THE OPEN BOOK SERIES 6

Motivic Geometry

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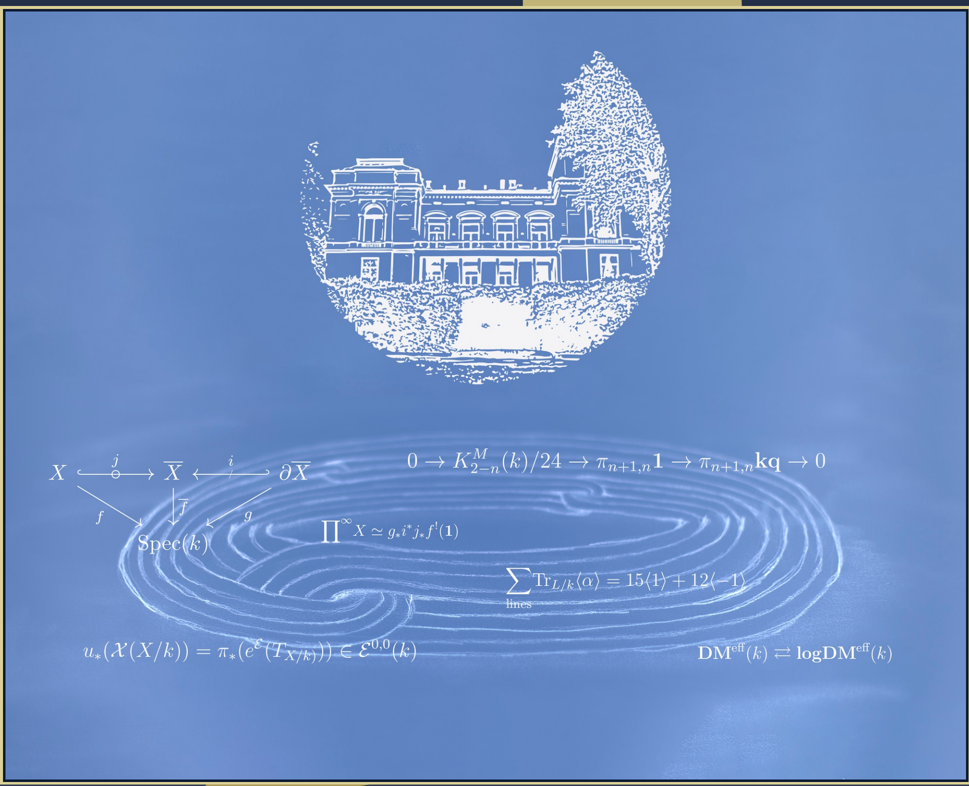

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THE OPEN BOOK SERIES **6** (2025)
Motivic Geometry (Oslo)
https://doi.org/10.2140/obs.2025.6.63

An introduction to six-functor formalisms

Martin Gallauer

These are notes for a mini-course given at the summer school and conference The six-functor formalism and motivic homotopy theory in Milan 9/2021. They provide an introduction to the formalism of Grothendieck's six operations in algebraic geometry and end with an excursion to rigid-analytic motives.

The notes do not correspond precisely to the lectures delivered but provide a more self-contained account for the benefit of the audience and others. No originality is claimed.

MSC2020: 14F08, 14F99.
Keywords: six operations, six-functor formalism, motivic homotopy theory, cohomology, rigid-analytic motives.

0. Introduction

The main goal of these lectures is to touch upon the following questions regarding six-functor formalisms:

(1) *Why care about them?*

(2) *What are they?*

(3) *How to construct them?*

Needless to say, our answers are far from complete. (We will try to give references for the reader who wants to venture further but they will certainly not be exhaustive either.) In short, they are:

(1) *Why?* If one cares about cohomology then one should care about the six operations because the latter *enhance* the former. Grothendieck's relative point of view is baked into the formalism, connecting it well with modern algebraic geometry. And, finally, the formalism has proven highly successful in the last decades. (This is particularly apparent in motivic homotopy theory, the other main topic of the summer school. Unfortunately, we will treat this last point only briefly and leave much to the other talks.)

(2) *What?* First the confession: there will be no definition of six-functor formalisms in these lectures. Just as 'cohomology' is arguably not a precisely defined term and varies from context to context, we cannot expect its enhancement to admit a definition pleasing everyone. Instead we will try to give a glimpse of the six functors in action, and we will describe a convenient and precise framework to think about them. This framework consists of *coefficient systems* which encode a minimal set of structure and axioms one would like a six-functor formalism to enjoy.

(3) *How?* Given the power of the formalism it is unsurprising that all known examples required major efforts, often by many mathematicians, until they were established. (And in several areas, a 'complete' formalism is still very much a work in progress.) We will focus on arguably the most common and serious stumbling block, the construction of the *exceptional functoriality*. In a slightly different direction, such difficulties can be circumvented altogether by constructing six-functor formalisms out of already-established ones. And finally, we will discuss by way of illustration, an example (from rigid-analytic geometry) of a recent new addition to the list of six-functor formalisms.

We assume that the reader is familiar with basic scheme theory and has seen derived categories before. Section 1 is written in the language of triangulated categories although the axioms are barely used. In Sections 2 and 3 we use the language of stable ∞-categories but some help is provided and much of it can also be understood just at the level of underlying triangulated categories. We imagine that the more exposition to the various cohomology theories for schemes (or for other geometric objects) the reader has had, the easier it will be to follow the text.

1. Why?

Here we will try to motivate the study and development of six-functor formalisms. The point of view we will try to convey is that

six-functor formalisms enhance cohomology.

Interspersedly, we will also make comments about the related question why six-functor formalisms arose historically in the first place although this is not our focus. There is little rigorous mathematics to be found here — for that we ask the reader to wait until Sections 2 and 3.

Remark 1.1. Another natural way to answer the question in the title would be to list applications of the theory and thereby argue for its importance. We will not do that here and, in any case, compiling an even approximately complete list would seem a daunting task. Indeed, the language and theory of six-functor formalisms permeates much of modern algebraic geometry and beyond, and has spawned entire fields of research. The development of, for example, étale cohomology, perverse sheaves, or motivic homotopy theory is quite unthinkable in the absence of the six operations.

1A. *A hierarchy of invariants.*

Example 1.2. If you are studying a topological space X, a useful invariant to know about is the sequence of Betti numbers $b_n(X)$, the latter measuring the number of n-dimensional holes in X. Famously, Noether explained how these numbers are just shadows of the homology of X, these being a sequence of abelian groups $H_n(X)$ measuring the difference between cycles and boundaries on X. Thus homology is a richer invariant than the Betti numbers since there is a way to go from the former to the latter but no way (in general) to reverse this process:

$$\begin{array}{c} H_n(X) \\ {\scriptstyle \text{rank}}\downarrow \quad \not\uparrow \\ b_n(X) \end{array}$$

Example 1.3. Now imagine instead a variety X over a finite field $k = \mathbb{F}_q$ (of cardinality $q = p^r$, say). If you are an arithmetic geometer, chances are you would like to know the number of rational points, that is, solutions to the polynomial equations defining X, possibly over finite extensions of k. The ζ-function of X conveniently packages this information:

$$\zeta_X(T) = \exp\left(\sum_{n\geq 1} \frac{\#X(\mathbb{F}_{q^n})}{n} T^n\right) \in \mathbb{Q}[\![T]\!].$$

If X is smooth and proper, Weil [49] predicted that this function is very nicely behaved: it should be a rational function, satisfy a certain functional equation, and one should have tight control over the zeroes and poles. (Weil also proved this for curves.) He suggested that these desirable properties would follow from a well-behaved cohomology theory for varieties over finite fields, a suggestion which was eventually realized by the concerted effort of many mathematicians, including Grothendieck, Serre and Deligne: the theory of ℓ-adic cohomology was at least partly developed to settle the Weil conjectures.

Here then we find a similar situation as in topology in that cohomology groups are richer invariants than individual numbers and that a certain behavior of the former implies a certain behavior of the latter:

$$\begin{array}{c} H^\bullet(X_{\bar{k}}; \mathbb{Q}_\ell) \\ \Big\downarrow {\scriptstyle \text{traces of (iterated) Frobenii}} \\ \zeta_X(T) \end{array}$$

What is of interest to us in this historical example is that the 'good behavior' of these cohomology groups $H^n(X_{\bar{k}}; \mathbb{Q}_\ell)$ was in turn deduced from properties of an even richer invariant, the ℓ-adic constructible derived category:

$$\begin{array}{cc} \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X_{\bar{k}}; \mathbb{Q}_\ell) & \text{category-level invariant} \\ \Big\downarrow {\scriptstyle \text{hom-groups}} & \\ H^\bullet(X_{\bar{k}}; \mathbb{Q}_\ell) & \text{set-level invariant} \\ \Big\downarrow {\scriptstyle \text{trace of Frobenius}} & \\ \zeta_X(T) & \text{element-level invariant} \end{array}$$

Summarizing, in order to prove certain things about element-level invariants mathematicians in this case have found themselves proving things about category-level invariants two levels up and deducing the former from the latter.

Remark 1.4. Jumping ahead of ourselves for a moment, we can say that

six-functor formalisms govern the behavior of certain category-level invariants.

Therefore we can expect that this formalism will be useful in proving things about certain set-level invariants, namely the cohomology of 'spaces' (these could be topological spaces or spaces appearing in algebraic geometry and even further beyond). To say something more precise we have to appreciate one important feature which arises in set-level and category-level invariants but is absent at the lower end of the hierarchy. This we will try to do in Section 1B.

Remark 1.5. The Weil conjectures (Example 1.3) are discussed from this point of view in an unpublished note of Voevodsky [47]. In the same note (from the year 2000) he expresses the view that the development of the six-functor formalism would become one of the main technical tools in advancing motivic homotopy theory: a view which over the last 20 years has certainly materialized!

1B. ***Relative point of view.*** Grothendieck famously stressed the 'relative point of view', replacing, for example, schemes by morphisms of schemes as the fundamental object of study. This shift is also apparent in the development of ℓ-adic cohomology and the proof of the Weil conjectures (Example 1.3).

Remark 1.6. Even if one is ultimately interested in the cohomology of a single variety X it is often necessary to invoke other, related varieties and their cohomologies in the process, for example, in arguments that proceed by induction on the dimension, or when covering X by simpler pieces. It then becomes important to study the cohomology groups not in isolation but together with the maps

$$f^* : H^\bullet(Y) \to H^\bullet(Y') \tag{1.7}$$

for all morphisms $f : Y' \to Y$.

And even if not passing through other varieties, the action of a morphism on cohomology provides additional, often very interesting information about the varieties involved. In the discussion of the Weil conjectures (Example 1.3) we already saw an example of this phenomenon: the action of the Frobenius endomorphism is used to express the number of rational points in terms of cohomology.

Remark 1.8. For the same reason, even if one is interested in proper varieties it is sometimes necessary to invoke nonproper varieties and their cohomologies in the process. The latter are typically less well behaved and to make up for that, the notion of *cohomology with compact support* was developed. Thus in addition to cohomology groups, we also want to study the groups $H^\bullet_c(X)$ and their dependence on X.

1C. ***The six functors in topology.*** In Section 1A we saw that moving up along the hierarchy of invariants, cohomology is replaced by sheaves, and in Section 1B we stressed the need to adopt a relative point of view. Putting the two together one arrives at the study of assignments

$$\text{spaces} \to \text{categories},$$

which send a space to some category of sheaves on that space, and where morphisms of spaces induce functors between the corresponding categories of 'sheaves'. (As we saw in Example 1.3, these are not necessarily literally sheaves but something related, such as derived categories of sheaves.) The latter are examples of the functors giving 'six-functor formalisms' their name. Sometimes they are also called *operations* since they operate on sheaves.

For Weil and Grothendieck, a good cohomology theory for varieties over finite fields was to behave similarly to the cohomology of topological spaces. It is therefore prudent to look at the topological situation first. This we do briefly here. References that include much more detail include [16; 27; 28].

Example 1.9. Let us go back to the topological Example 1.2. For a nice enough[1] space X the cohomology $H^\bullet(X)$ coincides with the sheaf cohomology of the constant sheaf on X. The most familiar operations on (abelian) sheaves associated with a continuous map $f : X \to Y$ are the *inverse image* (or *pull-back*) and *direct image* (or *push-forward*), respectively:

$$\mathrm{Sh}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Sh}(X)$$

Recall that $f_*\mathcal{F}$ is the sheaf whose sections on an open subset $U \subseteq Y$ are given by $\Gamma(f^{-1}(U), \mathcal{F})$. The functor f^* is left adjoint to f_*, and takes $\mathcal{G} \in \mathrm{Sh}(Y)$ to a sheaf satisfying $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)}$ for all $x \in X$. More explicitly, $f^*\mathcal{G}$ is the sheaf associated to the presheaf $V \mapsto \mathrm{colim}_{f(V)\subseteq U}\, \mathcal{G}(U)$, where U runs over the open neighborhoods of $f(V)$.

Note that if $Y = *$ is just a point, the functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(*) \simeq \mathrm{Mod}(\mathbb{Z})$ coincides with the global sections functor. We deduce that the right-derived functor coincides with sheaf cohomology:

$$\mathsf{R}^n f_*(\mathcal{F}) \cong H^n(X; \mathcal{F}). \tag{1.10}$$

Thus we may view the derived push-forward as a relative and enhanced version of cohomology.

Example 1.11. Continuing with Example 1.9, another familiar operation is the *direct image with compact support* (or *compactly supported push-forward*). It is defined as a subfunctor of the direct image functor:

$$\Gamma(U, f_!\mathcal{F}) := \{s \in \Gamma(U, f_*\mathcal{F}) = \Gamma(f^{-1}(U), \mathcal{F}) \mid s \text{ has compact support}\}.$$

Note that again, for $Y = *$ a point, the direct image with compact support recovers cohomology with compact support:

$$\mathsf{R}^n f_!(\mathcal{F}) \cong H^n_{\mathrm{c}}(X; \mathcal{F}). \tag{1.12}$$

[1] For example, cohomologically locally connected in the sense of [40].

Remark 1.13. The functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ does not admit an adjoint in general. This together with the fact that we are ultimately interested in derived functors leads us to consider derived categories of sheaves instead. It turns out that at least for locally compact Hausdorff spaces, the functor acquires a right adjoint:

$$\mathsf{D}(\mathrm{Sh}(X)) \underset{f^!}{\overset{\mathsf{R}f_!}{\rightleftarrows}} \mathsf{D}(\mathrm{Sh}(Y))$$

The functor $f^!$ is called the *exceptional inverse image* (or *pull-back*). Accordingly, $f_!$ is sometimes also called the *exceptional direct image* (or *push-forward*).[2]

Remark 1.14. Together with the tensor product and internal hom of sheaves we have collected all six functors:

$$\begin{array}{c} (\otimes^{\mathsf{L}}, \mathsf{R}\underline{\mathrm{Hom}}) \\ \mathsf{D}(\mathrm{Sh}(X)) \\ \mathsf{L}f^* \uparrow \; \downarrow \mathsf{R}f_* \qquad \mathsf{R}f_! \downarrow \; \uparrow f^! \\ \mathsf{D}(\mathrm{Sh}(Y)) \end{array}$$

It is customary to drop the symbols R and L for derived functors as the context usually makes it clear when derived functors are intended.

1D. *Enhancing cohomology: structure.*

Convention 1.15. Let us now abstract from the specific topological situation and instead assume that with each 'space' (topological space, scheme, stack, ...) X we are given a closed tensor triangulated category $(C(X), \otimes, \underline{\mathrm{Hom}})$ and with each morphism of spaces $f : X \to Y$ two adjunctions $f^* \dashv f_*$, $f_! \dashv f^!$ of exact functors:

$$\begin{array}{c} (\otimes, \underline{\mathrm{Hom}}) \\ C(X) \\ f^* \uparrow \; \downarrow f_* \Leftarrow f_! \downarrow \; \uparrow f^! \\ C(Y) \end{array}$$

The arrow $\Leftarrow$ indicates a natural transformation $f_! \Rightarrow f_*$ (which, in topology, is induced by the inclusion of sections with compact support). We also assume that

[2] Another common name is *f-upper-shriek* for $f^!$ and *f-lower-shriek* for $f_!$.

f^* is endowed with a symmetric monoidal structure. As a first approximation, the category $C(X)$ may be thought of as a 'derived category of sheaves on X' although we don't want to assume that this is literally the case. To have more neutral language we will refer to objects in $C(X)$ as *coefficients*, just as one speaks of cohomology with coefficients.

While we won't stress this aspect, it is important that the dependence on X and f is 'pseudofunctorial'. For example, we should in addition be given natural isomorphisms $g^*f^* \cong (fg)^*$ satisfying familiar cocycle conditions, and $\mathrm{id}^* \cong \mathrm{id}$. For a precise formulation we refer to [15, Definition 2.2].

Remark 1.16. We will now discuss how this basic setup allows us to recover structure present in cohomology. In Section 1E we will see some properties of the six functors and how these properties govern the behavior of cohomology.

Remark 1.17. The identifications in (1.10) and (1.12) show how the sheaf operations allow us to recover cohomology of spaces. In the basic setup of Convention 1.15 we may take this as our *definition*. Let $p : X \to B$ be a morphism of spaces, where we think of B as a 'base space', fixed by the context. For any coefficient $\mathscr{F} \in C(X)$, the *cohomology* (resp. *with compact support*) is

$$H^\bullet(X;\mathscr{F}) := p_*\mathscr{F} \in C(B) \quad (\text{resp. } H^\bullet_{\mathrm{c}}(X;\mathscr{F}) := p_!\mathscr{F} \in C(B)).$$

When $\mathscr{F} = \mathbb{1}$ is the tensor unit we denote these coefficients simply by $H^\bullet(X)$ and $H^\bullet_{\mathrm{c}}(X)$, respectively.

In order to obtain actual cohomology groups one may take appropriate homomorphism groups:

$$H^n(X;\mathscr{F}) := \hom_{C(B)}(\mathbb{1}, p_*\mathscr{F}[n]) \quad (\text{resp. } H^n_{\mathrm{c}}(X;\mathscr{F}) := \hom_{C(B)}(\mathbb{1}, p_!\mathscr{F}[n]))$$

Remark 1.18. One can also define *homology* and *Borel–Moore homology*, generalizing these theories from topology, like so:

cohomology	$p_*p^*\mathbb{1}$	$H^\bullet$
cohomology with compact support	$p_!p^*\mathbb{1}$	$H^\bullet_{\mathrm{c}}$
homology	$p_!p^!\mathbb{1}$	$H_\bullet$
Borel–Moore homology	$p_*p^!\mathbb{1}$	$H^{\mathrm{BM}}_\bullet$

Example 1.19. Let k be a field in which the prime ℓ is invertible and such that $\mathrm{cd}_\ell(k) < \infty$. Then one has a structure as described in Convention 1.15 which sends each finite-type k-scheme (or even algebraic stack) X to the ℓ-adic constructible derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X;\mathbb{Q}_\ell)$ (see, for example, [35], although much of it goes back to SGA, particularly [44; 45]). In this case the cohomology (resp. with compact support) as defined in Remark 1.17 recovers ℓ-adic cohomology (resp. with compact support).

Here are some more examples:[3]

coefficients	cohomology groups
$\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X; \mathbb{Q}_\ell)$ constructible ℓ-adic sheaves	ℓ-adic cohomology
$\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X(\mathbb{C}); \mathbb{Z})$ constructible analytic sheaves	Betti cohomology
$\mathsf{D}^{\mathsf{b}}_{\mathsf{h}}(\mathscr{D}_X)$ holonomic $\mathscr{D}$-modules	de Rham cohomology
$\mathsf{D}^{\mathsf{b}}(\mathrm{Coh}(X))$ coherent sheaves	coherent cohomology
$\mathsf{D}^{\mathsf{b}}(\mathsf{MHM}(X))$ mixed Hodge modules	absolute Hodge cohomology
$\mathsf{DM}(X)$ Voevodsky motivic sheaves	(weight-0) motivic cohomology
$\mathsf{SH}(X)$ stable motivic homotopy sheaves	stable motivic (weight-0) cohomotopy groups

Remark 1.20. Consider now a relative situation

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & * & \end{array}$$

The unit of the adjunction $f^* \dashv f_*$ induces a morphism

$$\eta : q_* \to q_* f_* f^* \cong p_* f^*$$

and thus a morphism in cohomology

$$H^\bullet(Y, \mathscr{F}) \to H^\bullet(X, f^*\mathscr{F}).$$

If $\mathscr{F} = \mathbb{1}_X$ one recovers the action of f on the cohomology of X as in (1.7).

Remark 1.21. With compactly supported cohomology the situation is more subtle. In the topological context, a natural map

$$\Gamma_{\mathrm{c}}(Y, F) \to \Gamma_{\mathrm{c}}(X, f^*F)$$

is defined when $f : X \to Y$ is proper. Namely, in that case pulling back sections restricts to those with compact support. This map is in turn induced by the same unit of the adjunction,

$$\eta : q_! \to q_! f_* f^* \cong p_! f^*,$$

using that $f_! = f_*$ as f is proper:

$$H^n_{\mathrm{c}}(Y, \mathscr{F}) \to H^n_{\mathrm{c}}(X, f^*\mathscr{F}).$$

Similar functoriality exists for proper morphisms of schemes and other ‘spaces’.

Exercise 1.22. Describe the functoriality of homology.

[3]Some of these are only partial examples, in others certain technical assumptions are required.

Remark 1.23. Let $p : X \to B$ be a morphism and recall that the inverse image $p^* : C(B) \to C(X)$ is symmetric monoidal. It follows formally that its right adjoint $p_* : C(X) \to C(B)$ sends commutative algebras to commutative algebras. In particular, the cohomology $H^\bullet(X) = p_*\mathbb{1} \in C(B)$ has the structure of a commutative algebra. We may view this as an enhancement of the *cup product* in cohomology. Indeed, evaluating the multiplication through appropriate hom-groups $\hom_{C(B)}(\mathbb{1}, -[n])$ one obtains a cup product at the level of cohomology groups:

$$\cup : H^a(X) \times H^b(X) \to H^{a+b}(X)$$

Remark 1.24. Needless to say, this section does not exhaust all structures of interest in cohomology. For example, *vanishing* and *nearby cycles* are additional concepts of interest. Another example will play a more important role in Section 2. The classical theorem of de Rham and its algebraic geometry version of Grothendieck identifies cohomology groups associated with different theories (de Rham and singular cohomology). Relatedly, *Chern classes* and *regulator maps* may be seen as morphisms from certain cohomology groups of one theory to those of another. We want to think of these as underlying '(iso)morphisms of six-functor formalisms'. For example, the Beilinson regulator maps algebraic K-theory classes to absolute Hodge cohomology, and this should arise from a family of *Hodge realization functors*

$$\rho^*_{\mathrm{H}}(X) : \mathsf{DM}^{\mathrm{c}}(X) \to \mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$$

from categories of (constructible) motivic sheaves that 'realize' the underlying Hodge cohomology of motives. Ideally we would like these functors to be suitably compatible with the six operations.[4]

1E. *Enhancing cohomology: properties.* We now turn to properties of the six functors and related properties in cohomology. We will discuss here only some of the many possibilities. Our selection is geared towards the approach to six-functor formalisms described in Section 2.

Remark 1.25. To avoid a possible confusion let us stress: the point is not (at least, not always) that results about a given cohomology theory come for free using six-functor formalisms. But the difficulty can sometimes be shifted from establishing them directly to establishing that the cohomology theory underlies a six-functor formalism. We will return to this in Sections 2 and 3.

1E1. *Proper push-forward.* We already mentioned in the topological context that $f_! = f_*$ whenever f is proper. The same is true for six-functor formalisms in general: whenever f is 'proper' (for example, a proper morphism of schemes), the transformation $f_! \to f_*$ is an isomorphism.

[4]This particular example is taken up again in Example 3.19.

1E2. *Duality.* An important impetus for developing the six-functor formalism was what is now sometimes called *Grothendieck duality*, as, for example, in [23; 24]. In a very limited sense, in our setup this can be viewed as the computation of $f^!\mathbb{1}$ for $f : X \to *$ smooth.

Example 1.26 (topology). Let X be a smooth manifold of dimension d and let $f : X \to *$ be the unique map. For a ring Λ, one finds that $f^!\Lambda = \omega_{X,\Lambda}[d]$ is the (shifted) Λ-orientation sheaf. Thus X is orientable if and only if $\omega_{X,\mathbb{Z}}$ is the constant sheaf with value $\mathbb{Z}$. In that case, $\omega_{X,\Lambda}$ is constant for every ring Λ.

Example 1.27 (coherent). Let X be a smooth k-variety of dimension d. If the map $f : X \to \operatorname{Spec}(k)$ denotes the structure morphism then $f^!k \cong \omega_X[d]$ is the (shifted) canonical sheaf on X.

Example 1.28 (ℓ-adic). Let X be a smooth k-variety of dimension d and ℓ a prime invertible in k. Then $f^!\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(d)[2d]$ where (d) denotes the d-th Tate twist.

Corollary 1.29. *With the assumptions of Examples 1.26–1.28, respectively, one has*:

(a) *Poincaré duality (topology): If X is orientable,* $H^n_c(X;\mathbb{Q})^* \cong H^{d-n}(X;\mathbb{Q})$.

(b) *Poincaré duality (ℓ-adic):* $H^n_c(X;\mathbb{Q}_\ell)^* \cong H^{2d-n}(X;\mathbb{Q}_\ell(d))$.

(c) *Serre duality: If X is proper,* $H^n(X;\mathcal{O}_X)^* \cong H^{d-n}(X;\omega_X)$.

Proof. This follows from the adjunction isomorphisms

$$H^n_c(X)^* \cong \hom(f_!f^*\mathbb{1}[n],\mathbb{1}) \cong \hom(\mathbb{1}, f_*f^!\mathbb{1}[-n])$$

together with the computations reported in Examples 1.26–1.28. □

Remark 1.30. The coefficient $f^!\mathbb{1}$ tries to be a dualizing object. Verdier duality is concerned with the functors $\mathbb{D} = \underline{\operatorname{Hom}}(-, f^!\mathbb{1})$ and asks under which conditions one has isomorphisms such as

$$\mathrm{id} \xrightarrow{\sim} \mathbb{D}\circ\mathbb{D}, \quad \mathbb{D}g_! \xrightarrow{\sim} g_*\mathbb{D}, \quad g^*\mathbb{D} \xrightarrow{\sim} \mathbb{D}g^!.$$

It provides a relative version and generalization of duality phenomena such as the ones of Corollary 1.29.

Exercise 1.31 (Atiyah duality). Let $f : X \to B$ be a smooth and proper morphism. Show that the coefficient $H^\bullet(X) = H^\bullet_{\mathrm{c}}(X)$ is *rigid*, with $\otimes$-dual given by $H_\bullet(X) = H^{\mathrm{BM}}_\bullet(X)$.[5]

You will want to use the following two fundamental properties:

[5]We use ‘rigid’ instead of ‘strongly dualizable’. Recall that an object a in a symmetric monoidal category is rigid if there is an object a^* (called its $\otimes$-*dual*) and morphisms $\mathbb{1} \to a\otimes a^*$ and $a^*\otimes a \to \mathbb{1}$ satisfying the identities familiar from adjunctions; see [17].

(a) **(Proper projection formula)** For arbitrary f,

$$f_!(f^*F\otimes G)\xrightarrow{\sim} F\otimes f_!G.$$

(b) If f is smooth then

$$f^*F\otimes f^!H\xrightarrow{\sim} f^!(F\otimes H).$$

Note that the two morphisms are related by adjunction. The second one is a form of relative purity, to which we now turn.

1E3. *Relative purity.*

Remark 1.32. Previously we 'computed' $f^!\mathbb{1}$ in the case where $f : X \to *$ is smooth. It is natural to want to generalize this to arbitrary *smooth* morphisms $f : X \to Y$,[6] and this is provided by *relative purity*. It implies:

(1) The difference between $f^!$ and f^* is measured yet again by $f^!\mathbb{1}$:

$$f^!\mathbb{1}\otimes f^*F\xrightarrow{\sim} f^!F.$$

(This is equivalent to Exercise 1.31(b).)

(2) The coefficient $f^!\mathbb{1}$ is $\otimes$-invertible.

The coefficient $f^!\mathbb{1}$ arises from the *Thom construction* (see below) applied to the relative tangent bundle T_f, and this information is very useful in computations. We interpret the equivalence $f^!\mathbb{1}\otimes(-) =: \{T_f\}$ as a 'twist' by the relative tangent bundle and may therefore rewrite

$$\{T_f\}f^*\simeq f^!. \tag{1.33}$$

Example 1.34. In the ℓ-adic setting[7] the Thom construction depends only on the rank of the vector bundle and the relative purity isomorphism reads as (see Example 1.44 below)

$$f^*(d)[2d]\simeq f^!. \tag{1.35}$$

Note how (1.35) generalizes Poincaré duality in ℓ-adic cohomology discussed in Section 1E2.

One often abbreviates the operation $(d)[2d]$ by $\{d\}$ and this is our inspiration for the notation in (1.33).

[6] In the topological context, this should be interpreted as a topological submersion [28, Definition 3.3.1].

[7] This is more generally true for *orientable* theories (that is, those with a good notion of Chern classes). Implicitly, we also used the canonical isomorphism $f^*\{d\}\simeq\{d\}f^*$ which, in the notation introduced in Remark 1.38, is a particular instance of $f^*\{V\}\simeq\{f^{-1}V\}f^*$ for any vector bundle V on Y.

Before discussing the Thom construction, let us note some important consequences of relative purity.

Remark 1.36. As a consequence we note that for f smooth, the inverse image functor f^* admits a left adjoint

$$f_\sharp = f_!\{T_f\} \quad \dashv \quad f^*.$$

It satisfies the (Smooth projection formula):

$$f_\sharp(f^*\mathscr{F} \otimes \mathscr{G}) \xrightarrow{\sim} \mathscr{F} \otimes f_\sharp\mathscr{G},$$

which arises, by adjunction, from the composite

$$f^*\mathscr{F} \otimes \mathscr{G} \to f^*\mathscr{F} \otimes f^* f_\sharp\mathscr{G} \xrightarrow{\sim} f^*(\mathscr{F} \otimes f_\sharp\mathscr{G})$$

of the unit of the adjunction $f_\sharp \dashv f^*$ and the monoidality of f^*.

Example 1.37. Note that for $f : X \to B$ smooth, the homology of X may be expressed alternatively as

$$H_\bullet(X) = f_! f^! \mathbb{1} \simeq f_\sharp f^* \mathbb{1}.$$

Remark 1.38. In terms of this left adjoint we can describe the Thom construction as follows. Let $p : V \to X$ be a vector bundle with zero section $s : X \hookrightarrow V$. The V-twist is defined as

$$\{V\} := p_\sharp s_* : C(X) \to C(X).$$

Then the *Thom construction* applied to V is defined as the evaluation of this functor at the unit, that is,

$$\mathrm{Th}(V) = \mathbb{1}\{V\} = p_\sharp s_* \mathbb{1}.$$

Example 1.39. As mentioned in Remark 1.32, for f smooth we have

$$f^! \mathbb{1} \simeq \mathrm{Th}(T_f).$$

If f is étale then the relative tangent bundle $T_f = X$ is of rank zero hence $\mathrm{Th}(T_f) \simeq \mathbb{1}$, and one deduces that $f^! \simeq f^*$. (Whatever 'étale' means in contexts other than schemes, we would expect this last property to hold.)

Exercise 1.40. Explain as a consequence that Borel–Moore homology is contravariantly functorial with respect to étale morphisms.

Remark 1.41. The Thom construction yields a morphism from the monoid (with respect to direct sums) of isomorphism classes of vector bundles on X to the Picard group of $C(X)$, which can be extended to virtual vector bundles [14; 41], perfect complexes and that passes to the level of K-theory. In particular, there is a group homomorphism

$$K_0(X) \xrightarrow{\mathrm{Th}(-)} \mathrm{Pic}(C(X)).$$

For a modern approach to this construction, see, for example, [7, §16], and for a more general discussion of purity we refer to [12; 13].

Remark 1.42. Continue with the setup of Remark 1.38 and denote by $j : V\backslash X \hookrightarrow V$ the inclusion of the complement of the zero section. From the localization property discussed just below in Section 1E4 we deduce an exact triangle

$$p_\sharp j_\sharp j^* p^* \mathbb{1} \to p_\sharp p^* \mathbb{1} \to p_\sharp s_* s^* p^* \mathbb{1} \tag{1.43}$$

in $C(X)$ which exhibits $\mathrm{Th}(V)$ as the cone of the canonical morphism between homologies relative to X,

$$H_\bullet(V\backslash X) \to H_\bullet(V).$$

These are the analogues of the sphere and disk bundle associated with V in topology, respectively, and this justifies labeling the construction 'Thom construction'.

Locally every vector bundle is trivial so we better understand these first. By Remark 1.41, it is enough to understand the rank-1 case.

Example 1.44. Let $V = \mathbb{A}^1_X$ be the trivial vector bundle of rank 1 on X. A fundamental property of the six-functor formalisms in algebraic geometry that we are interested in here is the contractibility of the affine line:

($\mathbb{A}^1$-homotopy) $p_\sharp p^* \xrightarrow{\sim} \mathrm{id}$.

This implies that $H_\bullet(\mathbb{A}^1_X) = H_\bullet(X)$. By Exercise 1.45 below, the exact triangle (1.43) in $C(X)$ becomes

$$\mathbb{1} \oplus \mathbb{1}\{1\}[-1] \to \mathbb{1} \to \mathrm{Th}(\mathbb{A}^1_X)$$

so that $\mathrm{Th}(\mathbb{A}^1_X) = \mathbb{1}\{1\}$.

Exercise 1.45. Show that the homology of $\mathbb{G}_m = \mathbb{A}^1\backslash 0$ splits:

$$H_\bullet(\mathbb{G}_m) = \mathbb{1} \oplus \tilde{H}_\bullet(\mathbb{G}_m)$$

for some coefficient $\tilde{H}_\bullet(\mathbb{G}_m)$ (which we think of as the *reduced* homology of $\mathbb{G}_m$). We define the *Tate twists* (and shifts thereof)

$$\mathbb{1}(1) := \tilde{H}_\bullet(\mathbb{G}_m)[-1], \quad \mathbb{1}\{1\} := \tilde{H}_\bullet(\mathbb{G}_m)[1].$$[8]

Using that open covers give rise to Mayer–Vietoris exact triangles,[9] show also that

$$H_\bullet(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{1}\{1\}.$$

Remark 1.46. We deduce from the preceding discussion the property

[8]There is a monoid morphism $\mathbb{N} \to K_0(X)$ that takes n to the class of the trivial vector bundle $\mathbb{A}^n_X$. Then this notation becomes compatible with the previous one; cf. Remark 1.41.

[9]This is a particular instance of Exercise 1.64 below.

(Tate stability) $\mathbb{1}\{1\} = \mathrm{Th}(\mathbb{A}^1)$ is $\otimes$-invertible.

We may then define, for any coefficient F and $n \in \mathbb{Z}$, $F\{n\} = F \otimes \mathbb{1}\{n\}$.

Remark 1.47. As soon as we have discussed smooth base change (Remark 1.61) we can establish that the Thom construction is Zariski local in a suitable sense. Together with the localization property we are about to discuss (Remark 1.51), (Tate stability) therefore is seen to imply the $\otimes$-invertibility of all Thom coefficients. In the same vein, ($\mathbb{A}^1$-homotopy) is enough to imply the contractibility of any vector bundle.

1E4. *Localization.* Let $i : Z \hookrightarrow X$ be a closed immersion with open complement $j : X \backslash Z \hookrightarrow X$.

Convention 1.48. One defines the *cohomology of X with support in Z* to be

$$H_Z^\bullet(X) := i_! i^! \mathbb{1} \in C(X).$$

(In other words, it is the homology of Z relative to X.)

Example 1.49. Applying $\hom_{C(X)}(\mathbb{1}, -[n])$ this recovers the corresponding notion in topology and in ℓ-adic cohomology. In the coherent context, these groups may be better known under the name of *local cohomology* (*with respect to* Z).

Remark 1.50. There is a so-called *localization triangle* of functors $C(X) \to C(X)$:

$$i_! i^! \to \mathrm{id} \to j_* j^*,$$

which we may apply to the tensor unit $\mathbb{1}$ to obtain (with base X)

$$H_Z^\bullet(X) \to H^\bullet(X) \to H^\bullet(X \backslash Z).$$

The associated long exact sequence is a well-known cohomological tool:

$$\cdots \to H_Z^n(X) \to H^n(X) \to H^n(X \backslash Z) \to H_Z^{n+1}(X) \to \cdots.$$

Remark 1.51. The localization triangle is in turn a consequence of the localization property of six-functor formalisms:

(Localization) The sequence of triangulated categories

$$C(Z) \xrightarrow{i_*} C(X) \xrightarrow{j^*} C(U)$$

is a localization sequence.

This means that one is in the situation of a recollement [8, § 1.4], and in particular that

(1) $i_* \simeq i_!$, j_*, $j_!$ are fully faithful,

(2) the composites $j^* i_*$, $i^! j_*$ and $i^* j_!$ all vanish,

(3) the pairs (i^*, j^*) and $(i^!, j^!)$ are each conservative,

(4) one has another localization sequence $j_! j^! \to \mathrm{id} \to i_* i^* \to^+$.

Remark 1.52. The last triangle may also be written as (with base space X)

$$H^\bullet_c(X \backslash Z) \to H^\bullet_c(X) \to H^\bullet_c(Z)$$

and gives rise to the usual long exact sequence of pairs in compactly supported cohomology. This follows from the identifications $j^* = j^!$ (Example 1.39) and $i_* = i_!$ (Section 1E1).

Example 1.53. (1) By (Localization), $C(\varnothing) \simeq 0$ (take $i = \mathrm{id} : \varnothing \to \varnothing$).

(2) Now let X_{red} be X with the reduced scheme structure, and $i : X_{\mathrm{red}} \hookrightarrow X$ the obvious closed immersion. It follows from (Localization) together with part (1) that $i_* : C(X_{\mathrm{red}}) \xrightarrow{\sim} C(X)$ is an equivalence. In other words, six-functor formalisms are insensitive to nilpotent thickenings.

Remark 1.54. In the ℓ-adic setting, localization is an easy property. In other contexts, however, it can be a substantial theorem. For example, for $\mathscr{D}$-modules fully faithfulness of i_* is known as *Kashiwara's lemma*. In motivic homotopy theory, the localization sequence is a fundamental result of Morel and Voevodsky which they call *the glueing theorem*.

1E5. *Blow-up.* The relation between the cohomology of a variety X and its blow-up $\tilde{X} = \mathrm{Bl}_Z(X)$ is as simple as one might hope but it encodes a fundamental property of six-functor formalisms, namely proper base change.

Convention 1.55. We place ourselves in a more general situation, with a commutative diagram of the following shape:

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{\tilde{i}} & \tilde{X} & \xleftarrow{\tilde{j}} & \tilde{X} \backslash \tilde{Z} \\ {\scriptstyle p'}\downarrow & \overset{w}{\searrow} & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle =} \\ Z & \xrightarrow[i]{} & X & \xleftarrow[j]{} & X \backslash Z \end{array} \tag{1.56}$$

We assume that p is proper and i a closed immersion, both squares are Cartesian and the right vertical arrow is an isomorphism. In this situation, the left part of the diagram is called an *abstract blow-up square*.

We now want to explain why there is an exact triangle (the *blow-up exact triangle*)

$$H^\bullet(X) \to H^\bullet(Z) \oplus H^\bullet(\tilde{X}) \to H^\bullet(\tilde{Z}). \tag{1.57}$$

"Proof". The functoriality of cohomology easily gives the two morphisms in (1.57) so that the composite is zero (this involves introducing a sign, as usual). We will

cheat a little bit and assume that cones are functorial so we get a canonical morphism from the cone of the first map to $H^\bullet(\tilde{Z})$ and it suffices to show this map is invertible. By (Localization), this in turn can be checked after applying each of i^* and j^*. The upshot of this little game is that we may prove $i^*(1.57)$ and $j^*(1.57)$ are exact triangles.

Let us write the two candidate triangles in terms of the six operations:

$$\begin{aligned} &i^*\mathbb{1} \to i^*i_*i^*\mathbb{1} \oplus i^*p_*p^*\mathbb{1} \to i^*w_*w^*\mathbb{1}, \\ &j^*\mathbb{1} \to j^*i_*i^*\mathbb{1} \oplus j^*p_*p^*\mathbb{1} \to j^*w_*w^*\mathbb{1}. \end{aligned}$$

By (Localization), i_* is fully faithful, $j^*i_* = 0$, and $j^*w_* = j^*i_*p'_* = 0$. Taking this into account the candidate triangles look as follows:

$$\begin{aligned} &\mathbb{1} \to \mathbb{1} \oplus i^*p_*\mathbb{1} \to p'_*\tilde{i}^*\mathbb{1}, \\ &\mathbb{1} \to j^*p_*\mathbb{1} \to 0. \end{aligned}$$

It is clear that what remains to do is to 'commute' p_* with i^* and j^*, respectively. This is precisely the content of proper base change (Remark 1.58). ☐

Remark 1.58. Let

$$\begin{array}{ccc} V & \xrightarrow{h} & X \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{g} & Y \end{array} \tag{1.59}$$

be a Cartesian square. Using the unit and counit of the adjunctions between inverse and direct image functors we deduce a canonical *Beck–Chevalley* (or, *push-pull*) transformation:

$$g^*f_* \to k_*k^*g^*f_* \simeq k_*h^*f^*f_* \to k_*h^*. \tag{1.60}$$

We note:

(Proper base change) If f is proper then (1.60) is invertible: $g^*f_* \simeq k_*h^*$.

Remark 1.61. Another instance in which the Beck-Chevalley transformation is invertible is:

(Smooth base change) If g is smooth then (1.60) is invertible: $g^*f_* \simeq k_*h^*$.

Exercise 1.62. Recall that by relative purity, the inverse image along a smooth morphism admits a left adjoint $(-)_\sharp$. Construct analogously a transformation

$$h_\sharp k^* \to f^*g_\sharp \tag{1.63}$$

and show that it is invertible if and only if (1.60) is. (This is a general phenomenon in 2-category theory: The transformations (1.60) and (1.63) are "mates".)

Exercise 1.64. Consider again the diagram (1.56). Assume now instead that p is étale, that i is an open immersion, and that the right vertical arrow is an isomorphism on the associated reduced schemes. The left part of the diagram in this case is called a *distinguished Nisnevich square*. Prove in a similar way that there is an associated exact triangle (1.57). (We might call this a *Nisnevich–Mayer–Vietoris triangle*.)

Exercise 1.65. Sometimes, proper and smooth base change instead refer to the following isomorphisms:

(a) Proper base change: $g^*f_! \simeq k_!h^*$.
(b) Smooth base change: $g^!f_* \simeq k_*h^!$.

At least morally, these are nothing but reformulations of (Proper base change) and (Smooth base change), respectively. More exactly, using properties discussed previously (for example, localization and relative purity):

(a) Assume f factors as an open immersion followed by a proper morphism (for example, f is separated and of finite type). Construct a zig-zag of push-pull transformations between $g^*f_!$ and $k_!h^*$. Show that it is an isomorphism if (Proper base change) holds.
(b) Assume g factors as a closed immersion followed by a smooth morphism (for example, g is quasiprojective). Construct a zig-zag of push-pull transformations between $g^!f_*$ and $k_*h^!$. Show that it is an isomorphism if (Smooth base change) holds.

What can you say about the converse statements?

2. What?

What is a six-functor formalism? As mentioned in the introduction, we will not try to give a definition. However, our main goal in this section is to describe an axiomatization of a convenient ‘stand-in’. It encodes a minimal set of structure and properties a six-functor formalism is commonly expected to enjoy. And we show how powerful this notion yet is. For example, most properties discussed in Section 1E are consequences, and the few remaining ones (related to duality) can still be studied within this framework.

This section’s results rely on the work of many mathematicians; see Remark 2.18.

2A. ***A convenient framework.*** From now on we officially restrict to schemes as our ‘spaces’. (But see Section 3C.)

Convention 2.1. Throughout we fix

- B: a base scheme, assumed Noetherian and finite dimensional,
- Sch_B: B-schemes, assumed separated and of finite type over B.

If B is clear from the context or doesn't play a role, we will refer to B-schemes as just schemes and write Sch instead of Sch_B. Note that all schemes considered are Noetherian and finite dimensional, and all morphisms are separated and of finite type. This will come in handy although more general setups are certainly possible.

Remark 2.2. For our framework to be flexible enough it is better to replace triangulated categories by a suitable enhancement. We saw a hint of this at a very basic level in the proof of the blow-up triangle (1.57). More serious uses of an enhancement will be made throughout Sections 2 and 3. We will work with *stable* ∞*-categories* as developed extensively in [38]. Nevertheless, a reader who is not familiar with this theory may replace them by triangulated categories (or another suitable enhancement) and still get the gist of the text. Most statements would still make sense and might even be true.

Convention 2.3. The ∞-category of stable ∞-categories and exact functors is denoted by $\mathrm{Cat}_\infty^{\mathrm{st}}$. This has a symmetric monoidal structure for which the tensor product $\mathscr{C} \otimes \mathscr{D}$ is the universal recipient of biexact functors from the Cartesian product $\mathscr{C} \times \mathscr{D}$. We identify commutative algebra objects therein with symmetric monoidal stable ∞-categories, and we write $\mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ for the ∞-category of these. (Note that by our convention, the tensor product is exact in both variables.) They are an enhancement of tensor-triangulated categories, where our coefficients lived in Section 1.

Here is the main definition. While a coefficient lives on a single B-scheme we are now interested in the system of all coefficients on all B-schemes. It seems natural to call this data a *coefficient system*. This terminology was introduced in [18]. Others have used different terms; see Section 2C.

Definition 2.4. A *coefficient system* (*over* B) is a functor $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ satisfying the following axioms (where we write $f^* = C(f)$ for f a morphism of B-schemes).

(1) **(Left)** For each smooth morphism $p : Y \to X$ contained in Sch_B, the functor $p^* : C(X) \to C(Y)$ admits a left adjoint $p_\sharp$, and:

(Smooth base change) For each Cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{p'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{p} & X \end{array}$$

in Sch_B, the Beck–Chevalley transformation $p'_\sharp(f')^* \to f^* p_\sharp$ is an equivalence.

(Smooth projection formula) The canonical transformation

$$p_\sharp(p^*(-) \otimes -) \to - \otimes p_\sharp(-)$$

is an equivalence.

(2) **(Right)** For every $X \in \mathrm{Sch}_B$ and every $f : Y \to X$:

(Internal hom) The symmetric monoidal structure on $C(X)$ is closed.

(Push-forward) The pull-back functor f^* admits a right adjoint $f_* : C(Y) \to C(X)$.

(3) **(Localization)** The ∞-category $C(\varnothing) = 0$ is trivial. And for each closed immersion $i : Z \hookrightarrow X$ in Sch_B with complementary open immersion $j : U \hookrightarrow X$, the square (see Remark 2.6 below)

$$\begin{array}{ccc} C(Z) & \xrightarrow{i_*} & C(X) \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ 0 & \longrightarrow & C(U) \end{array} \tag{2.5}$$

is Cartesian in $\mathrm{Cat}_\infty^{\mathrm{st}}$.

(4) For each $X \in \mathrm{Sch}_B$, if $p : \mathbb{A}^1_X \to X$ denotes the canonical projection with zero section $s : X \to \mathbb{A}^1_X$, then:

($\mathbb{A}^1$-homotopy) The functor $p^* : C(X) \to C(\mathbb{A}^1_X)$ is fully faithful.

(Tate stability) The composite $p_\sharp s_* : C(X) \to C(X)$ is an equivalence.

Remark 2.6. Let us comment on these axioms and relate them to what we've seen in Section 1.

(1) The existence of left adjoints $f_\sharp$ to inverse images along smooth morphisms is a consequence of relative purity, and we also discussed (Smooth projection formula) in this context. The (Smooth base change) is another fundamental property although typically formulated as base change of inverse images (along smooth morphisms) against *direct images*. It was shown in Exercise 1.62 that these two formulations are equivalent.

(2) The structure of a coefficient system encodes only inverse images and tensor products. The axiom (Right) ensures that direct images and internal homs exist as well.

(3) By applying (Smooth base change) to the Cartesian square (for i, j as in (Localization))

$$\begin{array}{ccc} \varnothing & \xrightarrow{j'} & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{j} & X \end{array}$$

we obtain the equivalence $j'_\sharp(i')^* \xrightarrow{\sim} i^* j_\sharp$ and the former composite is null-homotopic since $C(\varnothing) = 0$. Taking right adjoints provides the homotopy $j^* i_* \simeq 0$ that is used in (2.5).[10] In the presence of (Push-forward), the square (2.5) is

[10] I'm grateful to Ryomei Iwasa for pointing out that an explanation of the axiom was required.

Cartesian if and only if the sequence of underlying triangulated categories

$$\mathrm{Ho}(C(Z)) \xrightarrow{i_*} \mathrm{Ho}(C(X)) \xrightarrow{j^*} \mathrm{Ho}(C(U))$$

is a localization sequence so we recover the condition discussed in Section 1E4.

(4) The functor p^* in ($\mathbb{A}^1$-homotopy) is fully faithful if and only if the counit $p_\sharp p^* \to \mathrm{id}$ is an equivalence. As observed in Example 1.37, $p_\sharp p^* = p_! p^!$, and

$$p_\sharp p^* F = p_\sharp(\mathbb{1} \otimes p^* F) \xrightarrow{\sim} p_\sharp \mathbb{1} \otimes F = p_\sharp p^* \mathbb{1} \otimes F$$

by (Smooth projection formula). From which one deduces that ($\mathbb{A}^1$-homotopy) is equivalent to the $\mathbb{A}^1$-homotopy property considered in Section 1 (namely that the homology of the affine line is trivial).

(5) Recall that the functor $p_\sharp s_*$ in (Tate stability) was denoted by $\{1\}$ in Section 1.

Example 2.7. All theories mentioned in Section 1 'should' be examples of coefficient systems. This has been established for some of them, partially for others. The only exception to that statement is the bounded derived category of coherent sheaves as usually conceived. It is not invariant with respect to the affine line, and it does not admit $\sharp$-functoriality. Nevertheless, there is work in this direction too; see [43].

In fact, all these examples fall into two important special cases:

Convention 2.8. A coefficient system C is *small* (resp. *presentable*) if the functor takes values in symmetric monoidal small (resp. presentable) ∞-categories (resp. and symmetric monoidal left-adjoint functors).

Remark 2.9. Stable presentable ∞-categories can be viewed as the ∞-categorical version of well-generated triangulated categories. They satisfy a convenient adjoint functor theorem: a functor between presentable ∞-categories is a left adjoint if and only if it preserves colimits. The ∞-category of presentable ∞-categories and left adjoint functors is denoted by Pr^{L}. It is antiequivalent to the ∞-category of presentable ∞-categories and *right* adjoint functors, denoted Pr^{R}.

It follows that a functor $\mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\otimes}$ automatically satisfies (Right).[11]

Definition 2.10. (1) Let $C, D : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ be two coefficient systems. A natural transformation $\phi : C \to D$ is a *morphism of coefficient systems* if, for each smooth morphism f of B-schemes, the induced transformation $f_\sharp \phi \to \phi f_\sharp$ is an equivalence.

[11] By convention, symmetric monoidal presentable ∞-categories are *presentably symmetric monoidal*, that is, the tensor product commutes with colimits in each variable separately. (A better way of saying this is $\mathrm{Pr}^{\mathrm{L},\otimes} = \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ for a suitable symmetric monoidal structure on Pr^{L}, namely the Lurie tensor product.)

(2) We can then define the ∞-category of coefficient systems (over B) as a sub-∞-category of the functor category:

$$\mathrm{CoSy}_B \subseteq \mathrm{Fun}(\mathrm{Sch}_B^{\mathrm{op}}, \mathrm{Cat}_\infty^{\mathrm{st},\otimes}).$$

One has obvious variants for small and presentable coefficient systems, denoted $\mathrm{CoSy}_\mathrm{B}^{\mathrm{sm}}$ and $\mathrm{CoSy}_\mathrm{B}^{\mathrm{Pr}}$, respectively.

Exercise 2.11. Let $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$ satisfying both (Smooth base change) and (Push-forward). Show that the following are equivalent:

- (i) C satisfies (Localization).
- (ii) C satisfies the following three conditions:
  - (1) $C(\varnothing) = 0$.
  - (2) For each closed immersion i, the functor i_* is fully faithful.
  - (3) If j denotes the open immersion complementary to a closed immersion i then the pair (i^*, j^*) is conservative.

2B. ***Main result.*** The main result comes in two parts: The first wants to say that

coefficient systems underlie six-functor formalisms,

and the second wants to say that

morphisms of coefficient systems underlie morphisms of six-functor formalisms.

We refer to Remark 2.17 for the fine print.

Remark 2.12. If T is a small stable ∞-category then there is an associated presentable stable ∞-category $\mathrm{Ind}(T)$, its *Ind-completion*. As we will discuss in more detail below (Proposition 2.26), this process turns a small coefficient system into a presentable one. So while the main results in this section are stated in the presentable context, they are equally true for small, and therefore for 'all', coefficient systems (see Corollary 2.31).

Theorem 2.13. *Let C be a presentable coefficient system over B. Then there are functors (which are equal on objects)*

$$C = C^* : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}, \quad C_! : \mathrm{Sch}_B \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}},$$

with global right adjoints

$$C_* : \mathrm{Sch}_B \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{R}}, \quad C^! : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{R}},$$

and for each morphism f of B-schemes, a transformation $f_! := C_!(f) \to C_*(f) =: f_*$ which is invertible when f is proper, satisfying the projection formulae, smooth and proper base change, relative purity and 'the rest'.

Remark 2.14. We refer to [1, Scholie 1.4.2] or [11, Theorem 2.4.50] for more extensive (but still incomplete) lists of properties. Notably not included in this list is everything on duality, for which see Remark 2.17 and Section 2D3. Some aspects of the proof of Theorem 2.13 will be discussed in Section 3.

Theorem 2.15. *Let $\phi : C \to D$ be a morphism of presentable coefficient systems. Then there are natural transformations*

$$f^*\phi \xrightarrow{\sim} \phi f^*, \quad \phi(-)\otimes\phi(-) \xrightarrow{\sim} \phi((-)\otimes(-)), \qquad f_!\phi \xrightarrow{\sim} \phi f_!,$$
$$\phi f_* \to f_*\phi, \qquad \phi\underline{\mathrm{Hom}}(-,-) \to \underline{\mathrm{Hom}}(\phi(-),\phi(-)), \quad \phi f^! \to f^!\phi,$$

the first three of which are always equivalences, and the last three of which are so 'in good cases'.

Remark 2.16. For example, ϕ commutes with direct image along *proper* morphisms, and with exceptional inverse image along *smooth* morphisms. Further 'good cases' will be discussed in Section 2D3.

Remark 2.17. Theorems 2.13 and 2.15 are our main justification for viewing coefficient systems as a stand-in for six-functor formalisms. Let us repeat the caveats already alluded to:

(1) The results are on the face of it about presentable coefficient systems only. But analogous statements can be deduced for small coefficient systems (Corollary 2.31). And all known examples are either small or presentable.

(2) Theorem 2.13 does not say anything about duality, an important topic in the context of six-functor formalisms (as briefly discussed in Section 1). This is a consequence of our goal to be as encompassing as possible. For general coefficient systems, duality cannot be expected unless one restricts to coefficients that are 'small' in a certain sense. This will be taken up again in our short discussion of constructibility (Section 2D3).

(3) The last caveat is related. Namely, Theorem 2.15 does not quite say that a morphism of coefficient systems 'commutes' with the six operations. However, in good cases it does so when restricted to 'constructible coefficients'; see Section 2D3.

Remark 2.18. It is clear that in Theorem 2.13 the extension of a coefficient system $C = C^*$ to $C_!$ is essentially unique (see, for details, Section 3A). The importance of the axioms (Localization) and ($\mathbb{A}^1$-homotopy) in *constructing* the exceptional functoriality was first observed by Voevodsky and was formalized in his notion of *cross-functors* [15]. A version of Theorem 2.13 was first proved by Ayoub [1]. He worked at the level of triangulated categories and restricted to quasiprojective morphisms. The latter restriction was removed by Cisinski and Déglise [11] albeit with an additional axiom. The homotopy-theoretic difficulties in lifting these results to ∞-categories were addressed by another host of mathematicians, including Liu and Zheng [36; 37], Robalo [42] and Khan [31].

Many consequences can be deduced from Theorems 2.13 and 2.15. We refer to [1; 2; 11] for comprehensive treatments. As an example, we mention the following result. It can be viewed as a distillation of the properties discussed in Section 1E5 and in the language of ∞-categories it becomes arguably even more powerful.

Corollary 2.19. *Let C be a presentable coefficient system. The underlying functor*

$$C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

is a cdh-sheaf.

Remark 2.20. For the precise meaning of this statement for general topologies we refer to [18, §2] or [6, Definition 2.3.1]. In the particular case of the cdh-topology, there is a very convenient criterion, however, for which see the proof below.

In practice, this means that C can be studied locally for the cdh-topology. For example, if $B = \mathrm{Spec}(k)$ with k a characteristic-zero field, then C is uniquely determined by its restriction to Sm_k, the category of smooth k-varieties.

Proof. By (Localization), the ∞-category $C(\varnothing)$ is final. It then remains to check that C takes distinguished Nisnevich (resp. abstract blow-up) squares to Cartesian squares in Cat_∞. This amounts essentially to the existence of Nisnevich–Mayer–Vietoris triangles and (abstract) blow-up triangles, which we deduced in Section 1E5 from localization and smooth and proper base change. All of these hold in C, by Theorem 2.13.

For similar proofs, see [11, §3.3.a–b; 26, §6.3]. □

2C. *Other approaches.* The framework of coefficient systems is closely related to others in the literature. Let us summarize some of these relations, without trying to be exhaustive.

Remark 2.21. (1) A functor $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ is a coefficient system if and only if passing to homotopy categories gives a *closed symmetric monoidal stable homotopy* 2*-functor* $\mathrm{Ho}(C) : \mathrm{Sch}_B^{\mathrm{op}} \to \mathsf{TrCat}^\otimes$ in the sense of [1]. Similarly, a natural transformation $\phi : C \to D$ between coefficient systems is a morphism of coefficient systems if and only if passing to homotopy categories produces a morphism of symmetric monoidal stable homotopy 2-functors.

(2) A functor $C : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ is a coefficient system if and only if passing to homotopy categories gives a *motivic triangulated category* $\mathrm{Ho}(C) : \mathrm{Sch}_B^{\mathrm{op}} \to \mathsf{TrCat}^\otimes$ in the sense of [11]. This is not completely obvious since in loc. cit. an additional axiom (Adj) is assumed. It follows from Theorem 2.13 that this axiom is automatic.

(3) It follows from this last observation that presentable coefficient systems also have been considered before, under the name of *motivic categories of coefficients* [31].

(4) Ultimately, a more complete and thus satisfying framework for six-functor formalisms might be provided by the technology of [21], using the $(\infty, 2)$*-category of correspondences*. However, some of this technology rests on assumptions that are — as far as we are aware — not yet verified in the literature.[12]

2D. *Internal structure of framework.* From a bird's-eye view, the framework of coefficient systems consists of cohomology theories and their manifold relations. For example, Grothendieck's comparison isomorphism between algebraic de Rham cohomology and Betti cohomology should be reflected in an isomorphism of coefficient systems over $\mathrm{Spec}(\mathbb{C})$,

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}} \otimes \mathbb{C} \simeq \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}},$$

that is, by an enhanced version of the Riemann–Hilbert correspondence between (derived) constructible sheaves and regular holonomic $\mathscr{D}$-modules (cf. Remark 1.24). In particular, extending scalars at the level of cohomology groups is thus reflected by an operation at the level of coefficient systems.

This and many more phenomena should, in other words, be reflected in a rich internal structure of the ∞-category CoSy. We will be able to provide just a glimpse of this structure if only because mathematicians have barely started to investigate it systematically.

2D1. *Initial object.* Let's say we wanted to construct the 'universal' coefficient system, that is, the initial object of CoSy. We would probably start with the initial required structure and then try to freely enforce the axioms of coefficient systems one by one. As we will see, this can in fact be done, more or less, and the resulting coefficient system turns out to be SH, (stable) motivic homotopy theory!

Remark 2.22. One might find this result remarkable. Without mentioning SH in the definition, the ∞-category of coefficient systems knows about it in a strong sense. It is probably less remarkable once one remembers that the approach to six-functor formalisms axiomatized in the notion of coefficient systems goes back to Voevodsky's study of the functoriality of $\mathsf{SH}(X)$ in X.

We now put this into practice, trying to construct the universal coefficient system. For more details and generalizations, see [19].

Construction 2.23. The construction proceeds in several steps.

[12] In any case, presentable coefficient systems should extend uniquely to this framework; see [31, §4.2].

(1) Coefficient systems encode the $()^*$- and $\otimes$-structure. The initial (as well as final) functor doing so is

$$* : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$$

that sends every scheme to the final category with the only possible symmetric monoidal structure.[13]

(2) This is not the initial coefficient system because morphisms of coefficient systems are required to commute with $\sharp$-push-forwards. For example, given a coefficient system C and smooth morphism $p : P \to X$, we have an object

$$H_\bullet(P) := p_\sharp \mathbb{1} \in C(X)$$

and by (Smooth base change) and (Smooth projection formula) we have canonical equivalences

$$H_\bullet(P) \otimes H_\bullet(P') \cong H_\bullet(P \times_X P')$$

in $C(X)$. With some work one can show that

$$H_\bullet : \mathrm{Sm}_X \to C(X)$$

defines a functor, symmetric monoidal with respect to the Cartesian structure on smooth X-schemes, and this suggests that the functor

$$X \mapsto (\mathrm{Sm}_X)^\times \in \mathrm{Cat}_\infty^\otimes$$

is worth a closer look. In fact, with more work one can show that it is the initial functor satisfying (Left).

(3) Passing to the next axiom we see that (Right) is not satisfied by this functor. The only way we know of producing right adjoints in this context is to pass to presentable ∞-categories in order to invoke adjoint functor theorems. Thus,

$$X \mapsto \mathcal{P}(\mathrm{Sm}_X),$$

the category of presheaves on Sm_X with the pointwise symmetric monoidal structure (which is the Day convolution in this case). This forces us to work in the context of presentable ∞-categories from now on though. (Or at least ∞-categories admitting small colimits.)

(4) It is unclear how one would go about freely enforcing (Localization). On the other hand, the axiom seems to be saying that many questions about a coefficient system can be studied locally for the Zariski topology. And indeed, we saw that it plays an integral role in proving cdh-descent (Corollary 2.19). This suggests that we could get some way towards the axiom by restricting to sheaves for the

[13] Note that stability is a *condition* in the ∞-categorical world so we will not restrict to stable ∞-categories initially but rather enforce it eventually. (In fact, it will come for free.)

cdh-topology. And this 'works' except for the fact that the cdh topology isn't a very natural topology on *smooth* schemes.[14] It turns out that the Nisnevich topology, lying between the Zariski and the cdh-topology, works even better and eventually gives the 'same' result:

$$X \mapsto \mathsf{L}_{\mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_X).$$

Here we write $\mathsf{L}_{\mathrm{Nis}}$ for the (accessible) localization of presentable ∞-categories with respect to Nisnevich–Čech covers.

(5) Enforcing the next two axioms, ($\mathbb{A}^1$-homotopy) and (Tate stability), seems comparatively straightforward: we formally invert the canonical projection $\mathbb{A}^1_P \to P$ for each smooth $P \to X$, and we formally $\otimes$-invert the cofiber of $P \xrightarrow{\infty} \mathbb{P}^1_P$. To make sense of the cofiber it is necessary to pass freely to pointed ∞-categories.[15] This doesn't bother us in the least since in the end we want to end up in stable ∞-categories anyway. Thus we set

$$X \mapsto (\mathsf{L}_{\mathbb{A}^1 \cup \mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_X)_\bullet)[(\mathbb{P}^1, \infty)^{\otimes -1}].$$

The ∞-category on the right is $\mathsf{SH}(X)$, the *stable motivic* (*or* $\mathbb{A}^1$-)*homotopy category* on X. It is a presentable symmetric monoidal ∞-category. Note that since $(\mathbb{P}^1, \infty) = (\mathbb{G}_{\mathrm{m}}, 1) \otimes S^1$ (see Exercise 1.45), the ∞-category is automatically stable.

Remark 2.24. It follows that the resulting functor $\mathsf{SH} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L},\otimes}_{\mathrm{st}}$ has the required shape and it remains to verify the axioms of a coefficient system. All of them are formal except for (Localization). The latter is proved by Morel and Voevodsky [39] under the name of *the glueing theorem*.

Remark 2.25. Summarizing, there are at least three ways of thinking about stable motivic homotopy theory:

(a) Explicitly, the ∞-category $\mathsf{SH}(X)$ can be constructed as

$$(\mathsf{L}_{\mathbb{A}^1 \cup \mathrm{Nis}}\mathcal{P}(\mathrm{Sm}_X)_\bullet)[(\mathbb{P}^1, \infty)^{\otimes -1}].$$

(b) Robalo [42] shows that this symmetric monoidal presentable ∞-category also admits a characterization: any $\otimes$-functor $\mathrm{Sm}_X \to D$ into a stable presentable ∞-category factors uniquely through $\mathsf{SH}(X)$ as soon as it satisfies Nisnevich-excision, $\mathbb{A}^1$-invariance, and $\otimes$-inverts $(\mathbb{P}^1, \infty)$.

(c) By the discussion above [19], the coefficient system SH is the initial object of $\mathrm{CoSy}^{\mathrm{Pr}}$.

This ties back to Section 1A: While the second point concerns cohomology theories (at the set-level), the third point is the exact analogue at the category-level and concerns six-functor formalisms.

[14] See [32] for how to circumvent this problem.

[15] There are also subtle technical difficulties related to the $\otimes$-structure for which we refer to [42].

2D2. *Ind-completion.* Since Theorems 2.13 and 2.15 apply to presentable coefficient systems only, but many of the coefficient systems considered in Section 1 are small, it is very useful to have a process that takes a small coefficient system and outputs a presentable one. This process is simply Ind-completion (see Remark 2.12).

Proposition 2.26. *There is a functor*

$$\mathrm{Ind} : \mathrm{CoSy}_{\mathrm{B}}^{\mathrm{sm}} \to \mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}},$$

which takes a small coefficient system C to the functor $X \mapsto \mathrm{Ind}(C(X))$, the target being endowed with the Day convolution product. □

Exercise 2.27. Prove this result. (A useful fact is that if $f \dashv g$ is an adjunction between small stable ∞-categories, then their unique colimit-preserving extensions $\mathrm{Ind}(f) \dashv \mathrm{Ind}(g)$ again form an adjunction between their Ind-completions.)

Remark 2.28. So, given a small coefficient system C, we apply Theorem 2.13 to $\mathrm{Ind}(C)$ and obtain the full six operations on the system of ∞-categories $\mathrm{Ind}(C(X))$. However, at this point we do not know whether the exceptional functoriality restricts to the subsystem $C(X) \subset \mathrm{Ind}(C(X))$. It turns out that it does and the proof is not difficult.

Lemma 2.29. *Let $f : X \to Y$ be a morphism of B-schemes, let $M \in C(X)$ and $N \in C(Y)$. Then*

$$f_! M \in C(Y), \quad f^! N \in C(X).$$

Proof sketch. For the first statement we factor f as an open immersion followed by a proper morphism and reduce to proving each case separately. In the latter case we have $f_! M = f_* M \in C(Y)$ and we win. In the former we have $f_! M = f_\sharp M \in C(Y)$ and we win again.

For the second statement we use (Localization) to show that an object $L \in \mathrm{Ind}(C(X))$ belongs to $C(X)$ if (and only if) $L|_{U_i} \in C(U_i)$ for some open cover (U_i) of X. In other words, the question is local on X. In particular, we can assume that f is quasiprojective, and factor it as a closed immersion followed by a smooth morphism. The first case then follows from (Localization) and the second case follows from relative purity. □

Exercise 2.30. Fill in the details of this proof sketch.

Corollary 2.31. *Theorems 2.13 and 2.15 admit analogues for small coefficient systems.*[16]

[16] Of course, in this case the four functors $()^*, ()_*, ()_!, ()^!$ take values in *small* stable ∞-categories.

2D3. *Constructibility.* Let C be a coefficient system.

Convention 2.32. Denote by $C^{\mathrm{gm}}(X) \subset C(X)$ the smallest full sub-∞-category that

(1) contains $f_\sharp \mathbb{1}\{n\}$ for $f : Y \to X$ smooth, $n \in \mathbb{Z}$, and

(2) is stable and closed under direct factors (we call such subcategories *thick*).

This defines the subfunctor $C^{\mathrm{gm}} \subseteq C$ *of geometric origin* (see Lemma 2.33).

Lemma 2.33. *$C^{\mathrm{gm}} \subseteq C : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ is a subfunctor and the inclusion $C^{\mathrm{gm}} \to C$ commutes with $f_\sharp$ for f smooth.*

Proof. This follows immediately from the axioms (Smooth base change) and (Smooth projection formula). □

Example 2.34. (1) For $C = \mathsf{SH}$, the geometric part coincides with the compact part: the objects of $\mathsf{SH}^{\mathrm{gm}}(X)$ are precisely the compact objects in $\mathsf{SH}(X)$.[17] Also, SH is compactly generated so that $\mathrm{Ind}(\mathsf{SH}^{\mathrm{gm}}) = \mathsf{SH}$.

(2) The same is true for $C = \mathsf{DM}_{\mathrm{B}}$, Beilinson motives in the sense of [11] (or rather, their ∞-categorical enhancement). That is, $\mathsf{DM}_{\mathrm{B}}^{\mathrm{gm}}$ is the compact part and $\mathrm{Ind}(\mathsf{DM}_{\mathrm{B}}^{\mathrm{gm}}) = \mathsf{DM}_{\mathrm{B}}$. If k is a field there is a canonical equivalence

$$\mathsf{DM}_{\mathrm{B}}^{\mathrm{gm}}(\mathrm{Spec}(k)) = \mathsf{DM}^{\mathrm{gm}}(k; \mathbb{Q})$$

with (the ∞-categorical enhancement of) Voevodsky's category of geometric motives with rational coefficients [48].

(3) In the ℓ-adic setting, 'of geometric origin' is close to 'bounded-constructible'; see [10].

Remark 2.35. Let C be a presentable (or just cocomplete) coefficient system. By Section 2D1, there is a unique morphism of coefficient systems $\mathsf{SH} \to C$, and $C^{\mathrm{gm}} \subseteq C$ is exactly the thick subfunctor generated by the image of $\mathsf{SH}^{\mathrm{gm}}$.

We should now address the question whether C^{gm} is a coefficient system as well. We will not state sufficient conditions here and refer to the literature instead:

Theorem 2.36 [3, §3; 11, §4.2]. *In 'good cases', C^{gm} is a coefficient system.*

Example 2.37. An example to which the theorem applies is Beilinson motives (Example 2.34). In particular one obtains, in this case, a very satisfying picture

[17] Recall that in an ∞-category $\mathscr{C}$ with filtered colimits, an object M is compact if $\mathrm{Map}_{\mathscr{C}}(M, -) : \mathscr{C} \to \mathscr{S}$ to the ∞-category of spaces preserves filtered colimits. If $\mathscr{C}$ is a stable ∞-category, this can be tested at the level of homotopy categories and is equivalent to M being compact in the sense of triangulated categories: $\hom_{\mathrm{Ho}(\mathscr{C})}(M, -)$ preserves direct sums.

translating between small and compactly generated coefficient systems:

$$\mathrm{CoSy}^{\mathrm{sm}} \ni \mathrm{DM}^{\mathrm{gm}}_{\text{Б}} \underset{(-)^{\mathrm{gm}}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathrm{DM}_{\text{Б}} \in \mathrm{CoSy}^{\mathrm{Pr}}$$

Corollary 2.38. *In the same 'good cases' assume $\phi : C \to D$ is a morphism of coefficient systems. Then*

$$\phi|_{C^{\mathrm{gm}}} : C^{\mathrm{gm}} \to D$$

commutes with all six functors.

Remark 2.39. This improves on Theorem 2.15 in 'good cases'.

Remark 2.40. There is a more general notion of *constructibility* for coefficient systems over B. Instead of the generating set $\{f_\sharp \mathbb{1}\{n\}\}$ one may consider the set $\{f_\sharp p^* F\}$ where F runs through a specified set of coefficients on the base B, $f : Y \to X$ is smooth, and $p : Y \to B$ is the structure morphism. We recover the geometric part by allowing only Tate twists as coefficients on B.

The more general notion is useful in the study of duality phenomena, one of the topics of Section 1 which wasn't addressed by the notion of coefficient systems alone. We refer again to [3, §3; 11, §4.2] for in-depth discussions.

2D4. *Miscellanea.* Many other topics could be discussed in the framework of coefficient systems, for example:

(1) We saw in Corollary 2.19 that coefficient systems satisfy cdh-descent. Some of them satisfy *descent* with respect to stronger topologies, however, such as étale descent (and therefore eh-descent) or h-descent [11, §3]. This can be useful in extending coefficient systems from schemes to algebraic stacks via an atlas, say.

(2) It makes sense to consider *linear* coefficient systems and scalar extension. For example, in some cases being $\mathbb{Q}$-linear implies h-descent [11, §3.3.d]. A general discussion of scalar extension can be found in [18, §8], and we will discuss one application of this technique in Section 3B.

(3) *Orientable* coefficient systems are somewhat simpler to work with in the sense that 'all Thom twists are Tate twists' (see Example 1.34 and [11, §2.4.c]).

In these and many other cases there should be corresponding initial objects (similarly to Section 2D1).

Let us mention just two instances of possibly more surprising phenomena. As remarked at the beginning of this section, clearly, a lot remains to be explored!

Example 2.41. There is a functor

$$\exp : \mathrm{CoSy}_B \to \mathrm{CoSy}_B$$

which 'exponentiates' a coefficient system, and whose study we initiated in [22]. When applied to $\mathsf{DM}_{\mathbb{B}}$ it produces a new coefficient system $\mathsf{DM}_{\mathbb{B}}^{\exp}$ that should enhance Fresán and Jossen's theory of exponential motives [20].[18] And when applied to mixed Hodge modules, it should produce an enhancement of Kontsevich and Soibelman's exponential mixed Hodge structures [34]. An interesting aspect of this construction is that every exponentiated coefficient system comes with an additional 'seventh' operation, the Fourier transform familiar from the ℓ-adic theory as well as $\mathscr{D}$-modules (see, for example, [29; 30]).

Example 2.42. Let $\mathbb{F}_q$ be a finite field and choose an algebraic closure $\mathbb{F}$. If C is a coefficient system on $\mathbb{F}$-schemes, one can define a functor

$$C^{\mathrm{W}} : \mathrm{Sch}_{\mathbb{F}_q}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$$

by the formula, for any $\mathbb{F}_q$-scheme X,

$$C^{\mathrm{W}}(X) = \lim\big(C(X \times_{\mathbb{F}_q} \mathbb{F}) \underset{\mathrm{id}}{\overset{\mathrm{Fr}}{\rightrightarrows}} C(X \times_{\mathbb{F}_q} \mathbb{F})\big),$$

where Fr denotes the q-Frobenius on X and the limit is taken in $\mathrm{Cat}_\infty^{\mathrm{st},\otimes}$. The superscript is in honor of Weil since in the case of ℓ-adic cohomology, the ∞-category $C^{\mathrm{W}}(X)$ can be seen as a derived category of Weil sheaves [25]. With some work (see Exercise 2.43 below) one shows that this underlies a functor

$$(-)^{\mathrm{W}} : \mathrm{CoSy}_{\mathrm{Spec}(\mathbb{F})} \to \mathrm{CoSy}_{\mathrm{Spec}(\mathbb{F}_q)}.$$ [19]

Exercise 2.43. The goal of this extended exercise is to prove C^{W} of Example 2.42 is a coefficient system. This can be done as follows:

(1) Let $\omega : C \to D$ be a natural transformation of functors $\mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$ and assume that

(1.1) D is a coefficient system,

(1.2) C admits left adjoints $p_\sharp$ for smooth morphisms p, and ω commutes with them,

(1.3) $\omega_X : C(X) \to D(X)$ is conservative for each $X \in \mathrm{Sch}_B^{\mathrm{op}}$.

Show the functor C satisfies (Smooth base change), (Smooth projection formula), ($\mathbb{A}^1$-homotopy) as well as items (1) and (3) of Exercise 2.11.

[18] More precisely, $\mathsf{DM}_{\mathbb{B}}^{\exp}(k)$ bears to their theory the same relation as $\mathsf{DM}_{\mathbb{B}}(k)$ to Nori motives, for $k \subseteq \mathbb{C}$ a field.

[19] This example was brought to my attention by Joshua Lieber.

(2) Assume in addition that C satisfies (Right) and that ω commutes with f_* for all immersions f. Then C also satisfies (Localization) and (Tate stability), hence is a coefficient system.

(3) Use the previous point to show that C^{W} of Example 2.42 is a coefficient system. *Hint*: for any diagram $F : I \to \mathrm{Cat}_\infty^{\mathrm{st}}$ the canonical functor $\lim_I F \to \prod_{i \in I_0} F(i)$ is conservative.

3. How?

The question alluded to in the title can be understood in at least two ways:

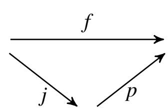
(A) How to construct six-functor formalisms in general?

(B) How to obtain six-functor formalisms from coefficient systems? That is, how to prove Theorem 2.13?

The two are related. Often the $\otimes$-structure and $*$-functoriality is produced without much effort and it is the !-functoriality that poses the most serious difficulties. Below we will focus on the problem of constructing exceptional direct and inverse images, and we will refer to the literature for the problem of proving the expected properties.

3A. ***Exceptional functoriality for coefficient systems.*** We start with question (B) and for this we want to follow the strategy employed by Deligne to produce exceptional functoriality in ℓ-adic cohomology [45, §XVII.3, 5.1]. As we will see, working in the generality we do, additional difficulties arise that need to be addressed.

Remark 3.1. Let f be a morphism of B-schemes. Since f is separated and of finite type (Convention 2.1), we may use Nagata compactification to find a factorization



(3.2)

with j an open immersion and p a proper morphism. We would then like to set

$$f_! := p_* j_\sharp$$

but this definition poses several difficulties:

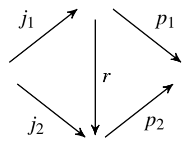
(1) *Well-definedness*: Is it ‘independent’ of the factorization?

(2) *Right-adjoint*: Why is there a right adjoint $f^!$?

(3) *Functoriality*: In what sense is it functorial in f?

We will address each of these difficulties in turn.

3A1. *Well-definedness.* Consider the category $\mathrm{Comp}(f)$ of compactifications of f: its objects are factorizations as in (3.2), with morphisms (necessarily proper) making the obvious diagram commute:

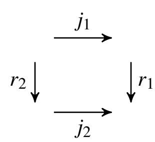


The category $\mathrm{Comp}(f)$ is easily seen to be cofiltered. In comparing $(p_1)_*(j_1)_\sharp$ with $(p_2)_*(j_2)_\sharp$ we may therefore assume a morphism r as above. We then find

$$(p_1)_*(j_1)_\sharp \simeq (p_2)_* r_* (j_1)_\sharp \overset{!}{\simeq} (p_2)_*(j_2)_\sharp,$$

where the last identification would follow if we could 'commute' $j_\sharp$'s with p_*'s. This is known as the *support property*:

(Support) Given a Cartesian square



with r_1 proper and j_2 an open immersion, the induced transformation $(j_2)_\sharp(r_2)_* \to (r_1)_*(j_1)_\sharp$ is an equivalence.

Exercise 3.3. Show that (Proper base change) $\Rightarrow$ (Support).

Remark 3.4. As a consequence of Exercise 3.3 it is natural to try to establish (Proper base change). We sketch the main ideas that go into deducing it from the axioms of a coefficient system. The same strategy will be employed in Section 3A2.

(1) Recall that we are given (1.59) with f proper, and would like to show the transformation $g^* f_* \to k_* h^*$ from (1.60) to be an equivalence. By (Localization) and Chow's lemma we reduce to f projective, $f = pi$ where i is a closed immersion and $p : \mathbb{P}^d_Y \to Y$ is the canonical projection. This reduction step is written out in detail in [6, 4.1.1.(1)].

(2) The case of i_* ('closed base change') follows easily from (Localization) so we further reduce to the case of p_*.

(3) Hence, in addition to being projective, p is also smooth of relative dimension d so we expect to observe Atiyah duality (Exercise 1.31). In other words, one ought to be able to show a canonical equivalence

$$p_* \simeq p_\sharp\{-T_p\}. \tag{3.5}$$

Although this is a rather explicit problem, the proof is long and involved. Moreover, constructing a candidate for the equivalence also involves proving some form of purity. We refer to [1, Théoréme 1.7.9] for details.

(4) Having established the equivalence (3.5), we are reduced to show that both $p_\sharp$ and $\{T_p\}$ 'commute with inverse images'. This is exactly (Smooth base change) and closed base change (recall Remark 1.38).

3A2. *Right-adjoint.* It is clear that the functor $j_\sharp$ admits a right adjoint, namely j^*. To show that p_* does as well (for p proper) we will use the adjoint functor theorem for presentable ∞-categories. In other words we will show that p_* preserves colimits. The advantage of this formulation of the problem is that it becomes amenable to the same attack as the one employed in proving (Proper base change) above: one reduces to projective and further to smooth projective morphisms and then obtains the identification (3.5). Both of the functors on the right are left adjoints and we conclude.

3A3. *Functoriality.* Well-definedness discussed in Section 3A1 is only one aspect of the problem that is posed by functoriality. Recall that we want to construct a functor $C_! : \mathrm{Sch}_B \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Deligne achieved this at the level of triangulated categories by setting, for $f : X \to Y$,

$$f_! := \varinjlim_{(p,j)\in\mathrm{Comp}(f)^{\mathrm{op}}} p_* j_\sharp, \tag{3.6}$$

using that $\mathrm{Comp}(f)$ is cofiltered and the functor

$$* \circ \sharp : \mathrm{Comp}(f)^{\mathrm{op}} \to \mathrm{Hom}(C(X), C(Y))$$

sends morphisms to isomorphisms, by (Support). But even constructing such a functor $* \circ \sharp$ is a daunting task in the context of ∞-categories as it would involve providing, in addition to the homotopies of (Support), homotopies between these and so on ad infinitum.

Remark 3.7. One solution to this homotopy theoretic problem was developed in [36], based on multisimplicial sets. It is very general but unfortunately rather complicated. We would like to describe a more elementary solution specific to the given problem. It is based on our recent collaboration with Ayoub and Vezzani [6].

Remark 3.8. The basic idea is very simple. Let $f : X \to Y$ be a morphism of B-schemes which admits a compactification $\bar{f}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \end{array} \tag{3.9}$$

Here, the square commutes, j and k are open immersions and $\bar{X}$ and $\bar{Y}$ are proper B-schemes. We obtain a diagram of solid arrows

$$\begin{array}{ccc} C(X) & \overset{f_!}{\dashrightarrow} & C(Y) \\ {\scriptstyle j_\sharp}\downarrow & & \downarrow{\scriptstyle k_\sharp} \\ C(\bar{X}) & \xrightarrow{\bar{f}_*} & C(\bar{Y}) \end{array}$$

and as $\bar{f}$ is proper we would like to define $f_!$ so that the square 'commutes'. Again, this is not a tenable strategy in the context of ∞-categories. Being commutative is not a property but a structure and we are back to the exact same issue as before.

However, we can avoid this issue with the following trick. Define a full sub-∞-category $C(X, \bar{X})_!$ of $C(\bar{X})$ as the essential image of $j_\sharp$, and similarly for k. (In particular, we have an equivalence $C(X) \simeq C(X, \bar{X})_!$.) (Support) implies that $\bar{f}_*$ restricts to a morphism $C(X, \bar{X})_! \to C(Y, \bar{Y})_!$. The gain is that the functor $\bar{f}_*$ is already part of a functor $C_* : \mathrm{Sch}_B \to \mathrm{Pr}^{\mathrm{R}}$ which encodes these higher homotopies.

After outlining the basic idea we can now summarize the construction of the functor $C_!$.

Construction 3.10. We will use the diagram

$$\begin{array}{ccc} & \mathrm{Comp}_B & \\ {\scriptstyle \omega}\swarrow & & \searrow{\scriptstyle \pi} \\ \mathrm{Sch}_B & & \mathrm{Sch}_B^{\mathrm{prop}} \end{array}$$

in which Comp_B denotes the category whose objects are pairs $(X, \bar{X})$ as above and whose morphisms are pairs $(f, \bar{f})$ as in (3.9). Forgetting $\bar{X}$ (resp. X) defines the functor ω (resp. π). (Here, $\mathrm{Sch}_B^{\mathrm{prop}}$ denotes the category of B-schemes and proper morphisms.)

Starting with C and passing to C_* as above we obtain the functor $C_* \circ \pi$: $\mathrm{Comp}_B \to \mathrm{Pr}^{\mathrm{R}}$ that informally can be described as

$$(X, \bar{X}) \mapsto C(\bar{X}), \quad (f, \bar{f}) \mapsto \bar{f}_*.$$

In fact, this functor takes values in Pr^{L} as well, by Section 3A2.

If $C(-,-)_! : \mathrm{Comp}_B \to \mathrm{Pr}^{\mathrm{L}}$ denotes the full subfunctor of $C_* \circ \pi$ considered in Remark 3.8 then we define

$$C_! := \mathrm{LKE}_\omega C(-,-)_! : \mathrm{Sch}_B \to \mathrm{Pr}^{\mathrm{L}},$$

the left-Kan extension along ω. This last step thus removes the dependency of the factorization in a similar way as in (3.6).

Remark 3.11. It is not difficult to prove that $C_!(f)$ recovers $p_* j_\sharp$ up to homotopy for any factorization as in (3.2). For details, we refer to [6, §4.3]. From there, one can go on and prove the expected properties of this six-functor formalism; see [1] or [11].

3B. ***Motivic coefficient systems.*** Once one has Theorem 2.13 at one's disposal, of course, question (A) becomes: how to construct coefficient systems? In this brief section we will describe an elegant and powerful procedure that has been employed in the literature to produce 'motivic' coefficient systems. This topic would have just as well fit in with Section 2D.

Remark 3.12. Let $C \in \mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}}$ be a coefficient system, say presentable to fix our ideas. As we saw in Section 2D1, there is an essentially unique morphism $\rho_C^* : \mathsf{SH} \to C$ from the initial object which can be viewed as the (homological) C-realization: For any B-scheme X, the functor $\rho_C^*(X) : \mathsf{SH}(X) \to C(X)$ sends a smooth X-scheme Y to its homology coefficient $H_\bullet(Y)$ in $C(X)$. This functor admits a right adjoint $\rho_*^C(X) : C(X) \to \mathsf{SH}(X)$ that has a canonical lax symmetric monoidal structure (since $\rho_C^*(X)$ underlies a symmetric monoidal functor). In particular we see that $\rho_*^C(B)\mathbb{1} \in \mathrm{CAlg}(\mathsf{SH}(B))$ is a motivic ring spectrum which we denote by $\mathscr{C}$.

This object represents C-cohomology in the sense that for any smooth B-scheme X, we have by adjunction

$$\pi_0 \mathrm{Map}_{\mathsf{SH}(B)}(X, \mathscr{C}(m)[n]) \simeq \pi_0 \mathrm{Map}_{C(B)}(H_\bullet(X), \mathbb{1}(m)[n]) \simeq H^n(X; \mathbb{1}(m)).$$

The observation we want to make now is that *every* motivic ring spectrum represents some cohomology theory.

Convention 3.13. Let $\mathscr{A} \in \mathrm{CAlg}(\mathsf{SH}(B))$ be a motivic ring spectrum and denote (abusively) by $\mathscr{A}_X := f^*\mathscr{A} \in \mathrm{CAlg}(\mathsf{SH}(X))$ its pull-back to any B-scheme $f : X \to B$. The association $X \mapsto \mathrm{Mod}_{\mathscr{A}_X}(\mathsf{SH}(X)) =: \mathsf{SH}(X; \mathscr{A})$ underlies a functor

$$\mathsf{SH}(-; \mathscr{A}) : \mathrm{Sch}_B^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\otimes} \tag{3.14}$$

that — in anticipation of the next theorem — we call the *motivic coefficient system represented by* $\mathscr{A}$.

Theorem 3.15. *The functor* $\mathsf{SH}(-; \mathscr{A})$ *of* (3.14) *is a presentable coefficient system and the canonical 'free functor'* $\rho_{\mathscr{A}}^* : \mathsf{SH} \to \mathsf{SH}(-; \mathscr{A})$ *is a morphism of presentable coefficient systems.*

A proof of this result can be found in [18, Theorem 8.10]; see also [11, §7.2, 17.1].

Remark 3.16. We may now combine the constructions of Remark 3.12 and Convention 3.13. That is, in the situation of Remark 3.12 we obtain a factorization of $\rho_C^* : \mathsf{SH} \to C$ through the *motivic coefficient system associated with C*:

$$\mathsf{SH} \xrightarrow{\rho_{\mathscr{C}}^*} \mathsf{SH}(-;\mathscr{C}) \xrightarrow{\tilde{\rho}_C^*} C.$$

The induced functor $\tilde{\rho}_C^*$ factors further through the localizing subfunctor $\tilde{C}$ of C generated by the part of geometric origin (Section 2D3). By a tilting argument, the resulting morphism

$$\tilde{\rho}_C^* : \mathsf{SH}(-;\mathscr{C}) \to \tilde{C}$$

is in fact an equivalence in some cases of interest; see [11, Theorem 17.1.5].

Remark 3.17. In summary, we have procedures which can be upgraded to functors:

$$\mathrm{CAlg}(\mathsf{SH}(B)) \underset{\rho_*(B)\mathbb{1}}{\overset{\mathsf{SH}(-;-)}{\rightleftarrows}} \mathrm{CoSy}_{\mathrm{B}}^{\mathrm{Pr}}$$

Example 3.18 [11, 17.1.7]. Consider the Betti realization functor

$$\rho_{\mathrm{B}}^* : \mathsf{SH}(\mathrm{Spec}(\mathbb{C})) \to \mathsf{D}(\mathbb{Q})$$

that sends a smooth complex scheme X to the rational singular chain complex $\mathrm{Sing}(X^{\mathrm{an}}) \otimes \mathbb{Q}$ on the underlying complex analytic space. It is naturally symmetric monoidal — in fact, it is part of a morphism of coefficient systems on complex schemes [3]:

$$\mathsf{SH} \to \mathsf{D}((-)^{\mathrm{an}};\mathbb{Q}).$$

The associated motivic ring spectrum $\mathscr{B} := \rho_*^B\mathbb{Q} \in \mathrm{CAlg}(\mathsf{SH}(\mathbb{C}))$ is the (*rational*) *Betti spectrum* that represents Betti cohomology. We will now describe the resulting coefficient system $\mathsf{SH}(-;\mathscr{B})$ more explicitly, following [5, §1.6].

First, observe that for a general complex scheme X, the functor

$$\tilde{\rho}_{\mathrm{B}}^*(X) : \mathsf{SH}(X;\mathscr{B}) \to \mathsf{D}(X^{\mathrm{an}};\mathbb{Q})$$

is far from an equivalence. Instead, it factors through

$$\mathsf{SH}(X;\mathscr{B}) \to \mathrm{Ind}(\mathsf{D}_{\mathrm{c}}^{\mathrm{b}}(X;\mathbb{Q})) \to \mathsf{D}(X^{\mathrm{an}};\mathbb{Q}),$$

where the second arrow is the colimit-preserving functor extending the identity on $\mathsf{D}_{\mathrm{c}}^{\mathrm{b}}(X;\mathbb{Q})$. The first functor in this factorization is in fact fully faithful, and the image is generated under colimits, desuspensions and truncations (with respect to the canonical t-structure) by sheaves of the form $f_*\mathbb{Q}$, where $f : Y \to X$ is proper; see [5, Theorem 1.93].

Example 3.19 [18]. Saito's derived categories of mixed Hodge modules do not, in an obvious way, admit an enhancement to a coefficient system. (As a result, the Hodge realization functors are not known to commute with the six functors on compact objects.) On the other hand, there is a Hodge realization functor

$$\rho_{\mathrm{H}}^* : \mathsf{SH}(\mathrm{Spec}(\mathbb{C})) \to \mathsf{D}(\mathrm{Ind}(\mathrm{MHS}_{\mathbb{Q}}^{\mathrm{p}}))$$

with values in the derived ∞-category of Ind-completed polarizable mixed Hodge structures over $\mathbb{Q}$. The associated motivic ring spectrum $\mathscr{H} := \rho_*^{\mathrm{H}}\mathbb{Q}(0)$ is the *absolute Hodge spectrum* that represents absolute Hodge cohomology. Drew calls the resulting coefficient system $\mathsf{SH}(-;\mathscr{H})$ *motivic Hodge modules*, and they satisfy many of the properties expected of a coefficient system that should capture mixed Hodge modules of geometric origin. In line with this, he conjectures that for each complex scheme X, the triangulated category of compact objects in $\mathrm{Ho}(\mathsf{SH}(X;\mathscr{H}))$ embeds fully faithfully into Saito's $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$.

Example 3.20 [46]. As in Example 3.19, until recently there was no known enhancement of Voevodsky's category of motives over a field, $\mathsf{DM}^{\mathrm{gm}}(\mathrm{Spec}(k);\mathbb{Z})$ to a coefficient system in mixed characteristic. (The situation was better understood with rational coefficients and/or in equal characteristic.) Spitzweck constructs a motivic ring spectrum $\mathscr{M} \in \mathsf{SH}(\mathrm{Spec}(\mathbb{Z}))$ that represents Bloch–Levine motivic cohomology and then defines

$$\mathsf{SH}(-;\mathscr{M}) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L},\otimes}$$

that can be seen as a coefficient system of integral motivic sheaves. Over a field k the compact part of $\mathsf{SH}(\mathrm{Spec}(k);\mathscr{M})$ is equivalent to $\mathsf{DM}^{\mathrm{gm}}(\mathrm{Spec}(k);\mathbb{Z})$, while with rational coefficients and for any scheme X one recovers Beilinson motives:

$$\mathsf{SH}(X;\mathscr{M}\otimes\mathbb{Q}) \simeq \mathsf{DM}_{\mathbb{B}}(X).$$

3C. *Exceptional functoriality for* RigSH. We have two goals for this last section. First, we want to say something regarding question (A) at the beginning of Section 3. And secondly, we want to give an example of a six-functor formalism outside the world of schemes (and topological spaces) that have dominated the discussion so far.

Remark 3.21. In the context of schemes, Theorem 2.13 provides a very useful criterion for recognizing six-functor formalisms. In contexts that are not too different from schemes one can hope to establish a similar criterion; see, for example, [33] for (certain) algebraic stacks. However, in general one shouldn't expect the axioms of coefficient systems — even if interpreted appropriately — to be sufficient to guarantee the existence of !-functoriality.

Rigid (or 'nonarchimedean') analytic geometry is arguably an example of a theory that is too different for a successful transfer. In the following pages we want to describe how a different kind of transfer allows one to construct !-functors (and prove the expected properties) on the 'universal' rigid-analytic theory, namely rigid-analytic stable motivic homotopy theory RigSH. This is a report on the work with Ayoub and Vezzani [6] already mentioned in Section 3A3.

Remark 3.22. Rigid-analytic geometry is the analogue of complex-analytic geometry over nonarchimedean fields, e.g., p-adic fields. The theory retains both algebraic and analytic aspects, and it has found many applications in arithmetic algebraic geometry, particularly in the wake of Scholze's work on perfectoid spaces and p-adic geometry.

Rigid-analytic spaces in the sense this term is used in [6] form a category RigSpc that encompasses both Tate's rigid-analytic varieties (and Berkovich spaces) as well as a large class of adic spaces (e.g., all 'stably uniform' ones [9]) in the sense of Huber. While ridding the treatment of unnecessary Noetherianity assumptions was a goal of [6], these technical details will not concern us in this short outline.

Remark 3.23. The construction of RigSH is originally due to Ayoub [4] and modeled on Morel and Voevodsky's construction of SH (see Section 2D1). In fact, the two are entirely parallel according to the 'dictionary'

$$\begin{aligned} \mathrm{Sch} &\leftrightsquigarrow \mathrm{RigSpc}, \\ \mathbb{A}^1 &\leftrightsquigarrow \mathbb{B}^1, \\ \mathbb{G}_{\mathrm{m}} &\leftrightsquigarrow \mathbb{T}. \end{aligned}$$

Here $\mathbb{B}^1$ is the closed unit ball and $\mathbb{T} \subset \mathbb{B}^1$ the annulus.

Unsurprisingly and in a completely parallel fashion, RigSH comes with a closed symmetric monoidal structure and $*$-functoriality. However, there is no analogue of Theorem 2.13 available, and Ayoub was able to construct the !-functoriality only for morphisms that arise as the analytification of algebraic morphisms (that is, those coming from Sch).[20] The original goal of [6] was to remedy this.

Remark 3.24. Let us explain why an analogue of Theorem 2.13 is not available and in fact might not be expected. Indeed, in following the strategy of Section 3A one encounters the following problems in the rigid-analytic world:

(a) The analogue of Exercise 3.3 does not hold (a priori), that is, proper base change does not imply the support property. The underlying reason is that while (Localization) holds for RigSH, it is of limited use since the complement of an open immersion is not, typically, a rigid-analytic space.

(b) At several places in Section 3A we used Chow's lemma to reduce questions about proper morphisms to projective ones. However, an analogue of Chow's lemma is not available in rigid-analytic geometry, thus making this strategy infeasible.

[20] In fact, he obtained this as an application of a version of Theorem 2.13.

Remark 3.25. On the other hand, morphisms locally of finite type between rigid-analytic spaces are still *weakly* compactifiable, at least locally. More precisely, every $f : X \to Y$ locally of finite type is, locally on X, the composition of a locally closed immersion followed by a proper morphism. Therefore, once one knows (Support) and the existence of right adjoints to proper push-forwards one can then follow essentially the same strategy in constructing the exceptional functoriality as in Section 3A3. The existence of the required right adjoints follows easily from the fact that the ∞-categories $\mathsf{RigSH}(X)$ are compactly generated and that the inverse image functors along proper morphisms preserve compact objects.

In the remainder of this section we will sketch how to prove (Support).

Remark 3.26. The proof still employs a transfer from algebraic to rigid-analytic geometry albeit in a very different way. It is based on Raynaud's approach to rigid-analytic geometry that can be roughly described by the picture

$$\begin{array}{ccc} & \mathrm{FSch} & \\ {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\rho} \\ \mathrm{Sch} & & \mathrm{RigSpc} \end{array} \tag{3.27}$$

Here, formal schemes sit at the top and admit two functors: the 'special fiber' that associates to $\mathscr{X}$ its underlying topological space with the reduced scheme structure $\sigma(X)$, and the 'generic fiber' ρ that is a categorical localization. More precisely, the category RigSpc is, as a first approximation, the localization of FSch with respect to so-called 'admissible blow-ups': blow-ups with center 'contained in the special fiber'. This approximation becomes correct if one imposes finiteness conditions on the formal schemes involved (adic with finitely generated ideals of definition) and if one allows rigid-analytic spaces to be glued along open immersions.

Remark 3.28. Passing to stable motivic homotopy theory in the three contexts in parallel gives rise to a roof like so,

$$\begin{array}{ccc} & \mathsf{FSH} & \\ {\scriptstyle\sigma^*}\swarrow\!\!\nearrow{\scriptstyle\sigma_*}\ (\sim) & & \searrow\!\!\nwarrow{\scriptstyle\rho^*,\ \rho_*} \\ \mathsf{SH} & \underset{\rho_*}{\overset{\rho^*}{\rightleftarrows}} & \mathsf{RigSH} \end{array}$$

where the components of the natural transformations $(-)^*$ are symmetric monoidal functors with right adjoints $(-)_*$, and where $\sigma^* \dashv \sigma_*$ is an adjoint equivalence, by localization for FSH. We continue to denote by ρ^* (resp. ρ_*) the functors at the bottom that make the triangle commute.

The basic idea in proving (Support) for RigSH is to apply ρ_* to the morphism $(j_2)_\sharp(r_2)_* \to (r_1)_*(j_1)_\sharp$ and use (Support) for SH to show it is an equivalence. This requires two inputs:

(a) The functor ρ_* needs to be sufficiently conservative. While it isn't on the nose, it is still true (and easy to prove) that the family (for fixed S)

$$\big(\mathsf{RigSH}(S) \xrightarrow{f^*} \mathsf{RigSH}(X) \xrightarrow{\rho_*^{\mathscr{X}}} \mathsf{SH}(\sigma(\mathscr{X}))\big)_{f,\mathscr{X}}$$

is jointly conservative, where $f : X \to S$ runs through smooth morphisms of rigid-analytic spaces and $\mathscr{X}$ is a chosen formal model of X.

(b) It is clear that f^* (resp. ρ_*) commutes with the $(j_i)_\sharp$ and the $(p_i)_*$ (resp. with the $(p_i)_*$) so it remains to prove that ρ_* commutes with the $(j_i)_\sharp$.

This last point turns out to be quite involved and required a systematic study of RigSH. We will not go into the details here and refer to [6, Theorem 4.1.3] instead. On the other hand, this systematic study leads to other results of independent interest which we do want to mention.

Theorem 3.29 [6, Theorem 3.3.3]. (1) *The components of the natural transformation* $\mathsf{SH}(\sigma(-), \rho_*\mathbb{1}) \to \mathsf{RigSH}(\rho(-))$ *are fully faithful.*

(2) *The natural transformation* $\mathsf{SH}^{\text{ét}}(\sigma(-), \rho_*\mathbb{Q}) \to \mathsf{RigSH}^{\text{ét}}(\rho(-), \mathbb{Q})$ *exhibits the latter as the rig-étale sheafification of the former.*

Here, the natural transformations in the statement are between $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$-valued functors on $\mathrm{RigSpc}^{\mathrm{op}}$ (viewed as having the same objects as FSch; see Remark 3.26). The notation $\mathsf{SH}(X, A)$ already employed in Section 3B is a shorthand for the ∞-category of A-modules, A being a commutative algebra object in $\mathsf{SH}(X)$. The first part of Theorem 3.29 can be read as saying that a whole chunk of RigSH admits a completely algebraic description. We call this chunk the part of good reduction and denote it by $\mathsf{RigSH}^{\mathrm{gr}}$. In fact, in good cases the commutative algebra $\rho_*\mathbb{1}$ can be computed. For example, over the p-adic integers $\rho_*^{\mathbb{Z}_p}\mathbb{1} \simeq H^\bullet(\mathbb{G}_{\mathrm{m}})$ and we deduce that

$$\mathsf{SH}^{\mathrm{uni}}(\mathbb{F}_p) \xrightarrow{\sim} \mathsf{RigSH}^{\mathrm{gr}}(\mathbb{Q}_p),$$

where the domain denotes the *unipotent motivic spectra*, that is, the localizing sub-∞-category of $\mathsf{SH}(\mathbb{G}_{\mathrm{m},\mathbb{F}_p})$ generated by the constant motivic spectra.

Finally, the second part of Theorem 3.29 gives a precise measure of the failure of all rigid-analytic motives to be of good reduction. In comparison to the first part, some additional hypotheses are necessary, for example étale-(hyper)sheafification and $\mathbb{Q}$-linearity are enough. All in all, Theorem 3.29 is a vast generalization of [4, Scholie 1.3.26.(1)] which inspired the strategy in the first place.

Acknowledgements

I would like to thank the organizers of the summer school and conference *The Six-Functor Formalism and Motivic Homotopy Theory* for the opportunity to speak about this subject and for their hard work in making the event a success. I would also like to thank all the participants of the summer school for their input during and after the lectures. Bastiaan Cnossen read a previous version of these notes and contributed many corrections and suggestions that greatly improved the document.

References

[1] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique*, *I*, Astérisque **314**, Soc. Math. France, Paris, 2007. MR Zbl

[2] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique*, *II*, Astérisque **315**, Soc. Math. France, Paris, 2007. MR Zbl

[3] J. Ayoub, "Note sur les opérations de Grothendieck et la réalisation de Betti", *J. Inst. Math. Jussieu* **9**:2 (2010), 225–263. MR Zbl

[4] J. Ayoub, *Motifs des variétés analytiques rigides*, Mém. Soc. Math. France (N.S.) **140-141**, Soc. Math. France, Paris, 2015. MR Zbl

[5] J. Ayoub, "Anabelian presentation of the motivic Galois group in characteristic zero", preprint, 2021, http://user.math.uzh.ch/ayoub/.

[6] J. Ayoub, M. Gallauer, and A. Vezzani, "The six-functor formalism for rigid analytic motives", *Forum Math. Sigma* **10** (2022), art. id. e61. MR Zbl

[7] T. Bachmann and M. Hoyois, *Norms in motivic homotopy theory*, Astérisque **425**, Soc. Math. France, Paris, 2021. MR Zbl

[8] A. A. Beilinson, J. Bernstein, and P. Deligne, "Faisceaux pervers", pp. 5–171 in *Analysis and topology on singular spaces*, *I* (Luminy, 1981), Astérisque **100**, Soc. Math. France, Paris, 1982. MR Zbl

[9] K. Buzzard and A. Verberkmoes, "Stably uniform affinoids are sheafy", *J. Reine Angew. Math.* **740** (2018), 25–39. MR Zbl

[10] D.-C. Cisinski and F. Déglise, "Étale motives", *Compos. Math.* **152**:3 (2016), 556–666. MR Zbl

[11] D.-C. Cisinski and F. Déglise, *Triangulated categories of mixed motives*, Springer, 2019. MR Zbl

[12] F. Déglise, "Bivariant theories in motivic stable homotopy", *Doc. Math.* **23** (2018), 997–1076. MR

[13] F. Déglise, F. Jin, and A. A. Khan, "Fundamental classes in motivic homotopy theory", *J. Eur. Math. Soc.* (*JEMS*) **23**:12 (2021), 3935–3993. MR Zbl

[14] P. Deligne, "Le déterminant de la cohomologie", pp. 93–177 in *Current trends in arithmetical algebraic geometry* (Arcata, CA, 1985), Contemp. Math. **67**, Amer. Math. Soc., Providence, RI, 1987. MR Zbl

[15] P. Deligne, "Voevodsky's lectures on cross functors", notes, 2001, https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/2015_transfer_from_ps_delnotes01.pdf.

[16] A. Dimca, *Sheaves in topology*, Springer, 2004. MR Zbl

[17] A. Dold and D. Puppe, "Duality, trace, and transfer", pp. 81–102 in *Proceedings of the International Conference on Geometric Topology* (Warsaw, 1978), PWN, Warsaw, 1980. MR Zbl

[18] B. Drew, "Motivic Hodge modules", preprint, 2018. arXiv 1801.10129

[19] B. Drew and M. Gallauer, "The universal six-functor formalism", *Ann. K-Theory* **7**:4 (2022), 599–649. MR Zbl

[20] J. Fresán and P. Jossen, "Exponential motives", notes, http://javier.fresan.perso.math.cnrs.fr/expmot.pdf.

[21] D. Gaitsgory and N. Rozenblyum, *A study in derived algebraic geometry, I: Correspondences and duality*, Mathematical Surveys and Monographs **221**, American Mathematical Society, Providence, RI, 2017. MR Zbl

[22] M. Gallauer and S. Pepin Lehalleur, "Exponentiation of coefficient systems and exponential motives", preprint, 2022. arXiv 2211.17247

[23] A. Grothendieck, "The cohomology theory of abstract algebraic varieties", pp. 103–118 in *Proc. Internat. Congress Math.* 1958, Cambridge Univ. Press, New York, 1960. MR Zbl

[24] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics **20**, Springer, 1966. MR Zbl

[25] T. Hemo, T. Richarz, and J. Scholbach, "Constructible sheaves on schemes", *Adv. Math.* **429** (2023), art. id. 109179. MR Zbl

[26] M. Hoyois, "The six operations in equivariant motivic homotopy theory", *Adv. Math.* **305** (2017), 197–279. MR Zbl

[27] B. Iversen, *Cohomology of sheaves*, Springer, 1986. MR Zbl

[28] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundl. Math. Wissen. **292**, Springer, 1990. MR Zbl

[29] N. M. Katz, *Exponential sums and differential equations*, Annals of Mathematics Studies **124**, Princeton University Press, 1990. MR Zbl

[30] N. M. Katz and G. Laumon, "Transformation de Fourier et majoration de sommes exponentielles", *Inst. Hautes Études Sci. Publ. Math.* 62 (1985), 361–418. MR

[31] A. Khan, *Motivic homotopy theory in derived algebraic geometry*, Ph.D. thesis, Universität Duisburg-Essen, 2016, https://www.preschema.com/papers/thesis.pdf.

[32] A. A. Khan, "The cdh-local motivic homotopy category", *J. Pure Appl. Algebra* **228**:5 (2024), art. id. 107562. MR Zbl

[33] A. A. Khan and C. Ravi, "Generalized cohomology theories for algebraic stacks", preprint, 2021. arXiv 2106.15001

[34] M. Kontsevich and Y. Soibelman, "Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants", *Commun. Number Theory Phys.* **5**:2 (2011), 231–352. MR Zbl

[35] Y. Laszlo and M. Olsson, "The six operations for sheaves on Artin stacks, II: Adic coefficients", *Publ. Math. Inst. Hautes Études Sci.* 107 (2008), 169–210. MR Zbl

[36] Y. Liu and W. Zheng, "Gluing restricted nerves of ∞-categories", preprint, 2012. arXiv 1211.5294

[37] Y. Liu and W. Zheng, "Enhanced six operations and base change theorem for higher Artin stacks", preprint, 2017. arXiv 1211.5948

[38] J. Lurie, "Higher algebra", notes, 2017, https://www.math.ias.edu/~lurie/papers/HA.pdf.

[39] F. Morel and V. Voevodsky, "$\mathbf{A}^1$-homotopy theory of schemes", *Inst. Hautes Études Sci. Publ. Math.* 90 (1999), 45–143. MR Zbl

[40] D. Petersen, "A remark on singular cohomology and sheaf cohomology", *Math. Scand.* **128**:2 (2022), 229–238. MR Zbl

[41] J. Riou, "Algebraic K-theory, $\mathbf{A}^1$-homotopy and Riemann–Roch theorems", *J. Topol.* **3**:2 (2010), 229–264. MR Zbl

[42] M. Robalo, *Théorie homotopique motivique des espaces noncommutatifs*, Ph.D. thesis, Institut de mathématiques de Jussieu - Paris Rive Gauche, 2014, https://webusers.imj-prg.fr/~marco.robalo/these.pdf.

[43] P. Scholze, "Lectures on condensed mathematics", lecture notes, 2019, http://tinyurl.com/Scholze-lecture-notes.

[44] P. Deligne, *Cohomologie étale* (Séminaire de Géométrie Algébrique du Bois Marie), Lecture Notes in Math. **569**, Springer, 1977. MR Zbl

[45] M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas, Tome 3: Exposés IX–XIX* (Séminaire de Géométrie Algébrique du Bois Marie 1963–1964), Lecture Notes in Math. **305**, Springer, 1973. MR Zbl

[46] M. Spitzweck, *A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains*, Mém. Soc. Math. Fr. (N.S.) **157**, Soc. Math. France, Paris, 2018. MR Zbl

[47] V. Voevodsky, "My view of the current state of motivic homotopy theory", notes, 2000, http://tinyurl.com/Voevodsky-Note.

[48] V. Voevodsky, "Triangulated categories of motives over a field", pp. 188–238 in *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud. **143**, Princeton Univ. Press, 2000. MR Zbl

[49] A. Weil, "Numbers of solutions of equations in finite fields", *Bull. Amer. Math. Soc.* **55** (1949), 497–508. MR Zbl

Received 20 Dec 2021. Revised 28 Dec 2022.

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Cover courtesy of Paul Arne Østvær.

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ISSN: 2329-9061 (print), 2329-907X (electronic)

ISBN: 978-1-935107-13-2 (print), 978-1-935107-14-9 (electronic)

First published 2025.

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