^{\overline{\square}}(G \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \widetilde{K^M_n})^{\mathrm{sp}}_{\mathrm{Nis}} \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$$

and

$$\gamma^n G := \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\widetilde{K}^M_n, G) \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}},$$

where K^M_n denotes the (improved) Milnor K-sheaf (see [33]) and the upper sp in the first formula denotes the semipurification functor $\mathbf{CI}^{\tau}_{\mathrm{Nis}} \to \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$, which is given by $G^{\mathrm{sp}} = \mathrm{Im}(G \to \underline{\omega}^*\underline{\omega}_!(G))$.

For F in $\mathbf{RSC}_{\mathrm{Nis}}$ we define

$$F\langle 1\rangle := (F, \mathbb{G}_m)_{\mathbf{RSC}_{\mathrm{Nis}}},$$

and recursively

$$F\langle n\rangle := (F\langle n-1\rangle)\langle 1\rangle \in \mathbf{RSC}_{\mathrm{Nis}}.$$

We also define

$$\gamma^n F := \underline{\mathrm{Hom}}_{\mathbf{PST}}(K^M_n, F) \in \mathbf{RSC}_{\mathrm{Nis}}.$$

Generalizing part of Voevodsky's cancellation theorem [53], Merici and S. Saito show the following:

[9] We use the transfers structure stemming from the decomposition $\mathcal{O}_{X\times_{\mathbb{Z}}X}/I^2_{\Delta_X} = \mathcal{O}_X \oplus \Omega^1_{/\mathbb{Z}}$.

Theorem 4.4 [38, Corollary 3.6]. *For F in $\mathbf{RSC}_{\mathrm{Nis}}$ we have*

$$\gamma^n(\widetilde{F}(n)) \cong \widetilde{F} \quad \text{and} \quad \gamma^n(F\langle n\rangle) = F.$$

Proposition 4.5 [6; 45]. *There are identities*

- $\widetilde{\mathbb{Z}}(n) = \underline{\omega}^* K_n^M \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,\mathrm{sp}}$,
- $\mathbb{Z}\langle n\rangle = K_n^M \in \mathbf{RSC}_{\mathrm{Nis}}$.

In char $k = 0$, *we have the identities*

- $\widetilde{\mathbb{G}_a}(n) = \widetilde{\Omega}^n_{/\mathbb{Z}} \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,\mathrm{sp}}$,
- $\mathbb{G}_a\langle n\rangle = \Omega^n_{/\mathbb{Z}} \in \mathbf{RSC}_{\mathrm{Nis}}$.

The proof of the latter identities uses the computation of $\widetilde{\Omega^n_{/\mathbb{Z}}}(\mathcal{O}_L, \mathfrak{m}_L^{-r})$.

Proposition 4.6. *Assume* char $k = p \geq 7$. *Then* (*see* (2-1) *for notation*)

$$(\mathbb{G}_a, K_n^M)_{\mathbf{RSC}_{\mathrm{Nis}}} = \Omega^n / B_\infty \Omega^n.$$

Proof. Since there is a natural morphism $\widetilde{\Omega^n} \to \widetilde{\Omega^n/B_\infty}$ in $\mathbf{CI}_{\mathrm{Nis}}^{\tau,\mathrm{sp}}$, and the inverse Cartier isomorphism induces an endomorphism $\bar{F} : \widetilde{\Omega^n/B_\infty} \to \widetilde{\Omega^n/B_\infty}$ (see (2-2)), we can use Theorems 3.2 and 3.3 to construct a map $(\mathbb{G}_a, K_n^M)_{\mathbf{RSC}_{\mathrm{Nis}}} \to \Omega^n/B_\infty$ in $\mathbf{RSC}_{\mathrm{Nis}}$ similar as in the proof of [45, Theorem 5.20]. By Corollary 1.18 and Theorem 1.19 it suffices to show that it is an isomorphism on any function field K. By [45, Proposition 5.18] there is a surjective map $\Omega^n_K \to (\mathbb{G}_a, K_n^M)_{\mathbf{RSC}_{\mathrm{Nis}}}(K)$ (here we use $p \geq 7$). The same proof as in [24, Corollary 5.4.12] shows that this map factors over the quotient Ω^n_K/B_∞. By construction of the maps, the composition

$$\Omega^n_K/B_\infty \to (\mathbb{G}_a, K_n^M)_{\mathbf{RSC}_{\mathrm{Nis}}}(K) \to \Omega^n_K/B_\infty$$

is the identity, which completes the proof. □

Remark 4.7. Similarly, one can also show $\mathbb{G}_a\langle n\rangle = \Omega^n/B_\infty$ (at least for $p \geq 7$). This is not immediate from the above. In the induction step $\Omega^n/B_\infty\langle 1\rangle = \Omega^{n+1}/B_\infty$, a description of $\widetilde{\Omega^n/B_\infty}(\mathcal{O}_L, \mathfrak{m}_L^{-n})$ is required. The latter group was computed by Rülling and will appear somewhere else.

Proposition 4.8 [6, Theorem 11.8]. *Assume $p > 0$. There is a natural isomorphism*

$$W_r\Omega^{q-n} \xrightarrow{\simeq} \gamma^n(W_r\Omega^q),$$

which sends a Witt-differential form $\omega \in W_r\Omega^{q-n}(X)$ to the map

$$\varphi_\omega \in \gamma^n(W_r\Omega^q)(X) = \mathrm{Hom}_{\mathbf{PST}}(K_n^M, \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), W_r\Omega^q)),$$

which on Y is given by

$$\varphi_\omega(Y) : K_n^M(Y) \to W_r\Omega^q(X \times Y), \quad a \mapsto p_X^*\omega \cdot d\log(p_Y^* a),$$

where p_X, p_Y denote the two projections from $X \times Y$ to X, Y, respectively.

As $W_r\Omega^q$ is a successive extension of certain subquotients of Ω^q, this is a consequence of the well-known equality $R^1\pi_*\Omega^q_{X\times\mathbb{P}^1} = \Omega^{q-1}_X$ and the fact that a reciprocity sheaf F satisfies

$$R^1\pi_* F_{X\times\mathbb{P}^1} = (\gamma^1 F)_X,$$

where $\pi : \mathbb{P}^1_X \to X$. This is a consequence of the cube-invariance of the cohomology of F; see [47, Theorem 9.3].

5. Cohomology of reciprocity sheaves

We explain some structural results about the cohomology of reciprocity sheaves such as a projective bundle formula, a blow-up formula, a Gysin sequence, the existence of a proper pushforward and the existence of an action of Chow correspondences. This has consequences outside the theory of reciprocity sheaves. For example, we obtain new birational invariants of smooth projective varieties and obstructions to the existence of zero-cycles of degree one. We survey these applications at the end of this section.

Structural results.

Theorem 5.1 (blow-up formula, [6, Corollary 7.3]). *Let $G \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ (see Definition 1.21 for notation). Assume that $i : Z \hookrightarrow X$ is a closed immersion of codimension c that is transversal to D (see 1.21). Let $\rho : \widetilde{X} \to X$ denote the blow-up of X in Z, and let $\widetilde{\mathcal{X}} := (\widetilde{X}, D_{|\widetilde{X}})$, and $\mathcal{Z} := (Z, D_{|Z})$. Then*

$$R\rho_* G_{\widetilde{\mathcal{X}}} \cong G_{\mathcal{X}} \oplus \bigoplus_{i=1}^{c-1} i_*\gamma^i G_{\mathcal{Z}}[-i].$$

Theorem 5.2 (projective bundle formula, [6, Theorem 6.3]). *Let $G \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$. Assume that $\pi : P \to X$ is a projective bundle of rank n, and let $\mathcal{P} := (P, D_{|P})$. Then*

$$R\pi_* G_{\mathcal{P}} \cong \bigoplus_{i=0}^{n} (\gamma^i G)_{\mathcal{X}}[-i].$$

The proofs of these two theorems are intertwined and go beyond the scope of these lectures. We just mention some points to compare with classical arguments in the $\mathbb{A}^1$-invariant case:

First one proves that there is a blow-up distinguished triangle (we comment on the proof below). The projective bundle formula is then proven by induction starting from the $\mathbb{P}^1$-invariance of the cohomology, which is a consequence of the cube-invariance proven in [47, Theorem 9.3], and using the blow-up triangle in the induction step. Using the projective bundle formula one can then, similar as Voevodsky, construct a splitting of the blow-up triangle, where one uses $\overline{\square}$-invariance instead of $\mathbb{A}^1$-invariance.

The essential point in the proof of the blow-up triangle is to show the vanishing

$$R^i\rho_* G_{(Y,\rho^* L)} = 0, \quad i \geq 1,$$

where $\rho : Y \to \mathbb{A}^2$ is the blow-up in the origin 0 and $L \subset \mathbb{A}^2$ is a line containing 0. Denote by $\pi : Y \to \mathbb{P}^1$ the projection to the exceptional divisor of the blow-up. Then it is not hard to see that the above vanishing is implied by the vanishing

$$H^1(\mathbb{P}^1, \pi_* G_{(Y,\rho^* L)}) = 0.$$

This is shown in [6, Lemma 2.13]. The proof is a bit technical. However, in the course of this proof one is confronted with certain modulus-related problems which do not come up in the $\mathbb{A}^1$-invariant story. This is why a crucial ingredient in the proof is the following modulus-descent result: Consider the morphism $\psi_0 : \mathbb{A}^1_y \times \mathbb{A}^1_s \to \mathbb{A}^1_x \times \mathbb{A}^1_s$ given by the $k[s]$-algebra morphism $k[x, s] \to k[y, s]$, $x \mapsto ys$. It induces a map

$$\psi : \overline{\square}^{(1)}_y \otimes \overline{\square}^{(2)}_s \to \overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s \quad \text{in } \underline{\mathbf{M}}\mathbf{Cor},$$

where $\overline{\square}^{(n)} = (\mathbb{P}^1, n\cdot\{0\}+n\cdot\{\infty\})$. Indeed to check this denote by $\Gamma \subset \mathbb{P}^1_y \times \mathbb{P}^1_s \times \mathbb{P}^1_x$ the closure of the graph of ψ_0 (as a morphism over $\mathbb{A}^1_s$). Then the claim holds by the following identities of divisors on Γ:

$$\begin{gathered} 2\cdot\{0_s\} + \{0_y\} = \{0_x\} + \{0_s\}, \quad 2\cdot\{\infty_s\} + \{\infty_y\} = \{\infty_x\} + \{\infty_s\}, \\ \{0_y\} = \{0_x\} + \{\infty_s\}, \quad \{\infty_y\} = \{\infty_x\} + \{0_s\}. \end{gathered}$$

In particular, the map ψ_0 does *not* define a modulus correspondence from $\overline{\square}^{(1)}_y \otimes \overline{\square}^{(1)}_s$ to $\overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s$.

Proposition 5.3 [6, Proposition 2.5]. *Let $G \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$. With the notation from above ψ^* factors for $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}_{\mathrm{ls}}$ as*

$$\begin{array}{ccc} & & G(\overline{\square}^{(1)}_y \otimes \overline{\square}^{(1)}_s \otimes \mathcal{X}) \\ & \nearrow\!\!\!\cdot\cdot\cdot & \big\downarrow \\ G(\overline{\square}^{(1)}_x \otimes \overline{\square}^{(1)}_s \otimes \mathcal{X}) & \xrightarrow{\ \psi^*\ } & G(\overline{\square}^{(1)}_y \otimes \overline{\square}^{(2)}_s \otimes \mathcal{X}), \end{array}$$

where the vertical map is induced by the natural morphism $\overline{\square}^{(2)}_s \to \overline{\square}^{(1)}_s$ and it is injective by the semipurity of G.

Let us illustrate Proposition 5.3 in the example $G = \widetilde{\Omega^2}_{/k}$ and $\mathcal{X} = (\operatorname{Spec} k, \varnothing)$. In this case $G(\overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)}) = k \cdot \mathrm{dlog}(x)\,\mathrm{dlog}(s)$ and we have

$$\psi^*(\mathrm{dlog}(x)\,\mathrm{dlog}(s)) = \mathrm{dlog}(ys)\,\mathrm{dlog}(s) = \mathrm{dlog}(y)\,\mathrm{dlog}(s) \in G(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(1)}).$$

Example 5.4. We spell out the projective bundle formula in two concrete cases (the blow-up formula is similar). Let the situation be as in Theorem 5.2.

- In char $k = 0$ we have

$$R\pi_*\Omega^j_{P/\mathbb{Z}}(\log D_{|P})(D_{|P} - D_{|P,\mathrm{red}}) = \bigoplus_{i=0}^{n} \Omega^{j-i}_{X/\mathbb{Z}}(\log D)(D - D_{\mathrm{red}})[-i].$$

  Taking $D = \varnothing$ recovers the classical projection bundle formula.

- In positive characteristic p with $D = \varnothing$, we have

$$R\pi_*(R^{j+1}\epsilon_*\mathbb{Z}/p^r(j))_P = \bigoplus_{i=0}^{n} (R^{j-i+1}\epsilon_*\mathbb{Z}/p^r(j-i))_X[-i].$$

  Note that this can also be deduced from the projective bundle formula for the Hodge–Witt cohomology by Gros [19, I, Theorem 4.1.11].

We remark, that in the second example we have to take the empty divisor since the formula $\gamma^i(R^{j+1}\epsilon_*\mathbb{Z}/p^r(j))_{(X,D)} = (R^{j-i+1}\epsilon_*\mathbb{Z}/p^r(j-i))_{(X,D)}$ is only known for $D = \varnothing$, in which case it follows from the exactness of γ^i and [6, Theorem 11.8]. In characteristic zero we have $\gamma^i\widetilde{\Omega^j_{/\mathbb{Z}}} = \widetilde{\Omega^{j-i}_{/\mathbb{Z}}}$ by [6, Corollary 11.2].

Similar to Voevodsky [52, Proposition 3.5.4] we obtain a Gysin triangle.

Theorem 5.5 (Gysin sequence [6, Theorem 7.16]). *Let $G \in \mathbf{CI}^{\tau,\mathrm{sp}}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{PST}_{\mathrm{ls}}$. Assume that $i : Z \hookrightarrow X$ is a closed immersion of codimension c that is transversal to D, in the sense of Definition 1.21. Let $\mathcal{Z} := (Z, D_{|Z})$, and let $\rho : \widetilde{X} \to X$ denote the blow-up of Z in X with $E = \rho^{-1}(Z)$ the exceptional divisor. Then there is an exact triangle*

$$i_*\gamma^c G_{\mathcal{Z}}[-c] \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} G_{\mathcal{X}} \to R\rho_* G_{(\widetilde{X}, D_{|\widetilde{X}+E})} \xrightarrow{\partial} i_*\gamma^c G_{\mathcal{Z}}[-c+1]$$

in $D(X_{\mathrm{Nis}})$.

Example 5.6. In char $k = 0$ and $c = 1$ we get an exact sequence

$$\begin{aligned} 0 &\to \widetilde{\mathrm{Conn}}^1(X, D) \to \widetilde{\mathrm{Conn}}^1(X, D+Z) \\ &\to H^0(Z, \mathcal{O}_Z(i^*D - (i^*D)_{\mathrm{red}}))/\mathbb{Z} \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} H^1\left(X, \frac{\Omega^1_{X/k}(\log D)(D - D_{\mathrm{red}})}{d\log(j_*\mathcal{O}^\times_{X\setminus D})}\right) \\ &\to H^1\left(X, \frac{\Omega^1_{X/k}(\log D + Z)(D - D_{\mathrm{red}})}{d\log(j_*\mathcal{O}^\times_{X\setminus(D+Z)})}\right) \end{aligned}$$

and for $c \geq 2$ we have

$$\widetilde{\mathrm{Conn}}^1(X, D) \cong \widetilde{\mathrm{Conn}}^1(\widetilde{X}, \rho^* D + E),$$

where $\mathrm{Conn}^1(X)$ denotes the group of isomorphism classes of rank-1 connections on X.

Example 5.7. Let $\mathrm{Lisse}^1 \in \mathbf{RSC}_{\mathrm{Nis}}$ be the sheaf whose sections over X are the lisse $\overline{\mathbb{Q}}_l$-sheaves of rank 1, and $\widetilde{\mathrm{Lisse}}^1(X, D)$ the lisse $\overline{\mathbb{Q}}_l$-sheaves of rank 1 on $X \setminus D$ with Artin conductor less than or equal to D. Then for $\mathrm{char}\, k = p > 0$, $l \neq p$, and $c \geq 2$, we have

$$\widetilde{\mathrm{Lisse}}^1(X, D) = \widetilde{\mathrm{Lisse}}^1(\widetilde{X}, \rho^* D + E).$$

We can now define a proper correspondence action on reciprocity sheaves.

Definition 5.8. Let S be a scheme of finite type over k and let C_S denote the category of *proper* (*Chow*) *correspondences*, i.e., its objects are S-schemes $X \xrightarrow{f} S$ such that X is quasiprojective and smooth over k, and f is a morphism of finite type. Morphisms in C_S between (connected) objects X and Y are elements in

$$C_S(X, Y) := \varinjlim_{Z \subset X \times_S Y} \mathrm{CH}_{\dim X}(Z),$$

where the direct limit runs over all closed subschemes $Z \subset X \times_S Y$ which are proper over X. The composition of morphisms in this category is defined using Fulton's refined intersections.

Definition 5.9. For $F \in \mathbf{RSC}_{\mathrm{Nis}}$, and objects $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ in C_S, together with an element $\alpha \in C_S(X, Y)$, we define the *proper correspondence action*

$$\alpha^* : Rg_* F_Y \to Rf_* F_X \quad \text{in } D(S_{\mathrm{Nis}})$$

by

(1) pulling back to $X \times Y$,

(2) cupping with α (minding the support),

(3) pushing forward to X (using the properness of the support over X).

For (2) we note that the Gersten resolution for Milnor K-theory yields an identification $\mathrm{CH}_{\dim X}(V) = H^e_V(X \times Y, K^M_e)$ (Bloch formula with support), where $V \subset X \times_S Y$ is proper over X and $e = \dim Y$, and hence $\alpha \in \mathrm{CH}_{\dim X}(V)$ corresponds to a map $\alpha : \mathbb{Z}[-e] \to R\underline{\Gamma}_V(K^M_e)$. The cupping with α is then defined as the composition

$$\begin{aligned}\gamma^e F[-e] &\xrightarrow{\alpha} \gamma^e F \otimes^L_{\mathbb{Z}} R\underline{\Gamma}_V(K^M_e)\\ &\to R\underline{\Gamma}_V(\gamma^e F \otimes_{\underline{\mathbf{M}}\mathbf{PST}} K^M_e)\\ &= R\underline{\Gamma}_V(\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(K^M_e, F) \otimes_{\underline{\mathbf{M}}\mathbf{PST}} K^M_e)\\ &\xrightarrow{\text{adj.}} R\underline{\Gamma}_V F.\end{aligned}$$

The construction of the pushforward in (3) follows the classical method, but we have to keep track of the support, as the projection $X \times Y \to X$ does not need to be projective: Take a closed embedding of Y into an open U of a projective space P over $\operatorname{Spec} k$. Then the pushforward is defined by using the Gysin map with support in V along the closed embedding $X \times Y \hookrightarrow X \times U$. By excision the cohomology of $X \times U$ with support in V agrees with the cohomology of $X \times P$ with support in V and we can use the projective bundle formula to pushforward to X. The cancellation theorem (Theorem 4.4) is used to cancel twists.

We obtain a functor $C_S \to D(S_{\mathrm{Nis}})$ given by $(X \xrightarrow{f} S) \mapsto Rf_* F$.

Applications. We survey applications of the results presented in the previous sections; see [6, Chapters 10 and 11] for more details.

Obstructions to the existence of zero-cycles in degree 1.

Theorem 5.10 [6, Corollary 10.2]. *Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Let $f : X \to S$ be a projective dominant map between smooth k-schemes. Assume there exists a degree-1 zero-cycle on X_K, where $K = k(S)$. Then*

$$f^* : H^i(S, F_S) \to H^i(X, F_X)$$

is split-injective.

Proof. Take $\bar{\xi}$ as a lift of $\xi \in \mathrm{CH}_0(X_K)^{\deg=1}$ under the map

$$C_S(S, X) = \mathrm{CH}_{\dim S}(S \times_S X) \to \mathrm{CH}_0(X_K).$$

Then $f_*\bar{\xi} = [S] = \mathrm{id}_S \in \mathrm{CH}_{\dim S}(S) = C_S(S, S)$. Hence, we have the splitting

$$\bar{\xi}^* \circ f^* = (f_*\bar{\xi})^* = \mathrm{id} : H^i(S, F) \to H^i(X, F) \to H^i(S, F). \qquad \square$$

Generalized Brauer–Manin obstruction for zero-cycles. Let S be a smooth projective curve over k with function field $K = k(S)$. Let $f : X \to S$ be a projective dominant map with X smooth, and choose $v \in S_{(0)}$. Then for $\alpha_v \in \mathrm{CH}_0(X_{K_v})$ with lift $\bar{\alpha}_v \in \mathrm{CH}_1(X_{S_v})$, where K_v and S_v are the henselization of K and S at v, the map

$$\Psi(\alpha_v) := \bar{\alpha}_v^* : f_* F_{X_{S_v}} \to F_{S_v}$$

depends only on α_v and not on the lift $\bar{\alpha}_v$. Indeed if α'_v is a different lift then $\beta = \bar{\alpha}_v - \alpha'_v$ can be represented by a cycle supported in the special fiber X_v, and it follows from the construction of the proper correspondence action in Definition 5.9 that $\beta^* : f_* F_{X_{S_v}} \to F_{S_v}$ factors via $\underline{\Gamma}_v(F_{S_v})$ which vanishes by S. Saito's purity theorem; see Theorem 1.19. Taking the first cohomology with support in v yields a map

$$H^1_v(S, f_* F_X) \xrightarrow{\Psi(\alpha_v)} H^1_v(S, F) \to H^1(S, F).$$

Thus we obtain a map

$$\Psi : \prod_{v \in S_{(0)}} \mathrm{CH}_0(X_{K_v}) \to \mathrm{Hom}\Big(\bigoplus_{v \in S_{(0)}} H^1_v(S, f_* F_X), H^1(S, F) \Big).$$

There is a map

$$\iota : F(X_K) \to \bigoplus_{v \in S_{(0)}} H^0(S_v \setminus \{v\}, f_* F_X) \xrightarrow{\partial} \bigoplus_{v \in S_{(0)}} H^1_v(S, f_* F_X).$$

Theorem 5.11 [6, Corollary 10.4]. *If* $\Psi((\alpha_v)_v) \circ \iota \neq 0$, *then there does not exist* $\alpha \in \mathrm{CH}_0(X_K)$ *such that* $\alpha \mapsto (\alpha_v)_v$.

Proof. Take $\alpha \mapsto \alpha_v$ and $\bar{\alpha} \in \mathrm{CH}_1(X)$ a lifting of α. We then get a diagram

$$\begin{array}{ccccc} & & \overset{0}{\frown\!\!\!\longrightarrow} & & \\ F(X_K) & \xrightarrow{\iota} & \bigoplus_v H^1_v(S, f_* F_X) & \longrightarrow & H^1(X, F) \\ & & \Big\downarrow {\scriptstyle \sum_v \Psi(\alpha_v)} & \swarrow {\scriptstyle \bar{\alpha}^*} & \\ & & H^1(S, F), & & \end{array}$$

where the second map in the horizontal sequence is the composition of summing the forget-support maps $H^1_v(S, f_* F_X) \to H^1(S, f_* F_X)$ with the natural map $H^1(S, f_* F_X) \to H^1(X, F)$ induced by $f_* F_X \to R f_* F_X$. $\square$

Remark 5.12. Assume $k = \mathbb{F}_q$ is a finite field with q elements and $F = \mathrm{Br}$. Using the Cousin resolution (1-5) of Br_S and the fact that in the case at hand we have

$$H^1_v(S, \mathrm{Br}_S) = \mathrm{Br}(K_v) / \mathrm{Br}(S_v) = \mathrm{Br}(K_v),$$

the Brauer–Hasse–Noether theorem in the function field case yields

$$H^1_{\mathrm{Nis}}(S, \mathrm{Br}) = \mathrm{Coker}(\mathrm{Br}(K) \to \bigoplus_{v \in S_{(0)}} \mathrm{Br}(K_v)) = \mathbb{Q}/\mathbb{Z};$$

see [56, XIII, §3, Theorem 2 and §6, Theorem 4]. Thus Ψ equals the map

$$\prod_{v \in S_{(0)}} \mathrm{CH}_0(X_{K_v}) \to \mathrm{Hom}\Big(\bigoplus_{v \in S_{(0)}} \frac{\mathrm{Br}(X_{K_v})}{\mathrm{Br}(X_{S_v})}, \mathbb{Q}/\mathbb{Z} \Big),$$

which is the classical Brauer–Manin obstruction for zero-cycles in the function field case.

Stably birational invariance.

Definition 5.13. Let $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ be objects of C_S and assume that X and Y are integral. We say that f and g are *properly birational over* S if there exists proper birational S-maps $Z \to X$ and $Z \to Y$ (Z could be singular).

The maps f and g are said to be *stably properly birational over* S if there exist vector bundles V over X and W over Y such that $\mathbb{P}(V)$ and $\mathbb{P}(W)$ are properly birational over S.

Example 5.14. If S is singular, and X and Y are two different resolutions of S, then they are properly birational over S.

If f is proper and we take $S = Y = \operatorname{Spec} k$, then f and $\mathrm{id}_{\operatorname{Spec} k}$ are stably properly birational if and only if X is stably rational over k.

Theorem 5.15 [6, Theorem 10.7]. *Any $F \in \mathbf{RSC}_{\mathrm{Nis}}$ is a stably properly birational invariant over S. In other words, for every stably properly birational S-schemes $X \xrightarrow{f} S$, $Y \xrightarrow{g} S \in C_S$, we have an isomorphism*

$$f_* F_X \xrightarrow{\simeq} g_* F_Y.$$

Proof. This follows from the projective bundle formula, purity, and the correspondence action. ☐

Theorem 5.16 [6, Theorem 10.10]. *Let $X \xrightarrow{f} S$, $Y \xrightarrow{g} S \in C_S$ be properly birational over S and let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Assume that $F(K) = 0$ for all function fields K/k of transcendence degree $\leq d-1$, where $d = \dim X = \dim Y$. Then*

$$Rg_* F_Y \xrightarrow{\simeq} Rf_* F_X.$$

Proof. Take a closed subscheme $Z \subset X \times_S Y$ mapping properly and birationally to X and Y. Then $Z \circ Z^t = \Delta_Y + \epsilon$ with $p_{Y*}\epsilon \in \mathrm{CH}^{\geq 1}(Y)$, and the condition on F implies that $\epsilon^* = 0$ on $Rg_* F_Y$; see [6, Proposition 9.13]. This implies that $Z \circ Z^t$ acts as the identity on $Rg_* F_Y$; similarly with $Z^t \circ Z$. ☐

Remark 5.17. Taking $Y = S$, and g the identity, yields the vanishing result

$$Rf_* F_X \cong 0.$$

Example 5.18. Assume that $\dim X = \dim Y = d$. Then Theorem 5.16 applies to the sheaves

$$\Omega^d_{/k}, \quad \Omega^d_{/k}/d\log K^M_d,$$

and, if $\operatorname{char} k = p \neq 0$, also to the sheaves

- $W_n\Omega^d/B_\infty$, $R^i\epsilon_*(\mathbb{Z}/p^n(d))$,

- $G\langle d\rangle$, for G a smooth unipotent group,
- $H^1(G)\langle d\rangle$, for G a finite p-group over k,
- $R^d\epsilon_*\mathbb{Q}/\mathbb{Z}(d)$ if k is algebraically closed.

Remark 5.19. The case for $\Omega^d_{/k}$ was known before by [11, Theorem 1]. It was later generalized to regular schemes in [13; 35].

In the first case in positive characteristic we use Geisser–Levine [18, Theorem 8.3], in the second and third case we use the Bloch–Kato–Gabber theorem [8, Theorem 2.1] and in the last case we use additionally the Milnor–Bloch–Kato conjecture, proven by Rost and Voevodsky [54, Theorem 6.16], to check that the condition $F(K)=0$ for $\mathrm{trdeg}(K/k)<d$ is satisfied in the cases at hand.

There is a version of Theorem 5.16 where the vanishing $F(K)=0$ for $\mathrm{trdeg}(K/k)$ less than d is replaced by the vanishing $\gamma^1F=0$ (which is, for example, satisfied if F is any smooth commutative unipotent group), but this requires at the moment resolution of singularities in dimension $d-1$.

Corollary 5.20 [6, Corollary 11.24]. *Let $X\xrightarrow{f}S$ and $Y\xrightarrow{g}S$ be flat, geometrically integral, and projective morphisms between smooth connected k-schemes. Assume that the generic fiber has index* 1, (*implying that the Picard schemes* $\mathrm{Pic}_{X/S}$ *and* $\mathrm{Pic}_{Y/S}$ *are representable*). *If X and Y are stably properly birational over S, then*

$$\mathrm{Pic}_{X/S}[n]\cong\mathrm{Pic}_{Y/S}[n]$$

on S_{Nis} for all n.

Remark 5.21. The above result is classical for $S=\operatorname{Spec}k$, with k algebraically closed.

Decomposition of the diagonal.

Definition 5.22. Let K be a function field over k and X a smooth scheme over K with $\dim X=d$. We say that the *diagonal of X decomposes* if

$$[\Delta_X]=p_2^*\xi+(i\times\mathrm{id})_*\beta\in\mathrm{CH}^d(X\times_K X), \tag{5-1}$$

where $\xi\in\mathrm{CH}_0(X)$ and $\beta\in\mathrm{CH}_d(Z\times_K X)$ for some closed subscheme $i:Z\hookrightarrow X$ with $\mathrm{codim}(Z,X)\geq 1$.

This condition was first considered by Bloch and Srinivas for rational coefficients in [9]. By [14, Lemma 1.5] an integral smooth projective k-scheme X, which is *retract rational* (i.e., there exist open dense subsets $U\subset X$ and $V\subset\mathbb{P}^n_K$ together with a map $V\to U$ which has a section), has the property that its diagonal decomposes. Hence implications of (5-1) on cohomology yield obstructions to X being retract rational over K.

Theorem 5.23 [6, Theorem 10.13]. *Let S be the henselization of a smooth k-scheme in a 1-codimensional point or a regular connected affine scheme of dimension ≤ 1 and of finite type over a function field K over k. Let $f : X \to S$ be a smooth and projective morphism, and assume that the diagonal of the generic fiber of f decomposes. Then*

$$F(S) = F(X),$$

for any $F \in \mathbf{RSC}_{\mathrm{Nis}}$.

Remark 5.24. In [3, Problem 1.2] the following problem is posed:

Let k be algebraically closed with $\operatorname{char} k = p > 0$ and X a smooth and proper scheme over k with decomposition of the diagonal. Do we then have that

$$H^0(X, R^i\epsilon_*\mathbb{Z}/p(j)) = 0 \quad \text{for all } i \neq 0?$$

Theorem 5.23 gives a positive answer to the problem if X is projective over k. Indeed, if $S = \operatorname{Spec} k$ and $F = R^i\epsilon_*\mathbb{Z}/p(j)$ we observe that $F(k) = 0$; see (2-2).

Theorem 5.25 [6, Corollary 11.22]. *Let $f : X \to S$ be a projective morphism between smooth integral and quasiprojective k-schemes. Let $\dim S = e$ and $\dim X = d$. Assume the diagonal of the generic fiber of f decomposes. Then*

$$f_* : Rf_*F^d_X \xrightarrow{\cong} F^e_S[e-d]$$

is an isomorphism, where F^d is one of the sheaves

- $\Omega^d_{/k}$, *or*
- $\Omega^d_{/k}/d\log K^M_d$, *or*
- $R^d\epsilon_*\mathbb{Q}/\mathbb{Z}(d)$ *if k is algebraically closed,*

or, if $\operatorname{char} k = p > 0$, one of the sheaves

- $W_n\Omega^d/B_\infty$, *or*
- $R^i\epsilon_*(\mathbb{Z}/p^n(d))$, *or*
- $G\langle d\rangle$, *for G a smooth commutative unipotent k-group, or*
- $H^1(G)\langle d\rangle$, *for G a finite p-group over k.*

Example 5.26. If k is algebraically closed and X is smooth and projective over k of dimension d such that the diagonal of X decomposes, then

$$H^i(X, R^{d+1}\epsilon_*\mathbb{Z}/p^n(d)) = 0 \quad \text{for all } i.$$

6. Further results

We give an overview of some more recent results obtained in [40; 42]. This section was not part of the original lecture series given at CAS.

Zariski–Nagata purity. Let U be a smooth k-scheme over a perfect base field k and let K be a function field over k. For any presheaf with transfers we have a pairing

$$F(U_K) \otimes \mathbf{Cor}_K(\operatorname{Spec} K, U_K) \to F(K), \quad (a, \gamma) \mapsto \gamma^* a, \tag{6-1}$$

where $U_K = U \otimes_k K$ and $\mathbf{Cor}_K$ denotes the finite correspondences on smooth K-schemes. For X an integral finite-type k-scheme of dimension d, and $D \subset X$ a closed subscheme with ideal sheaf I_D and open complement $U = X \setminus D$, we have by [32, Theorem 2.5] a surjective map

$$\mathbf{Cor}_K(\operatorname{Spec} K, U_K) \twoheadrightarrow H^d(X_{K,\mathrm{Nis}}, K_d^M(\mathcal{O}_{X_K}, I_{D_K})), \tag{6-2}$$

where $K_d^M(\mathcal{O}_{X_K}, I_{D_K}) = \operatorname{Ker}(K_d^M(\mathcal{O}_{X_K}) \to K_d^M(\mathcal{O}_{D_K}))$. The map is induced by the isomorphism $\mathbb{Z} \cong H_x^d(X_{K,\mathrm{Nis}}, K_d^M(\mathcal{O}_{X_K}, I_{D_K}))$, for x a closed point in U_K, stemming from the Gersten resolution; see [33, Proposition 10 (8)].

Theorem 6.1 [42, Theorem 1.6]. *Let X be a smooth integral projective k-scheme of dimension d and let $\sum_{i=1}^r D_i$ be an SNC divisor with complement $U = X \setminus \bigcup_{i=1}^r D_i$. Let $\mathfrak{n} = (n_1, \dots, n_r) \in (\mathbb{N}_{\geq 1})^r$ and set $D_{\mathfrak{n}} = \sum_{i=1}^r n_i D_i$. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Then the following are equivalent for $a \in F(U)$:*

(i) $a \in \widetilde{F}(X, D_{\mathfrak{n}})$.

(ii) $a \in \widetilde{F}(\mathcal{O}_{X,\eta_i}, \mathfrak{m}_{\eta_i}^{-n_i})$, *for all i, where η_i is the generic point of D_i.*

(iii) *For any function field K the map $(a_K, -)_{U_K/K} : \mathbf{Cor}_K(\operatorname{Spec} K, U_K) \to F(K)$ induced by* (6-1) *factors via* (6-2) *to give a map*

$$(a_K, -)_{(X_K, D_{\mathfrak{n},K})/K} : H^d(X_{K,\mathrm{Nis}}, K_d^M(\mathcal{O}_{X_K}, I_{D_{\mathfrak{n},K}})) \to F(K), \tag{6-3}$$

where a_K denotes the pullback of a to $F(U_K)$.

Note that the equivalence of (i) and (ii) is a statement of Zariski–Nagata type (“purity of the branch locus”). It continues to hold if X is assumed to be quasiprojective and $\big(X, \sum_i D_i\big)$ has a projective SNC-compactification (which is always the case in characteristic zero); see the proof of [42, Corollary 6.10]. In case $r = 1$ (i.e., $D_{\mathfrak{n}}$ has just one component) the equivalence of (i) and (ii) also follows from [47, Corollary 8.6(2)]. For $\mathfrak{n} = (1, \dots, 1)$ (i.e., $D_{\mathfrak{n}}$ is reduced SNCD), it follows from [48, Corollary 2.4].

Observe that from the equivalence of (i) and (iii) we obtain a map

$$\mathrm{CH}_0(X \mid D_{\mathfrak{n}}) \twoheadrightarrow H^d(X_{\mathrm{Nis}}, K_d^M(\mathcal{O}_X, I_{D_{\mathfrak{n}}})) \to \operatorname{Hom}(\widetilde{F}(X, D_{\mathfrak{n}}), F(k)), \tag{6-4}$$

where $\mathrm{CH}_0(X \mid D_{\mathfrak{n}})$ denotes the Chow group of zero-cycles with modulus introduced in [34] and the first map is induced from (6-2). (This factorization is proved in various cases by, for example, Krishna as well as Gupta and Krishna. For the situation at hand, see [41]). Taking the limit of the composition we get a natural map

$$C(U) := \varprojlim_{\mathfrak{n}} \mathrm{CH}_0(X \mid D_{\mathfrak{n}}) \to \mathrm{Hom}(F(U), F(k)).$$

If k is a finite field and $F = \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\mathrm{ab}}(-), \mathbb{Q}/\mathbb{Z})$, then $F(k) \cong \mathbb{Q}/\mathbb{Z}$ and this map is the *reciprocity homomorphism* constructed in [34, Proposition 3.2]. Similarly the limit over $\mathfrak{n}$ of the second map in (6-4) is the *reciprocity homomorphism* constructed in [32].

The formula of Abbes and T. Saito. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. We ask the following:

(1) Is it possible to give a more computable description of $\widetilde{F}$, in particular without using the transfers structure of F?

(2) For X smooth and $D \subset X$ a smooth divisor, we get from the Gysin sequence of Theorem 5.5 an isomorphism

$$\widetilde{F}(X, D)/F(X) \cong \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{G}_m, F)(D).$$

Can we also describe the quotients $\widetilde{F}(X, nD)/\widetilde{F}(X, (n-1)D)$, for $n \geq 2$?

In [40] it is shown that these questions can be approached using a method introduced by Abbes and Saito in [2] and Saito [46] to study the ramification of Galois torsors by means of dilatations. For simplicity we assume in the following that:

(∗) X is smooth, D is a smooth divisor on X, $U = X \setminus D$, and (X, D) has a projective SNC compactification, that is, there exists an open embedding $X \hookrightarrow \overline{X}$ into a smooth projective scheme $\overline{X}$ such that $\overline{X} \setminus U$ is the support of a divisor with simple normal crossings.[10]

The dilatation $P_X^{(nD)}$, for $n \geq 1$, is the blow-up of $X \times X$ in nD diagonally embedded and with the strict transforms of $X \times nD$ and $nD \times X$ removed. It comes with two maps

$$p_1, p_2 : P_X^{(nD)} \rightrightarrows X$$

induced by the two projection maps $X \times X \to X$. Note that the open immersion $U \times U \hookrightarrow X \times X$ extends to an open immersion $U \times U \hookrightarrow P_X^{(nD)}$.

Theorem 6.2 [40, Theorem 1]. *Assume* (∗) *and let* $n \geq 1$. *Then*

$$\widetilde{F}(X, nD) = \{a \in F(U) \mid p_1^* a = p_2^* a \text{ in } F(U \times U)/F(P_X^{(nD)})\}.$$

In particular the theorem applies to $F = H^1_{\mathrm{fppf}}(-, G)$,[11] where G is a commutative finite k-group scheme (not necessarily étale). A version of this formula was proved

[10] In [40] more generally the case where D is a SNCD is considered.

[11] In this case even without the assumption on the existence of a projective SNC compactification.

by Abbes and Saito for G any étale k-group (not necessarily commutative). For more details and precise references see [40, Example 2.12].

The proof of the above theorem uses heavily the theory of higher local symbols along Paršin chains for reciprocity sheaves developed in [43], which in turn relies on Theorem 6.1 and Section 5.

Using Theorem 6.2 we obtain the following partial description of the quotients considered in (2) above.

Theorem 6.3 [40, Theorem 4.12]. *Assume* $(*)$ *and let* $n \geq 2$. *Then there is an injective map*

$$\mathrm{char}_F^{(nD)} : \frac{\widetilde{F}(X, nD)}{\widetilde{F}(X, (n-1)D)} \hookrightarrow H^0(D, \Omega^1_X(nD)_{|D} \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{Sh}_D}(\mathcal{O}_D, F_D)),$$

where $\underline{\mathrm{Hom}}_{\mathrm{Sh}_D}$ *denotes the internal hom in the category of Nisnevich sheaves of abelian groups on smooth schemes over* D.

Some comments:

- If $\mathrm{char}\, k = p > 0$, it follows that the quotient on the left-hand side is p-torsion. This can be seen as an analogue of [46, Corollary 2.28] for reciprocity sheaves.
- The characteristic form for

$$F = H^1_{\text{ét}}(-, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{ab}}(-), \mathbb{Q}/\mathbb{Z})$$

factors via

$$H^0(D, \Omega^1(nD)_{|D}) \to H^0(D, \Omega^1_X(nD)_{|D} \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{Sh}_D}(\mathcal{O}_D, H^1_{\text{ét},D}))$$

induced by the natural map $\mathcal{O}_D \to H^1_{\text{ét},D}$ stemming from the Artin–Schreier sequence, and yields a global version of the characteristic form of Matsuda–Yatagawa (which is a nonlog version of the refined Swan conductor of Kato); see [40, Section 5] for details.

- The characteristic form for differential forms is computed in [40, Theorem 6.6 and Corollary 6.8]. These computations are also used to prove the formula in Theorem 3.2 in positive characteristic.
- It is an intriguing problem to give a general (motivic) description of the image of $\mathrm{char}_F^{(nD)}$. For example, the images of the characteristic forms of differential forms and Witt vectors of finite length in positive characteristic are rather complicated and do not give a direct hint towards a general formula.

Finally, we give an exemplary application of how a local form of Theorem 6.3 reveals an interesting structure of Chow groups of zero-cycles with modulus over local fields of equicharacteristic.

Example 6.4. Let Y be a proper k-scheme with an effective Cartier divisor E, such that $V = Y \setminus E$ is smooth. By [28, Corollary 2.3.5; 47, Theorem 0.1] the Nisnevich sheafification of $\underline{\omega}_! h_0^{\overline{\square}}(Y, E)$ (see Definition 1.5) is a reciprocity sheaf $h_0(Y, E)_{\mathrm{Nis}}$. For a field K over k we have

$$h_0(Y, E)_{\mathrm{Nis}}(K) = \mathrm{CH}_0(Y_K, E_K),$$

where the right-hand side denotes the Chow group of zero-cycles with modulus and $Y_K = Y \otimes_k K$.

Assume L is a henselian discrete valuation field of geometric type over k with ring of integers $\mathcal{O}_L$, maximal ideal $\mathfrak{m}_L$, and residue field $K = \mathcal{O}_L/\mathfrak{m}_L$. For simplicity assume the transcendence degree of L/k is 1, so that all geometric models of $(\mathcal{O}_L, \mathfrak{m}_L)$ have a projective SNC-compactification and $\Omega^1_{\mathcal{O}_L/k} \otimes_{\mathcal{O}_L} K \cong K$. Then $\mathrm{fil}_n := h_0(Y, E)_{\mathrm{Nis}}(S, ns)$, where $S = \operatorname{Spec} \mathcal{O}_L$ and $s \in S$ is the closed point, defines a filtration

$$\mathrm{fil}_0 \subset \mathrm{fil}_1 \subset \cdots \subset \mathrm{fil}_n \subset \cdots \subset \mathrm{CH}_0(Y_L, E_L),$$

where fil_0 is the subgroup of $\mathrm{CH}_0(Y_L, E_L)$ generated by closed points in V_L whose closure in $V \times_k S$ is finite over S. By Theorem 6.3 we have an injection (which depends on the choice of a local parameter; more precisely, we use the isomorphism $\mathfrak{m}^{-n}\Omega^1_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} K = \mathfrak{m}^{-\mathfrak{n}}/\mathfrak{m}^{-\mathfrak{n}+1} \cong K$)

$$\mathrm{fil}_n/\mathrm{fil}_{n-1} \hookrightarrow \underline{\mathrm{Hom}}_{\mathrm{Sh}_K}(\mathcal{O}_K, h_0(Y, E)_{\mathrm{Nis}})(K) \quad \text{for } n \geq 2.$$

However the internal hom on the right is not well understood and it would be interesting to describe the image of the map. Evaluating at 1 induces a canonical map

$$\mathrm{fil}_n/\mathrm{fil}_{n-1} \to \mathrm{CH}_0(Y_K, E_K) \quad \text{for } n \geq 2.$$

This is a new map and it is tempting to view it as a specialization map (depending on the choice of a local parameter). It remains to study its properties more closely, e.g., if it happens to be injective for certain pairs (Y, E).

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The Grothendieck ring of varieties and algebraic K-theory of spaces

Oliver Röndigs

Waldhausen's algebraic K-theory machinery is applied to Morel–Voevodsky $\mathbb{A}^1$-homotopy, producing an interesting $\mathbb{A}^1$-homotopy type. Over a field F of characteristic zero, its path components receive a surjective ring homomorphism from the Grothendieck ring of varieties over F.

1. Introduction

Waldhausen's approach to algebraic K-theory [46] is of such a generality (though not for its own sake) that it applies to a wide class of homotopy theories. The choice here, as suggested by Waldhausen in the last century, is the $\mathbb{A}^1$-homotopy theory over a Noetherian finite-dimensional base scheme S introduced by Morel and Voevodsky [28]. Subject to an appropriate finiteness condition (for which there are several choices), the resulting homotopy type $A(S)$ is nontrivial; for example, it contains Waldhausen's algebraic K-theory of a point, $A(*)$, as a retract up to homotopy. It can be viewed as an $\mathbb{A}^1$-homotopy type in a natural way. The present paper is an admittedly rather meager attempt to advertise this $\mathbb{A}^1$-homotopy type to algebraic geometers, although it might be more attractive to homotopy theorists. Recall that almost by construction, the path components of Waldhausen's K-theory provide the universal Euler characteristic.

Theorem 1. *Let F be a field of characteristic zero. Sending a smooth projective variety to its natural class in $\pi_0 A(F)$ defines a surjective ring homomorphism*

$$K_0(\mathrm{Var}_F) \to \pi_0 A(F)$$

from the Grothendieck ring of varieties over F.

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See Theorem 5.2 for a precise version including the appropriate finiteness condition. This ring homomorphism refines several other motivic measures, such as the topological Euler characteristic, the Hodge motivic measure, and the Gillet–Soulé motivic measure. In these cases the ring homomorphism is naturally induced on path components by a map from the homotopy type $A(F)$, thereby solving at least problems 7.3–7.5 in [10] in a natural way. The constructions [9; 49] supply a homotopy type whose path components form the Grothendieck ring of varieties over F. However, the significance of its higher homotopy groups is not clear. In the case of Waldhausen's original application of algebraic K-theory to the geometry of manifolds, the higher homotopy groups yield interesting information on their automorphism groups [15; 34; 35], thanks to the following statement from [47].

Theorem 2 (Waldhausen). *Let M be a smooth manifold with possibly empty boundary. The homotopy type $A(M)$ is defined as the Waldhausen K-theory of the category of finite cell complexes retractive over M. There is a splitting*

$$A(M) \simeq \Sigma^\infty M_+ \times \mathrm{Wh}(M)$$

up to homotopy, *where $\Sigma^\infty M_+$ is the stable homotopy type of M*, *and* $\mathrm{Wh}(M)$ *is the Whitehead spectrum of M, a double delooping of the spectrum of stable smooth pseudoisotopies of M.*

Taking path components of this splitting recovers the s-cobordism theorem of Smale, Barden, Mazur, and Stallings. Perhaps an $\mathbb{A}^1$-s-cobordism theorem for smooth varieties over a field can be produced via the trace given in Section 6 on the $\mathbb{A}^1$-homotopy type obtained from the algebraic K-theory of dualizable motivic T-spectra.

2. K-theory of model categories

In [46], Waldhausen generalized Quillen's K-theory machinery to the setup of categories with cofibrations and weak equivalences, henceforth called *Waldhausen categories*. A Waldhausen category is a quadruple $(\mathbf{C}, *, \mathrm{w}\mathbf{C}, \mathrm{cof}\mathbf{C})$, where $\mathbf{C}$ is a pointed category with zero object $*$, a subcategory $\mathrm{w}\mathbf{C}$ of weak equivalences and a subcategory $\mathrm{cof}\mathbf{C}$ of cofibrations. Also, $* \to A$ is always a cofibration, cobase changes along cofibrations exist in $\mathbf{C}$, and the weak equivalences satisfy the gluing lemma. The *algebraic K-theory* of a Waldhausen category $(\mathbf{C}, *, \mathrm{w}\mathbf{C}, \mathrm{cof}\mathbf{C})$ is the spectrum

$$A(\mathbf{C}) = (\mathrm{w}\mathbf{C}, \mathrm{w}\mathcal{S}_\bullet\mathbf{C}, \dots, \mathrm{w}\mathcal{S}^{(n)}_\bullet\mathbf{C}, \dots) \tag{1}$$

of pointed simplicial sets obtained by the diagonal of the nerve of n-fold simplicial categories; the latter produced by iterated applications of Waldhausen's

$\mathcal{S}_\bullet$-construction. The structure maps of (1) are induced by the inclusion of the 1-skeleton.

Definition 2.1. An exact functor $F : \mathbf{C} \to \mathbf{D}$ of Waldhausen categories is a *K-theory equivalence* if the induced map $A(F) : A(\mathbf{C}) \to A(\mathbf{D})$ of spectra is a stable equivalence.

All the categories with cofibrations and weak equivalences in the following are obtained as full subcategories of a Quillen model category, where the weak equivalences are determined by the model structure. The cofibrations are either determined by the model structure, or by a slight variation. References for model categories are [17; 18]. Algebraic K-theory requires finiteness conditions, and suitable cofibrantly generated model categories, as defined in [13, Definition 3.4; 19, Definition 4.1], provide a convenient setup for these.

Definition 2.2. A model category $\mathbf{M}$ is *weakly finitely generated* if it is cofibrantly generated and satisfies the following further requirements:

(1) There exists a set I of generating cofibrations with finitely presentable domains and codomains.

(2) There exists a set J of acyclic cofibrations with finitely presentable domains and codomains detecting fibrations with fibrant codomain.

Examples of weakly finitely generated model categories are the usual model categories of (pointed) simplicial sets (denoted $\mathbf{sSet}_\bullet$), spectra of such (denoted $\mathbf{Spt}$), chain complexes over a ring, and suitable model structures for $\mathbb{A}^1$-homotopy theory. The latter is essentially a consequence of the following statement.

Proposition 2.3. *Let $\mathbf{M}$ be a weakly finitely generated simplicial model category, and let Z be a set of morphisms in $\mathbf{M}$ with finitely presentable domains and codomains. Suppose that tensoring with a finite simplicial set L preserves finitely presentable objects. If the left-Bousfield localization $\mathrm{L}_Z\mathbf{M}$ exists, it is weakly finitely generated.*

Proof. The proof of [19, Proposition 4.2] applies. □

In a weakly finitely generated model category $\mathbf{M}$, a fibrant replacement functor

$$\mathrm{fib} : \mathbf{M} \to \mathbf{M},$$

which commutes with filtered colimits, can be constructed by attaching cells from a set of acyclic cofibrations J with finitely presentable domains and codomains. It follows that the natural transformation $\mathrm{Id}_{\mathbf{M}} \to \mathrm{fib}$ is an acyclic cofibration.

For applications in algebraic K-theory recall that, given an object $B \in \mathbf{M}$ in a category, an object *retractive over* B is a pair $(B \xrightarrow{s} D, D \xrightarrow{r} B)$ of morphisms in $\mathbf{M}$ such that $r \circ s = \mathrm{id}_B$. Such a pair will often be abbreviated as "D". With the obvious notion of morphism, these form a category $\mathbf{R}(\mathbf{M}, B)$. A morphism $\phi : B \to C$ in $\mathbf{M}$ induces a functor

$$\phi_! : \mathbf{R}(\mathbf{M}, B) \to \mathbf{R}(\mathbf{M}, C), \quad D \mapsto D \cup_B C$$

having the functor

$$\phi^! : \mathbf{R}(\mathbf{M}, C) \to \mathbf{R}(\mathbf{M}, B), \quad E \mapsto E \times_C B$$

as right adjoint, provided pushouts and pullbacks exist.

Proposition 2.4. *Let $\mathbf{M}$ be a weakly finitely generated model category, and B an object of $\mathbf{M}$. The category $\mathbf{R}(\mathbf{M}, B)$ of objects retractive over B is a weakly finitely generated model category in a natural way. For every $\phi : B \to C$, the pair $(\phi_!, \phi^!)$ is Quillen. If $\mathbf{M}$ is simplicial, then so is $\mathbf{R}(\mathbf{M}, B)$. If $\mathbf{M}$ is a monoidal model category under the cartesian product, then $\mathbf{R}(\mathbf{M}, B)$ is a $\mathbf{R}(\mathbf{M}, *)$-model category.*

Proof. The statement regarding the (simplicial) model structure is [39, Proposition 1.2.2], which implies the Quillen pair property. The statement regarding the generators can be deduced from [39, Lemma 1.3.4]. For later reference, if $I = \{s_i \hookrightarrow t_i\}_{i \in I}$ is the set of generating cofibrations in $\mathbf{M}$, then

$$\left\{B \coprod (s_i \hookrightarrow t_i)\right\}_{i \in I, \psi \in \mathrm{Hom}_{\mathbf{M}}(t_i, B)}$$

is the set of generating cofibration in $\mathbf{R}(\mathbf{M}, B)$, where the maps $\psi : t_i \to B$ define the required retractions. Note that $\phi_!$ preserves this set of generating cofibrations. The final statement follows from [18, Proposition 4.2.9] and the standard pairing

$$\mathbf{R}(\mathbf{M}, B) \times \mathbf{R}(\mathbf{M}, C) \to \mathbf{R}(\mathbf{M}, B \times C), \quad (D, E) \mapsto D \times E \cup_{(B \times E \cup_{B \times C}, D \times C)} B \times C,$$

of retractive objects, which is a Quillen bifunctor. □

Definition 2.5. Let $\mathbf{M}$ be a symmetric monoidal model category, let $\mathbf{N}$ be an $\mathbf{M}$-model category, and let B be an object of $\mathbf{M}$. A *B-spectrum* E in $\mathbf{N}$ consists of a sequence $(\mathsf{E}_0, \mathsf{E}_1, \dots)$ of objects in $\mathbf{N}$ together with a sequence of structure maps

$$\sigma_n^{\mathsf{E}} : \Sigma_B \mathsf{E}_n := \mathsf{E}_n \wedge B \to \mathsf{E}_{n+1}.$$

The category of B-spectra in $\mathbf{N}$ is denoted by $\mathbf{Spt}_B(\mathbf{N})$. Set $\mathbf{Spt}(\mathbf{N}) := \mathbf{Spt}_{S^1}(\mathbf{N})$, where $S^1 \in \mathbf{sSet}_\bullet = \mathbf{M}$ is the category of pointed simplicial sets.

Proposition 2.6. *Let $\mathbf{M}$ be a symmetric monoidal $\mathbf{sSet}_\bullet$-model category, let $\mathbf{N}$ be an $\mathbf{M}$-model category, and let B be a finitely presentable and cofibrant object of $\mathbf{M}$. Suppose that tensoring with a finite simplicial set L preserves finitely presentable objects in $\mathbf{N}$. If $\mathbf{N}$ is weakly finitely generated, then $\mathbf{Spt}_B(\mathbf{N})$ is a weakly finitely generated model category such that $\Sigma_B^\infty : \mathbf{N} \to \mathbf{Spt}_B(\mathbf{N})$ is a left-Quillen functor.*

Proof. The proof given for [19, Theorem 4.12] applies. For later reference, the required sets are listed explicitly. The levelwise model structure on $\mathbf{Spt}_B(\mathbf{N})$ is weakly finitely generated with the sets

$$\mathrm{Fr}\, I := \{\mathrm{Fr}_n\, i\}_{n\geq 0, i\in I} \quad \text{and} \quad \mathrm{Fr}\, J := \{\mathrm{Fr}_n\, j\}_{n\geq 0, j\in J},$$

where Fr_n is the left adjoint of the evaluation functor sending E to E_n, and I and J are sets of maps in $\mathbf{N}$ satisfying Definition 2.2. The statement follows from Proposition 2.3 because the B-stable model structure is a left-Bousfield localization of the levelwise model structure with respect to the set

$$\{\mathrm{Fr}_{n+1}(C \wedge B) \to \mathrm{Fr}_n\, C\}_{n\in\mathbb{N}, C \text{ domain or codomain in } I}$$

of morphisms with finitely presentable domains and codomains. □

As soon as B is a suspension (for example, S^1 itself), the model structure from Proposition 2.6 on B-spectra is stable in the sense of [18, Definition 7.1.1]. In particular, the weak equivalences of B-spectra then satisfy Waldhausen's extension axiom [46, p. 327]. However, since $\mathbf{Spt}_B(\mathbf{M})$ usually does not inherit any monoidality properties from $\mathbf{M}$, one has to use Jeff Smith's symmetric B-spectra instead. Consider [19, Theorem 8.11, Corollary 10.4] for the following.

Proposition 2.7. *Let* $\mathbf{M}$ *be a symmetric monoidal* $\mathbf{sSet}_\bullet$*-model category, let* $\mathbf{N}$ *be an* $\mathbf{M}$*-model category, and let* B *be a finitely presentable and cofibrant object of* $\mathbf{M}$*. Suppose that tensoring with a finite simplicial set* L *preserves finitely presentable objects. If* $\mathbf{N}$ *is weakly finitely generated, then* $\mathbf{SymSpt}_B(\mathbf{N})$ *is a weakly finitely generated* $\mathbf{SymSpt}_B(\mathbf{M})$*-model category such that* $\Sigma_B^\infty : \mathbf{N} \to \mathbf{SymSpt}_B(\mathbf{N})$ *is a left-Quillen functor. If additionally the cyclic permutation on* $B \wedge B \wedge B$ *is homotopic to the identity, the model categories* $\mathbf{SymSpt}_B(\mathbf{N})$ *and* $\mathbf{Spt}_B(\mathbf{N})$ *are Quillen equivalent.*

Proof. The proof given for [19, Theorem 8.11, Corollary 10.4] applies. For later reference, the required sets are listed explicitly. The levelwise model structure on $\mathbf{SymSpt}_B(\mathbf{M})$ is weakly finitely generated with the sets

$$\mathrm{Fr}^{\mathrm{sym}}\, I := \{\mathrm{Fr}^{\mathrm{sym}}_n\, i\}_{n\geq 0, i\in I} \quad \text{and} \quad \mathrm{Fr}^{\mathrm{sym}}\, J := \{\mathrm{Fr}^{\mathrm{sym}}_n\, j\}_{n\geq 0, j\in J},$$

where $\mathrm{Fr}^{\mathrm{sym}}_n$ is the left adjoint of the evaluation functor sending E to E_n, and I and J are sets of maps in $\mathbf{M}$ satisfying Definition 2.2. The statement follows from Proposition 2.3 because the B-stable model structure is a left-Bousfield localization of the levelwise model structure with respect to the set

$$\{\mathrm{Fr}^{\mathrm{sym}}_{n+1}(C \wedge B) \to \mathrm{Fr}^{\mathrm{sym}}_n\, C\}_{n\in\mathbb{N}, C \text{ domain or codomain in } I}$$

of morphisms with finitely presentable domains and codomains. □

Lemma 2.8. *Let* $\mathbf{M}$ *be a symmetric monoidal* $\mathbf{sSet}_\bullet$*-model category, let* $\mathbf{N}$ *be an* $\mathbf{M}$*-model category, and let* B *be a finitely presentable and cofibrant object of* $\mathbf{M}$*. Suppose that tensoring with a finite simplicial set* L *preserves finitely presentable objects in* $\mathbf{N}$*. Suppose that* $\mathbf{N}$ *is weakly finitely generated, and let* $\mathbf{Spt}_B(\mathbf{N})$ *be the stable model category of* B*-spectra in* $\mathbf{N}$*. If* E *is a cofibrant finitely presentable* B*-spectrum in* $\mathbf{N}$ *which is stably contractible, then there exists a natural number* N *such that* E_n *is contractible for every* $n \geq N$*.*

Proof. Let E be finitely presentable and cofibrant. Then E_n is cofibrant and finitely presentable for every n. Moreover, there exists a natural number M such that the structure maps σ_m are isomorphisms for every $m \geq M$. Since the canonical map

$$\mathrm{Fr}_M(\mathsf{E}_M) \to \mathsf{E}$$

from the shifted suspension spectrum of E_M to E is a stable equivalence, one may work with $\mathrm{Fr}_M(\mathsf{E}_M)$ directly. Moreover, one may choose $M = 0$. Thus E_0 is a cofibrant finitely presentable object with the property that $\mathrm{colim}_n \, \Omega^n \, \mathrm{fib}(\Sigma^n \mathsf{E}_0)$ is contractible. Here $\mathrm{fib} : \mathbf{N} \to \mathbf{N}$ is a fibrant replacement functor. Equivalently, the class of the canonical map $\mathsf{E}_0 \to \Omega^n \, \mathrm{fib}(\Sigma^n \mathsf{E}_0)$ becomes the class of the constant map in the colimit

$$[\mathsf{E}_0, \mathrm{fib}(\mathsf{E}_0)] \to [\mathsf{E}_0, \Omega \, \mathrm{fib}(\Sigma_B \mathsf{E}_0)] \to \cdots \to [\mathsf{E}_0, \Omega^n \, \mathrm{fib}(\Sigma_B^n \mathsf{E}_0)] \to \cdots$$

of sets of pointed homotopy classes of maps. As E_0 and $\mathsf{E}_0 \otimes \Delta^1$ are finitely presentable, there exists a natural number N such that the homotopy class of the canonical map $\mathsf{E}_0 \to \Omega^n \, \mathrm{fib}(\Sigma_B^n \mathsf{E}_0)$ coincides with the homotopy class of the constant map. By adjointness, the canonical map $\Sigma_B^n \mathsf{E}_0 \to \mathrm{fib}(\Sigma_B^n \mathsf{E}_0)$ is homotopic to the constant map for every $n \geq N$. Thus $\Sigma_B^n \mathsf{E}_0$ is contractible for every $n \geq N$. □

Definition 2.2 leads to the following finiteness notions. More variations, such as being finitely dominated, are possible.

Definition 2.9. Let $\mathbf{M}$ be a weakly finitely generated pointed model category, and choose a set of generating cofibrations I with finitely presentable domains and codomains. Let $B \in \mathbf{M}$.

(1) The object B is *finite* if it is cofibrant and finitely presentable; in other words, $\mathrm{Hom}_{\mathbf{M}}(B, -)$ commutes with filtered colimits.

(2) The object B is *homotopy finite* if it is cofibrant and weakly equivalent to a finite object.

(3) The object B is I*-finite* if the map $* \to B$ is obtained by attaching finitely many maps from I.

(4) The object B is I*-homotopy finite* if it is cofibrant and weakly equivalent to an I-finite object.

The resulting full subcategories are denoted $\mathbf{M}^{\text{fin}}$, $\mathbf{M}^{\text{hfin}}$, $\mathbf{M}^{\text{ifin}}$, and $\mathbf{M}^{\text{ihfin}}$, respectively.

The category $\mathbf{M}^{\text{ifin}}$ is essentially small, and so is $\mathbf{M}^{\text{fin}}$, at least if $\mathbf{M}$ is locally finitely presentable. This will usually not be the case for $\mathbf{M}^{\text{ihfin}}$ and $\mathbf{M}^{\text{hfin}}$. This set-theoretical issue can be resolved in several ways, but will be ignored in the present approach, following [46, Remark on page 379]. Homotopy finite objects in a weakly finitely generated pointed model category $\mathbf{M}$ are compact in the homotopy category of $\mathbf{M}$, as the proof of [18, Theorem 7.4.3] shows. In the case where $\mathbf{M}$ is a symmetric monoidal model category, another finiteness notion is quite natural and will be used eventually.

Definition 2.10. Let $(\mathbf{M}, \wedge, \mathbb{I})$ be a symmetric monoidal model category. A cofibrant object B is *dualizable* if there exists a cofibrant object C and morphisms $\phi : \mathbb{I} \to B \wedge C$, $\psi : C \wedge B \to \mathbb{I}$ in the homotopy category of $\mathbf{M}$, such that the compositions

$$B \xrightarrow{\phi \wedge B} B \wedge C \wedge B \xrightarrow{B \wedge \psi} B \quad \text{and} \quad C \xrightarrow{C \wedge \phi} C \wedge B \wedge C \xrightarrow{\psi \wedge C} C$$

are the respective identities. The full subcategory of cofibrant and dualizable objects is denoted $\mathbf{M}^{\text{dual}}$.

The categories introduced in Definitions 2.9 and 2.10 are equipped with a subcategory of weak equivalences by intersecting with $\mathrm{w}\mathbf{M}$, and with a subcategory of cofibrations by intersecting with $\mathrm{cof}\mathbf{M}$ in the cases of $\mathbf{M}^{\text{fin}}$, $\mathbf{M}^{\text{hfin}}$, and $\mathbf{M}^{\text{dual}}$. In the cases of $\mathbf{M}^{\text{ifin}}$ and $\mathbf{M}^{\text{ihfin}}$ this may lead to trouble with the required existence of cobase changes. The subcategory of cofibrations in $\mathbf{M}^{\text{ifin}}$ consists of those maps obtained by attaching finitely many cells from I, and in $\mathbf{M}^{\text{ihfin}}$ it is simply maps obtained by attaching cells from I.

Lemma 2.11. *Let* $\mathbf{M}$ *be a weakly finitely generated model category, and choose a set of generating cofibrations I with finitely presentable domains and codomains. If $B \hookrightarrow E$ is a cofibration of finitely presentable objects in* $\mathbf{M}$, *there exists a finite I-cofibration $B \hookrightarrow C$ in* $\mathbf{M}$ *such that $B \hookrightarrow E$ is a retract of $B \hookrightarrow C$.*

Proof. Factor $B \hookrightarrow E$ via the small object argument applied to I, to obtain a lifting problem

$$\begin{array}{ccc} B & \xrightarrow{i} & D \\ \downarrow & & \downarrow \sim \\ E & \xrightarrow{\text{id}} & E \end{array}$$

which can be solved. The object D is a sequential colimit of a diagram

$$B = D_{-1} \hookrightarrow D_0 \hookrightarrow \cdots \hookrightarrow D_n \hookrightarrow D_{n+1} \hookrightarrow \cdots$$

such that D_{n+1} is obtained by attaching I-cells to D_n indexed by a specific subset of the disjoint union $\coprod_{i\in I}\operatorname{Hom}_{\mathbf{M}}(\operatorname{dom}(i), D_n)$ for every n. Since E is finitely presentable, a lift $E \to D$ factors via a morphism $E \to D_{n+1}$. This object is the filtered colimit of objects $D_{n,\alpha}$ which are obtained by attaching finitely many cells to D_n. This colimit is indexed over certain finite subsets $\alpha \subset \coprod_{i\in I}\operatorname{Hom}_{\mathbf{M}}(\operatorname{dom}(i), D_n)$. Again since E is finitely presentable, there exists a finite subset $\beta \subset \coprod_{i\in I}\operatorname{Hom}_{\mathbf{M}}(\operatorname{dom}(i), D_n)$ and a factorization over $E \to D_{n,\beta}$. Since the domains of the morphisms in I are finitely presentable, one may proceed in the same fashion for every domain in β inductively to obtain a factorization $E \to C$ where C is obtained by attaching finitely many I-cells. □

Already algebraic examples such as chain complexes $\mathbf{Ch}(R)$ over a ring R show that the classes of finite and I-finite objects can be different, for example, on the level of algebraic K-theory. Bounded chain complexes of finitely generated projective R-modules are the finite objects in the standard model category of all chain complexes of R-modules given in [18, Definition 2.3.3], whereas the I-finite objects are the bounded chain complexes of finitely generated free R-modules. One may use [48, Corollary II.26.3 and Theorem II.9.2.2] to conclude that $K_0(\mathbf{Ch}^{\mathrm{ifin}}(\mathbb{Z}[\sqrt{-5}])) \cong \mathbb{Z}$ and $K_0(\mathbf{Ch}^{\mathrm{fin}}(\mathbb{Z}[\sqrt{-5}])) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proposition 2.12. *Let* $\mathbf{M}$ *be a weakly finitely generated model category, and choose a set* I *of generating cofibrations with finitely presentable domains and codomains. Then the following hold.*

(1) *With these choices,* $\mathbf{M}^{\mathrm{fin}}$, $\mathbf{M}^{\mathrm{hfin}}$, $\mathbf{M}^{\mathrm{ifin}}$ *and* $\mathbf{M}^{\mathrm{ihfin}}$ *are categories with cofibrations and weak equivalences.*

(2) *The horizontal functors in the commutative diagram*

$$\begin{array}{ccc} \mathbf{M}^{\mathrm{ifin}} & \longrightarrow & \mathbf{M}^{\mathrm{ihfin}} \\ \downarrow & & \downarrow \\ \mathbf{M}^{\mathrm{fin}} & \longrightarrow & \mathbf{M}^{\mathrm{hfin}} \end{array} \tag{2}$$

of exact inclusion functors are K*-theory equivalences.*

(3) *The path components of the homotopy fiber of the map* $A(\mathbf{M}^{\mathrm{if}} \hookrightarrow \mathbf{M}^{\mathrm{fin}})$ *are all contractible.*

Proof. Consider statement (1) first. The gluing lemma follows from the cube lemma for model categories [18, Lemma 5.2.6]. It remains to check in each case that a cobase change in $\mathbf{M}$ along the cofibration in question does not lead outside of the category in question. For $\mathbf{M}^{\mathrm{fin}}$ this follows, since a pushout of finitely presentable objects is again finitely presentable, and cofibrancy is preserved.

In the case of $\mathbf{M}^{\text{hfin}}$, let $C \leftarrow B \hookrightarrow D$ be a diagram in $\mathbf{M}^{\text{hfin}}$ such that $B \hookrightarrow D$ is a cofibration. Let $B \to \text{fib}(B)$ be a fibrant replacement obtained by attaching cells from a set J with finitely presentable domains and codomains. Choose a weak equivalence $B' \xrightarrow{\sim} \text{fib}(B)$ from a finite object. The gluing lemma then implies that one may choose B to be finite. Similarly to the above, choose a finite object D' and weak equivalences $D \xrightarrow{\sim} \text{fib}(D') \xleftarrow{\sim} D'$. Note that $D' \xrightarrow{\sim} \text{fib}(D')$ is the filtered colimit of certain maps $D' \xrightarrow{\sim} D''$ which are obtained by attaching finitely many maps from J. Since B is finitely presentable, $B \hookrightarrow D \xrightarrow{\sim} \text{fib}(D')$ lifts to such a D'', and analogously for C. By assumption on J and the gluing lemma, statement (1) follows.

Statement (1) follows for $\mathbf{M}^{\text{ifin}}$ basically by definition. The argument in the case of $\mathbf{M}^{\text{ihfin}}$ is similar to the argument in the case of $\mathbf{M}^{\text{hfin}}$, which concludes the proof of statement (1).

Diagram (2) exists because the domains and codomains of the maps in I are finitely presentable. The statements for $A(\mathbf{M}^{\text{fin}}) \to A(\mathbf{M}^{\text{hfin}})$ and for $A(\mathbf{M}^{\text{ifin}}) \to A(\mathbf{M}^{\text{ihfin}})$ follow from [38, Theorem 2.8]. For statement (3) observe that every object in $\mathbf{M}^{\text{fin}}$ is a retract of an object in $\mathbf{M}^{\text{ifin}}$ by Lemma 2.11. One then concludes with [40, Theorem 1.10.1]. □

Proposition 2.13. *Let* $\mathbf{M}$ *be a symmetric monoidal stable model category. Then* $\mathbf{M}^{\text{dual}}$ *is a Waldhausen category in the natural way described above.*

Proof. It remains to prove that if $C \leftarrow B \hookrightarrow D$ is a diagram in $\mathbf{M}^{\text{dual}}$, then its pushout $C \cup_B D$ is again dualizable. This follows, for example, from [26, Theorem 0.1]. □

The next statement is quite useful, since it implies that through the eyes of algebraic K-theory, restriction to stable model categories is acceptable. This in turn allows the full applicability of Waldhausen's theory, since the weak equivalences then satisfy the extension axiom. Its proof goes back to [45] which led to [36].

Theorem 2.14. *Let* $\mathbf{M}$ *be a pointed simplicial model category such that tensoring with a finite simplicial set preserves finitely presentable objects. Suppose that* $\mathbf{M}$ *is weakly finitely generated, and let* $\mathbf{Spt}(\mathbf{M})$ *be the stable model category of* S^1*-spectra in* $\mathbf{M}$*. Then the suspension spectrum functor induces a* K*-theory equivalence*

$$A(\mathbf{M}^{\mathrm{g}}) \to A(\mathbf{Spt}^{\mathrm{g}}(\mathbf{M})),$$

where $\mathrm{g} \in \{\text{fin}, \text{hfin}\}$*. If* Σ *preserves cofibrations in* $\mathbf{M}^{\text{ifin}}$*, then the same is true for* $\mathrm{g} \in \{\text{ifin}, \text{ihfin}\}$*.*

Proof. Consider first $\mathrm{g} = \text{fin}$. By the additivity theorem [46, Theorem 1.4.2] the suspension functor $\Sigma = - \wedge S^1$ induces a K-theory equivalence. Consider the colimit of

$$\mathbf{M}^{\text{fin}} \xrightarrow{\Sigma} \mathbf{M}^{\text{fin}} \xrightarrow{\Sigma} \cdots$$

in the category of Waldhausen categories. There is an isomorphism

$$\operatorname*{colim}_{\Sigma} \mathcal{S}_{\bullet}\mathbf{M}^{\mathrm{fin}} \cong \mathcal{S}_{\bullet} \operatorname*{colim}_{\Sigma} \mathbf{M}^{\mathrm{fin}},$$

which implies that the canonical functor $\mathbf{M}^{\mathrm{fin}} \to \operatorname{colim}_{\Sigma} \mathbf{M}^{\mathrm{fin}}$ is a K-theory equivalence. A S^1-spectrum E is *strictly finite* if there exists a natural number $N = N(\mathsf{E})$ such that E_N is finite and for every $n \geq N$ the structure map $\sigma_n : \Sigma\mathsf{E}_n \to \mathsf{E}_{n+1}$ is the identity. Let $\mathbf{Spt}^{\mathrm{sf}}(\mathbf{M})$ denote the full subcategory of strictly finite S^1-spectra which are also cofibrant. It is a Waldhausen category in a natural way, and the inclusion $\mathbf{Spt}^{\mathrm{sf}}(\mathbf{M}) \hookrightarrow \mathbf{Spt}^{\mathrm{fin}}(\mathbf{M})$ is an exact equivalence. In particular, the inclusion is a K-theory equivalence.

Let $\Phi : \mathbf{Spt}^{\mathrm{sf}}(\mathbf{M}) \to \operatorname{colim}_{\Sigma} \mathbf{M}^{f}$ denote the functor sending E to the equivalence class of (E_n, n), where $n \geq N(\mathsf{E})$. This functor is well defined, preserves cofibrations, and pushouts essentially by construction. Moreover, it preserves weak equivalences by Lemma 2.8. It is straightforward to verify that Φ satisfies the conditions of Waldhausen's approximation theorem [46, Theorem 1.6.7]. Thus Φ is a K-theory equivalence. Lastly, note the suspension spectrum functor $\Sigma^{\infty} : \mathbf{M}^{\mathrm{fin}} \to \mathbf{Spt}^{\mathrm{fin}}(\mathbf{M})$ factors as

$$\mathbf{M}^{\mathrm{fin}} \to \mathbf{Spt}^{\mathrm{sf}}(\mathbf{M}) \hookrightarrow \mathbf{Spt}^{\mathrm{fin}}(\mathbf{M}),$$

which completes the proof for $\mathrm{g} = \mathrm{fin}$. The case $\mathrm{g} = \mathrm{hfin}$ then follows from Proposition 2.12. The extra assumption on Σ implies that these arguments apply also to $\mathrm{g} \in \{\mathrm{ifin}, \mathrm{ihfin}\}$. □

Theorem 2.14 provides many examples of nonequivalent homotopy theories having the same K-theory.

3. $\mathbb{A}^1$-homotopy theory

Motivic or $\mathbb{A}^1$-homotopy theory was introduced in [28]. Its stabilization is considered in [22]. For technical reasons, the unstable projective version (which is the basis of [14]) is more convenient, although the closed motivic model structure described in [31, Appendix] seems to be quite ideal for the comparison with the Grothendieck ring of varieties.

A *base scheme* is a Noetherian separated scheme of finite Krull dimension. A *motivic space* over S is a presheaf on the site Sm_S of smooth separated S-schemes with values in the category of simplicial sets. Let $\mathbf{M}(S)$ denote the category of pointed motivic spaces.

Example 3.1. Any scheme X in Sm_S defines a discrete representable motivic space over S which is also denoted X, and a discrete representable pointed motivic space X_+ over S. One has $X_+(Y) = \mathbf{Set}_{\mathrm{Sm}_S}(Y, X)_+$, where B_+ denotes the set B with a disjoint basepoint. Any (pointed) simplicial set L defines a constant (pointed) motivic space which is also denoted L.

Many model structures exist on $\mathbf{M}(S)$ having the Morel–Voevodsky $\mathbb{A}^1$-homotopy category of S as its homotopy category. Waldhausen's setup of algebraic K-theory requires specific choices. The following model structure is well suited for base change (see [28, Example 3.1.22]).

Definition 3.2. *Cofibrations* in $\mathbf{M}(S)$ are generated by the set

$$\{(X \times (\partial\Delta^n \hookrightarrow \Delta^n))_+\}_{X \in \mathrm{Sm}_S,\, n \geq 0}. \tag{3}$$

Applying the small object argument to this set produces a cofibrant replacement functor $\kappa : (-)^c \to \mathrm{Id}_{\mathbf{M}(S)}$. A pointed motivic space B is *fibrant* if

- $B(X)$ is a fibrant simplicial set for all $X \in \mathrm{Sm}_S$,
- the image of every Nisnevich elementary distinguished square

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

in Sm_S under B is a homotopy pullback square of simplicial sets,
- $B(\varnothing)$ is contractible, and
- for every $X \in \mathrm{Sm}_S$, the map $B(X \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}} X)$ is a weak equivalence of simplicial sets.

A map $\phi : D \to B$ of pointed motivic spaces over S is a *weak equivalence* if, for every fibrant motivic space C, the induced map

$$\mathbf{sSet}_{\mathbf{M}(S)}(\phi^c, C) : \mathbf{sSet}_{\mathbf{M}(S)}(B^c, C) \to \mathbf{sSet}_{\mathbf{M}(S)}(D^c, C)$$

is a weak equivalence of simplicial sets. A map of motivic spaces is a *fibration* if it has the right lifting property with respect to all cofibrations which are also weak equivalences (the *acyclic* cofibrations).

Theorem 3.3. *The classes from Definition 3.2 define a symmetric monoidal* $\mathbf{sSet}_\bullet$*-model structure on* $\mathbf{M}(S)$ *which is weakly finitely generated. It is Quillen equivalent to the Morel–Voevodsky model.*

Proof. See [14, Section 2.1] for a proof. □

Remark 3.4. The smash product of pointed motivic spaces is defined sectionwise. The smash product of a weak equivalence with an arbitrary pointed motivic space is a weak equivalence. Since the domains and codomains of the generating cofibrations are finitely presentable, a filtered colimit of weak equivalences is again a weak equivalence. A filtered colimit of fibrant motivic spaces is again fibrant.

Proposition 2.6 applies to the model structure from Theorem 3.3. The two relevant examples are S^1-spectra $\mathbf{Spt}(S) := \mathbf{Spt}_{S^1}(\mathbf{M}(S))$ and T-spectra $\mathbf{Spt}_T(S) := \mathbf{Spt}_T(\mathbf{M}(S))$, as well as their symmetric analogues $\mathbf{SymSpt}(S)$ and $\mathbf{SymSpt}_T(S)$. Here $S^1 = \Delta^1/\partial\Delta^1$ is the constant simplicial circle, and $T = S^1 \wedge S^{1,1}$, where $S^{1,1}$ is the simplicial mapping cylinder of the unit $S \hookrightarrow \mathbb{G}_m$ in the multiplicative group scheme over S.

Definition 3.5. Let S be a base scheme. Then I_S denotes the set of generating cofibrations in $\mathbf{M}(S)$ given in (3), or (if no confusion can arise) the corresponding set of generating cofibrations in (symmetric) B-spectra over S as introduced in the proof of Proposition 2.6 and 2.7, respectively.

If $f : X \to Y$ is a morphism of base schemes, pullback along f defines a functor $\mathrm{Sm}_Y \to \mathrm{Sm}_X$. Precomposition with this functor yields another functor, denoted $f_* : \mathbf{M}(X) \to \mathbf{M}(Y)$. On objects

$$(f_*B)(Z) = B(X \times_Y Z) \tag{4}$$

for any $Z \in \mathrm{Sm}_Y$. Via left-Kan extension, f_* has a left adjoint $f^* : \mathbf{M}(Y) \to \mathbf{M}(X)$ which is strict symmetric monoidal. Since every motivic space is a colimit of representable ones, f^* is characterized by the formula

$$f^*(Z_+) = (X \times_Y Z)_+ \tag{5}$$

for every $Z \in \mathrm{Sm}_Y$.

Example 3.6. Base change describes the internal hom in $\mathbf{M}(X)$ as

$$\mathbf{M}(X)(C, B)(Z \xrightarrow{z} X) = \mathbf{sSet}_{\mathbf{M}(X)}(C, z_*z^*B).$$

Note that if f is smooth, the canonical natural transformation

$$f^*\mathbf{M}(Y)(C, B) \to \mathbf{M}(X)(f^*C, f^*B) \tag{6}$$

is a natural isomorphism.

If $f : X \to Y$ is a smooth morphism of base schemes, composition with f defines a functor $\mathrm{Sm}_X \to \mathrm{Sm}_Y$. Precomposition with this functor defines the functor $f^\star : \mathbf{M}(Y) \to \mathbf{M}(X)$, which then has a left adjoint $f_\sharp : \mathbf{M}(X) \to \mathbf{M}(Y)$ by (enriched) Kan extension. Since every motivic space is a colimit of representable ones, $f_\sharp$ is characterized by the formula

$$f_\sharp(Z \xrightarrow{z} X)_+ = (Z \xrightarrow{z} X \xrightarrow{f} Y)_+ \tag{7}$$

for every $Z \in \mathrm{Sm}_X$. If $Z \to Y$ is in Sm_Y, the canonical Y-morphism $X \times_Y Z \to Z$ defines a map $B(Z) \to f_*f^\star B(Z)$ which is natural in Z and $B \in \mathbf{M}(Y)$, and hence a natural transformation $\mathrm{Id}_{\mathbf{M}(Y)} \to f_* \circ f^\star$.

Lemma 3.7. *If $f : X \to Y$ is a smooth morphism of base schemes, the adjoint*

$$f^* \to f^\star$$

of the natural transformation $\mathrm{Id}_{\mathbf{M}(Y)} \to f_* \circ f^\star$ *is a natural isomorphism.*

Proof. This is straightforward. □

In the following, $f^\star$ will be used implicitly as a concrete description for the left-Kan extension f^* whenever f is smooth. It has the advantage that it is strictly functorial. These base change functors can be extended to the category of (symmetric) B-spectra by levelwise application in the case $B \in \{S^1, T\}$. This extension involves the identification $f^*(B_Y) \xrightarrow{\cong} B_X$, where $f : X \to Y$ and B_S indicates that B is a pointed motivic space over S. They are still denoted by $f_* : \mathbf{Spt}_B(X) \to \mathbf{Spt}_B(Y)$, etc.

Proposition 3.8. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of base schemes.*

(1) *There is an equality $(g \circ f)_* = g_* \circ f_*$ and a unique natural isomorphism $(g \circ f)^* \xrightarrow{\cong} f^* \circ g^*$.*

(2) *If f and g are smooth, the unique natural isomorphism $(g \circ f)^* \xrightarrow{\cong} f^* \circ g^*$ is the identity, and there is a unique natural isomorphism $(g \circ f)_\sharp \xrightarrow{\cong} g_\sharp \circ f_\sharp$.*

(3) *There are equalities* $\mathrm{id}_* = \mathrm{Id}$, $\mathrm{id}^* = \mathrm{Id}$, *and a natural isomorphism* $\mathrm{id}_\sharp \xrightarrow{\cong} \mathrm{Id}$.

(4) *The diagrams*

$$\begin{array}{ccc} \mathbf{M}(X) & \xrightarrow{f_*} & \mathbf{M}(Y) \\ \downarrow{\scriptstyle \Sigma^\infty_B} & & \downarrow{\scriptstyle \Sigma^\infty_B} \\ \mathbf{Spt}(X, B) & \xrightarrow{f_*} & \mathbf{Spt}(Y, B) \end{array} \qquad \begin{array}{ccc} \mathbf{M}(Y) & \xrightarrow{f^*} & \mathbf{M}(X) \\ \downarrow{\scriptstyle \Sigma^\infty_B} & & \downarrow{\scriptstyle \Sigma^\infty_B} \\ \mathbf{Spt}(Y, B) & \xrightarrow{f^*} & \mathbf{Spt}(X, B) \end{array}$$

commute, and similarly for $f_\sharp$, and for symmetric spectra.

Proof. This is straightforward. See also [1, Chapitre 4]. □

Lemma 3.9. *Let $f : X \to Y$ be a morphism of base schemes. Then $f^*(I_Y) \subseteq I_X$, and $f_\sharp(I_X) \subseteq I_Y$ if f is smooth.*

Proof. This follows from direct inspection. □

Proposition 3.10. *Let $f : X \to Y$ be a morphism of base schemes. Then (f^*, f_*) is a Quillen adjoint pair. If f is smooth, then $(f_\sharp, f^*)$ is a Quillen adjoint pair.*

Proof. Consider the case of pointed motivic spaces first. Lemma 3.9 implies that f^* and (if f is smooth) $f_\sharp$ preserve cofibrations. To prove the first statement, it remains to show that f_* preserves fibrations. By Dugger's lemma [12, A.2], it suffices to prove that f_* preserves fibrations between fibrant motivic spaces. These fibrations are detected by the set of acyclic cofibrations described in Remark 3.4.

Hence it suffices to prove that f^* maps each of these special acyclic cofibrations in $\mathbf{M}(Y)$ to an acyclic cofibration in $\mathbf{M}(X)$. This is straightforward by (5). The proof for the second statement is similar, using (7).

By Proposition 3.10, f^* preserves cofibrations. However, one can see directly that $f^*(\mathrm{Fr}\, I_Y) \subseteq \mathrm{Fr}\, I_X$ since f^* commutes with the functors Fr_n. As in the proof of the preceding case, it remains to prove that f_* preserves stable fibrations of stably fibrant motivic B_X-spectra. Those coincide with the levelwise fibrations. Since f_* preserves fibrations, it suffices to prove that f_* preserves stably fibrant motivic B_X-spectra. This in turn follows from the preceding case, since f_* preserves weak equivalences of fibrant pointed motivic spaces.

If $f : X \to Y$ is smooth, $f_\sharp$ preserves cofibrations of motivic B_X-spectra. However, one can see directly that $f_\sharp(\mathrm{Fr}\, I_X) \subseteq \mathrm{Fr}\, I_Y$ since $f_\sharp$ commutes with the functors Fr_n. It remains to prove that f^* preserves fibrations of stably fibrant motivic B_Y-spectra. As above, it suffices to check that f^* preserves stably fibrant motivic B_Y-spectra. This follows from isomorphism (6), together with the fact that f^* preserves all weak equivalences of pointed motivic spaces. The latter is implied by the fact that f^* is both a left- and a right-Quillen functor. □

Lemma 3.11. *Let $f : X \to Y$ be a morphism of base schemes. The functor f^* preserves finite objects, I-finite objects, and dualizable objects. If f is smooth, $f_\sharp$ preserves finite objects and I-finite objects.*

Proof. The statements about I-finiteness appear in the proof of Proposition 3.10. The implicit statements about cofibrancy follow from Proposition 3.10. Observe that f^* preserves finitely presentable objects since its right adjoint f_* preserves filtered (even all!) colimits, and similarly for $f_\sharp$. Since f^* is strict symmetric monoidal, it preserves dualizable objects. □

Let $\widetilde{\mathrm{Sm}}_S$ be the subcategory of Sm_S having the same objects, but only smooth S-morphisms as morphisms. One may summarize some of the results above by saying that the model categories considered so far are Quillen functors on $\widetilde{\mathrm{Sm}}_S$, but only Quillen pseudofunctors on Sm_S. It is possible to strictify these Quillen pseudofunctors to a naturally (not just Quillen) equivalent Quillen functor on Sm_S by the categorical result [32]. This will be assumed from now on, without applying notational changes.

4. Algebraic K-theory of $\mathbb{A}^1$-homotopy theory

Let $\mathbf{M}(S)$ be the model category of pointed motivic spaces over S, equipped with the $\mathbb{A}^1$-local Nisnevich model structure given in Definition 3.2. Before applying Waldhausen's K-theory construction, one of the finiteness notions introduced in Definition 2.9 will be imposed, indicated by the respective superscript $\mathbf{M}^{\mathrm{g}}(S)$ for

$\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. Unless otherwise specified, the I-finiteness notions always refer to the set of generating cofibrations listed in Definition 3.2.

Definition 4.1. Let $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. Then $A(\mathbf{M}^{\mathrm{g}}(S))$ denotes the spectrum obtained by applying Waldhausen's $\mathcal{S}_\bullet$-construction to the Waldhausen category $\mathbf{M}^{\mathrm{g}}(S)$.

Technically speaking, $A(\mathbf{M}^{\mathrm{g}}(S))$ is the algebraic K-theory of the one-point motivic space over S. It is possible to consider the algebraic K-theory of an arbitrary motivic space B over S by viewing the canonical Waldhausen category of g-finite motivic spaces over S which are retractive over B, as mentioned abstractly in Proposition 2.4.

Proposition 4.2 (Waldhausen). *Suppose* $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. *The spectrum* $A(\mathbf{M}^{\mathrm{g}}(\mathbb{C}))$ *contains* $A(*)$ *as a retract. In particular, it is nontrivial.*

Proof. The constant pointed motivic space functor $\mathbf{sSet}_\bullet \to \mathbf{M}(S)$ and the complex realization functor $\mathbf{M}(\mathbb{C}) \to \mathbf{Top}_\bullet$ are left-Quillen functors. The constant pointed motivic space sends (homotopy) finite pointed simplicial sets to $I_{\mathbb{C}}$-finite pointed motivic spaces. The complex realization functor sends representables to homotopy finite pointed topological spaces, and hence $I_{\mathbb{C}}$-finite pointed motivic spaces to homotopy finite pointed topological spaces. A finite pointed motivic space is a retract of an $I_{\mathbb{C}}$-finite pointed motivic space. Since homotopy finite pointed topological spaces are closed under retracts, the complex realization functor preserves homotopy finiteness. Hence both functors induce maps on Waldhausen K-theory spectra. Their composition coincides with the geometric realization functor

$$|-| : \mathbf{sSet}_\bullet \to \mathbf{Top}_\bullet,$$

which induces an equivalence on Waldhausen K-theory by [46, Theorem 2.1.5]. The statement follows. □

Proposition 4.3. *Let S be a base scheme and* $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. *The spectrum* $A(\mathbf{M}^{\mathrm{g}}(S))$ *contains* $A(*)$ *as a retract. In particular, it is nontrivial.*

Proof. It suffices to consider a connected base scheme S. Let $\mathbf{M}_{\mathrm{hell}}(S)$ be the left-Bousfield localization of the Nisnevich local projective model structure with respect to the maps $X \to S$ in Sm_S such that X is connected. It is a left-Bousfield localization of the $\mathbb{A}^1$-Nisnevich local projective model structure, and again weakly finitely generated. The identity functor is a left-Quillen functor from $\mathbf{M}(S)$ to $\mathbf{M}_{\mathrm{hell}}(S)$ preserving finitely presentable cofibrant pointed motivic spaces. If $B \in \mathbf{M}_{\mathrm{hell}}(S)$ is fibrant, it is locally constant, since $B(S) \to B(X)$ is a weak equivalence for every smooth morphism $X \to S$ such that X is connected.

The constant pointed motivic space functor $\mathbf{sSet}_\bullet \to \mathbf{M}_{\mathrm{hell}}(S)$ is a left-Quillen functor, but also a Quillen equivalence. Its right adjoint is the evaluation at the terminal scheme. A map of fibrant objects in $\mathbf{M}_{\text{hell}}(S)$ is a weak equivalence if and only if it is a levelwise weak equivalence. Let $B \to C$ be a map of motivic spaces over S which are fibrant in $\mathbf{M}_{\text{hell}}(S)$. If the map $B(S) \to C(S)$ is a weak equivalence, then $B(X) \to C(X)$ is a weak equivalence for every connected S-scheme X. Since every smooth S-scheme admits a Zariski open cover by smooth connected S-schemes, $B(X) \to C(X)$ is then a weak equivalence for every smooth S-scheme. It follows that evaluation at the terminal scheme S preserves and detects weak equivalences of fibrant objects in $\mathbf{M}_{\text{hell}}(S)$. If L is a pointed simplicial set, considered as a constant motivic space over S, its fibrant replacement in $\mathbf{M}_{\text{hell}}(S)$ sends X to the product of L indexed over the connected components of X. In particular, the derived unit of the adjunction is the identity. This concludes the proof that the constant motivic space functor $\mathbf{sSet}_\bullet \to \mathbf{M}_{\text{hell}}(S)$ is a Quillen equivalence.

Both the constant motivic space functor $\mathbf{sSet}_\bullet \to \mathbf{M}(S)$ and the identity functor $\mathbf{M}(S) \to \mathbf{M}_{\text{hell}}(S)$ preserve finite and I-finite objects, hence induce maps on suitable Waldhausen categories. Since $\mathbf{sSet}_\bullet \to \mathbf{M}_{\text{hell}}(S)$ is a Quillen equivalence, it is a K-theory equivalence by [38, Theorem 3.3]. The result follows. □

The full technology of Waldhausen's algebraic K-theory of spaces requires that the weak equivalences satisfy the extension axiom. This axiom is not satisfied in the category of pointed motivic spaces (the counterexample given for pointed simplicial sets in [46, Section 1.2] right after the definition of the extension axiom extends). However, it is satisfied in the category of S^1-spectra of pointed motivic spaces over S. The suspension spectrum functor induces a K-theory equivalence, as one deduces from the following theorem.

Remark 4.4. Theorem 2.14, including its assertion for the I-finiteness notions, shows that

$$A^{\mathrm{g}}(S) := A(\mathbf{Spt}^{\mathrm{g}}(S)) \leftarrow A(\mathbf{M}^{g}(S))$$

is a K-theory equivalence for every $\mathrm{g} \in \{\text{fin, hfin, ifin, ihfin}\}$. Moreover, it turns out to be natural in the base scheme S. Thus in the discussion below Waldhausen's fibration theorem may be applied to $A^{\mathrm{g}}(S)$.

As a consequence of Lemma 3.11 and Proposition 3.10, the functors f^* and (if applicable) $f_\sharp$ induce exact functors on Waldhausen categories.

Proposition 4.5. *Let $j : U \hookrightarrow S$ be an open embedding of base schemes, with reduced closed complement $i : Z \hookrightarrow S$. Then the functors j^* and i^* induce a splitting*

$$A^{\mathrm{g}}(S) \xrightarrow{\sim} A^{\mathrm{g}}(U) \times A^{\mathrm{g}}(Z)$$

for $\mathrm{g} \in \{\text{fin, hfin, ifin, ihfin}\}$.

Proof. Consider the homotopy fiber sequence

$$\mathrm{hofib}(j^*) \to A^{\mathrm{g}}(S) \xrightarrow{j^*} A^{\mathrm{g}}(U)$$

of spectra. In order to identify the homotopy fiber of j^*, the map $A^{\mathrm{g}}(S) \xrightarrow{j^*} A^{\mathrm{g}}(U)$ is factored as follows. Let $\mathrm{v}\mathbf{Spt}^{\mathrm{g}}(S)$ denote the subcategory of maps f such that $j^*(f)$ is a weak equivalence in $\mathbf{Spt}(U)$. Let $\mathbf{Spt}^{\mathrm{g}}(S|U)$ denote the resulting Waldhausen category $(\mathbf{Spt}^{\mathrm{g}}(S), *, \mathrm{v}\mathbf{Spt}^{\mathrm{g}}(S), \mathrm{cof}\mathbf{Spt}^{\mathrm{g}}(S))$. The identity can then be regarded as an exact functor $\Phi : \mathbf{Spt}^{\mathrm{g}}(S) \to \mathbf{Spt}^{\mathrm{g}}(S|U)$. Almost by definition, $j^* : \mathbf{Spt}^{\mathrm{g}}(S|U) \to \mathbf{Spt}^{\mathrm{g}}(M(U))$ satisfies the conditions of Waldhausen's approximation theorem [46, Theorem 1.6.7]. In fact, j^* detects and preserves weak equivalences by definition. If E is a g-finite S^1-spectrum over S and $j^*(\mathsf{E}) \to \mathsf{D}$ is a map of g-finite S^1-spectra over U, consider it as the map

$$j^*(\mathsf{E}) = j^* j_\sharp j^*(\mathsf{E}) \to j^* j_\sharp(\mathsf{D}) = (\mathsf{D}).$$

Here the fact that the unit $\mathrm{Id} \to j^* j_\sharp$ is the identity enters. The map $j_\sharp j^*(\mathsf{E}) \to j_\sharp(\mathsf{D})$ can be factored via the simplicial mapping cylinder as a cofibration of g-finite S^1-spectra over U, followed by a weak equivalence. Therefore, the functor $j^* : \mathbf{Spt}^{\mathrm{g}}(S|U) \to \mathbf{Spt}^{\mathrm{g}}(M(U))$ satisfies the second approximation property, whence it is a K-theory equivalence by the approximation theorem [46, Theorem 1.6.7]. Thus $\mathrm{hofib}(j^*)$ is weakly equivalent to the homotopy fiber of Φ. The latter may be identified, by the fibration theorem [46, Theorem 1.6.4], with the algebraic K-theory of the sub-Waldhausen category $\mathbf{Spt}^{\mathrm{g}, j^* \simeq *}(S)$ of g-finite S^1-spectra E over S such that $j^*(\mathsf{E})$ is (weakly) contractible. The homotopy cofiber sequence

$$j_\sharp j^*(\mathsf{E}) \to \mathsf{E} \to i_* i^*(\mathsf{E}),$$

which is an S^1-spectrum version of [28, Theorem 3.2.21], implies that the induced functor $i^* : \mathbf{Spt}^{\mathrm{g}, j^* \simeq *}(S) \to \mathbf{Spt}^{\mathrm{g}}(Z)$ satisfies the special approximation property. Thus i^* induces a K-theory equivalence by [38, Theorem 2.8]. It remains to observe the splitting, which is induced by $j_\sharp : \mathbf{M}(U) \to \mathbf{M}(S)$, the left adjoint of j^*. It is a left-Quillen functor preserving the set of generating cofibrations defined in Definition 3.2. The unit $\mathrm{Id} \to j^* j_\sharp$ is the identity, since j is an open embedding. □

In particular, the map $A^{\mathrm{g}}(S) \to A^{\mathrm{g}}(\mathbb{A}^1_S)$ induced by the projection is not a weak equivalence for every $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$, because $A^{\mathrm{g}}(\mathbb{A}^1_S \smallsetminus \{0\})$ is not contractible by Proposition 4.3.

Corollary 4.6. *Let* $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. *There is a natural weak equivalence*

$$\Omega_T A^{\mathrm{g}}(S) \xrightarrow{\simeq} A^{\mathrm{g}}(S).$$

Proof. This follows from the Yoneda lemma and Proposition 4.5. □

Corollary 4.7. *Let* $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. *Assume that the pullback square*

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & Y \end{array} \tag{8}$$

in Sm_S *is either a Nisnevich distinguished square or an abstract blow-up square. Then the square*

$$\begin{array}{ccc} A^{\mathrm{g}}(Y) & \xrightarrow{j^*} & A^{\mathrm{g}}(U) \\ {\scriptstyle p^*}\downarrow & & \downarrow \\ A^{\mathrm{g}}(X) & \longrightarrow & A^{\mathrm{g}}(V) \end{array} \tag{9}$$

is a homotopy pullback square.

Proof. This is a straightforward consequence of Proposition 4.5. □

Proposition 4.8. *Let* S *be a base scheme and* $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. *Then Waldhausen* K*-theory provides a presheaf*

$$A^{\mathrm{g}} : \mathrm{Sm}_S^{\mathrm{op}} \to \mathbf{SymSpt}$$

of symmetric S^1*-spectra which is almost sectionwise fibrant.*

In fact, the symmetric spectrum $A^{\mathrm{g}}(X)$ is an Ω-spectrum beyond the first term. Corollary 4.7 then implies that up to a sectionwise fibrant replacement, the symmetric S^1-spectrum A^{g} over S is fibrant in the Nisnevich local projective model structure. Its $\mathbb{A}^1$-fibrant replacement can be determined fairly explicitly via Suslin's singular functor [28]. Recall the standard cosimplicial smooth scheme $\Delta^\bullet_S$ over S given by $[n] \mapsto \Delta^n_S = \mathrm{Spec}\big(\mathcal{O}_S[t_0, \ldots, t_n]/\sum_{i=0}^n t_i = 1\big)$. Realizing the simplicial motivic S^1-spectrum $[n] \mapsto A^{\mathrm{g}}(- \times_S \Delta^n)$ produces a motivic S^1-spectrum $\mathrm{Sing}\, A^{\mathrm{g}}$ over S. As Bjørn Ian Dundas pointed out to me, it is not very interesting.

Proposition 4.9. *Let* $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. *A sectionwise fibrant replacement of the motivic* S^1*-spectrum* $\mathrm{Sing}\, A^{\mathrm{g}}$ *over* S *is fibrant and sectionwise contractible.*

Proof. Recall first that the sectionwise fibrant replacement is fairly harmless, as the adjoint structure maps of $A^{\mathrm{g}}(X)$ are weak equivalences, except for the first. The standard argument from [28] implies that $\mathrm{Sing}\, A^{\mathrm{g}}(X \times \mathbb{A}^1_S \to X)$ is a stable equivalence. It remains to show that $\mathrm{Sing}\, A^{\mathrm{g}}$ is still Nisnevich fibrant. Since $\varnothing \times_S \Delta^n_S$ is the empty scheme, $\mathrm{Sing}\, A^{\mathrm{g}}(\varnothing)$ is the realization of a degreewise contractible spectrum, hence contractible. The value of $\mathrm{Sing}\, A^{\mathrm{g}}$ at a distinguished square Q as displayed in (8) is a homotopy pullback square by [8, Appendix B]. In fact, Corollary 4.7 implies that in every simplicial degree $A^{\mathrm{g}}(Q \times_S \Delta^n_S)$ is a homotopy pullback square. To apply [8, Appendix B], it remains to observe that, by construction, every pointed simplicial set occurring in $A^{g}(Q \times_S \Delta^n_S)$ is connected.

For the final statement, observe that for every closed embedding $i : Z \hookrightarrow X$ in Sm_X, with open complement $j : U = X \smallsetminus Z \hookrightarrow X$, the induced map

$$(i^*, j^*) : \mathrm{Sing}\, A^{g}(X) \to \mathrm{Sing}\, A^{g}(Z) \times \mathrm{Sing}\, A^{g}(U)$$

is a weak equivalence, by realizing the weak equivalences from Proposition 4.5. Suppose that $B : \mathrm{Sm}_S \to \mathbf{Spt}$ is any presheaf of spectra with this property which is also $\mathbb{A}^1$-invariant. Then $B(X) \xrightarrow{\sim} B(\mathbb{A}^1_X) \sim B(\mathbb{A}^1_X \smallsetminus \{0\}) \times B(X)$ is an equivalence, as well as the identity on the second factor. Hence its cofiber $B(\mathbb{A}^1_X \smallsetminus \{0\})$ is contractible. It contains $B(X)$ as a retract included via $1 : X \to \mathbb{A}^1_X \smallsetminus \{0\}$. Thus $B(X)$ is also contractible. □

The contractibility of the $\mathbb{A}^1$-homotopy type associated with the Waldhausen K-theory of any of the finiteness notions on motivic homotopy theory stated in Proposition 4.9 calls for an adjustment. A solution is the finiteness notion of dualizability, as introduced in Definition 2.10. Since dualizability results for smooth varieties in motivic homotopy theory involve invertibility of Thom spaces of vector bundles, passage to T-spectra is required.

Definition 4.10. Let $g \in \{\mathrm{dual}, \mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$ be one of the finiteness notions introduced above, and let S be a base scheme. Set

$$A^{g}_T(S) := A(\mathbf{SymSpt}^{g}_T\mathbf{M}(S))$$

the algebraic K-theory of the category of g-finite symmetric T-spectra over S.

It is straightforward to verify that $A^{g}_T(S)$ satisfies similar properties as $A^{g}(S)$: Proposition 4.2 holds as well for $A^{g}_T(S)$ and any $g \in \{\mathrm{dual}, \mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$. For any $g \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$, Propositions 4.5, 4.8, and 4.9, and Corollaries 4.6 and 4.7 hold with A^{g}_T replacing A^{g}. However, A^{dual}_T does not lead to a contractible $\mathbb{A}^1$-homotopy type, as Theorem 6.6 implies. Nevertheless, the global sections of A^{dual}_T and A^{hfin}_T coincide over fields of characteristic zero.

Proposition 4.11. *Let F be a field of characteristic zero. Then the Waldhausen categories* $\mathbf{SymSpt}_T(F)^{\mathrm{hfin}}$ *and* $\mathbf{SymSpt}_T(F)^{\mathrm{dual}}$ *coincide.*

Proof. This follows from the main result in [33]. Here are some details. Smooth projective schemes are dualizable over a base scheme [1, Chapitre 4; 21, Theorem A.1]. If F is a field of characteristic zero, resolution of singularities provides that then also smooth quasiprojective F-varieties are dualizable [37, Theorem 52; 42]. Every smooth F-variety admits a Zariski open cover by smooth quasiprojective F-varieties, which implies that smooth F-varieties are dualizable. Hence every I_F-finite symmetric T-spectrum is dualizable. Since the property of being dualizable is closed under retracts and weak equivalences, and every finite cofibrant symmetric T-spectrum is a retract of an I_F-finite symmetric T-spectrum by Lemma 2.11, every homotopy finite symmetric T-spectrum is dualizable.

Conversely, every dualizable symmetric T-spectrum is compact as an object of $\mathrm{SH}(F)$. It is an easy consequence of basic properties of dualizable objects in a symmetric monoidal stable model category with a compact unit [26]. Any compact T-spectrum in $\mathrm{SH}(F)$ is a retract of a cofibrant T-spectrum which is weakly equivalent to an I-finite T-spectrum. □

Proposition 4.11 holds also over fields of positive characteristic, provided that the characteristic is inverted. However, it does not hold for base schemes of positive dimension. For example, the T-suspension spectrum of $\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{0\}$ is not dualizable in $\mathrm{SH}(\mathbb{A}^1_{\mathbb{C}})$, although it is finite; see [29, Remark 8.2] for a more general statement.

5. Grothendieck rings

Let S be a base scheme. The *Grothendieck ring of S* is the free abelian group on isomorphism classes of finite-type S-schemes, denoted $[X]$, modulo the relations $[X] = [Z] + [X \smallsetminus Z]$ whenever Z is a closed subscheme, and $[\varnothing] = 0$. The ring structure is induced by the product $X \times_S Y$. The ring $K_0(\mathrm{Var}_S)$ is commutative and has $[S]$ as a unit. Note that $[X] = [X_{\mathrm{red}}]$, where $X_{\mathrm{red}} \hookrightarrow X$ denotes the maximal reduced closed subscheme. Weak factorization is used in the following main result of [3], which gives a much simpler presentation of the Grothendieck ring of fields of characteristic zero.

Theorem 5.1 (F. Heinloth née Bittner). *Suppose that F is a field of characteristic zero. Then $K_0(\mathrm{Var}_F)$ is generated by isomorphism classes of connected smooth projective F-schemes, modulo the relations $[X] - [f^{-1}(Z)] = [Y] - [Z]$ whenever $f : X \to Y$ is the blow-up of the smooth projective variety Y along the smooth center $Z \hookrightarrow Y$, and $[\varnothing] = 0$.*

A *motivic measure* on S is a ring homomorphism

$$K_0(\mathrm{Var}_S) \to A$$

to some commutative ring. Main examples of motivic measures are the Euler characteristic $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ on the complex numbers, point counting $K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$ on a finite field, and the Gillet–Soulé motivic measure [16]. Theorem 5.1 simplifies the construction of motivic measures. For example, the motivic measure on fields of characteristic zero obtained by sending a smooth projective variety to its stable birational class constructed in [24] can be deduced from Theorem 5.1. In order to provide a new motivic measure, recall that if $\mathbf{C}$ is a Waldhausen category, the abelian group $\pi_0 A(\mathbf{C})$ is generated by the objects in $\mathbf{C}$, subject to the following two relations:

(1) $\langle B \rangle = \langle C \rangle$ if there exists a weak equivalence $B \xrightarrow{\sim} C$.

(2) $\langle B \rangle + \langle D \rangle = \langle C \rangle$ if there exists a cofibration $B \hookrightarrow C$ with cofiber D.

Theorem 5.2. *Let F be a field of characteristic zero. Sending the isomorphism class $[X]$ of a smooth projective F-scheme X to its class $\langle X_+ \rangle \in \pi_0 A^{\mathrm{ifin}}(F)$ defines a surjective motivic measure*

$$\Phi_F : K_0(\mathrm{Var}_F) \to A^{\mathrm{ifin}}(F).$$

Proof. The relations given in Theorem 5.1 are fulfilled in $\pi_0 A^{\mathrm{ifin}}(F)$ by [28, Remark 3.2.30]. Hence $[X] \mapsto \langle X_+ \rangle$ defines a group homomorphism

$$\Phi_F : K_0(\mathrm{Var}_F) \to \pi_0 A^{\mathrm{ifin}}(F),$$

which is compatible with the multiplicative structure. It remains to prove its surjectivity. However, $\pi_0 A^{\mathrm{ifin}}(F)$ is generated as an abelian group by I-finite S^1-spectra over F, and hence by the domains and codomains of the maps in I. These are of the form $\mathrm{Fr}_m\, X_+ \wedge \partial \Delta^n_+$ and $\mathrm{Fr}_m\, X_+ \wedge \Delta^n_+$, where X is a smooth F-variety and $m, n \in \mathbb{N}$. Since Fr_m corresponds to a simplicial desuspension and suspension induces multiplication by -1 on $\pi_0 A^{\mathrm{ifin}}(F)$, one may restrict to $m = 0$. Induction on n and the cofiber sequence

$$X_+ \wedge \partial \Delta^n_+ \hookrightarrow X_+ \wedge \Delta^n_+ \to X_+ \wedge S^n$$

imply that $\pi_0 A^{\mathrm{ifin}}(F)$ is generated as an abelian group by S^1-suspension spectra $\mathrm{Fr}_0\, X_+ = \Sigma^\infty X_+$ of smooth F-varieties. Resolution of singularities implies that it suffices to restrict to S^1-suspension spectra of smooth projective F-varieties, similar to the argument in the proof of Proposition 4.11. This concludes the proof. □

The formula $\Phi_F([X]) = \langle X_+ \rangle$ does not apply to nonprojective varieties in general. For example,

$$\Phi_F([\mathbb{A}^1]) = \Phi_F([\mathbb{P}^1] - [\mathrm{Spec}(F)]) = \langle \mathbb{P}^1_+ \rangle - \langle \mathrm{Spec}(F)_+ \rangle = \langle \mathbb{P}^1, \infty \rangle \neq \langle \mathbb{A}^1_+ \rangle = \langle \mathrm{Spec}(F)_+ \rangle,$$

where the inequality follows — at least in the case $k \subseteq \mathbb{R}$ — from the left-Quillen functor which takes a smooth k-scheme to the topological space of its complex points, together with the conjugation action. The fixed points of the action on the left-hand side of the inequality yield the class of $\mathbb{RP}^1$ having reduced Euler characteristic -1, while the fixed points of the action on the right-hand side of the inequality have reduced Euler characteristic 1.

Since I-finiteness is the smallest of the finiteness notions $\mathrm{g} \in \{\mathrm{fin}, \mathrm{hfin}, \mathrm{ifin}, \mathrm{ihfin}\}$ considered on motivic spaces over a field, there is a motivic measure

$$\Phi_F : K_0(\mathrm{Var}_F) \to \pi_0 A^{\mathrm{g}}(F)$$

as well; however, it may not be surjective in the case $g \in \{\text{fin}, \text{hfin}\}$. Proposition 4.2 shows that it refines the Euler characteristic if F is a subfield of $\mathbb{C}$. It also refines the Gillet–Soulé "motivic" motivic measure [16].

Proposition 5.3. *Let F be a field of characteristic zero. There is a commutative diagram*

$$\begin{array}{ccc} K_0(\mathrm{Var}_F) & \xrightarrow{\Phi_F} & \pi_0 A^{\mathrm{ifin}}(F) \\ {\scriptstyle\Psi_F}\downarrow & & \downarrow \\ K_0(\mathrm{ChMot}_F^{\mathrm{eff}}) & \xrightarrow{\cong} & K_0(\mathrm{DM}_F^{\mathrm{eff,hfin}}) \end{array}$$

of ring homomorphisms, where Ψ_F maps the class of a smooth projective F-scheme to the class of its effective Chow motive.

Proof. Voevodsky's derived category of effective motives may be obtained as the homotopy category of S^1-spectra of motivic spaces with transfers; see [37]. This implies an identification of $K_0(\mathrm{DM}_F^{\mathrm{eff,hfin}})$ with the ring of path components of $A(\mathbf{M}^{\mathrm{tr,hfin}}(F))$, where $\mathbf{M}^{\mathrm{tr}}(F)$ is the model category of motivic spaces with transfers defined via the functor $\mathbf{M}^{\mathrm{tr}}(F) \to \mathbf{M}(F)$ forgetting transfers. Its left adjoint induces the vertical arrow on the right-hand side of the diagram displayed above. Similar to the argument in the proof of Proposition 4.11 is an argument proving that homotopy finite and compact motives coincide in $\mathrm{DM}_F^{\mathrm{eff}}$. Furthermore, compact objects and geometric motives coincide by [11, Theorem 11.1.13], both in the effective and the noneffective case. The lower horizontal morphism in Proposition 5.3 is thus an isomorphism by [6, Theorem 6.4.2]. □

Theorem 5.4. *Let F be a field of characteristic zero. The ring homomorphism Φ_F extends to a surjective ring homomorphism*

$$\Phi_F : K_0(\mathrm{Var}_F)[\mathbb{L}^{-1}] \to \pi_0 A_T^{\mathrm{ifin}}(F),$$

where $\mathbb{L} = [\mathbb{A}^1_S]$.

Proof. This follows from Theorem 5.2, the equality $\Phi_F(\mathbb{L}) = \langle \mathbb{P}^1, \infty \rangle$, and the fact that $\Sigma_T^\infty(\mathbb{P}^1, \infty)$ is invertible in $\mathrm{SH}(F)$, hence also in $\pi_0 A_T^{\mathrm{ifin}}(F)$. □

In particular, all relations that hold in the Grothendieck ring $K_0(\mathrm{Var}_F)$ or its localization $K_0(\mathrm{Var}_F)[\mathbb{L}^{-1}]$ also hold in $\pi_0 A(F)$ or $\pi_0 A_T(F)$, respectively. The ring $K_0(\mathrm{Var}_F)[\mathbb{L}^{-1}]$ is relevant to the theory of motivic integration, and also to the construction of the duality involution induced by $[X] \mapsto \mathbb{L}^{-\dim X}[X]$. In the ring $\pi_0 A_T^{\mathrm{g}}(F)$ (which from a certain perspective still consists of algebro-geometric objects), the class of the pointed projective line is naturally invertible. Also the duality involution has a natural interpretation in $\pi_0 A_T^{\mathrm{g}}(F)$, since the dual of a smooth projective F-variety X is the Thom T-spectrum of its negative tangent bundle, considered as a class in $K_0(X)$. The equality

$$\langle \mathrm{Th}(-\mathcal{T}(X))\rangle = \langle \mathbb{P}^1, \infty\rangle^{-\dim X} \cdot \langle X_+\rangle$$

follows from the Zariski local triviality of vector bundles. However, the localization passage $K_0(\mathrm{Var}_F) \to K_0(\mathrm{Var}_F)[\mathbb{L}^{-1}]$ involves a loss of information. Lev Borisov proved that $\mathbb{L}$ is a zero divisor in $K_0(\mathrm{Var}_F)$ [7]. In particular, the composition

$$K_0(\mathrm{Var}_F) \to \pi_0 A^{\mathrm{ifin}}(F) \to \pi_0 A_T^{\mathrm{ifin}}(F)$$

is not injective.

Proposition 5.5. *The canonical homomorphism* $\pi_0 A^{\mathrm{g}}(F) \to \pi_0 A_T^{\mathrm{g}}(F)$ *induces a surjective homomorphism* $\pi_0 A^{\mathrm{g}}(F)[\langle \mathbb{P}^1, \infty\rangle^{-1}] \to \pi_0 A_T^{\mathrm{g}}(F)$ *for* g *contained in* {ifin, fin, ihfin, hfin}.

Proof. In fact, this holds for any base scheme S. The abelian group $\pi_0 A_T^{\mathrm{g}}(F)$ is generated by shifted T-suspension spectra $\mathrm{Fr}_m\, X_+$ of smooth S-schemes; the contribution from the simplicial direction can be ignored, as the proof of Theorem 5.2 implies. The symmetric T-spectrum $\mathrm{Fr}_m\, X^+ \wedge T^m$ is stably equivalent to $\mathrm{Fr}_0\, X_+ = \Sigma_T^\infty X_+$, showing that

$$\pi_0 A^{\mathrm{g}}(S)[\langle \mathbb{P}^1, \infty\rangle^{-1}] \to \pi_0 A_T^{\mathrm{g}}(S)$$

is surjective. □

More can be and has been said on the relationship between the Grothendieck ring of varieties and of motives. Proposition 5.3 and Theorem 5.4 provide the commutative diagram

$$\begin{array}{ccccc}
K_0(\mathrm{Var}_F) & \xrightarrow{\Phi_F} & \pi_0 A^{\mathrm{ifin}}(F) & \longrightarrow & \pi_0 A_T^{\mathrm{ifin}}(F) \\
{\scriptstyle \Psi_F}\downarrow & & \downarrow & & \downarrow \\
K_0(\mathrm{ChMot}_F^{\mathrm{eff}}) & \xrightarrow{\cong} & K_0(\mathrm{DM}_F^{\mathrm{eff,hfin}}) & \longrightarrow & K_0(\mathrm{DM}_F^{\mathrm{hfin}})
\end{array}$$

in which the homomorphism $K_0(\mathrm{DM}_F^{\mathrm{eff,hfin}}) \to K_0(\mathrm{DM}_F^{\mathrm{hfin}})$ corresponds to inverting $\mathbb{L}$, the class of the Lefschetz motive. The latter can be deduced from Voevodsky's cancellation theorem [43]. If one imposes rational instead of integral coefficients on the T-spectra and motives above, the canonical functor becomes a Quillen equivalence for all fields in which -1 is a sum of squares, by a theorem of Morel's. It is known that the canonical homomorphism

$$K_0(\mathrm{Var}_F)[\mathbb{L}^{-1}] \to K_0(\mathrm{DM}_F^{\mathrm{hfin}} \otimes \mathbb{Q})$$

is not injective [30, Proposition 7.9]. Already inverting 2 in the homotopy category of T-spectra splits it as a product $\mathrm{SH}(F)_+ \times \mathrm{SH}(F)_-$ corresponding to the two idempotents $\frac{1-\varepsilon}{2}$, $\frac{1+\varepsilon}{2}$. Here ε is induced by the twist isomorphism on $T \wedge T$. If F is formally real, the category $\mathrm{SH}(F)_-$ maps nontrivially to the derived category of $\mathbb{Z}\left[\frac{1}{2}\right]$-modules. After rationalizing, the category $\mathrm{SH}(F)_+$ is equivalent to the derived category of motives over any field F [11, Theorem 16.2.13]. In particular, the canonical homomorphism

$$\pi_0 A(\mathbf{Sp}_T^{\mathrm{hfin}}(F) \otimes \mathbb{Q}) \to K_0(\mathrm{DM}_F^{\mathrm{hfin}} \otimes \mathbb{Q})$$

is always surjective, but not injective if F is formally real. See [23, Theorem 1.5] for an identification of $\pi_0 A(\mathbf{Sp}_T^{\mathrm{hfin}}(X) \otimes \mathbb{Q})$ in terms of Chow motives and the real étale site of the excellent and separated scheme X of finite Krull dimension.

6. A trace map

The next goal is to produce a trace map on the $\mathbb{A}^1$-homotopy type $A_T^{\mathrm{dual}} : \mathrm{Sm}_F^{\mathrm{op}} \to \mathbf{Spt}$ for a field F of characteristic zero and the duality finiteness notion. The general result [20, Theorem 6.5] essentially provides such a trace. However, when the existence of the motivic trace was announced at a talk in Heidelberg in 2014, the argument proceeded along the lines of [44]. For the sake of concreteness, this construction of the trace will be sketched as follows. In principle, it suffices to fatten the Waldhausen category $\mathbf{SymSpt}_T^{\mathrm{dual}}(F) = \mathbf{SymSpt}_T^{\mathrm{hfin}}(F)$ slightly as in [44]. The fattened Waldhausen category consists of additional duality data.

Definition 6.1. For a base scheme S, let $\mathbf{DSp}(S)$ be the category whose objects are triples $(\mathsf{E}^+, \mathsf{E}^-, e^-)$, where E^+ is a dualizable symmetric T-spectrum, E^- is fibrant symmetric T-spectrum, and $e^- : \mathsf{E}^- \wedge \mathsf{E}^+ \to \mathrm{fib}(\mathbb{I})$ is a map whose adjoint $\mathsf{E}^- \to \underline{\mathrm{Hom}}(\mathsf{E}^+, \mathrm{fib}(\mathbb{I}))$ is a weak equivalence. A morphism of triples from $(\mathsf{D}^+, \mathsf{D}^-, d^-)$ to $(\mathsf{E}^+, \mathsf{E}^-, e^-)$ is a pair $\phi^+ : \mathsf{D}^+ \to \mathsf{E}^+$, $\phi^- : \mathsf{E}^- \to \mathsf{D}^-$ of maps such that the diagram

$$\begin{array}{ccc} \mathsf{D}^- \wedge \mathsf{E}^+ & \xrightarrow{\phi^- \wedge \mathsf{E}^+} & \mathsf{E}^- \wedge \mathsf{E}^+ \\ {\scriptstyle \mathsf{D}^- \wedge \phi^+} \downarrow & & \downarrow {\scriptstyle e^-} \\ \mathsf{D}^- \wedge \mathsf{D}^+ & \xrightarrow{d^-} & \mathrm{fib}(\mathbb{I}) \end{array}$$

commutes. Such a morphism (ϕ^+, ϕ^-) is a *weak equivalence* if both ϕ^+ and ϕ^- are weak equivalences, and a *cofibration* if ϕ^+ is a cofibration and ϕ^- is a fibration.

Since smashing with a cofibrant symmetric T-spectrum preserves weak equivalences [22, Proposition 4.19], the symmetric T-spectrum $\mathsf{E}^- \wedge \mathsf{E}^+$ has the correct homotopy type, even if E^- is not cofibrant.

Proposition 6.2. *The category* $\mathbf{DSp}(S)$ *is a Waldhausen category.*

Proof. The category $\mathbf{DSp}(S)$ is pointed by $(*, *, * \to \mathrm{fib}(\mathbb{I}))$. Weak equivalences in $\mathbf{DSp}(S)$ form a subcategory, and so do the cofibrations. Every triple is then cofibrant, using that the second entry is fibrant. Suppose that

$$(\mathsf{B}^+, \mathsf{B}^-, b^-) \xleftarrow{(\psi^+,\psi^-)} (\mathsf{D}^+, \mathsf{D}^-, d^-) \xrightarrow{(\phi^+,\phi^-)} (\mathsf{E}^+, \mathsf{E}^-, e^-)$$

is a diagram. Its pushout is defined as the triple $(\mathsf{E}^+ \cup_{\mathsf{D}^+} \mathsf{B}^+, \mathsf{E}^- \times_{\mathsf{D}^-} \mathsf{B}^-, c)$ where c is adjoint to the map

$$\mathsf{E}^- \times_{\mathsf{D}^-} \mathsf{B}^- \to \underline{\mathrm{Hom}}(\mathsf{E}^+ \cup_{\mathsf{D}^+} \mathsf{B}^+, \mathrm{fib}\,\mathbb{I}) \cong \underline{\mathrm{Hom}}(\mathsf{E}^+, \mathrm{fib}\,\mathbb{I}) \times_{\underline{\mathrm{Hom}}(\mathsf{D}^+, \mathrm{fib}\,\mathbb{I})} \underline{\mathrm{Hom}}(\mathsf{B}^+, \mathrm{fib}\,\mathbb{I})$$

induced by the adjoints of b^-, d^-, and e^-. The dual of the gluing lemma implies that the map above is a weak equivalence. □

Lemma 6.3. *The forgetful functor* $\mathbf{DSp}(S) \to \mathbf{SymSpt}_T^{\mathrm{dual}}(S)$ *is a K-theory equivalence.*

Proof. The forgetful functor sends the triple $(\mathsf{E}^+, \mathsf{E}^-, e^-)$ to E^+ and is exact by definition. It admits the exact section sending E to the triple $(\mathsf{E}, \underline{\mathrm{Hom}}(\mathsf{E}, \mathrm{fib}\,\mathbb{I}), \mathrm{ev})$ where $\mathrm{ev} : \underline{\mathrm{Hom}}(\mathsf{E}, \mathrm{fib}\,\mathbb{I}) \wedge \mathsf{E} \to \mathrm{fib}\,\mathbb{I}$ is the evaluation map, adjoint to the identity. Moreover, there is a natural weak equivalence

$$(\mathsf{E}^+, \underline{\mathrm{Hom}}(\mathsf{E}^+, \mathrm{fib}\,\mathbb{I}), \mathrm{ev}) \xrightarrow{(\mathrm{id}, \flat(e^-))} (\mathsf{E}^+, \mathsf{E}^-, e^-), \tag{10}$$

where $\flat(e^-)$ is the adjoint of e^-. Hence applying the section to the forgetful functor induces a map on algebraic K-theory which is homotopic to the identity map. □

Recall that $\mathbf{SymSpt}_T(S)$ admits the structure of a pointed simplicial model category in which the n-simplices of morphisms are given by the maps $\mathsf{D} \wedge \Delta^n_+ \to \mathsf{E}$ of symmetric T-spectra over S. The functoriality listed in Proposition 3.10 is simplicial. It follows that the assignment $[n] \mapsto \mathbf{SymSpt}_T^{\mathrm{dual}}(S)_n$ is a simplicial category with constant objects, and so is the assignment $[n] \mapsto \mathrm{w}\mathbf{SymSpt}_T^{\mathrm{dual}}(S)_n$ restricted to the subcategories of weak equivalences.

Proposition 6.4. *The category* $\mathbf{DSp}(S)_\bullet$ *is a simplicial category, and so is the restriction to* $\mathrm{w}\mathbf{DSp}(S)_\bullet$*, the subcategory of weak equivalences.*

Proof. The n-simplices of morphisms are given by pairs

$$(\mathsf{D}^+ \wedge \Delta^n_+ \to \mathsf{E}^+, \mathsf{E}^- \wedge \Delta^n_+ \to \mathsf{D}^-)$$

satisfying the appropriate compatibility condition. The required axioms are straightforward to check. □

Lemma 6.5. *The natural inclusion*

$$\mathrm{w}\mathbf{DSp}(S) \xrightarrow{\kappa} \mathrm{w}\mathbf{DSp}(S)_\bullet$$

induces a weak equivalence after geometric realization.

Proof. Lemma 6.3, or rather its proof, implies that the forgetful functor induces the following diagram

$$\begin{array}{ccc} \mathrm{w}\mathbf{DSp}(S) & \xrightarrow{\quad\kappa\quad} & \mathrm{w}\mathbf{DSp}_\bullet(S) \\ \downarrow & & \downarrow \\ \mathrm{w}\mathbf{SymSp}_T^{\mathrm{dual}}(S) & \xrightarrow{\kappa} & \mathrm{w}\mathbf{SymSp}_T^{\mathrm{dual}}(S)_\bullet \end{array}$$

whose vertical arrows induce weak equivalences after geometric realization. It suffices to prove that the lower horizontal arrow has the same property. The inclusion κ is induced by the collection of degeneracy maps $s_m : \Delta^m \to \Delta^0$. Let $d_m : \Delta^0 \to \Delta^m$ be the inclusion of the m-th vertex. By the realization lemma, it suffices to prove that the composition

$$\mathrm{w}\mathbf{SymSp}_T^{\mathrm{dual}}(S)_m \xrightarrow{d_m^*} \mathrm{w}\mathbf{SymSp}_T^{\mathrm{dual}}(S) \xrightarrow{s_m^*} \mathrm{w}\mathbf{SymSp}_T^{\mathrm{dual}}(S)_m$$

is homotopic to the identity for every m. This follows from the fact that Δ^m simplicially contracts onto its last vertex. □

Let $(\mathsf{E}^+, \mathsf{E}^-, e^-)$ be an object in $\mathbf{DSp}(S)$. Consider the simplicial set of maps of symmetric T-spectra over S from $\mathbb{1}$ to $\mathrm{fib}(\mathsf{E}^+ \wedge \mathsf{E}^-)$. The aim is to modify this simplicial set to consist of only those maps which — together with e^- — express E^+ and E^- as dual objects in the stable homotopy category $\mathrm{SH}(S)$. A map $\mathbb{1} \to \mathrm{fib}(\mathsf{E}^+ \wedge \mathsf{E}^-)$ induces a composition

$$\mathsf{E}^+ = \mathbb{1} \wedge \mathsf{E}^+ \to \mathrm{fib}(\mathsf{E}^+ \wedge \mathsf{E}^-) \wedge \mathsf{E}^+ \xrightarrow{\sim} \mathrm{fib}(\mathsf{E}^+ \wedge \mathsf{E}^- \wedge \mathsf{E}^+) \xrightarrow{\mathrm{fib}(\mathsf{E}^+ \wedge e^-)} \mathrm{fib}(\mathsf{E}^+ \wedge \mathrm{fib}(\mathbb{1}))$$

(as well as a similar composition $\mathrm{cof}(\mathsf{E}^-) \to \mathrm{fib}(\mathrm{fib}(\mathbb{1}) \wedge \mathsf{E}^-)$ where cof is a cofibrant replacement functor). There is a preferred map $z : \mathsf{E}^+ \to \mathrm{fib}(\mathsf{E}^+ \wedge \mathrm{fib}(\mathbb{1}))$, given by unit and replacement natural transformations. Let

$$H(\mathsf{E}^+, \mathsf{E}^-, e^-) = \{H : \mathsf{E}^+ \wedge \Delta^1_+ \to \mathrm{fib}(\mathsf{E}^+ \wedge \mathrm{fib}(\mathbb{1})) \text{ with } H|_{\mathsf{E}^+ \wedge 0_+} = z\}$$

be the simplicial set of simplicial homotopies starting at the respective preferred map. By construction, $H(\mathsf{E}^+, \mathsf{E}^-, e^-)$ is a fibrant simplicial set which simplicially contracts to the zero simplex given by the preferred map. Moreover, it maps via an "endpoint" Kan fibration to the simplicial set of maps in the terminal corner of the following diagram whose pullback is the desired modification:

$$\begin{array}{ccc} D(\mathsf{E}^+, \mathsf{E}^-, e^-) & \longrightarrow & \mathbf{sSet}(\mathbb{1}, \mathrm{fib}(\mathsf{E}^+ \wedge \mathsf{E}^-)) \\ \downarrow & & \downarrow \\ H(\mathsf{E}^+, \mathsf{E}^-, e^-) & \longrightarrow & \mathbf{sSet}(\mathsf{E}^+, \mathrm{fib}(\mathsf{E}^+ \wedge \mathrm{fib}\, \mathbb{1})) \end{array} \tag{11}$$

The condition on e^- guarantees that the vertical map on the right-hand side of diagram (11) is a weak equivalence, and the horizontal arrows depict fibrations. Hence $D(\mathsf{E}^+, \mathsf{E}^-, e^-)$ is a contractible fibrant simplicial set. Its zero simplices are maps $e^+ : \mathbb{I} \to \mathrm{fib}(\mathsf{E}^+ \wedge \mathsf{E}^-)$, together with a simplicial homotopy providing that $\mathsf{E}^+ \wedge e^- \circ e^+ \wedge \mathsf{E}^+$ coincides with $\mathrm{id}_{\mathsf{E}^+}$ in the motivic stable homotopy category of S. Such a zero simplex represented by the tuple

$$((\mathsf{E}^+, \mathsf{E}^-, e^-), e^+, H)$$

maps naturally to the composition

$$\mathbb{I} \xrightarrow{e^+} \mathrm{fib}(\mathsf{E}^+ \wedge \mathsf{E}^-) \xrightarrow{\cong} \mathrm{fib}(\mathsf{E}^- \wedge \mathsf{E}^+) \xrightarrow{\mathrm{fib}(e^-)} \mathrm{fib}(\mathrm{fib}(\mathbb{I})),$$

which is a zero simplex in $\mathrm{fib}(\mathrm{fib}(\mathbb{I}))(S)$ representing the Euler characteristic of E^+ [26]. More generally, an n-simplex maps to an n-simplex in $\mathrm{fib}(\mathrm{fib}(\mathbb{I}))(S)$. A similar variant $D(\underline{\mathsf{E}})$ exists for an n-simplex

$$\underline{\mathsf{E}} = (\mathsf{E}_0 \xrightarrow{\sim} \mathsf{E}_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathsf{E}_n)$$

of the nerve of $\mathrm{w}\mathbf{DSp}_\bullet(S)$, starting with maps from $\mathbb{I}$ to $\mathrm{fib}(\mathsf{E}_0^+ \wedge \mathsf{E}_n^-)$ and using the duality datum $\mathsf{E}_0^+ \wedge \mathsf{E}_n^- \to \mathrm{fib}(\mathbb{I})$ obtained via compositions instead. The map to $\mathrm{fib}(\mathrm{fib}(\mathbb{I}))$ described above for $D(\mathsf{E}^+, \mathsf{E}^-, e^-)$ extends to $D(\underline{\mathsf{E}})$ for $\underline{\mathsf{E}}$ such an n-simplex. A map $\alpha : [m] \to [n]$ in Δ induces a map $D(\underline{\mathsf{E}}) \to D(\alpha^*(\underline{\mathsf{E}}))$. Consider the induced map of bisimplicial sets

$$\coprod_{\underline{\mathsf{E}} \in \mathrm{w}\mathbf{DSp}_\bullet(S)} D(\underline{\mathsf{E}}) \to \coprod_{\underline{\mathsf{E}} \in \mathrm{w}\mathbf{DSp}_\bullet(S)} \{\underline{\mathsf{E}}\}.$$

Since $D(\underline{\mathsf{E}})$ is contractible, this map is a weak equivalence by the realization lemma. It maps to $\mathrm{fib}(\mathrm{fib}\,\mathbb{I})(S)$, and this map is natural in S. It may be regarded as the zeroth level of a map of S^1-spectra

$$A_T^{\mathrm{dual}}(S) \to \mathrm{fib}(\mathrm{fib}\,\mathbb{I})(S),$$

which is natural in S. Instead of providing a spectrum-level map by explicit constructions similar to those appearing in the proof of [26, Theorem 0.1], however, [20, Theorem 6.5], which in turn refers to [41], will be invoked.

Theorem 6.6. *Let S be a Noetherian finite-dimensional base scheme. There exists a multiplicative map $A_T^{\mathrm{dual}}(S) \to \mathrm{fib}(\mathbb{I})(S)$ of symmetric ring S^1-spectra which is natural in S. On path components it sends the class given by the dualizable T-spectrum E to the Euler characteristic of E, considered as an endomorphism $\chi(\mathsf{E}) : \mathbb{I} \to \mathrm{fib}(\mathbb{I})$.*

Proof. The category $\mathbf{SymSp}_T^{\mathrm{dual}}(S)$ of cofibrant dualizable symmetric T-spectra over S gives rise to a small, stable, idempotent-complete, rigid symmetric monoidal

∞-category $\mathcal{E}(S)$, naturally in S. Hence it fits into the general framework of [20]. More specifically, [20, Theorem 6.5 and Remark 6.6] apply to give a map of symmetric ring spectra

$$A(\mathcal{E}(S)) \to \mathrm{fib}(\mathbb{1})(S),$$

whose domain is the ∞-categorical K-theory of $\mathcal{E}(S)$ as introduced in [25, Remark 1.2.2.5], and whose target is regarded as the endomorphism S^1-spectrum of the unit in $\mathcal{E}(S)$. The Waldhausen K-theory $A_T^{\mathrm{dual}}(S)$ of $\mathbf{SymSp}^{\mathrm{dual}}(S)$ maps via a natural weak equivalence to $A(\mathcal{E}(S))$ [4, Theorem 7.8], and this so as a symmetric ring spectrum [5, Proposition 5.8]. This provides the desired multiplicative map of S^1-spectra. For the statement regarding path components, see [20, Remark 6.6]. □

It would be very interesting to relate the homotopy fiber of the trace from Theorem 6.6 with geometrical data.

Corollary 6.7. *Let F be a field of characteristic zero. There exists a multiplicative trace map*

$$A_T^{\mathrm{hfin}}(F) \to \mathrm{fib}(\mathbb{1})(F),$$

which induces a ring homomorphism

$$\pi_0 A_T^{\mathrm{hfin}}(F) \to \pi_0\, \mathrm{fib}(\mathbb{1})(F) \cong \mathrm{GW}(F)$$

to the Grothendieck–Witt ring of F.

Proof. This follows from Theorem 6.6 and Proposition 4.11. The statement on path components involves Morel's theorem [27, Corollary 1.25] computing the path components of the endomorphism spectrum of the sphere T-spectrum. More precisely, the global sections of $\mathrm{fib}(\mathbb{1})$ coincide with the infinite T-loop space (or spectrum) associated with a fibrant replacement of the T-suspension spectrum of the zero sphere S_F^0 over F. Hence by construction its path components form the endomorphism ring of $\mathbb{1} \in \mathrm{SH}(F)$, which Morel computed to be naturally isomorphic to the Grothendieck–Witt ring of F. □

The composition of the motivic measure

$$K_0(\mathrm{Var}_F) \to \pi_0 A_T^{\mathrm{hfin}}(F) = \pi_0 A_T^{\mathrm{dual}}(F)$$

induced by Theorem 5.2 and the ring homomorphism from Corollary 6.7 provides a motivic measure to the Grothendieck–Witt ring on F. For formal reasons — see, for example, [26] — it extends the categorical Euler characteristic $K_0(\mathrm{SH}^{\mathrm{dual}}(F)) \to [\mathbb{1}, \mathbb{1}]_{\mathrm{SH}(F)} \cong \mathrm{GW}(F)$ used in refined enumerative geometry by Levine, Wickelgren, and others [2]. As another application of Theorem 6.6, an interesting $\mathbb{A}^1$-homotopy type results.

Corollary 6.8. *Let S be a Noetherian finite-dimensional base scheme. The fibrant replacement of the presheaf $A_T^{\mathrm{dual}} \in \mathbf{SymSpt}(S)$ factors the unit map*

$$S^0 \to \Omega_T^\infty \Sigma_T^\infty S^0 \to \Omega_T^\infty \operatorname{fib}(\mathbb{1}) \in \mathbf{SymSpt}(S).$$

In particular, the $\mathbb{A}^1$-homotopy type associated with A_T^{dual} is nontrivial, and the global sections of its $\mathbb{A}^1$-path components $\pi_0^{\mathbb{A}^1} A_T^{\mathrm{dual}}$ factor the motivic measure

$$K_0(\mathrm{Var}_F) \to \pi_0^{\mathbb{A}^1} A_T^{\mathrm{dual}}(F) \to \mathrm{GW}(F)$$

from Corollary 6.7 in the case $S = \mathrm{Spec}(F)$ is the spectrum of a field F of characteristic zero.

Proof. The proof of Corollary 6.7 already mentioned that the global sections of $\operatorname{fib}(\mathbb{1})$ over any Noetherian finite-dimensional base scheme such as $X \in \mathrm{Sm}_S$ is the infinite T-loop space of the sphere spectrum $\mathbb{1}_X \in \mathrm{SH}(X)$, viewed as a symmetric S^1-spectrum. Hence the naturality in Theorem 6.6 provides a map

$$A_T^{\mathrm{dual}} \to \Omega_T^\infty \operatorname{fib}(\mathbb{1}) \in \mathbf{SymSpt}(S),$$

which sectionwise is a multiplicative map of symmetric ring spectra. In particular, this map is naturally compatible with the unit. Since its target $\Omega_T^\infty \operatorname{fib}(\mathbb{1})$ is fibrant by construction, the map factors over a fibrant replacement of A_T^{dual}, giving rise to the claimed factorization of the unit map. The remaining statement then follows from Corollary 6.7. □

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Stable homotopy groups of motivic spheres

Oliver Röndigs and Markus Spitzweck

These lecture notes are based on lectures given by the authors at the autumn school "Computations in motivic homotopy theory" at Regensburg University during September 16–20, 2019. Main results include a computation of the first Milnor–Witt stem of stable homotopy groups of motivic spheres over a field, presented differently than Röndigs, Spitzweck and Østvær (2019), and a partial computation of the zeroth Milnor–Witt stem of stable homotopy groups of motivic spheres over discrete valuation rings of mixed characteristic after inverting the residue characteristic.

1. Introduction

These lecture notes connect fundamental results in motivic or $\boldsymbol{A}^1$-homotopy theory, as developed by Vladimir Voevodsky, Fabien Morel, and others, with concrete computations of stable homotopy groups of motivic spheres given in [43; 55]. They are based on lectures given during the autumn school "Computations in motivic homotopy theory" at Regensburg University during September 16–20, 2019. Given the circumstances, we decided to refer to the literature for certain results and arguments, whereas other simple results are exercises for the reader; therefore these lecture notes do not contain complete computations of the zeroth and first Milnor–Witt stem of stable homotopy groups of motivic spheres over a field. Instead, we offer with Theorem 7.13 a partial extension of Morel's computation of the zeroth Milnor–Witt stem of stable homotopy groups of spheres to the case of a discrete valuation ring of mixed characteristic subject to inverting the positive residue characteristic.

Besides the foundations [47] of unstable $\boldsymbol{A}^1$-homotopy theory, the sources [25; 35; 39; 64] may serve as accompanying reading material. We thank the organizers (Denis-Charles Cisinski, Markus Land, Florian Strunk, and Georg Tamme), the

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2. Fundamental properties

One viewpoint on motivic or $\boldsymbol{A}^1$-homotopy theory, as heavily advertised by Fabien Morel, is to supply a homotopy theory for smooth varieties over a field which enjoys similar properties as classical homotopy theory does with respect to smooth manifolds. More precisely, the following theorems hold, based on the notational convention that, for every Noetherian separated scheme S of finite Krull dimension (to be abbreviated as *base scheme*), $\mathbf{Sm}_S$ denotes the category of smooth finite-type S-schemes, and (not necessarily smooth) morphisms of such. Let $\mathbf{Spc}(S)$ denote the category of *spaces over* S, that is, presheaves (or Nisnevich sheaves) on $\mathbf{Sm}_S$ with values in simplicial sets. The pointed version, $\mathbf{Spc}_\bullet(S)$, is the category of pointed spaces over S, that is, presheaves (or Nisnevich sheaves) on $\mathbf{Sm}_S$ with values in pointed simplicial sets. The $\boldsymbol{A}^1$-homotopy theory on $\mathbf{Spc}(S)$ (and therefore also on its pointed version $\mathbf{Spc}_\bullet(S)$) is determined by the following two properties:

Nisnevich excision: Every elementary distinguished square

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\;j\;} & X \end{array}$$

(that is, every pullback square in which j is an open embedding and p is an étale morphism inducing an isomorphism on reduced closed subschemes $Y \smallsetminus V \cong X \smallsetminus U$) in $\mathbf{Sm}_S$ induces a homotopy pushout square in $\mathbf{Spc}(S)$.

Homotopy invariance: The affine line parametrizes homotopies in the sense that the projection $X \times_S \boldsymbol{A}^1 \to X$ is a weak equivalence for all $X \in \mathbf{Sm}_S$.

In these properties, the Yoneda embedding $\mathbf{Sm}_S \hookrightarrow \mathbf{Spc}(S)$ is used for the passage from smooth S-schemes to spaces over S without appearing in the notation. It sends a smooth S-scheme to the discrete simplicial presheaf it represents. If a smooth S-scheme X admits a rational point $x : S \to X$, the resulting pair (X, x) will be viewed as a pointed space over S. For any smooth S-scheme X, the disjoint union $X \coprod S$ comes with a canonical rational point, producing the pointed space X_+ over S.

Example 2.1. The canonical covering of the projective line $\boldsymbol{P}^1_S$ over S by two copies of the affine line over S supplies a canonical identification of $(\boldsymbol{P}^1_S, [1, 1])$ with the reduced suspension $\Sigma(\boldsymbol{G}_{\mathrm{m}}, 1)$ of the multiplicative group scheme over S, as well as a canonical identification $\mathsf{Th}(\boldsymbol{A}^1_S) := \boldsymbol{A}^1_S/(\boldsymbol{A}^1_S \smallsetminus \{0\}) \simeq \boldsymbol{P}^1_S$ with the Thom space of the trivial line bundle over S.

Definition 2.2. Let $S^0 = S_+ = S \coprod S$ be the zero sphere over S. It is the unit for the closed symmetric monoidal smash product in $\mathbf{Spc}_\bullet(S)$. The basic *circles* over S are the simplicial circle $\Delta^1/\partial\Delta^1$, considered as a constant pointed (pre)sheaf, and the Tate circle $\boldsymbol{G}_\mathsf{m} = (\boldsymbol{A}^1_S \smallsetminus \{0\}, 1)$ over S. Smash products of these produce, for every $s, w \in \mathbb{N}$, a sphere

$$\Sigma^{s+(w)}_S := \Sigma^{s+w,w}_S := (\Delta^1/\partial^1)^{\wedge s} \wedge \boldsymbol{G}_\mathsf{m}^{\wedge w},$$

which is a pointed space over S. The base scheme may be removed from the notation. For example, there are canonical identifications

$$\boldsymbol{A}^n \smallsetminus \{0\} \simeq \Sigma^{n-1+(n)} = \Sigma^{2n-1,n} \quad \text{and} \quad \mathsf{Th}(\boldsymbol{A}^n) \simeq \boldsymbol{P}^n/\boldsymbol{P}^{n-1} \simeq \Sigma^{n+(n)} = \Sigma^{2n,n}.$$

(Exercise.)

Remark 2.3. If $S = \mathrm{Spec}(\mathbb{C})$ (or more generally if S maps to $\mathrm{Spec}(\mathbb{C})$), sending a smooth $\mathbb{C}$-scheme to the underlying topological space of its associated complex analytic manifold induces a functor of homotopy theories preserving (homotopy) colimits and (smash) products. It sends the sphere $\Sigma^{s+(w)}_\mathbb{C} = \Sigma^{s+w,w}_\mathbb{C}$ to the topological sphere S^{s+w}. If $S = \mathrm{Spec}(\mathbb{R})$ (or more generally if S maps to $\mathrm{Spec}(\mathbb{R})$), sending a smooth $\mathbb{R}$-scheme to the underlying topological space of its associated real analytic manifold induces another functor of homotopy theories preserving (homotopy) colimits and (smash) products. It sends the sphere $\Sigma^{s+(w)}_\mathbb{R} = \Sigma^{s+w,w}_\mathbb{R}$ to the topological sphere S^s. (Exercise.)

Several arguments will employ the homotopy theory of spaces over S, which is of a very complicated nature. Inverting (smashing with) the sphere $\boldsymbol{P}^1 \simeq \Sigma^{1+(1)} = \Sigma^{2,1}$ produces a simpler homotopy theory.

Theorem 2.4. *For every base scheme S, there exists a symmetric monoidal functor $\Sigma^\infty_{\boldsymbol{P}^1_S} : \mathbf{Spc}_\bullet(S) \to \mathbf{SH}(S)$ to a closed symmetric monoidal triangulated category. The object $\Sigma^\infty_{\boldsymbol{P}^1_S}(\boldsymbol{P}^1_S, \infty)$ is invertible with respect to the symmetric monoidal structure. Every morphism $f : S \to T$ of base schemes induces a strong symmetric monoidal triangulated functor $f^* : \mathbf{SH}(T) \to \mathbf{SH}(S)$, having a right adjoint f_*, and even a left adjoint $f_\sharp$ if f is smooth.*

The functoriality alluded to in Theorem 2.4 is in fact part of an encompassing setup often referred to as the *six functor formalism*. More properties and details regarding this formalism will be provided in Theorems 2.7 and 2.10, and in Section 7 below, while the comprehensive motivic reference is [7; 8].

The classical analog of $\mathbf{SH}(S)$ is the stable homotopy category $\mathbf{SH}$ of spectra. Objects in $\mathbf{SH}(S)$ are called *motivic spectra over S*. One may interpret $\mathbf{SH}(S)$ as the homotopy category of a suitable symmetric monoidal model category, or as a suitable symmetric monoidal stable ∞-category. Being triangulated, there exists a shift endofunctor Σ which is an equivalence and, for formal reasons, coincides with smash product (from the right) with $\Delta^1/\partial\Delta^1 = \Sigma^{1+(0)} = \Sigma^{1,0}$. The bigraded notation is explained by the existence of another canonically invertible endofunctor, the smash product with $\Sigma_S^{1+(1)} = \Sigma_S^{2,1} \simeq (\boldsymbol{P}_S^1, \infty)$. Other bigrading conventions exist in the literature. Slightly abusing notation, the sphere $\Sigma_S^{s+(w)}$, its image $\Sigma^\infty_{\boldsymbol{P}^1_S}\Sigma_S^{s+(w)} \in \mathbf{SH}(S)$, and the smash product with either (from the right) will be denoted as $\Sigma_S^{s+(w)} = \Sigma_S^{s+w,w}$. Using invertibility of $\Sigma_S^{1+(1)}$, given integers s, w and $\mathsf{E} \in \mathbf{SH}(S)$, the motivic spectrum $\Sigma^{s+(w)}\mathsf{E} = \Sigma^{s+w,w}\mathsf{E}$ is well defined, as are the homotopy groups

$$\pi_{s+(w)}\mathsf{E} = \pi_{s+w,w}\mathsf{E} = [\Sigma^{s+(w)}, \mathsf{E}]$$

in simplicial degree s and weight w. Assembling all weights together yields

$$\pi_s\mathsf{E} := \pi_{s+(\star)}\mathsf{E} := \bigoplus_{w\in\mathbb{Z}} \pi_{s+(w)}\mathsf{E},$$

where "$(\star)$" may be carried around to indicate the weight grading. The Nisnevich sheafification of the presheaf $X \mapsto [\Sigma^{s+(w)}X_+, \mathsf{E}]$ on $\mathbf{Sm}_S$ with values in abelian groups will be denoted by $\underline{\pi}_{s+(w)}\mathsf{E} = \underline{\pi}_{s+w,w}\mathsf{E}$.

Lemma 2.5. *A map $\phi : \mathsf{D} \to \mathsf{E}$ is an equivalence if and only if $\underline{\pi}_{s+(w)}(\phi)$ is an isomorphism of Nisnevich sheaves for all $s, w \in \mathbb{Z}$.*

The functor $\Sigma^\infty_{\boldsymbol{P}^1_S}$ factors other prominent functors, for example, the functor sending a smooth variety over a field F to its motive in Voevodsky's category $\mathbf{DM}(F)$.[1] This will, in particular, imply the nontriviality of the category $\mathbf{SH}(S)$ (unless S is the empty scheme). Both can be deduced from the following statement, a motivic analog of the representability of singular cohomology through the Eilenberg–MacLane spectrum.

Theorem 2.6 (Voevodsky). *Let F be a field. There exists a motivic spectrum $\mathbf{M}\mathbb{Z}_F \in \mathbf{SH}(F)$ representing motivic cohomology in the following sense: For every pair of integers s, w and every smooth S-scheme X, there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{SH}(S)}(\Sigma^\infty_{\boldsymbol{P}^1_S}X, \Sigma^{s+(w)}\mathbf{M}\mathbb{Z}_F) \cong H^{s+w,w}(X, \mathbb{Z}) = H^{s+w}(X, \mathbb{Z}(w))$$

of abelian groups.

This isomorphism may be promoted to an isomorphism of graded rings: the degree-d part of the Chow ring is naturally given as $\mathrm{Hom}_{\mathbf{SH}(F)}(\Sigma^\infty_{\boldsymbol{P}^1_S}X, \Sigma^{d+(d)}\mathbf{M}\mathbb{Z}_F)$ as a particular case. Isomorphisms in $\mathbf{SH}(S)$ can be detected locally in the following rather strong sense.

[1] Removing "Spec" from the notation will often occur in the hope that it simplifies the text and does not confuse the reader.

Theorem 2.7 (localization). *Let $i : Z \hookrightarrow S$ be a closed embedding of base schemes, with open complement $j : S \smallsetminus Z \hookrightarrow S$. The natural maps define a homotopy cofiber sequence*

$$j_\sharp j^* \mathsf{E} \to \mathsf{E} \to i_* i^* \mathsf{E} \to \Sigma j_\sharp j^* \mathsf{E}$$

in $\mathbf{SH}(S)$. *In particular, a map $\phi : \mathsf{D} \to \mathsf{E}$ in $\mathbf{SH}(S)$ is an isomorphism if and only if $i^*(\phi)$ and $j^*(\phi)$ are isomorphisms.*

Given an open embedding $j : U \hookrightarrow X$ in $\mathbf{Sm}_S$, let $\Sigma^\infty_{\boldsymbol{P}^1_S} X/U$ denote the canonical cone appearing in the homotopy cofiber sequence

$$\Sigma^\infty_{\boldsymbol{P}^1_S} U \to \Sigma^\infty_{\boldsymbol{P}^1_S} X \to \Sigma^\infty_{\boldsymbol{P}^1_S} X/U \to \Sigma \Sigma^\infty_{\boldsymbol{P}^1_S} U$$

in $\mathbf{SH}(S)$. In the special case where $V \to X$ is a vector bundle with zero section $z : X \hookrightarrow V$, abbreviate $\mathsf{Th}(V \to X) = \Sigma^\infty_{\boldsymbol{P}^1_S} V/(V \smallsetminus z(X))$. It serves to formulate the following analog of the tubular neighborhood construction, given by Morel and Voevodsky:

Theorem 2.8 (homotopy purity). *Let $i : Z \hookrightarrow X$ be a closed embedding in $\mathbf{Sm}_S$, with normal bundle $Ni \to Z$. There exists a suitably natural identification*

$$\Sigma^\infty_{\boldsymbol{P}^1_S} X/(X \smallsetminus i(Z)) \simeq \mathsf{Th}(Ni \to Z)$$

in $\mathbf{SH}(S)$.

Example 2.9. Let $i : \boldsymbol{P}^{n-1} \hookrightarrow \boldsymbol{P}^n$ be the closed embedding given by $x \mapsto (x, 0)$. To compute its homotopy cofiber with the help of Theorem 2.8, it helps to replace i up to $\boldsymbol{A}^1$-equivalence by an open embedding. In this very special situation, i factors as $\boldsymbol{P}^{n-1} \hookrightarrow \boldsymbol{P}^n \smallsetminus \{(0, 1)\} \subset \boldsymbol{P}^n$. The first map is the zero section of the line bundle $\boldsymbol{P}^n \smallsetminus \{(0, 1)\} \to \boldsymbol{P}^{n-1}$ forgetting the last coordinate, and hence an $\boldsymbol{A}^1$-homotopy equivalence. The homotopy cofiber of $\boldsymbol{P}^n \smallsetminus \{(0, 1)\} \hookrightarrow \boldsymbol{P}^n$ is the Thom space of the (trivial) normal bundle of the point $(0, 1) \in \boldsymbol{P}^n$ by Theorem 2.8.

The homotopy purity theorem allows us to describe a duality in $\mathbf{SH}(S)$ modeled on the classical Spanier–Whitehead duality. In order to state it, let $\mathbf{1}_S = \Sigma^\infty_{\boldsymbol{P}^1_S} S \in \mathbf{SH}(S)$ be the unit of the closed symmetric monoidal structure, usually denoted as $(\mathsf{D}, \mathsf{E}) \mapsto \mathsf{D} \wedge \mathsf{E}$, with internal hom denoted $\underline{\mathrm{Hom}}_S(\mathsf{D}, \mathsf{E})$.

Theorem 2.10 (Spanier–Whitehead duality). *Let $X \in \mathbf{Sm}_S$ be projective, with structure morphism $f : X \to S$ and tangent bundle $\mathscr{T}_f \to X$. Then $\Sigma^\infty_{\boldsymbol{P}^1_S} X$ admits a strong dual, given as*

$$\underline{\mathrm{Hom}}_S(\Sigma^\infty_{\boldsymbol{P}^1_S} X, \mathbf{1}_S) \simeq f_\sharp \,\underline{\mathrm{Hom}}_X(\mathsf{Th}(\mathscr{T}_f \to X), \Sigma^\infty_{\boldsymbol{P}^1_X} X).$$

While the category $\mathbf{SH}(S)$ is always generated by compact objects (namely $\boldsymbol{P}^1_S$-(de)suspensions of smooth S-schemes, or even smooth affine S-schemes), it is unclear if it is generated by strongly dualizable objects. If F is a field of characteristic zero, one may use resolution of singularities as an ingredient to prove that $\mathbf{SH}(F)$ is generated by $\boldsymbol{P}^1_S$-(de)suspensions of smooth projective F-schemes, which are also strongly dualizable by Theorem 2.10. Already compact generation can be very helpful, as the following continuity statement shows, which could be formulated in greater generality.

Lemma 2.11. *Let $\mathcal{I} \to$ Rings, $\beta \mapsto R_\beta$, be a diagram of Noetherian rings of finite Krull dimension, where $\mathcal{I}$ is a filtered category with initial object α. Suppose its colimit R_ω is Noetherian of finite Krull dimension. For a motivic spectrum $\mathsf{E} = \mathsf{E}_\alpha \in \mathbf{SH}(R_\alpha)$, let E_β denote its pullback to $\mathbf{SH}(R_\beta)$, and similarly for E_ω. Then for every compact motivic spectrum $\mathsf{D} \in \mathbf{SH}(R_\alpha)$ the canonical map*

$$\operatorname*{colim}_{\beta \in \mathcal{I}} \mathrm{Hom}_{\mathbf{SH}(R_\beta)}(\mathsf{D}_\beta, \mathsf{E}_\beta) \to \mathrm{Hom}_{\mathbf{SH}(R_\omega)}(\mathsf{D}_\omega, \mathsf{E}_\omega)$$

is an isomorphism.

Proof. This can be derived as a special case of a statement on filtered diagrams of stable ∞-categories. The construction of the $\boldsymbol{P}^1$-stable $\boldsymbol{A}^1$-homotopy theory involving the Nisnevich topology implies that the compactly generated homotopy category $\mathbf{SH}(R_\omega)$ at the colimit R_ω is equivalent to the filtered colimit of the categories $\mathbf{SH}(R_\beta)$. Modulo some details, adjointness translates the comparison map in question to a homomorphism of Hom-groups in $\mathbf{SH}(R_\alpha)$. Here the right adjoints of the pullback functors commute with filtered colimits, as does the Hom-functor represented by a compact object. □

A further representability result recovers the first known generalized cohomology theory for schemes, as invented by Quillen.

Theorem 2.12 (Morel–Voevodsky). *Let S be a regular base scheme. There exists a motivic spectrum $\mathbf{KGL}_S \in \mathbf{SH}(S)$ representing Quillen's higher algebraic K-groups in the following sense: For every pair of integers s, w and every smooth F-scheme X, there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{SH}(F)}(\Sigma^\infty_{\boldsymbol{P}^1_S} X, \Sigma^{s+(w)} \mathbf{KGL}_S) \cong K^Q_{w-s}(X)$$

of abelian groups.

Also this isomorphism may be promoted to an isomorphism of graded rings. The classical analog is Bott and Atiyah's representability of complex topological K-theory. In principle, every motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ gives rise to a generalized motivic cohomology theory, with potentially interesting geometric applications. The initial generalized motivic cohomology theory is then motivic stable cohomotopy, represented by the unit object $\mathbf{1}_S$. As in the classical situation, only limited information about the represented theory is available.

3. Maps of spheres and Milnor–Witt K-theory

Recall that $\mathbf{1} = \mathbf{1}_S$ denotes the motivic sphere spectrum over the base scheme S. It is the unit for the symmetric monoidal structure of $\mathbf{SH}(S)$ given by the smash product $(\mathsf{D}, \mathsf{E}) \mapsto \mathsf{D} \wedge \mathsf{E}$. In particular, given elements $\alpha \in \pi_{s+(w)}\mathbf{1}$ and $\beta \in \pi_{t+(x)}\mathbf{1}$, the smash product defines the element

$$\alpha \cdot \beta : \Sigma^{s+t+(w+x)}\mathbf{1} \xrightarrow{\text{commutativity iso.}} \Sigma^{s+(w)} \wedge \Sigma^{t+(x)} \xrightarrow{\alpha\wedge\beta} \mathbf{1} \wedge \mathbf{1} \xrightarrow{\text{unit iso.}} \mathbf{1}$$

in $\pi_{s+t+(w+x)}\mathbf{1}$. Alternatively, one may define the element $\alpha \circ \beta$ as the composition

$$\alpha \circ \beta : \Sigma^{s+t+(w+x)}\mathbf{1} \xrightarrow{\text{commutativity iso.}} \Sigma^{s+(w)}\Sigma^{t+(x)} \xrightarrow{\Sigma^{s+(w)}\beta} \Sigma^{s+(w)}\mathbf{1} \xrightarrow{\alpha} \mathbf{1}$$

in $\pi_{s+t+(w+x)}\mathbf{1}$. A careful discussion of these two structures and their relation can be found in [22].

Lemma 3.1 (ring structure). *For every $\alpha \in \pi_{s+(w)}\mathbf{1}$ and $\beta \in \pi_{t+(x)}\mathbf{1}$, there is an equality $\alpha \cdot \beta = \alpha \circ \beta$. The graded group*

$$\pi_{*+(\star)}\mathbf{1} = \bigoplus_{(s,w)\in\mathbb{Z}\times\mathbb{Z}} \pi_{s+(w)}\mathbf{1}$$

forms a graded ring under this multiplication, with the identity as unit.

Proof. Exercise. The references [21; 24] might help. □

Hence the homotopy groups and homotopy sheaves of $\mathbf{1}$ act on the homotopy groups and homotopy sheaves of any motivic spectrum, respectively. We start by writing down obvious elements in $\pi_{*+(\star)}\mathbf{1}$. Every invertible element $u \in \mathcal{O}_S^\times$ defines a morphism $[u] : S_+ \to \boldsymbol{G}_{\mathsf{m}}$ of pointed schemes over S, sending the nonbasepoint to u. It induces a map $[u] : \Sigma^{(-1)} \to \mathbf{1}$ in $\mathbf{SH}(S)$. Since 1 is the basepoint of $\boldsymbol{G}_{\mathsf{m}}$, $[1]$ is the zero element in $\pi_{(-1)}\mathbf{1}$. The following statement was proved first in [33].

Lemma 3.2 (Steinberg relation). *Let S be a base scheme. Then $[u] \cdot [1-u] = 0 \in \pi_{(-2)}\mathbf{1}$ for every $u \in \mathcal{O}_S^\times$ such that $1-u \in \mathcal{O}_S^\times$.*

Proof. Consider the morphism $\boldsymbol{A}^1 \smallsetminus \{0, 1\} \to \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}}$, $u \mapsto (u, 1-u)$, of unpointed smooth schemes. Its image in the affine plane with coordinate axes removed is the line through the (removed) points $(1, 0)$ and $(0, 1)$. Adjoining a disjoint basepoint to its source provides a morphism

$$f : (\boldsymbol{A}^1 \smallsetminus \{0, 1\})_+ \to \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}}, \quad u \mapsto (u, 1-u),$$

of pointed smooth schemes. Postcomposing the morphism f with the canonical map $\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \to \boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}}$ and precomposing with $[u] : S_+ \to (\boldsymbol{A}^1 \smallsetminus \{0, 1\})_+$ for any given $u \in \boldsymbol{A}^1 \smallsetminus \{0, 1\}(S)$ provides the map in question. The basic observation is that the aforementioned image, a line with two points removed, is sent in the smash product to a looped line at the basepoint, which now fills the two holes. In

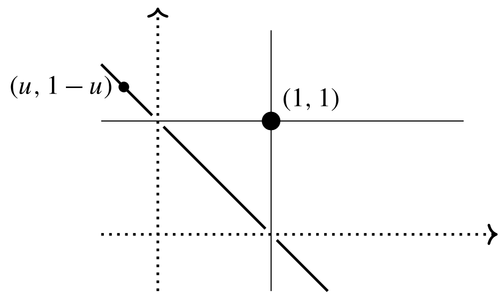

Figure 1. The Steinberg relation.

particular, this line connects the point $(u, 1-u)$ to the basepoint. See Figure 1. More precisely, the canonical map $\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \to \boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}}$ factors as

$$\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \hookrightarrow \boldsymbol{A} \times \{1\} \cup_{\boldsymbol{G}_{\mathsf{m}} \times \{1\}} \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \cup_{\{1\} \times \boldsymbol{G}_{\mathsf{m}}} \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{A}^1 \to \boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}},$$

where the last map collapses the $\boldsymbol{A}^1$-contractible subvariety $\boldsymbol{A} \times \{1\} \cup_{(1,1)} \{1\} \times \boldsymbol{A}^1$ to the basepoint, and hence is an equivalence. (Exercise: write down a contracting $\boldsymbol{A}^1$-homotopy which is constant on the basepoint.) The composition

$$(\boldsymbol{A}^1 \smallsetminus \{0, 1\})_+ \to \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \to \boldsymbol{A} \times \{1\} \cup_{\boldsymbol{G}_{\mathsf{m}} \times \{1\}} \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \cup_{\{1\} \times \boldsymbol{G}_{\mathsf{m}}} \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{A}^1$$

factors over the union $\boldsymbol{A}^1 \times \{1\} \cup_{\{(0,1)\}} \{(t, 1-t)\}$, which is also $\boldsymbol{A}^1$-contractible as a union of two affine lines. The result follows. □

As a consequence, the Steinberg relation holds in the homotopy groups of any motivic spectrum. One particular instance is Voevodsky's Eilenberg–MacLane spectrum $\mathbf{M}\mathbb{Z}_F$, where one has $\pi_{(-n)}\mathbf{M}\mathbb{Z}_F \cong \mathbf{K}^{\mathsf{M}}_n(F)$, the latter being defined as the degree-n component of the quotient of the tensor algebra on the units in F subject to the Steinberg relation (see Definition 3.8 below for details). Note that the unit $\mathbf{1}_F \to \mathbf{M}\mathbb{Z}_F$ maps $[u]$ to the symbol $\{u\} \in \mathbf{K}^{\mathsf{M}}_1(F)$ under this identification. Contrary to the identity $\{u\}+\{v\}=\{uv\} \in \mathbf{K}^{\mathsf{M}}_1(F)$, the term $[u]+[v]-[uv] \in \pi_{(-1)}\mathbf{1}_F$ is not always zero. In order to describe it, consider the multiplication $\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \to \boldsymbol{G}_{\mathsf{m}}$. After passing to motivic spectra, the canonical map $\Sigma^\infty(\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}}) \to \Sigma^\infty(\boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}})$ admits a canonical section. This is in fact true for pointed simplicial presheaves after a single simplicial suspension. The resulting composition

$$\Sigma^\infty(\boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}}) \to \Sigma^\infty(\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}}) \xrightarrow{\text{multiplication}} \Sigma^\infty \boldsymbol{G}_{\mathsf{m}}$$

defines an element $\eta \in \pi_{(1)}\mathbf{1}$.

Lemma 3.3 (logarithm). *For every $u, v \in \mathcal{O}_S^\times$ the equality $[uv]=[u]+[v]+\eta\cdot[u]\cdot[v]$ holds.*

Proof. Exercise. □

Lemma 3.4 (commutativity). *For every $u \in \mathcal{O}_S^\times$ the equality $\eta \cdot [u] = [u] \cdot \eta$ holds.*

Proof. Exercise. □

Let $\varepsilon \in \pi_{0+(0)}\mathbf{1}$ denote the element which is induced by the commutativity isomorphism $\boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}} \cong \boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}}$. By definition and Lemma 3.1, $\varepsilon^2 = 1 \in \pi_{0+(0)}\mathbf{1}$.

Lemma 3.5 (hyperbolic plane). *The equality $\eta \circ \varepsilon = \eta$ holds in $\pi_{(1)}\mathbf{1}$.*

Proof. This follows because the multiplication on $\boldsymbol{G}_{\mathsf{m}}$ is commutative. □

Lemma 3.6. *The ring structure on $\pi_{*+(\star)}\mathbf{1}$ is ε-graded commutative in the sense that for every $\alpha \in \pi_{s+(w)}\mathbf{1}$ and $\beta \in \pi_{t+(x)}\mathbf{1}$, there is an equality $\alpha \cdot \beta = (-1)^{st}\varepsilon^{wx}\beta \cdot \alpha$.*

There is another canonical element in $\pi_{0+(0)}\mathbf{1}$, namely the element ϵ which is induced by the inverse $\boldsymbol{G}_{\mathsf{m}} \xrightarrow{u \mapsto u^{-1}} \boldsymbol{G}_{\mathsf{m}}$. Again by definition and Lemma 3.1, $\epsilon^2 = 1 \in \pi_{0+(0)}\mathbf{1}$. Yet another canonical element is $\tilde{\eta} \in \pi_{(1)}\mathbf{1}$ induced by the composition

$$\Sigma^{1+(2)} \simeq \boldsymbol{A}^2 \smallsetminus \{0\} \xrightarrow{\text{canonical}} \boldsymbol{P}^1 \simeq \Sigma^{1+(1)},$$

where the identifications are the canonical ones.

Lemma 3.7. *There are equalities $\varepsilon = \epsilon$, $\eta = \tilde{\eta}$, and $\varepsilon = -(1 + \eta[-1])$.*

Proof. Consider the map $\boldsymbol{P}^1 \to \boldsymbol{P}^1$, $[x, y] \mapsto [y, x]$. It restricts to the inverse map $u \mapsto u^{-1}$ on $\boldsymbol{G}_{\mathsf{m}}$. Since it also interchanges the two copies of the affine line, it corresponds to $-\Sigma^1\epsilon : \Sigma^1\boldsymbol{G}_{\mathsf{m}} \to \Sigma^1\boldsymbol{G}_{\mathsf{m}}$ via the canonical identification $\Sigma^1\boldsymbol{G}_{\mathsf{m}} \simeq \boldsymbol{P}^1$. The map $[x, y] \mapsto [y, x]$ can equivalently be described via the action of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and since the last matrix is a product of elementary matrices, the resulting map is $\boldsymbol{A}^1$-homotopic to the map induced by $[x, y] \mapsto [-x, y]$. Via the canonical identification $\boldsymbol{A}^1/(\boldsymbol{A}^1 \smallsetminus \{0\}) \simeq \boldsymbol{P}^1$, it thus corresponds to the map induced by multiplication with -1.

The commutativity isomorphism $\boldsymbol{P}^1 \wedge \boldsymbol{P}^1 \cong \boldsymbol{P}^1 \wedge \boldsymbol{P}^1$ induces via the canonical equivalences

$$\boldsymbol{P}^1 \wedge \boldsymbol{P}^1 \simeq \boldsymbol{A}^1/(\boldsymbol{A}^1 \smallsetminus \{0\}) \wedge \boldsymbol{A}^1/(\boldsymbol{A}^1 \smallsetminus \{0\}) \cong \boldsymbol{A}^2/(\boldsymbol{A}^2 \smallsetminus \{0\})$$

the map $\boldsymbol{A}^2/(\boldsymbol{A}^2 \smallsetminus \{0\}) \to \boldsymbol{A}^2/(\boldsymbol{A}^2 \smallsetminus \{0\})$ induced by $(x, y) \mapsto (y, x)$. Also this map is induced by the action of the matrix given above, hence is $\boldsymbol{A}^1$-homotopic to the map induced by $(x, y) \mapsto (-x, y)$. The latter corresponds to the $\boldsymbol{A}^1/(\boldsymbol{A}^1 \smallsetminus \{0\})$-suspension of the map induced by multiplication with -1, and hence to $-\Sigma^{2+(1)}\epsilon$.

The commutativity isomorphism on $S^1 \wedge S^1$ coincides with -1 by topology, whence the commutativity isomorphism on $\boldsymbol{P}^1 \wedge \boldsymbol{P}^1$ equals an appropriate suspension of $-\varepsilon$. The equality $\epsilon = \varepsilon$ follows. Moreover, the identification $\varepsilon = -(1+\eta[-1])$ results from the intermediate step of this argument.

The equality $\eta = \tilde{\eta}$ is left as an exercise. □

Another canonical element can be obtained from the cell filtration on projective spaces, namely the composition

$$\phi_n : \Sigma^{n+(n+1)} \simeq \boldsymbol{A}^{n+1} \smallsetminus \{0\} \to \boldsymbol{P}^n \to \boldsymbol{P}^n/\boldsymbol{P}^{n-1} \simeq \Sigma^{n+(n)}$$

of canonical maps, which gives an element $\phi_n \in \pi_{0+(1)}\mathbf{1}$. Lemma 3.7 implies that $\phi_1 = \eta$. Exercise: identify ϕ_n for other n.

For a unit u, set $\langle u\rangle := 1 + \eta[u]$; the notation reflects that this element corresponds to the one-dimensional symmetric bilinear form represented by u which is established essentially in Theorem 3.10 below. The proof of Lemma 3.7 above implies that $-\langle u\rangle$ can be described as the endomorphism $[x, y] \mapsto [ux, y]$ on $\boldsymbol{P}^1$, provided either $[0, 1]$ or $[1, 0]$ are chosen as the basepoint.

Definition 3.8. The Milnor–Witt K-theory $\mathbf{K}^{\mathsf{MW}}(F)$ of a field F has the Hopkins-Morel presentation as the associative graded ring (with unit) whose generators are the field units $[u]$, $u \in F^\times$ in degree 1, and a generator η in degree -1, subject to four relations:

Steinberg: $[u]\cdot[v] = 0$ if $u + v = 1$.

Logarithm: $[u\cdot v] = [u] + [v] + \eta\cdot[u]\cdot[v]$.

Commutativity: $[u]\cdot\eta = \eta\cdot[u]$.

Hyperbolic plane: $\mathsf{h}\cdot\eta = 0$, where $\mathsf{h} := 2 + \eta[-1]$.

Corollary 3.9. *The canonical map defines a "degree inverting" ring homomorphism* $\mathbf{K}^{\mathsf{MW}}_\star(F) \to \pi_{0-(\star)}\mathbf{1}$.

Proof. This follows from Lemmata 3.2–3.5. □

Theorem 3.10 (Morel). *Let F be a field. Then $\pi_s\mathbf{1}_F = 0$ for $s < 0$. The canonical ring homomorphism $\mathbf{K}^{\mathsf{MW}}_\star(F) \to \pi_{0-(\star)}\mathbf{1}$ is an isomorphism.*

Proof. See [46] for the unstable result implying the $\boldsymbol{P}^1$-stable one, as well as [43] for a sketch. The injectivity will be clarified as a consequence of Theorem 4.19 below. □

The Milnor–Witt K-theory of F is closely related to Milnor K-theory (by sending η to 0), and to quadratic form theory. In fact, $\mathbf{K}^{\mathsf{MW}}_0(F)$ is the Grothendieck–Witt ring of F; the 1-dimensional form $\langle u\rangle$ corresponds to $1+\eta[u]$. If $n < 0$, then $\mathbf{K}^{\mathsf{MW}}_n(F)$ is isomorphic to the Witt ring of F and multiplication with η induces an isomorphism $\mathbf{K}^{\mathsf{MW}}_n(F) \to \mathbf{K}^{\mathsf{MW}}_{n-1}(F)$. (Exercise; see [46, Lemma 3.10].)

4. Filtrations

Various filtrations exist on $\mathbf{SH}(S)$. Let n be an integer. A motivic spectrum E is said to be *n-connective* if it is contained in the full subcategory $\mathbf{SH}_{\geq n}(S)$ generated under extensions and homotopy colimits by the shifted suspension spectra $\{\Sigma^{s+(w)}X_+\}_{s\geq n, w\in\mathbb{Z}, X\in\mathbf{Sm}_S}$. This produces the so-called *homotopy t-structure*, as studied by Morel in [45]. We might refer to it as the *connectivity filtration*.

Remark 4.1. If S is a field, E is n-connective if and only if its Nisnevich sheaves of homotopy groups $\pi_{s+(w)}\mathsf{E}$ equal 0 for $s < n$; see [45, Theorem 6.1.8] for the case of perfect fields and [32, Theorem 2.3] for all fields.

A motivic spectrum $\mathsf{E}\in\mathbf{SH}(S)$ is said to be *n-effective* if it is contained in the full subcategory $\mathbf{SH}_{\geq(n)}(S)$ generated under extensions and homotopy colimits by the shifted suspension spectra $\{\Sigma^{s+(w)}X_+\}_{s\in\mathbb{Z}, w\geq n, X\in\mathbf{Sm}_S}$. This produces Voevodsky's *slice filtration*, as introduced in [65]. Let $\mathsf{f}_n := i_n\circ r_n : \mathbf{SH}(S)\to\mathbf{SH}(S)$, where $r_n:\mathbf{SH}(S)\to\mathbf{SH}_{\geq(n)}(S)$ is the right adjoint to the inclusion $i_n:\mathbf{SH}_{\geq(n)}(S)\to\mathbf{SH}(S)$.

Lemma 4.2. *The canonical transformation $\mathsf{f}_{n+1}\to\mathsf{f}_n$ admits a canonical extension to a homotopy cofiber sequence*

$$\mathsf{f}_{n+1}\to\mathsf{f}_n\to\mathsf{s}_n\to\Sigma\mathsf{f}_{n+1}$$

of triangulated (*in fact homotopy-colimit preserving*) *functors defining the n-th slice s_n.*

Proof. Exercise. □

A motivic spectrum $\mathsf{E}\in\mathbf{SH}(S)$ is said to be *n-very effective* if it is contained in the full subcategory $\mathbf{SH}_{\geq n+(n)}(S)$ generated under extensions and homotopy colimits by the shifted suspension spectra $\{\Sigma^{s+(w)}X_+\}_{s\geq n, w\geq n, X\in\mathbf{Sm}_S}$. This produces the *very effective slice filtration*, as introduced by Spitzweck and Østvær in [62]. Let $\mathsf{vf}_n := vi_n\circ vr_n : \mathbf{SH}(S)\to\mathbf{SH}(S)$, where $vr_n:\mathbf{SH}(S)\to\mathbf{SH}_{\geq n+(n)}(S)$ is the right adjoint to the inclusion $vi_n:\mathbf{SH}_{\geq n+(n)}(S)\to\mathbf{SH}(S)$. Again the canonical natural transformation completes to a homotopy cofiber sequence

$$\mathsf{vf}_{n+1}\to\mathsf{vf}_n\to\mathsf{vs}_n\to\Sigma\mathsf{vf}_{n+1}$$

defining the n-th very effective slice functor.

An n-very effective motivic spectrum is n-effective and n-connective. Instead of 0-(very) effective or 0-connective, one simply says "(very) effective" or "connective".

Example 4.3. If $V\to Y$ is a vector bundle of rank r over $Y\in\mathbf{Sm}_S$, then the Thom space $\mathsf{Th}(V)$ is r-very effective. More generally, if $a\in K^0(Y)$ is a virtual vector bundle of rank $r\in\mathbb{Z}$, then $\mathsf{Th}(a)$ is r-very effective.

Example 4.4. The motivic spectra $\mathbf{1}$, $\mathbf{MGL}$, $\mathbf{MSp}$, and $\mathbf{MSL}$ are very effective over any base scheme. Because of the periodicities they satisfy, the motivic spectra $\mathbf{KGL}$ and $\mathbf{KQ}$ representing algebraic and hermitian K-theory, respectively, are neither n-effective nor n-connective for any $n \in \mathbb{Z}$. The motivic spectrum $\mathbf{1}[\eta^{-1}]$ is not n-effective for any n, but connective. The effective cover $\mathsf{f}_0\mathbf{KQ}$ is not n-connective for any n.

Let S be ind-smooth over a field or a Dedekind domain. Then [61] provides a highly structured motivic Eilenberg–MacLane spectrum $\mathbf{M}A$ for any abelian group A (which in the following will mostly be cyclic) representing motivic cohomology in the sense that there is an identification

$$\pi_{s+(w)}\mathbf{M}A = \pi_{s+w,w}\mathbf{M}A = H^{-s-w,-w}(S; A)$$

and in particular

$$\pi_{0+(w)}\mathbf{M}\mathbb{Z} = \mathbf{K}^{\mathsf{M}}_{-w}(S) \tag{4-1}$$

in the case that S is additionally local. Note that $\pi_{s+(w)}\mathbf{M}A = 0$ for $w > 0$ and for $s < -\dim(S)$.

The starting point for many determinations of slices is the following.

Theorem 4.5 (Levine, Voevodsky, Bachmann–Hoyois). *Let S be ind-smooth over a Dedekind domain or a field. Then $\mathbf{M}\mathbb{Z}$ is effective and the canonical map $\mathbf{M}\mathbb{Z} = \mathsf{f}_0\mathbf{M}\mathbb{Z} \to \mathsf{s}_0\mathbf{M}\mathbb{Z}$ is an isomorphism in $\mathbf{SH}(S)$. The unit map $\upsilon : \mathbf{1} \to \mathbf{M}\mathbb{Z}$ coincides with $\mathbf{1} = \mathsf{f}_0\mathbf{1} \to \mathsf{s}_0\mathbf{1} \xrightarrow{\mathsf{s}_0\upsilon} \mathsf{s}_0\mathbf{M}\mathbb{Z}$.*

Proof. Note first that $[\Sigma^{s+(w)}X_+, \mathbf{M}\mathbb{Z}] = 0$ whenever $w > 0$. Hence if $\mathbf{M}\mathbb{Z}$ is effective, the identification $\mathbf{M}\mathbb{Z} = \mathsf{f}_0\mathbf{M}\mathbb{Z} = \mathsf{s}_0\mathbf{M}\mathbb{Z}$ follows. A proof of this effectivity for a field of characteristic zero may be obtained by modeling $\mathbf{M}\mathbb{Z}$ via infinite symmetric powers of spheres. This proof, due to Voevodsky, proceeds along a filtration on $\mathbf{M}\mathbb{Z}$ which identifies the sphere spectrum as its starting level, and hence filters the unit map $\mathbf{1} \to \mathbf{M}\mathbb{Z}$. It in fact implies that the cofiber of the unit map is 1-effective, whence the unit map induces an isomorphism on zero slices. The proof for a field of any characteristic, as provided by Levine, gives a "reverse cycle map" $\mathbf{M}\mathbb{Z} \to \mathsf{s}_0\mathbf{1}$ via a homotopy coniveau tower. A proof up to inverting the exponential characteristic e of the base field is supplied by the Hopkins–Morel isomorphism. The latter expresses $\mathbf{M}\mathbb{Z}\big[\frac{1}{e}\big]$ as the quotient of $\mathbf{MGL}\big[\frac{1}{e}\big]$ with respect to the standard generators x_i of the Lazard ring. Passage to a Dedekind scheme then follows by a base change argument detecting effectivity on residue fields. □

Example 4.6. Let F be a field of characteristic not two, and let $\tau \in h^{0,1}$, where $h^{0,1} := H^{0,1}(F, \mathbb{Z}/2)$, be the unique nontrivial element given by $-1 \in F$. Then $\mathbf{M}\mathbb{F}_2[\tau^{-1}]$ is n-effective for every $n \in \mathbb{Z}$, but not n-connective for any n. This motivic spectrum represents étale cohomology with coefficients in μ_2 [65].

As Example 4.6 drastically reveals, the slice filtration is not separated in general. More precisely, if the *slice completion* of E is defined via the canonical cofiber sequence

$$\operatorname*{holim}_{q\to\infty} \mathsf{f}_q\mathsf{E} \to \mathsf{E} \to \mathsf{sc}(\mathsf{E}) \tag{4-2}$$

(so that the slice completion of E is the natural target of the slice spectral sequence of E), then $\mathbf{M}\mathbb{F}_2[\tau^{-1}] \simeq \operatorname{holim}_q \mathsf{f}_q\mathbf{M}\mathbb{F}_2[\tau^{-1}]$ and $\mathsf{sc}(\mathbf{M}\mathbb{F}_2[\tau^{-1}]) \simeq *$. One advantage of the very effective slice filtration is, for every $\mathsf{E} \in \mathbf{SH}(S)$, one has $\operatorname{holim}_q \mathsf{vf}_q\mathsf{E} \simeq *$. In favorable cases (for example, for $\mathbf{MGL}$, as a consequence of Corollary 4.8 below), the slice filtration coincides with the very effective slice filtration.

Let $\mathbb{L} = \mathbb{Z}[x_1, x_2, \dots]$ denote the Lazard ring classifying formal group laws, with $\deg(x_n) = n$. Its universal property provides a ring homomorphism

$$\mathbb{L} \to \bigoplus_{n\in\mathbb{Z}} \pi_{n+(n)}\mathbf{MGL}$$

and the images of the polynomial generators in $\pi_{n+(n)}\mathbf{MGL}$ are denoted by the same name. They describe the very effective and effective slice filtration on $\mathbf{MGL}$, at least under suitable restrictions on the base scheme.

Theorem 4.7 (Hopkins–Morel, Hoyois). *Let S be the spectrum of a field or a discrete valuation ring of mixed characteristic. The canonical map* $\mathbf{MGL} \to \mathbf{MGL}/(x_1, x_2, \dots)$ *coincides with both the canonical map* $\mathbf{MGL} = \mathsf{f}_0\mathbf{MGL} = \mathsf{vf}_0\mathbf{MGL} \to \mathsf{s}_0\mathbf{MGL} = \mathsf{vs}_0\mathbf{MGL}$ *and the canonical map* $\mathbf{MGL} \to \mathbf{M}\mathbb{Z}$, *at least after inverting the exponential characteristic of S.*

Proof. See [32; 61]. □

Work by Spitzweck then provides all (very) effective slices of $\mathbf{MGL}$.

Corollary 4.8 (Spitzweck). *There is an identification* $\mathsf{vs}_n\mathbf{MGL} = \mathsf{s}_n\mathbf{MGL} = \Sigma^{n+(n)}\mathbf{M}\mathbb{L}_n$ *with the Eilenberg–MacLane spectrum given by the degree-n part of the Lazard ring*, *at least after inverting the exponential characteristic of S.*

Proof. See [60]. □

For example, $\mathsf{s}_2\mathbf{MGL} = \Sigma^{2+(2)}(\mathbf{M}\mathbb{Z}\{x_1^2\} \vee \mathbf{M}\mathbb{Z}\{x_2\})$.

Let $\mathbf{KGL}$ denote the motivic spectrum representing (homotopy) algebraic K-theory over S in the sense that

$$\pi_{s+(w)}\mathbf{KGL} \cong K^Q_{s-w}(S),$$

where the superscript "Q" stands for "Quillen". It is a motivic ring spectrum in a natural and unique way. The difference of the tautological and the trivial line bundle on the projective line $\boldsymbol{P}^1$ define an element $\beta \in \pi_{1+(1)}\mathbf{KGL}$ such that multiplication with it yields an equivalence $\Sigma^{1+(1)}\mathbf{KGL} \simeq \mathbf{KGL}$ often called "Bott periodicity".

Theorem 4.9 (Levine, Voevodsky). *Let S be the spectrum of a field. The unit* $\mathbf{1} \to \mathbf{KGL}$ *induces an isomorphism on zero slices. Bott periodicity thus describes the graded slice as*

$$\mathsf{s}_*\mathbf{KGL} = \mathsf{s}_0\mathbf{KGL}[\beta^{\pm}] = \mathbf{M}\mathbb{Z}[\beta^{\pm}],$$

where $\deg(\beta) = 1+(1)$. *Also*, $\mathbf{KGL}$ *is slice complete*: $\operatorname{holim} \mathsf{f}_q\mathbf{KGL} \simeq *$.

Proof. Levine's proof via the homotopy coniveau tower works over any field. Since it is going to be used anyhow, instead a presentation of $\mathbf{kgl} := \mathsf{f}_0\mathbf{KGL}$, the effective cover of $\mathbf{KGL}$, will be used. The projective bundle formula in K-theory supplies a canonical map $\mathbf{MGL} \to \mathbf{KGL}$ over any base scheme, thanks to the universal property of $\mathbf{MGL}$. It factors over $\mathbf{MGL}/(x_2, x_3, \dots)$, and the latter coincides with $\mathbf{kgl}$ up to inverting the exponential characteristic. A description of $\mathsf{s}_0\mathbf{KGL} = \mathsf{s}_0\mathbf{kgl} = \mathsf{s}_0\mathbf{MGL}$ follows. Bott periodicity provides an identification

$$\mathsf{f}_q\mathbf{KGL} \simeq \mathsf{f}_q(\Sigma^{q+(q)}\mathbf{KGL}) \simeq \Sigma^{q+(q)}(\mathsf{f}_0\mathbf{KGL}) = \Sigma^{q+(q)}\mathbf{kgl}$$

and similarly for the slices. This identification also shows that $\mathsf{f}_q\mathbf{KGL} = \mathsf{vf}_q\mathbf{KGL}$ for all q. Since $\operatorname{holim}_q \mathsf{vf}_q\mathsf{E} \simeq *$ for all E, convergence as stated follows. Additionally the columns of the first page of the slice spectral sequence of $\mathbf{KGL}$ are finite. □

Remark 4.10. Theorem 4.9 implies that the slice spectral sequence for $\mathbf{KGL}$ converges strongly. Recent work of Bachmann [11] shows that Theorem 4.9 holds over any Dedekind domain. Even more recent work implies the same over any quasicompact quasiseparated scheme.

Example 4.11. Let S be the spectrum of a field. For all $w \geq 0$, $\pi_{0+(-w)}\mathbf{kgl} \cong \mathbf{K}^{\mathsf{M}}_w$. The canonical map $\mathbf{kgl} \to \mathbf{KGL}$ induces the canonical map from Milnor to Quillen K-theory. Given that the first slice differential $\mathsf{s}_0\mathbf{kgl} \to \Sigma^1\mathsf{s}_1\mathbf{kgl}$ coincides with the Steenrod operation

$$\mathbf{M}\mathbb{Z} \xrightarrow{\mathsf{pr}_2^\infty} \mathbf{M}\mathbb{Z}/2 \xrightarrow{\mathsf{Sq}^2} \Sigma^{1+(1)}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\partial^2_\infty} \Sigma^{2+(1)}\mathbf{M}\mathbb{Z}$$

(which will follow from the corresponding differential for $\mathbf{kq}$ stated in Theorem 5.2 using notation introduced immediately before), there results an exact sequence

$$H^{w-2,w} \xrightarrow{\partial^2_\infty\mathsf{Sq}^2\mathsf{pr}_2^\infty} H^{w+1,w+1} \to \pi_{1-(w)}\mathbf{kgl} \to H^{w-1,w} \to 0$$

in which the surjection $\pi_{1-(w)}\mathbf{kgl} \to H^{w-1,w}$ usually does not split. In the case $S = \operatorname{Spec}(\mathbb{Q})$ this sequence has the form

$$H^{0,2}(\mathbb{Q}) \xrightarrow{0} H^{3,3}(\mathbb{Q}) = \mathbb{Z}/2 \to \mathbb{Z}/48 \to H^{1,2}(\mathbb{Q}) \to 0$$

for $w = -2$, as [36] implies.

Convergence of the slice spectral sequence is often a subtle issue, as the following example will show.

Definition 4.12. Let S be a scheme containing $\frac{1}{2}$. Then $\mathbf{KQ} \in \mathbf{SH}(S)$ denotes the motivic spectrum representing hermitian K-theory. Let $\mathbf{kq} := \mathsf{vf}_0\mathbf{KQ}$ denote its very effective cover.

A few of the homotopy groups or sheaves of $\mathbf{KQ}$ are known explicitly. For example, if S is the spectrum of a field or a discrete valuation ring in which 2 is invertible, $\pi_{0+(0)}\mathbf{KQ} \cong \pi_{0+(0)}\mathbf{kq} \cong \mathbf{GW}(S)$ and $\pi_{0+(w)}\mathbf{KQ} \cong \pi_{0+(w)}\mathbf{kq} \cong \boldsymbol{W}(S)$ for $w > 0$.

Proposition 4.13. *Over a field of characteristic $\neq 2$, multiplication with the Hopf map η induces cofiber sequences*

$$\Sigma^{(1)}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{\text{forget}} \mathbf{kgl} \to \Sigma^{1+(1)}\mathbf{kq}, \tag{4-3}$$

$$\Sigma^{(1)}\mathbf{KQ} \xrightarrow{\eta} \mathbf{KQ} \xrightarrow{\text{forget}} \mathbf{KGL} \xrightarrow{\Sigma^{1+(1)}\text{hyper}\circ\beta} \Sigma^{1+(1)}\mathbf{KQ}. \tag{4-4}$$

Here forget *and* hyper *are induced by the forgetful and hyperbolic maps between algebraic and hermitian K-theory, respectively.*

Proof. See [54; 57] □

There exists a periodicity element $\alpha \in \pi_{4+(4)}\mathbf{KQ}$ such that multiplication with α induces an equivalence $\Sigma^{4+(4)}\mathbf{KQ} \simeq \mathbf{KQ}$. One has $\text{forget}(\alpha) = \beta^4 \in \pi_{4+(4)}\mathbf{KGL}$, relating the two periodicities.

Remark 4.14. As a consequence of Proposition 4.13 and the slice completeness of $\mathbf{kgl}$, which in turn follows from Theorem 4.9, the slice completion of $\mathbf{kq}$ coincides with its η-completion:

$$\mathsf{sc}(\mathbf{kq}) \simeq \mathbf{kq}^{\wedge}_{\eta}. \tag{4-5}$$

This identification is helpful, because for any motivic spectrum E the canonical η-arithmetic square

$$\begin{array}{ccc} \mathsf{E} & \longrightarrow & \mathsf{E}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \mathsf{E}^{\wedge}_{\eta} & \longrightarrow & \mathsf{E}^{\wedge}_{\eta}[\eta^{-1}] \end{array} \tag{4-6}$$

is a homotopy pullback square.[2]

Another consequence of Proposition 4.13 is a determination of the slices of $\mathbf{kq}$ and $\mathbf{KQ}$. For comparing with the sphere spectrum, it suffices to focus on $\mathbf{kq}$. Note that the periodicity element $\alpha \in \pi_{4+(4)}\mathbf{KQ}$ induces an equivalence $\mathsf{s}_q\mathbf{KQ} \simeq$

[2]There is nothing special about η; this holds for any endomorphism of the sphere spectrum.

$\Sigma^{4+(4)}\mathsf{s}_{q-4}\mathbf{KQ}$ of period 4. Funnily enough, on the level of slices a periodicity of period 2 holds, induced by an element $\sqrt{\alpha}$ of degree $2+(2)$.

Theorem 4.15. *Over a field of characteristic not two the nonnegative slices of* $\mathbf{kq}$ *are given as*

$$\mathsf{s}_q\mathbf{kq} = \begin{cases} \Sigma^{(2n)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(2n)}\mathbf{M}\mathbb{Z}/2 \vee \\ \qquad \cdots \vee \Sigma^{2n-2+(2n)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+(2n)}\mathbf{M}\mathbb{Z}, & q=2n, \\ \Sigma^{(2n+1)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(2n+1)}\mathbf{M}\mathbb{Z}/2 \vee \\ \qquad \cdots \vee \Sigma^{2n+(2n+1)}\mathbf{M}\mathbb{Z}/2, & q=2n+1, \end{cases}$$

and in closed form

$$\mathsf{s}_*\mathbf{kq} = \mathbf{M}\mathbb{Z}[\eta, \sqrt{\alpha}]/(2\eta = 0, \eta^2 \xrightarrow{\partial^2_\infty} \sqrt{\alpha}),$$

where η *of degree* (1) *is induced by the Hopf map. The negative slices of* $\mathbf{kq}$ *are zero. The canonical map* $\mathbf{kq} \to \mathbf{KQ}$ *induces a natural inclusion as a direct summand on slices, and respects the multiplicative structure. The multiplicative relation* $\eta^2 \xrightarrow{\partial^2_\infty} \sqrt{\alpha}$ *says that the product*

$$\mathsf{s}_1\mathbf{kq} \wedge_{\mathsf{s}_0\mathbf{kq}} \mathsf{s}_1\mathbf{kq} \cong \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{1+(2)}\mathbf{M}\mathbb{Z}/2 \\ \to \mathsf{s}_2\mathbf{kq} \cong \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2\{\eta^2\} \vee \Sigma^{2+(2)}\mathbf{M}\mathbb{Z}\{\sqrt{\alpha}\}$$

maps via the (*unique*) *nontrivial map to the integral summand.*

Proof. Since $\mathbf{kq} = \mathsf{f}_0(\mathbf{KQ}_{\geq 0})$ is (very) effective, its negative slices are zero. Applying the slice functor to (4-3) yields a cofiber sequence. The natural isomorphism $\mathsf{s}_q \circ \Sigma^{(1)} \cong \Sigma^{(1)} \circ \mathsf{s}_{q-1}$ of [54, Lemma 2.1] shows the forgetful map forget : $\mathbf{kq} \to \mathbf{kgl}$ induces, on zero slices, an isomorphism

$$\mathsf{s}_0\mathbf{kq} \xrightarrow{\cong} \mathsf{s}_0\mathbf{kgl},$$

and likewise for the unit map $\mathbf{1} \to \mathbf{kq}$. For the 1-slices (4-3) induces a cofiber sequence

$$\Sigma^{(1)}\mathsf{s}_0\mathbf{kq} = \Sigma^{(1)}\mathbf{M}\mathbb{Z} \xrightarrow{\eta} \mathsf{s}_1\mathbf{kq} \\ \xrightarrow{\mathsf{s}_1\text{forget}} \mathsf{s}_1\mathbf{kgl} = \Sigma^{1+(1)}\mathbf{M}\mathbb{Z} \to \Sigma^{1+(1)}\mathsf{s}_0\mathbf{kq} = \Sigma^{1+(1)}\mathbf{M}\mathbb{Z}.$$

Hence $\mathsf{s}_1$hyper can be identified with an integer $n \in \mathbb{Z}$. Note that as an endomorphism of $\mathbf{KGL}$, the composition forget $\circ$ hyper $= 1 + \psi^{-1}$ is the sum of the identity $\mathrm{id}_{\mathbf{KGL}}$ and the map induced by sending a vector bundle to its dual. Note further that as an endomorphism of $\mathbf{KQ}$, the composition hyper $\circ$ forget coincides with multiplication by h. It follows that $n = 2$, so that $\mathsf{s}_1\mathbf{kq} = \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2$. For the 2-slices then (4-3) induces a cofiber sequence

$$\Sigma^{(1)}\mathsf{s}_1\mathbf{kq} = \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\eta} \mathsf{s}_2\mathbf{kq} \\ \xrightarrow{\mathsf{s}_2\text{forget}} \mathsf{s}_2\mathbf{kgl} = \Sigma^{2+(2)}\mathbf{M}\mathbb{Z} \to \Sigma^{1+(1)}\mathsf{s}_1\mathbf{kq} = \Sigma^{1+(2)}\mathbf{M}\mathbb{Z}/2.$$

Hence the last map is zero for simplicial degree reasons, the cofiber sequence splits, and we get $\mathsf{s}_2\mathbf{kq} = \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(2)}\mathbf{M}\mathbb{Z}$. Also, $\mathsf{s}_2$forget is the projection map onto $\Sigma^{2+(2)}\mathbf{M}\mathbb{Z}$. On 3-slices (4-3) induces a cofiber sequence

$$\Sigma^{(1)}\mathsf{s}_2\mathbf{kq} = \Sigma^{(3)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(3)}\mathbf{M}\mathbb{Z} \xrightarrow{\eta} \mathsf{s}_3\mathbf{kq} \xrightarrow{\mathsf{s}_3\text{forget}} \mathsf{s}_3\mathbf{kgl} = \Sigma^{3+(3)}\mathbf{M}\mathbb{Z} \to \Sigma^{1+(1)}\mathsf{s}_2\mathbf{kq}.$$

Here the last map lands trivially in $\Sigma^{1+(3)}\mathbf{M}\mathbb{Z}/2$ for simplicial degree reasons, while its component mapping to $\Sigma^{3+(3)}\mathbf{M}\mathbb{Z}$ can be identified with an integer $n \in \mathbb{Z}$. We deduce $n = 2$ by comparison with the hyperbolic map $\mathbf{KGL} \to \mathbf{KQ}$ in [54, §4.3]. Hence we obtain $\mathsf{s}_3\mathbf{kq} \cong \Sigma^{(3)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(3)}\mathbf{M}\mathbb{Z}/2$. Iterating these arguments produces the claimed additive calculation. The statement regarding $\mathbf{kq} \to \mathbf{KQ}$ follows from applying the main result of [30]. The "polynomial part" of the multiplicative structure follows from the periodicity of $\mathbf{KQ}$ and $\mathbf{KQ}[\eta^{-1}]$. The relation between η^2 and $\sqrt{\alpha}$ follows from the commutative diagram

$$\begin{array}{ccc} \mathsf{s}_1\mathbf{kq} \wedge_{\mathsf{s}_0\mathbf{kq}} \mathsf{s}_1\mathbf{kq} & \longrightarrow & \mathsf{s}_2\mathbf{kq} \\ \downarrow{\scriptstyle \mathsf{s}_1\text{forget} \wedge \mathsf{s}_1\text{forget}} & & \downarrow{\scriptstyle \mathsf{s}_2\text{forget}} \\ \mathsf{s}_1\mathbf{kgl} \wedge_{\mathsf{s}_0\mathbf{kgl}} \mathsf{s}_1\mathbf{kgl} & \xrightarrow{\simeq} & \mathsf{s}_2\mathbf{kgl} \end{array}$$

and the above identification of the vertical maps, using the bottom horizontal equivalence given by Theorem 4.9. □

Remark 4.16. Contrary to the calculation of the slices of $\mathbf{KQ}$ in [54] there is no "mysterious summand" appearing in Theorem 4.15, thanks to the connectivity of $\mathbf{kq}$. Each slice of $\mathbf{kq}$ is a finite sum of motivic Eilenberg–MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$. The odd slices of $\mathbf{kq}$ are cellular of finite type for every F [55, §3.3], and likewise for all the slices when $\mathsf{char}(F) = 0$.

Corollary 4.17. *When* $\mathsf{char}(F) \neq 2$ *the slices of* $\mathbf{kq}[\frac{1}{\eta}] = \mathbf{KW}_{\geq 0}$ *are given by*

$$\mathsf{s}_q(\mathbf{KW}_{\geq 0}) = \Sigma^{(q)}(\mathbf{M}\mathbb{Z}/2 \vee \Sigma^2\mathbf{M}\mathbb{Z}/2 \vee \Sigma^4\mathbf{M}\mathbb{Z}/2 \vee \cdots),$$

and

$$\mathsf{s}_*(\mathbf{KW}_{\geq 0}) \cong \mathbf{M}\mathbb{Z}[\eta^{\pm 1}, \sqrt{\alpha}]/(2\eta = 2\sqrt{\alpha} = 0, \eta^2 \xrightarrow{\mathsf{Sq}^1} \sqrt{\alpha}).$$

The canonical map $\mathbf{KW}_{\geq 0} \to \mathbf{KW}$ *induces the natural inclusion on slices, and respects the multiplicative structure.*

As in the case of $\mathsf{s}_*\mathbf{kq}$, the multiplicative structure is not quite polynomial and, because of the multiplicative relation involving Sq^1 (which is similar to the multiplicative relation in $\mathsf{s}_*\mathbf{kq}$ from Theorem 4.15), not $\mathbf{M}\mathbb{Z}/2$-linear.

Lemma 4.18. *Let $n \in \mathbb{Z}$. The sequence*

$$\mathbf{K}^{\mathsf{M}}_{n+1} \xrightarrow{\mathsf{h}} \mathbf{K}^{\mathsf{MW}}_{n+1} \xrightarrow{\eta} \mathbf{K}^{\mathsf{MW}}_{n} \to \mathbf{K}^{\mathsf{M}}_{n} \to 0$$

is exact.

Proof. Multiplication with $\mathsf{h} = 2+\eta[-1] \in \mathbf{K}^{\mathsf{MW}}_0$ on $\mathbf{K}^{\mathsf{MW}}_{n+1}$ factors over the projection $\mathbf{K}^{\mathsf{MW}}_{n+1} \to \mathbf{K}^{\mathsf{M}}_{n+1} = \mathbf{K}^{\mathsf{MW}}_{n+1}/\eta\mathbf{K}^{\mathsf{MW}}_{n+2}$, because $\eta\mathsf{h} = 0$. This explains the choice of the first homomorphism in the sequence above. It remains to prove that an element in the kernel of $\eta : \mathbf{K}^{\mathsf{MW}}_{n+1} \to \mathbf{K}^{\mathsf{MW}}_{n}$ is a multiple of h. The exercise at the end of Section 3 says that multiplication with η is an isomorphism for $n < -1$. For $n = -1$, multiplication with η corresponds to the canonical map from the Grothendieck–Witt ring to the Witt ring. By definition, its kernel is a copy of the integers generated by the hyperbolic plane h, thus giving exactness for $n = -1$.

Define Witt K-theory as the quotient $\mathbf{K}^{\mathsf{W}}_n := \mathbf{K}^{\mathsf{MW}}_n/\mathsf{h}\mathbf{K}^{\mathsf{MW}}_n \cong \mathbf{K}^{\mathsf{MW}}_n/\mathsf{h}\mathbf{K}^{\mathsf{M}}_n$, where the last identification follows from the relation $\eta\mathsf{h} = 0$. This relation also implies that the surjection $\mathbf{K}^{\mathsf{MW}}_n \to \mathbf{K}^{\mathsf{W}}_n$ is injective as well for $n < 0$. Let $\boldsymbol{I} \subset \boldsymbol{W}$ denote the fundamental ideal, that is, the kernel of the mod 2 rank homomorphism $\boldsymbol{W} \to \mathbb{Z}/2$, or, equivalently, the kernel of the rank homomorphism $\mathbf{GW} \to \mathbb{Z}$. Sending the symbol for a unit $[u]$ to the additive inverse (in $\boldsymbol{I}$) of the class of the quadratic form $\langle 1, -u\rangle$ of rank 2 and η to $\langle 1\rangle$ defines a graded homomorphism $\mathbf{K}^{\mathsf{MW}}_n \to \boldsymbol{I}^n$ for all n where $\boldsymbol{I}^n = \boldsymbol{W}$ for $n \leq 0$. As it sends h to zero, there results a homomorphism $\mathbf{K}^{\mathsf{W}}_n \to \boldsymbol{I}^n$ which is an isomorphism by [44] (see also [28, Theorem 3.8]). As explained in [28, Theorem 5.4], one may then conclude that the canonical homomorphism $\mathbf{K}^{\mathsf{MW}}_{n+1}/\mathsf{h}\mathbf{K}^{\mathsf{M}}_{n+1} \to \mathbf{K}^{\mathsf{MW}}_n$ is an injection whose image coincides with $\boldsymbol{I}^{n+1}$. $\square$

Theorem 4.19. *Let $n \in \mathbb{Z}$. The canonical map $\mathbf{K}^{\mathsf{MW}}_n \to \pi_{(-n)}\mathbf{kq}$ is an isomorphism.*

Proof. The proof is by induction on n. If $n \leq 0$, then $\pi_{(-n)}\mathbf{kq} = \pi_{(-n)}\mathbf{KQ}$ by definition of $\mathbf{kq} \to \mathbf{KQ}$. Since the canonical map $\mathbf{K}^{\mathsf{MW}}_n \to \pi_{(-n)}\mathbf{KQ}$ is an isomorphism for $n \leq 0$, the induction may start at $n = 0$. Suppose $n \geq 0$ is such that the statement is true for n. For the induction step consider the long exact sequence of homotopy sheaves induced by the cofiber sequence

$$\Sigma^{(1)}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{\text{forget}} \mathbf{kgl} \xrightarrow{\text{hyper}} \Sigma^{1+(1)}\mathbf{kq}.$$

It fits into a commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathbf{K}^{\mathsf{M}}_{n+1} & \xrightarrow{\mathsf{h}} & \mathbf{K}^{\mathsf{MW}}_{n+1} & \xrightarrow{\eta} & \mathbf{K}^{\mathsf{MW}}_{n} & \longrightarrow & \mathbf{K}^{\mathsf{M}}_{n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \\
\cdots & \xrightarrow{\text{forget}} & \pi_{1-(n)}\mathbf{kgl} & \xrightarrow{\text{hyper}} & \pi_{-(n+1)}\mathbf{kq} & \xrightarrow{\eta} & \pi_{-(n)}\mathbf{kq} & \xrightarrow{\text{forget}} & \pi_{-(n)}\mathbf{kgl} & \longrightarrow & 0
\end{array}$$

of exact sequences (using Lemma 4.18 and Proposition 4.13) in which the leftmost vertical map is the composition

$$\mathbf{K}^{\mathsf{M}}_{n+1} \cong \pi_{1-(n+1)}\mathsf{s}_1\mathbf{kgl} \xleftarrow{\cong} \mathsf{f}_1\mathbf{kgl} \to \pi_{1-(n+1)}\mathbf{kgl}. \tag{4-7}$$

Let $z \in \mathbf{K}^{\mathsf{MW}}_{n+1}$ such that $\phi(z) = 0$. Then by the induction assumption, $z\eta = 0$. Lemma 4.18 implies that there is an element $y \in \mathbf{K}^{\mathsf{M}}_{n+1}$ with $z = y\mathsf{h}$. Since $\psi(y) \in \pi_{1-(n)}\mathbf{kgl}$ is such that $\mathsf{h}(\psi(y)) = \phi(y\mathsf{h}) = 0$, there exists $x \in \pi_{1-(n)}\mathbf{kq}$ such that $\text{forget}(x) = \psi(y)$. The forgetful map $\mathbf{kq} \to \mathbf{kgl}$ induces an isomorphism on zero slices. Hence the element $x \in \pi_{1-(n)}\mathbf{kq}$ lifts to an element $w \in \pi_{1-(n)}\mathsf{f}_1\mathbf{kq}$. The element $\mathsf{f}_1(\text{forget})(w) \in \pi_{1-(n)}\mathsf{f}_1\mathbf{kgl}$ may not coincide with y, but their images in $\pi_{1-(n)}\mathbf{kgl}$ do, being $\text{forget}(x)$ and $\psi(y)$, respectively. Hence there exists an element $t \in \pi_{2-(n)}\mathsf{s}_0\mathbf{kgl} \cong H^{n-2,n}$ which is mapped to the difference $y - \mathsf{f}_1(\text{forget})(w)$. Since the forgetful map induces an isomorphism of zero slices, subtracting the image of t from w gives w' such that $\mathsf{f}_1(\text{forget})(w') = y$. As the map $\pi_{1-(n)}\mathsf{f}_1\mathbf{kgl} \to \pi_{1-(n)}\mathsf{s}_1\mathbf{kgl} \cong \mathbf{K}^{\mathsf{M}}_{n+1}$ is an isomorphism and $\mathsf{s}_1(\text{forget})$ factors over the map $\partial^2_\infty : \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2 \to \Sigma^{1+(1)}\mathbf{M}\mathbb{Z}$, the element y is in the image of ∂^2_∞. Hence, $y = y' \cdot \{-1\}$ for some $y' \in \mathbf{K}^{\mathsf{M}}_n$. Thus $z = y\mathsf{h} = y' \cdot \{-1\} \cdot \mathsf{h} = y' \cdot 0 = 0$, whence ϕ is injective.

Surjectivity of ϕ can be proven as follows. Let $z \in \pi_{-(n+1)}\mathbf{kq}$. By the induction assumption, there exists $y \in \mathbf{K}^{\mathsf{MW}}_{n+1}$ with $\eta(\phi(y) - z) = 0$. The forgetful map induces an isomorphism on zero slices, whence the canonical diagram

$$\begin{array}{ccc} \mathsf{f}_1\mathbf{kq} & \xrightarrow{\mathsf{f}_1(\text{forget})} & \mathsf{f}_1\mathbf{kgl} \\ \downarrow & & \downarrow \\ \mathbf{kq} & \xrightarrow{\mathsf{f}_1(\text{forget})} & \mathbf{kgl} \end{array}$$

is a homotopy pushout diagram. In particular, the homotopy cofiber of $\mathsf{f}_1(\text{forget})$ is $\Sigma^{1+(1)}\mathbf{kq}$, whose $\pi_{1-(n)}$ contains $\phi(y) - z$. Since $\eta(\phi(y) - z) = 0$, there exists $x \in \pi_{1-(n)}\mathsf{f}_1\mathbf{kgl}$ with $\text{hyper}(x) = \phi(y) - z$. Viewing $x \in \mathbf{K}^{\mathsf{M}}_{n+1}$ via the isomorphism (4-7), the equation $z = \phi(y - \mathsf{h} \cdot x)$, and thus surjectivity of ϕ, follows. □

As a consequence of Theorem 4.19, the canonical map $\mathbf{K}^{\mathsf{MW}} \to \pi_{0+(\star)}\mathbf{1}$ is injective, thereby proving part of Theorem 3.10. Its surjectivity requires further arguments which will not be discussed here.

5. The slice filtration on the sphere spectrum

Corollary 4.8 implies a description of all slices of the sphere spectrum, as suggested by Voevodsky in [65], provided in [38] and, in slightly different form and with multiplicative structure, in [55].

Theorem 5.1. *Suppose S is ind-smooth over a field or a Dedekind domain of mixed characteristic. Let P denote the set of positive residue characteristics in S. Then* *the slices of the P-inverted sphere spectrum over S are*

$$\mathsf{s}_q(\mathbf{1}[P^{-1}]) \cong \bigvee_{p\geq 0} \Sigma^{2q-p,q}\mathbf{M}(\mathrm{Ext}^{p,2q}_{\mathbf{MU}_*\mathbf{MU}}(\mathbf{MU}_*,\mathbf{MU}_*)[P^{-1}]).$$

Sketch of proof. The cosimplicial **MGL**-based Adams resolution of the sphere spectrum

$$\mathbf{1} \longrightarrow \mathbf{MGL} \rightrightarrows \mathbf{MGL}\wedge\mathbf{MGL} \rightrightarrows \cdots$$

induces natural equivalences

$$\mathsf{s}_q(\mathbf{1}_\Lambda) \xrightarrow{\cong} \operatorname*{holim}_{\Delta} \mathsf{s}_q(\mathbf{MGL}^{\wedge\bullet}_\Lambda)$$

for every q by downward induction and the 1-effectivity of the cofiber of $\mathbf{1}\to\mathbf{MGL}$, the latter holding over every base scheme (exercise). It remains to identify the latter with the corresponding motivic Eilenberg–MacLane spectra associated to $\mathrm{Ext}^{p,2q}_{\mathbf{MU}_*\mathbf{MU}}(\mathbf{MU}_*,\mathbf{MU}_*)$, the E^2-page of the Adams–Novikov spectral sequence for the topological sphere spectrum. This uses Corollary 4.8 and the fact that a perfect chain complex of modules over a principal ideal domain is quasi-isomorphic to its homology. □

Although only a finite portion of $\mathrm{Ext}^{p,2q}_{\mathbf{MU}_*\mathbf{MU}}(\mathbf{MU}_*,\mathbf{MU}_*)$, the E^2-page of the Adams–Novikov spectral sequence for the topological sphere spectrum, is known explicitly, certain infinite families are well understood, as explained very well in [50]. Funnily, some properties of the so-called α-family and their powers can be discovered by comparison with the slices of **kq** given in Theorem 4.15 via Lemma 5.10 below. More concretely, over a field of exponential characteristic $e\neq 2$, one has $s_1\mathbf{1}[e^{-1}]\simeq\Sigma^{(1)}\mathbf{M}\mathbb{Z}/2\{\alpha_1\}$ and $s_3\mathbf{1}[e^{-1}]\simeq\Sigma^{(3)}\mathbf{M}\mathbb{Z}/2\{\alpha_1^3\}\vee\Sigma^{2+(3)}\mathbf{M}\mathbb{Z}/2\{\alpha_3\}$. If also $e\neq 3$, then $s_2\mathbf{1}[e^{-1}]\simeq\Sigma^{(2)}\mathbf{M}\mathbb{Z}/2\{\alpha_1^2\}\vee\Sigma^{1+(2)}\mathbf{M}\mathbb{Z}/12\{\alpha_{2/2}\}$.

In order to use the information given by the form of the slices, a description of the first slice differential often helps. If E is a motivic spectrum, its *first slice differential* (as a map of motivic spectra) in weight q is the composition

$$\mathsf{d}^1_{\mathsf{E}}(q):\mathsf{s}_q\mathsf{E}\to\Sigma\mathsf{f}_{q+1}\mathsf{E}\to\Sigma\mathsf{s}_{q+1}\mathsf{E} \tag{5-1}$$

involving canonical maps from the slice filtration on E. The induced homomorphisms on the first page of the slice spectral sequence are denoted

$$d^1_{\mathsf{E}}:E^1_{s+(w),q}(\mathsf{E}):=\pi_{s+(w)}\mathsf{s}_q\mathsf{E}\to\pi_{s-1+(w)}\mathsf{s}_{q+1}\mathsf{E}=E^1_{s-1+(w),q+1}(\mathsf{E}). \tag{5-2}$$

Higher pages and differential homomorphisms $d^r_{\mathsf{E}}:E^r_{s+(w),q}(\mathsf{E})\to E^r_{s-1+(w),q+r}(\mathsf{E})$ are then produced as usual [15]. The advantage of using the first slice differential as a map of motivic spectra instead of just the induced homomorphisms is that the possibilities are much more restricted. For example, $\mathsf{d}^1_{\mathbf{kq}}(q):\mathsf{s}_q\mathbf{kq}\to\Sigma\mathsf{s}_{q+1}\mathbf{kq}$, the first slice differential as a map of motivic spectra for **kq**, is a map between finite sums of motivic Eilenberg–MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$ by Theorem 4.15. Thus $\mathsf{d}^1_{\mathbf{kq}}(q)$ can be described via its restriction $\mathsf{d}^1_{\mathbf{kq}}(q,i)$ to the summand corresponding to the unique suspension $\Sigma^{i+(q)}$. Furthermore, $\mathsf{d}^1_{\mathbf{kq}}(q,i)$ splits into at most three nontrivial components. Voevodsky's work on the motivic Steenrod algebra [66] implies in particular that nonzero cohomology operations increasing the weight by one can only increase the simplicial degree by numbers in $\{0, 1, 2, 3, 4\}$, which limits the possible components.

To describe these, let $\tau \in h^{0,1} \cong \mu_2(F)$ and $\rho \in h^{1,1} \cong F^\times/2$ denote the classes represented by $-1 \in F$; $h^{p,q}$ is shorthand for the mod-2 motivic cohomology group of F in degree p and weight q. There are canonical maps $\mathrm{pr} = \mathrm{pr}^\infty_2 : \mathbf{M}\mathbb{Z} \to \mathbf{M}\mathbb{Z}/2$ and $\partial = \partial^2_\infty : \mathbf{M}\mathbb{Z}/2 \to \Sigma^{1,0}\mathbf{M}\mathbb{Z}$ such that the first motivic Steenrod square Sq^1 equals $\mathrm{pr}^\infty_2 \circ \partial^2_\infty$.

Theorem 5.2. *When* $\mathsf{char}(F) \neq 2$ *the* d^1*-differential in the slice spectral sequence for* $\mathbf{kq}$ *is given by*

$$\mathsf{d}^1_{\mathbf{kq}}(q,i) = \begin{cases} (0, \mathsf{Sq}^2, \mathsf{Sq}^3\mathsf{Sq}^1), & q-1 > i \equiv 0 \bmod 4, \\ (\tau, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \mathsf{Sq}^3\mathsf{Sq}^1), & q-1 > i \equiv 2 \bmod 4, \end{cases}$$

$$\mathsf{d}^1_{\mathbf{kq}}(q,q) = \begin{cases} (0, \mathsf{Sq}^2 \circ \mathrm{pr}, 0), & q \equiv 0 \bmod 4, \\ (\tau \circ \mathrm{pr}, \mathsf{Sq}^2 \circ \mathrm{pr}), & q \equiv 2 \bmod 4, \end{cases}$$

$$\mathsf{d}^1_{\mathbf{kq}}(q,q-1) = \begin{cases} (0, \mathsf{Sq}^2, \partial\mathsf{Sq}^2\mathsf{Sq}^1), & q \equiv 1 \bmod 4, \\ (\tau, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \partial\mathsf{Sq}^2\mathsf{Sq}^1), & q \equiv 3 \bmod 4. \end{cases}$$

Proof. Whenever it is nontrivial, the first slice differential for $\mathbf{kgl}/2$ coincides with the first slice differential for $\mathbf{KGL}/2$. The latter is $1 + (1)$-periodic, and hence amounts to a map

$$\mathsf{d}^1_{\mathbf{KGL}/2} : \mathbf{M}\mathbb{Z}/2 \to \Sigma^{2+(1)}\mathbf{M}\mathbb{Z}/2,$$

which squares to zero. Voevodsky's computation of the motivic Steenrod algebra at the prime two [67] implies that $\mathsf{d}^1_{\mathbf{KGL}/2} \in \{0, \mathsf{Sq}^2\mathsf{Sq}^1 + \mathsf{Sq}^1\mathsf{Sq}^2\}$. The convergence result for the slice filtration on $\mathbf{KGL}$, the computation of $K^Q_*(\mathbb{R}; \mathbb{Z}/2)$ due to Suslin, and base change imply that $\mathsf{d}^1_{\mathbf{KGL}/2} = \mathsf{Sq}^1\mathsf{Sq}^2 + \mathsf{Sq}^2\mathsf{Sq}^1$. Naturality of slices with respect to the composition forget : $\mathbf{kq} \to \mathbf{kgl} \to \mathbf{kgl}/2$ implies that

$$\mathsf{Sq}^2 \circ \mathrm{pr} = \mathsf{d}^1_{\mathbf{kq}} : \mathsf{s}_0\mathbf{kq} \to \Sigma\mathsf{s}_1\mathbf{kq}.$$

Periodicity with respect to η then implies occurrences of Sq^2 on the slice summands generated by powers of η. The Adem relations $\mathsf{Sq}^2\mathsf{Sq}^2 = \tau\mathsf{Sq}^3\mathsf{Sq}^1$ and $\mathsf{Sq}^3\mathsf{Sq}^1\tau = (\mathsf{Sq}^2 + \rho\mathsf{Sq}^1)(\mathsf{Sq}^2 + \rho\mathsf{Sq}^1)$ then basically imply the rest (at least in the region of the slices where η acts invertibly). In this region the slices inject into the slices of $\mathbf{KW}_{\geq 0}$ given by Corollary 4.17; compare with the identification of $\mathsf{d}^1_{\mathbf{KW}}$ recorded in [54, Theorem 5.3]. □

Theorem 5.3. *When* $\mathsf{char}(F) \neq 2$ *the* d^1*-differential in the slice spectral sequence for* $\mathbf{KW}_{\geq 0}$ *is given by*

$$\mathsf{d}^1_{\mathbf{KW}_{\geq 0}}(q,i) = \begin{cases} (0, \mathsf{Sq}^2, \mathsf{Sq}^3\mathsf{Sq}^1), & i \equiv 0 \bmod 4, \\ (\tau, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \mathsf{Sq}^3\mathsf{Sq}^1), & i \equiv 2 \bmod 4. \end{cases}$$

Proof. This follows essentially from Theorem 5.2. □

Before getting back to the topic of this section, the slice filtration of the sphere spectrum, the case of **kq** will be elaborated further, in order to demonstrate the applicability of the slice spectral sequence.

Lemma 5.4. *The groups* $E^1_{0-(n),m}(\mathbf{kq})$ *and* $E^1_{1-(n),m}(\mathbf{kq})$ *consist of permanent cycles.*

Proof. As an exercise, one should compute the second page of the slice spectral sequence for $\mathbf{KW}_{\geq 0}$ using Theorem 5.3. It is concentrated in columns divisible by 4. Hence all higher slice differential homomorphisms are zero, whence $E^2(\mathbf{KW}_{\geq 0}) = E^\infty(\mathbf{KW}_{\geq 0})$. Since $E^1_{0-(n),m}(\mathbf{kq}) \cong E^1_{0-(n),m}(\mathbf{KW}_{\geq 0})$ naturally for $m > 0$, the result follows. □

Remark 5.5. The computation of the abutment of the slice spectral sequence for $\mathbf{KW}_{\geq 0}$ from the proof of Lemma 5.4 is compatible with Milnor's conjecture on quadratic forms. More precisely, the known coefficients

$$\pi_{s+(w)}\mathbf{KW}_{\geq 0} \cong \begin{cases} W(F), & s \equiv 0 \bmod 4, \\ 0, & \text{else}, \end{cases}$$

and passage to an algebraic closure of the base field imply that

$$\pi_{s+(w)}\mathsf{f}_1\mathbf{KW}_{\geq 0} \cong \begin{cases} \boldsymbol{I}(F), & s \equiv 0 \bmod 4, \\ 0, & \text{else}, \end{cases}$$

where $\boldsymbol{I}(F) \subset \boldsymbol{W}(F)$ is the fundamental ideal, that is, the kernel of the rank homomorphism. The multiplicativity of the slice filtration and the form of the abutment then implies that the slice filtration on $\pi_{0+(\star)}\mathbf{KW}_{\geq 0}$ coincides with the fundamental ideal filtration on the Witt ring. Moreover, Milnor's conjecture on quadratic forms holds for any field of characteristic not two. Details can be found in [54]. Since the fundamental ideal filtration on the Witt ring is separated by the main result of [3], the slice filtration for $\mathbf{KW}_{\geq 0}$ converges. Nevertheless, $\mathbf{KW}_{\geq 0}$ is not slice complete in general. The canonical map $\mathbf{KW}_{\geq 0} \to \mathsf{sc}(\mathbf{KW}_{\geq 0})$ induces the canonical homomorphism

$$\pi_{0+(0)}\mathbf{KW}_{\geq 0} \cong \boldsymbol{W}(F) \to \boldsymbol{W}(F)^\wedge_{\boldsymbol{I}} \cong \pi_{0+(0)}\mathsf{sc}(\mathbf{KW}_{\geq 0})$$

from the Witt ring to its completion at the fundamental ideal. This homomorphism is always injective. However, it is surjective if and only if the fundamental ideal filtration is finite, which is equivalent to the 2-cohomological dimension of F being finite. Every formally real field has infinite 2-cohomological dimension.

Lemma 5.6. *The possibly nontrivial groups in the first column $E^2_{1-(n),m}(\mathbf{kq})$ are*

$$E^2_{1-(n),0}(\mathbf{kq}) \cong H^{n-1,n},$$
$$E^2_{1-(n),1}(\mathbf{kq}) \cong h^{n,n+1}/\mathsf{Sq}^2\mathrm{pr}H^{n-2,n},$$
$$E^2_{1-(n),2}(\mathbf{kq}) \cong h^{n+1,n+2}/\mathsf{Sq}^2 h^{n-1,n+1}.$$

Proof. To prove that the group $E^2_{1-(n),m}(\mathbf{kq})$ is trivial for $m \geq 3$, one has to show that the d^1-differential entering $E^1_{1-(n),m}(\mathbf{kq}) = h^{n+m-1,n+m}$ is surjective. For $m \geq 4$ the differential is given by

$$E^1_{2-(n),m-1}(\mathbf{kq}) = h^{n+m-3,n+m-1} \oplus h^{n+m-1,n+m-1} \to h^{n+m-1,n+m} = E^1_{1-(n),m}(\mathbf{kq}),$$
$$(b_2, b_0) \mapsto \mathsf{Sq}^2 b_2 + \tau b_0,$$

as stated in Theorem 5.2. The claim follows since multiplying with the map $\tau : h^{n+m-1,n+m-1} \to h^{n+m-1,n+m}$ is surjective. For $m = 3$,

$$E^1_{-n+2,2,-n}(\mathbf{kq}) = h^{n,n+2} \oplus H^{n+2,n+2} \to h^{n+2,n+3} = E^1_{-n+1,3,-n}(\mathbf{kq}),$$
$$(b_2, B_0) \mapsto \mathsf{Sq}^2 b_2 + \tau\mathrm{pr}B_0,$$

is surjective, since τ and $\mathrm{pr}^\infty_2 : H^{n+2,n+2} \to h^{n+2,n+2}$ are both surjective maps. The remaining identifications follow from the determination of the slice d^1-differential in Theorem 5.2. □

Remark 5.7. Lemma 5.6 implies that the slice spectral sequence for $\mathbf{kq}$ admits at most one further nonzero differential to the first column, but Lemma 5.8 below shows that it is trivial. Furthermore, Lemma 5.6 indicates that a triple iteration of η on $\pi_1\mathsf{sc}(\mathbf{kq})$ is zero, the reason being on the one hand that the slice spectral sequence for E computes the homotopy groups of $\mathsf{sc}(\mathsf{E})$, and on the other hand that the Hopf map $\Sigma^{(1)}\mathsf{E} \to \mathsf{E}$ induces maps

$$\mathsf{s}_q(\eta \wedge \mathsf{E}) : \mathsf{s}_q\Sigma^{(1)}\mathsf{E} \simeq \Sigma^{(1)}\mathsf{s}_{q-1}\mathsf{E} \to \mathsf{s}_q\mathsf{E}. \tag{5-3}$$

In particular, $\pi_1\mathsf{sc}(\mathbf{kq})[\eta^{-1}] = 0$. Similarly, one can show $\pi_2\mathsf{sc}(\mathbf{kq})[\eta^{-1}] = 0$ by proving $E^2_{2-(n),m}(\mathbf{kq}) = 0$ for $m > 3$ (exercise involving Theorem 5.2), thereby implying $\pi_1\mathbf{kq} = \pi_1\mathsf{sc}(\mathbf{kq})$. Hence the first column of the slice spectral sequence for $\mathbf{kq}$ computes $\pi_1\mathbf{kq}$, not just $\pi_1\mathsf{sc}(\mathbf{kq})$.

The next statement, which was employed in the previous proof, uses the element $\eta_{\mathsf{top}} \in \pi_{1+(0)}\mathbf{1}_{\mathbb{Z}}$ which is the image of the topological Hopf map associated with the Hopf construction on the topological group S^1 under the "constant sheaf" map $\pi_1\mathbb{S} \to \pi_{1+(0)}\mathbf{1}_{\mathbb{Z}}$. It is necessary to distinguish it from the element $\eta \in \pi_{0+(1)}\mathbf{1}_{\mathbb{Z}}$, which can be obtained as the Hopf construction on the algebraic group $\boldsymbol{G}_{\mathsf{m}}$. The Hopf construction on the algebraic group $\mathrm{SL}_2 \simeq \boldsymbol{A}^2 \smallsetminus \{0\} \simeq \Sigma^{1+(2)}$ gives an unstable representative of ν over the integers which complex-realizes to the topological second Hopf map $\nu_{\mathsf{top}} \in \pi_3\mathbb{S}$, and real-realizes to the topological first Hopf map[3] $\eta_{\mathsf{top}} \in \pi_1\mathbb{S}$.

Lemma 5.8. *The second differential*

$$E^2_{2-(n),0}(\mathbf{kq}) \to E^2_{1-(n),2}(\mathbf{kq})$$

in the slice spectral sequence for $\mathbf{kq}$ *is trivial.*

Proof. The cofiber sequence

$$\mathsf{f}_1\mathbf{kq} \to \mathbf{kq} \to \mathsf{s}_0\mathbf{kq} \to \Sigma\mathsf{f}_1\mathbf{kq}$$

induces a long exact sequence of homotopy modules

$$\cdots \to \pi_2\mathsf{s}_0\mathbf{kq} \to \pi_1\mathsf{f}_1\mathbf{kq} \to \pi_1\mathbf{kq} \to \pi_1\mathsf{s}_0\mathbf{kq} \to 0.$$

Since η acts trivially on $\mathsf{s}_0\mathbf{kq} \cong \mathbf{M}\mathbb{Z}$, the homotopy module $\pi_2\mathsf{s}_0\mathbf{kq}$ is a $\mathbf{K}^{\mathsf{M}}$-module. Hence the $\mathbf{K}^{\mathsf{MW}}$-module homomorphism $\pi_2\mathsf{s}_0\mathbf{kq} \to \pi_1\mathsf{f}_1\mathbf{kq}$ factors as

$$\pi_2\mathsf{s}_0\mathbf{kq} \to {}_{\eta}(\pi_1\mathsf{f}_1\mathbf{kq}) \to \pi_1\mathsf{f}_1\mathbf{kq},$$

where the first map is a $\mathbf{K}^{\mathsf{M}}$-module homomorphism to the η-torsion in $\pi_1\mathsf{f}_1\mathbf{kq}$. Its target is identified as a consequence of the $\mathbf{K}^{\mathsf{MW}}$-module presentation of $\pi_1\mathsf{f}_1\mathbf{kq}$ given in Lemma 5.9 below. Set $\mathbf{k}^{\mathsf{M}} := \mathbf{K}^{\mathsf{M}}/2 \cong \pi_0\mathbf{M}\mathbb{Z}/2$ to be Milnor K-theory modulo 2, and more generally $\mathbf{k}^{\mathsf{M}}(w) \cong \pi_0\Sigma^{(w)}\mathbf{M}\mathbb{Z}/2$ its w-fold weight shift, where $w \in \mathbb{Z}$. The image of the $\mathbf{K}^{\mathsf{MW}}$-module

$${}_{\eta}(\pi_1\mathsf{f}_1\mathbf{kq}) \cong \mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1) \oplus \rho^2(\mathbf{k}^{\mathsf{M}}(-2))$$

in $\pi_1\mathsf{s}_1\mathbf{kq}$ coincides with $\rho^2(\mathbf{k}^{\mathsf{M}}(-2))$ generated by the element $[-1]^2\eta_{\mathsf{top}}$, because $\eta\eta_{\mathsf{top}}$ maps trivially to $\pi_{1+(1)}\mathsf{s}_1\mathbf{kq}$. The image of the map $\pi_2\mathbf{M}\mathbb{Z}/2 \to \pi_1\mathsf{s}_1\mathbf{kq}$ given by the first slice differential $\mathrm{Sq}^2\mathrm{pr}_2^{\infty} : \mathsf{s}_0\mathbf{kq} \to \Sigma\mathsf{s}_1\mathbf{kq}$ is (strictly) contained in $\rho^2(\mathbf{k}^{\mathsf{M}}(-2))$. It remains to observe that $\pi_2\mathbf{M}\mathbb{Z} \to \mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1)$ maps trivially to the first summand of ${}_{\eta}(\pi_1\mathsf{f}_1\mathbf{kq})$. Since the target is 2-torsion by Lemma 5.9 below, ψ factors over $(\pi_2\mathbf{M}\mathbb{Z})/2$. This occurs in the $\mathbf{K}^{\mathsf{MW}}$-module short exact sequence

$$0 \to (\pi_2\mathbf{M}\mathbb{Z})/2 \to \pi_2\mathbf{M}\mathbb{Z}/2 \to {}_2\pi_1\mathbf{M}\mathbb{Z} \to 0,$$

where the first map is induced by $\mathrm{pr}_2^{\infty} : \mathbf{M}\mathbb{Z} \to \mathbf{M}\mathbb{Z}/2$. It coincides with the map $\mathsf{s}_0(\mathbf{kq} \xrightarrow{\text{canonical}} \mathbf{kq}/2)$ up to equivalence. Since $\pi_1\mathsf{f}_1\mathbf{kq}$ is 2-torsion, the map $\pi_1\mathsf{f}_1\mathbf{kq} \to \mathsf{f}_1\mathbf{kq}/2$ is injective. Owing to the commutative diagram

[3] Pardon the slight abuse of notation.

$$\begin{array}{ccccc}
\pi_2\mathbf{M}\mathbb{Z} & \longrightarrow & (\pi_2\mathbf{M}\mathbb{Z})/2 & \longrightarrow & \pi_2\mathbf{M}\mathbb{Z}/2 \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1\mathsf{f}_1\mathbf{kq} & \xrightarrow{\text{id}} & \pi_1\mathsf{f}_1\mathbf{kq} & \longrightarrow & \pi_1\mathsf{f}_1\mathbf{kq}/2
\end{array}$$

it suffices to prove $\pi_2\mathbf{M}\mathbb{Z}/2 \cong \mathbf{k}^{\mathsf{M}}(-2) \to \pi_1\mathsf{f}_1\mathbf{kq}/2$ maps trivially to the summand $\mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1)$. The latter is determined by the image of the (unique) $\mathbf{K}^{\mathsf{MW}}$-module generator $g = g_F \in \pi_{2+(-2)}\mathbf{M}\mathbb{Z}/2$. The generator g_F is the image of the generator $g_{F_0} \in \pi_{2+(-2)}\mathbf{M}\mathbb{Z}/2$, where F_0 is the prime field of F. The commutative diagram

$$\begin{array}{ccc}
\pi_{2+(-2)}(\mathbf{M}\mathbb{Z}/2)_{F_0} & \xrightarrow{\text{id}} & \pi_{2+(-2)}(\mathbf{M}\mathbb{Z}/2)_F \\
\downarrow & & \downarrow \\
\mathbf{k}^{\mathsf{M}}_3(F_0)/\rho^2\mathbf{k}^{\mathsf{M}}_1(F_0) & \longrightarrow & \mathbf{k}^{\mathsf{M}}_3(F)/\rho^2\mathbf{k}^{\mathsf{M}}_1(F)
\end{array}$$

implies the right-hand side vertical map is zero, because $\mathbf{k}^{\mathsf{M}}_3(F_0)/\rho^2\mathbf{k}^{\mathsf{M}}_1(F_0) = 0$ due to [42, Example 1.5, Appendix]. □

Lemma 5.9. *The $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1\mathsf{f}_1\mathbf{kq} \cong \mathbf{K}^{\mathsf{MW}}/(2, \eta^2)$ is generated by the image of the topological Hopf map $\eta_{\mathsf{top}} \in \pi_{1+(0)}\mathbf{1}$ in $\pi_{1+(0)}\mathsf{f}_1\mathbf{kq}$. The image of $\pi_1\mathsf{f}_2\mathbf{kq} \to \pi_1\mathsf{f}_1\mathbf{kq}$ is the submodule generated by $\eta\eta_{\mathsf{top}}$ and it is isomorphic to $\mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1)$.*

Proof. The column on the second page of the slice spectral sequence for $\mathsf{f}_1\mathbf{kq}$ computing π_1 is concentrated in s_1 and s_2. The d^1-differential $\mathsf{Sq}^2 : \mathsf{s}_1\mathbf{kq} \to \Sigma\mathsf{f}_2\mathbf{kq} \to \Sigma\mathsf{s}_2\mathbf{kq}$ induces $\mathsf{Sq}^2 : \mathbf{k}^{\mathsf{M}}(-1) \cong \pi_2\mathsf{s}_1\mathbf{kq} \to \pi_1\mathsf{f}_2\mathbf{kq} \cong \pi_1\mathsf{s}_2\mathbf{kq} \cong \mathbf{k}^{\mathsf{M}}(1)$, whence the image in $\pi_1\mathsf{f}_2\mathbf{kq}$ coincides with $\rho^2\mathbf{k}^{\mathsf{M}}(-1)$. There is no room for higher differentials. Thus the long exact sequence

$$\cdots \to \pi_2\mathsf{s}_1\mathbf{kq} \to \pi_1\mathsf{f}_2\mathbf{kq} \to \pi_1\mathsf{f}_1\mathbf{kq} \to \pi_1\mathsf{s}_1\mathbf{kq} \to 0$$

induced by the cofiber sequence

$$\mathsf{f}_2\mathbf{kq} \to \mathsf{f}_1\mathbf{kq} \to \mathsf{s}_1\mathbf{kq} \to \Sigma\mathsf{f}_2\mathbf{kq}$$

yields a $\mathbf{K}^{\mathsf{MW}}$-module short exact sequence

$$0 \to \mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1) \to \pi_1\mathsf{f}_1\mathbf{kq} \to \pi_1\mathsf{s}_1\mathbf{kq} \cong \mathbf{k}^{\mathsf{M}} \to 0. \tag{5-4}$$

Étale or complex realization implies the rightmost term in (5-4) is generated by the image of the topological Hopf map η_{top} in $\pi_{1+(0)}\mathsf{s}_1\mathbf{kq}$. The multiplicative structure of the slices shows the image of $\eta\eta_{\mathsf{top}}$ in $\pi_{1+(1)}\mathsf{f}_1\mathbf{kq}$ is nontrivial, hence it coincides with the (unique) generator of $\mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1)$. In particular, the map $\mathbf{K}^{\mathsf{MW}} \to \pi_1\mathsf{f}_1\mathbf{kq}$ sending 1 to the image of η_{top} is surjective. As in the proof of [52, Lemma 2.3], one concludes the desired isomorphisms. □

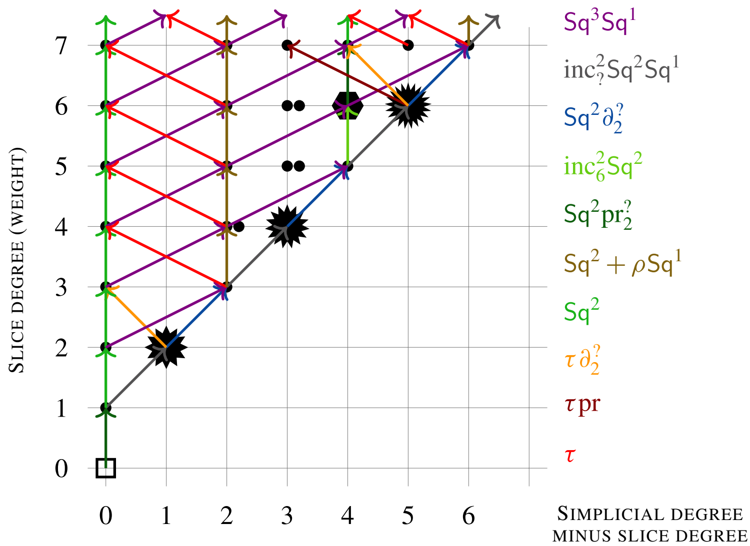

Figure 2. The first slice differential for **1**.

Lemma 5.8 implies the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1\mathbf{kq}$ is given by the short exact sequence

$$0 \to \pi_1\mathsf{f}_1\mathbf{kq}/\mathsf{Sq}^2\mathrm{pr}_2^\infty\pi_2\mathbf{M}\mathbb{Z} \to \pi_1\mathbf{kq} \to \pi_1\mathbf{M}\mathbb{Z} \to 0, \tag{5-5}$$

which does not split in general, neither as an extension of $\mathbf{K}^{\mathsf{MW}}$-modules nor degreewise as an extension of abelian groups. Since η^2 acts as zero on $\pi_1\mathsf{f}_1\mathbf{kq}$ by Lemma 5.9, η^3 acts as zero on $\pi_1\mathbf{kq}$.

To return to the topic at hand, substantial information on the first slice differential for **1**, at least over fields of characteristic not two, can be deduced from the first slice differential on **kq** given in Theorem 5.2 along the unit map $\mathbf{1} \to \mathbf{kq}$. Instead of a proper "theorem", the result will be displayed as Figure 2. It is based on Lemma 5.10 about the behavior of the unit map $\mathbf{1} \to \mathbf{kq}$ on slices. A more complete description can be found in [55, Lemmas 4.1 and 4.2]. In Figure 2 each slice is displayed along a horizontal line indexed by its weight. Every direct summand of a slice is placed on the vertical line indexed by its simplicial suspension degree. An open square refers to a motivic Eilenberg–MacLane spectrum with $\mathbb{Z}$-coefficients and solid dots to $\mathbb{Z}/2$-coefficients. The solid polygons indicate coefficients in $\mathbb{Z}/12$, $\mathbb{Z}/240$, $\mathbb{Z}/6$, and $\mathbb{Z}/504$, respectively. The colors of the d^1-differentials are split according to the respective direct sum decomposition and refer to elements of the motivic Steenrod algebra ordered by simplicial degree. Note that over a field of odd characteristic p, every occurrence of "$\mathbb{Z}$" should be replaced with "$\mathbb{Z}[\frac{1}{p}]$".

Lemma 5.10. *Let F be a field of characteristic not two. On* 0*-slices and* 1*-slices the unit map* $\mathbf{1} \to \mathbf{kq}$ *induces the identity, on* 2*-slices a rather canonical map, and* *for $q \geq 3$ the induced map $\mathsf{s}_q(\mathbf{1} \to \mathbf{kq})$ is the identity on the summands generated by α_1^q and $\alpha_3\alpha_1^{q-3}$. In particular, the unit map $\mathbf{1} \to \mathbf{kq}$ induces the identity on 3-slices. In other words,*

$$\begin{aligned}
&\mathsf{s}_0(\mathbf{1}) = \mathbf{M}\mathbb{Z} \xrightarrow{1} \mathbf{M}\mathbb{Z} = \mathsf{s}_0(\mathbf{kq}),\\
&\mathsf{s}_1(\mathbf{1}) = \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2\{\alpha_1\} \xrightarrow{1} \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2 = \mathsf{s}_1(\mathbf{kq}),\\
&\mathsf{s}_2(\mathbf{1}) = \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2\{\alpha_1^2\} \vee \Sigma^{1+(2)}\mathbf{M}\mathbb{Z}/12\{\alpha_{2/2}\}\\
&\qquad \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & \partial_\infty^{12}\end{pmatrix}} \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(2)}\mathbf{M}\mathbb{Z} = \mathsf{s}_2(\mathbf{kq}),\\
&s_q(\mathbf{1}) \hookleftarrow \Sigma^{(q)}\mathbf{M}\mathbb{Z}/2\{\alpha_1^q\} \vee \Sigma^{2+(q)}\mathbf{M}\mathbb{Z}/2\{\alpha_3\alpha_1^{q-3}\}\\
&\qquad \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} \Sigma^{(q)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(q)}\mathbf{M}\mathbb{Z}/2 \hookrightarrow \mathsf{s}_q\mathbf{kq}.
\end{aligned}$$

Proof. The zero slice functor preserves the ring structure; see also [55, Lemma 2.29]. More generally, the graded slice functor preserves the ring structure, which implies the statements on the summands generated by α_1^q and $\alpha_1^{q-3}\alpha_3$. By [55, Lemma 2.30] the second diagonal entry for $\mathsf{s}_2(\mathbf{1} \to \mathbf{kq})$ has the form $n \cdot \partial_\infty^{12}$ for n an odd integer. The commutative diagram of motivic ring maps

$$\begin{array}{ccc}
\mathbf{1} & \longrightarrow & \mathbf{MGL}\\
\downarrow & & \downarrow\\
\mathbf{KQ} & \longrightarrow & \mathbf{KGL}
\end{array}$$

implies that n can be identified from the map on 2-slices induced by the unit map $\mathbf{1} \to \mathbf{MGL}$. This computation can be derived from the proof of Theorem 5.1 and shows that n is not divisible by 3. The result follows by applying a suitable isomorphism to $\mathsf{s}_2(\mathbf{1})$. □

Lemma 5.11. *For all $n \in \mathbb{Z}$ the unit map $\mathbf{1} \to \mathbf{kq}$ induces a surjection on $E^1_{1-(n),2}$ and $E^1_{2-(n),4}$, and for all $m, k \in \mathbb{Z}$ an isomorphism on*

$$\begin{aligned}
&E^1_{0-(n),m},\\
&E^1_{1-(n),m}, \quad m \neq 2,\\
&E^1_{2-(n),m}, \quad m \neq 2, 4,\\
&E^1_{k-(n),m}, \quad m \leq 1.
\end{aligned}$$

Proof. This follows from Lemma 5.10. While each slice may contribute many homotopy classes, these occur always on the same row and to the right of the respective slice. □

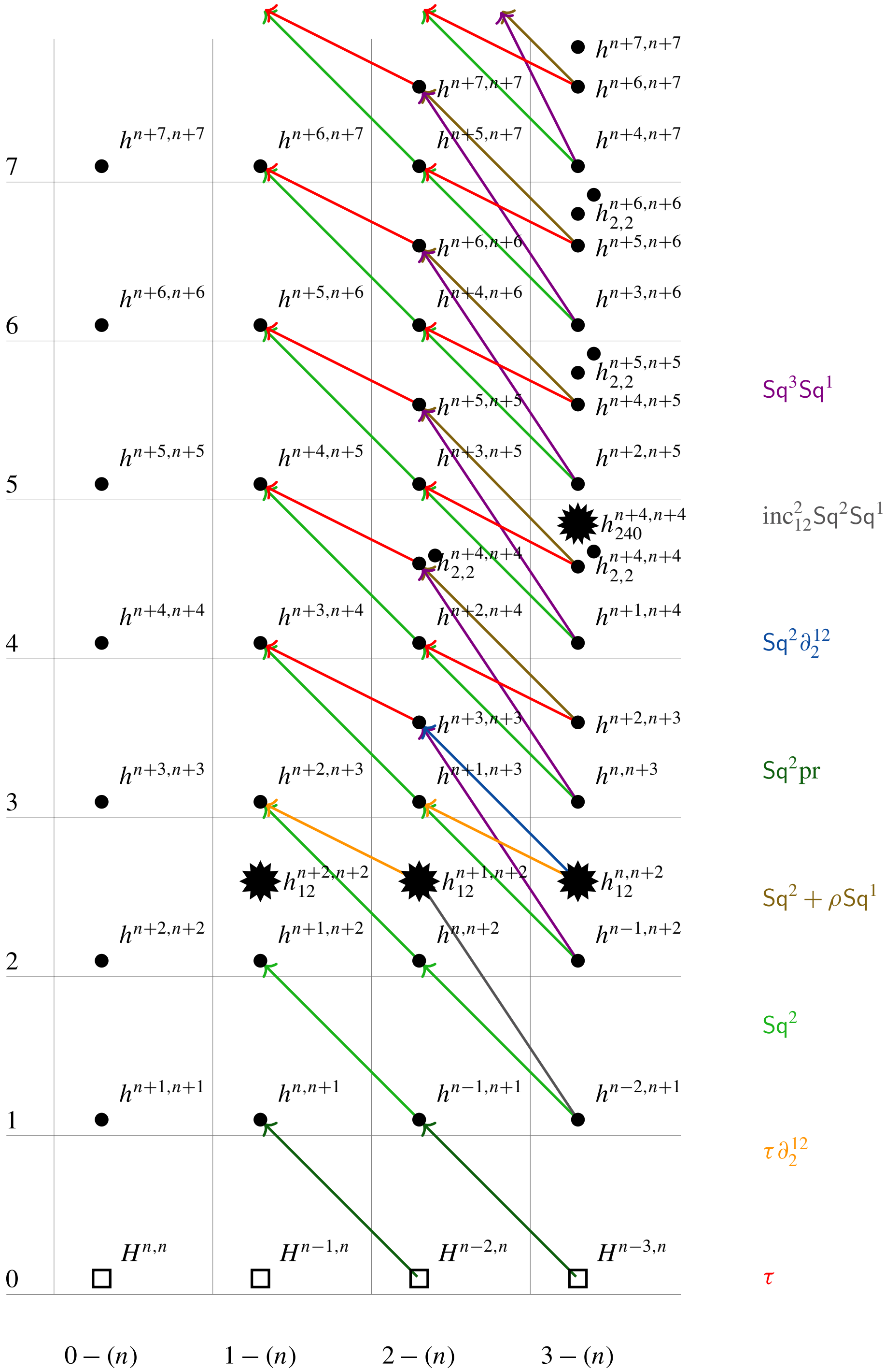

Figure 3. E^1-page of the weight $-n$-th slice spectral sequence for $\mathbf{1}_\Lambda$.

The description of the first four nontrivial columns of the first page of the slice spectral sequence of the sphere spectrum — together with the first slice differential — in Figure 3 allows a computation of the first three nontrivial columns of the second page.

Proposition 5.12. *Let F be a field of characteristic not two, and $n \in \mathbb{Z}$. The second page of the slice spectral sequence for the sphere spectrum in weight $-n$ contains the following groups*:

q	$E^2_{0-(n),q}(\mathbf{1})$	$E^2_{1-(n),q}(\mathbf{1})$	$E^2_{2-(n),q}(\mathbf{1})$
5	$h^{n+5,n+5}$	0	0
4	$h^{n+4,n+4}$	0	$h^{n+4,n+4}$
3	$h^{n+3,n+3}$	$h^{n+2,n+3}/\tau\partial^{12}_2 h^{n+1,n+2}_{12}$	$h^{n+1,n+3}/\mathsf{Sq}^2 h^{n-1,n+2}$
2	$h^{n+2,n+2}$	$h^{n+2,n+2}_{12} \oplus h^{n+1,n+2}/\mathsf{Sq}^2 h^{n-1,n+1}$	$\ker(h^{n+1,n+2}_{12} \xrightarrow{\partial^{12}_2} h^{n+2,n+2}) \oplus h^{n,n+2}/\mathsf{Sq}^2 h^{n-2,n+1}$
1	$h^{n+1,n+1}$	$h^{n,n+1}/\mathsf{Sq}^2\mathrm{pr}^\infty_2 H^{n-2,n}$	$\ker(h^{n-1,n+1} \xrightarrow{\mathsf{Sq}^2} h^{n+1,n+2})$
0	$H^{n,n}$	$H^{n-1,n}$	$\ker(H^{n-2,n} \xrightarrow{\mathsf{Sq}^2\mathrm{pr}^\infty_2} h^{n,n+1})$

In particular, $E^2_{1-(n),q}(\mathbf{1}) = 0$ *for* $q > 3$ *and* $E^2_{2-(n),q}(\mathbf{1}) = 0$ *for* $q > 4$.

Proof. As mentioned right before the statement, this is a direct consequence of Figure 3. ☐

In order to draw consequences for actual homotopy groups, a short interlude on convergence is required. By construction, the slice spectral sequence of E converges to the slice completion $\mathsf{sc}(\mathsf{E})$. Recall from Remark 5.5 that $\mathbf{KW}_{\geq 0}$ is not slice complete in general, which implies the same for $\mathbf{kq}$: the slice filtration on $\pi_0\mathbf{kq} \cong \mathbf{K}^{\mathsf{MW}}$ coincides with the fundamental ideal filtration, and if this filtration is infinite, $\mathbf{K}^{\mathsf{MW}} \to (\mathbf{K}^{\mathsf{MW}})^\wedge_{\boldsymbol{I}}$ is not surjective. Since $\pi_0\mathbf{1} \to \pi_0\mathbf{kq}$ is an isomorphism as a consequence of Theorems 3.10 and 4.19, the same is true for $\mathbf{1}$. On the other hand, Remark 4.14 identifies $\mathsf{sc}(\mathbf{kq}) \simeq \mathbf{kq}^\wedge_\eta$, which was used in the identification $\pi_1\mathbf{kq} \cong \pi_1\mathsf{sc}(\mathbf{kq})$. The same works for the sphere spectrum, with more effort.

Theorem 5.13. *Let F be a field of exponential characteristic e. Then* $\mathsf{sc}(\mathbf{1}[e^{-1}])$ *and* $\mathbf{1}[e^{-1}]^\wedge_\eta$ *are naturally equivalent.*

Proof. This is [55, Theorem 3.50], and its proof occupies most of [55, Section 3]. The basic idea, modeled on [34], is to prove the statement for motivic spectra which are cellular of finite type. It is then necessary to see that slices of such are again cellular of finite type. Theorem 5.1 implies that it suffices to prove this for motivic Eilenberg–MacLane spectra for $\mathbb{Z}$ and for finite abelian groups. This in turn follows from Theorem 4.7, giving two reasons for inverting the exponential characteristic.

Extending cellularity of finite type to slice completions requires the connectivity to increase with the slice. Theorem 5.1 shows that it does not hold for the sphere spectrum $\mathbf{1}$: every positive slice $\mathsf{s}_q(\mathbf{1})$ contains $\Sigma^{(q)}\mathbf{M}\mathbb{Z}/2$ as a direct summand, which satisfies $\pi_0\Sigma^{(q)}\mathbf{M}\mathbb{Z}/2 \cong h^{q,q}$, a nontrivial group for many fields, such as formally real fields. Nevertheless, these summands are generated by powers of α_1, the element detecting the Hopf map. The main result of [2] implies that this pattern persists: summands affecting increasing connectivity of slices of the sphere spectrum are always generated by a product of an element in the α-family with a suitable power of α_1. As a consequence, the connectivity of $\mathsf{s}_q(\mathbf{1}[e^{-1}]/\eta)$ tends to ∞ with q, and the same is then true for $\mathsf{s}_q(\mathsf{E}[e^{-1}]/\eta^\ell)$ for all ℓ and for all E which are cellular of finite type. □

Remark 5.14. In case F is a field of finite cohomological dimension in the sense that the cohomology of its absolute Galois cohomology is nonzero only in finitely many degrees, Levine showed in [37] that $\mathbf{1}[e^{-1}]$ is slice complete. Here e is the exponential characteristic of F. In fact, he proved the rather amazing and much more general statement that for every compact motivic spectrum E over a field of finite cohomological dimension the canonical map $\mathsf{E}[e^{-1}] \to \mathsf{sc}\mathsf{E}[e^{-1}]$ is an equivalence. See [58] for details on Galois cohomology and examples of such fields.

Recall from Remark 4.14 that the canonical square

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{1}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \mathbf{1}^\wedge_\eta & \longrightarrow & \mathbf{1}^\wedge_\eta[\eta^{-1}] \end{array}$$

is a homotopy pullback square. Theorem 5.13 implies that the slice spectral sequence for $\mathbf{1}[e^{-1}]$ converges to the homotopy groups of $\mathbf{1}[e^{-1}]^\wedge_\eta$. The finiteness of the first and second column of the second page of the slice spectral sequence stated in Proposition 5.12 yields the vanishing

$$\pi_1(\mathbf{1}[e^{-1}])^\wedge_\eta[\eta^{-1}] = \pi_2(\mathbf{1}[e^{-1}])^\wedge_\eta[\eta^{-1}] = 0. \tag{5-6}$$

On the one hand, this implies the aforementioned injectivity of $\pi_0\mathbf{1} \to \pi_0\mathbf{1}^\wedge_\eta$. On the other hand, there results an isomorphism

$$\pi_1\mathbf{1}[e^{-1}] \xrightarrow{\cong} \pi_1(\mathbf{1}[e^{-1}])^\wedge_\eta \oplus \pi_1\mathbf{1}[e^{-1},\eta^{-1}]$$

of homotopy groups from the long exact sequence associated to (4-6) for $\mathsf{E} = \mathbf{1}[e^{-1}]$. The second summand has been identified in [51].

Theorem 5.15. *Let F be a field of characteristic not two. Then $\pi_1\mathbf{1}[\eta^{-1}] = \pi_2\mathbf{1}[\eta^{-1}] = 0$.*

Proof. To sketch the argument, Morel's Theorem 3.10 implies that the canonical map $\mathbf{1}[\eta^{-1}] \to \mathbf{KW}_{\geq 0}$ induces an isomorphism on π_s in degrees $s < 1$; π_0 being isomorphic to $W(F)[\eta, \eta^{-1}]$. The vanishing $\pi_1\mathbf{KW}_{\geq 0} = \pi_2\mathbf{KW}_{\geq 0} = 0$ is known and can also be deduced from the slice spectral sequence, as in Remark 5.5. Hence the canonical map $\mathbf{1}[\eta^{-1}] \to \mathbf{KW}_{\geq 0}$ is a 1-connected map in the sense that $\pi_s\mathsf{C} = 0$ for $s \leq 1$, where C is its homotopy cofiber. Then $\pi_s\mathsf{C} \wedge \mathsf{C} = 0$ for $s \leq 3$, whence $\mathsf{C} \to \mathbf{KW}_{\geq 0} \wedge \mathsf{C}$ is 3-connected. A rather involved comparison of $\mathbf{KW}_{\geq 0} \to \mathbf{KW}_{\geq 0} \wedge \mathbf{KW}_{\geq 0}$ with the corresponding map $\mathrm{ko} \to \mathrm{ko} \wedge \mathrm{ko}$ for connective real topological K-theory ko implies that this map, and hence its cofiber $\mathbf{KW}_{\geq 0} \wedge \mathsf{C}$, is 3-connected. It follows that C is even 2-connected, whence $\mathsf{C} \wedge \mathsf{C}$ is 5-connected, and thus C is indeed 3-connected. □

A lot more is known about the homotopy of $\mathbf{1}[\eta^{-1}]$; see [29] over the complex numbers, [69] over the rational numbers, [49] over various "small" fields, and [10] over fields of characteristic not two. In any case, Theorem 5.15 suffices to conclude the following.

Corollary 5.16. *Let F be a field of exponential characteristic $e \neq 2$. Then the map $\pi_1\mathbf{1}[e^{-1}] \to \pi_1\mathsf{sc}(\mathbf{1}[e^{-1}])$ is an isomorphism.*

Proof. As explained above, this follows from Theorem 5.15 and the vanishing in (5-6). □

Theorem 5.17. *Let F be a field of exponential characteristic $e \neq 2$. The unit map $\mathbf{1} \to \mathbf{kq}$ induces a surjection $\pi_{1+(\star)}\mathbf{1} \to \pi_{1+(\star)}\mathbf{kq}$ of $\mathbf{K}^{\mathsf{MW}}$-modules, whose kernel coincides with $\mathbf{K}^{\mathsf{M}}_{2-\star}/24$ after inverting e. In particular, since $\pi_{1+(n)}\mathbf{kq} = 0$ for $n \geq 2$, there are isomorphisms $\pi_{1+(2)}\mathbf{1}[e^{-1}] \cong \mathbb{Z}/24[e^{-1}]$ and $\pi_{1+(n)}\mathbf{1}[e^{-1}] = 0$ for $n \geq 3$. Also, $\mathbf{K}^{\mathsf{M}}_{2-\star}/24$ is generated by the second Hopf map $\nu \in \pi_{1+(2)}\mathbf{1}$. The relations $\eta\nu = 0 \in \pi_{1+(3)}\mathbf{1}$ and $\eta^2\eta_{\mathsf{top}} = 12\nu \in \pi_{1+(2)}\mathbf{1}$ hold.*

Proof. Lemma 5.9 implies that the unit map induces a surjection $\pi_1\mathsf{f}_1\mathbf{1} \to \pi_1\mathsf{f}_1\mathbf{kq}$, since the target is generated as a $\mathbf{K}^{\mathsf{MW}}$-module by the image of η_{top}. This element naturally lives in $\pi_{1+(0)}\mathbf{1}$ and lifts uniquely to $\pi_{1+(0)}\mathsf{f}_1\mathbf{1}$ because $\pi_{2+(0)}\mathsf{s}_0\mathbf{1} = \pi_{1+(0)}\mathsf{s}_0\mathbf{1} = 0$. Since $\mathsf{s}_0\mathbf{1} \xrightarrow{\simeq} \mathsf{s}_0\mathbf{kq}$ is an equivalence by Lemma 5.10, surjectivity for $\pi_1\mathbf{1} \to \pi_1\mathbf{kq}$ follows as soon as the connecting map $\pi_1\mathsf{s}_0\mathbf{1} \to \pi_0\mathsf{f}_1\mathbf{1}$ is zero. As explained in Remark 5.5 for $\mathbf{KW}_{\geq 0}$, the group $\pi_0\mathsf{f}_1\mathbf{1}$ injects into $\pi_0\mathbf{1}$; see also Lemma 5.4.

In order to identify the kernel of $\pi_1\mathbf{1}[e^{-1}] \to \pi_1\mathbf{kq}[e^{-1}]$, observe that both $\mathbf{K}^{\mathsf{MW}}$-modules are computed by the respective slice spectral sequence, thanks to Remark 5.7 and Corollary 5.16. The column of the second page responsible for $\pi_1\mathbf{1}[e^{-1}]$ has been determined in Proposition 5.12. The major problem is to show that all further differentials ending in this column are zero. Basic techniques are shrinking the base field, a short exact sequence from [48, Theorem 3.2], real realization, and — for differentials originating in the zero slice — passing to suitable quotients of the sphere spectrum.

Possibly nonzero targets for these differentials are the terms $E^2_{1-(n),q}(\mathbf{1}[e^{-1}])$ for $q \in \{2, 3\}$. As a warm-up, consider the second differential $E^2_{2-(n),1}(\mathbf{1}[e^{-1}]) \to E^2_{1-(n),3}(\mathbf{1}[e^{-1}])$. The connecting homomorphism

$$\mathbf{k}^{\mathsf{M}}(-1) \cong \pi_2 \mathsf{s}_1 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{f}_2 \mathbf{1}[e^{-1}] \tag{5-7}$$

is defined on a $\mathbf{K}^{\mathsf{MW}}$-module having a (unique) generator, here denoted g, in $\mathbf{k}^{\mathsf{M}}(-1)_1 = \mathbf{k}^{\mathsf{M}}_0$. To describe the image of g, observe that the long exact sequence

$$\cdots \to \pi_1 \mathsf{f}_4 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{f}_3 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{s}_3 \mathbf{1}[e^{-1}] \to \pi_0 \mathsf{f}_4 \mathbf{1}[e^{-1}] \to \cdots$$

induces an isomorphism $\pi_1 \mathsf{f}_3 \mathbf{1}[e^{-1}] \cong \pi_1 \mathsf{s}_3 \mathbf{1}[e^{-1}] \cong \mathbf{k}^{\mathsf{M}}(2)$. This $\mathbf{K}^{\mathsf{MW}}$-module has a (unique) generator given by the element $\eta^2 \eta_{\mathsf{top}} \in \pi_{1+(2)} \mathbf{1}$, which naturally lifts to $\pi_1 \mathsf{f}_3 \mathbf{1}$. The short exact sequence

$$0 \to \pi_1 \mathsf{f}_3 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{f}_2 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{s}_2 \mathbf{1}[e^{-1}] \to 0$$

of $\mathbf{K}^{\mathsf{MW}}$-modules can be determined as follows. Ignoring the possible 3-primary component of $\pi_1 \mathsf{s}_2 \mathbf{1}[e^{-1}]$, the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1 \mathsf{f}_2 \mathbf{1}[e^{-1}]$ is classified by an element in the group

$$\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{k}^{\mathsf{M}}(1) \oplus \mathbf{K}^{\mathsf{M}}/4(2), \mathbf{k}^{\mathsf{M}}(2)) \cong \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{k}^{\mathsf{M}}, \mathbf{k}^{\mathsf{M}}(1)) \oplus \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}/4, \mathbf{k}^{\mathsf{M}}).$$

The multiplicative structure on the slices of $\mathbf{1}[e^{-1}]$ (or a suitable étale or complex realization) identifies the first component of this element as the unique nonzero element in $\mathbf{k}^{\mathsf{M}}(1)_{-1} = \mathbf{k}^{\mathsf{M}}_0$ represented by $\eta\eta_{\mathsf{top}}$, because $\eta \cdot \eta\eta_{\mathsf{top}} = \eta^2\eta_{\mathsf{top}}$. For the same reason, the second component of this element is the unique nonzero element in $\mathbf{k}^{\mathsf{M}}_0$, because $\nu \in \pi_{1+(2)}\mathbf{1}$ naturally lifts to $\pi_{1+(2)}\mathsf{f}_2\mathbf{1}$ and satisfies $4\nu \neq 0$. Thus $\pi_1 \mathsf{f}_2 \mathbf{1}[e^{-1}]$ is generated, as a $\mathbf{K}^{\mathsf{MW}}$-module, by the elements $\eta\eta_{\mathsf{top}}$ and ν, subject to the relations $2\eta\eta_{\mathsf{top}} = 0 = \eta\nu$ and $12\nu = \eta^2\eta_{\mathsf{top}}$; see also [22]. The image of the generator g under the connecting homomorphism (5-7) is thus of the form $x\eta\eta_{\mathsf{top}} + y\nu$, where $x \in \mathbf{K}^{\mathsf{MW}}_2$ and $y \in \mathbf{K}^{\mathsf{MW}}_3$. The form of the first differential originating in the 1-slice given in Figure 2 then implies $x = \rho^2$ and $y = 0$. In particular, over fields where $\rho^2 = 0$ (such as fields of positive characteristic), the homomorphism (5-7) is zero, and hence so is the second differential induced by it. Suppose now that F is a field (of characteristic zero) in which $\rho^2 \neq 0$. Being a pure symbol of degree 2, [48, Theorem 3.2] shows that every element in

$$E^2_{2-(\star),1}(\mathbf{1}) = {}_{\rho^2}\mathbf{k}^{\mathsf{M}}(-1)$$

is in the image of the transfer map for an étale field extension $f : F \hookrightarrow E$ such that $\rho^2 = 0 \in \mathbf{k}^{\mathsf{M}}_2(E)$. By [31, Theorem 1.9] this transfer map is induced by a map $f_\sharp \mathbf{1}_E \to \mathbf{1}_F$ of motivic spectra. Since the slice spectral sequence is natural with respect to maps of motivic spectra, and the second differential homomorphism in question is zero for E, it is also zero for F. At this stage, one can deduce that the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1\mathsf{f}_1\mathbf{1}[e^{-1}]$ is generated by $\eta_{\mathsf{top}} \in \pi_{1+(0)}\mathsf{f}_1\mathbf{1}$ and $\nu \in \pi_{1+(2)}\mathsf{f}_1\mathbf{1}$, subject to the relations $2\eta_{\mathsf{top}} = 0 = \eta\nu$ and $12\nu = \eta^2\eta_{\mathsf{top}}$; see also [52, Theorem 2.5].

One cannot argue similarly for the second differential $E^2_{2-(n),0}(\mathbf{1}) \to E^2_{1-(n),2}(\mathbf{1})$, because no manageable set of $\mathbf{K}^{\mathsf{M}}$-generators for $\pi_2\mathsf{s}_0\mathbf{1}$ in small degrees is known. This is resolved by passage to quotient spectra $\mathbf{1} \to \mathbf{1}/\beta$ for suitable $\beta \in \mathbf{K}^{\mathsf{MW}}$; here "suitable" implies that $\mathbf{1} \to \mathbf{1}/\beta$ induces an injection on the target group for the differential in question. To illustrate this, observe that the second differential $E^2_{2-(n),0}(\mathbf{1}[e^{-1}]) \to E^2_{1-(n),2}(\mathbf{1}[e^{-1}])$ lands in a direct sum. The component ending in $h^{n+1,n+2}/\mathsf{Sq}^2h^{n-1,n+1}$ is zero by comparison with $\mathbf{kq}$, as Lemma 5.8 shows.

The component ending in $h^{n+2,n+2}_{12}$ can be split further into the direct sum of the 3-primary component $h^{n+2,n+2}_3$ and the 2-primary component $h^{n+2,n+2}_4$. As the 3-primary component of $\pi_1\mathsf{s}_2(\mathbf{1}[3^{-1}])$ is zero, without loss of generality the base field F has characteristic different from 3. The canonical map $\mathbf{1} \to \mathbf{1}/3$ then induces an injection on the 3-primary component of $\pi_1\mathsf{s}_2$. Hence it suffices to prove that the second differential

$$\begin{aligned} &E^2_{2-(n),0}(\mathbf{1}/3) \\ &\quad = \pi_{2-(n)}\mathsf{s}_0\mathbf{1}/3 = \pi_{2-(n)}\mathbf{M}\mathbb{Z}/3 \to \pi_{1-(n)}\mathsf{s}_2\mathbf{1}/3 = E^1_{1-(n),2}(\mathbf{1}/3) = E^2_{1-(n),2}(\mathbf{1}/3) \end{aligned}$$

is zero. If F contains a primitive third root of unity ω, $\pi_{2-(n)}\mathbf{M}\mathbb{Z}/3$ is generated by $\omega \in h^{0,2}_3(F)$ as a $\mathbf{K}^{\mathsf{M}}(F)$-module. The second differential maps this generator to the group $h^{4,4}_3(F)$. To conclude that the image of this generator is zero, consider the subfield $G := F_0(\omega)$, where $F_0 \subset F$ is the prime field of F. Functoriality with respect to field extensions shows that the diagram

$$\begin{array}{ccc} E^2_{2-(n),0}(\mathbf{1}_G/3) & \longrightarrow & E^2_{1-(n),2}(\mathbf{1}_G/3) = 0 \\ \downarrow & & \downarrow \\ E^2_{2-(n),0}(\mathbf{1}_F/3) & \longrightarrow & E^2_{1-(n),2}(\mathbf{1}_F/3) \end{array}$$

commutes. The group $h^{4,4}_3(G) = 0$ is zero, because the cohomological dimension of G is at most 2. Hence the image of the generator, which lifts to the initial corner, is zero. If F does not contain a primitive third root of unity, transfer with respect to the quadratic field extension $F \hookrightarrow F(\omega)$ induces an injection on the relevant 3-primary component, whence this case follows from the previous one.

To analyze the component ending in $h^{n+2,n+2}_4$ requires a different quotient spectrum. The projection $\mathbf{1} \to \mathbf{1}/4\mathsf{h}$ works; consider the proof of [55, Lemma 4.15] for details. One interesting feature is that the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_2\mathsf{s}_0(\mathbf{1}/4\mathsf{h}) \cong \pi_2\mathbf{M}\mathbb{Z}/8$ is generated by a single element in $\pi_{2-(2)}\mathbf{M}\mathbb{Z}/8 = h^{0,2}_8$, which lifts to the prime field. Another interesting feature is that, after eliminating the easy case of odd characteristic, shrinking to the prime field $\mathbb{Q}$ allows the usage of real realization. The aforementioned generator hits an element in $h^{4,4}_4(\mathbb{Q})$, a group whose unique nonzero element real-realizes to the nonzero element $\eta_{\mathsf{top}} \in \pi_1\mathbb{S}$, and hence cannot be hit. Thus also this second differential is zero.

With all second differentials ending in the first column being zero, there remains a single third differential

$$E^3_{2-(n),0}(\mathbf{1}[e^{-1}]) = E^2_{2-(n),0}(\mathbf{1}[e^{-1}]) \to E^2_{1-(n),3}(\mathbf{1}[e^{-1}]) = E^3_{1-(n),3}(\mathbf{1}[e^{-1}])$$

to consider. It is treated as before (exercise; which $\beta \in \mathbf{K}^{\mathsf{MW}}$ will work?). Hence the kernel A_n of the unit map $\pi_{1+(n)}\mathbf{1}[e^{-1}] \to \pi_{1+(n)}\mathbf{kq}[e^{-1}]$ admits a filtration whose associated graded consists of the two terms listed in Proposition 5.12 which map trivially to the E^∞-page for $\mathbf{kq}$. In other words, the slice filtration induces a short exact sequence

$$0 \to h^{n+2,n+2}/\partial^{12}_2 h^{n+1,n+2}_{12} \to A_{-n} \to h^{n+2,n+2}_{12}[e^{-1}] \to 0. \tag{5-8}$$

Removing τ from the corresponding entry in Proposition 5.12 is justified, because multiplication with τ is an isomorphism on $h^{*,\star}$ for $\star \geq 0$. Assembling all weights in (5-8) together, there results a short exact sequence

$$0 \to \mathbf{k}^{\mathsf{M}}(2)/\partial^{12}_2\pi_1\Sigma^{(2)}\mathbf{M}\mathbb{Z}/12 \to A_{-\star} \to \mathbf{K}^{\mathsf{M}}(2)/12[e^{-1}] \to 0 \tag{5-9}$$

of $\mathbf{K}^{\mathsf{MW}}$-modules, whence $A_{-\star}$ is classified by an element in

$$\begin{aligned}\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}(2)/12[e^{-1}], \mathbf{k}^{\mathsf{M}}(2)/\partial^{12}_2\pi_1\Sigma^{(2)}\mathbf{M}\mathbb{Z}/12)& \\ \cong \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}/12, \mathbf{k}^{\mathsf{M}}/\partial^{12}_2\pi_1\mathbf{M}\mathbb{Z}/12).&\end{aligned}$$

The outer terms in (5-9) are in fact $\mathbf{K}^{\mathsf{M}}$-modules, with canonical generators in degree $\star = -2$. It follows that also $A_{-\star}$ is a $\mathbf{K}^{\mathsf{M}}$-module, because the action of η on $A_{-\star}$ is determined by its effect on a lift $\tilde{g} \in A_2$ of a generator $g \in \mathbf{K}^{\mathsf{M}}_0/12[e^{-1}]$. However, $\eta\tilde{g} \in A_3 = 0$ necessarily lands in the trivial group. Hence $A_{-\star}$ is classified by an element in

$$\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}/12, \mathbf{k}^{\mathsf{M}}/\partial^{12}_2\pi_1\mathbf{M}\mathbb{Z}/12) \cong \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(\mathbf{K}^{\mathsf{M}}/12, \mathbf{k}^{\mathsf{M}}/\partial^{12}_2\pi_1\mathbf{M}\mathbb{Z}/12) \cong \mathbf{k}^{\mathsf{M}}_0 = \mathbb{Z}/2$$

and is thus the split extension or the unique nontrivial extension $\mathbf{K}^{\mathsf{M}}(2)/24[e^{-1}]$. That $A_{-\star}$ is the latter follows either by contemplating the multiplicative structure of the slice spectral sequence, or by complex or étale realization $\pi_{1+(2)}\mathbf{1}[e^{-1}] \to \pi_3\mathbb{S}[e^{-1}]$, which hits an element of order 8. Viewed as an element in $\pi_{1+(2)}\mathbf{1}$, it thus generates the kernel of $\pi_1\mathbf{1}[e^{-1}] \to \pi_1\mathbf{kq}[e^{-1}]$ as a $\mathbf{K}^{\mathsf{MW}}$-module. The equation $\eta\nu = 0$ follows from the above argument, but is already known by [22, Theorem

1.4]. The element $\eta^2\eta_{\mathsf{top}}$ is a nontrivial element of order 2, and since e is odd by assumption, has to coincide with 12ν already in $\pi_{1+(2)}\mathbf{1}$. □

Remark 5.18. With a bit more effort, the second Milnor–Witt stem $\pi_{2+(\star)}\mathbf{1}_F$ over a field F of characteristic $p \neq 2$ can be determined in the following sense: the kernel of the (nonsurjective!) unit map

$$\pi_{2+(\star)}\mathbf{1} \to \pi_{2+(\star)}\mathbf{kq}$$

is isomorphic to the $\mathbf{K}^{\mathsf{M}}$-module $\mathbf{k}^{\mathsf{M}}(4) \oplus \pi_1\Sigma^{(2)}\mathbf{M}\mathbb{Z}/24$ after inverting p if $p > 0$. The (unique) generator for $\mathbf{k}^{\mathsf{M}}(4)$ is $\nu^2 \in \pi_{2+(4)}\mathbf{1}$. See [56] for details and the proof. Also in this case the first slice differential is nontrivial, but the higher ones are zero. This is different for the third Milnor–Witt stem $\pi_{3+(\star)}\mathbf{1}_F$, as [23] implies in the case $F = \mathbb{R}$.

6. Applications

While Morel's identification $\pi_{0+(0)}\mathbf{1}_F \cong \mathbf{GW}(F)$ has spawned the area of refined enumerative geometry, pioneered by Levine [40] and Wickelgren [68]; see also [14], the computation of $\pi_1\mathbf{1}$ gave rise to several applications in geometry and K-theory.

In [5], Asok and Fasel determine the unstable $\boldsymbol{A}^1$-homotopy sheaf $\underline{\pi}_3^{\boldsymbol{A}^1}\boldsymbol{A}^3 \smallsetminus \{0\}$ over infinite fields of characteristic not two. Their statement involves Voevodsky's contraction, an operation sending a sheaf A of groups on $\mathbf{Sm}_S$ to the sheaf defined as the kernel of the map $A(X \times_S \boldsymbol{G}_{\mathsf{m}}) \to A(X)$ given by restriction along the morphism $X \xrightarrow{X\times_S 1} X \times_X \boldsymbol{G}_{\mathsf{m}}$ at every $X \in \mathbf{Sm}_S$.

Theorem 6.1 (Asok–Fasel). *Let F be an infinite perfect field of characteristic not two. There is a canonical "Suslin matrix" map inducing a surjection in a short exact sequence*

$$0 \to \boldsymbol{F}_5 \to \underline{\pi}_3^{\boldsymbol{A}^1}\boldsymbol{A}^3 \smallsetminus \{0\} \to \underline{\pi}_{1-(3)}\mathbf{KQ} \to 0.$$

The fifth contraction of the kernel $\boldsymbol{F}_5$ is isomorphic to the pullback $\mathbf{K}_0^{\mathsf{M}}/24 \times_{\mathbf{K}_0^{\mathsf{M}}/2} \boldsymbol{W}$ as a sheaf.

While [5] says more about $\boldsymbol{F}_5$ and in particular suggests that it is generated by an appropriate suspension of the unstable Hopf map $\nu : S^{3+(4)} \to S^{2+(2)}$, Theorem 6.1 already indicates a close relation to the stable computation in Theorem 5.17. The geometric consequence is a proof of a case of a conjecture of Murthy's.

Corollary 6.2. *Let X be a smooth affine variety of dimension four over an algebraically closed field of characteristic not two. If $V \to X$ is a vector bundle of rank 3 over X, then E splits off a trivial rank-1 summand if and only if $c_3(E) = 0$ in $CH^3(X) = H^{6,3}(X, \mathbb{Z})$.*

The advantage of the computation in Theorem 6.1 is that it paves a way to prove Murthy's conjecture in general, provided the following unstable computation of a homotopy sheaf holds.

Conjecture 6.3 (Asok–Fasel). *Let F be an infinite perfect field of characteristic not two. There is a canonical "Suslin matrix" map inducing an exact sequence*

$$\mathbf{K}^{\mathsf{M}}_{2+n}/24 \to \underline{\pi}_n^{\boldsymbol{A}^1}\boldsymbol{A}^n \smallsetminus \{0\} \to \underline{\pi}_{1-(n)}\mathbf{KQ}$$

of Nisnevich sheaves which becomes short exact after n-fold contraction.

This conjecture was stated in [4] and implies Murthy's conjecture in any dimension. However, it is in fact much stronger, as it gives the complete secondary obstruction to splitting a free rank-1 summand of a vector bundle on a smooth affine scheme over an infinite perfect field. The unstable "Suslin matrix" map stabilizes to the $\boldsymbol{P}^1$-stable unit map $\mathbf{1} \to \mathbf{KQ}$, whose factorization over $\mathbf{1} \to \mathbf{kq}$ figures in Theorem 5.17. However, while the unstable Conjecture 6.3 implies a $\boldsymbol{P}^1$-stable exact sequence

$$\mathbf{K}^{\mathsf{M}}_{2+n}/24 \to \pi_{1-(n)}\mathbf{1} \to \pi_{1-(n)}\mathbf{KQ}$$

over any infinite perfect field, Theorem 5.17 does not imply Conjecture 6.3 even after inverting the exponential characteristic, due to a lack of a $\boldsymbol{P}^1$-Freudenthal suspension theorem. Nevertheless, it is possible to derive unstable information from Theorem 5.17 and reprove the part of Theorem 6.1 which is relevant for Murthy's conjecture. This part turns out to be useful for a proper application of Theorem 5.17 provided in [6].

Theorem 6.4 (Asok–Fasel–Williams). *Let F be an infinite field of characteristic different from two and three, and A an essentially smooth local F-algebra. The image of the Suslin–Hurewicz homomorphism $K_5^{\text{Quillen}}(A) \to \mathbf{K}_5^{\mathsf{M}}(A)$ coincides with $24\mathbf{K}_5^{\mathsf{M}}(A)$.*

Suslin described the homomorphism $K_n^{\text{Quillen}}(F) \to \mathbf{K}_n^{\mathsf{M}}(F)$ for infinite fields in [63] and showed that the image for $n = 3$ coincides with $2\mathbf{K}_3^{\mathsf{M}}(F)$ if and only if the Milnor conjectures on Galois cohomology and on quadratic forms hold in degree 3, which he settled with Merkurjev in [41]. The case of $n = 4$ was recently proven in [53].

A final application, provided in [1], concerns the question whether vector bundles, perhaps equipped with a special linear or symplectic structure, induce Thom isomorphisms for generalized cohomology theories of algebraic varieties. One particularly interesting cohomology theory is the universal stable $\boldsymbol{A}^1$-derived cohomology $\mathsf{H}_{\boldsymbol{A}^1}\mathbb{Z}$, obtained by linearizing the sphere spectrum. It resides naturally as the unit object in the symmetric monoidal stable $\boldsymbol{A}^1$-derived category of S, obtained from the category of chain complexes of Nisnevich sheaves of abelian groups on $\mathbf{Sm}_S$ by inverting the affine line and stabilizing with respect to the projective line. See [16, Chapter 5] for a detailed definition, and [9] for the étale version. The latter admits Thom isomorphisms, whereas the former does not, by the following theorem.

Theorem 6.5 (Ananyevskiy). *Let F be a field. Then there is a short exact sequence*

$$\pi_0\mathbf{1} \xrightarrow{\eta_{\mathsf{top}}} \pi_1\mathbf{1} \to \pi_1\mathsf{H}_{\mathbb{A}^1}\mathbb{Z} \to 0$$

of $\mathbf{K}^{\mathsf{MW}}$*-modules.*

In particular, if $\mathsf{char}(F) \neq 2$, the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1\mathsf{H}_{\mathbb{A}^1}\mathbb{Z}$ contains an element of order 4 in weight (2), given by the image of the second Hopf map ν, as Theorem 5.17 implies. Étale realization proves that this image is nonzero of order 3 if $\mathsf{char}(F) = 2$. As a consequence of this property, $\mathsf{H}_{\mathbb{A}^1}\mathbb{Z}$ cannot be equipped with Thom isomorphisms associated with special linear or symplectic vector bundles. An equivalent formulation is that $\mathsf{H}_{\mathbb{A}^1}\mathbb{Z}$ is neither a module over **MSL** nor over **MSp**, special linear or symplectic algebraic bordism.

As another application of Theorem 5.17, Morel's Theorem 3.10 will be partially extended to other bases than fields. Due to the length of the arguments involved, a new section is in order.

7. The sphere spectrum over discrete valuation rings

Recall from Theorem 2.7 that, given a closed embedding $i : Z \hookrightarrow S$ of base schemes, with open complement $j : S \smallsetminus Z \hookrightarrow S$, the natural maps define a localization homotopy cofiber sequence

$$j_\sharp j^*\mathsf{E} \to \mathsf{E} \to i_*i^*\mathsf{E} \to \Sigma j_\sharp j^*\mathsf{E}$$

in $\mathbf{SH}(S)$ for any motivic spectrum E over S. As an exercise, one may check that i_* commutes with infinite direct sums, which implies it admits a right adjoint $i^! : \mathbf{SH}(S) \to \mathbf{SH}(Z)$. Taking right adjoints in the localization homotopy cofiber sequence above yields another distinguished localization homotopy cofiber sequence

$$i_*i^!\mathsf{E} \to \mathsf{E} \to j_*j^*\mathsf{E} \to \Sigma i_*i^!\mathsf{E}, \tag{7-1}$$

which in turn induces a long exact sequence

$$\cdots \to \pi_s i_*i^!\mathsf{E} \to \pi_s\mathsf{E} \to \pi_s j_*j^*\mathsf{E} \to \pi_{s-1}i_*i^!\mathsf{E} \to \cdots \tag{7-2}$$

of homotopy modules in $\mathbf{SH}(S)$. By adjointness, $\pi_{s+(w)}f_*\mathsf{E} \cong \pi_{s+(w)}\mathsf{E}$ for any morphism $f : S \to R$ of base schemes, where the first homotopy group is computed in $\mathbf{SH}(R)$, and the second in $\mathbf{SH}(S)$. In favorable cases (such as algebraic bordism **MGL** and its variants, homotopy algebraic K-theory **KGL**, and the sphere spectrum $\mathbf{1}$), there exists a version E_S over any base scheme S, such that E_S is canonically isomorphic to $f^*\mathsf{E}_R$ for any morphism $f : S \to R$ of base schemes. One calls such a collection an *absolute* motivic spectrum.

Example 7.1. Any motivic spectrum over $\mathrm{Spec}(\mathbb{Z})$ defines an absolute motivic spectrum. In particular, Spitzweck's motivic Eilenberg–MacLane spectrum $\mathbf{M}\mathbb{Z}$, which is constructed in [61] over Dedekind rings, defines an absolute motivic spectrum. Part of the construction is a verification that its pullback along a morphism from the spectrum of a field to the spectrum of the Dedekind ring coincides with Voevodsky's motivic Eilenberg–MacLane spectrum, as cited in Theorem 2.6. As indicated already, $\mathbf{1}$, $\mathbf{MSp}$, $\mathbf{MSL}$, $\mathbf{MGL}$, $\mathbf{KGL}$ are absolute motivic spectra. Hermitian K-theory may be represented by an absolute motivic spectrum; it certainly is over base schemes in which 2 is invertible.

Hence for an absolute motivic spectrum $\{\mathsf{E}_S\}$, the long exact sequence (7-2) contains the homotopy groups of E_S, the homotopy groups of E_U, and the homotopy groups of $i^!\mathsf{E}_S$, a motivic spectrum over Z. In even more favorable cases, this spectrum can be described in terms of E_Z. One case is given in Theorem 2.8. If $i : Z \hookrightarrow S$ is a closed embedding of smooth R-schemes, then $\Sigma^\infty_{\mathbf{P}^1_R} S/U \simeq \mathsf{Th}(Ni)$. The target of this canonical equivalence is the motivic Thom spectrum of the normal bundle of $i : Z \hookrightarrow S$, and the source is the canonical cone of the counit $j_\sharp j^* \mathbf{1}_S \to \mathbf{1}_S$, which by Theorem 2.7 is equivalent to $i_* i^* \mathbf{1}_S$, both viewed as objects in $\mathbf{SH}(R)$. This equivalence induces an equivalence $i^*\mathsf{E} \simeq i^!\mathsf{E} \wedge \mathsf{Th}(Ni)$ in $\mathbf{SH}(Z)$ for every motivic spectrum E over S [7]. Since motivic Thom spectra are invertible with respect to the smash product, there results an identification of the functor $i^!$ as

$$i^!\mathsf{E} \simeq \underline{\mathrm{Hom}}_Z(\mathsf{Th}(Ni), i^*\mathsf{E}) \tag{7-3}$$

for every closed embedding of schemes which are smooth over a base scheme. As a consequence of the six-functor formalism alluded to in and right after Theorem 2.4, [19] generalizes this equivalence as follows.

Theorem 7.2 (Déglise–Jin–Khan). *For every separated morphism $f : R \to S$ which can be factored as a regular closed embedding, followed by a smooth morphism, there exists an element $a \in K^0(R)$ and a natural transformation*

$$\mathrm{pur}_f(\mathsf{E}) : f^*\mathsf{E} \to f^!\mathsf{E} \wedge \mathsf{Th}(a) \tag{7-4}$$

of functors $\mathbf{SH}(S) \to \mathbf{SH}(R)$ which is suitably natural.

The natural transformation (7-4) listed in Theorem 7.2 specializes to the homotopy purity equivalence from Theorem 2.8 in case f already is smooth; then a is the inverse of the class of the tangent bundle of f in $K^0(R)$. Hence the "interesting" case of Theorem 7.2 is where $f = i : Z \to S$ is a regular closed embedding. The obvious candidate for the K-theory class a is then the class of the normal bundle $Ni \to Z$. Before pursuing this, an example might be helpful.

Example 7.3. Let $i : Z \hookrightarrow S$ be the zero scheme of a single regular function $\phi \in \mathcal{O}_S$. Then ϕ generates the conormal ideal sheaf $(\phi)/(\phi)^2$ and hence defines a trivialization of the normal bundle of rank 1. The restriction of ϕ to the complement $U := S \smallsetminus i(Z)$ along the open embedding $j : U \hookrightarrow S$ is an invertible element in $\mathcal{O}_U$, and thus defines an element $j^*(\phi) \in \pi_{(-1)}\mathbf{1}_U$. Its image in $\pi_{-1-(1)}i^!\mathbf{1}_S$ under the connecting map in the localization sequence (7-2) can be viewed as an incarnation of the purity transformation

$$i^*\mathbf{1}_S = \mathbf{1}_Z \to \Sigma^{1+(1)}i^!\mathbf{1}_S \simeq i^!\mathbf{1}_S \wedge \mathsf{Th}(Ni)$$

evaluated at $\mathbf{1}_S$.

Definition 7.4. Let $i : Z \hookrightarrow S$ be a regular closed immersion. A motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ is *i-pure* if the purity transformation

$$\mathrm{pur}_i : i^*\mathsf{E} \to i^!\mathsf{E} \wedge \mathsf{Th}(Ni)$$

is an equivalence.

Example 7.5. The absolute motivic spectrum **KGL** is i-pure for every regular closed immersion i of regular schemes, and the same is true for the motivic Eilenberg–MacLane spectrum with rational coefficients [16]. Both motivic spectra are orientable, whence their smash product with a Thom spectrum for a K-theory class of rank r is equivalent to a suspension with $S^{r+(r)}$.

With integral coefficients, the situation is more complicated, as usual, and "absolute purity" is currently not known to hold. However, based on [61], the integral motivic Eilenberg–MacLane spectrum satisfies the following, by [27, Proposition A.3].

Theorem 7.6 (Spitzweck). *Let $S = \mathrm{Spec}(D)$ be the spectrum of a Dedekind ring and $i : Z \hookrightarrow S$ the inclusion of a closed point. Then the motivic Eilenberg–MacLane spectrum $\mathbf{M}\mathbb{Z}_S$ with integral coefficients is i-pure.*

Theorem 7.6, whose proof is too involved for these lecture notes, will be used as the starting point for the proof of a special case of the following conjecture from [18].

Conjecture 7.7 (Déglise). *The absolute motivic spectra* $\mathbf{1}$ *and* **MGL** *are i-pure for every regular closed immersion i of regular schemes.*

The rationalized versions of $\mathbf{1}$ and **MGL** are i-pure for every regular closed immersion of regular schemes, thus supplying the current evidence for Conjecture 7.7. Further evidence is provided by the following theorem.

Theorem 7.8. *Let D be a discrete valuation ring with residue field k of characteristic $p > 0$ and fraction field F of characteristic zero. Let $i : \mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(D)$ denote the inclusion of the closed point. Then the purity transformation*

$$\mathrm{pur}_i(\mathbf{1}_D[p^{-1}]) : \mathbf{1}_k[p^{-1}] \to \Sigma^{1+(1)} i^! \mathbf{1}_D[p^{-1}]$$

is an equivalence.

The proof of Theorem 7.8 requires some preparatory lemmata.

Lemma 7.9. *Let $i : Z \to S$ be a closed embedding. The class of i-pure motivic spectra over S is closed under homotopy colimits.*

Proof. By construction, i^* and $- \wedge \mathsf{Th}(Ni)$ commute with homotopy colimits. It is also true that $i^!$ preserves homotopy colimits [27, Lemma 7.8], thereby implying the result by naturality of the purity transformation. Here is the argument. Applying i^* to the localization homotopy cofiber sequence (7-1) and using the counit equivalence $i^* i_* \simeq \mathrm{id}$ provides a homotopy cofiber sequence

$$i^! \to i^* \to i^* j_* j^* \to \Sigma i^!$$

in which i^*, j^*, j_* all commute with homotopy colimits. Hence so does $i^!$. □

Dealing with homotopy limits requires more effort. For a base scheme X, let X_{Nis} denote the small Nisnevich site of X. Standard properties of the Nisnevich topology imply that the cohomological dimension of X_{Nis} coincides with the Krull dimension of X. Let Sp denote the classical category of spectra of pointed simplicial sets, and let $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$ denote the category of Nisnevich sheaves with values in spectra. Every motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ restricts canonically to a Nisnevich sheaf $\mathsf{E}|_{X_{\mathrm{Nis}}} \in \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$ for every $X \in \mathbf{Sm}_S$ by viewing it as an S^1-$\boldsymbol{G}_{\mathrm{m}}$-bispectrum, taking the zeroth S^1-spectrum, and restricting the resulting Nisnevich sheaf of S^1-spectra on the big site $\mathbf{Sm}_S$ to the small site X_{Nis}. A closed inclusion $i : W \hookrightarrow X$ induces a base change functor $i^\diamond : \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp}) \to \mathrm{Sh}(W_{\mathrm{Nis}}, \mathrm{Sp})$ which commutes with homotopy colimits and preserves connected sheaves of spectra. It is unclear whether the diagram

$$\begin{array}{ccc} \mathbf{SH}(S) & \longrightarrow & \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp}) \\ \big\downarrow{\scriptstyle i^*} & & \big\downarrow{\scriptstyle (i \times_S X)^\diamond} \\ \mathbf{SH}(Z) & \longrightarrow & \mathrm{Sh}((Z \times_S X)_{\mathrm{Nis}},) \end{array}$$

commutes. Nevertheless, commuting $i^\diamond$ with homotopy limits works on the small site X_{Nis} under suitable connectivity assumptions.

Lemma 7.10. *Let $i : W \hookrightarrow X$ be a closed inclusion of base schemes. If*

$$\cdots \to E_{n+1} \to E_n \to \cdots \to E_1 \to E_0$$

is a tower in the category $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$ *such that the connectivity of the homotopy fibers* $\mathrm{hofib}(E_n \to E_{n-1})$ *tends to* ∞ *as n does, then the canonical map*

$$i^\diamond \operatorname*{holim}_n E_n \to \operatorname*{holim}_n i^\diamond(E_n)$$

is an equivalence in $\mathrm{Sh}(W_{\mathrm{Nis}}, \mathrm{Sp})$.

Proof. Testing on schemes in X_{Nis} and using finite cohomological dimension of X_{Nis} it follows from the assumption on the tower and the $\lim^1$-exact sequence that the canonical map

$$[\Sigma^s Y_+, \operatorname*{holim}_{m>n} \mathrm{hofib}(E_m \to E_n)] \to \lim_{m>n} [\Sigma^s Y_+, \mathrm{hofib}(E_m \to E_n)]$$

is an isomorphism for every $s \in \mathbb{Z}$ and every $Y \in X_{\mathrm{Nis}}$; here "$[-,-]$" denotes the Hom groups in the homotopy category of $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$. In particular also the connectivity of

$$\operatorname*{holim}_{m>n} \mathrm{hofib}(E_m \to E_n)$$

tends to ∞ with n. Moreover $i^\diamond : \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp}) \to \mathrm{Sh}(W_{\mathrm{Nis}}, \mathrm{Sp})$ preserves connected objects. For formal reasons, both holim and $i^\diamond$ preserve homotopy fiber sequences. Hence for every $s \in \mathbb{Z}$ and every $V \in W_{\mathrm{Nis}}$, there exists a natural number n such that both nonhorizontal maps in the diagram

$$\begin{array}{ccc} [\Sigma^s V_+, i^\diamond \operatorname{holim}_{m>n} E_m] & \longrightarrow & [\Sigma^s V_+, \operatorname{holim}_{m>n} i^\diamond E_m] \\ & \searrow \quad \swarrow & \\ & [\Sigma^s V_+, E_n] & \end{array}$$

are isomorphisms. Hence the horizontal map is an isomorphism for every $s \in \mathbb{Z}$ and every $V \in W_{\mathrm{Nis}}$, showing the claim. □

Given the right assumptions on a tower of motivic spectra, Lemma 7.10 suffices to commute i^* with the homotopy limit of the tower.

Lemma 7.11. *Let* $i : Z \hookrightarrow S$ *be a closed inclusion of base schemes. Let*

$$\cdots \to \mathsf{E}_{n+1} \to \mathsf{E}_n \to \cdots \to \mathsf{E}_1 \to \mathsf{E}_0$$

be a tower in $\mathbf{SH}(S)$ *with associated homotopy fibers* $\mathsf{D}_n := \mathrm{hofib}(\mathsf{E}_n \to \mathsf{E}_{n-1})$ *for every* $n > 0$. *Assume that*

(1) *for every* $w \in \mathbb{Z}$ *and* $X \in \mathbf{Sm}_S$ *the connectivity of the restriction of* $\Sigma^{(w)}\mathsf{D}_n$ *to* $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$ *tends to* ∞ *with* n, *and*

(2) *for every* $w \in \mathbb{Z}$ *there is an* $N \in \mathbb{N}$ *such that for every* $X \in \mathbf{Sm}_S$ *and* $n > N$ *the natural map*

$$(i \times_S X)^\diamond((\Sigma^{(w)}\mathsf{D}_n)|_{X_{\mathrm{Nis}}}) \to (i^*\Sigma^{(w)}\mathsf{D}_n)|_{(Z\times_S X)_{\mathrm{Nis}}}$$

is an equivalence in $\mathrm{Sh}((Z\times_S X)_{\mathrm{Nis}},\mathrm{Sp})$.

Then the natural map

$$i^*\operatorname*{holim}_n \mathsf{E}_n \to \operatorname*{holim}_n i^*\mathsf{E}_n$$

is an equivalence in $\mathbf{SH}(Z)$.

Proof. For $w\in\mathbb{Z}$ and sufficiently large n, the restriction of

$$\Sigma^{(w)}\operatorname*{holim}_m \operatorname{hofib}(\mathsf{E}_m\to\mathsf{E}_n)$$

to $\mathrm{Sh}(X_{\mathrm{Nis}},\mathrm{Sp})$ and the restriction of

$$\Sigma^{(w)}\operatorname*{holim}_m i^*\operatorname{fib}(\mathsf{E}_m\to\mathsf{E}_n)$$

to $\mathrm{Sh}((Z\times_S X)_{\mathrm{Nis}},\mathrm{Sp})$ satisfy the assumptions of [61, Proposition 8.6]. The result then follows from Lemma 7.10 and [61, Proposition 8.6]. □

Remark 7.12. In order to apply Lemma 7.11 to the slice filtration, one has to verify the relevant assumptions on the tower. Suppose S is the spectrum of a Dedekind domain whose cohomological dimension is finite. If each D_n is of the form $\bigvee_{j\in J_n}\Sigma^{s_j+(w_j)}\mathsf{M}A_j$, with each A_j a finite abelian group and $s_j+w_j\geq\varphi(n)$, where $\varphi:\mathbb{N}_{>0}\to\mathbb{Z}$ is a function with $\lim_{n\to\infty}\varphi(n)=\infty$, then the assumptions of Lemma 7.11 are satisfied. The first condition follows from [61, Theorem 3.9], and the second condition can be deduced from the proof of [61, Proposition 8.7]. In the case of the slice filtration for $\mathsf{f}_1\mathbf{1}_D[p^{-1}]$, where D is a discrete valuation ring whose fraction field has characteristic zero and whose residue field has characteristic $p>0$, one may use $\varphi(n)=n-1$ by Theorem 5.1.

Proof of Theorem 7.8. The argument will use the slice filtration. In the situation at hand, Theorem 5.1 provides the slices of $\mathbf{1}_D[p^{-1}]$. In particular, every slice of $\mathbf{1}_D[p^{-1}]$ is a (finite) coproduct of suspensions of motivic Eilenberg–MacLane spectra with coefficients in (finitely generated) modules over $\mathbb{Z}[p^{-1}]$. Thus every slice of $\mathbf{1}_D[p^{-1}]$ is i-pure, because of Lemma 7.9 and the fact that $\mathbf{M}\mathbb{Z}_D$ is i-pure by Theorem 7.6. Furthermore, the n-th effective cover $\mathsf{f}_n\to\mathrm{id}$ sits in a natural homotopy cofiber sequence

$$\mathsf{f}_n\to\mathrm{id}\to\mathsf{f}^{n-1}\to\Sigma\mathsf{f}_n$$

such that $\mathsf{f}^0\mathbf{1}=\mathsf{s}_0\mathbf{1}$ (and more generally for all effective motivic spectra). One obtains natural homotopy cofiber sequences

$$\mathsf{s}_n\to\mathsf{f}^n\to\mathsf{f}^{n-1}\to\Sigma\mathsf{s}_n$$

for all n. Note that the slice completion defined in (4-2) identifies with $\mathsf{sc}\mathsf{E}=\operatorname{holim}_{n\to\infty}\mathsf{f}^{n-1}\mathsf{E}$.

Theorem 5.1 and Example 7.1 then imply that the canonical natural transformation $i^* \circ \mathsf{s}_n \to \mathsf{s}_n \circ i^*$ is an equivalence when evaluated at $\mathbf{1}_D[p^{-1}]$; hence the same is true for the natural transformation $i^* \circ \mathsf{f}^{n-1} \to \mathsf{f}^{n-1} \circ i^*$. Suppose now that the residue field k of D is perfect and has finite cohomological dimension. Then [37, Theorem 4] says that the composition

$$\mathbf{1}_k[p^{-1}] = i^*\mathbf{1}_D[p^{-1}] \to i^*\mathsf{sc}(\mathbf{1}_D[p^{-1}]) \to \mathsf{sc}(i^*\mathbf{1}_D[p^{-1}]) = \mathsf{sc}(\mathbf{1}_k[p^{-1}])$$

is an equivalence. However, at this stage it is unclear whether i^* commutes with sc; there is no reason to assume that i^* preserves homotopy limits.

If additionally the fraction field F of D has finite cohomological dimension (a condition which is automatic according to [59, Dimension cohomologique: Theorem 2.2] in the case D is Henselian), [37, Theorem 4] implies that $\mathbf{1}_F$ and $\mathbf{1}_F[p^{-1}]$ are slice convergent. Hence the canonical map $\mathbf{1}_D[p^{-1}] \to \mathsf{sc}(\mathbf{1}_D[p^{-1}])$ is such that its image under j^* is an equivalence. The functor j^*, having a left adjoint $j_\sharp$, commutes with homotopy limits. Thus the image of the canonical map $\mathbf{1}_D[p^{-1}] \to \mathsf{sc}(\mathbf{1}_D[p^{-1}])$ under j^* is, up to equivalence, the canonical map $\mathbf{1}_F[p^{-1}] \to \mathsf{sc}(\mathbf{1}_F[p^{-1}])$, and in particular an equivalence.

Lemma 7.11 and Remark 7.12 imply that i^* commutes with the homotopy limit over the slice filtration of $\mathbf{1}_D[p^{-1}]$. Hence the canonical map $\mathbf{1}_D[p^{-1}] \to \mathsf{sc}(\mathbf{1}_D[p^{-1}])$ is such that its image under i^* is an equivalence as well, just as for j^*, and also coincides with the canonical map $\mathbf{1}_k[p^{-1}] \to \mathsf{sc}(\mathbf{1}_k[p^{-1}])$ up to equivalence. Theorem 2.7 implies that $\mathbf{1}_D[p^{-1}]$ is slice convergent. More importantly for the discussion at hand, the purity transformation

$$\mathrm{pur}_i(\mathbf{1}_D[p^{-1}]) : i^*\mathbf{1}_D[p^{-1}] \to i^!\mathbf{1}_D[p^{-1}] \wedge \mathsf{Th}(Ni) \tag{7-5}$$

is then, up to natural equivalence, the homotopy limit of the purity transformations

$$\mathrm{pur}_i(\mathsf{f}^n\mathbf{1}_D[p^{-1}]) : i^*\mathsf{f}^n\mathbf{1}_D[p^{-1}] \to i^!\mathsf{f}^n\mathbf{1}_D[p^{-1}] \wedge \mathsf{Th}(Ni),$$

which are equivalences. Thus also (7-5) is an equivalence if D is a discrete valuation ring whose perfect residue and fraction field have finite cohomological dimension.

The condition on the fraction field is unnecessary for the following reason. The essentially étale passage $f : D \to D_{\mathrm{Hens}}$ from a discrete valuation ring D to its Henselization induces an equality on residue fields. Since f is essentially étale, properties of the six-functor formalism imply that $\mathbf{1}_D[p^{-1}]$ is i-pure if and only if $\mathbf{1}_{D_{\mathrm{Hens}}}[p^{-1}]$ is i_{Hens}-pure. Note that D_{Hens} has finite cohomological dimension if and only if its residue field k does, by the aforementioned [59, Dimension cohomologique: Theorem 2.2].

Furthermore, if $k \hookrightarrow \ell$ is a purely inseparable field extension of characteristic $p > 0$, it induces an equivalence $\mathbf{SH}(k)[p^{-1}] \xrightarrow{\simeq} \mathbf{SH}(\ell)[p^{-1}]$ by [26]. This holds in particular for the passage from a field of characteristic $p > 0$ to its perfection.

Hence the assumption on the residue field of D being perfect may be dropped (using that $\mathbf{1}_k[p^{-1}]$ is slice complete if and only if $\mathbf{1}_\ell[p^{-1}]$ is).

Finally, if D is a discrete valuation ring with residue field k of characteristic $p > 0$ and fraction field F of characteristic zero, express F as the filtered colimit of subfields $E \subset F$ having finite transcendence degree over $\mathbb{Q}$. For every such E, the intersection $D \cap E$ is again a discrete valuation ring, simply by restricting the valuation, with fraction field E. The residue field ℓ of $D \cap E$ necessarily has finite transcendence degree over $\mathbb{F}_p$. The reason is that any lift of a family of elements in ℓ generating a subfield of transcendence degree at least d to $D \cap E$ generates a subfield of E of transcendence degree at least d. Since ℓ then has finite cohomological dimension by [58], this concludes the proof by the case already completed. □

As a consequence of Theorem 7.8, Morel's computation of $\pi_0\mathbf{1}_F$ for F a field can be extended to discrete valuation rings of mixed characteristic, subject to inverting the residue characteristic and restricting the weight.[4] Here are some details. Still remaining over the discrete valuation ring D of mixed characteristic with residue characteristic $p > 0$, Theorem 7.8 gives the localization long exact sequence (7-2) the following form for the p-inverted sphere spectrum:

$$\begin{aligned} \cdots \to \pi_{s+1+(\star+1)}\mathbf{1}_k[p^{-1}] \to \pi_{s+(\star)}\mathbf{1}_D[p^{-1}] \\ \to \pi_{s+(\star)}\mathbf{1}_F[p^{-1}] \to \pi_{s+(\star+1)}\mathbf{1}_k[p^{-1}] \to \cdots \end{aligned} \tag{7-6}$$

This long exact sequence contains a homomorphism

$$\mathbf{K}^{\mathsf{MW}}(F)[p^{-1}] \cong \pi_{0+(\star)}\mathbf{1}_F[p^{-1}] \to \pi_{0+(\star+1)}\mathbf{1}_k[p^{-1}] \cong \mathbf{K}^{\mathsf{MW}}(k)[p^{-1}],$$

which commutes with multiplication by η and units in D. Moreover, since the purity transformation used implicitly can be made explicit by choosing a uniformizing element, as explained in Example 7.3, one can check that it coincides with a homomorphism constructed by Morel [46, Theorem 3.15].

The maps of spheres discussed in Section 3 give rise to a ring homomorphism from $\mathbf{K}^{\mathsf{MW}}(D)$ as discussed in [46, Theorem 3.22] and defined — by a verbatim copy of Definition 3.8 — in [28] to $\pi_0\mathbf{1}_D$.[5] The statements at the end of Section 3 extend to D by [28, Theorem 5.4].

Theorem 7.13. *Let D be a discrete valuation ring whose residue field k has characteristic $p > 2$ and at least five elements. Suppose further that its fraction field F has characteristic zero. For all $s < 0$, the group $\pi_s\mathbf{1}_D[p^{-1}]$ equals 0. For all $w < 2$ the canonical map $\mathbf{K}^{\mathsf{MW}}_w(D) \to \pi_{0-(w)}\mathbf{1}_D$ is an isomorphism after inverting p.*

[4] In the case of a 2-regular number ring R, [12, Theorem 5.2] provides an identification of $\pi_{0+(0)}\mathbf{1}_{R[1/2]}$ in weight zero with the Grothendieck–Witt ring of $R[1/2]$ after localizing at two.

[5] Strictly speaking, [28] restricts attention to local rings in which 2 is invertible and whose residue field contains at least 5 elements.

Proof. Of course Morel's Theorem 3.10 is used here. The connectivity statement gives $\pi_s \mathbf{1}_F = \pi_s \mathbf{1}_k$ for $s < 0$, whence $\pi_s \mathbf{1}_D[p^{-1}] = 0$ for $s < -1$ by the long exact sequence (7-6). The homomorphism

$$\mathbf{K}^{\mathsf{MW}}(F) \cong \pi_{0+(\star)} \mathbf{1}_F \to \pi_{0+(\star+1)} \mathbf{1}_k \cong \mathbf{K}^{\mathsf{MW}}(k)$$

is surjective, because every unit in k lifts, as does the Hopf map. Hence it is also surjective after inverting p, which implies $\pi_{-1} \mathbf{1}_D[p^{-1}] = 0$ by the long exact sequence (7-6). The second statement concerns the surjectivity of the homomorphism

$$\delta_1 : \pi_{1+(\star)} \mathbf{1}_F[p^{-1}] \to \pi_{1+(\star+1)} \mathbf{1}_k[p^{-1}].$$

It fits into a commutative diagram

$$\begin{array}{ccccccc}
\pi_{1-(w)} \mathsf{f}_1 \mathbf{1}_F[p^{-1}] & \longrightarrow & \pi_{1-(w)} \mathbf{1}_F[p^{-1}] & \longrightarrow & \pi_{1-(w)} \mathbf{M}\mathbb{Z}_F[p^{-1}] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_{1-(w-1)} \mathsf{f}_1 \mathbf{1}_k[p^{-1}] & \longrightarrow & \pi_{1-(w-1)} \mathbf{1}_k[p^{-1}] & \longrightarrow & \pi_{1-(w-1)} \mathbf{M}\mathbb{Z}_k[p^{-1}] & \longrightarrow & 0
\end{array}$$

whose horizontal sequence is exact by construction and [55, Corollary 5.6]. Note that $\mathsf{f}_1 \mathbf{1}_D[p^{-1}]$ is i-pure by Theorem 7.6, Theorem 7.8 and Lemma 7.9. As a $\mathbf{K}^{\mathsf{MW}}(k)[p^{-1}]$-module, $\pi_{1+(\star)} \mathsf{f}_1 \mathbf{1}[p^{-1}]$ is generated by η_{top} and ν [52, Theorem 2.5]. These elements are in the image of δ_1, namely $\delta_1([u]\eta_{\mathsf{top}}) = \eta_{\mathsf{top}}$ and $\delta_1([u]\nu) = \nu$, where u is a uniformizing element in D. Hence the middle vertical arrow in the diagram above will be surjective as soon as the one right next to it is. It remains to see that the connecting map

$$\delta_1^{\mathbf{M}\mathbb{Z}} : \pi_{1-(w)} \mathbf{M}\mathbb{Z}_F[p^{-1}] \to \pi_{1-(w-1)} \mathbf{M}\mathbb{Z}_k[p^{-1}]$$

is surjective. This is definitely the case for $w < 3$, because $\pi_{1-(w-1)} \mathbf{M}\mathbb{Z}_k = H^{w-2,w-1}(k, \mathbb{Z})$ vanishes for $w < 3$. Hence the sequence

$$0 \to \pi_{0-(w)} \mathbf{1}_D[p^{-1}] \to \pi_{0-(w)} \mathbf{1}_F[p^{-1}] \to \pi_{0-(w-1)} \mathbf{1}_k[p^{-1}] \to 0$$

is exact for $w < 3$. In weight $w < 0$, this sequence is obtained by inverting p on the Gersten sequence

$$0 \to \boldsymbol{W}(D) \to \boldsymbol{W}(F) \to \boldsymbol{W}(k) \to 0$$

for the Witt group, because it is exact by direct computation, or by [13]. This result extends to the Grothendieck–Witt group, giving the desired result in weight $w = 0$. With the help of [28, Theorem 5.4] and the exactness of the sequence

$$0 \to D^\times = \mathbf{K}_1^{\mathsf{M}}(D) \to F^\times = \mathbf{K}_1^{\mathsf{M}}(F) \xrightarrow{\delta} \mathbb{Z} = \mathbf{K}_0^{\mathsf{M}}(k) \to 0$$

the result follows in weight $w = 1$ as well. □

The methods of [17] allow an extension of Theorem 7.13 to all weights, identifying $\pi_0 \mathbf{1}_D[p^{-1}]$ with the so-called unramified Milnor–Witt K-theory defined in [28, Section 6.2]. Also, [20] might lead to an identification of $\mathbf{K}_2^{\mathsf{MW}}(D)[p^{-1}]$ with $\pi_{0-(2)}\mathbf{1}_D[p^{-1}]$.

8. Notation

S, R, U, Z	base (Noetherian separated of finite Krull dimension) schemes
F, E, G	base fields
$\mathbf{Sm}_S$	category of smooth separated S-schemes
$\boldsymbol{A}^d_S$, $\boldsymbol{P}^d_S$	affine and projective space of dimension d over S
$\mathbf{Spc}_\bullet(S)$	pointed simplicial presheaves on $\mathbf{Sm}_S$
$\mathbf{SH}(S)$	$\boldsymbol{P}^1$-stable $\boldsymbol{A}^1$-homotopy category of motivic spectra over S
D, E	generic motivic spectra
$\Sigma^{s+(w)}$	sphere of dimension $s+w$ and weight w, suspension functor
$\pi_{s+(w)}\mathsf{E}$	homotopy group $[S^{s+(w)}, \mathsf{E}] = \mathrm{Hom}_{\mathbf{SH}(S)}(S^{s+(w)}, \mathsf{E})$
$\underline{\pi}_{s+(w)}\mathsf{E}$	Nisnevich sheaf associated to $X \mapsto [\Sigma^{s+(w)}X_+, \mathsf{E}]$
$\pi_s\mathsf{E} = \pi_{s+(\star)}\mathsf{E}$	homotopy module $\bigoplus_{w\in\mathbb{Z}} \pi_{s+(w)}\mathsf{E}$
$\mathbf{M}A$	motivic Eilenberg–MacLane-spectrum for the abelian group A
$\mathbf{1}_S$	motivic sphere spectrum over S
MGL, **MSL**, **MSp**	algebraic bordism spectra of various flavors
KGL, **kgl**	algebraic K-theory spectrum and its (very) effective cover
KQ, **kq**	hermitian K-theory spectrum and its (very) effective cover
KW, $\mathbf{KW}_{\geq 0}$	Witt theory spectrum and its connective cover
f_n, s_n	n-th effective cover and slice
vf_n, vs_n	n-th very effective cover and slice
$\mathbf{K}^{\mathsf{MW}}(F)$	Milnor–Witt K-theory of F
$\mathbf{K}^{\mathsf{M}}(F)$	Milnor K-theory of F
$\mathbf{GW}(F)$, $\boldsymbol{W}(F)$	Grothendieck–Witt ring and Witt ring of F
$[u] : S^{0+(0)} \to S^{0+(1)}$	map of spheres associated with unit u
$\mathsf{h} = 2 + \eta[-1]$	hyperbolic plane; zeroth algebraic Hopf map
$\eta : S^{1+(2)} \to S^{1+(1)}$	first algebraic Hopf map
$\nu : S^{3+(4)} \to S^{2+(2)}$	second algebraic Hopf map
$\eta_{\mathsf{top}} : S^{3+(0)} \to S^{2+(0)}$	first topological Hopf map

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