

Motivic Geometry

Lectures on the cohomology of reciprocity sheaves

Nikolai Opdan and Kay Rülling



$$\begin{array}{ccc}
 X \xleftarrow{j} \circ \xrightarrow{j} \overline{X} \xleftarrow{j} \partial \overline{X} & & 0 \rightarrow K_{2-n}^M(k)/24 \rightarrow \pi_{n+1,n} \mathbf{1} \rightarrow \pi_{n+1,n} \mathbf{kq} \rightarrow 0 \\
 \downarrow f & \downarrow \overline{f} & \downarrow g \\
 \text{Spec}(k) & & \prod_{\infty} X \simeq g_* f^* j_* f^! (1)
 \end{array}$$

$\sum_{\text{lines}} \text{Tr}_{L/k}(\alpha) = 15\langle 1 \rangle + 12\langle -1 \rangle$

$u_*(\mathcal{X}(X/k)) = \pi_*(e^*(T_{X/k})) \in \mathcal{E}^{0,0}(k)$
 $\text{DM}^{\text{eff}}(k) \doteq \text{logDM}^{\text{eff}}(k)$

Lectures on the cohomology of reciprocity sheaves

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These are the notes accompanying three lectures given by K. Rülling at the Motivic Geometry program at CAS, which aim to give an introduction and an overview of some recent developments in the field of reciprocity sheaves. We begin by introducing the theory of reciprocity sheaves and the necessary background of modulus sheaves with transfers as developed by B. Kahn, H. Miyazaki, S. Saito, and T. Yamazaki. We then explain some basic examples of reciprocity sheaves with a special emphasis on Kähler differentials and the de Rham–Witt complex. After an overview of some fundamental results, we survey the recent work of F. Binda, K. Rülling, and S. Saito on the cohomology of reciprocity sheaves. In particular, we discuss a projective bundle formula, a blow-up formula, and a Gysin sequence, which generalizes work of Voevodsky on homotopy invariant sheaves with transfers. From this, pushforwards along projective morphisms can be constructed, which give rise to an action of projective Chow correspondences on the cohomology of reciprocity sheaves. This generalizes several constructions which originally relied on Grothendieck duality for coherent sheaves and gives a motivic view towards these results.

We then survey some applications which include the birational invariance of the cohomology of certain classes of reciprocity sheaves, many of which were not considered before. Finally, we outline some recent results which were not part of the lecture series.

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Foreword. These are the notes accompanying three lectures¹ given by K. Rülling in October 2020 at the Motivic Geometry program at CAS, which aim to give an introduction and an overview of some recent developments in the field of reciprocity sheaves. We stress that the focus of this lecture series, and the present notes, is on the properties of reciprocity sheaves, on their cohomology, and on applications of the theory, with a particular emphasis on de Rham–Witt sheaves. We do not stress categorical constructions such as the triangulated category of motives with modulus and we do not intend to give a complete overview of the whole theory, which was first and foremost developed by B. Kahn, H. Miyazaki, S. Saito, and T. Yamazaki. We try to keep the informal style of the lectures also in these notes and we do not claim any originality.

1. Reciprocity sheaves

By work of Voevodsky and many others, the general theory of the cohomology of \mathbb{A}^1 -invariant sheaves with transfers is fully developed. Among the most fundamental properties are the projective bundle formula, the blow-up formula, the Gysin sequence, Gersten resolution, action of proper Chow correspondences, and representability of cohomology theories.

However, the theory has a drawback: Many interesting non- \mathbb{A}^1 -invariant sheaves share the same properties as above, such as Kähler differentials, smooth commutative unipotent group schemes, étale motivic cohomology with \mathbb{Z}/p^n -coefficients (in char $p > 0$), etc. Despite this, they are not representable in the classical motivic theory and hence cannot be studied by motivic methods. This is in part because the \mathbb{A}^1 -invariant theory only detects log poles, regular singularities and tame ramifications. In order to study more general theories, we need a more general theory than the classical theory provided by Voevodsky.

One approach was recently introduced by Binda, Park, and Østvær in [5]. The basic idea is to generalize the classical motivic homotopy theory by replacing \mathbb{A}^1 with the “cube”

$$\overline{\square} := (\mathbb{P}^1, \infty).$$

It is the log scheme whose underlying scheme is \mathbb{P}^1 and whose log structure is induced by the inclusion of the divisor $\infty \hookrightarrow \mathbb{P}^1$. Working with log smooth log schemes and a suitable topology, the authors construct in loc. cit. the triangulated category $\log\mathrm{DM}^{\mathrm{eff}}(k)$ of effective logarithmic motives. A cohomology theory representable in $\log\mathrm{DM}^{\mathrm{eff}}(k)$ has the nice properties listed above (at least under the assumption of the existence of resolutions of singularities). An example of such a theory is the sheaf of log-Kähler differentials which becomes representable in

¹The lecture series can be found on Youtube at <http://tinyurl.com/reciprocity-sheaves-lectures>.

this new triangulated category. Embedding the classical triangulated category of motives $\mathrm{DM}^{\mathrm{eff}}(k)$ fully faithfully in $\log\mathrm{DM}^{\mathrm{eff}}(k)$, they construct an enlargement of the classical \mathbb{A}^1 -invariant theory. So far there is however no pole order or ramification filtration on the sheaves in this category.

The theory of reciprocity sheaves [25; 28] provides another solution: The basic idea, which goes back to Kahn in the 1990s, is to consider only those sheaves whose sections behave in a controlled way at infinity, i.e., replace \mathbb{A}^1 -invariance by a *modulus condition*. This is a similar condition to the one considered by Rosenlicht and Serre to define the generalized Jacobian for curves.

Modulus à la Rosenlicht and Serre. The definition of a modulus condition goes back to Rosenlicht [39] and Serre [49, III]. They considered the modulus of a rational map from a curve to a commutative algebraic group.

Definition 1.1. Let k be a perfect field, C a smooth projective curve over k with an effective divisor D , $U := C \setminus D$ the complement of D , and G a smooth commutative k -group. Then a k -morphism $a : U \rightarrow G$ has *modulus D* if

$$\sum_{x \in U} v_x(f) \cdot \mathrm{Tr}_{x/k}(a(x)) = 0,$$

for all $f \in k(C)^\times$ with $f \equiv 1 \pmod{D}$, i.e., $f \in \bigcap_{x \in D} \mathrm{Ker}(\mathcal{O}_{X,x}^\times \rightarrow \mathcal{O}_{D,x}^\times)$, where v_x denotes the discrete valuation defined by the point x and $\mathrm{Tr}_{x/k} : G(x) \rightarrow G(k)$ is the trace;² see, for example, [51, Exp XVII, Appendice 2].

The choice of a rational point $x \in U(k)$ gives a universal morphism called the Albanese map

$$\mathrm{alb}_{(C,D)} : U \rightarrow \mathrm{Alb}(C, D),$$

with the property that any map $a : U \rightarrow G$ to a smooth commutative group scheme G , which satisfies the modulus condition, factors via $\mathrm{alb}_{(C,D)}$.

Let Sm denote the category of smooth separated k -schemes of finite type, in the following simply called smooth k -schemes. For $X, Y \in \mathrm{Sm}$ denote by $\mathbf{Cor}(X, Y)$ the group of finite correspondences from X to Y as introduced by Suslin and Voevodsky, i.e., it is the free abelian group generated by integral closed subschemes of $X \times Y$, which are finite and surjective over a connected component of X . There is a category of finite correspondences \mathbf{Cor} whose objects are the objects of Sm and with morphisms the finite correspondences. A *presheaf with transfers* is an additive contravariant functor from \mathbf{Cor} to the category of abelian groups. The category of presheaves with transfers is denoted by \mathbf{PST} .

²If $G = \mathbb{G}_a$ is the additive group, then this is the usual trace. If $G = \mathbb{G}_m$ is the multiplicative group, then this is the norm.

Using the trace construction alluded to above, any smooth commutative k -group admits a structure of a presheaf with transfers by means of which the modulus condition may be reformulated as follows:

Definition 1.2. An element $a \in G(U)$ has *modulus* D if

$$\gamma^* a = 0,$$

for all prime correspondences $\Gamma \in \mathbf{Cor}(\mathbb{P}^1 \setminus \{1\}, U)$ such that

$$\{1\}|_{\Gamma^N} \geq D|_{\Gamma^N},$$

where $\Gamma^N \rightarrow \mathbb{P}^1 \times C$ is the normalization of the closure of Γ , and

$$\gamma := i_0^* \Gamma - i_\infty^* \Gamma \in \mathbf{Cor}(\mathrm{Spec} k, U).$$

Modulus pairs. The framework of modulus presheaves with transfers was introduced by Kahn, Miyazaki, S. Saito, and Yamazaki in [26; 27], as a fundamental tool in the construction of their triangulated category of motives with modulus.

Definition 1.3 [26]. Fix a perfect field k . A *modulus pair* \mathcal{X} is a pair (X, D) , where X is a separated scheme of finite type over k and D is an effective Cartier divisor (or the empty scheme) on X such that the complement of the support of D in X is smooth. A modulus pair (X, D) is *proper* if X is proper over $\mathrm{Spec} k$.

The group of *modulus correspondences* from (X, D) to (Y, E) , denoted by $\mathbf{MCor}((X, D), (Y, E))$, is the subgroup of $\mathbf{Cor}(X \setminus D, Y \setminus E)$ generated by finite prime correspondences

$$V \subset (X \setminus D) \times (Y \setminus E)$$

such that

- (i) the projection $V^N \rightarrow X$ is proper,
- (ii) $D|_{V^N} \geq E|_{V^N}$,

where $V^N \rightarrow X \times Y$ is the normalization of the closure of V in $X \times Y$. An element of

$$\mathbf{MCor}((X, D), (Y, E))$$

is called a *modulus correspondence* from (X, D) to (Y, E) .

The composition of finite correspondences restricts to a composition of modulus correspondences.³ Hence we can define the category \mathbf{MCor} as the *category of modulus pairs*, i.e., objects are modulus pairs and morphisms are modulus correspondences.

³We remark that condition (i) is essential for this; see [26, Proposition 1.2.3].

The category $\underline{\mathbf{MCor}}$ has a monoidal structure given by

$$(X, D) \otimes (Y, E) = (X \times Y, p_X^* D + p_Y^* E),$$

where p_X (resp. p_Y) denotes the projection onto the first (resp. second) factor.

Let $\underline{\mathbf{MPST}}$ denote the category of presheaves of abelian groups on $\underline{\mathbf{MCor}}$. It comes with a monoidal structure $\otimes_{\underline{\mathbf{MPST}}}$ which via the Yoneda embedding extends the one on $\underline{\mathbf{MCor}}$. There is an adjoint functor pair

$$\omega_! : \underline{\mathbf{MPST}} \rightleftarrows \mathbf{PST} : \omega^*$$

such that $\omega_! G(X) = G(X, \emptyset)$ and $\omega^* F(X, D) = F(X \setminus D)$.

The modulus presheaf represented by a modulus pair (X, D) in $\underline{\mathbf{MPST}}$ is denoted by

$$\mathbb{Z}_{\text{tr}}(X, D) := \underline{\mathbf{MCor}}(-, (X, D)).$$

Let $\bar{\square}$ denote the modulus pair (\mathbb{P}^1, ∞) , and set

$$h_0^{\bar{\square}}(X, D) := \text{Coker}(\mathbb{Z}_{\text{tr}}(X, D)(-\otimes \bar{\square}) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}_{\text{tr}}(X, D)),$$

which we can consider as the *cubical modulus* version of h_0 of the Suslin complex.

Remark 1.4. Note that we have surjections

$$\mathbb{Z}_{\text{tr}}(X \setminus D) \twoheadrightarrow \omega_! h_0^{\bar{\square}}(X, D) \twoheadrightarrow h_0^{\mathbb{A}^1}(X \setminus D),$$

where $\mathbb{Z}_{\text{tr}}(X \setminus D) = \mathbf{Cor}(-, X \setminus D)$ and

$$h_0^{\mathbb{A}^1}(X \setminus D) = \text{Coker}(\mathbb{Z}_{\text{tr}}(X \setminus D)(-\otimes \mathbb{A}^1) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}_{\text{tr}}(X \setminus D))$$

is its maximal \mathbb{A}^1 -invariant quotient.

Indeed, the surjectivity of the first map follows from the right exactness of $\omega_!$ and the equality

$$\omega_! \mathbb{Z}_{\text{tr}}(X, D) = \mathbb{Z}_{\text{tr}}(X \setminus D).$$

To see this, observe if $V \in \mathbf{Cor}(S, X \setminus D)$ is a finite prime correspondence, then V is already closed in $S \times X$ since it is finite over S . Hence $V \in \underline{\mathbf{MCor}}((S, \emptyset), (X, D))$. The second surjection follows directly from the fact that $\mathbb{Z}_{\text{tr}}(X, D)(-\otimes \bar{\square})$ is a subpresheaf of $\mathbb{Z}_{\text{tr}}(X \setminus D)(-\otimes \mathbb{A}^1)$.

We remark that by the above we have, for $S \in \text{Sm}$,

$$\omega_! h_0^{\bar{\square}}(X, D)(S) = \text{Coker}(\underline{\mathbf{MCor}}((S, \emptyset) \otimes \bar{\square}, (X, D)) \xrightarrow{i_0^* - i_1^*} \mathbf{Cor}(S, X \setminus D)).$$

Definition 1.5 [28, Definition 2.2.4]. Let (X, D) be a proper modulus pair with $U := X \setminus D$, F a presheaf with transfers, and $a \in F(U)$ a section. We say that a has *modulus* (X, D) if the Yoneda map defined by a factors through $\omega_! h_0^{\bar{\square}}(X, D)$, that is, there exists a map that makes the following diagram commute:

$$\begin{array}{ccc}
 \mathbb{Z}_{\text{tr}}(U) & \xrightarrow[\text{Yoneda}]{a} & F \\
 \searrow & & \nearrow \\
 & \omega_! h_0^{\square}(X, D) &
 \end{array}$$

Remark 1.6. (1) In [28] the pair (X, D) above is called an SC-modulus of a in order to distinguish it from a slightly different notion of modulus which was introduced before in [25]. In [28, Theorem 3.2.1], it is proven that the two notions of modulus coincide as long as $X \setminus D$ is quasiaffine. In the following we will only work with the above definition of modulus and therefore simply say *modulus* instead of *SC-modulus*.

- (2) If $F = G$ is a smooth commutative k -group and X is a smooth projective curve, then evaluating the diagram above at k gives back the definition of modulus introduced by Rosenlicht and Serre which was reformulated in Definition 1.2.

Reciprocity sheaves. We are now in the position to define reciprocity sheaves.

Definition 1.7 [28, Definition 2.2.4]. We say that a presheaf with transfers F is a *reciprocity presheaf* if for any smooth k -scheme U , and for all $a \in F(U)$, there exists a proper modulus pair (X, D) such that $U = X \setminus D$ and a has modulus (X, D) .

Let \mathbf{RSC} denote the full subcategory of \mathbf{PST} consisting of reciprocity presheaves. The category of *reciprocity sheaves* is $\mathbf{RSC}_{\text{Nis}} := \mathbf{RSC} \cap \mathbf{NST}$, where \mathbf{NST} denotes the subcategory of \mathbf{PST} consisting of presheaves with transfers which are Nisnevich sheaves on Sm .

Remark 1.8. (1) In loc. cit. the term *presheaves with transfers with SC-reciprocity* is used for what above is called reciprocity presheaf in order to distinguish them from the reciprocity presheaves introduced in [25]. This difference is however not relevant for us since we mostly work with Nisnevich sheaves with transfers, for which the two notions coincide; see [28, Corollary 3.2.3].

- (2) A precursor of reciprocity sheaves are the reciprocity functors defined in [24]. These are defined only on function fields and regular curves over such. It follows from the injectivity theorem [25, Theorem 6] that the restriction of any reciprocity sheaf in the sense of Definition 1.7 to fields and regular curves defines a reciprocity functor; see [45, Theorem 5.7].

Example 1.9. Smooth commutative k -groups and homotopy invariant Nisnevich sheaves provide important examples of reciprocity sheaves. For the homotopy invariant sheaves this follows directly from Remark 1.4; for the smooth commutative groups the argument is essentially given by Rosenlicht and Serre;⁴ see [49, Chapter III, Theorem 1].

⁴The argument of Rosenlicht and Serre works for curves over an algebraically closed field; see [25, Theorem 4.1.1] for an extension of the argument to the general case.

Proposition 1.10. *The (absolute) Kähler differentials are reciprocity sheaves, i.e.,*

$$\Omega^j \in \mathbf{RSC}_{\text{Nis}} \quad \text{for all } j \geq 0.$$

Proof. As a corollary of [11, Theorem 3.1.8] Kähler differentials have an action of finite transfers. It remains to show that they satisfy the modulus condition in Definition 1.5.

For any form $a \in \Omega^j(U)$ there is a proper modulus pair (X, D) such that $a \in H^0(X, \Omega_X^j \otimes_{\mathcal{O}_X} \mathcal{O}_X(D))$. We claim that $(X, 2D)$ is a modulus for a in the sense of Definition 1.5. For an integral smooth k -scheme S the restriction $\Omega_S^j \rightarrow \Omega_{k(S)}^j$ is injective, hence we reduce to show the following: If C is a regular projective curve over a k -function field K which comes with a map to X , such that its image is not contained in D , and $f \in K(C)$ satisfies the condition $f \equiv 1 \pmod{2D_C}$, where D_C denotes the pullback of D to C , then we have to show

$$\text{div}_C(f)^* a = 0. \quad (1-1)$$

To this end, observe that the modulus condition for f and the choice of (X, D) imply

$$\text{Res}_x(a|_C \text{dlog}(f)) = 0,$$

for all $x \in D_C$, where $\text{Res}_x : \Omega_{K(C)}^{j+1} \rightarrow \Omega_K^j$ denotes the residue symbol. Since the pullback of a to $C \setminus D_C$ is regular, we have

$$\text{Res}_x(a|_C \text{dlog}(f)) = v_x(f) \text{tr}_{K(x)/K}(a(x)) \quad \text{for all } x \in C \setminus D_C,$$

with $\text{tr}_{K(x)/K} : \Omega_{K(x)}^j \rightarrow \Omega_K^j$ the trace. Thus the reciprocity law yields

$$0 = \sum_{x \in C} \text{Res}_x(a \text{dlog}(f)) = \sum_{x \in C \setminus D_C} v_x(f) \text{tr}_{K(x)/K}(a(x)),$$

which is a reformulation of (1-1). \square

Definition 1.11. For a reciprocity presheaf $F \in \mathbf{RSC}$ we form the *modulus presheaf* \tilde{F} by defining

$$\tilde{F}(X, D) := \{a \in F(X \setminus D) : a \text{ has modulus } (\bar{X}, \bar{D} + N \cdot B), \text{ for some } N \gg 0\},$$

where $(\bar{X}, \bar{D} + B)$ is a compactification of (X, D) , in the sense of [26, Definition 1.8.1], i.e., it is a proper modulus pair with $X = \bar{X} \setminus B$ and $D = \bar{D}|_X$. This definition is independent of the choice of the compactification, by, e.g., [47, Remark 1.5].

We have $\tilde{F} \in \mathbf{MPST}$ and it satisfies:

- \square -invariance: $\tilde{F}(\mathcal{X} \otimes \square) \cong \tilde{F}(\mathcal{X})$.
- M-reciprocity: $\tilde{F}(X, D) = \varinjlim_N \tilde{F}(\bar{X}, \bar{D} + N \cdot B)$.
- Semipurity: $\tilde{F}(X, D) \subset \tilde{F}(X \setminus D, \emptyset)$.

In fact, \tilde{F} is a $\bar{\square}$ -invariant modulus presheaf by [28, Propositions 2.3.7 and 2.4.1], and the other two properties follow directly from the definition.

Definition 1.12. We define \mathbf{CI}^τ as the full subcategory of \mathbf{MPST} consisting of presheaves satisfying cube-invariance and M-reciprocity. We let $\mathbf{CI}^{\tau, \text{sp}}$ denote the full subcategory of \mathbf{CI}^τ consisting of semipure presheaves.

This gives an adjoint functor pair (see [28, Proposition 2.3.7])

$$\underline{\omega}_! : \mathbf{CI}^{\tau, \text{sp}} \rightleftarrows \mathbf{RSC} : \underline{\omega}^{\mathbf{CI}},$$

where $\underline{\omega}^{\mathbf{CI}}(F) = \tilde{F} \in \mathbf{CI}^{\tau, \text{sp}}$. For a proper modulus pair \mathcal{X} we obtain $\underline{\omega}_! h_0^{\bar{\square}}(\mathcal{X}) \in \mathbf{RSC}$; see [28, Corollary 2.3.5].

Modulus sheaves.

Definition 1.13. A presheaf $G \in \mathbf{MPST}$ is a *modulus sheaf* if for all modulus pairs $\mathcal{X} = (X, D)$ the presheaf on the category of étale X -schemes

$$(U \xrightarrow{\text{ét}} X) \mapsto G(U, D|_U) =: G_{\mathcal{X}}(U)$$

is a Nisnevich sheaf.

We let \mathbf{MNST} denote the category of Nisnevich modulus sheaves. Note that $\underline{\omega}_!$ restricts to a functor $\mathbf{MNST} \rightarrow \mathbf{NST}$ which we also denote by $\underline{\omega}_!$.

Remark 1.14. Note that \mathbf{MNST} is not the category of sheaves on a site whose underlying category is \mathbf{MCor} . However, there is a subcategory of \mathbf{MCor} which underlies a site associated to a regular and complete cd-structure, such that $G \in \mathbf{MPST}$ is a sheaf in the above sense if and only if the restriction of G is a sheaf on this site; see [26, Proposition 3.2.3].

Theorem 1.15 [26; 27]. *The natural inclusion $\mathbf{MNST} \rightarrow \mathbf{MPST}$ admits an exact left adjoint, the so-called sheafification,*

$$a_{\text{Nis}} : \mathbf{MPST} \rightarrow \mathbf{MNST},$$

which sends presheaves with M-reciprocity to sheaves with M-reciprocity.

The existence of the sheafification functor is proven in [26]. It follows from [27, Theorem 2] that it is compatible with M-reciprocity. By [26, Theorem 2] the sheafification functor is determined by the formula

$$a_{\text{Nis}}(G)_{(X, D)} = \varinjlim f_* (G_{(Y, f^*D), \text{Nis}}),$$

where the colimit is taken over a directed set of proper morphisms $f : Y \rightarrow X$ which induce an isomorphism on the complement $Y \setminus f^*D \cong X \setminus D$ and the index Nis on the right denotes the Nisnevich sheafification on the category of étale Y -schemes. Thus,

$$\underline{\omega}_!(a_{\text{Nis}}(G)) = (\underline{\omega}_! G)_{\text{Nis}}, \quad (1-2)$$

where the index Nis on the right denotes the Nisnevich sheafification on Sm , which by a result of Voevodsky restricts to a functor $\mathbf{PST} \rightarrow \mathbf{NST}$. Also,

$$\text{Ext}_{\underline{\mathbf{MNST}}}^i(\mathbb{Z}_{\text{tr}}(X, D), G) = \varinjlim H^i(Y_{\text{Nis}}, G_{(Y, f^*D)}). \quad (1-3)$$

This leads to the following question [26, Question 1]:

Question 1.16. Does (1-3) stabilize for $G \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}} := \mathbf{CI}^{\tau, \text{sp}} \cap \underline{\mathbf{MNST}}$?⁵

A fundamental result by S. Saito generalizes Voevodsky's strict homotopy invariance theorem by proving that Nisnevich sheafification preserves \square -invariance. The proof requires in particular a delicate extension of Voevodsky's theory of standard triples (see [55, Chapters 3 and 4]) to the setup of modulus pairs.

Theorem 1.17 [47, Theorem 10.1]. *We have*

$$\underline{a}_{\text{Nis}}(\mathbf{CI}^{\tau, \text{sp}}) \subset \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}.$$

Corollary 1.18. *For every $F \in \mathbf{RSC}$, the Nisnevich sheafification F_{Nis} belongs to $\mathbf{RSC}_{\text{Nis}}$. In particular, $\mathbf{RSC}_{\text{Nis}} \subset \mathbf{NST}$ is a full abelian subcategory.*

Proof. By Theorem 1.17 we have

$$G := \underline{a}_{\text{Nis}}(\tilde{F}) \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}} = \mathbf{CI}^{\tau, \text{sp}} \cap \underline{\mathbf{MNST}}.$$

Together with (1-2) we get

$$F_{\text{Nis}} = \underline{\omega}_!(G) \in \mathbf{RSC} \cap \mathbf{NST} = \mathbf{RSC}_{\text{Nis}}. \quad \square$$

The following purity theorem by S. Saito generalizes the \mathbb{A}^1 -invariant purity theorem by Voevodsky and will be essential in Section 5.

Theorem 1.19 [47, Theorem 0.2]. *For $F \in \mathbf{RSC}_{\text{Nis}}$ and $x \in X^{(c)}$ a c -codimensional point in X we have*

$$H_x^i(X, F) = 0 \quad \text{for } i \neq c,$$

and

$$H_x^c(X, F) \simeq F_{-c}(X) := \frac{F((\mathbb{A}^1 \setminus \{0\})^c \times x)}{\sum_{i=1}^n F((\mathbb{A}^1 \setminus \{0\})^{i-1} \times \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{c-i} \times x)}. \quad (1-4)$$

Remark 1.20. The isomorphism in (1-4) depends on the choice of a k -isomorphism

$$k(x)\{t_1, \dots, t_c\} \xrightarrow{\simeq} \mathcal{O}_{X,x}^h,$$

⁵It is shown in [40, no. 6.9] that this question has a negative answer if the base field has positive characteristic p , the divisor of the modulus pair (X, D) has a component of multiplicity divisible by p , and $G = \widehat{\Omega}^q$ with $q \neq 0$ and $q \neq \dim X$. But it remains an interesting question for which G and (X, D) one has a positive answer. Assuming resolutions of singularity this is, for example, the case if $q = \dim X$; see [40, Corollary 7.5].

where $\mathcal{O}_{X,x}^h$ denotes the henselization of $\mathcal{O}_{X,x}$. In particular, it is not functorial (as one directly sees by considering the case $F = \mathbb{G}_a$ and $c = 1$). Note that if F is \mathbb{A}^1 -invariant the isomorphism is independent of this choice; see [55, Chapter 3, Lemma 4.36].

Using Theorem 1.19 we find that the E_1 -complex of the coniveau spectral sequence has vanishing cohomology except in degree zero, hence we obtain the *Cousin resolution*

$$0 \rightarrow F_X \rightarrow \bigoplus_{x \in X^{(0)}} i_{x*} H_x^0(F) \rightarrow \cdots \rightarrow \bigoplus_{x \in X^{(c)}} i_{x*} H_x^c(F) \rightarrow \cdots \quad (1-5)$$

on X_{Nis} as a generalization of the Gersten resolution in the \mathbb{A}^1 -invariant case. The injectivity of the first morphism was already proven in [25].

Relation with the logarithmic theory. Further results by S. Saito give the relation with the logarithmic theory of Binda, Park, and Østvær.

Definition 1.21. We say that $\mathcal{X} = (X, D)$ is an *ls modulus pair* if $X \in \text{Sm}$ and D_{red} is a strict normal crossing divisor on X . (Note that D is allowed to be nonreduced.) Let $\underline{\mathbf{MCor}}_{\text{ls}}$ denote the full subcategory of $\underline{\mathbf{MCor}}$ of ls modulus pairs.

A morphism $f : Y \rightarrow X$ of smooth schemes is *transversal* to D if

$$f^{-1}(D_1 \cap \cdots \cap D_r) \hookrightarrow Y$$

is a regular closed embedding of codimension equal to r , for any irreducible components D_1, \dots, D_r of D_{red} .

Definition 1.22. Let (X, \mathcal{M}) be a smooth log smooth scheme, where \mathcal{M} is a monoid sheaf with a multiplicative map $\mathcal{M} \rightarrow \mathcal{O}_X$, which is an isomorphism over \mathcal{O}_X^\times , defining the log structure. By definition $\text{supp } \mathcal{M}$ denotes the support of the monoid sheaf $\mathcal{M}/\mathcal{O}_X^\times$. We have $(X, \text{supp } \mathcal{M}) \in \underline{\mathbf{MCor}}_{\text{ls}}$ (see, for example, [5, Lemma A.5.10]) and define, for $F \in \mathbf{RSC}_{\text{Nis}}$,

$$F^{\log}(X, \mathcal{M}) := \widetilde{F}(X, \text{supp } \mathcal{M}).$$

Theorem 1.23 [48, Theorems 6.1 and 6.3]. *Let $\mathbf{Shv}_{\text{dNis}}^{\text{ltr}}$ denote the category of dividing Nisnevich sheaves with log-transfers on log smooth fs log schemes, in the sense of [5, Definition 4.2.1]. Then there exists a functor*

$$\mathbf{RSC}_{\text{Nis}} \longrightarrow \mathbf{Shv}_{\text{dNis}}^{\text{ltr}}$$

sending $F \mapsto F^{\log}$, which is exact and fully faithful. Also, F^{\log} is strictly $\overline{\square}$ -invariant, that is, for smooth log smooth schemes (X, \mathcal{M}) we have

$$H_{\text{dNis}}^i((X, \mathcal{M}), F^{\log}) = H_{\text{dNis}}^i((X, \mathcal{M}) \times \overline{\square}, F^{\log})$$

and the Nisnevich cohomology of $F_{(X, \text{supp } \mathcal{M})}$ (with the notation from Definition 1.13) is representable in the triangulated category of logarithmic motives $\log\text{DM}^{\text{eff}}(k)$ constructed in [5]:

$$H^i(X, F_{(X, \text{supp } \mathcal{M})}) \cong \text{Hom}_{\log\text{DM}^{\text{eff}}(k)}(M(X, \mathcal{M}), F^{\log}[i]).$$

2. De Rham–Witt sheaves as reciprocity sheaves

We give a short introduction to the de Rham–Witt sheaves introduced by Bloch [7] and Illusie [22], describe some basic properties, define transfers, and show that they are reciprocity sheaves. From this one obtains many interesting reciprocity sheaves, which are useful, for example, in the study of crystalline cohomology, the Brauer group, and étale motivic cohomology with p -primary torsion coefficients.

Motivation. Let X be a smooth projective scheme over \mathbb{F}_{p^n} and set

$$H^i := H_{\text{crys}}^i(X/W(\mathbb{F}_{p^n}))\left[\frac{1}{p}\right],$$

the i -th crystalline cohomology group, where $W(\mathbb{F}_{p^n})$ denotes the Witt vectors of \mathbb{F}_{p^n} . Then H^i is a finite-dimensional vector space over the field $W(\mathbb{F}_{p^n})[1/p]$, and considering the action of Frobenius on it, it becomes a $F := (F_X^n)^*$ -crystal. Such crystals have a slope decomposition

$$H^i \cong \bigoplus H_\lambda^i,$$

where λ ranges through the nonnegative rational numbers. Here H_λ^i is a subvector space on which the Frobenius acts with eigenvalues having p -adic valuation equal to λ .

One of the main motivations behind the construction of the de Rham–Witt complex is the wish to understand this slope decomposition from a cohomological point of view. And indeed, the de Rham–Witt complex $W\Omega^*$ computes crystalline cohomology by

$$H^i = H^i(X_{\text{Zar}}, W\Omega^*)\left[\frac{1}{p}\right].$$

The Hodge-to-de Rham spectral sequence yields the *slope spectral sequence* given by

$$E_1^{j,i} = H^i(X, W\Omega^j)\left[\frac{1}{p}\right] \implies H^*,$$

which degenerates to give

$$\bigoplus_{j \leq \lambda < j+1} H_\lambda^i = H^{i-j}(X, W\Omega^j)\left[\frac{1}{p}\right].$$

Witt vectors. Let A be an \mathbb{F}_p -algebra. We recall that, for $n \geq 1$, the *Witt vectors* of length n of A form a ring whose underlying set is equal to

$$W_n(A) = \{(a_0, \dots, a_{n-1}) : a_i \in A\},$$

and whose ring structure is defined in such a way that the following properties hold: The map

$$R : W_{n+1}(A) \rightarrow W_n(A), \quad (a_0, \dots, a_n) \mapsto (a_0, \dots, a_{n-1}),$$

is a ring-map, called the *restriction*. The map

$$F : W_{n+1}(A) \rightarrow W_n(A), \quad (a_0, \dots, a_n) \mapsto (a_0^p, \dots, a_{n-1}^p),$$

is a ring-map, called the *Frobenius*. The map

$$V : W_n(A) \rightarrow W_{n+1}(A), \quad (a_0, \dots, a_{n-1}) \mapsto (0, a_0, \dots, a_{n-1}),$$

is a group-map, called the *Verschiebung* (or shift). The map

$$[-] : A \rightarrow W_n(A), \quad a \mapsto [a] := (a, 0, \dots, 0),$$

is multiplicative and is called the Teichmüller lift. Also,

- $W_1(A) = A$ as a ring,
- $(a_0, \dots, a_{n-1}) = \sum_{i=0}^{n-1} V^i([a_i])$, where $V^i = \underbrace{V \circ \dots \circ V}_{i\text{-times}}$,
- $FV = VF$ is multiplication by p ,
- $V(a) \cdot b = V(a \cdot F(b))$.

Passing to the limit

$$W(A) := \varprojlim_n W_n(A),$$

where the transition maps are given by the restriction, we get a ring which is p -torsion free if A is reduced. For details, see, e.g., [50, Chapter II, §6].

Example 2.1. The Witt vectors have the following properties:

- $W(\mathbb{F}_p) = \varprojlim_n W_n(\mathbb{F}_p) = \mathbb{Z}_p$.
- If A is perfect, that is, the Frobenius is an isomorphism, then $W_n(A)$ is the unique flat $\mathbb{Z}/p^n\mathbb{Z}$ -lift of A/\mathbb{F}_p .
- The contravariant functor

$$(\text{Schemes}/\mathbb{F}_p)^\circ \rightarrow (\text{Ab-groups}), \quad X \mapsto H^0(X, W_n\mathcal{O}_X),$$

is represented by a ring scheme W_n . Thus the ring $W_n(A)$ is equal to the A -rational points of W_n .

- Any commutative unipotent \mathbb{F}_p -group scheme can be embedded into $\bigoplus_{n_i} W_{n_i}$, viewed as a group scheme.

The de Rham–Witt complex. The *de Rham–Witt complex* of an \mathbb{F}_p -scheme X , as defined by Bloch [7], Illusie [22], or Kato [29] is a pro-differential graded algebra

$$((W_n \Omega^*, d)_{n \geq 1}, R),$$

where d is the differential and R is the restriction map, such that

$$W_n \Omega^0 = W_n \mathcal{O}_X.$$

It is equipped with an extension of the Frobenius map

$$F : W_{\bullet+1} \Omega_X^* \rightarrow W_{\bullet} \Omega_X^*,$$

and an extension of the Verschiebung map on $W_{\bullet} \mathcal{O}_X$

$$V : W_{\bullet} \Omega_X^* \rightarrow W_{\bullet+1} \Omega_X^*,$$

which satisfy the following conditions:

- F is a map of graded rings, and V is a map of graded groups.
- The composition of maps FV is given by multiplication by p .
- F -linearity: $V(\alpha) \cdot \beta = V(\alpha \cdot F(\beta))$.
- $FdV = d$.
- $Fd[a] = [a]^{p-1}d[a]$, for $a \in W_{\bullet} \mathcal{O}_X$.

In fact, in [21] Hesselholt and Madsen show that $W_{\bullet} \Omega_X^*$ is the initial object in the category of pro-differentially graded algebras with the above properties. They also extend the definition to all $\mathbb{Z}_{(p)}$ -algebras.

We have $W_1 \Omega_X^* = \Omega_{X/\mathbb{F}_p}^*$, and there is a commutative square

$$\begin{array}{ccc} W_{n+1} \Omega_X^j & \xrightarrow{F} & W_n \Omega_X^j \\ \downarrow & & \downarrow \\ \Omega_X^j & \xrightarrow{C^{-1}} & \Omega_X^j / d \Omega_X^{j-1}, \end{array}$$

i.e., F lifts the inverse Cartier operator C^{-1} , which is determined by the formula $C^{-1}(ad \log b) = a^p d \log b$.

Remark 2.2. (1) For an \mathbb{F}_p -algebra A , Bloch constructed the de Rham–Witt complex in [7] as the pro-object

$$W_{\bullet} \Omega_A^q \cong TS \ker(K_{q+1}(A[T]/T^{\bullet}) \xrightarrow{T \mapsto 0} K_{q+1}(A)),$$

where T denotes the p -typical part and S the symbolic part of Quillen K -theory. Bloch's construction was originally limited to the case $\dim X < p$ and $p \neq 2$. This restriction was removed in [29].

(2) Following an idea of Deligne, Illusie constructed the de Rham–Witt complex in [22] as a quotient of $\Omega_{W_n \mathcal{O}_X / W_n(\mathbb{F}_p)}^q$, such that it is the universal example of a pro-dga with a Verschiebung V satisfying certain properties. Then he proves that on this complex an F as above exists.

(3) If X is a smooth scheme over k with a smooth lift over $X_n / W_n(k)$, then

$$W_n \Omega_X^q \cong \mathcal{H}^q(\Omega_{X_n / W_n(k)}^*),$$

and one can show that this isomorphism is independent of the lift; see [23, Chapter III, (1.5)]. As Illusie and Raynaud explain in loc. cit., it was observed by N. Katz that one can take the right-hand side of the above isomorphism as the definition of the de Rham–Witt sheaves (using local lifts of X over $W_n(k)$ and glue) and that it is possible to construct the structure of a pro-dga with maps F and V using this description.

(4) We mention that there are other constructions of the de Rham–Witt complex by Bhatt, Lurie, and Mathew [4], Cuntz and Deninger [16], Hesselholt [20], Hesselholt and Madsen [21], and Langer and Zink [36], each of which works in a different generality, but they all agree for smooth schemes over a perfect field of positive characteristic.

Theorem 2.3 [22, Chapter II, Theorem 1.4]. *Let X be a smooth scheme over a perfect field k of characteristic $p > 0$ and let $u : (X / W_n(k))_{\text{crys}} \rightarrow X_{\text{Zar}}$ be the change of sites map. Then there is an isomorphism*

$$Ru_* \mathcal{O}_{X / W_n(k), \text{crys}} \cong W_n \Omega_X^\bullet.$$

We remark that Bloch proved in [7, Chapter III, Theorem (2.1)] such an isomorphism in the limit over n using his K -theoretic construction of the de Rham–Witt complex (as a pro-object) under the additional assumption that $\dim X < p$ and $p \neq 2$. These assumptions were later removed by Kato; see [29, p. 635, Remark 2].

Theorem 2.4 [17, Chapter I, Theorem 4.1 and Chapter II, Theorem 2.2]. *Let X be a smooth scheme over k and let*

$$\pi : W_n X = (|X|, W_n \mathcal{O}_X) \rightarrow \text{Spec } W_n(k)$$

be the finite-type morphism of schemes induced by the structure map of X . Then

$$\pi^! W_n(k) \cong W_n \Omega_X^{\dim X}[\dim X],$$

where $\pi^!$ denotes the exceptional inverse image in the derived category of \mathcal{O} -modules from Grothendieck duality. There is a canonical isomorphism

$$W_n \Omega_X^j \xrightarrow{\simeq} R\mathcal{H}om_{W_n \mathcal{O}_X}(W_n \Omega_X^{\dim X - j}, W_n \Omega_X^{\dim X}).$$

Using the above theorem, Gros constructed in [19] a pushforward

$$f_* : Rf_* W_n \Omega_Y^j \rightarrow W_n \Omega_X^{j-r}[-r],$$

for a proper morphism $f : Y \rightarrow X$ of relative dimension r between smooth schemes.

Proposition 2.5. *The de Rham–Witt sheaves are reciprocity sheaves, i.e.,*

$$W_n \Omega^j \in \mathbf{RSC}_{\text{Nis}}$$

for all $j \geq 0$. The maps d, R, F, V are compatible with the transfer structure and hence are morphisms of reciprocity sheaves.

Proof. The finite transfers structure on $W_n \Omega^j$ and its compatibility with d, R, V, F is a consequence of [12, Theorem 3.4.6]. We recall the definition: for $Z \in \mathbf{Cor}(X, Y)$ the correspondence action is given by the composition

$$\begin{aligned} Z^* : W_n \Omega^j(Y) &\xrightarrow{p_Y^*} W_n \Omega^j(X \times Y) \\ &\xrightarrow{\cup \text{cl}_Z} H_Z^{\dim Y}(X \times Y, W_n \Omega^{j+\dim Y}) \\ &\xrightarrow{p_{X*}} W_n \Omega^j(X), \end{aligned}$$

where cl_Z denotes the cycle class and p_{X*} denotes the pushforward with supports from [12, 2.3]. Note that since Y does not need to be proper, this pushforward only exists with support in the finite X -scheme Z . We also want to point out the compatibility of the pushforward, and hence the correspondence action with d, R, F, V , is not obvious and requires Ekedahl's careful analysis of the behavior of these maps under duality; see [17, Chapter III]. We find $W_n \Omega^j \in \mathbf{NST}$. It remains to show that any form $a \in W_n \Omega^j(X)$ has a modulus. This is similar to the case of Kähler differentials in the proof of Proposition 1.10; see [25, Theorem B.2.2] for details. \square

Since $\mathbf{RSC}_{\text{Nis}}$ is an abelian category (see Corollary 1.18), Proposition 2.5 gives us many more examples of reciprocity sheaves by taking kernels and quotients of the maps d, R, F, V . In particular we obtain:

(1) $W_n \Omega^* \in \text{Comp}^+(\mathbf{RSC}_{\text{Nis}})$ represents the complex of sheaves sending a smooth scheme X to $Ru_* \mathcal{O}_{X/W_n, \text{crys}}$.

(2) We have

$$B_\infty W_n \Omega^j := \bigcup_{r \geq 0} F^r d W_{n+r} \Omega^{j-1} \in \mathbf{RSC}_{\text{Nis}}. \quad (2-1)$$

(3) The *generalized Artin–Schreier–Witt* sequence is the exact sequence

$$0 \rightarrow W_n \Omega_{X, \log}^j \rightarrow W_n \Omega_X^j / B_\infty \xrightarrow{\bar{F}-1} W_n \Omega_X^j / B_\infty \rightarrow 0$$

on $X_{\text{ét}}$, where $W_n \Omega_{X, \log}^j$ is the subsheaf of $W_n \Omega_X^j$ which locally is generated by dlog-forms [22, Chapter I, 5.7], $B_\infty = B_\infty W_n \Omega_X^j$, and $\bar{F} : W_n \Omega_X^j / B_\infty \rightarrow W_n \Omega_X^j / B_\infty$ is induced by “lifting to level $n+1$ and applying $F : W_{n+1} \Omega_X^j \rightarrow W_n \Omega_X^j$ ” (this operation is well defined on the quotient modulo B_∞). The exactness of this sequence can be deduced from [15, §1, Lemma 2; 22, I, (3.21.1.1), (3.21.1.3) and Proposition 3.26 and its proof]. By a famous theorem of Geisser and Levine [18] we have

$$W_n \Omega_{X, \log}^j[-j] \cong \mathbb{Z}/p^n(j)_{X_{\text{ét}}},$$

where the right-hand side denotes the étale motivic complex of weight j with \mathbb{Z}/p^n -coefficients. Let $\epsilon : \text{Sm}_{\text{ét}} \rightarrow \text{Sm}_{\text{Nis}}$ be the change of sites map. Since $W_n \Omega_X^j / B_\infty$ is a direct limit of sheaves which are successive extensions of coherent \mathcal{O} -modules, it is acyclic for $R\epsilon_*$. Thus we obtain

$$R\epsilon_* \mathbb{Z}/p^n(j) \cong (W_n \Omega^j / B_\infty \xrightarrow{\bar{F}-1} W_n \Omega^j / B_\infty)[-j] \in D^b(\mathbf{RSC}_{\text{Nis}}). \quad (2-2)$$

(4) By a result of Voevodsky the prime-to- p part of $R^i \epsilon_*(\mathbb{Q}/\mathbb{Z}(j))$ is homotopy invariant, and combining this with the above yields

$$R^i \epsilon_*(\mathbb{Q}/\mathbb{Z}(j)) \in \mathbf{RSC}_{\text{Nis}} \quad \text{for all } i, j.$$

By the above this is not homotopy invariant only for $i = j + 1$. In particular, the Brauer group defines a reciprocity sheaf:

$$X \mapsto \text{Br}(X) = H^0(X, R^2 \epsilon_*(\mathbb{Q}/\mathbb{Z}(1))) \in \mathbf{RSC}_{\text{Nis}}.$$

3. Computation of the modulus in examples

We give some computations of the modulus in certain examples. We will see that the modulus detects higher poles and ramifications, which is not captured by the classical \mathbb{A}^1 -invariant theory.

We let L be a henselian discrete valuation field of geometric type over the perfect base field k , i.e.,

$$L = \text{Frac } \mathcal{O}_{U, x}^h,$$

where U is a smooth k -scheme and $x \in U^{(1)}$.

For $F \in \mathbf{RSC}_{\text{Nis}}$ we let $F(L) := F(\text{Spec } L)$ and

$$\tilde{F}(\mathcal{O}_L, \mathfrak{m}^{-n}) := \tilde{F}(\text{Spec } \mathcal{O}_L, n \cdot \{\text{closed point}\}).$$

By [44, Theorem 4.15(4)] we obtain for any proper modulus pair (X, D) that $\widetilde{F}(X, D)$ equals

$$\{a \in F(X \setminus D) \mid \rho^* a \in \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-v_L(\rho^* D)}) \quad \forall L \quad \forall \rho \in (X \setminus D)(L)\}. \quad (3-1)$$

In order to understand the modulus sheaf \widetilde{F} we have to study the filtration

$$F(\mathcal{O}_L) \subset \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) \subset \cdots \subset \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) \subset \cdots \subset F(L)$$

for all L . For \mathbb{A}^1 -invariant Nisnevich sheaves we have $\widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) = F(L)$. For a non- \mathbb{A}^1 -invariant reciprocity sheaf this is an exhaustive increasing filtration, which for varying L is infinite, in the sense that there exists no natural number $n \geq 0$ such that $F(L)$ is equal to $\widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n})$ for all L ; see [45, Lemma 5.2].

Definition 3.1. The reciprocity sheaf F has level $n \geq 0$, if for any smooth k -scheme X and any $a \in F(\mathbb{A}^1 \times X)$ the following implication holds:

$$a_{\mathbb{A}_z^1} \in F(z) \subset F(\mathbb{A}_z^1) \text{ for all } z \in X_{(\leq n-1)} \implies a \in F(X) \subset F(\mathbb{A}^1 \times X),$$

where $a_{\mathbb{A}_z^1}$ denotes the restriction of a to \mathbb{A}_z^1 and $X_{(\leq n-1)}$ denotes the set of points in X whose closure has dimension $\leq n - 1$.⁶

Clearly, \mathbb{A}^1 -invariant sheaves have level 0. Any commutative algebraic group G over k has level 1 by [44, Theorem 5.2]. If F has level n it suffices to consider in (3-1) those L which have transcendence degree $\leq n$ over k . For example, if the level is $n = 1$, this can be interpreted as a *cut-by-curves criterion* for determining the modulus of an element $a \in F(U)$. If the level is $n = 2$ we have a *cut-by-surfaces criterion*, etc.

Differential forms and rank-1 connections.

Theorem 3.2 [40, Chapter 6; 44, Chapter 6]. *Let $\text{char } k = p \geq 0$ and $j \geq 1$. Then the modulus sheaf $\widetilde{\Omega}_{/Z}^j$ has level $j + 1$ and*

$$\widetilde{\Omega}_{/Z}^j(\mathcal{O}_L, \mathfrak{m}_L^{-n}) = \begin{cases} \frac{1}{t^{n-1}} \cdot \Omega_{\mathcal{O}_L/\mathbb{Z}}^j(\log t) & \text{if } p = 0 \text{ or } (n, p) = 1, \\ \frac{1}{t^n} \cdot \Omega_{\mathcal{O}_L/\mathbb{Z}}^j & \text{if } p > 0 \text{ and } p \mid n, \end{cases}$$

where $t \in \mathfrak{m}_L$ is a local parameter.

If $p = 0$ and $\text{Conn}^1(X)$ (resp. $\text{Conn}_{\text{int}}^1(X)$) denotes the group of isomorphism classes of (resp. integrable) rank-1 connections on X/k , we have:

- $\text{Conn}^1 \in \mathbf{RSC}_{\text{Nis}}$ has level 2 and $\text{Conn}_{\text{int}}^1(X) \in \mathbf{RSC}_{\text{Nis}}$ has level 1.
- $\widetilde{\text{Conn}}_{\text{int}}^1(X, D)$ is the group of isomorphism classes of integrable rank-1 connections on $U = X \setminus D$ whose nonlog irregularity⁷ is bounded by D .

⁶This is equivalent to the motivic conductor of F having level n in the language of [44].

⁷The nonlog irregularity of a rank-1 connection E on $\text{Spec } L$ is zero if this connection extends to $\text{Spec } \mathcal{O}_L$ and else is equal to $\text{irr}(E) + 1$, where $\text{irr}(E)$ denotes the usual irregularity.

- $h_{\mathbb{A}^1}^0(\text{Conn}_{\text{int}}^1(U))$ is the group of isomorphism classes of regular singular rank-1 connections on U , where we denote by $h_{\mathbb{A}^1}^0(F)$ the maximal \mathbb{A}^1 -invariant subsheaf of F .

Using the above formulas and the birational invariance of $\widetilde{\Omega}_{/Z}^j$ (see (1-3) with $i = 0$) it is shown in [40, Corollary 7.3] (see also [42, Theorem 7.1]) that an integral normal Cohen–Macaulay scheme Y of dimension d and of finite type over k has *pseudorational singularities* if for each effective Cartier divisor R , such that $Y \setminus R$ is smooth, the sheaf $\widetilde{\Omega}_{(Y,R)}^d$ is $S2$, i.e., is completely determined by its stalks at the zero- and one-codimensional points of Y . Note that in view of (3-1), this condition can be rephrased as a condition on the local filtrations $\widetilde{\Omega}^d(\mathcal{O}_L, \mathfrak{m}^{-n})$ for various L mapping to Y .

Witt vectors and characters of the abelianized fundamental group. Let $\text{char } k = p > 0$. In order to define the Albanese with modulus in higher dimension, Kato and Russell defined in [31] the filtration

$$\text{fil}_r^F W_n(L) := \sum_{j \geq 0} F^j (\text{fil}_{r-1}^{\log} W_n(L) + V^{n-s} (\text{fil}_r^{\log} W_s(L))), \quad r \geq 0,$$

where $\text{fil}_r^{\log} W_n(L) = \{(a_0, \dots, a_{n-1}) \mid p^{n-1-i} v_L(a_i) \geq -r \text{ for all } i\}$, and $s = \min\{n, \text{ord}_p(r)\}$.

Theorem 3.3 [44, Theorem 7.20]. *The Witt sheaf W_n has level 1, and*

$$\widetilde{W}_n(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \text{fil}_r^F W_n(L).$$

In particular,

$$\widetilde{G}_a(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \begin{cases} \mathcal{O}_L, & r \leq 1, \\ \sum_j F^j \left(\frac{1}{r-1} \mathcal{O}_L\right), & (p, r) = 1, \\ \sum_j F^j \left(\frac{1}{r} \mathcal{O}_L\right), & p \mid r. \end{cases}$$

Let $H_{\text{ét}}^1(L, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\text{cts}}(G_L, \mathbb{Q}/\mathbb{Z})$, where G_L denotes the absolute Galois group of L . As a variant of the Brylinski–Kato filtration [10; 30] Matsuda introduced in [37], on $H_{\text{ét}}^1(L, \mathbb{Q}/\mathbb{Z})$, the filtration

$$\text{fil}_r H_{\text{ét}}^1(L, \mathbb{Q}/\mathbb{Z}) := \bigoplus_{l \neq p} H_{\text{ét}}^1(L, \mathbb{Q}_l/\mathbb{Z}_l) \oplus \bigcup_n \text{im}(\text{fil}_r^F W_n(L) \rightarrow H_{\text{ét}}^1(L, \mathbb{Q}/\mathbb{Z})),$$

where the maps $\text{fil}_r^F W_n(L) \rightarrow H_{\text{ét}}^1(L, \mathbb{Q}/\mathbb{Z})$ are induced by the isomorphism $H_{\text{ét}}^1(L, \mathbb{Q}/\mathbb{Z}) = \varinjlim_V W_n(L)/(F-1)W_n(L)$ stemming from Artin–Schreier–Witt sequence.⁸ This filtration was originally introduced to generalize the Artin-conductor to the case of imperfect residue fields.

⁸Matsuda does not consider the F -saturated filtration, but note that the images of fil^F and fil in the quotient $W_n(L)/(F-1)$ coincide.

Theorem 3.4 [44, Theorem 8.10]. *Let $\epsilon : \text{Sm}_{\text{ét}} \rightarrow \text{Sm}_{\text{Nis}}$ denote the change of sites. Then $R^1\epsilon_*\mathbb{Q}/\mathbb{Z} \in \mathbf{RSC}_{\text{Nis}}$ has level 1, and*

$$\overline{R^1\epsilon_*(\mathbb{Q}/\mathbb{Z})}(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \text{fil}_r H_{\text{ét}}^1(L, \mathbb{Q}/\mathbb{Z}).$$

Remark 3.5. (1) By work of A. Abbes and T. Saito [1] and Y. Yatagawa [57],

$$\text{fil}_r H_{\text{ét}}^1(L, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\text{cts}}(G_L/G_L^{r+}, \mathbb{Q}/\mathbb{Z}),$$

where $\{G_L^j\}_{j \in \mathbb{Q}_{\geq 0}}$ is the Abbes–Saito ramification filtration of G_L and $G_L^{r+} = \bigcup_{s>r} G_L^s$.

(2) Similarly as above one can use (2-2) to determine $\overline{R^{j+1}\epsilon_*(\mathbb{Q}/\mathbb{Z}(j))}$ for $j \geq 1$. This is work in progress.

Torsors under finite group schemes in positive characteristic. Let $\text{char } k = p > 0$, and let G be a finite commutative k -group scheme. We can write

$$G = G_{\text{em}} \times G_{\text{eu}} \times G_{\text{im}} \times G_{\text{iu}},$$

where G_{em} is an étale multiplicative group (e.g., \mathbb{Z}/l), G_{eu} is an étale unipotent group (e.g., \mathbb{Z}/p), G_{im} is an infinitesimal multiplicative group (e.g., μ_p), and G_{iu} is an infinitesimal unipotent group (e.g., α_p). Consider the presheaf on Sm

$$X \mapsto H^1(G)(X) := H_{\text{fppf}}^1(X, G),$$

which classifies isomorphism classes of fppf- G -torsors over X .

Theorem 3.6 [44, Theorem 9.12]. • *The presheaf $H^1(G)$ belongs to $\mathbf{RSC}_{\text{Nis}}$ and has level 2, except for the case when $G_{\text{iu}} = 0$, in which case it has level 1.*

- $H^1(G_{\text{em}} \times G_{\text{im}}) \in \mathbf{HI}_{\text{Nis}}$.
- *The map $L \rightarrow H^1(\alpha_p)(L)$ induced by the exact sequence of fppf-sheaves $0 \rightarrow \alpha_p \rightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \rightarrow 0$ restricts to surjections*

$$\widetilde{\mathbb{G}}_a(\mathcal{O}_L, \mathfrak{m}_L^{-r}) \twoheadrightarrow \widetilde{H^1(\alpha_p)}(\mathcal{O}_L, \mathfrak{m}_L^{-r}), \quad r \geq 0.$$

4. Tensor products and twists

The Lax monoidal structure on $\mathbf{RSC}_{\text{Nis}}$.

Definition 4.1. For two reciprocity sheaves F and G we define a new reciprocity sheaf by

$$(F, G)_{\mathbf{RSC}_{\text{Nis}}} := \omega_!(h_0^{\square}(\widetilde{F} \otimes_{\mathbf{MPST}} \widetilde{G}))_{\text{Nis}} \in \mathbf{RSC}_{\text{Nis}}.$$

It is not clear that this induces a monoidal structure since it is not clear that this construction is associative. Also, it is not clear that this is right exact since $\omega^{\mathbf{CI}}$ is not right exact. However, it induces a lax monoidal structure; see [45, Corollary 4.18].

Theorem 4.2 [45]. (1) Let \mathbf{HI}_{Nis} denote the category of \mathbb{A}^1 -invariant Nisnevich sheaves with transfers. By Voevodsky it has a symmetric monoidal structure denoted by $\otimes_{\mathbf{HI}_{\text{Nis}}}$. For $F, G \in \mathbf{HI}_{\text{Nis}}$ we have

$$(F, G)_{\mathbf{RSC}_{\text{Nis}}} = F \otimes_{\mathbf{HI}_{\text{Nis}}} G.$$

(2) If $\text{char } k = 0$, then

$$(G, A)_{\mathbf{RSC}_{\text{Nis}}} = 0,$$

for any commutative unipotent group G and abelian variety A .

(3) If $\text{char } k = 0$, then there are isomorphisms

$$(\mathbb{G}_a, \mathbb{G}_m)_{\mathbf{RSC}_{\text{Nis}}} \xrightarrow{\cong} \Omega_{/\mathbb{Z}}^1, \quad \gamma \otimes a \otimes u \mapsto \gamma^*(p^*a \wedge d \log q^*u),$$

and if we denote by $I_{\Delta_X} \subset \mathcal{O}_{X \times_{\mathbb{Z}} X}$ the ideal sheaf of the diagonal, then

$$(\mathbb{G}_a, \mathbb{G}_a)_{\mathbf{RSC}_{\text{Nis}}}(X) \xrightarrow{\cong} H^0(X, \mathcal{O}_{X \times_{\mathbb{Z}} X}/I_{\Delta_X}^2), \quad \gamma \otimes a \otimes b \mapsto \gamma^*(p^*a \otimes q^*b),^9$$

where $\gamma \in \mathbf{Cor}(X, Y \times Z)$, $a \in \mathbb{G}_a(Y)$, $b \in \mathbb{G}_a(Z)$, $u \in \mathbb{G}_m(Z)$, p and q denote the projections from $Y \times Z$ to Y and $Y \times Z$ to Z , respectively, and the element $\gamma \otimes a \otimes u$ denotes the image of the corresponding element in $(\mathbb{G}_a \otimes_{\mathbf{PST}} \mathbb{G}_m)(X)$ under the natural map $\mathbb{G}_a \otimes_{\mathbf{PST}} \mathbb{G}_m \rightarrow (\mathbb{G}_a, \mathbb{G}_m)_{\mathbf{RSC}_{\text{Nis}}}$, similarly with $\gamma \otimes a \otimes b$.

Definition 4.3. For G in $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ we define

$$G(n) := h_0^{\square}(G \otimes_{\mathbf{MPST}} \widetilde{K}_n^M)_{\text{Nis}}^{\text{sp}} \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$$

and

$$\gamma^n G := \underline{\mathbf{Hom}}_{\mathbf{MPST}}(\widetilde{K}_n^M, G) \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}},$$

where K_n^M denotes the (improved) Milnor K -sheaf (see [33]) and the upper sp in the first formula denotes the semipurification functor $\mathbf{CI}_{\text{Nis}}^{\tau} \rightarrow \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, which is given by $G^{\text{sp}} = \text{Im}(G \rightarrow \underline{\omega}^* \underline{\omega}!(G))$.

For F in $\mathbf{RSC}_{\text{Nis}}$ we define

$$F(1) := (F, \mathbb{G}_m)_{\mathbf{RSC}_{\text{Nis}}},$$

and recursively

$$F(n) := (F(n-1))(1) \in \mathbf{RSC}_{\text{Nis}}.$$

We also define

$$\gamma^n F := \underline{\mathbf{Hom}}_{\mathbf{PST}}(K_n^M, F) \in \mathbf{RSC}_{\text{Nis}}.$$

Generalizing part of Voevodsky's cancellation theorem [53], Merici and S. Saito show the following:

⁹We use the transfers structure stemming from the decomposition $\mathcal{O}_{X \times_{\mathbb{Z}} X}/I_{\Delta_X}^2 = \mathcal{O}_X \oplus \Omega_{/\mathbb{Z}}^1$.

Theorem 4.4 [38, Corollary 3.6]. *For F in $\mathbf{RSC}_{\text{Nis}}$ we have*

$$\gamma^n(\widetilde{F}(n)) \cong \widetilde{F} \quad \text{and} \quad \gamma^n(F\langle n \rangle) = F.$$

Proposition 4.5 [6; 45]. *There are identities*

- $\widetilde{\mathbb{Z}}(n) = \underline{\omega}^* K_n^M \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$,
- $\mathbb{Z}\langle n \rangle = K_n^M \in \mathbf{RSC}_{\text{Nis}}$.

In char $k = 0$, we have the identities

- $\widetilde{\mathbb{G}}_a(n) = \widetilde{\Omega}_{/\mathbb{Z}}^n \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$,
- $\mathbb{G}_a\langle n \rangle = \Omega_{/\mathbb{Z}}^n \in \mathbf{RSC}_{\text{Nis}}$.

The proof of the latter identities uses the computation of $\widetilde{\Omega}_{/\mathbb{Z}}^n(\mathcal{O}_L, \mathfrak{m}_L^{-r})$.

Proposition 4.6. *Assume char $k = p \geq 7$. Then (see (2-1) for notation)*

$$(\mathbb{G}_a, K_n^M)_{\mathbf{RSC}_{\text{Nis}}} = \Omega^n / B_\infty \Omega^n.$$

Proof. Since there is a natural morphism $\widetilde{\Omega}^n \rightarrow \widetilde{\Omega}^n / B_\infty$ in $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, and the inverse Cartier isomorphism induces an endomorphism $\overline{F} : \widetilde{\Omega}^n / B_\infty \rightarrow \widetilde{\Omega}^n / B_\infty$ (see (2-2)), we can use Theorems 3.2 and 3.3 to construct a map $(\mathbb{G}_a, K_n^M)_{\mathbf{RSC}_{\text{Nis}}} \rightarrow \Omega^n / B_\infty$ in $\mathbf{RSC}_{\text{Nis}}$ similar as in the proof of [45, Theorem 5.20]. By Corollary 1.18 and Theorem 1.19 it suffices to show that it is an isomorphism on any function field K . By [45, Proposition 5.18] there is a surjective map $\Omega_K^n \rightarrow (\mathbb{G}_a, K_n^M)_{\mathbf{RSC}_{\text{Nis}}}(K)$ (here we use $p \geq 7$). The same proof as in [24, Corollary 5.4.12] shows that this map factors over the quotient Ω_K^n / B_∞ . By construction of the maps, the composition

$$\Omega_K^n / B_\infty \rightarrow (\mathbb{G}_a, K_n^M)_{\mathbf{RSC}_{\text{Nis}}}(K) \rightarrow \Omega_K^n / B_\infty$$

is the identity, which completes the proof. \square

Remark 4.7. Similarly, one can also show $\mathbb{G}_a\langle n \rangle = \Omega^n / B_\infty$ (at least for $p \geq 7$). This is not immediate from the above. In the induction step $\Omega^n / B_\infty\langle 1 \rangle = \Omega^{n+1} / B_\infty$, a description of $\widetilde{\Omega}^n / B_\infty(\mathcal{O}_L, \mathfrak{m}_L^{-n})$ is required. The latter group was computed by Rülling and will appear somewhere else.

Proposition 4.8 [6, Theorem 11.8]. *Assume $p > 0$. There is a natural isomorphism*

$$W_r \Omega^{q-n} \xrightarrow{\sim} \gamma^n(W_r \Omega^q),$$

which sends a Witt-differential form $\omega \in W_r \Omega^{q-n}(X)$ to the map

$$\varphi_\omega \in \gamma^n(W_r \Omega^q)(X) = \mathbf{Hom}_{\mathbf{PST}}(K_n^M, \mathbf{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), W_r \Omega^q)),$$

which on Y is given by

$$\varphi_\omega(Y) : K_n^M(Y) \rightarrow W_r \Omega^q(X \times Y), \quad a \mapsto p_X^* \omega \cdot d \log(p_Y^* a),$$

where p_X, p_Y denote the two projections from $X \times Y$ to X, Y , respectively.

As $W_r \Omega^q$ is a successive extension of certain subquotients of Ω^q , this is a consequence of the well-known equality $R^1 \pi_* \Omega_{X \times \mathbb{P}^1}^q = \Omega_X^{q-1}$ and the fact that a reciprocity sheaf F satisfies

$$R^1 \pi_* F_{X \times \mathbb{P}^1} = (\gamma^1 F)_X,$$

where $\pi : \mathbb{P}_X^1 \rightarrow X$. This is a consequence of the cube-invariance of the cohomology of F ; see [47, Theorem 9.3].

5. Cohomology of reciprocity sheaves

We explain some structural results about the cohomology of reciprocity sheaves such as a projective bundle formula, a blow-up formula, a Gysin sequence, the existence of a proper pushforward and the existence of an action of Chow correspondences. This has consequences outside the theory of reciprocity sheaves. For example, we obtain new birational invariants of smooth projective varieties and obstructions to the existence of zero-cycles of degree one. We survey these applications at the end of this section.

Structural results.

Theorem 5.1 (blow-up formula, [6, Corollary 7.3]). *Let $G \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ and $\mathcal{X} = (X, D) \in \mathbf{MCor}_{\text{Is}}$ (see Definition 1.21 for notation). Assume that $i : Z \hookrightarrow X$ is a closed immersion of codimension c that is transversal to D (see 1.21). Let $\rho : \tilde{X} \rightarrow X$ denote the blow-up of X in Z , and let $\tilde{\mathcal{X}} := (\tilde{X}, D|_{\tilde{X}})$, and $\mathcal{Z} := (Z, D|_Z)$. Then*

$$R\rho_* G_{\tilde{\mathcal{X}}} \cong G_{\mathcal{X}} \oplus \bigoplus_{i=1}^{c-1} i_* \gamma^i G_{\mathcal{Z}}[-i].$$

Theorem 5.2 (projective bundle formula, [6, Theorem 6.3]). *Let $G \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ and $\mathcal{X} = (X, D) \in \mathbf{MCor}_{\text{Is}}$. Assume that $\pi : P \rightarrow X$ is a projective bundle of rank n , and let $\mathcal{P} := (P, D|_P)$. Then*

$$R\pi_* G_{\mathcal{P}} \cong \bigoplus_{i=0}^n (\gamma^i G)_{\mathcal{X}}[-i].$$

The proofs of these two theorems are intertwined and go beyond the scope of these lectures. We just mention some points to compare with classical arguments in the \mathbb{A}^1 -invariant case:

First one proves that there is a blow-up distinguished triangle (we comment on the proof below). The projective bundle formula is then proven by induction starting from the \mathbb{P}^1 -invariance of the cohomology, which is a consequence of the cube-invariance proven in [47, Theorem 9.3], and using the blow-up triangle in

the induction step. Using the projective bundle formula one can then, similar as Voevodsky, construct a splitting of the blow-up triangle, where one uses $\overline{\square}$ -invariance instead of \mathbb{A}^1 -invariance.

The essential point in the proof of the blow-up triangle is to show the vanishing

$$R^i \rho_* G_{(Y, \rho^* L)} = 0, \quad i \geq 1,$$

where $\rho : Y \rightarrow \mathbb{A}^2$ is the blow-up in the origin 0 and $L \subset \mathbb{A}^2$ is a line containing 0 . Denote by $\pi : Y \rightarrow \mathbb{P}^1$ the projection to the exceptional divisor of the blow-up. Then it is not hard to see that the above vanishing is implied by the vanishing

$$H^1(\mathbb{P}^1, \pi_* G_{(Y, \rho^* L)}) = 0.$$

This is shown in [6, Lemma 2.13]. The proof is a bit technical. However, in the course of this proof one is confronted with certain modulus-related problems which do not come up in the \mathbb{A}^1 -invariant story. This is why a crucial ingredient in the proof is the following modulus-descent result: Consider the morphism $\psi_0 : \mathbb{A}_y^1 \times \mathbb{A}_s^1 \rightarrow \mathbb{A}_x^1 \times \mathbb{A}_s^1$ given by the $k[s]$ -algebra morphism $k[x, s] \rightarrow k[y, s]$, $x \mapsto ys$. It induces a map

$$\psi : \overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(2)} \rightarrow \overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)} \quad \text{in } \underline{\mathbf{MCor}},$$

where $\overline{\square}^{(n)} = (\mathbb{P}^1, n \cdot \{0\} + n \cdot \{\infty\})$. Indeed to check this denote by $\Gamma \subset \mathbb{P}_y^1 \times \mathbb{P}_s^1 \times \mathbb{P}_x^1$ the closure of the graph of ψ_0 (as a morphism over \mathbb{A}_s^1). Then the claim holds by the following identities of divisors on Γ :

$$\begin{aligned} 2 \cdot \{0_s\} + \{0_y\} &= \{0_x\} + \{0_s\}, & 2 \cdot \{\infty_s\} + \{\infty_y\} &= \{\infty_x\} + \{\infty_s\}, \\ \{0_y\} &= \{0_x\} + \{\infty_s\}, & \{\infty_y\} &= \{\infty_x\} + \{0_s\}. \end{aligned}$$

In particular, the map ψ_0 does *not* define a modulus correspondence from $\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(1)}$ to $\overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)}$.

Proposition 5.3 [6, Proposition 2.5]. *Let $G \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$. With the notation from above ψ^* factors for $\mathcal{X} \in \underline{\mathbf{MCor}}_{\text{Is}}$ as*

$$\begin{array}{ccc} & G(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(1)} \otimes \mathcal{X}) & \\ & \nearrow \text{dotted arrow} & \downarrow \\ G(\overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)} \otimes \mathcal{X}) & \xrightarrow{\psi^*} & G(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(2)} \otimes \mathcal{X}), \end{array}$$

where the vertical map is induced by the natural morphism $\overline{\square}_s^{(2)} \rightarrow \overline{\square}_s^{(1)}$ and it is injective by the semipurity of G .

Let us illustrate Proposition 5.3 in the example $G = \widetilde{\Omega}_{/k}^2$ and $\mathcal{X} = (\text{Spec } k, \emptyset)$. In this case $G(\overline{\square}_x^{(1)} \otimes \overline{\square}_s^{(1)}) = k \cdot \text{dlog}(x) \text{dlog}(s)$ and we have

$$\psi^*(\text{dlog}(x) \text{dlog}(s)) = \text{dlog}(ys) \text{dlog}(s) = \text{dlog}(y) \text{dlog}(s) \in G(\overline{\square}_y^{(1)} \otimes \overline{\square}_s^{(1)}).$$

Example 5.4. We spell out the projective bundle formula in two concrete cases (the blow-up formula is similar). Let the situation be as in Theorem 5.2.

- In char $k = 0$ we have

$$R\pi_* \Omega_{P/\mathbb{Z}}^j(\log D|_P)(D|_P - D|_{P,\text{red}}) = \bigoplus_{i=0}^n \Omega_{X/\mathbb{Z}}^{j-i}(\log D)(D - D_{\text{red}})[-i].$$

Taking $D = \emptyset$ recovers the classical projection bundle formula.

- In positive characteristic p with $D = \emptyset$, we have

$$R\pi_*(R^{j+1}\epsilon_*\mathbb{Z}/p^r(j))_P = \bigoplus_{i=0}^n (R^{j-i+1}\epsilon_*\mathbb{Z}/p^r(j-i))_X[-i].$$

Note that this can also be deduced from the projective bundle formula for the Hodge–Witt cohomology by Gros [19, I, Theorem 4.1.11].

We remark, that in the second example we have to take the empty divisor since the formula $\gamma^i(R^{j+1}\epsilon_*\mathbb{Z}/p^r(j))_{(X,D)} = (R^{j-i+1}\epsilon_*\mathbb{Z}/p^r(j-i))_{(X,D)}$ is only known for $D = \emptyset$, in which case it follows from the exactness of γ^i and [6, Theorem 11.8]. In characteristic zero we have $\gamma^i\Omega_{/\mathbb{Z}}^j = \Omega_{/\mathbb{Z}}^{j-i}$ by [6, Corollary 11.2].

Similar to Voevodsky [52, Proposition 3.5.4] we obtain a Gysin triangle.

Theorem 5.5 (Gysin sequence [6, Theorem 7.16]). *Let $G \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{MPST}}_{\text{Is}}$. Assume that $i : Z \hookrightarrow X$ is a closed immersion of codimension c that is transversal to D , in the sense of Definition 1.21. Let $\mathcal{Z} := (Z, D|_Z)$, and let $\rho : \widetilde{X} \rightarrow X$ denote the blow-up of Z in X with $E = \rho^{-1}(Z)$ the exceptional divisor. Then there is an exact triangle*

$$i_*\gamma^c G_{\mathcal{Z}}[-c] \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} G_{\mathcal{X}} \rightarrow R\rho_* G_{(\widetilde{X}, D|_{\widetilde{X}+E})} \xrightarrow{\partial} i_*\gamma^c G_{\mathcal{Z}}[-c+1]$$

in $D(X_{\text{Nis}})$.

Example 5.6. In char $k = 0$ and $c = 1$ we get an exact sequence

$$\begin{aligned} 0 &\rightarrow \widetilde{\text{Conn}}^1(X, D) \rightarrow \widetilde{\text{Conn}}^1(X, D+Z) \\ &\rightarrow H^0(Z, \mathcal{O}_Z(i^*D - (i^*D)_{\text{red}}))/\mathbb{Z} \xrightarrow{g_{\mathcal{Z}/\mathcal{X}}} H^1\left(X, \frac{\Omega_{X/k}^1(\log D)(D - D_{\text{red}})}{d \log(j_*\mathcal{O}_{X \setminus D}^\times)}\right) \\ &\rightarrow H^1\left(X, \frac{\Omega_{X/k}^1(\log D+Z)(D - D_{\text{red}})}{d \log(j_*\mathcal{O}_{X \setminus (D+Z)}^\times)}\right) \end{aligned}$$

and for $c \geq 2$ we have

$$\widetilde{\text{Conn}}^1(X, D) \cong \widetilde{\text{Conn}}^1(\tilde{X}, \rho^*D + E),$$

where $\text{Conn}^1(X)$ denotes the group of isomorphism classes of rank-1 connections on X .

Example 5.7. Let $\text{Lisse}^1 \in \mathbf{RSC}_{\text{Nis}}$ be the sheaf whose sections over X are the lisse $\overline{\mathbb{Q}}_l$ -sheaves of rank 1, and $\widetilde{\text{Lisse}}^1(X, D)$ the lisse $\overline{\mathbb{Q}}_l$ -sheaves of rank 1 on $X \setminus D$ with Artin conductor less than or equal to D . Then for $\text{char } k = p > 0$, $l \neq p$, and $c \geq 2$, we have

$$\widetilde{\text{Lisse}}^1(X, D) = \widetilde{\text{Lisse}}^1(\tilde{X}, \rho^*D + E).$$

We can now define a proper correspondence action on reciprocity sheaves.

Definition 5.8. Let S be a scheme of finite type over k and let C_S denote the category of *proper (Chow) correspondences*, i.e., its objects are S -schemes $X \xrightarrow{f} S$ such that X is quasiprojective and smooth over k , and f is a morphism of finite type. Morphisms in C_S between (connected) objects X and Y are elements in

$$C_S(X, Y) := \varinjlim_{Z \subset X \times_S Y} \text{CH}_{\dim X}(Z),$$

where the direct limit runs over all closed subschemes $Z \subset X \times_S Y$ which are proper over X . The composition of morphisms in this category is defined using Fulton's refined intersections.

Definition 5.9. For $F \in \mathbf{RSC}_{\text{Nis}}$, and objects $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ in C_S , together with an element $\alpha \in C_S(X, Y)$, we define the *proper correspondence action*

$$\alpha^* : Rg_*F_Y \rightarrow Rf_*F_X \quad \text{in } D(S_{\text{Nis}})$$

by

- (1) pulling back to $X \times Y$,
- (2) cupping with α (minding the support),
- (3) pushing forward to X (using the properness of the support over X).

For (2) we note that the Gersten resolution for Milnor K -theory yields an identification $\text{CH}_{\dim X}(V) = H_V^e(X \times Y, K_e^M)$ (Bloch formula with support), where $V \subset X \times_S Y$ is proper over X and $e = \dim Y$, and hence $\alpha \in \text{CH}_{\dim X}(V)$ corresponds to a map $\alpha : \mathbb{Z}[-e] \rightarrow R\Gamma_V(K_e^M)$. The cupping with α is then defined as

the composition

$$\begin{aligned}
\gamma^e F[-e] &\xrightarrow{\alpha} \gamma^e F \otimes_{\mathbb{Z}}^L R\Gamma_V(K_e^M) \\
&\rightarrow R\Gamma_V(\gamma^e F \otimes_{\mathbf{MPST}} K_e^M) \\
&= R\Gamma_V(\underline{\mathbf{Hom}}_{\mathbf{MPST}}(K_e^M, F) \otimes_{\mathbf{MPST}} K_e^M) \\
&\xrightarrow{\text{adj.}} R\Gamma_V F.
\end{aligned}$$

The construction of the pushforward in (3) follows the classical method, but we have to keep track of the support, as the projection $X \times Y \rightarrow X$ does not need to be projective: Take a closed embedding of Y into an open U of a projective space P over $\text{Spec } k$. Then the pushforward is defined by using the Gysin map with support in V along the closed embedding $X \times Y \hookrightarrow X \times U$. By excision the cohomology of $X \times U$ with support in V agrees with the cohomology of $X \times P$ with support in V and we can use the projective bundle formula to pushforward to X . The cancellation theorem (Theorem 4.4) is used to cancel twists.

We obtain a functor $C_S \rightarrow D(S_{\text{Nis}})$ given by $(X \xrightarrow{f} S) \mapsto Rf_* F$.

Applications. We survey applications of the results presented in the previous sections; see [6, Chapters 10 and 11] for more details.

Obstructions to the existence of zero-cycles in degree 1.

Theorem 5.10 [6, Corollary 10.2]. *Let $F \in \mathbf{RSC}_{\text{Nis}}$. Let $f : X \rightarrow S$ be a projective dominant map between smooth k -schemes. Assume there exists a degree-1 zero-cycle on X_K , where $K = k(S)$. Then*

$$f^* : H^i(S, F_S) \rightarrow H^i(X, F_X)$$

is split-injective.

Proof. Take $\bar{\xi}$ as a lift of $\xi \in \text{CH}_0(X_K)^{\text{deg}=1}$ under the map

$$C_S(S, X) = \text{CH}_{\dim S}(S \times_S X) \rightarrow \text{CH}_0(X_K).$$

Then $f_* \bar{\xi} = [S] = \text{id}_S \in \text{CH}_{\dim S}(S) = C_S(S, S)$. Hence, we have the splitting

$$\bar{\xi}^* \circ f^* = (f_* \bar{\xi})^* = \text{id} : H^i(S, F) \rightarrow H^i(X, F) \rightarrow H^i(S, F). \quad \square$$

Generalized Brauer–Manin obstruction for zero-cycles. Let S be a smooth projective curve over k with function field $K = k(S)$. Let $f : X \rightarrow S$ be a projective dominant map with X smooth, and choose $v \in S_{(0)}$. Then for $\alpha_v \in \text{CH}_0(X_{K_v})$ with lift $\bar{\alpha}_v \in \text{CH}_1(X_{S_v})$, where K_v and S_v are the henselization of K and S at v , the map

$$\Psi(\alpha_v) := \bar{\alpha}_v^* : f_* F_{X_{S_v}} \rightarrow F_{S_v}$$

depends only on α_v and not on the lift $\bar{\alpha}_v$. Indeed if α'_v is a different lift then $\beta = \bar{\alpha}_v - \alpha'_v$ can be represented by a cycle supported in the special fiber X_v , and it follows from the construction of the proper correspondence action in Definition 5.9 that $\beta^* : f_* F_{X_{S_v}} \rightarrow F_{S_v}$ factors via $\Gamma_v(F_{S_v})$ which vanishes by S. Saito's purity theorem; see Theorem 1.19. Taking the first cohomology with support in v yields a map

$$H_v^1(S, f_* F_X) \xrightarrow{\Psi(\alpha_v)} H_v^1(S, F) \rightarrow H^1(S, F).$$

Thus we obtain a map

$$\Psi : \prod_{v \in S(0)} \text{CH}_0(X_{K_v}) \rightarrow \text{Hom} \left(\bigoplus_{v \in S(0)} H_v^1(S, f_* F_X), H^1(S, F) \right).$$

There is a map

$$\iota : F(X_K) \rightarrow \bigoplus_{v \in S(0)} H^0(S_v \setminus \{v\}, f_* F_X) \xrightarrow{\partial} \bigoplus_{v \in S(0)} H_v^1(S, f_* F_X).$$

Theorem 5.11 [6, Corollary 10.4]. *If $\Psi((\alpha_v)_v) \circ \iota \neq 0$, then there does not exist $\alpha \in \text{CH}_0(X_K)$ such that $\alpha \mapsto (\alpha_v)_v$.*

Proof. Take $\alpha \mapsto \alpha_v$ and $\bar{\alpha} \in \text{CH}_1(X)$ a lifting of α . We then get a diagram

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \text{---} & \searrow & \\ F(X_K) & \xrightarrow{\iota} & \bigoplus_v H_v^1(S, f_* F_X) & \longrightarrow & H^1(X, F) \\ & & \downarrow \sum_v \Psi(\alpha_v) & \swarrow \bar{\alpha}^* & \\ & & H^1(S, F) & & \end{array}$$

where the second map in the horizontal sequence is the composition of summing the forget-support maps $H_v^1(S, f_* F_X) \rightarrow H^1(S, f_* F_X)$ with the natural map $H^1(S, f_* F_X) \rightarrow H^1(X, F)$ induced by $f_* F_X \rightarrow Rf_* F_X$. \square

Remark 5.12. Assume $k = \mathbb{F}_q$ is a finite field with q elements and $F = \text{Br}$. Using the Cousin resolution (1-5) of Br_S and the fact that in the case at hand we have

$$H_v^1(S, \text{Br}_S) = \text{Br}(K_v) / \text{Br}(S_v) = \text{Br}(K_v),$$

the Brauer–Hasse–Noether theorem in the function field case yields

$$H_{\text{Nis}}^1(S, \text{Br}) = \text{Coker}(\text{Br}(K) \rightarrow \bigoplus_{v \in S(0)} \text{Br}(K_v)) = \mathbb{Q}/\mathbb{Z};$$

see [56, XIII, §3, Theorem 2 and §6, Theorem 4]. Thus Ψ equals the map

$$\prod_{v \in S(0)} \text{CH}_0(X_{K_v}) \rightarrow \text{Hom} \left(\bigoplus_{v \in S(0)} \frac{\text{Br}(X_{K_v})}{\text{Br}(X_{S_v})}, \mathbb{Q}/\mathbb{Z} \right),$$

which is the classical Brauer–Manin obstruction for zero-cycles in the function field case.

Stably birational invariance.

Definition 5.13. Let $X \xrightarrow{f} S, Y \xrightarrow{g} S$ be objects of C_S and assume that X and Y are integral. We say that f and g are *properly birational over S* if there exists proper birational S -maps $Z \rightarrow X$ and $Z \rightarrow Y$ (Z could be singular).

The maps f and g are said to be *stably properly birational over S* if there exist vector bundles V over X and W over Y such that $\mathbb{P}(V)$ and $\mathbb{P}(W)$ are properly birational over S .

Example 5.14. If S is singular, and X and Y are two different resolutions of S , then they are properly birational over S .

If f is proper and we take $S = Y = \text{Spec } k$, then f and $\text{id}_{\text{Spec } k}$ are stably properly birational if and only if X is stably rational over k .

Theorem 5.15 [6, Theorem 10.7]. *Any $F \in \mathbf{RSC}_{\text{Nis}}$ is a stably properly birational invariant over S . In other words, for every stably properly birational S -schemes $X \xrightarrow{f} S, Y \xrightarrow{g} S \in C_S$, we have an isomorphism*

$$f_*F_X \xrightarrow{\cong} g_*F_Y.$$

Proof. This follows from the projective bundle formula, purity, and the correspondence action. □

Theorem 5.16 [6, Theorem 10.10]. *Let $X \xrightarrow{f} S, Y \xrightarrow{g} S \in C_S$ be properly birational over S and let $F \in \mathbf{RSC}_{\text{Nis}}$. Assume that $F(K) = 0$ for all function fields K/k of transcendence degree $\leq d - 1$, where $d = \dim X = \dim Y$. Then*

$$Rg_*F_Y \xrightarrow{\cong} Rf_*F_X.$$

Proof. Take a closed subscheme $Z \subset X \times_S Y$ mapping properly and birationally to X and Y . Then $Z \circ Z^t = \Delta_Y + \epsilon$ with $p_{Y*}\epsilon \in \text{CH}^{\geq 1}(Y)$, and the condition on F implies that $\epsilon^* = 0$ on Rg_*F_Y ; see [6, Proposition 9.13]. This implies that $Z \circ Z^t$ acts as the identity on Rg_*F_Y ; similarly with $Z^t \circ Z$. □

Remark 5.17. Taking $Y = S$, and g the identity, yields the vanishing result

$$Rf_*F_X \cong 0.$$

Example 5.18. Assume that $\dim X = \dim Y = d$. Then Theorem 5.16 applies to the sheaves

$$\Omega_{/k}^d, \quad \Omega_{/k}^d/d \log K_d^M,$$

and, if $\text{char } k = p \neq 0$, also to the sheaves

- $W_n \Omega^d / B_\infty, R^i \epsilon_*(\mathbb{Z}/p^n(d)),$

- $G\langle d \rangle$, for G a smooth unipotent group,
- $H^1(G)\langle d \rangle$, for G a finite p -group over k ,
- $R^d \epsilon_* \mathbb{Q}/\mathbb{Z}(d)$ if k is algebraically closed.

Remark 5.19. The case for $\Omega_{/k}^d$ was known before by [11, Theorem 1]. It was later generalized to regular schemes in [13; 35].

In the first case in positive characteristic we use Geisser–Levine [18, Theorem 8.3], in the second and third case we use the Bloch–Kato–Gabber theorem [8, Theorem 2.1] and in the last case we use additionally the Milnor–Bloch–Kato conjecture, proven by Rost and Voevodsky [54, Theorem 6.16], to check that the condition $F(K) = 0$ for $\text{trdeg}(K/k) < d$ is satisfied in the cases at hand.

There is a version of Theorem 5.16 where the vanishing $F(K) = 0$ for $\text{trdeg}(K/k)$ less than d is replaced by the vanishing $\gamma^1 F = 0$ (which is, for example, satisfied if F is any smooth commutative unipotent group), but this requires at the moment resolution of singularities in dimension $d - 1$.

Corollary 5.20 [6, Corollary 11.24]. *Let $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ be flat, geometrically integral, and projective morphisms between smooth connected k -schemes. Assume that the generic fiber has index 1, (implying that the Picard schemes $\text{Pic}_{X/S}$ and $\text{Pic}_{Y/S}$ are representable). If X and Y are stably properly birational over S , then*

$$\text{Pic}_{X/S}[n] \cong \text{Pic}_{Y/S}[n]$$

on S_{Nis} for all n .

Remark 5.21. The above result is classical for $S = \text{Spec } k$, with k algebraically closed.

Decomposition of the diagonal.

Definition 5.22. Let K be a function field over k and X a smooth scheme over K with $\dim X = d$. We say that the *diagonal of X decomposes* if

$$[\Delta_X] = p_2^* \xi + (i \times \text{id})_* \beta \in \text{CH}^d(X \times_K X), \quad (5-1)$$

where $\xi \in \text{CH}_0(X)$ and $\beta \in \text{CH}_d(Z \times_K X)$ for some closed subscheme $i : Z \hookrightarrow X$ with $\text{codim}(Z, X) \geq 1$.

This condition was first considered by Bloch and Srinivas for rational coefficients in [9]. By [14, Lemma 1.5] an integral smooth projective k -scheme X , which is *retract rational* (i.e., there exist open dense subsets $U \subset X$ and $V \subset \mathbb{P}_K^n$ together with a map $V \rightarrow U$ which has a section), has the property that its diagonal decomposes. Hence implications of (5-1) on cohomology yield obstructions to X being retract rational over K .

Theorem 5.23 [6, Theorem 10.13]. *Let S be the henselization of a smooth k -scheme in a 1-codimensional point or a regular connected affine scheme of dimension ≤ 1 and of finite type over a function field K over k . Let $f : X \rightarrow S$ be a smooth and projective morphism, and assume that the diagonal of the generic fiber of f decomposes. Then*

$$F(S) = F(X),$$

for any $F \in \mathbf{RSC}_{\text{Nis}}$.

Remark 5.24. In [3, Problem 1.2] the following problem is posed:

Let k be algebraically closed with $\text{char } k = p > 0$ and X a smooth and proper scheme over k with decomposition of the diagonal. Do we then have that

$$H^0(X, R^i \epsilon_* \mathbb{Z}/p(j)) = 0 \quad \text{for all } i \neq 0?$$

Theorem 5.23 gives a positive answer to the problem if X is projective over k . Indeed, if $S = \text{Spec } k$ and $F = R^i \epsilon_* \mathbb{Z}/p(j)$ we observe that $F(k) = 0$; see (2-2).

Theorem 5.25 [6, Corollary 11.22]. *Let $f : X \rightarrow S$ be a projective morphism between smooth integral and quasiprojective k -schemes. Let $\dim S = e$ and $\dim X = d$. Assume the diagonal of the generic fiber of f decomposes. Then*

$$f_* : Rf_* F_X^d \xrightarrow{\cong} F_S^e[e-d]$$

is an isomorphism, where F^d is one of the sheaves

- $\Omega_{/k}^d$, or
- $\Omega_{/k}^d/d \log K_d^M$, or
- $R^d \epsilon_* \mathbb{Q}/\mathbb{Z}(d)$ if k is algebraically closed,

or, if $\text{char } k = p > 0$, one of the sheaves

- $W_n \Omega^d/B_\infty$, or
- $R^i \epsilon_*(\mathbb{Z}/p^n(d))$, or
- $G\langle d \rangle$, for G a smooth commutative unipotent k -group, or
- $H^1(G)\langle d \rangle$, for G a finite p -group over k .

Example 5.26. If k is algebraically closed and X is smooth and projective over k of dimension d such that the diagonal of X decomposes, then

$$H^i(X, R^{d+1} \epsilon_* \mathbb{Z}/p^n(d)) = 0 \quad \text{for all } i.$$

6. Further results

We give an overview of some more recent results obtained in [40; 42]. This section was not part of the original lecture series given at CAS.

Zariski–Nagata purity. Let U be a smooth k -scheme over a perfect base field k and let K be a function field over k . For any presheaf with transfers we have a pairing

$$F(U_K) \otimes \mathbf{Cor}_K(\mathrm{Spec} K, U_K) \rightarrow F(K), \quad (a, \gamma) \mapsto \gamma^* a, \quad (6-1)$$

where $U_K = U \otimes_k K$ and \mathbf{Cor}_K denotes the finite correspondences on smooth K -schemes. For X an integral finite-type k -scheme of dimension d , and $D \subset X$ a closed subscheme with ideal sheaf I_D and open complement $U = X \setminus D$, we have by [32, Theorem 2.5] a surjective map

$$\mathbf{Cor}_K(\mathrm{Spec} K, U_K) \twoheadrightarrow H^d(X_{K, \mathrm{Nis}}, K_d^M(\mathcal{O}_{X_K}, I_{D_K})), \quad (6-2)$$

where $K_d^M(\mathcal{O}_{X_K}, I_{D_K}) = \mathrm{Ker}(K_d^M(\mathcal{O}_{X_K}) \rightarrow K_d^M(\mathcal{O}_{D_K}))$. The map is induced by the isomorphism $\mathbb{Z} \cong H_x^d(X_{K, \mathrm{Nis}}, K_d^M(\mathcal{O}_{X_K}, I_{D_K}))$, for x a closed point in U_K , stemming from the Gersten resolution; see [33, Proposition 10 (8)].

Theorem 6.1 [42, Theorem 1.6]. *Let X be a smooth integral projective k -scheme of dimension d and let $\sum_{i=1}^r D_i$ be an SNC divisor with complement $U = X \setminus \bigcup_{i=1}^r D_i$. Let $\mathbf{n} = (n_1, \dots, n_r) \in (\mathbb{N}_{\geq 1})^r$ and set $D_{\mathbf{n}} = \sum_{i=1}^r n_i D_i$. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Then the following are equivalent for $a \in F(U)$:*

- (i) $a \in \tilde{F}(X, D_{\mathbf{n}})$.
- (ii) $a \in \tilde{F}(\mathcal{O}_{X, \eta_i}, \mathfrak{m}_{\eta_i}^{-n_i})$, for all i , where η_i is the generic point of D_i .
- (iii) For any function field K the map $(a_K, -)_{U_K/K} : \mathbf{Cor}_K(\mathrm{Spec} K, U_K) \rightarrow F(K)$ induced by (6-1) factors via (6-2) to give a map

$$(a_K, -)_{(X_K, D_{\mathbf{n}, K})/K} : H^d(X_{K, \mathrm{Nis}}, K_d^M(\mathcal{O}_{X_K}, I_{D_{\mathbf{n}, K}})) \rightarrow F(K), \quad (6-3)$$

where a_K denotes the pullback of a to $F(U_K)$.

Note that the equivalence of (i) and (ii) is a statement of Zariski–Nagata type (“purity of the branch locus”). It continues to hold if X is assumed to be quasiprojective and $(X, \sum_i D_i)$ has a projective SNC-compactification (which is always the case in characteristic zero); see the proof of [42, Corollary 6.10]. In case $r = 1$ (i.e., $D_{\mathbf{n}}$ has just one component) the equivalence of (i) and (ii) also follows from [47, Corollary 8.6(2)]. For $\mathbf{n} = (1, \dots, 1)$ (i.e., $D_{\mathbf{n}}$ is reduced SNCD), it follows from [48, Corollary 2.4].

Observe that from the equivalence of (i) and (iii) we obtain a map

$$\mathrm{CH}_0(X | D_{\mathbf{n}}) \twoheadrightarrow H^d(X_{\mathrm{Nis}}, K_d^M(\mathcal{O}_X, I_{D_{\mathbf{n}}})) \rightarrow \mathrm{Hom}(\tilde{F}(X, D_{\mathbf{n}}), F(k)), \quad (6-4)$$

where $\mathrm{CH}_0(X | D_{\mathbf{n}})$ denotes the Chow group of zero-cycles with modulus introduced in [34] and the first map is induced from (6-2). (This factorization is proved in

various cases by, for example, Krishna as well as Gupta and Krishna. For the situation at hand, see [41]). Taking the limit of the composition we get a natural map

$$C(U) := \varprojlim_n \mathrm{CH}_0(X \mid D_n) \rightarrow \mathrm{Hom}(F(U), F(k)).$$

If k is a finite field and $F = \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\mathrm{ab}}(-), \mathbb{Q}/\mathbb{Z})$, then $F(k) \cong \mathbb{Q}/\mathbb{Z}$ and this map is the *reciprocity homomorphism* constructed in [34, Proposition 3.2]. Similarly the limit over n of the second map in (6-4) is the *reciprocity homomorphism* constructed in [32].

The formula of Abbes and T. Saito. Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. We ask the following:

- (1) Is it possible to give a more computable description of \tilde{F} , in particular without using the transfers structure of F ?
- (2) For X smooth and $D \subset X$ a smooth divisor, we get from the Gysin sequence of Theorem 5.5 an isomorphism

$$\tilde{F}(X, D)/F(X) \cong \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{G}_m, F)(D).$$

Can we also describe the quotients $\tilde{F}(X, nD)/\tilde{F}(X, (n-1)D)$, for $n \geq 2$?

In [40] it is shown that these questions can be approached using a method introduced by Abbes and Saito in [2] and Saito [46] to study the ramification of Galois torsors by means of dilatations. For simplicity we assume in the following that:

- (*) X is smooth, D is a smooth divisor on X , $U = X \setminus D$, and (X, D) has a projective SNC compactification, that is, there exists an open embedding $X \hookrightarrow \bar{X}$ into a smooth projective scheme \bar{X} such that $\bar{X} \setminus U$ is the support of a divisor with simple normal crossings.¹⁰

The dilatation $P_X^{(nD)}$, for $n \geq 1$, is the blow-up of $X \times X$ in nD diagonally embedded and with the strict transforms of $X \times nD$ and $nD \times X$ removed. It comes with two maps

$$p_1, p_2 : P_X^{(nD)} \rightrightarrows X$$

induced by the two projection maps $X \times X \rightarrow X$. Note that the open immersion $U \times U \hookrightarrow X \times X$ extends to an open immersion $U \times U \hookrightarrow P_X^{(nD)}$.

Theorem 6.2 [40, Theorem 1]. *Assume (*) and let $n \geq 1$. Then*

$$\tilde{F}(X, nD) = \{a \in F(U) \mid p_1^*a = p_2^*a \text{ in } F(U \times U)/F(P_X^{(nD)})\}.$$

In particular the theorem applies to $F = H_{\mathrm{fppf}}^1(-, G)$,¹¹ where G is a commutative finite k -group scheme (not necessarily étale). A version of this formula was proved

¹⁰In [40] more generally the case where D is a SNCD is considered.

¹¹In this case even without the assumption on the existence of a projective SNC compactification.

by Abbes and Saito for G any étale k -group (not necessarily commutative). For more details and precise references see [40, Example 2.12].

The proof of the above theorem uses heavily the theory of higher local symbols along Paršin chains for reciprocity sheaves developed in [43], which in turn relies on Theorem 6.1 and Section 5.

Using Theorem 6.2 we obtain the following partial description of the quotients considered in (2) above.

Theorem 6.3 [40, Theorem 4.12]. *Assume $(*)$ and let $n \geq 2$. Then there is an injective map*

$$\mathrm{char}_F^{(nD)} : \frac{\tilde{F}(X, nD)}{\tilde{F}(X, (n-1)D)} \hookrightarrow H^0(D, \Omega_X^1(nD)|_D \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{Sh}_D}(\mathcal{O}_D, F_D)),$$

where $\underline{\mathrm{Hom}}_{\mathrm{Sh}_D}$ denotes the internal hom in the category of Nisnevich sheaves of abelian groups on smooth schemes over D .

Some comments:

- If $\mathrm{char} k = p > 0$, it follows that the quotient on the left-hand side is p -torsion. This can be seen as an analogue of [46, Corollary 2.28] for reciprocity sheaves.
- The characteristic form for

$$F = H_{\mathrm{ét}}^1(-, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{ab}}(-), \mathbb{Q}/\mathbb{Z})$$

factors via

$$H^0(D, \Omega^1(nD)|_D) \rightarrow H^0(D, \Omega_X^1(nD)|_D \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{Sh}_D}(\mathcal{O}_D, H_{\mathrm{ét}, D}^1))$$

induced by the natural map $\mathcal{O}_D \rightarrow H_{\mathrm{ét}, D}^1$ stemming from the Artin–Schreier sequence, and yields a global version of the characteristic form of Matsuda–Yatagawa (which is a nonlog version of the refined Swan conductor of Kato); see [40, Section 5] for details.

- The characteristic form for differential forms is computed in [40, Theorem 6.6 and Corollary 6.8]. These computations are also used to prove the formula in Theorem 3.2 in positive characteristic.
- It is an intriguing problem to give a general (motivic) description of the image of $\mathrm{char}_F^{(nD)}$. For example, the images of the characteristic forms of differential forms and Witt vectors of finite length in positive characteristic are rather complicated and do not give a direct hint towards a general formula.

Finally, we give an exemplary application of how a local form of Theorem 6.3 reveals an interesting structure of Chow groups of zero-cycles with modulus over local fields of equicharacteristic.

Example 6.4. Let Y be a proper k -scheme with an effective Cartier divisor E , such that $V = Y \setminus E$ is smooth. By [28, Corollary 2.3.5; 47, Theorem 0.1] the Nisnevich sheafification of $\omega_! h_0^\square(Y, E)$ (see Definition 1.5) is a reciprocity sheaf $h_0(Y, E)_{\text{Nis}}$. For a field K over k we have

$$h_0(Y, E)_{\text{Nis}}(K) = \text{CH}_0(Y_K, E_K),$$

where the right-hand side denotes the Chow group of zero-cycles with modulus and $Y_K = Y \otimes_k K$.

Assume L is a henselian discrete valuation field of geometric type over k with ring of integers \mathcal{O}_L , maximal ideal \mathfrak{m}_L , and residue field $K = \mathcal{O}_L/\mathfrak{m}_L$. For simplicity assume the transcendence degree of L/k is 1, so that all geometric models of $(\mathcal{O}_L, \mathfrak{m}_L)$ have a projective SNC-compactification and $\Omega_{\mathcal{O}_L/k}^1 \otimes_{\mathcal{O}_L} K \cong K$. Then $\text{fil}_n := h_0(Y, E)_{\text{Nis}}(S, ns)$, where $S = \text{Spec } \mathcal{O}_L$ and $s \in S$ is the closed point, defines a filtration

$$\text{fil}_0 \subset \text{fil}_1 \subset \cdots \subset \text{fil}_n \subset \cdots \subset \text{CH}_0(Y_L, E_L),$$

where fil_0 is the subgroup of $\text{CH}_0(Y_L, E_L)$ generated by closed points in V_L whose closure in $V \times_k S$ is finite over S . By Theorem 6.3 we have an injection (which depends on the choice of a local parameter; more precisely, we use the isomorphism $\mathfrak{m}^{-n} \Omega_{\mathcal{O}_L}^1 \otimes_{\mathcal{O}_L} K = \mathfrak{m}^{-n}/\mathfrak{m}^{-n+1} \cong K$)

$$\text{fil}_n/\text{fil}_{n-1} \hookrightarrow \underline{\text{Hom}}_{\text{Sh}_K}(\mathcal{O}_K, h_0(Y, E)_{\text{Nis}}(K)) \quad \text{for } n \geq 2.$$

However the internal hom on the right is not well understood and it would be interesting to describe the image of the map. Evaluating at 1 induces a canonical map

$$\text{fil}_n/\text{fil}_{n-1} \rightarrow \text{CH}_0(Y_K, E_K) \quad \text{for } n \geq 2.$$

This is a new map and it is tempting to view it as a specialization map (depending on the choice of a local parameter). It remains to study its properties more closely, e.g., if it happens to be injective for certain pairs (Y, E) .

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References

- [1] A. Abbes and T. Saito, ‘‘Analyse micro-locale l -adique en caractéristique $p > 0$: le cas d’un trait’’, *Publ. Res. Inst. Math. Sci.* **45**:1 (2009), 25–74. MR Zbl

- [2] A. Abbes and T. Saito, “Ramification and cleanliness”, *Tohoku Math. J. (2)* **63**:4 (2011), 775–853. MR Zbl
- [3] A. Auel, A. Bigazzi, C. Böhnig, and H.-C. G. von Bothmer, “Universal triviality of the Chow group of 0-cycles and the Brauer group”, *Int. Math. Res. Not.* **2021**:4 (2021), 2479–2496. MR Zbl
- [4] B. Bhatt, J. Lurie, and A. Mathew, *Revisiting the de Rham–Witt complex*, Astérisque **424**, Société Mathématique de France, Paris, 2021. MR Zbl
- [5] F. Binda, D. Park, and P. A. Østvær, *Triangulated categories of logarithmic motives over a field*, Astérisque **433**, Société Mathématique de France, Paris, 2022. MR Zbl
- [6] F. Binda, K. Rülling, and S. Saito, “On the cohomology of reciprocity sheaves”, *Forum Math. Sigma* **10** (2022), art. id. e72. MR Zbl
- [7] S. Bloch, “Algebraic K -theory and crystalline cohomology”, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 187–268. MR Zbl
- [8] S. Bloch and K. Kato, “ p -adic étale cohomology”, *Inst. Hautes Études Sci. Publ. Math.* **63** (1986), 107–152. MR Zbl
- [9] S. Bloch and V. Srinivas, “Remarks on correspondences and algebraic cycles”, *Amer. J. Math.* **105**:5 (1983), 1235–1253. MR Zbl
- [10] J.-L. Brylinski, “Théorie du corps de classes de Kato et revêtements abéliens de surfaces”, *Ann. Inst. Fourier (Grenoble)* **33**:3 (1983), 23–38. MR Zbl
- [11] A. Chatzistamatiou and K. Rülling, “Higher direct images of the structure sheaf in positive characteristic”, *Algebra Number Theory* **5**:6 (2011), 693–775. MR Zbl
- [12] A. Chatzistamatiou and K. Rülling, “Hodge–Witt cohomology and Witt-rational singularities”, *Doc. Math.* **17** (2012), 663–781. MR Zbl
- [13] A. Chatzistamatiou and K. Rülling, “Vanishing of the higher direct images of the structure sheaf”, *Compos. Math.* **151**:11 (2015), 2131–2144. MR Zbl
- [14] J.-L. Colliot-Thélène and A. Pirutka, “Hypersurfaces quartiques de dimension 3: non-rationalité stable”, *Ann. Sci. Éc. Norm. Supér. (4)* **49**:2 (2016), 371–397. MR Zbl
- [15] J.-L. Colliot-Thélène, J.-J. Sansuc, and C. Soulé, “Torsion dans le groupe de Chow de codimension deux”, *Duke Math. J.* **50**:3 (1983), 763–801. MR Zbl
- [16] J. Cuntz and C. Deninger, “Witt vector rings and the relative de Rham Witt complex”, *J. Algebra* **440** (2015), 545–593. MR Zbl
- [17] T. Ekedahl, “On the multiplicative properties of the de Rham–Witt complex, I”, *Ark. Mat.* **22**:2 (1984), 185–239. MR Zbl
- [18] T. Geisser and M. Levine, “The K -theory of fields in characteristic p ”, *Invent. Math.* **139**:3 (2000), 459–493. MR
- [19] M. Gros, *Classes de Chern et classes de cycles en cohomologie de Hodge–Witt logarithmique*, Mém. Soc. Math. France (N.S.) **21**, 1985. MR Zbl
- [20] L. Hesselholt, “The big de Rham–Witt complex”, *Acta Math.* **214**:1 (2015), 135–207. MR Zbl
- [21] L. Hesselholt and I. Madsen, “On the De Rham–Witt complex in mixed characteristic”, *Ann. Sci. École Norm. Sup. (4)* **37**:1 (2004), 1–43. MR Zbl
- [22] L. Illusie, “Complexe de de Rham–Witt et cohomologie cristalline”, *Ann. Sci. École Norm. Sup. (4)* **12**:4 (1979), 501–661. MR Zbl
- [23] L. Illusie and M. Raynaud, “Les suites spectrales associées au complexe de de Rham–Witt”, *Inst. Hautes Études Sci. Publ. Math.* **57** (1983), 73–212. MR Zbl

- [24] F. Ivorra and K. Rülling, “K-groups of reciprocity functors”, *J. Algebraic Geom.* **26**:2 (2017), 199–278. MR Zbl
- [25] B. Kahn, S. Saito, and T. Yamazaki, “Reciprocity sheaves”, *Compos. Math.* **152**:9 (2016), 1851–1898. MR Zbl
- [26] B. Kahn, H. Miyazaki, S. Saito, and T. Yamazaki, “Motives with modulus, I: Modulus sheaves with transfers for non-proper modulus pairs”, *Épjournal Géom. Algébrique* **5** (2021), art. id. 1. MR Zbl
- [27] B. Kahn, H. Miyazaki, S. Saito, and T. Yamazaki, “Motives with modulus, II: Modulus sheaves with transfers for proper modulus pairs”, *Épjournal Géom. Algébrique* **5** (2021), art. id. 2. MR Zbl
- [28] B. Kahn, S. Saito, and T. Yamazaki, “Reciprocity sheaves, II”, *Homology Homotopy Appl.* **24**:1 (2022), 71–91. MR Zbl
- [29] K. Kato, “A generalization of local class field theory by using K -groups, II”, *Proc. Japan Acad. Ser. A Math. Sci.* **54**:8 (1978), 250–255. MR Zbl
- [30] K. Kato, “Swan conductors for characters of degree one in the imperfect residue field case”, pp. 101–131 in *Algebraic K-theory and algebraic number theory* (Honolulu, HI, 1987), Contemp. Math. **83**, Amer. Math. Soc., Providence, RI, 1989. MR Zbl
- [31] K. Kato and H. Russell, “Modulus of a rational map into a commutative algebraic group”, *Kyoto J. Math.* **50**:3 (2010), 607–622. Zbl
- [32] K. Kato and S. Saito, “Global class field theory of arithmetic schemes”, pp. 255–331 in *Applications of algebraic K-theory to algebraic geometry and number theory, Part I* (Boulder, CO, 1983), Contemp. Math. **55**, Amer. Math. Soc., Providence, RI, 1986. MR Zbl
- [33] M. Kerz, “Milnor K -theory of local rings with finite residue fields”, *J. Algebraic Geom.* **19**:1 (2010), 173–191. MR Zbl
- [34] M. Kerz and S. Saito, “Chow group of 0-cycles with modulus and higher-dimensional class field theory”, *Duke Math. J.* **165**:15 (2016), 2811–2897. MR Zbl
- [35] S. J. Kovács, “Rational singularities”, preprint, 2017. arXiv 1703.02269
- [36] A. Langer and T. Zink, “De Rham–Witt cohomology for a proper and smooth morphism”, *J. Inst. Math. Jussieu* **3**:2 (2004), 231–314. MR Zbl
- [37] S. Matsuda, “On the Swan conductor in positive characteristic”, *Amer. J. Math.* **119**:4 (1997), 705–739. MR Zbl
- [38] A. Merici and S. Saito, “Cancellation theorems for reciprocity sheaves”, preprint, 2020. arXiv 1703.02269v13
- [39] M. Rosenlicht, “A universal mapping property of generalized jacobian varieties”, *Ann. of Math.* (2) **66** (1957), 80–88. MR Zbl
- [40] K. Rülling and S. Saito, “Ramification theory for reciprocity sheaves, III: Abbes–Saito formula”, preprint, 2022. arXiv 2204.10637v1
- [41] K. Rülling and S. Saito, “Cycle class maps for Chow groups of zero-cycles with modulus”, *J. Pure Appl. Algebra* **227**:5 (2023), art. id. 107282. MR Zbl
- [42] K. Rülling and S. Saito, “Ramification theory of reciprocity sheaves, I: Zariski–Nagata purity”, *J. Reine Angew. Math.* **797** (2023), 41–78. MR Zbl
- [43] K. Rülling and S. Saito, “Ramification theory of reciprocity sheaves, II: higher local symbols”, *Eur. J. Math.* **9**:3 (2023), art. id. 56. MR Zbl
- [44] K. Rülling and S. Saito, “Reciprocity sheaves and their ramification filtrations”, *J. Inst. Math. Jussieu* **22**:1 (2023), 71–144. MR Zbl

- [45] K. Rülling, R. Sugiyama, and T. Yamazaki, “Tensor structures in the theory of modulus presheaves with transfers”, *Math. Z.* **300**:1 (2022), 929–977. MR Zbl
- [46] T. Saito, “Wild ramification and the cotangent bundle”, *J. Algebraic Geom.* **26**:3 (2017), 399–473. MR Zbl
- [47] S. Saito, “Purity of reciprocity sheaves”, *Adv. Math.* **366** (2020), art. id. 107067. MR Zbl
- [48] S. Saito, “Reciprocity sheaves and logarithmic motives”, *Compos. Math.* **159**:2 (2023), 355–379. MR Zbl
- [49] J.-P. Serre, *Groupes algébriques et corps de classes*, Publ. Inst. Math. Univ. Nancago **7**, Hermann, Paris, 1959. MR Zbl
- [50] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, 1979. MR Zbl
- [51] M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas, Tome 3: Exposés IX–XIX* (Séminaire de Géométrie Algébrique du Bois Marie 1963–1964), Lecture Notes in Math. **305**, Springer, 1973. MR Zbl
- [52] V. Voevodsky, “Triangulated categories of motives over a field”, pp. 188–238 in *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud. **143**, Princeton University Press, 2000. MR Zbl
- [53] V. Voevodsky, “Cancellation theorem”, *Doc. Math.* Suslin birthday volume (2010), 671–685. MR Zbl
- [54] V. Voevodsky, “On motivic cohomology with \mathbf{Z}/l -coefficients”, *Ann. of Math. (2)* **174**:1 (2011), 401–438. MR Zbl
- [55] V. Voevodsky, A. Suslin, and E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies **143**, Princeton University Press, 2000. MR Zbl
- [56] A. Weil, *Basic number theory*, Springer, 1995. MR Zbl
- [57] Y. Yatagawa, “Equality of two non-logarithmic ramification filtrations of abelianized Galois group in positive characteristic”, *Doc. Math.* **22** (2017), 917–952. MR Zbl

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Motivic Geometry

Based on lectures given at the Centre for Advanced Study (CAS) of the Norwegian Academy of Science and Letters, this book provides a panorama of developments in motivic homotopy theory and related fields.

A common goal of the research program underlying this volume is the understanding of the geometric nature of spaces, revealed through algebraic and homotopical invariants. The articles in this volume, contributed by leading experts, together touch on an extensive network of related topics in algebraic geometry, homotopy theory, K -theory and related areas.

The volume has a significant expository component, making it accessible to students, while also containing information and in-depth discussion of interest to all practitioners including specialists.

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