

Motivic Geometry

The Grothendieck ring of varieties and algebraic K -theory of spaces

Oliver Röndigs



$$\begin{array}{ccc}
 X & \xleftarrow{j} \bar{X} & \xrightarrow{i} \partial \bar{X} \\
 & \searrow f & \downarrow \bar{f} \\
 & & \text{Spec}(k)
 \end{array}
 \quad
 \begin{array}{c}
 \prod^{\infty} X \simeq g_* i^* j_* f^!(1) \\
 \sum_{\text{lines}} \text{Tr}_{L/k}(\alpha) = 15\langle 1 \rangle + 12\langle -1 \rangle \\
 u_* (\mathcal{X}(X/k)) = \pi_* (e^{\mathcal{E}}(T_{X/k})) \in \mathcal{E}^{0,0}(k) \\
 \text{DM}^{\text{eff}}(k) \simeq \log \text{DM}^{\text{eff}}(k)
 \end{array}
 \quad
 \begin{array}{c}
 0 \rightarrow K_{2-n}^M(k)/24 \rightarrow \pi_{n+1,n} \mathbf{1} \rightarrow \pi_{n+1,n} \mathbf{kq} \rightarrow 0
 \end{array}$$

The Grothendieck ring of varieties and algebraic K -theory of spaces

Oliver Röndigs

Waldhausen's algebraic K -theory machinery is applied to Morel–Voevodsky \mathbb{A}^1 -homotopy, producing an interesting \mathbb{A}^1 -homotopy type. Over a field F of characteristic zero, its path components receive a surjective ring homomorphism from the Grothendieck ring of varieties over F .

1. Introduction

Waldhausen's approach to algebraic K -theory [46] is of such a generality (though not for its own sake) that it applies to a wide class of homotopy theories. The choice here, as suggested by Waldhausen in the last century, is the \mathbb{A}^1 -homotopy theory over a Noetherian finite-dimensional base scheme S introduced by Morel and Voevodsky [28]. Subject to an appropriate finiteness condition (for which there are several choices), the resulting homotopy type $A(S)$ is nontrivial; for example, it contains Waldhausen's algebraic K -theory of a point, $A(*)$, as a retract up to homotopy. It can be viewed as an \mathbb{A}^1 -homotopy type in a natural way. The present paper is an admittedly rather meager attempt to advertise this \mathbb{A}^1 -homotopy type to algebraic geometers, although it might be more attractive to homotopy theorists. Recall that almost by construction, the path components of Waldhausen's K -theory provide the universal Euler characteristic.

Theorem 1. *Let F be a field of characteristic zero. Sending a smooth projective variety to its natural class in $\pi_0 A(F)$ defines a surjective ring homomorphism*

$$K_0(\mathrm{Var}_F) \rightarrow \pi_0 A(F)$$

from the Grothendieck ring of varieties over F .

MSC2020: primary 14F42, 19D10; secondary 55P42.

Keywords: Grothendieck ring of varieties, motivic homotopy theory, Waldhausen K -theory of spaces.

See [Theorem 5.2](#) for a precise version including the appropriate finiteness condition. This ring homomorphism refines several other motivic measures, such as the topological Euler characteristic, the Hodge motivic measure, and the Gillet–Soulé motivic measure. In these cases the ring homomorphism is naturally induced on path components by a map from the homotopy type $A(F)$, thereby solving at least problems 7.3–7.5 in [10] in a natural way. The constructions [9; 49] supply a homotopy type whose path components form the Grothendieck ring of varieties over F . However, the significance of its higher homotopy groups is not clear. In the case of Waldhausen’s original application of algebraic K -theory to the geometry of manifolds, the higher homotopy groups yield interesting information on their automorphism groups [15; 34; 35], thanks to the following statement from [47].

Theorem 2 (Waldhausen). *Let M be a smooth manifold with possibly empty boundary. The homotopy type $A(M)$ is defined as the Waldhausen K -theory of the category of finite cell complexes retractive over M . There is a splitting*

$$A(M) \simeq \Sigma^\infty M_+ \times \text{Wh}(M)$$

up to homotopy, where $\Sigma^\infty M_+$ is the stable homotopy type of M , and $\text{Wh}(M)$ is the Whitehead spectrum of M , a double delooping of the spectrum of stable smooth pseudoisotopies of M .

Taking path components of this splitting recovers the s -cobordism theorem of Smale, Barden, Mazur, and Stallings. Perhaps an \mathbb{A}^1 - s -cobordism theorem for smooth varieties over a field can be produced via the trace given in [Section 6](#) on the \mathbb{A}^1 -homotopy type obtained from the algebraic K -theory of dualizable motivic T -spectra.

2. K -theory of model categories

In [46], Waldhausen generalized Quillen’s K -theory machinery to the setup of categories with cofibrations and weak equivalences, henceforth called *Waldhausen categories*. A Waldhausen category is a quadruple $(\mathbf{C}, *, \text{wC}, \text{cofC})$, where \mathbf{C} is a pointed category with zero object $*$, a subcategory wC of weak equivalences and a subcategory cofC of cofibrations. Also, $* \rightarrow A$ is always a cofibration, cobase changes along cofibrations exist in \mathbf{C} , and the weak equivalences satisfy the gluing lemma. The *algebraic K -theory* of a Waldhausen category $(\mathbf{C}, *, \text{wC}, \text{cofC})$ is the spectrum

$$A(\mathbf{C}) = (\text{wC}, \text{wS}_\bullet \mathbf{C}, \dots, \text{wS}_\bullet^{(n)} \mathbf{C}, \dots) \quad (1)$$

of pointed simplicial sets obtained by the diagonal of the nerve of n -fold simplicial categories; the latter produced by iterated applications of Waldhausen’s

S_\bullet -construction. The structure maps of (1) are induced by the inclusion of the 1-skeleton.

Definition 2.1. An exact functor $F : \mathbf{C} \rightarrow \mathbf{D}$ of Waldhausen categories is a *K-theory equivalence* if the induced map $A(F) : A(\mathbf{C}) \rightarrow A(\mathbf{D})$ of spectra is a stable equivalence.

All the categories with cofibrations and weak equivalences in the following are obtained as full subcategories of a Quillen model category, where the weak equivalences are determined by the model structure. The cofibrations are either determined by the model structure, or by a slight variation. References for model categories are [17; 18]. Algebraic *K*-theory requires finiteness conditions, and suitable cofibrantly generated model categories, as defined in [13, Definition 3.4; 19, Definition 4.1], provide a convenient setup for these.

Definition 2.2. A model category \mathbf{M} is *weakly finitely generated* if it is cofibrantly generated and satisfies the following further requirements:

- (1) There exists a set I of generating cofibrations with finitely presentable domains and codomains.
- (2) There exists a set J of acyclic cofibrations with finitely presentable domains and codomains detecting fibrations with fibrant codomain.

Examples of weakly finitely generated model categories are the usual model categories of (pointed) simplicial sets (denoted \mathbf{sSet}_\bullet), spectra of such (denoted \mathbf{Spt}), chain complexes over a ring, and suitable model structures for \mathbb{A}^1 -homotopy theory. The latter is essentially a consequence of the following statement.

Proposition 2.3. *Let \mathbf{M} be a weakly finitely generated simplicial model category, and let Z be a set of morphisms in \mathbf{M} with finitely presentable domains and codomains. Suppose that tensoring with a finite simplicial set L preserves finitely presentable objects. If the left-Bousfield localization $L_Z\mathbf{M}$ exists, it is weakly finitely generated.*

Proof. The proof of [19, Proposition 4.2] applies. □

In a weakly finitely generated model category \mathbf{M} , a fibrant replacement functor

$$\text{fib} : \mathbf{M} \rightarrow \mathbf{M},$$

which commutes with filtered colimits, can be constructed by attaching cells from a set of acyclic cofibrations J with finitely presentable domains and codomains. It follows that the natural transformation $\text{Id}_{\mathbf{M}} \rightarrow \text{fib}$ is an acyclic cofibration.

For applications in algebraic *K*-theory recall that, given an object $B \in \mathbf{M}$ in a category, an object *retractive over B* is a pair $(B \xrightarrow{s} D, D \xrightarrow{r} B)$ of morphisms in \mathbf{M} such that $r \circ s = \text{id}_B$. Such a pair will often be abbreviated as “ D ”. With

the obvious notion of morphism, these form a category $\mathbf{R}(\mathbf{M}, B)$. A morphism $\phi : B \rightarrow C$ in \mathbf{M} induces a functor

$$\phi_! : \mathbf{R}(\mathbf{M}, B) \rightarrow \mathbf{R}(\mathbf{M}, C), \quad D \mapsto D \cup_B C$$

having the functor

$$\phi^! : \mathbf{R}(\mathbf{M}, C) \rightarrow \mathbf{R}(\mathbf{M}, B), \quad E \mapsto E \times_C B$$

as right adjoint, provided pushouts and pullbacks exist.

Proposition 2.4. *Let \mathbf{M} be a weakly finitely generated model category, and B an object of \mathbf{M} . The category $\mathbf{R}(\mathbf{M}, B)$ of objects retractive over B is a weakly finitely generated model category in a natural way. For every $\phi : B \rightarrow C$, the pair $(\phi_!, \phi^!)$ is Quillen. If \mathbf{M} is simplicial, then so is $\mathbf{R}(\mathbf{M}, B)$. If \mathbf{M} is a monoidal model category under the cartesian product, then $\mathbf{R}(\mathbf{M}, B)$ is a $\mathbf{R}(\mathbf{M}, *)$ -model category.*

Proof. The statement regarding the (simplicial) model structure is [39, Proposition 1.2.2], which implies the Quillen pair property. The statement regarding the generators can be deduced from [39, Lemma 1.3.4]. For later reference, if $I = \{s_i \hookrightarrow t_i\}_{i \in I}$ is the set of generating cofibrations in \mathbf{M} , then

$$\left\{ B \coprod_{i \in I, \psi \in \text{Hom}_{\mathbf{M}}(t_i, B)} (s_i \hookrightarrow t_i) \right\}$$

is the set of generating cofibration in $\mathbf{R}(\mathbf{M}, B)$, where the maps $\psi : t_i \rightarrow B$ define the required retractions. Note that $\phi_!$ preserves this set of generating cofibrations. The final statement follows from [18, Proposition 4.2.9] and the standard pairing

$$\mathbf{R}(\mathbf{M}, B) \times \mathbf{R}(\mathbf{M}, C) \rightarrow \mathbf{R}(\mathbf{M}, B \times C), \quad (D, E) \mapsto D \times E \cup_{(B \times E \cup_{B \times C}, D \times C)} B \times C,$$

of retractive objects, which is a Quillen bifunctor. □

Definition 2.5. Let \mathbf{M} be a symmetric monoidal model category, let \mathbf{N} be an \mathbf{M} -model category, and let B be an object of \mathbf{M} . A B -spectrum E in \mathbf{N} consists of a sequence (E_0, E_1, \dots) of objects in \mathbf{N} together with a sequence of structure maps

$$\sigma_n^E : \Sigma_B E_n := E_n \wedge B \rightarrow E_{n+1}.$$

The category of B -spectra in \mathbf{N} is denoted by $\mathbf{Spt}_B(\mathbf{N})$. Set $\mathbf{Spt}(\mathbf{N}) := \mathbf{Spt}_{S^1}(\mathbf{N})$, where $S^1 \in \mathbf{sSet}_\bullet = \mathbf{M}$ is the category of pointed simplicial sets.

Proposition 2.6. *Let \mathbf{M} be a symmetric monoidal \mathbf{sSet}_\bullet -model category, let \mathbf{N} be an \mathbf{M} -model category, and let B be a finitely presentable and cofibrant object of \mathbf{M} . Suppose that tensoring with a finite simplicial set L preserves finitely presentable objects in \mathbf{N} . If \mathbf{N} is weakly finitely generated, then $\mathbf{Spt}_B(\mathbf{N})$ is a weakly finitely generated model category such that $\Sigma_B^\infty : \mathbf{N} \rightarrow \mathbf{Spt}_B(\mathbf{N})$ is a left-Quillen functor.*

Proof. The proof given for [19, Theorem 4.12] applies. For later reference, the required sets are listed explicitly. The levelwise model structure on $\mathbf{Spt}_B(\mathbf{N})$ is weakly finitely generated with the sets

$$\mathrm{Fr} I := \{\mathrm{Fr}_n i\}_{n \geq 0, i \in I} \quad \text{and} \quad \mathrm{Fr} J := \{\mathrm{Fr}_n j\}_{n \geq 0, j \in J},$$

where Fr_n is the left adjoint of the evaluation functor sending E to E_n , and I and J are sets of maps in \mathbf{N} satisfying Definition 2.2. The statement follows from Proposition 2.3 because the B -stable model structure is a left-Bousfield localization of the levelwise model structure with respect to the set

$$\{\mathrm{Fr}_{n+1}(C \wedge B) \rightarrow \mathrm{Fr}_n C\}_{n \in \mathbb{N}, C \text{ domain or codomain in } I}$$

of morphisms with finitely presentable domains and codomains. \square

As soon as B is a suspension (for example, S^1 itself), the model structure from Proposition 2.6 on B -spectra is stable in the sense of [18, Definition 7.1.1]. In particular, the weak equivalences of B -spectra then satisfy Waldhausen's extension axiom [46, p. 327]. However, since $\mathbf{Spt}_B(\mathbf{M})$ usually does not inherit any monoidality properties from \mathbf{M} , one has to use Jeff Smith's symmetric B -spectra instead. Consider [19, Theorem 8.11, Corollary 10.4] for the following.

Proposition 2.7. *Let \mathbf{M} be a symmetric monoidal \mathbf{sSet}_\bullet -model category, let \mathbf{N} be an \mathbf{M} -model category, and let B be a finitely presentable and cofibrant object of \mathbf{M} . Suppose that tensoring with a finite simplicial set L preserves finitely presentable objects. If \mathbf{N} is weakly finitely generated, then $\mathbf{SymSpt}_B(\mathbf{N})$ is a weakly finitely generated $\mathbf{SymSpt}_B(\mathbf{M})$ -model category such that $\Sigma_B^\infty : \mathbf{N} \rightarrow \mathbf{SymSpt}_B(\mathbf{N})$ is a left-Quillen functor. If additionally the cyclic permutation on $B \wedge B \wedge B$ is homotopic to the identity, the model categories $\mathbf{SymSpt}_B(\mathbf{N})$ and $\mathbf{Spt}_B(\mathbf{N})$ are Quillen equivalent.*

Proof. The proof given for [19, Theorem 8.11, Corollary 10.4] applies. For later reference, the required sets are listed explicitly. The levelwise model structure on $\mathbf{SymSpt}_B(\mathbf{M})$ is weakly finitely generated with the sets

$$\mathrm{Fr}^{\mathrm{sym}} I := \{\mathrm{Fr}_n^{\mathrm{sym}} i\}_{n \geq 0, i \in I} \quad \text{and} \quad \mathrm{Fr}^{\mathrm{sym}} J := \{\mathrm{Fr}_n^{\mathrm{sym}} j\}_{n \geq 0, j \in J},$$

where $\mathrm{Fr}_n^{\mathrm{sym}}$ is the left adjoint of the evaluation functor sending E to E_n , and I and J are sets of maps in \mathbf{M} satisfying Definition 2.2. The statement follows from Proposition 2.3 because the B -stable model structure is a left-Bousfield localization of the levelwise model structure with respect to the set

$$\{\mathrm{Fr}_{n+1}^{\mathrm{sym}}(C \wedge B) \rightarrow \mathrm{Fr}_n^{\mathrm{sym}} C\}_{n \in \mathbb{N}, C \text{ domain or codomain in } I}$$

of morphisms with finitely presentable domains and codomains. \square

Lemma 2.8. *Let \mathbf{M} be a symmetric monoidal \mathbf{sSet}_\bullet -model category, let \mathbf{N} be an \mathbf{M} -model category, and let B be a finitely presentable and cofibrant object of \mathbf{M} . Suppose that tensoring with a finite simplicial set L preserves finitely presentable objects in \mathbf{N} . Suppose that \mathbf{N} is weakly finitely generated, and let $\mathbf{Spt}_B(\mathbf{N})$ be the stable model category of B -spectra in \mathbf{N} . If E is a cofibrant finitely presentable B -spectrum in \mathbf{N} which is stably contractible, then there exists a natural number N such that E_n is contractible for every $n \geq N$.*

Proof. Let E be finitely presentable and cofibrant. Then E_n is cofibrant and finitely presentable for every n . Moreover, there exists a natural number M such that the structure maps σ_m are isomorphisms for every $m \geq M$. Since the canonical map

$$\mathrm{Fr}_M(E_M) \rightarrow E$$

from the shifted suspension spectrum of E_M to E is a stable equivalence, one may work with $\mathrm{Fr}_M(E_M)$ directly. Moreover, one may choose $M = 0$. Thus E_0 is a cofibrant finitely presentable object with the property that $\mathrm{colim}_n \Omega^n \mathrm{fib}(\Sigma^n E_0)$ is contractible. Here $\mathrm{fib} : \mathbf{N} \rightarrow \mathbf{N}$ is a fibrant replacement functor. Equivalently, the class of the canonical map $E_0 \rightarrow \Omega^n \mathrm{fib}(\Sigma^n E_0)$ becomes the class of the constant map in the colimit

$$[E_0, \mathrm{fib}(E_0)] \rightarrow [E_0, \Omega \mathrm{fib}(\Sigma_B E_0)] \rightarrow \cdots \rightarrow [E_0, \Omega^n \mathrm{fib}(\Sigma_B^n E_0)] \rightarrow \cdots$$

of sets of pointed homotopy classes of maps. As E_0 and $E_0 \otimes \Delta^1$ are finitely presentable, there exists a natural number N such that the homotopy class of the canonical map $E_0 \rightarrow \Omega^n \mathrm{fib}(\Sigma_B^n E_0)$ coincides with the homotopy class of the constant map. By adjointness, the canonical map $\Sigma_B^n E_0 \rightarrow \mathrm{fib}(\Sigma_B^n E_0)$ is homotopic to the constant map for every $n \geq N$. Thus $\Sigma_B^n E_0$ is contractible for every $n \geq N$. \square

Definition 2.2 leads to the following finiteness notions. More variations, such as being finitely dominated, are possible.

Definition 2.9. Let \mathbf{M} be a weakly finitely generated pointed model category, and choose a set of generating cofibrations I with finitely presentable domains and codomains. Let $B \in \mathbf{M}$.

- (1) The object B is *finite* if it is cofibrant and finitely presentable; in other words, $\mathrm{Hom}_{\mathbf{M}}(B, -)$ commutes with filtered colimits.
- (2) The object B is *homotopy finite* if it is cofibrant and weakly equivalent to a finite object.
- (3) The object B is *I -finite* if the map $* \rightarrow B$ is obtained by attaching finitely many maps from I .
- (4) The object B is *I -homotopy finite* if it is cofibrant and weakly equivalent to an I -finite object.

The resulting full subcategories are denoted \mathbf{M}^{fin} , \mathbf{M}^{hfin} , \mathbf{M}^{ifin} , and $\mathbf{M}^{\text{ihfin}}$, respectively.

The category \mathbf{M}^{ifin} is essentially small, and so is \mathbf{M}^{fin} , at least if \mathbf{M} is locally finitely presentable. This will usually not be the case for $\mathbf{M}^{\text{ihfin}}$ and \mathbf{M}^{hfin} . This set-theoretical issue can be resolved in several ways, but will be ignored in the present approach, following [46, Remark on page 379]. Homotopy finite objects in a weakly finitely generated pointed model category \mathbf{M} are compact in the homotopy category of \mathbf{M} , as the proof of [18, Theorem 7.4.3] shows. In the case where \mathbf{M} is a symmetric monoidal model category, another finiteness notion is quite natural and will be used eventually.

Definition 2.10. Let $(\mathbf{M}, \wedge, \mathbb{1})$ be a symmetric monoidal model category. A cofibrant object B is *dualizable* if there exists a cofibrant object C and morphisms $\phi : \mathbb{1} \rightarrow B \wedge C$, $\psi : C \wedge B \rightarrow \mathbb{1}$ in the homotopy category of \mathbf{M} , such that the compositions

$$B \xrightarrow{\phi \wedge B} B \wedge C \wedge B \xrightarrow{B \wedge \psi} B \quad \text{and} \quad C \xrightarrow{C \wedge \phi} C \wedge B \wedge C \xrightarrow{\psi \wedge C} C$$

are the respective identities. The full subcategory of cofibrant and dualizable objects is denoted \mathbf{M}^{dual} .

The categories introduced in Definitions 2.9 and 2.10 are equipped with a subcategory of weak equivalences by intersecting with $\text{w}\mathbf{M}$, and with a subcategory of cofibrations by intersecting with $\text{cof}\mathbf{M}$ in the cases of \mathbf{M}^{fin} , \mathbf{M}^{hfin} , and \mathbf{M}^{dual} . In the cases of \mathbf{M}^{ifin} and $\mathbf{M}^{\text{ihfin}}$ this may lead to trouble with the required existence of cobase changes. The subcategory of cofibrations in \mathbf{M}^{ifin} consists of those maps obtained by attaching finitely many cells from I , and in $\mathbf{M}^{\text{ihfin}}$ it is simply maps obtained by attaching cells from I .

Lemma 2.11. *Let \mathbf{M} be a weakly finitely generated model category, and choose a set of generating cofibrations I with finitely presentable domains and codomains. If $B \hookrightarrow E$ is a cofibration of finitely presentable objects in \mathbf{M} , there exists a finite I -cofibration $B \hookrightarrow C$ in \mathbf{M} such that $B \hookrightarrow E$ is a retract of $B \hookrightarrow C$.*

Proof. Factor $B \hookrightarrow E$ via the small object argument applied to I , to obtain a lifting problem

$$\begin{array}{ccc} B & \xrightarrow{i} & D \\ \downarrow & & \downarrow \sim \\ E & \xrightarrow{\text{id}} & E \end{array}$$

which can be solved. The object D is a sequential colimit of a diagram

$$B = D_{-1} \hookrightarrow D_0 \hookrightarrow \dots \hookrightarrow D_n \hookrightarrow D_{n+1} \hookrightarrow \dots$$

such that D_{n+1} is obtained by attaching I -cells to D_n indexed by a specific subset of the disjoint union $\coprod_{i \in I} \text{Hom}_{\mathbf{M}}(\text{dom}(i), D_n)$ for every n . Since E is finitely presentable, a lift $E \rightarrow D$ factors via a morphism $E \rightarrow D_{n+1}$. This object is the filtered colimit of objects $D_{n,\alpha}$ which are obtained by attaching finitely many cells to D_n . This colimit is indexed over certain finite subsets $\alpha \subset \coprod_{i \in I} \text{Hom}_{\mathbf{M}}(\text{dom}(i), D_n)$. Again since E is finitely presentable, there exists a finite subset $\beta \subset \coprod_{i \in I} \text{Hom}_{\mathbf{M}}(\text{dom}(i), D_n)$ and a factorization over $E \rightarrow D_{n,\beta}$. Since the domains of the morphisms in I are finitely presentable, one may proceed in the same fashion for every domain in β inductively to obtain a factorization $E \rightarrow C$ where C is obtained by attaching finitely many I -cells. \square

Already algebraic examples such as chain complexes $\mathbf{Ch}(R)$ over a ring R show that the classes of finite and I -finite objects can be different, for example, on the level of algebraic K -theory. Bounded chain complexes of finitely generated projective R -modules are the finite objects in the standard model category of all chain complexes of R -modules given in [18, Definition 2.3.3], whereas the I -finite objects are the bounded chain complexes of finitely generated free R -modules. One may use [48, Corollary II.26.3 and Theorem II.9.2.2] to conclude that $K_0(\mathbf{Ch}^{\text{fin}}(\mathbb{Z}[\sqrt{-5}])) \cong \mathbb{Z}$ and $K_0(\mathbf{Ch}^{\text{fin}}(\mathbb{Z}[\sqrt{-5}])) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proposition 2.12. *Let \mathbf{M} be a weakly finitely generated model category, and choose a set I of generating cofibrations with finitely presentable domains and codomains. Then the following hold.*

- (1) *With these choices, \mathbf{M}^{fin} , \mathbf{M}^{hfin} , \mathbf{M}^{ifin} and $\mathbf{M}^{\text{ihfin}}$ are categories with cofibrations and weak equivalences.*
- (2) *The horizontal functors in the commutative diagram*

$$\begin{array}{ccc}
 \mathbf{M}^{\text{ifin}} & \longrightarrow & \mathbf{M}^{\text{ihfin}} \\
 \downarrow & & \downarrow \\
 \mathbf{M}^{\text{fin}} & \longrightarrow & \mathbf{M}^{\text{hfin}}
 \end{array} \tag{2}$$

of exact inclusion functors are K -theory equivalences.

- (3) *The path components of the homotopy fiber of the map $A(\mathbf{M}^{\text{if}} \hookrightarrow \mathbf{M}^{\text{fin}})$ are all contractible.*

Proof. Consider statement (1) first. The gluing lemma follows from the cube lemma for model categories [18, Lemma 5.2.6]. It remains to check in each case that a cobase change in \mathbf{M} along the cofibration in question does not lead outside of the category in question. For \mathbf{M}^{fin} this follows, since a pushout of finitely presentable objects is again finitely presentable, and cofibrancy is preserved.

In the case of \mathbf{M}^{hfin} , let $C \leftarrow B \hookrightarrow D$ be a diagram in \mathbf{M}^{hfin} such that $B \hookrightarrow D$ is a cofibration. Let $B \rightarrow \text{fib}(B)$ be a fibrant replacement obtained by attaching cells from a set J with finitely presentable domains and codomains. Choose a weak equivalence $B' \xrightarrow{\sim} \text{fib}(B)$ from a finite object. The gluing lemma then implies that one may choose B to be finite. Similarly to the above, choose a finite object D' and weak equivalences $D \xrightarrow{\sim} \text{fib}(D') \xleftarrow{\sim} D'$. Note that $D' \xrightarrow{\sim} \text{fib}(D')$ is the filtered colimit of certain maps $D' \xrightarrow{\sim} D''$ which are obtained by attaching finitely many maps from J . Since B is finitely presentable, $B \hookrightarrow D \xrightarrow{\sim} \text{fib}(D')$ lifts to such a D'' , and analogously for C . By assumption on J and the gluing lemma, statement (1) follows.

Statement (1) follows for \mathbf{M}^{ifin} basically by definition. The argument in the case of $\mathbf{M}^{\text{ihfin}}$ is similar to the argument in the case of \mathbf{M}^{hfin} , which concludes the proof of statement (1).

Diagram (2) exists because the domains and codomains of the maps in I are finitely presentable. The statements for $A(\mathbf{M}^{\text{fin}}) \rightarrow A(\mathbf{M}^{\text{hfin}})$ and for $A(\mathbf{M}^{\text{ifin}}) \rightarrow A(\mathbf{M}^{\text{ihfin}})$ follow from [38, Theorem 2.8]. For statement (3) observe that every object in \mathbf{M}^{fin} is a retract of an object in \mathbf{M}^{ifin} by Lemma 2.11. One then concludes with [40, Theorem 1.10.1]. \square

Proposition 2.13. *Let \mathbf{M} be a symmetric monoidal stable model category. Then \mathbf{M}^{dual} is a Waldhausen category in the natural way described above.*

Proof. It remains to prove that if $C \leftarrow B \hookrightarrow D$ is a diagram in \mathbf{M}^{dual} , then its pushout $C \cup_B D$ is again dualizable. This follows, for example, from [26, Theorem 0.1]. \square

The next statement is quite useful, since it implies that through the eyes of algebraic K -theory, restriction to stable model categories is acceptable. This in turn allows the full applicability of Waldhausen’s theory, since the weak equivalences then satisfy the extension axiom. Its proof goes back to [45] which led to [36].

Theorem 2.14. *Let \mathbf{M} be a pointed simplicial model category such that tensoring with a finite simplicial set preserves finitely presentable objects. Suppose that \mathbf{M} is weakly finitely generated, and let $\mathbf{Spt}(\mathbf{M})$ be the stable model category of S^1 -spectra in \mathbf{M} . Then the suspension spectrum functor induces a K -theory equivalence*

$$A(\mathbf{M}^g) \rightarrow A(\mathbf{Spt}^g(\mathbf{M})),$$

where $g \in \{\text{fin}, \text{hfin}\}$. If Σ preserves cofibrations in \mathbf{M}^{ifin} , then the same is true for $g \in \{\text{ifin}, \text{ihfin}\}$.

Proof. Consider first $g = \text{fin}$. By the additivity theorem [46, Theorem 1.4.2] the suspension functor $\Sigma = - \wedge S^1$ induces a K -theory equivalence. Consider the colimit of

$$\mathbf{M}^{\text{fin}} \xrightarrow{\Sigma} \mathbf{M}^{\text{fin}} \xrightarrow{\Sigma} \dots$$

in the category of Waldhausen categories. There is an isomorphism

$$\operatorname{colim}_{\Sigma} \mathcal{S}_{\bullet} \mathbf{M}^{\operatorname{fin}} \cong \mathcal{S}_{\bullet} \operatorname{colim}_{\Sigma} \mathbf{M}^{\operatorname{fin}},$$

which implies that the canonical functor $\mathbf{M}^{\operatorname{fin}} \rightarrow \operatorname{colim}_{\Sigma} \mathbf{M}^{\operatorname{fin}}$ is a K -theory equivalence. A S^1 -spectrum E is *strictly finite* if there exists a natural number $N = N(E)$ such that E_N is finite and for every $n \geq N$ the structure map $\sigma_n : \Sigma E_n \rightarrow E_{n+1}$ is the identity. Let $\mathbf{Spt}^{\operatorname{sf}}(\mathbf{M})$ denote the full subcategory of strictly finite S^1 -spectra which are also cofibrant. It is a Waldhausen category in a natural way, and the inclusion $\mathbf{Spt}^{\operatorname{sf}}(\mathbf{M}) \hookrightarrow \mathbf{Spt}^{\operatorname{fin}}(\mathbf{M})$ is an exact equivalence. In particular, the inclusion is a K -theory equivalence.

Let $\Phi : \mathbf{Spt}^{\operatorname{sf}}(\mathbf{M}) \rightarrow \operatorname{colim}_{\Sigma} \mathbf{M}^f$ denote the functor sending E to the equivalence class of (E_n, n) , where $n \geq N(E)$. This functor is well defined, preserves cofibrations, and pushouts essentially by construction. Moreover, it preserves weak equivalences by Lemma 2.8. It is straightforward to verify that Φ satisfies the conditions of Waldhausen's approximation theorem [46, Theorem 1.6.7]. Thus Φ is a K -theory equivalence. Lastly, note the suspension spectrum functor $\Sigma^{\infty} : \mathbf{M}^{\operatorname{fin}} \rightarrow \mathbf{Spt}^{\operatorname{fin}}(\mathbf{M})$ factors as

$$\mathbf{M}^{\operatorname{fin}} \rightarrow \mathbf{Spt}^{\operatorname{sf}}(\mathbf{M}) \hookrightarrow \mathbf{Spt}^{\operatorname{fin}}(\mathbf{M}),$$

which completes the proof for $g = \operatorname{fin}$. The case $g = \operatorname{hfin}$ then follows from Proposition 2.12. The extra assumption on Σ implies that these arguments apply also to $g \in \{\operatorname{ifin}, \operatorname{ihfin}\}$. \square

Theorem 2.14 provides many examples of nonequivalent homotopy theories having the same K -theory.

3. \mathbb{A}^1 -homotopy theory

Motivic or \mathbb{A}^1 -homotopy theory was introduced in [28]. Its stabilization is considered in [22]. For technical reasons, the unstable projective version (which is the basis of [14]) is more convenient, although the closed motivic model structure described in [31, Appendix] seems to be quite ideal for the comparison with the Grothendieck ring of varieties.

A *base scheme* is a Noetherian separated scheme of finite Krull dimension. A *motivic space* over S is a presheaf on the site Sm_S of smooth separated S -schemes with values in the category of simplicial sets. Let $\mathbf{M}(S)$ denote the category of pointed motivic spaces.

Example 3.1. Any scheme X in Sm_S defines a discrete representable motivic space over S which is also denoted X , and a discrete representable pointed motivic space X_+ over S . One has $X_+(Y) = \mathbf{Set}_{\operatorname{Sm}_S}(Y, X)_+$, where B_+ denotes the set B with

a disjoint basepoint. Any (pointed) simplicial set L defines a constant (pointed) motivic space which is also denoted L .

Many model structures exist on $\mathbf{M}(S)$ having the Morel–Voevodsky \mathbb{A}^1 -homotopy category of S as its homotopy category. Waldhausen’s setup of algebraic K -theory requires specific choices. The following model structure is well suited for base change (see [28, Example 3.1.22]).

Definition 3.2. *Cofibrations* in $\mathbf{M}(S)$ are generated by the set

$$\{(X \times (\partial \Delta^n \hookrightarrow \Delta^n))_+ \}_{X \in \text{Sm}_S, n \geq 0}. \quad (3)$$

Applying the small object argument to this set produces a cofibrant replacement functor $\kappa : (-)^c \rightarrow \text{Id}_{\mathbf{M}(S)}$. A pointed motivic space B is *fibrant* if

- $B(X)$ is a fibrant simplicial set for all $X \in \text{Sm}_S$,
- the image of every Nisnevich elementary distinguished square

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

in Sm_S under B is a homotopy pullback square of simplicial sets,

- $B(\emptyset)$ is contractible, and
- for every $X \in \text{Sm}_S$, the map $B(X \times \mathbb{A}^1 \xrightarrow{\text{pt}} X)$ is a weak equivalence of simplicial sets.

A map $\phi : D \rightarrow B$ of pointed motivic spaces over S is a *weak equivalence* if, for every fibrant motivic space C , the induced map

$$\mathbf{sSet}_{\mathbf{M}(S)}(\phi^c, C) : \mathbf{sSet}_{\mathbf{M}(S)}(B^c, C) \rightarrow \mathbf{sSet}_{\mathbf{M}(S)}(D^c, C)$$

is a weak equivalence of simplicial sets. A map of motivic spaces is a *fibration* if it has the right lifting property with respect to all cofibrations which are also weak equivalences (the *acyclic* cofibrations).

Theorem 3.3. *The classes from Definition 3.2 define a symmetric monoidal \mathbf{sSet}_- -model structure on $\mathbf{M}(S)$ which is weakly finitely generated. It is Quillen equivalent to the Morel–Voevodsky model.*

Proof. See [14, Section 2.1] for a proof. □

Remark 3.4. The smash product of pointed motivic spaces is defined sectionwise. The smash product of a weak equivalence with an arbitrary pointed motivic space is a weak equivalence. Since the domains and codomains of the generating cofibrations are finitely presentable, a filtered colimit of weak equivalences is again a weak equivalence. A filtered colimit of fibrant motivic spaces is again fibrant.

Proposition 2.6 applies to the model structure from **Theorem 3.3**. The two relevant examples are S^1 -spectra $\mathbf{Spt}(S) := \mathbf{Spt}_{S^1}(\mathbf{M}(S))$ and T -spectra $\mathbf{Spt}_T(S) := \mathbf{Spt}_T(\mathbf{M}(S))$, as well as their symmetric analogues $\mathbf{SymSpt}(S)$ and $\mathbf{SymSpt}_T(S)$. Here $S^1 = \Delta^1/\partial\Delta^1$ is the constant simplicial circle, and $T = S^1 \wedge S^{1,1}$, where $S^{1,1}$ is the simplicial mapping cylinder of the unit $S \hookrightarrow \mathbb{G}_m$ in the multiplicative group scheme over S .

Definition 3.5. Let S be a base scheme. Then I_S denotes the set of generating cofibrations in $\mathbf{M}(S)$ given in (3), or (if no confusion can arise) the corresponding set of generating cofibrations in (symmetric) B -spectra over S as introduced in the proof of **Proposition 2.6** and **2.7**, respectively.

If $f : X \rightarrow Y$ is a morphism of base schemes, pullback along f defines a functor $\mathrm{Sm}_Y \rightarrow \mathrm{Sm}_X$. Precomposition with this functor yields another functor, denoted $f_* : \mathbf{M}(X) \rightarrow \mathbf{M}(Y)$. On objects

$$(f_*B)(Z) = B(X \times_Y Z) \quad (4)$$

for any $Z \in \mathrm{Sm}_Y$. Via left-Kan extension, f_* has a left adjoint $f^* : \mathbf{M}(Y) \rightarrow \mathbf{M}(X)$ which is strict symmetric monoidal. Since every motivic space is a colimit of representable ones, f^* is characterized by the formula

$$f^*(Z_+) = (X \times_Y Z)_+ \quad (5)$$

for every $Z \in \mathrm{Sm}_Y$.

Example 3.6. Base change describes the internal hom in $\mathbf{M}(X)$ as

$$\mathbf{M}(X)(C, B)(Z \xrightarrow{z} X) = \mathbf{sSet}_{\mathbf{M}(X)}(C, z_*z^*B).$$

Note that if f is smooth, the canonical natural transformation

$$f^*\mathbf{M}(Y)(C, B) \rightarrow \mathbf{M}(X)(f^*C, f^*B) \quad (6)$$

is a natural isomorphism.

If $f : X \rightarrow Y$ is a smooth morphism of base schemes, composition with f defines a functor $\mathrm{Sm}_X \rightarrow \mathrm{Sm}_Y$. Precomposition with this functor defines the functor $f^* : \mathbf{M}(Y) \rightarrow \mathbf{M}(X)$, which then has a left adjoint $f_{\sharp} : \mathbf{M}(X) \rightarrow \mathbf{M}(Y)$ by (enriched) Kan extension. Since every motivic space is a colimit of representable ones, f_{\sharp} is characterized by the formula

$$f_{\sharp}(Z \xrightarrow{z} X)_+ = (Z \xrightarrow{z} X \xrightarrow{f} Y)_+ \quad (7)$$

for every $Z \in \mathrm{Sm}_X$. If $Z \rightarrow Y$ is in Sm_Y , the canonical Y -morphism $X \times_Y Z \rightarrow Z$ defines a map $B(Z) \rightarrow f_*f^*B(Z)$ which is natural in Z and $B \in \mathbf{M}(Y)$, and hence a natural transformation $\mathrm{Id}_{\mathbf{M}(Y)} \rightarrow f_* \circ f^*$.

Lemma 3.7. *If $f : X \rightarrow Y$ is a smooth morphism of base schemes, the adjoint*

$$f^* \rightarrow f^*$$

of the natural transformation $\text{Id}_{\mathbf{M}(Y)} \rightarrow f_ \circ f^*$ is a natural isomorphism.*

Proof. This is straightforward. □

In the following, f^* will be used implicitly as a concrete description for the left-Kan extension f^* whenever f is smooth. It has the advantage that it is strictly functorial. These base change functors can be extended to the category of (symmetric) B -spectra by levelwise application in the case $B \in \{S^1, T\}$. This extension involves the identification $f^*(B_Y) \xrightarrow{\cong} B_X$, where $f : X \rightarrow Y$ and B_S indicates that B is a pointed motivic space over S . They are still denoted by $f_* : \mathbf{Spt}_B(X) \rightarrow \mathbf{Spt}_B(Y)$, etc.

Proposition 3.8. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of base schemes.*

- (1) *There is an equality $(g \circ f)_* = g_* \circ f_*$ and a unique natural isomorphism $(g \circ f)^* \xrightarrow{\cong} f^* \circ g^*$.*
- (2) *If f and g are smooth, the unique natural isomorphism $(g \circ f)^* \xrightarrow{\cong} f^* \circ g^*$ is the identity, and there is a unique natural isomorphism $(g \circ f)_\# \xrightarrow{\cong} g_\# \circ f_\#$.*
- (3) *There are equalities $\text{id}_* = \text{Id}$, $\text{id}^* = \text{Id}$, and a natural isomorphism $\text{id}_\# \xrightarrow{\cong} \text{Id}$.*
- (4) *The diagrams*

$$\begin{array}{ccc}
 \mathbf{M}(X) & \xrightarrow{f_*} & \mathbf{M}(Y) & & \mathbf{M}(Y) & \xrightarrow{f^*} & \mathbf{M}(X) \\
 \Sigma_B^\infty \downarrow & & \downarrow \Sigma_B^\infty & & \Sigma_B^\infty \downarrow & & \downarrow \Sigma_B^\infty \\
 \mathbf{Spt}(X, B) & \xrightarrow{f_*} & \mathbf{Spt}(Y, B) & & \mathbf{Spt}(Y, B) & \xrightarrow{f^*} & \mathbf{Spt}(X, B)
 \end{array}$$

commute, and similarly for $f_\#$, and for symmetric spectra.

Proof. This is straightforward. See also [1, Chapitre 4]. □

Lemma 3.9. *Let $f : X \rightarrow Y$ be a morphism of base schemes. Then $f^*(I_Y) \subseteq I_X$, and $f_\#(I_X) \subseteq I_Y$ if f is smooth.*

Proof. This follows from direct inspection. □

Proposition 3.10. *Let $f : X \rightarrow Y$ be a morphism of base schemes. Then (f^*, f_*) is a Quillen adjoint pair. If f is smooth, then $(f_\#, f^*)$ is a Quillen adjoint pair.*

Proof. Consider the case of pointed motivic spaces first. Lemma 3.9 implies that f^* and (if f is smooth) $f_\#$ preserve cofibrations. To prove the first statement, it remains to show that f_* preserves fibrations. By Dugger’s lemma [12, A.2], it suffices to prove that f_* preserves fibrations between fibrant motivic spaces. These fibrations are detected by the set of acyclic cofibrations described in Remark 3.4.

Hence it suffices to prove that f^* maps each of these special acyclic cofibrations in $\mathbf{M}(Y)$ to an acyclic cofibration in $\mathbf{M}(X)$. This is straightforward by (5). The proof for the second statement is similar, using (7).

By Proposition 3.10, f^* preserves cofibrations. However, one can see directly that $f^*(\mathrm{Fr} I_Y) \subseteq \mathrm{Fr} I_X$ since f^* commutes with the functors Fr_n . As in the proof of the preceding case, it remains to prove that f_* preserves stable fibrations of stably fibrant motivic B_X -spectra. Those coincide with the levelwise fibrations. Since f_* preserves fibrations, it suffices to prove that f_* preserves stably fibrant motivic B_X -spectra. This in turn follows from the preceding case, since f_* preserves weak equivalences of fibrant pointed motivic spaces.

If $f : X \rightarrow Y$ is smooth, f_{\sharp} preserves cofibrations of motivic B_X -spectra. However, one can see directly that $f_{\sharp}(\mathrm{Fr} I_X) \subseteq \mathrm{Fr} I_Y$ since f_{\sharp} commutes with the functors Fr_n . It remains to prove that f^* preserves fibrations of stably fibrant motivic B_Y -spectra. As above, it suffices to check that f^* preserves stably fibrant motivic B_Y -spectra. This follows from isomorphism (6), together with the fact that f^* preserves all weak equivalences of pointed motivic spaces. The latter is implied by the fact that f^* is both a left- and a right-Quillen functor. \square

Lemma 3.11. *Let $f : X \rightarrow Y$ be a morphism of base schemes. The functor f^* preserves finite objects, I -finite objects, and dualizable objects. If f is smooth, f_{\sharp} preserves finite objects and I -finite objects.*

Proof. The statements about I -finiteness appear in the proof of Proposition 3.10. The implicit statements about cofibrancy follow from Proposition 3.10. Observe that f^* preserves finitely presentable objects since its right adjoint f_* preserves filtered (even all!) colimits, and similarly for f_{\sharp} . Since f^* is strict symmetric monoidal, it preserves dualizable objects. \square

Let $\widetilde{\mathrm{Sm}}_S$ be the subcategory of Sm_S having the same objects, but only smooth S -morphisms as morphisms. One may summarize some of the results above by saying that the model categories considered so far are Quillen functors on $\widetilde{\mathrm{Sm}}_S$, but only Quillen pseudofunctors on Sm_S . It is possible to strictify these Quillen pseudofunctors to a naturally (not just Quillen) equivalent Quillen functor on Sm_S by the categorical result [32]. This will be assumed from now on, without applying notational changes.

4. Algebraic K -theory of \mathbb{A}^1 -homotopy theory

Let $\mathbf{M}(S)$ be the model category of pointed motivic spaces over S , equipped with the \mathbb{A}^1 -local Nisnevich model structure given in Definition 3.2. Before applying Waldhausen's K -theory construction, one of the finiteness notions introduced in Definition 2.9 will be imposed, indicated by the respective superscript $\mathbf{M}^g(S)$ for

$g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. Unless otherwise specified, the I -finiteness notions always refer to the set of generating cofibrations listed in [Definition 3.2](#).

Definition 4.1. Let $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. Then $A(\mathbf{M}^g(S))$ denotes the spectrum obtained by applying Waldhausen's \mathcal{S}_\bullet -construction to the Waldhausen category $\mathbf{M}^g(S)$.

Technically speaking, $A(\mathbf{M}^g(S))$ is the algebraic K -theory of the one-point motivic space over S . It is possible to consider the algebraic K -theory of an arbitrary motivic space B over S by viewing the canonical Waldhausen category of g -finite motivic spaces over S which are retractive over B , as mentioned abstractly in [Proposition 2.4](#).

Proposition 4.2 (Waldhausen). *Suppose $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. The spectrum $A(\mathbf{M}^g(\mathbb{C}))$ contains $A(*)$ as a retract. In particular, it is nontrivial.*

Proof. The constant pointed motivic space functor $\mathbf{sSet}_\bullet \rightarrow \mathbf{M}(S)$ and the complex realization functor $\mathbf{M}(\mathbb{C}) \rightarrow \mathbf{Top}_\bullet$ are left-Quillen functors. The constant pointed motivic space sends (homotopy) finite pointed simplicial sets to $I_{\mathbb{C}}$ -finite pointed motivic spaces. The complex realization functor sends representables to homotopy finite pointed topological spaces, and hence $I_{\mathbb{C}}$ -finite pointed motivic spaces to homotopy finite pointed topological spaces. A finite pointed motivic space is a retract of an $I_{\mathbb{C}}$ -finite pointed motivic space. Since homotopy finite pointed topological spaces are closed under retracts, the complex realization functor preserves homotopy finiteness. Hence both functors induce maps on Waldhausen K -theory spectra. Their composition coincides with the geometric realization functor

$$|-| : \mathbf{sSet}_\bullet \rightarrow \mathbf{Top}_\bullet,$$

which induces an equivalence on Waldhausen K -theory by [\[46, Theorem 2.1.5\]](#). The statement follows. \square

Proposition 4.3. *Let S be a base scheme and $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. The spectrum $A(\mathbf{M}^g(S))$ contains $A(*)$ as a retract. In particular, it is nontrivial.*

Proof. It suffices to consider a connected base scheme S . Let $\mathbf{M}_{\text{hell}}(S)$ be the left-Bousfield localization of the Nisnevich local projective model structure with respect to the maps $X \rightarrow S$ in Sm_S such that X is connected. It is a left-Bousfield localization of the \mathbb{A}^1 -Nisnevich local projective model structure, and again weakly finitely generated. The identity functor is a left-Quillen functor from $\mathbf{M}(S)$ to $\mathbf{M}_{\text{hell}}(S)$ preserving finitely presentable cofibrant pointed motivic spaces. If $B \in \mathbf{M}_{\text{hell}}(S)$ is fibrant, it is locally constant, since $B(S) \rightarrow B(X)$ is a weak equivalence for every smooth morphism $X \rightarrow S$ such that X is connected.

The constant pointed motivic space functor $\mathbf{sSet}_\bullet \rightarrow \mathbf{M}_{\text{hell}}(S)$ is a left-Quillen functor, but also a Quillen equivalence. Its right adjoint is the evaluation at the

terminal scheme. A map of fibrant objects in $\mathbf{M}_{\text{hell}}(S)$ is a weak equivalence if and only if it is a levelwise weak equivalence. Let $B \rightarrow C$ be a map of motivic spaces over S which are fibrant in $\mathbf{M}_{\text{hell}}(S)$. If the map $B(S) \rightarrow C(S)$ is a weak equivalence, then $B(X) \rightarrow C(X)$ is a weak equivalence for every connected S -scheme X . Since every smooth S -scheme admits a Zariski open cover by smooth connected S -schemes, $B(X) \rightarrow C(X)$ is then a weak equivalence for every smooth S -scheme. It follows that evaluation at the terminal scheme S preserves and detects weak equivalences of fibrant objects in $\mathbf{M}_{\text{hell}}(S)$. If L is a pointed simplicial set, considered as a constant motivic space over S , its fibrant replacement in $\mathbf{M}_{\text{hell}}(S)$ sends X to the product of L indexed over the connected components of X . In particular, the derived unit of the adjunction is the identity. This concludes the proof that the constant motivic space functor $\mathbf{sSet}_{\bullet} \rightarrow \mathbf{M}_{\text{hell}}(S)$ is a Quillen equivalence.

Both the constant motivic space functor $\mathbf{sSet}_{\bullet} \rightarrow \mathbf{M}(S)$ and the identity functor $\mathbf{M}(S) \rightarrow \mathbf{M}_{\text{hell}}(S)$ preserve finite and I -finite objects, hence induce maps on suitable Waldhausen categories. Since $\mathbf{sSet}_{\bullet} \rightarrow \mathbf{M}_{\text{hell}}(S)$ is a Quillen equivalence, it is a K -theory equivalence by [38, Theorem 3.3]. The result follows. \square

The full technology of Waldhausen's algebraic K -theory of spaces requires that the weak equivalences satisfy the extension axiom. This axiom is not satisfied in the category of pointed motivic spaces (the counterexample given for pointed simplicial sets in [46, Section 1.2] right after the definition of the extension axiom extends). However, it is satisfied in the category of S^1 -spectra of pointed motivic spaces over S . The suspension spectrum functor induces a K -theory equivalence, as one deduces from the following theorem.

Remark 4.4. [Theorem 2.14](#), including its assertion for the I -finiteness notions, shows that

$$A^{\mathfrak{g}}(S) := A(\mathbf{Spt}^{\mathfrak{g}}(S)) \leftarrow A(\mathbf{M}^{\mathfrak{g}}(S))$$

is a K -theory equivalence for every $\mathfrak{g} \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. Moreover, it turns out to be natural in the base scheme S . Thus in the discussion below Waldhausen's fibration theorem may be applied to $A^{\mathfrak{g}}(S)$.

As a consequence of [Lemma 3.11](#) and [Proposition 3.10](#), the functors f^* and (if applicable) f_{\sharp} induce exact functors on Waldhausen categories.

Proposition 4.5. *Let $j : U \hookrightarrow S$ be an open embedding of base schemes, with reduced closed complement $i : Z \hookrightarrow S$. Then the functors j^* and i^* induce a splitting*

$$A^{\mathfrak{g}}(S) \xrightarrow{\sim} A^{\mathfrak{g}}(U) \times A^{\mathfrak{g}}(Z)$$

for $\mathfrak{g} \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$.

Proof. Consider the homotopy fiber sequence

$$\text{hofib}(j^*) \rightarrow A^g(S) \xrightarrow{j^*} A^g(U)$$

of spectra. In order to identify the homotopy fiber of j^* , the map $A^g(S) \xrightarrow{j^*} A^g(U)$ is factored as follows. Let $\mathbf{vSpt}^g(S)$ denote the subcategory of maps f such that $j^*(f)$ is a weak equivalence in $\mathbf{Spt}(U)$. Let $\mathbf{Spt}^g(S|U)$ denote the resulting Waldhausen category $(\mathbf{Spt}^g(S), *, \mathbf{vSpt}^g(S), \text{cof}\mathbf{Spt}^g(S))$. The identity can then be regarded as an exact functor $\Phi : \mathbf{Spt}^g(S) \rightarrow \mathbf{Spt}^g(S|U)$. Almost by definition, $j^* : \mathbf{Spt}^g(S|U) \rightarrow \mathbf{Spt}^g(M(U))$ satisfies the conditions of Waldhausen’s approximation theorem [46, Theorem 1.6.7]. In fact, j^* detects and preserves weak equivalences by definition. If E is a g -finite S^1 -spectrum over S and $j^*(E) \rightarrow D$ is a map of g -finite S^1 -spectra over U , consider it as the map

$$j^*(E) = j^* j_{\#} j^*(E) \rightarrow j^* j_{\#}(D) = (D).$$

Here the fact that the unit $\text{Id} \rightarrow j^* j_{\#}$ is the identity enters. The map $j_{\#} j^*(E) \rightarrow j_{\#}(D)$ can be factored via the simplicial mapping cylinder as a cofibration of g -finite S^1 -spectra over U , followed by a weak equivalence. Therefore, the functor $j^* : \mathbf{Spt}^g(S|U) \rightarrow \mathbf{Spt}^g(M(U))$ satisfies the second approximation property, whence it is a K -theory equivalence by the approximation theorem [46, Theorem 1.6.7]. Thus $\text{hofib}(j^*)$ is weakly equivalent to the homotopy fiber of Φ . The latter may be identified, by the fibration theorem [46, Theorem 1.6.4], with the algebraic K -theory of the sub-Waldhausen category $\mathbf{Spt}^{g, j^* \simeq *}(S)$ of g -finite S^1 -spectra E over S such that $j^*(E)$ is (weakly) contractible. The homotopy cofiber sequence

$$j_{\#} j^*(E) \rightarrow E \rightarrow i_* i^*(E),$$

which is an S^1 -spectrum version of [28, Theorem 3.2.21], implies that the induced functor $i^* : \mathbf{Spt}^{g, j^* \simeq *}(S) \rightarrow \mathbf{Spt}^g(Z)$ satisfies the special approximation property. Thus i^* induces a K -theory equivalence by [38, Theorem 2.8]. It remains to observe the splitting, which is induced by $j_{\#} : \mathbf{M}(U) \rightarrow \mathbf{M}(S)$, the left adjoint of j^* . It is a left-Quillen functor preserving the set of generating cofibrations defined in Definition 3.2. The unit $\text{Id} \rightarrow j^* j_{\#}$ is the identity, since j is an open embedding. \square

In particular, the map $A^g(S) \rightarrow A^g(\mathbb{A}_S^1)$ induced by the projection is not a weak equivalence for every $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$, because $A^g(\mathbb{A}_S^1 \setminus \{0\})$ is not contractible by Proposition 4.3.

Corollary 4.6. *Let $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. There is a natural weak equivalence*

$$\Omega_T A^g(S) \xrightarrow{\sim} A^g(S).$$

Proof. This follows from the Yoneda lemma and Proposition 4.5. \square

Corollary 4.7. *Let $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. Assume that the pullback square*

$$\begin{array}{ccc}
 V & \longrightarrow & X \\
 \downarrow & & \downarrow p \\
 U & \xrightarrow{j} & Y
 \end{array} \tag{8}$$

in Sm_g is either a Nisnevich distinguished square or an abstract blow-up square. Then the square

$$\begin{array}{ccc}
 A^g(Y) & \xrightarrow{j^*} & A^g(U) \\
 p^* \downarrow & & \downarrow \\
 A^g(X) & \longrightarrow & A^g(V)
 \end{array} \tag{9}$$

is a homotopy pullback square.

Proof. This is a straightforward consequence of [Proposition 4.5](#). □

Proposition 4.8. *Let S be a base scheme and $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. Then Waldhausen K -theory provides a presheaf*

$$A^g : \text{Sm}_S^{\text{op}} \rightarrow \mathbf{SymSpt}$$

of symmetric S^1 -spectra which is almost sectionwise fibrant.

In fact, the symmetric spectrum $A^g(X)$ is an Ω -spectrum beyond the first term. [Corollary 4.7](#) then implies that up to a sectionwise fibrant replacement, the symmetric S^1 -spectrum A^g over S is fibrant in the Nisnevich local projective model structure. Its \mathbb{A}^1 -fibrant replacement can be determined fairly explicitly via Suslin’s singular functor [\[28\]](#). Recall the standard cosimplicial smooth scheme Δ_S^\bullet over S given by $[n] \mapsto \Delta_S^n = \text{Spec}(\mathcal{O}_S[t_0, \dots, t_n] / \sum_{i=0}^n t_i = 1)$. Realizing the simplicial motivic S^1 -spectrum $[n] \mapsto A^g(- \times_S \Delta^n)$ produces a motivic S^1 -spectrum $\text{Sing } A^g$ over S . As Bjørn Ian Dundas pointed out to me, it is not very interesting.

Proposition 4.9. *Let $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. A sectionwise fibrant replacement of the motivic S^1 -spectrum $\text{Sing } A^g$ over S is fibrant and sectionwise contractible.*

Proof. Recall first that the sectionwise fibrant replacement is fairly harmless, as the adjoint structure maps of $A^g(X)$ are weak equivalences, except for the first. The standard argument from [\[28\]](#) implies that $\text{Sing } A^g(X \times_{\mathbb{A}_S^1} \rightarrow X)$ is a stable equivalence. It remains to show that $\text{Sing } A^g$ is still Nisnevich fibrant. Since $\emptyset \times_S \Delta_S^n$ is the empty scheme, $\text{Sing } A^g(\emptyset)$ is the realization of a degreewise contractible spectrum, hence contractible. The value of $\text{Sing } A^g$ at a distinguished square Q as displayed in [\(8\)](#) is a homotopy pullback square by [\[8, Appendix B\]](#). In fact, [Corollary 4.7](#) implies that in every simplicial degree $A^g(Q \times_S \Delta_S^n)$ is a

homotopy pullback square. To apply [8, Appendix B], it remains to observe that, by construction, every pointed simplicial set occurring in $A^g(Q \times_S \Delta_S^n)$ is connected.

For the final statement, observe that for every closed embedding $i : Z \hookrightarrow X$ in Sm_X , with open complement $j : U = X \setminus Z \hookrightarrow X$, the induced map

$$(i^*, j^*) : \text{Sing } A^g(X) \rightarrow \text{Sing } A^g(Z) \times \text{Sing } A^g(U)$$

is a weak equivalence, by realizing the weak equivalences from Proposition 4.5. Suppose that $B : \text{Sm}_S \rightarrow \mathbf{Spt}$ is any presheaf of spectra with this property which is also \mathbb{A}^1 -invariant. Then $B(X) \xrightarrow{\sim} B(\mathbb{A}_X^1) \sim B(\mathbb{A}_X^1 \setminus \{0\}) \times B(X)$ is an equivalence, as well as the identity on the second factor. Hence its cofiber $B(\mathbb{A}_X^1 \setminus \{0\})$ is contractible. It contains $B(X)$ as a retract included via $1 : X \rightarrow \mathbb{A}_X^1 \setminus \{0\}$. Thus $B(X)$ is also contractible. \square

The contractibility of the \mathbb{A}^1 -homotopy type associated with the Waldhausen K -theory of any of the finiteness notions on motivic homotopy theory stated in Proposition 4.9 calls for an adjustment. A solution is the finiteness notion of dualizability, as introduced in Definition 2.10. Since dualizability results for smooth varieties in motivic homotopy theory involve invertibility of Thom spaces of vector bundles, passage to T -spectra is required.

Definition 4.10. Let $g \in \{\text{dual}, \text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$ be one of the finiteness notions introduced above, and let S be a base scheme. Set

$$A_T^g(S) := A(\mathbf{SymSpt}_T^g \mathbf{M}(S))$$

the algebraic K -theory of the category of g -finite symmetric T -spectra over S .

It is straightforward to verify that $A_T^g(S)$ satisfies similar properties as $A^g(S)$: Proposition 4.2 holds as well for $A_T^g(S)$ and any $g \in \{\text{dual}, \text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$. For any $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$, Propositions 4.5, 4.8, and 4.9, and Corollaries 4.6 and 4.7 hold with A_T^g replacing A^g . However, A_T^{dual} does not lead to a contractible \mathbb{A}^1 -homotopy type, as Theorem 6.6 implies. Nevertheless, the global sections of A_T^{dual} and A_T^{hfin} coincide over fields of characteristic zero.

Proposition 4.11. *Let F be a field of characteristic zero. Then the Waldhausen categories $\mathbf{SymSpt}_T(F)^{\text{hfin}}$ and $\mathbf{SymSpt}_T(F)^{\text{dual}}$ coincide.*

Proof. This follows from the main result in [33]. Here are some details. Smooth projective schemes are dualizable over a base scheme [1, Chapitre 4; 21, Theorem A.1]. If F is a field of characteristic zero, resolution of singularities provides that then also smooth quasiprojective F -varieties are dualizable [37, Theorem 52; 42]. Every smooth F -variety admits a Zariski open cover by smooth quasiprojective F -varieties, which implies that smooth F -varieties are dualizable. Hence every I_F -finite symmetric T -spectrum is dualizable. Since the property of being dualizable is

closed under retracts and weak equivalences, and every finite cofibrant symmetric T -spectrum is a retract of an I_F -finite symmetric T -spectrum by [Lemma 2.11](#), every homotopy finite symmetric T -spectrum is dualizable.

Conversely, every dualizable symmetric T -spectrum is compact as an object of $\mathrm{SH}(F)$. It is an easy consequence of basic properties of dualizable objects in a symmetric monoidal stable model category with a compact unit [\[26\]](#). Any compact T -spectrum in $\mathrm{SH}(F)$ is a retract of a cofibrant T -spectrum which is weakly equivalent to an I -finite T -spectrum. \square

[Proposition 4.11](#) holds also over fields of positive characteristic, provided that the characteristic is inverted. However, it does not hold for base schemes of positive dimension. For example, the T -suspension spectrum of $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$ is not dualizable in $\mathrm{SH}(\mathbb{A}_{\mathbb{C}}^1)$, although it is finite; see [\[29, Remark 8.2\]](#) for a more general statement.

5. Grothendieck rings

Let S be a base scheme. The *Grothendieck ring of S* is the free abelian group on isomorphism classes of finite-type S -schemes, denoted $[X]$, modulo the relations $[X] = [Z] + [X \setminus Z]$ whenever Z is a closed subscheme, and $[\emptyset] = 0$. The ring structure is induced by the product $X \times_S Y$. The ring $K_0(\mathrm{Var}_S)$ is commutative and has $[S]$ as a unit. Note that $[X] = [X_{\mathrm{red}}]$, where $X_{\mathrm{red}} \hookrightarrow X$ denotes the maximal reduced closed subscheme. Weak factorization is used in the following main result of [\[3\]](#), which gives a much simpler presentation of the Grothendieck ring of fields of characteristic zero.

Theorem 5.1 (F. Heinloth née Bittner). *Suppose that F is a field of characteristic zero. Then $K_0(\mathrm{Var}_F)$ is generated by isomorphism classes of connected smooth projective F -schemes, modulo the relations $[X] - [f^{-1}(Z)] = [Y] - [Z]$ whenever $f : X \rightarrow Y$ is the blow-up of the smooth projective variety Y along the smooth center $Z \hookrightarrow Y$, and $[\emptyset] = 0$.*

A *motivic measure* on S is a ring homomorphism

$$K_0(\mathrm{Var}_S) \rightarrow A$$

to some commutative ring. Main examples of motivic measures are the Euler characteristic $K_0(\mathrm{Var}_{\mathbb{C}}) \rightarrow \mathbb{Z}$ on the complex numbers, point counting $K_0(\mathrm{Var}_{\mathbb{F}_q}) \rightarrow \mathbb{Z}$ on a finite field, and the Gillet–Soulé motivic measure [\[16\]](#). [Theorem 5.1](#) simplifies the construction of motivic measures. For example, the motivic measure on fields of characteristic zero obtained by sending a smooth projective variety to its stable birational class constructed in [\[24\]](#) can be deduced from [Theorem 5.1](#). In order to provide a new motivic measure, recall that if \mathbf{C} is a Waldhausen category, the abelian group $\pi_0 A(\mathbf{C})$ is generated by the objects in \mathbf{C} , subject to the following two relations:

- (1) $\langle B \rangle = \langle C \rangle$ if there exists a weak equivalence $B \xrightarrow{\sim} C$.
- (2) $\langle B \rangle + \langle D \rangle = \langle C \rangle$ if there exists a cofibration $B \hookrightarrow C$ with cofiber D .

Theorem 5.2. *Let F be a field of characteristic zero. Sending the isomorphism class $[X]$ of a smooth projective F -scheme X to its class $\langle X_+ \rangle \in \pi_0 A^{\text{ifin}}(F)$ defines a surjective motivic measure*

$$\Phi_F : K_0(\text{Var}_F) \rightarrow A^{\text{ifin}}(F).$$

Proof. The relations given in [Theorem 5.1](#) are fulfilled in $\pi_0 A^{\text{ifin}}(F)$ by [\[28, Remark 3.2.30\]](#). Hence $[X] \mapsto \langle X_+ \rangle$ defines a group homomorphism

$$\Phi_F : K_0(\text{Var}_F) \rightarrow \pi_0 A^{\text{ifin}}(F),$$

which is compatible with the multiplicative structure. It remains to prove its surjectivity. However, $\pi_0 A^{\text{ifin}}(F)$ is generated as an abelian group by I -finite S^1 -spectra over F , and hence by the domains and codomains of the maps in I . These are of the form $\text{Fr}_m X_+ \wedge \partial \Delta_+^n$ and $\text{Fr}_m X_+ \wedge \Delta_+^n$, where X is a smooth F -variety and $m, n \in \mathbb{N}$. Since Fr_m corresponds to a simplicial desuspension and suspension induces multiplication by -1 on $\pi_0 A^{\text{ifin}}(F)$, one may restrict to $m = 0$. Induction on n and the cofiber sequence

$$X_+ \wedge \partial \Delta_+^n \hookrightarrow X_+ \wedge \Delta_+^n \rightarrow X_+ \wedge S^n$$

imply that $\pi_0 A^{\text{ifin}}(F)$ is generated as an abelian group by S^1 -suspension spectra $\text{Fr}_0 X_+ = \Sigma^\infty X_+$ of smooth F -varieties. Resolution of singularities implies that it suffices to restrict to S^1 -suspension spectra of smooth projective F -varieties, similar to the argument in the proof of [Proposition 4.11](#). This concludes the proof. \square

The formula $\Phi_F([X]) = \langle X_+ \rangle$ does not apply to nonprojective varieties in general. For example,

$$\Phi_F([\mathbb{A}^1]) = \Phi_F([\mathbb{P}^1] - [\text{Spec}(F)]) = \langle \mathbb{P}_+^1 \rangle - \langle \text{Spec}(F)_+ \rangle = \langle \mathbb{P}^1, \infty \rangle \neq \langle \mathbb{A}_+^1 \rangle = \langle \text{Spec}(F)_+ \rangle,$$

where the inequality follows — at least in the case $k \subseteq \mathbb{R}$ — from the left-Quillen functor which takes a smooth k -scheme to the topological space of its complex points, together with the conjugation action. The fixed points of the action on the left-hand side of the inequality yield the class of $\mathbb{R}\mathbb{P}^1$ having reduced Euler characteristic -1 , while the fixed points of the action on the right-hand side of the inequality have reduced Euler characteristic 1 .

Since I -finiteness is the smallest of the finiteness notions $g \in \{\text{fin}, \text{hfin}, \text{ifin}, \text{ihfin}\}$ considered on motivic spaces over a field, there is a motivic measure

$$\Phi_F : K_0(\text{Var}_F) \rightarrow \pi_0 A^g(F)$$

as well; however, it may not be surjective in the case $g \in \{\text{fin}, \text{hfin}\}$. [Proposition 4.2](#) shows that it refines the Euler characteristic if F is a subfield of \mathbb{C} . It also refines the Gillet–Soulé “motivic” motivic measure [\[16\]](#).

Proposition 5.3. *Let F be a field of characteristic zero. There is a commutative diagram*

$$\begin{array}{ccc}
 K_0(\text{Var}_F) & \xrightarrow{\Phi_F} & \pi_0 A^{\text{ifin}}(F) \\
 \Psi_F \downarrow & & \downarrow \\
 K_0(\text{ChMot}_F^{\text{eff}}) & \xrightarrow{\cong} & K_0(\text{DM}_F^{\text{eff, hfin}})
 \end{array}$$

of ring homomorphisms, where Ψ_F maps the class of a smooth projective F -scheme to the class of its effective Chow motive.

Proof. Voevodsky’s derived category of effective motives may be obtained as the homotopy category of S^1 -spectra of motivic spaces with transfers; see [\[37\]](#). This implies an identification of $K_0(\text{DM}_F^{\text{eff, hfin}})$ with the ring of path components of $A(\mathbf{M}^{\text{tr, hfin}}(F))$, where $\mathbf{M}^{\text{tr}}(F)$ is the model category of motivic spaces with transfers defined via the functor $\mathbf{M}^{\text{tr}}(F) \rightarrow \mathbf{M}(F)$ forgetting transfers. Its left adjoint induces the vertical arrow on the right-hand side of the diagram displayed above. Similar to the argument in the proof of [Proposition 4.11](#) is an argument proving that homotopy finite and compact motives coincide in DM_F^{eff} . Furthermore, compact objects and geometric motives coincide by [\[11, Theorem 11.1.13\]](#), both in the effective and the noneffective case. The lower horizontal morphism in [Proposition 5.3](#) is thus an isomorphism by [\[6, Theorem 6.4.2\]](#). \square

Theorem 5.4. *Let F be a field of characteristic zero. The ring homomorphism Φ_F extends to a surjective ring homomorphism*

$$\Phi_F : K_0(\text{Var}_F)[\mathbb{L}^{-1}] \rightarrow \pi_0 A_T^{\text{ifin}}(F),$$

where $\mathbb{L} = [\mathbb{A}_S^1]$.

Proof. This follows from [Theorem 5.2](#), the equality $\Phi_F(\mathbb{L}) = \langle \mathbb{P}^1, \infty \rangle$, and the fact that $\Sigma_T^\infty(\mathbb{P}^1, \infty)$ is invertible in $\text{SH}(F)$, hence also in $\pi_0 A_T^{\text{ifin}}(F)$. \square

In particular, all relations that hold in the Grothendieck ring $K_0(\text{Var}_F)$ or its localization $K_0(\text{Var}_F)[\mathbb{L}^{-1}]$ also hold in $\pi_0 A(F)$ or $\pi_0 A_T(F)$, respectively. The ring $K_0(\text{Var}_F)[\mathbb{L}^{-1}]$ is relevant to the theory of motivic integration, and also to the construction of the duality involution induced by $[X] \mapsto \mathbb{L}^{-\dim X}[X]$. In the ring $\pi_0 A_T^{\mathbb{G}}(F)$ (which from a certain perspective still consists of algebro-geometric objects), the class of the pointed projective line is naturally invertible. Also the duality involution has a natural interpretation in $\pi_0 A_T^{\mathbb{G}}(F)$, since the dual of a

smooth projective F -variety X is the Thom T -spectrum of its negative tangent bundle, considered as a class in $K_0(X)$. The equality

$$\langle \mathrm{Th}(-\mathcal{T}(X)) \rangle = \langle \mathbb{P}^1, \infty \rangle^{-\dim X} \cdot \langle X_+ \rangle$$

follows from the Zariski local triviality of vector bundles. However, the localization passage $K_0(\mathrm{Var}_F) \rightarrow K_0(\mathrm{Var}_F)[\mathbb{L}^{-1}]$ involves a loss of information. Lev Borisov proved that \mathbb{L} is a zero divisor in $K_0(\mathrm{Var}_F)$ [7]. In particular, the composition

$$K_0(\mathrm{Var}_F) \rightarrow \pi_0 A^{\mathrm{ifin}}(F) \rightarrow \pi_0 A_T^{\mathrm{ifin}}(F)$$

is not injective.

Proposition 5.5. *The canonical homomorphism $\pi_0 A^{\mathfrak{g}}(F) \rightarrow \pi_0 A_T^{\mathfrak{g}}(F)$ induces a surjective homomorphism $\pi_0 A^{\mathfrak{g}}(F)[\langle \mathbb{P}^1, \infty \rangle^{-1}] \rightarrow \pi_0 A_T^{\mathfrak{g}}(F)$ for \mathfrak{g} contained in $\{\mathrm{ifin}, \mathrm{fin}, \mathrm{ihfin}, \mathrm{hfin}\}$.*

Proof. In fact, this holds for any base scheme S . The abelian group $\pi_0 A_T^{\mathfrak{g}}(F)$ is generated by shifted T -suspension spectra $\mathrm{Fr}_m X_+$ of smooth S -schemes; the contribution from the simplicial direction can be ignored, as the proof of [Theorem 5.2](#) implies. The symmetric T -spectrum $\mathrm{Fr}_m X^+ \wedge T^m$ is stably equivalent to $\mathrm{Fr}_0 X_+ = \Sigma_T^\infty X_+$, showing that

$$\pi_0 A^{\mathfrak{g}}(S)[\langle \mathbb{P}^1, \infty \rangle^{-1}] \rightarrow \pi_0 A_T^{\mathfrak{g}}(S)$$

is surjective. □

More can be and has been said on the relationship between the Grothendieck ring of varieties and of motives. [Proposition 5.3](#) and [Theorem 5.4](#) provide the commutative diagram

$$\begin{array}{ccccc} K_0(\mathrm{Var}_F) & \xrightarrow{\Phi_F} & \pi_0 A^{\mathrm{ifin}}(F) & \longrightarrow & \pi_0 A_T^{\mathrm{ifin}}(F) \\ \Psi_F \downarrow & & \downarrow & & \downarrow \\ K_0(\mathrm{ChMot}_F^{\mathrm{eff}}) & \xrightarrow{\cong} & K_0(\mathrm{DM}_F^{\mathrm{eff}, \mathrm{hfin}}) & \longrightarrow & K_0(\mathrm{DM}_F^{\mathrm{hfin}}) \end{array}$$

in which the homomorphism $K_0(\mathrm{DM}_F^{\mathrm{eff}, \mathrm{hfin}}) \rightarrow K_0(\mathrm{DM}_F^{\mathrm{hfin}})$ corresponds to inverting \mathbb{L} , the class of the Lefschetz motive. The latter can be deduced from Voevodsky’s cancellation theorem [43]. If one imposes rational instead of integral coefficients on the T -spectra and motives above, the canonical functor becomes a Quillen equivalence for all fields in which -1 is a sum of squares, by a theorem of Morel’s. It is known that the canonical homomorphism

$$K_0(\mathrm{Var}_F)[\mathbb{L}^{-1}] \rightarrow K_0(\mathrm{DM}_F^{\mathrm{hfin}} \otimes \mathbb{Q})$$

is not injective [30, Proposition 7.9]. Already inverting 2 in the homotopy category of T -spectra splits it as a product $\mathrm{SH}(F)_+ \times \mathrm{SH}(F)_-$ corresponding to the two idempotents $\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}$. Here ε is induced by the twist isomorphism on $T \wedge T$. If F is formally real, the category $\mathrm{SH}(F)_-$ maps nontrivially to the derived category of $\mathbb{Z}[\frac{1}{2}]$ -modules. After rationalizing, the category $\mathrm{SH}(F)_+$ is equivalent to the derived category of motives over any field F [11, Theorem 16.2.13]. In particular, the canonical homomorphism

$$\pi_0 A(\mathbf{Sp}_T^{\mathrm{hfin}}(F) \otimes \mathbb{Q}) \rightarrow K_0(\mathrm{DM}_F^{\mathrm{hfin}} \otimes \mathbb{Q})$$

is always surjective, but not injective if F is formally real. See [23, Theorem 1.5] for an identification of $\pi_0 A(\mathbf{Sp}_T^{\mathrm{hfin}}(X) \otimes \mathbb{Q})$ in terms of Chow motives and the real étale site of the excellent and separated scheme X of finite Krull dimension.

6. A trace map

The next goal is to produce a trace map on the \mathbb{A}^1 -homotopy type $A_T^{\mathrm{dual}} : \mathrm{Sm}_F^{\mathrm{op}} \rightarrow \mathbf{Spt}$ for a field F of characteristic zero and the duality finiteness notion. The general result [20, Theorem 6.5] essentially provides such a trace. However, when the existence of the motivic trace was announced at a talk in Heidelberg in 2014, the argument proceeded along the lines of [44]. For the sake of concreteness, this construction of the trace will be sketched as follows. In principle, it suffices to fatten the Waldhausen category $\mathbf{SymSpt}_T^{\mathrm{dual}}(F) = \mathbf{SymSpt}_T^{\mathrm{hfin}}(F)$ slightly as in [44]. The fattened Waldhausen category consists of additional duality data.

Definition 6.1. For a base scheme S , let $\mathbf{DSp}(S)$ be the category whose objects are triples (E^+, E^-, e^-) , where E^+ is a dualizable symmetric T -spectrum, E^- is fibrant symmetric T -spectrum, and $e^- : E^- \wedge E^+ \rightarrow \mathrm{fib}(\mathbb{1})$ is a map whose adjoint $E^- \rightarrow \underline{\mathrm{Hom}}(E^+, \mathrm{fib}(\mathbb{1}))$ is a weak equivalence. A morphism of triples from (D^+, D^-, d^-) to (E^+, E^-, e^-) is a pair $\phi^+ : D^+ \rightarrow E^+, \phi^- : E^- \rightarrow D^-$ of maps such that the diagram

$$\begin{array}{ccc} D^- \wedge E^+ & \xrightarrow{\phi^- \wedge E^+} & E^- \wedge E^+ \\ D^- \wedge \phi^+ \downarrow & & \downarrow e^- \\ D^- \wedge D^+ & \xrightarrow{d^-} & \mathrm{fib}(\mathbb{1}) \end{array}$$

commutes. Such a morphism (ϕ^+, ϕ^-) is a *weak equivalence* if both ϕ^+ and ϕ^- are weak equivalences, and a *cofibration* if ϕ^+ is a cofibration and ϕ^- is a fibration.

Since smashing with a cofibrant symmetric T -spectrum preserves weak equivalences [22, Proposition 4.19], the symmetric T -spectrum $E^- \wedge E^+$ has the correct homotopy type, even if E^- is not cofibrant.

Proposition 6.2. *The category $\mathbf{DSp}(S)$ is a Waldhausen category.*

Proof. The category $\mathbf{DSp}(S)$ is pointed by $(*, *, * \rightarrow \text{fib}(\mathbb{I}))$. Weak equivalences in $\mathbf{DSp}(S)$ form a subcategory, and so do the cofibrations. Every triple is then cofibrant, using that the second entry is fibrant. Suppose that

$$(B^+, B^-, b^-) \xleftarrow{(\psi^+, \psi^-)} (D^+, D^-, d^-) \xrightarrow{(\phi^+, \phi^-)} (E^+, E^-, e^-)$$

is a diagram. Its pushout is defined as the triple $(E^+ \cup_{D^+} B^+, E^- \times_{D^-} B^-, c)$ where c is adjoint to the map

$$E^- \times_{D^-} B^- \rightarrow \underline{\text{Hom}}(E^+ \cup_{D^+} B^+, \text{fib } \mathbb{I}) \cong \underline{\text{Hom}}(E^+, \text{fib } \mathbb{I}) \times_{\underline{\text{Hom}}(D^+, \text{fib } \mathbb{I})} \underline{\text{Hom}}(B^+, \text{fib } \mathbb{I})$$

induced by the adjoints of b^- , d^- , and e^- . The dual of the gluing lemma implies that the map above is a weak equivalence. \square

Lemma 6.3. *The forgetful functor $\mathbf{DSp}(S) \rightarrow \mathbf{SymSpt}_T^{\text{dual}}(S)$ is a K -theory equivalence.*

Proof. The forgetful functor sends the triple (E^+, E^-, e^-) to E^+ and is exact by definition. It admits the exact section sending E to the triple $(E, \underline{\text{Hom}}(E, \text{fib } \mathbb{I}), \text{ev})$ where $\text{ev} : \underline{\text{Hom}}(E, \text{fib } \mathbb{I}) \wedge E \rightarrow \text{fib } \mathbb{I}$ is the evaluation map, adjoint to the identity. Moreover, there is a natural weak equivalence

$$(E^+, \underline{\text{Hom}}(E^+, \text{fib } \mathbb{I}), \text{ev}) \xrightarrow{(\text{id}, b(e^-))} (E^+, E^-, e^-), \quad (10)$$

where $b(e^-)$ is the adjoint of e^- . Hence applying the section to the forgetful functor induces a map on algebraic K -theory which is homotopic to the identity map. \square

Recall that $\mathbf{SymSpt}_T(S)$ admits the structure of a pointed simplicial model category in which the n -simplices of morphisms are given by the maps $D \wedge \Delta_+^n \rightarrow E$ of symmetric T -spectra over S . The functoriality listed in [Proposition 3.10](#) is simplicial. It follows that the assignment $[n] \mapsto \mathbf{SymSpt}_T^{\text{dual}}(S)_n$ is a simplicial category with constant objects, and so is the assignment $[n] \mapsto \mathbf{wSymSpt}_T^{\text{dual}}(S)_n$ restricted to the subcategories of weak equivalences.

Proposition 6.4. *The category $\mathbf{DSp}(S)_\bullet$ is a simplicial category, and so is the restriction to $\mathbf{wDSp}(S)_\bullet$, the subcategory of weak equivalences.*

Proof. The n -simplices of morphisms are given by pairs

$$(D^+ \wedge \Delta_+^n \rightarrow E^+, E^- \wedge \Delta_+^n \rightarrow D^-)$$

satisfying the appropriate compatibility condition. The required axioms are straightforward to check. \square

Lemma 6.5. *The natural inclusion*

$$\mathbf{wDSp}(S) \xrightarrow{\kappa} \mathbf{wDSp}(S)_\bullet$$

induces a weak equivalence after geometric realization.

Proof. [Lemma 6.3](#), or rather its proof, implies that the forgetful functor induces the following diagram

$$\begin{array}{ccc} \mathbf{wDSp}(S) & \xrightarrow{\kappa} & \mathbf{wDSp}_\bullet(S) \\ \downarrow & & \downarrow \\ \mathbf{wSymSp}_T^{\text{dual}}(S) & \xrightarrow{\kappa} & \mathbf{wSymSp}_T^{\text{dual}}(S)_\bullet \end{array}$$

whose vertical arrows induce weak equivalences after geometric realization. It suffices to prove that the lower horizontal arrow has the same property. The inclusion κ is induced by the collection of degeneracy maps $s_m : \Delta^m \rightarrow \Delta^0$. Let $d_m : \Delta^0 \rightarrow \Delta^m$ be the inclusion of the m -th vertex. By the realization lemma, it suffices to prove that the composition

$$\mathbf{wSymSp}_T^{\text{dual}}(S)_m \xrightarrow{d_m^*} \mathbf{wSymSp}_T^{\text{dual}}(S) \xrightarrow{s_m^*} \mathbf{wSymSp}_T^{\text{dual}}(S)_m$$

is homotopic to the identity for every m . This follows from the fact that Δ^m simplicially contracts onto its last vertex. \square

Let (E^+, E^-, e^-) be an object in $\mathbf{DSp}(S)$. Consider the simplicial set of maps of symmetric T -spectra over S from $\mathbb{1}$ to $\text{fib}(E^+ \wedge E^-)$. The aim is to modify this simplicial set to consist of only those maps which — together with e^- — express E^+ and E^- as dual objects in the stable homotopy category $\text{SH}(S)$. A map $\mathbb{1} \rightarrow \text{fib}(E^+ \wedge E^-)$ induces a composition

$$E^+ = \mathbb{1} \wedge E^+ \rightarrow \text{fib}(E^+ \wedge E^-) \wedge E^+ \xrightarrow{\sim} \text{fib}(E^+ \wedge E^- \wedge E^+) \xrightarrow{\text{fib}(E^+ \wedge e^-)} \text{fib}(E^+ \wedge \text{fib}(\mathbb{1}))$$

(as well as a similar composition $\text{cof}(E^-) \rightarrow \text{fib}(\text{fib}(\mathbb{1}) \wedge E^-)$ where cof is a cofibrant replacement functor). There is a preferred map $z : E^+ \rightarrow \text{fib}(E^+ \wedge \text{fib}(\mathbb{1}))$, given by unit and replacement natural transformations. Let

$$H(E^+, E^-, e^-) = \{H : E^+ \wedge \Delta_+^1 \rightarrow \text{fib}(E^+ \wedge \text{fib}(\mathbb{1})) \text{ with } H|_{E^+ \wedge 0_+} = z\}$$

be the simplicial set of simplicial homotopies starting at the respective preferred map. By construction, $H(E^+, E^-, e^-)$ is a fibrant simplicial set which simplicially contracts to the zero simplex given by the preferred map. Moreover, it maps via an “endpoint” Kan fibration to the simplicial set of maps in the terminal corner of the following diagram whose pullback is the desired modification:

$$\begin{array}{ccc} D(E^+, E^-, e^-) & \longrightarrow & \mathbf{sSet}(\mathbb{1}, \text{fib}(E^+ \wedge E^-)) \\ \downarrow & & \downarrow \\ H(E^+, E^-, e^-) & \longrightarrow & \mathbf{sSet}(E^+, \text{fib}(E^+ \wedge \text{fib}(\mathbb{1}))) \end{array} \quad (11)$$

The condition on e^- guarantees that the vertical map on the right-hand side of diagram (11) is a weak equivalence, and the horizontal arrows depict fibrations. Hence $D(E^+, E^-, e^-)$ is a contractible fibrant simplicial set. Its zero simplices are maps $e^+ : \mathbb{1} \rightarrow \text{fib}(E^+ \wedge E^-)$, together with a simplicial homotopy providing that $E^+ \wedge e^- \circ e^+ \wedge E^+$ coincides with id_{E^+} in the motivic stable homotopy category of S . Such a zero simplex represented by the tuple

$$((E^+, E^-, e^-), e^+, H)$$

maps naturally to the composition

$$\mathbb{1} \xrightarrow{e^+} \text{fib}(E^+ \wedge E^-) \xrightarrow{\cong} \text{fib}(E^- \wedge E^+) \xrightarrow{\text{fib}(e^-)} \text{fib}(\text{fib}(\mathbb{1})),$$

which is a zero simplex in $\text{fib}(\text{fib}(\mathbb{1}))(S)$ representing the Euler characteristic of E^+ [26]. More generally, an n -simplex maps to an n -simplex in $\text{fib}(\text{fib}(\mathbb{1}))(S)$. A similar variant $D(\underline{E})$ exists for an n -simplex

$$\underline{E} = (E_0 \xrightarrow{\sim} E_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} E_n)$$

of the nerve of $\mathbf{wDSp}_\bullet(S)$, starting with maps from $\mathbb{1}$ to $\text{fib}(E_0^+ \wedge E_n^-)$ and using the duality datum $E_0^+ \wedge E_n^- \rightarrow \text{fib}(\mathbb{1})$ obtained via compositions instead. The map to $\text{fib}(\text{fib}(\mathbb{1}))$ described above for $D(E^+, E^-, e^-)$ extends to $D(\underline{E})$ for \underline{E} such an n -simplex. A map $\alpha : [m] \rightarrow [n]$ in Δ induces a map $D(\underline{E}) \rightarrow D(\alpha^*(\underline{E}))$. Consider the induced map of bisimplicial sets

$$\coprod_{\underline{E} \in \mathbf{wDSp}_\bullet(S)} D(\underline{E}) \rightarrow \coprod_{\underline{E} \in \mathbf{wDSp}_\bullet(S)} \{\underline{E}\}.$$

Since $D(\underline{E})$ is contractible, this map is a weak equivalence by the realization lemma. It maps to $\text{fib}(\text{fib}(\mathbb{1}))(S)$, and this map is natural in S . It may be regarded as the zeroth level of a map of S^1 -spectra

$$A_T^{\text{dual}}(S) \rightarrow \text{fib}(\text{fib}(\mathbb{1}))(S),$$

which is natural in S . Instead of providing a spectrum-level map by explicit constructions similar to those appearing in the proof of [26, Theorem 0.1], however, [20, Theorem 6.5], which in turn refers to [41], will be invoked.

Theorem 6.6. *Let S be a Noetherian finite-dimensional base scheme. There exists a multiplicative map $A_T^{\text{dual}}(S) \rightarrow \text{fib}(\mathbb{1})(S)$ of symmetric ring S^1 -spectra which is natural in S . On path components it sends the class given by the dualizable T -spectrum E to the Euler characteristic of E , considered as an endomorphism $\chi(E) : \mathbb{1} \rightarrow \text{fib}(\mathbb{1})$.*

Proof. The category $\mathbf{SymSp}_T^{\text{dual}}(S)$ of cofibrant dualizable symmetric T -spectra over S gives rise to a small, stable, idempotent-complete, rigid symmetric monoidal

∞ -category $\mathcal{E}(S)$, naturally in S . Hence it fits into the general framework of [20]. More specifically, [20, Theorem 6.5 and Remark 6.6] apply to give a map of symmetric ring spectra

$$A(\mathcal{E}(S)) \rightarrow \text{fib}(\mathbb{1})(S),$$

whose domain is the ∞ -categorical K -theory of $\mathcal{E}(S)$ as introduced in [25, Remark 1.2.2.5], and whose target is regarded as the endomorphism S^1 -spectrum of the unit in $\mathcal{E}(S)$. The Waldhausen K -theory $A_T^{\text{dual}}(S)$ of $\mathbf{SymSp}^{\text{dual}}(S)$ maps via a natural weak equivalence to $A(\mathcal{E}(S))$ [4, Theorem 7.8], and this so as a symmetric ring spectrum [5, Proposition 5.8]. This provides the desired multiplicative map of S^1 -spectra. For the statement regarding path components, see [20, Remark 6.6]. \square

It would be very interesting to relate the homotopy fiber of the trace from Theorem 6.6 with geometrical data.

Corollary 6.7. *Let F be a field of characteristic zero. There exists a multiplicative trace map*

$$A_T^{\text{hfin}}(F) \rightarrow \text{fib}(\mathbb{1})(F),$$

which induces a ring homomorphism

$$\pi_0 A_T^{\text{hfin}}(F) \rightarrow \pi_0 \text{fib}(\mathbb{1})(F) \cong \text{GW}(F)$$

to the Grothendieck–Witt ring of F .

Proof. This follows from Theorem 6.6 and Proposition 4.11. The statement on path components involves Morel’s theorem [27, Corollary 1.25] computing the path components of the endomorphism spectrum of the sphere T -spectrum. More precisely, the global sections of $\text{fib}(\mathbb{1})$ coincide with the infinite T -loop space (or spectrum) associated with a fibrant replacement of the T -suspension spectrum of the zero sphere S_F^0 over F . Hence by construction its path components form the endomorphism ring of $\mathbb{1} \in \text{SH}(F)$, which Morel computed to be naturally isomorphic to the Grothendieck–Witt ring of F . \square

The composition of the motivic measure

$$K_0(\text{Var}_F) \rightarrow \pi_0 A_T^{\text{hfin}}(F) = \pi_0 A_T^{\text{dual}}(F)$$

induced by Theorem 5.2 and the ring homomorphism from Corollary 6.7 provides a motivic measure to the Grothendieck–Witt ring on F . For formal reasons — see, for example, [26] — it extends the categorical Euler characteristic $K_0(\text{SH}^{\text{dual}}(F)) \rightarrow [\mathbb{1}, \mathbb{1}]_{\text{SH}(F)} \cong \text{GW}(F)$ used in refined enumerative geometry by Levine, Wickelgren, and others [2]. As another application of Theorem 6.6, an interesting \mathbb{A}^1 -homotopy type results.

Corollary 6.8. *Let S be a Noetherian finite-dimensional base scheme. The fibrant replacement of the presheaf $A_T^{\text{dual}} \in \mathbf{SymSpt}(S)$ factors the unit map*

$$S^0 \rightarrow \Omega_T^\infty \Sigma_T^\infty S^0 \rightarrow \Omega_T^\infty \text{fib}(\mathbb{1}) \in \mathbf{SymSpt}(S).$$

In particular, the \mathbb{A}^1 -homotopy type associated with A_T^{dual} is nontrivial, and the global sections of its \mathbb{A}^1 -path components $\pi_0^{\mathbb{A}^1} A_T^{\text{dual}}$ factor the motivic measure

$$K_0(\text{Var}_F) \rightarrow \pi_0^{\mathbb{A}^1} A_T^{\text{dual}}(F) \rightarrow \text{GW}(F)$$

from Corollary 6.7 in the case $S = \text{Spec}(F)$ is the spectrum of a field F of characteristic zero.

Proof. The proof of Corollary 6.7 already mentioned that the global sections of $\text{fib}(\mathbb{1})$ over any Noetherian finite-dimensional base scheme such as $X \in \text{Sm}_S$ is the infinite T -loop space of the sphere spectrum $\mathbb{1}_X \in \text{SH}(X)$, viewed as a symmetric S^1 -spectrum. Hence the naturality in Theorem 6.6 provides a map

$$A_T^{\text{dual}} \rightarrow \Omega_T^\infty \text{fib}(\mathbb{1}) \in \mathbf{SymSpt}(S),$$

which sectionwise is a multiplicative map of symmetric ring spectra. In particular, this map is naturally compatible with the unit. Since its target $\Omega_T^\infty \text{fib}(\mathbb{1})$ is fibrant by construction, the map factors over a fibrant replacement of A_T^{dual} , giving rise to the claimed factorization of the unit map. The remaining statement then follows from Corollary 6.7. \square

Acknowledgements

Friedhelm Waldhausen suggested to consider algebraic K -theory of motivic spaces to me a long time ago. I thank him for his support during many years. Basically all results except the ones listed in Section 6 were sketched during a short presentation at the Union College Mathematics Conference 2003. I thank Lars Hesselholt and Marc Levine for discussions and encouragement after further talks I gave on the subject in 2011 and 2015, respectively. Finally I thank Bjørn Ian Dundas and an anonymous referee for very helpful comments.

References

- [1] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, II*, Astérisque **315**, Soc. Math. France, Paris, 2007. MR Zbl
- [2] F. Binda, M. Levine, M. T. Nguyen, and O. Röndigs (editors), *Motivic homotopy theory and refined enumerative geometry*, Contemporary Mathematics **745**, American Mathematical Society, Providence, RI, 2020. MR Zbl
- [3] F. Bittner, “The universal Euler characteristic for varieties of characteristic zero”, *Compos. Math.* **140**:4 (2004), 1011–1032. MR Zbl

- [4] A. J. Blumberg, D. Gepner, and G. Tabuada, “A universal characterization of higher algebraic K -theory”, *Geom. Topol.* **17**:2 (2013), 733–838. [MR](#) [Zbl](#)
- [5] A. J. Blumberg, D. Gepner, and G. Tabuada, “Uniqueness of the multiplicative cyclotomic trace”, *Adv. Math.* **260** (2014), 191–232. [MR](#) [Zbl](#)
- [6] M. V. Bondarko, “Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky versus Hanamura”, *J. Inst. Math. Jussieu* **8**:1 (2009), 39–97. [MR](#) [Zbl](#)
- [7] L. A. Borisov, “The class of the affine line is a zero divisor in the Grothendieck ring”, *J. Algebraic Geom.* **27**:2 (2018), 203–209. [MR](#) [Zbl](#)
- [8] A. K. Bousfield and E. M. Friedlander, “Homotopy theory of Γ -spaces, spectra, and bisimplicial sets”, pp. 80–130 in *Geometric applications of homotopy theory, II* (Evanston, IL, 1977), Lecture Notes in Math. **658**, Springer, 1978. [MR](#) [Zbl](#)
- [9] J. A. Campbell, “The K -theory spectrum of varieties”, *Trans. Amer. Math. Soc.* **371**:11 (2019), 7845–7884. [MR](#) [Zbl](#)
- [10] J. Campbell, J. Wolfson, and I. Zakharevich, “Derived ℓ -adic zeta functions”, *Adv. Math.* **354** (2019), art. id. 106760. [MR](#) [Zbl](#)
- [11] D.-C. Cisinski and F. Déglise, *Triangulated categories of mixed motives*, Springer, 2019. [MR](#) [Zbl](#)
- [12] D. Dugger, “Replacing model categories with simplicial ones”, *Trans. Amer. Math. Soc.* **353**:12 (2001), 5003–5027. [MR](#) [Zbl](#)
- [13] B. I. Dundas, O. Röndigs, and P. A. Østvær, “Enriched functors and stable homotopy theory”, *Doc. Math.* **8** (2003), 409–488. [MR](#) [Zbl](#)
- [14] B. I. Dundas, O. Röndigs, and P. A. Østvær, “Motivic functors”, *Doc. Math.* **8** (2003), 489–525. [MR](#) [Zbl](#)
- [15] F. T. Farrell and W. C. Hsiang, “On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds”, pp. 325–337 in *Algebraic and geometric topology, I* (Stanford, CA, 1976), Proc. Sympos. Pure Math. **32**, Amer. Math. Soc., Providence, RI, 1978. [MR](#) [Zbl](#)
- [16] H. Gillet and C. Soulé, “Descent, motives and K -theory”, *J. Reine Angew. Math.* **478** (1996), 127–176. [MR](#) [Zbl](#)
- [17] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs **99**, American Mathematical Society, Providence, RI, 2003. [MR](#) [Zbl](#)
- [18] M. Hovey, *Model categories*, Mathematical Surveys and Monographs **63**, American Mathematical Society, Providence, RI, 1999. [MR](#) [Zbl](#)
- [19] M. Hovey, “Spectra and symmetric spectra in general model categories”, *J. Pure Appl. Algebra* **165**:1 (2001), 63–127. [MR](#) [Zbl](#)
- [20] M. Hoyois, S. Scherotzke, and N. Sibilla, “Higher traces, noncommutative motives, and the categorified Chern character”, *Adv. Math.* **309** (2017), 97–154. [MR](#) [Zbl](#)
- [21] P. Hu, “On the Picard group of the stable \mathbb{A}^1 -homotopy category”, *Topology* **44**:3 (2005), 609–640. [MR](#) [Zbl](#)
- [22] J. F. Jardine, “Motivic symmetric spectra”, *Doc. Math.* **5** (2000), 445–552. [MR](#) [Zbl](#)
- [23] F. Jin, “On some finiteness results in real étale cohomology”, *Bull. Lond. Math. Soc.* **54**:2 (2022), 389–403. [MR](#) [Zbl](#)
- [24] M. Larsen and V. A. Lunts, “Motivic measures and stable birational geometry”, *Mosc. Math. J.* **3**:1 (2003), 85–95, 259. [MR](#) [Zbl](#)

- [25] J. Lurie, “Higher algebra”, notes, 2017, available at <https://www.math.ias.edu/~lurie/papers/HA.pdf>.
- [26] J. P. May, “The additivity of traces in triangulated categories”, *Adv. Math.* **163**:1 (2001), 34–73. [MR](#) [Zbl](#)
- [27] F. Morel, \mathbb{A}^1 -algebraic topology over a field, Lecture Notes in Mathematics **2052**, Springer, 2012. [MR](#) [Zbl](#)
- [28] F. Morel and V. Voevodsky, “ \mathbb{A}^1 -homotopy theory of schemes”, *Inst. Hautes Études Sci. Publ. Math.* **90** (1999), 45–143. [MR](#) [Zbl](#)
- [29] N. Naumann, M. Spitzweck, and P. A. Østvær, “Motivic Landweber exactness”, *Doc. Math.* **14** (2009), 551–593. [MR](#) [Zbl](#)
- [30] J. Nicaise, “A trace formula for varieties over a discretely valued field”, *J. Reine Angew. Math.* **650** (2011), 193–238. [MR](#) [Zbl](#)
- [31] I. Panin, K. Pimenov, and O. Röndigs, “On Voevodsky’s algebraic K -theory spectrum”, pp. 279–330 in *Algebraic topology*, Abel Symp. **4**, Springer, 2009. [MR](#) [Zbl](#)
- [32] A. J. Power, “A general coherence result”, *J. Pure Appl. Algebra* **57**:2 (1989), 165–173. [MR](#) [Zbl](#)
- [33] J. Riou, “Dualité de Spanier–Whitehead en géométrie algébrique”, *C. R. Math. Acad. Sci. Paris* **340**:6 (2005), 431–436. [MR](#) [Zbl](#)
- [34] J. Rognes, “Two-primary algebraic K -theory of pointed spaces”, *Topology* **41**:5 (2002), 873–926. [MR](#) [Zbl](#)
- [35] J. Rognes, “The smooth Whitehead spectrum of a point at odd regular primes”, *Geom. Topol.* **7** (2003), 155–184. [MR](#) [Zbl](#)
- [36] O. Röndigs, “Algebraic K -theory of spaces in terms of spectra”, Diplomarbeit, Universität Bielefeld, 1999.
- [37] O. Röndigs and P. A. Østvær, “Modules over motivic cohomology”, *Adv. Math.* **219**:2 (2008), 689–727. [MR](#) [Zbl](#)
- [38] S. Sagave, “On the algebraic K -theory of model categories”, *J. Pure Appl. Algebra* **190**:1–3 (2004), 329–340. [MR](#) [Zbl](#)
- [39] S. Schwede, “Spectra in model categories and applications to the algebraic cotangent complex”, *J. Pure Appl. Algebra* **120**:1 (1997), 77–104. [MR](#) [Zbl](#)
- [40] R. W. Thomason and T. Trobaugh, “Higher algebraic K -theory of schemes and of derived categories”, pp. 247–435 in *The Grothendieck Festschrift, III*, Progr. Math. **88**, Birkhäuser, Boston, 1990. [MR](#) [Zbl](#)
- [41] B. Toën and G. Vezzosi, “Caractères de Chern, traces équivariantes et géométrie algébrique dérivée”, *Selecta Math. (N.S.)* **21**:2 (2015), 449–554. [MR](#) [Zbl](#)
- [42] V. Voevodsky, “Open problems in the motivic stable homotopy theory, I”, pp. 3–34 in *Motives, polylogarithms and Hodge theory, Part I* (Irvine, CA, 1998), Int. Press Lect. Ser. **3**, International Press, Somerville, MA, 2002. [MR](#) [Zbl](#)
- [43] V. Voevodsky, “Cancellation theorem”, *Doc. Math.* Extra vol.: Andrei A. Suslin sixtieth birthday (2010), 671–685. [MR](#) [Zbl](#)
- [44] W. Vogell, “The involution in the algebraic K -theory of spaces”, pp. 277–317 in *Algebraic and geometric topology* (New Brunswick, NJ, 1983), Lecture Notes in Math. **1126**, Springer, 1985. [MR](#) [Zbl](#)

- [45] F. Waldhausen, “Algebraic K -theory of spaces, localization, and the chromatic filtration of stable homotopy”, pp. 173–195 in *Algebraic topology* (Aarhus, 1982), Lecture Notes in Math. **1051**, Springer, 1984. [MR](#) [Zbl](#)
- [46] F. Waldhausen, “Algebraic K -theory of spaces”, pp. 318–419 in *Algebraic and geometric topology* (New Brunswick, NJ, 1983), Lecture Notes in Math. **1126**, Springer, 1985. [MR](#) [Zbl](#)
- [47] F. Waldhausen, “Algebraic K -theory of spaces, concordance, and stable homotopy theory”, pp. 392–417 in *Algebraic topology and algebraic K -theory* (Princeton, NJ, 1983), Ann. of Math. Stud. **113**, Princeton Univ. Press, 1987. [MR](#) [Zbl](#)
- [48] C. A. Weibel, *The K -book: an introduction to algebraic K -theory*, Graduate Studies in Mathematics **145**, American Mathematical Society, 2013. [MR](#) [Zbl](#)
- [49] I. Zakharevich, “Perspectives on scissors congruence”, *Bull. Amer. Math. Soc. (N.S.)* **53**:2 (2016), 269–294. [MR](#) [Zbl](#)

Received 6 Jan 2022. Revised 27 Nov 2023.

OLIVER RÖNDIGS: oliver.roendigs@uni-osnabrueck.de

Institut für Mathematik, Universität Osnabrück, Osnabrück, Germany

Volume Editor:

Paul Arne Østvær
University of Oslo
Oslo, Norway

Cover courtesy of Paul Arne Østvær.

The contents of this work are copyrighted by MSP or the respective authors.
All rights reserved.

Electronic copies can be obtained free of charge from <http://msp.org/obs/6> and
printed copies can be ordered from MSP (contact@msp.org).

The Open Book Series is a trademark of Mathematical Sciences Publishers.

ISSN: 2329-9061 (print), 2329-907X (electronic)

ISBN: 978-1-935107-13-2 (print), 978-1-935107-14-9 (electronic)

First published 2025.



MATHEMATICAL SCIENCES PUBLISHERS

2000 Allston Way # 59
Berkeley CA 94701-4004

contact@msp.org
<https://msp.org>

Motivic Geometry

Based on lectures given at the Centre for Advanced Study (CAS) of the Norwegian Academy of Science and Letters, this book provides a panorama of developments in motivic homotopy theory and related fields.

A common goal of the research program underlying this volume is the understanding of the geometric nature of spaces, revealed through algebraic and homotopical invariants. The articles in this volume, contributed by leading experts, together touch on an extensive network of related topics in algebraic geometry, homotopy theory, K -theory and related areas.

The volume has a significant expository component, making it accessible to students, while also containing information and in-depth discussion of interest to all practitioners including specialists.

TABLE OF CONTENTS

Preface — Paul Arne Østvær	ix
Notes on motivic infinite loop space theory — Tom Bachmann and Elden Elmanto	1
An introduction to six-functor formalisms — Martin Gallauer	63
Introduction to framed correspondences — Marc Hoyois and Nikolai Opdan	107
Lectures on the cohomology of reciprocity sheaves — Nikolai Opdan and Kay Rülling	127
The Grothendieck ring of varieties and algebraic K -theory — Oliver Röndigs	165
Stable homotopy groups of motivic spheres — Oliver Röndigs and Markus Spitzweck	197