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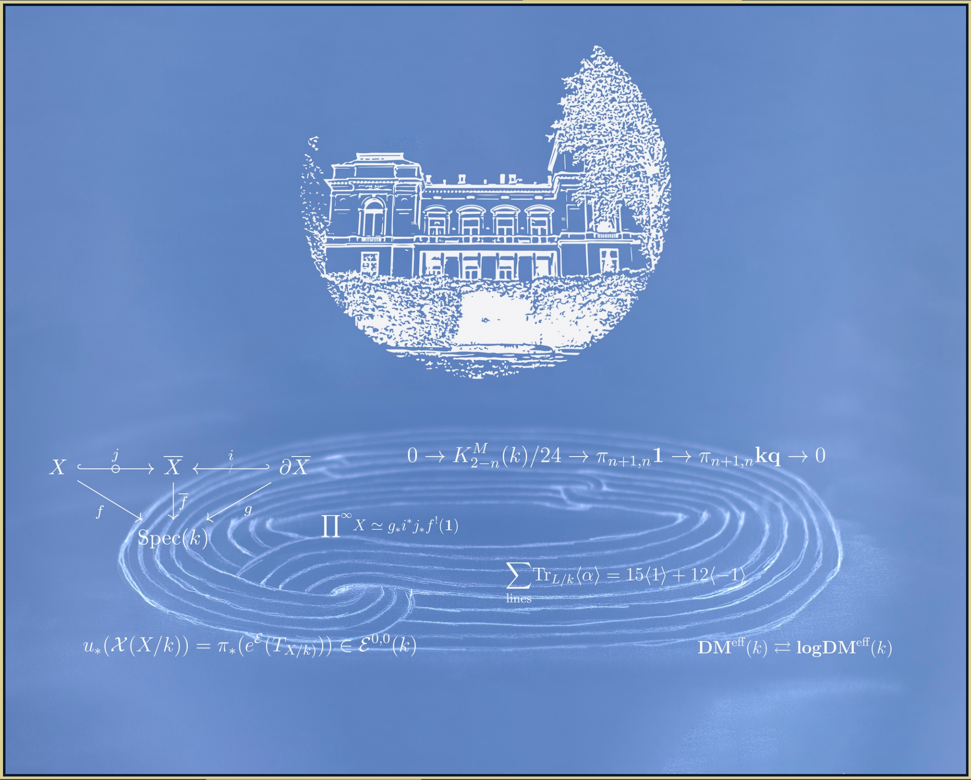
Motivic Geometry

Stable homotopy groups of motivic spheres

Oliver Röndigs and Markus Spitzweck



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Stable homotopy groups of motivic spheres

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These lecture notes are based on lectures given by the authors at the autumn school "Computations in motivic homotopy theory" at Regensburg University during September 16–20, 2019. Main results include a computation of the first Milnor–Witt stem of stable homotopy groups of motivic spheres over a field, presented differently than Röndigs, Spitzweck and Østvær (2019), and a partial computation of the zeroth Milnor–Witt stem of stable homotopy groups of motivic spheres over discrete valuation rings of mixed characteristic after inverting the residue characteristic.

1. Introduction

These lecture notes connect fundamental results in motivic or $\boldsymbol{A}^1$-homotopy theory, as developed by Vladimir Voevodsky, Fabien Morel, and others, with concrete computations of stable homotopy groups of motivic spheres given in [43; 55]. They are based on lectures given during the autumn school "Computations in motivic homotopy theory" at Regensburg University during September 16–20, 2019. Given the circumstances, we decided to refer to the literature for certain results and arguments, whereas other simple results are exercises for the reader; therefore these lecture notes do not contain complete computations of the zeroth and first Milnor–Witt stem of stable homotopy groups of motivic spheres over a field. Instead, we offer with Theorem 7.13 a partial extension of Morel's computation of the zeroth Milnor–Witt stem of stable homotopy groups of spheres to the case of a discrete valuation ring of mixed characteristic subject to inverting the positive residue characteristic.

Besides the foundations [47] of unstable $\boldsymbol{A}^1$-homotopy theory, the sources [25; 35; 39; 64] may serve as accompanying reading material. We thank the organizers (Denis-Charles Cisinski, Markus Land, Florian Strunk, and Georg Tamme), the

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other speakers (Marc Hoyois and Kirsten Wickelgren), and all participants for a thoroughly pleasant and motivating event.

2. Fundamental properties

One viewpoint on motivic or $\boldsymbol{A}^1$-homotopy theory, as heavily advertised by Fabien Morel, is to supply a homotopy theory for smooth varieties over a field which enjoys similar properties as classical homotopy theory does with respect to smooth manifolds. More precisely, the following theorems hold, based on the notational convention that, for every Noetherian separated scheme S of finite Krull dimension (to be abbreviated as *base scheme*), $\mathbf{Sm}_S$ denotes the category of smooth finite-type S-schemes, and (not necessarily smooth) morphisms of such. Let $\mathbf{Spc}(S)$ denote the category of *spaces over* S, that is, presheaves (or Nisnevich sheaves) on $\mathbf{Sm}_S$ with values in simplicial sets. The pointed version, $\mathbf{Spc}_\bullet(S)$, is the category of pointed spaces over S, that is, presheaves (or Nisnevich sheaves) on $\mathbf{Sm}_S$ with values in pointed simplicial sets. The $\boldsymbol{A}^1$-homotopy theory on $\mathbf{Spc}(S)$ (and therefore also on its pointed version $\mathbf{Spc}_\bullet(S)$) is determined by the following two properties:

Nisnevich excision: Every elementary distinguished square

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

(that is, every pullback square in which j is an open embedding and p is an étale morphism inducing an isomorphism on reduced closed subschemes $Y \smallsetminus V \cong X \smallsetminus U$) in $\mathbf{Sm}_S$ induces a homotopy pushout square in $\mathbf{Spc}(S)$.

Homotopy invariance: The affine line parametrizes homotopies in the sense that the projection $X \times_S \boldsymbol{A}^1 \to X$ is a weak equivalence for all $X \in \mathbf{Sm}_S$.

In these properties, the Yoneda embedding $\mathbf{Sm}_S \hookrightarrow \mathbf{Spc}(S)$ is used for the passage from smooth S-schemes to spaces over S without appearing in the notation. It sends a smooth S-scheme to the discrete simplicial presheaf it represents. If a smooth S-scheme X admits a rational point $x : S \to X$, the resulting pair (X, x) will be viewed as a pointed space over S. For any smooth S-scheme X, the disjoint union $X \coprod S$ comes with a canonical rational point, producing the pointed space X_+ over S.

Example 2.1. The canonical covering of the projective line $\boldsymbol{P}^1_S$ over S by two copies of the affine line over S supplies a canonical identification of $(\boldsymbol{P}^1_S, [1, 1])$ with the reduced suspension $\Sigma(\boldsymbol{G}_{\mathrm{m}}, 1)$ of the multiplicative group scheme over S, as well as a canonical identification $\mathsf{Th}(\boldsymbol{A}^1_S) := \boldsymbol{A}^1_S/(\boldsymbol{A}^1_S \smallsetminus \{0\}) \simeq \boldsymbol{P}^1_S$ with the Thom space of the trivial line bundle over S.

Definition 2.2. Let $S^0 = S_+ = S \coprod S$ be the zero sphere over S. It is the unit for the closed symmetric monoidal smash product in $\mathbf{Spc}_\bullet(S)$. The basic *circles* over S are the simplicial circle $\Delta^1/\partial\Delta^1$, considered as a constant pointed (pre)sheaf, and the Tate circle $\boldsymbol{G}_\mathsf{m} = (\boldsymbol{A}^1_S \smallsetminus \{0\}, 1)$ over S. Smash products of these produce, for every $s, w \in \mathbb{N}$, a sphere

$$\Sigma_S^{s+(w)} := \Sigma_S^{s+w,w} := (\Delta^1/\partial^1)^{\wedge s} \wedge \boldsymbol{G}_\mathsf{m}^{\wedge w},$$

which is a pointed space over S. The base scheme may be removed from the notation. For example, there are canonical identifications

$$\boldsymbol{A}^n \smallsetminus \{0\} \simeq \Sigma^{n-1+(n)} = \Sigma^{2n-1,n} \quad \text{and} \quad \mathsf{Th}(\boldsymbol{A}^n) \simeq \boldsymbol{P}^n/\boldsymbol{P}^{n-1} \simeq \Sigma^{n+(n)} = \Sigma^{2n,n}.$$

(Exercise.)

Remark 2.3. If $S = \mathrm{Spec}(\mathbb{C})$ (or more generally if S maps to $\mathrm{Spec}(\mathbb{C})$), sending a smooth $\mathbb{C}$-scheme to the underlying topological space of its associated complex analytic manifold induces a functor of homotopy theories preserving (homotopy) colimits and (smash) products. It sends the sphere $\Sigma_\mathbb{C}^{s+(w)} = \Sigma_\mathbb{C}^{s+w,w}$ to the topological sphere S^{s+w}. If $S = \mathrm{Spec}(\mathbb{R})$ (or more generally if S maps to $\mathrm{Spec}(\mathbb{R})$), sending a smooth $\mathbb{R}$-scheme to the underlying topological space of its associated real analytic manifold induces another functor of homotopy theories preserving (homotopy) colimits and (smash) products. It sends the sphere $\Sigma_\mathbb{R}^{s+(w)} = \Sigma_\mathbb{R}^{s+w,w}$ to the topological sphere S^s. (Exercise.)

Several arguments will employ the homotopy theory of spaces over S, which is of a very complicated nature. Inverting (smashing with) the sphere $\boldsymbol{P}^1 \simeq \Sigma^{1+(1)} = \Sigma^{2,1}$ produces a simpler homotopy theory.

Theorem 2.4. *For every base scheme S, there exists a symmetric monoidal functor $\Sigma^\infty_{\boldsymbol{P}^1_S} : \mathbf{Spc}_\bullet(S) \to \mathbf{SH}(S)$ to a closed symmetric monoidal triangulated category. The object $\Sigma^\infty_{\boldsymbol{P}^1_S}(\boldsymbol{P}^1_S, \infty)$ is invertible with respect to the symmetric monoidal structure. Every morphism $f : S \to T$ of base schemes induces a strong symmetric monoidal triangulated functor $f^* : \mathbf{SH}(T) \to \mathbf{SH}(S)$, having a right adjoint f_*, and even a left adjoint $f_\sharp$ if f is smooth.*

The functoriality alluded to in Theorem 2.4 is in fact part of an encompassing setup often referred to as the *six functor formalism*. More properties and details regarding this formalism will be provided in Theorems 2.7 and 2.10, and in Section 7 below, while the comprehensive motivic reference is [7; 8].

The classical analog of $\mathbf{SH}(S)$ is the stable homotopy category $\mathbf{SH}$ of spectra. Objects in $\mathbf{SH}(S)$ are called *motivic spectra over S*. One may interpret $\mathbf{SH}(S)$ as the homotopy category of a suitable symmetric monoidal model category, or as a suitable symmetric monoidal stable ∞-category. Being triangulated, there exists a shift endofunctor Σ which is an equivalence and, for formal reasons, coincides with smash product (from the right) with $\Delta^1/\partial\Delta^1 = \Sigma^{1+(0)} = \Sigma^{1,0}$. The bigraded notation is explained by the existence of another canonically invertible endofunctor, the smash product with $\Sigma_S^{1+(1)} = \Sigma_S^{2,1} \simeq (\boldsymbol{P}_S^1, \infty)$. Other bigrading conventions exist in the literature. Slightly abusing notation, the sphere $\Sigma_S^{s+(w)}$, its image $\Sigma_{\boldsymbol{P}_S^1}^\infty \Sigma_S^{s+(w)} \in \mathbf{SH}(S)$, and the smash product with either (from the right) will be denoted as $\Sigma_S^{s+(w)} = \Sigma_S^{s+w,w}$. Using invertibility of $\Sigma_S^{1+(1)}$, given integers s, w and $\mathsf{E} \in \mathbf{SH}(S)$, the motivic spectrum $\Sigma^{s+(w)}\mathsf{E} = \Sigma^{s+w,w}\mathsf{E}$ is well defined, as are the homotopy groups

$$\pi_{s+(w)}\mathsf{E} = \pi_{s+w,w}\mathsf{E} = [\Sigma^{s+(w)}, \mathsf{E}]$$

in simplicial degree s and weight w. Assembling all weights together yields

$$\pi_s\mathsf{E} := \pi_{s+(\star)}\mathsf{E} := \bigoplus_{w\in\mathbb{Z}} \pi_{s+(w)}\mathsf{E},$$

where "$(\star)$" may be carried around to indicate the weight grading. The Nisnevich sheafification of the presheaf $X \mapsto [\Sigma^{s+(w)}X_+, \mathsf{E}]$ on $\mathbf{Sm}_S$ with values in abelian groups will be denoted by $\underline{\pi}_{s+(w)}\mathsf{E} = \underline{\pi}_{s+w,w}\mathsf{E}$.

Lemma 2.5. *A map $\phi : \mathsf{D} \to \mathsf{E}$ is an equivalence if and only if $\underline{\pi}_{s+(w)}(\phi)$ is an isomorphism of Nisnevich sheaves for all $s, w \in \mathbb{Z}$.*

The functor $\Sigma_{\boldsymbol{P}_S^1}^\infty$ factors other prominent functors, for example, the functor sending a smooth variety over a field F to its motive in Voevodsky's category $\mathbf{DM}(F)$.[1] This will, in particular, imply the nontriviality of the category $\mathbf{SH}(S)$ (unless S is the empty scheme). Both can be deduced from the following statement, a motivic analog of the representability of singular cohomology through the Eilenberg–MacLane spectrum.

Theorem 2.6 (Voevodsky). *Let F be a field. There exists a motivic spectrum $\mathbf{M}\mathbb{Z}_F \in \mathbf{SH}(F)$ representing motivic cohomology in the following sense: For every pair of integers s, w and every smooth S-scheme X, there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{SH}(S)}(\Sigma_{\boldsymbol{P}_S^1}^\infty X, \Sigma^{s+(w)}\mathbf{M}\mathbb{Z}_F) \cong H^{s+w,w}(X, \mathbb{Z}) = H^{s+w}(X, \mathbb{Z}(w))$$

of abelian groups.

This isomorphism may be promoted to an isomorphism of graded rings: the degree-d part of the Chow ring is naturally given as $\mathrm{Hom}_{\mathbf{SH}(F)}(\Sigma_{\boldsymbol{P}_S^1}^\infty X, \Sigma^{d+(d)}\mathbf{M}\mathbb{Z}_F)$ as a particular case. Isomorphisms in $\mathbf{SH}(S)$ can be detected locally in the following rather strong sense.

[1] Removing "Spec" from the notation will often occur in the hope that it simplifies the text and does not confuse the reader.

Theorem 2.7 (localization). *Let $i : Z \hookrightarrow S$ be a closed embedding of base schemes, with open complement $j : S \smallsetminus Z \hookrightarrow S$. The natural maps define a homotopy cofiber sequence*

$$j_\sharp j^* \mathsf{E} \to \mathsf{E} \to i_* i^* \mathsf{E} \to \Sigma j_\sharp j^* \mathsf{E}$$

in $\mathbf{SH}(S)$. In particular, a map $\phi : \mathsf{D} \to \mathsf{E}$ in $\mathbf{SH}(S)$ is an isomorphism if and only if $i^*(\phi)$ and $j^*(\phi)$ are isomorphisms.

Given an open embedding $j : U \hookrightarrow X$ in $\mathbf{Sm}_S$, let $\Sigma^\infty_{\boldsymbol{P}^1_S} X/U$ denote the canonical cone appearing in the homotopy cofiber sequence

$$\Sigma^\infty_{\boldsymbol{P}^1_S} U \to \Sigma^\infty_{\boldsymbol{P}^1_S} X \to \Sigma^\infty_{\boldsymbol{P}^1_S} X/U \to \Sigma\Sigma^\infty_{\boldsymbol{P}^1_S} U$$

in $\mathbf{SH}(S)$. In the special case where $V \to X$ is a vector bundle with zero section $z : X \hookrightarrow V$, abbreviate $\mathsf{Th}(V \to X) = \Sigma^\infty_{\boldsymbol{P}^1_S} V/(V \smallsetminus z(X))$. It serves to formulate the following analog of the tubular neighborhood construction, given by Morel and Voevodsky:

Theorem 2.8 (homotopy purity). *Let $i : Z \hookrightarrow X$ be a closed embedding in $\mathbf{Sm}_S$, with normal bundle $Ni \to Z$. There exists a suitably natural identification*

$$\Sigma^\infty_{\boldsymbol{P}^1_S} X/(X \smallsetminus i(Z)) \simeq \mathsf{Th}(Ni \to Z)$$

in $\mathbf{SH}(S)$.

Example 2.9. Let $i : \boldsymbol{P}^{n-1} \hookrightarrow \boldsymbol{P}^n$ be the closed embedding given by $x \mapsto (x, 0)$. To compute its homotopy cofiber with the help of Theorem 2.8, it helps to replace i up to $\boldsymbol{A}^1$-equivalence by an open embedding. In this very special situation, i factors as $\boldsymbol{P}^{n-1} \hookrightarrow \boldsymbol{P}^n \smallsetminus \{(0, 1)\} \subset \boldsymbol{P}^n$. The first map is the zero section of the line bundle $\boldsymbol{P}^n \smallsetminus \{(0, 1)\} \to \boldsymbol{P}^{n-1}$ forgetting the last coordinate, and hence an $\boldsymbol{A}^1$-homotopy equivalence. The homotopy cofiber of $\boldsymbol{P}^n \smallsetminus \{(0, 1)\} \hookrightarrow \boldsymbol{P}^n$ is the Thom space of the (trivial) normal bundle of the point $(0, 1) \in \boldsymbol{P}^n$ by Theorem 2.8.

The homotopy purity theorem allows us to describe a duality in $\mathbf{SH}(S)$ modeled on the classical Spanier–Whitehead duality. In order to state it, let $\mathbf{1}_S = \Sigma^\infty_{\boldsymbol{P}^1_S} S \in \mathbf{SH}(S)$ be the unit of the closed symmetric monoidal structure, usually denoted as $(\mathsf{D}, \mathsf{E}) \mapsto \mathsf{D} \wedge \mathsf{E}$, with internal hom denoted $\underline{\mathrm{Hom}}_S(\mathsf{D}, \mathsf{E})$.

Theorem 2.10 (Spanier–Whitehead duality). *Let $X \in \mathbf{Sm}_S$ be projective, with structure morphism $f : X \to S$ and tangent bundle $\mathscr{T}_f \to X$. Then $\Sigma^\infty_{\boldsymbol{P}^1_S} X$ admits a strong dual, given as*

$$\underline{\mathrm{Hom}}_S(\Sigma^\infty_{\boldsymbol{P}^1_S} X, \mathbf{1}_S) \simeq f_\sharp\, \underline{\mathrm{Hom}}_X(\mathsf{Th}(\mathscr{T}_f \to X), \Sigma^\infty_{\boldsymbol{P}^1_X} X).$$

While the category $\mathbf{SH}(S)$ is always generated by compact objects (namely $\boldsymbol{P}^1_S$-(de)suspensions of smooth S-schemes, or even smooth affine S-schemes), it is unclear if it is generated by strongly dualizable objects. If F is a field of characteristic zero, one may use resolution of singularities as an ingredient to prove that $\mathbf{SH}(F)$ is generated by $\boldsymbol{P}^1_S$-(de)suspensions of smooth projective F-schemes, which are also strongly dualizable by Theorem 2.10. Already compact generation can be very helpful, as the following continuity statement shows, which could be formulated in greater generality.

Lemma 2.11. *Let $\mathcal{I} \to$ Rings, $\beta \mapsto R_\beta$, be a diagram of Noetherian rings of finite Krull dimension, where $\mathcal{I}$ is a filtered category with initial object α. Suppose its colimit R_ω is Noetherian of finite Krull dimension. For a motivic spectrum $\mathsf{E} = \mathsf{E}_\alpha \in \mathbf{SH}(R_\alpha)$, let E_β denote its pullback to $\mathbf{SH}(R_\beta)$, and similarly for E_ω. Then for every compact motivic spectrum $\mathsf{D} \in \mathbf{SH}(R_\alpha)$ the canonical map*

$$\operatorname*{colim}_{\beta\in\mathcal{I}} \mathrm{Hom}_{\mathbf{SH}(R_\beta)}(\mathsf{D}_\beta, \mathsf{E}_\beta) \to \mathrm{Hom}_{\mathbf{SH}(R_\omega)}(\mathsf{D}_\omega, \mathsf{E}_\omega)$$

is an isomorphism.

Proof. This can be derived as a special case of a statement on filtered diagrams of stable ∞-categories. The construction of the $\boldsymbol{P}^1$-stable $\boldsymbol{A}^1$-homotopy theory involving the Nisnevich topology implies that the compactly generated homotopy category $\mathbf{SH}(R_\omega)$ at the colimit R_ω is equivalent to the filtered colimit of the categories $\mathbf{SH}(R_\beta)$. Modulo some details, adjointness translates the comparison map in question to a homomorphism of Hom-groups in $\mathbf{SH}(R_\alpha)$. Here the right adjoints of the pullback functors commute with filtered colimits, as does the Hom-functor represented by a compact object. □

A further representability result recovers the first known generalized cohomology theory for schemes, as invented by Quillen.

Theorem 2.12 (Morel–Voevodsky). *Let S be a regular base scheme. There exists a motivic spectrum $\mathbf{KGL}_S \in \mathbf{SH}(S)$ representing Quillen's higher algebraic K-groups in the following sense: For every pair of integers s, w and every smooth F-scheme X, there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{SH}(F)}(\Sigma^\infty_{\boldsymbol{P}^1_S} X, \Sigma^{s+(w)}\mathbf{KGL}_S) \cong K^Q_{w-s}(X)$$

of abelian groups.

Also this isomorphism may be promoted to an isomorphism of graded rings. The classical analog is Bott and Atiyah's representability of complex topological K-theory. In principle, every motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ gives rise to a generalized motivic cohomology theory, with potentially interesting geometric applications. The initial generalized motivic cohomology theory is then motivic stable cohomotopy, represented by the unit object $\mathbf{1}_S$. As in the classical situation, only limited information about the represented theory is available.

3. Maps of spheres and Milnor–Witt K-theory

Recall that $\mathbf{1} = \mathbf{1}_S$ denotes the motivic sphere spectrum over the base scheme S. It is the unit for the symmetric monoidal structure of $\mathbf{SH}(S)$ given by the smash product $(\mathsf{D}, \mathsf{E}) \mapsto \mathsf{D} \wedge \mathsf{E}$. In particular, given elements $\alpha \in \pi_{s+(w)}\mathbf{1}$ and $\beta \in \pi_{t+(x)}\mathbf{1}$, the smash product defines the element

$$\alpha \cdot \beta : \Sigma^{s+t+(w+x)}\mathbf{1} \xrightarrow{\text{commutativity iso.}} \Sigma^{s+(w)} \wedge \Sigma^{t+(x)} \xrightarrow{\alpha\wedge\beta} \mathbf{1} \wedge \mathbf{1} \xrightarrow{\text{unit iso.}} \mathbf{1}$$

in $\pi_{s+t+(w+x)}\mathbf{1}$. Alternatively, one may define the element $\alpha \circ \beta$ as the composition

$$\alpha \circ \beta : \Sigma^{s+t+(w+x)}\mathbf{1} \xrightarrow{\text{commutativity iso.}} \Sigma^{s+(w)}\Sigma^{t+(x)} \xrightarrow{\Sigma^{s+(w)}\beta} \Sigma^{s+(w)}\mathbf{1} \xrightarrow{\alpha} \mathbf{1}$$

in $\pi_{s+t+(w+x)}\mathbf{1}$. A careful discussion of these two structures and their relation can be found in [22].

Lemma 3.1 (ring structure). *For every $\alpha \in \pi_{s+(w)}\mathbf{1}$ and $\beta \in \pi_{t+(x)}\mathbf{1}$, there is an equality $\alpha \cdot \beta = \alpha \circ \beta$. The graded group*

$$\pi_{*+(\star)}\mathbf{1} = \bigoplus_{(s,w)\in\mathbb{Z}\times\mathbb{Z}} \pi_{s+(w)}\mathbf{1}$$

forms a graded ring under this multiplication, with the identity as unit.

Proof. Exercise. The references [21; 24] might help. □

Hence the homotopy groups and homotopy sheaves of $\mathbf{1}$ act on the homotopy groups and homotopy sheaves of any motivic spectrum, respectively. We start by writing down obvious elements in $\pi_{*+(\star)}\mathbf{1}$. Every invertible element $u \in \mathcal{O}_S^\times$ defines a morphism $[u] : S_+ \to \boldsymbol{G}_\mathsf{m}$ of pointed schemes over S, sending the nonbasepoint to u. It induces a map $[u] : \Sigma^{(-1)} \to \mathbf{1}$ in $\mathbf{SH}(S)$. Since 1 is the basepoint of $\boldsymbol{G}_\mathsf{m}$, $[1]$ is the zero element in $\pi_{(-1)}\mathbf{1}$. The following statement was proved first in [33].

Lemma 3.2 (Steinberg relation). *Let S be a base scheme. Then* $[u] \cdot [1-u] = 0 \in \pi_{(-2)}\mathbf{1}$ *for every $u \in \mathcal{O}_S^\times$ such that $1-u \in \mathcal{O}_S^\times$.*

Proof. Consider the morphism $\boldsymbol{A}^1 \smallsetminus \{0, 1\} \to \boldsymbol{G}_\mathsf{m} \times \boldsymbol{G}_\mathsf{m}$, $u \mapsto (u, 1-u)$, of unpointed smooth schemes. Its image in the affine plane with coordinate axes removed is the line through the (removed) points $(1, 0)$ and $(0, 1)$. Adjoining a disjoint basepoint to its source provides a morphism

$$f : (\boldsymbol{A}^1 \smallsetminus \{0, 1\})_+ \to \boldsymbol{G}_\mathsf{m} \times \boldsymbol{G}_\mathsf{m}, \quad u \mapsto (u, 1-u),$$

of pointed smooth schemes. Postcomposing the morphism f with the canonical map $\boldsymbol{G}_\mathsf{m} \times \boldsymbol{G}_\mathsf{m} \to \boldsymbol{G}_\mathsf{m} \wedge \boldsymbol{G}_\mathsf{m}$ and precomposing with $[u] : S_+ \to (\boldsymbol{A}^1 \smallsetminus \{0, 1\})_+$ for any given $u \in \boldsymbol{A}^1 \smallsetminus \{0, 1\}(S)$ provides the map in question. The basic observation is that the aforementioned image, a line with two points removed, is sent in the smash product to a looped line at the basepoint, which now fills the two holes. In

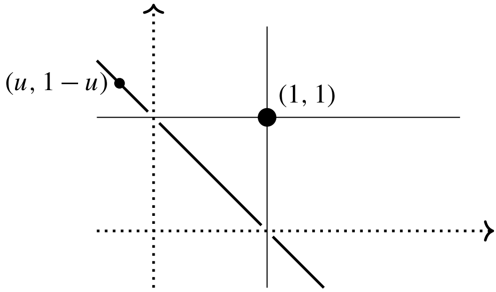

Figure 1. The Steinberg relation.

particular, this line connects the point $(u, 1-u)$ to the basepoint. See Figure 1. More precisely, the canonical map $\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \to \boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}}$ factors as

$$\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \hookrightarrow \boldsymbol{A} \times \{1\} \cup_{\boldsymbol{G}_{\mathsf{m}} \times \{1\}} \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \cup_{\{1\} \times \boldsymbol{G}_{\mathsf{m}}} \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{A}^1 \to \boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}},$$

where the last map collapses the $\boldsymbol{A}^1$-contractible subvariety $\boldsymbol{A} \times \{1\} \cup_{(1,1)} \{1\} \times \boldsymbol{A}^1$ to the basepoint, and hence is an equivalence. (Exercise: write down a contracting $\boldsymbol{A}^1$-homotopy which is constant on the basepoint.) The composition

$$(\boldsymbol{A}^1 \smallsetminus \{0, 1\})_+ \to \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \to \boldsymbol{A} \times \{1\} \cup_{\boldsymbol{G}_{\mathsf{m}} \times \{1\}} \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \cup_{\{1\} \times \boldsymbol{G}_{\mathsf{m}}} \boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{A}^1$$

factors over the union $\boldsymbol{A}^1 \times \{1\} \cup_{\{(0,1)\}} \{(t, 1-t)\}$, which is also $\boldsymbol{A}^1$-contractible as a union of two affine lines. The result follows. □

As a consequence, the Steinberg relation holds in the homotopy groups of any motivic spectrum. One particular instance is Voevodsky's Eilenberg–MacLane spectrum $\mathbf{M}\mathbb{Z}_F$, where one has $\pi_{(-n)}\mathbf{M}\mathbb{Z}_F \cong \mathbf{K}_n^{\mathsf{M}}(F)$, the latter being defined as the degree-n component of the quotient of the tensor algebra on the units in F subject to the Steinberg relation (see Definition 3.8 below for details). Note that the unit $\mathbf{1}_F \to \mathbf{M}\mathbb{Z}_F$ maps $[u]$ to the symbol $\{u\} \in \mathbf{K}_1^{\mathsf{M}}(F)$ under this identification. Contrary to the identity $\{u\}+\{v\}=\{uv\} \in \mathbf{K}_1^{\mathsf{M}}(F)$, the term $[u]+[v]-[uv] \in \pi_{(-1)}\mathbf{1}_F$ is not always zero. In order to describe it, consider the multiplication $\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}} \to \boldsymbol{G}_{\mathsf{m}}$. After passing to motivic spectra, the canonical map $\Sigma^\infty(\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}}) \to \Sigma^\infty(\boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}})$ admits a canonical section. This is in fact true for pointed simplicial presheaves after a single simplicial suspension. The resulting composition

$$\Sigma^\infty(\boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}}) \to \Sigma^\infty(\boldsymbol{G}_{\mathsf{m}} \times \boldsymbol{G}_{\mathsf{m}}) \xrightarrow{\text{multiplication}} \Sigma^\infty \boldsymbol{G}_{\mathsf{m}}$$

defines an element $\eta \in \pi_{(1)}\mathbf{1}$.

Lemma 3.3 (logarithm). *For every $u, v \in \mathcal{O}_S^\times$ the equality $[uv]=[u]+[v]+\eta\cdot[u]\cdot[v]$ holds.*

Proof. Exercise. □

Lemma 3.4 (commutativity). *For every $u \in \mathcal{O}_S^\times$ the equality $\eta \cdot [u] = [u] \cdot \eta$ holds.*

Proof. Exercise. □

Let $\varepsilon \in \pi_{0+(0)}\mathbf{1}$ denote the element which is induced by the commutativity isomorphism $\boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}} \cong \boldsymbol{G}_{\mathsf{m}} \wedge \boldsymbol{G}_{\mathsf{m}}$. By definition and Lemma 3.1, $\varepsilon^2 = 1 \in \pi_{0+(0)}\mathbf{1}$.

Lemma 3.5 (hyperbolic plane). *The equality $\eta \circ \varepsilon = \eta$ holds in $\pi_{(1)}\mathbf{1}$.*

Proof. This follows because the multiplication on $\boldsymbol{G}_{\mathsf{m}}$ is commutative. □

Lemma 3.6. *The ring structure on $\pi_{*+(\star)}\mathbf{1}$ is ε-graded commutative in the sense that for every $\alpha \in \pi_{s+(w)}\mathbf{1}$ and $\beta \in \pi_{t+(x)}\mathbf{1}$, there is an equality $\alpha \cdot \beta = (-1)^{st}\varepsilon^{wx}\beta \cdot \alpha$.*

There is another canonical element in $\pi_{0+(0)}\mathbf{1}$, namely the element ϵ which is induced by the inverse $\boldsymbol{G}_{\mathsf{m}} \xrightarrow{u \mapsto u^{-1}} \boldsymbol{G}_{\mathsf{m}}$. Again by definition and Lemma 3.1, $\epsilon^2 = 1 \in \pi_{0+(0)}\mathbf{1}$. Yet another canonical element is $\tilde{\eta} \in \pi_{(1)}\mathbf{1}$ induced by the composition

$$\Sigma^{1+(2)} \simeq \boldsymbol{A}^2 \smallsetminus \{0\} \xrightarrow{\text{canonical}} \boldsymbol{P}^1 \simeq \Sigma^{1+(1)},$$

where the identifications are the canonical ones.

Lemma 3.7. *There are equalities $\varepsilon = \epsilon$, $\eta = \tilde{\eta}$, and $\varepsilon = -(1 + \eta[-1])$.*

Proof. Consider the map $\boldsymbol{P}^1 \to \boldsymbol{P}^1$, $[x, y] \mapsto [y, x]$. It restricts to the inverse map $u \mapsto u^{-1}$ on $\boldsymbol{G}_{\mathsf{m}}$. Since it also interchanges the two copies of the affine line, it corresponds to $-\Sigma^1\epsilon : \Sigma^1\boldsymbol{G}_{\mathsf{m}} \to \Sigma^1\boldsymbol{G}_{\mathsf{m}}$ via the canonical identification $\Sigma^1\boldsymbol{G}_{\mathsf{m}} \simeq \boldsymbol{P}^1$. The map $[x, y] \mapsto [y, x]$ can equivalently be described via the action of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and since the last matrix is a product of elementary matrices, the resulting map is $\boldsymbol{A}^1$-homotopic to the map induced by $[x, y] \mapsto [-x, y]$. Via the canonical identification $\boldsymbol{A}^1/(\boldsymbol{A}^1 \smallsetminus \{0\}) \simeq \boldsymbol{P}^1$, it thus corresponds to the map induced by multiplication with -1.

The commutativity isomorphism $\boldsymbol{P}^1 \wedge \boldsymbol{P}^1 \cong \boldsymbol{P}^1 \wedge \boldsymbol{P}^1$ induces via the canonical equivalences

$$\boldsymbol{P}^1 \wedge \boldsymbol{P}^1 \simeq \boldsymbol{A}^1/(\boldsymbol{A}^1 \smallsetminus \{0\}) \wedge \boldsymbol{A}^1/(\boldsymbol{A}^1 \smallsetminus \{0\}) \cong \boldsymbol{A}^2/(\boldsymbol{A}^2 \smallsetminus \{0\})$$

the map $\boldsymbol{A}^2/(\boldsymbol{A}^2 \smallsetminus \{0\}) \to \boldsymbol{A}^2/(\boldsymbol{A}^2 \smallsetminus \{0\})$ induced by $(x, y) \mapsto (y, x)$. Also this map is induced by the action of the matrix given above, hence is $\boldsymbol{A}^1$-homotopic to the map induced by $(x, y) \mapsto (-x, y)$. The latter corresponds to the $\boldsymbol{A}^1/(\boldsymbol{A}^1 \smallsetminus \{0\})$-suspension of the map induced by multiplication with -1, and hence to $-\Sigma^{2+(1)}\epsilon$.

The commutativity isomorphism on $S^1 \wedge S^1$ coincides with -1 by topology, whence the commutativity isomorphism on $\boldsymbol{P}^1 \wedge \boldsymbol{P}^1$ equals an appropriate suspension of $-\varepsilon$. The equality $\epsilon = \varepsilon$ follows. Moreover, the identification $\varepsilon = -(1+\eta[-1])$ results from the intermediate step of this argument.

The equality $\eta = \tilde{\eta}$ is left as an exercise. □

Another canonical element can be obtained from the cell filtration on projective spaces, namely the composition

$$\phi_n : \Sigma^{n+(n+1)} \simeq \boldsymbol{A}^{n+1} \smallsetminus \{0\} \to \boldsymbol{P}^n \to \boldsymbol{P}^n/\boldsymbol{P}^{n-1} \simeq \Sigma^{n+(n)}$$

of canonical maps, which gives an element $\phi_n \in \pi_{0+(1)}\mathbf{1}$. Lemma 3.7 implies that $\phi_1 = \eta$. Exercise: identify ϕ_n for other n.

For a unit u, set $\langle u \rangle := 1 + \eta[u]$; the notation reflects that this element corresponds to the one-dimensional symmetric bilinear form represented by u which is established essentially in Theorem 3.10 below. The proof of Lemma 3.7 above implies that $-\langle u \rangle$ can be described as the endomorphism $[x, y] \mapsto [ux, y]$ on $\boldsymbol{P}^1$, provided either $[0, 1]$ or $[1, 0]$ are chosen as the basepoint.

Definition 3.8. The Milnor–Witt K-theory $\mathbf{K}^{\mathsf{MW}}(F)$ of a field F has the Hopkins-Morel presentation as the associative graded ring (with unit) whose generators are the field units $[u]$, $u \in F^\times$ in degree 1, and a generator η in degree -1, subject to four relations:

Steinberg: $[u] \cdot [v] = 0$ if $u + v = 1$.

Logarithm: $[u \cdot v] = [u] + [v] + \eta \cdot [u] \cdot [v]$.

Commutativity: $[u] \cdot \eta = \eta \cdot [u]$.

Hyperbolic plane: $\mathsf{h} \cdot \eta = 0$, where $\mathsf{h} := 2 + \eta[-1]$.

Corollary 3.9. *The canonical map defines a "degree inverting" ring homomorphism* $\mathbf{K}^{\mathsf{MW}}_\star(F) \to \pi_{0-(\star)}\mathbf{1}$.

Proof. This follows from Lemmata 3.2–3.5. □

Theorem 3.10 (Morel). *Let F be a field. Then $\pi_s \mathbf{1}_F = 0$ for $s < 0$. The canonical ring homomorphism $\mathbf{K}^{\mathsf{MW}}_\star(F) \to \pi_{0-(\star)}\mathbf{1}$ is an isomorphism.*

Proof. See [46] for the unstable result implying the $\boldsymbol{P}^1$-stable one, as well as [43] for a sketch. The injectivity will be clarified as a consequence of Theorem 4.19 below. □

The Milnor–Witt K-theory of F is closely related to Milnor K-theory (by sending η to 0), and to quadratic form theory. In fact, $\mathbf{K}^{\mathsf{MW}}_0(F)$ is the Grothendieck–Witt ring of F; the 1-dimensional form $\langle u \rangle$ corresponds to $1 + \eta[u]$. If $n < 0$, then $\mathbf{K}^{\mathsf{MW}}_n(F)$ is isomorphic to the Witt ring of F and multiplication with η induces an isomorphism $\mathbf{K}^{\mathsf{MW}}_n(F) \to \mathbf{K}^{\mathsf{MW}}_{n-1}(F)$. (Exercise; see [46, Lemma 3.10].)

4. Filtrations

Various filtrations exist on $\mathbf{SH}(S)$. Let n be an integer. A motivic spectrum E is said to be *n-connective* if it is contained in the full subcategory $\mathbf{SH}_{\geq n}(S)$ generated under extensions and homotopy colimits by the shifted suspension spectra $\{\Sigma^{s+(w)}X_+\}_{s\geq n, w\in\mathbb{Z}, X\in\mathbf{Sm}_S}$. This produces the so-called *homotopy t-structure*, as studied by Morel in [45]. We might refer to it as the *connectivity filtration*.

Remark 4.1. If S is a field, E is n-connective if and only if its Nisnevich sheaves of homotopy groups $\pi_{s+(w)}\mathsf{E}$ equal 0 for $s < n$; see [45, Theorem 6.1.8] for the case of perfect fields and [32, Theorem 2.3] for all fields.

A motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ is said to be *n-effective* if it is contained in the full subcategory $\mathbf{SH}_{\geq (n)}(S)$ generated under extensions and homotopy colimits by the shifted suspension spectra $\{\Sigma^{s+(w)}X_+\}_{s\in\mathbb{Z}, w\geq n, X\in\mathbf{Sm}_S}$. This produces Voevodsky's *slice filtration*, as introduced in [65]. Let $\mathsf{f}_n := i_n \circ r_n : \mathbf{SH}(S) \to \mathbf{SH}(S)$, where $r_n : \mathbf{SH}(S) \to \mathbf{SH}_{\geq (n)}(S)$ is the right adjoint to the inclusion $i_n : \mathbf{SH}_{\geq (n)}(S) \to \mathbf{SH}(S)$.

Lemma 4.2. *The canonical transformation* $\mathsf{f}_{n+1} \to \mathsf{f}_n$ *admits a canonical extension to a homotopy cofiber sequence*

$$\mathsf{f}_{n+1} \to \mathsf{f}_n \to \mathsf{s}_n \to \Sigma\mathsf{f}_{n+1}$$

of triangulated (in fact homotopy-colimit preserving) functors defining the n-th slice s_n.

Proof. Exercise. □

A motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ is said to be *n-very effective* if it is contained in the full subcategory $\mathbf{SH}_{\geq n+(n)}(S)$ generated under extensions and homotopy colimits by the shifted suspension spectra $\{\Sigma^{s+(w)}X_+\}_{s\geq n, w\geq n, X\in\mathbf{Sm}_S}$. This produces the *very effective slice filtration*, as introduced by Spitzweck and Østvær in [62]. Let $\mathsf{vf}_n := vi_n \circ vr_n : \mathbf{SH}(S) \to \mathbf{SH}(S)$, where $vr_n : \mathbf{SH}(S) \to \mathbf{SH}_{\geq n+(n)}(S)$ is the right adjoint to the inclusion $vi_n : \mathbf{SH}_{\geq n+(n)}(S) \to \mathbf{SH}(S)$. Again the canonical natural transformation completes to a homotopy cofiber sequence

$$\mathsf{vf}_{n+1} \to \mathsf{vf}_n \to \mathsf{vs}_n \to \Sigma\mathsf{vf}_{n+1}$$

defining the n-th very effective slice functor.

An n-very effective motivic spectrum is n-effective and n-connective. Instead of 0-(very) effective or 0-connective, one simply says "(very) effective" or "connective".

Example 4.3. If $V \to Y$ is a vector bundle of rank r over $Y \in \mathbf{Sm}_S$, then the Thom space $\mathsf{Th}(V)$ is r-very effective. More generally, if $a \in K^0(Y)$ is a virtual vector bundle of rank $r \in \mathbb{Z}$, then $\mathsf{Th}(a)$ is r-very effective.

Example 4.4. The motivic spectra $\mathbf{1}$, $\mathbf{MGL}$, $\mathbf{MSp}$, and $\mathbf{MSL}$ are very effective over any base scheme. Because of the periodicities they satisfy, the motivic spectra $\mathbf{KGL}$ and $\mathbf{KQ}$ representing algebraic and hermitian K-theory, respectively, are neither n-effective nor n-connective for any $n \in \mathbb{Z}$. The motivic spectrum $\mathbf{1}[\eta^{-1}]$ is not n-effective for any n, but connective. The effective cover $\mathsf{f}_0\mathbf{KQ}$ is not n-connective for any n.

Let S be ind-smooth over a field or a Dedekind domain. Then [61] provides a highly structured motivic Eilenberg–MacLane spectrum $\mathbf{M}A$ for any abelian group A (which in the following will mostly be cyclic) representing motivic cohomology in the sense that there is an identification

$$\pi_{s+(w)}\mathbf{M}A = \pi_{s+w,w}\mathbf{M}A = H^{-s-w,-w}(S; A)$$

and in particular

$$\pi_{0+(w)}\mathbf{M}\mathbb{Z} = \mathbf{K}^{\mathsf{M}}_{-w}(S) \tag{4-1}$$

in the case that S is additionally local. Note that $\pi_{s+(w)}\mathbf{M}A = 0$ for $w > 0$ and for $s < -\dim(S)$.

The starting point for many determinations of slices is the following.

Theorem 4.5 (Levine, Voevodsky, Bachmann–Hoyois). *Let S be ind-smooth over a Dedekind domain or a field. Then $\mathbf{M}\mathbb{Z}$ is effective and the canonical map $\mathbf{M}\mathbb{Z} = \mathsf{f}_0\mathbf{M}\mathbb{Z} \to \mathsf{s}_0\mathbf{M}\mathbb{Z}$ is an isomorphism in $\mathbf{SH}(S)$. The unit map $\upsilon : \mathbf{1} \to \mathbf{M}\mathbb{Z}$ coincides with $\mathbf{1} = \mathsf{f}_0\mathbf{1} \to \mathsf{s}_0\mathbf{1} \xrightarrow{\mathsf{s}_0\upsilon} \mathsf{s}_0\mathbf{M}\mathbb{Z}$.*

Proof. Note first that $[\Sigma^{s+(w)}X_+, \mathbf{M}\mathbb{Z}] = 0$ whenever $w > 0$. Hence if $\mathbf{M}\mathbb{Z}$ is effective, the identification $\mathbf{M}\mathbb{Z} = \mathsf{f}_0\mathbf{M}\mathbb{Z} = \mathsf{s}_0\mathbf{M}\mathbb{Z}$ follows. A proof of this effectivity for a field of characteristic zero may be obtained by modeling $\mathbf{M}\mathbb{Z}$ via infinite symmetric powers of spheres. This proof, due to Voevodsky, proceeds along a filtration on $\mathbf{M}\mathbb{Z}$ which identifies the sphere spectrum as its starting level, and hence filters the unit map $\mathbf{1} \to \mathbf{M}\mathbb{Z}$. It in fact implies that the cofiber of the unit map is 1-effective, whence the unit map induces an isomorphism on zero slices. The proof for a field of any characteristic, as provided by Levine, gives a "reverse cycle map" $\mathbf{M}\mathbb{Z} \to \mathsf{s}_0\mathbf{1}$ via a homotopy coniveau tower. A proof up to inverting the exponential characteristic e of the base field is supplied by the Hopkins–Morel isomorphism. The latter expresses $\mathbf{M}\mathbb{Z}\left[\frac{1}{e}\right]$ as the quotient of $\mathbf{MGL}\left[\frac{1}{e}\right]$ with respect to the standard generators x_i of the Lazard ring. Passage to a Dedekind scheme then follows by a base change argument detecting effectivity on residue fields. □

Example 4.6. Let F be a field of characteristic not two, and let $\tau \in h^{0,1}$, where $h^{0,1} := H^{0,1}(F, \mathbb{Z}/2)$, be the unique nontrivial element given by $-1 \in F$. Then $\mathbf{M}\mathbb{F}_2[\tau^{-1}]$ is n-effective for every $n \in \mathbb{Z}$, but not n-connective for any n. This motivic spectrum represents étale cohomology with coefficients in μ_2 [65].

As Example 4.6 drastically reveals, the slice filtration is not separated in general. More precisely, if the *slice completion* of E is defined via the canonical cofiber sequence

$$\operatorname*{holim}_{q\to\infty} \mathsf{f}_q\mathsf{E} \to \mathsf{E} \to \mathsf{sc}(\mathsf{E}) \tag{4-2}$$

(so that the slice completion of E is the natural target of the slice spectral sequence of E), then $\mathbf{M}\mathbb{F}_2[\tau^{-1}] \simeq \operatorname{holim}_q \mathsf{f}_q\mathbf{M}\mathbb{F}_2[\tau^{-1}]$ and $\mathsf{sc}(\mathbf{M}\mathbb{F}_2[\tau^{-1}]) \simeq *$. One advantage of the very effective slice filtration is, for every $\mathsf{E} \in \mathbf{SH}(S)$, one has $\operatorname{holim}_q \mathsf{vf}_q\mathsf{E} \simeq *$. In favorable cases (for example, for $\mathbf{MGL}$, as a consequence of Corollary 4.8 below), the slice filtration coincides with the very effective slice filtration.

Let $\mathbb{L} = \mathbb{Z}[x_1, x_2, \dots]$ denote the Lazard ring classifying formal group laws, with $\deg(x_n) = n$. Its universal property provides a ring homomorphism

$$\mathbb{L} \to \bigoplus_{n\in\mathbb{Z}} \pi_{n+(n)}\mathbf{MGL}$$

and the images of the polynomial generators in $\pi_{n+(n)}\mathbf{MGL}$ are denoted by the same name. They describe the very effective and effective slice filtration on $\mathbf{MGL}$, at least under suitable restrictions on the base scheme.

Theorem 4.7 (Hopkins–Morel, Hoyois). *Let S be the spectrum of a field or a discrete valuation ring of mixed characteristic. The canonical map $\mathbf{MGL} \to \mathbf{MGL}/(x_1, x_2, \dots)$ coincides with both the canonical map $\mathbf{MGL} = \mathsf{f}_0\mathbf{MGL} = \mathsf{vf}_0\mathbf{MGL} \to \mathsf{s}_0\mathbf{MGL} = \mathsf{vs}_0\mathbf{MGL}$ and the canonical map $\mathbf{MGL} \to \mathbf{M}\mathbb{Z}$, at least after inverting the exponential characteristic of S.*

Proof. See [32; 61]. □

Work by Spitzweck then provides all (very) effective slices of $\mathbf{MGL}$.

Corollary 4.8 (Spitzweck). *There is an identification $\mathsf{vs}_n\mathbf{MGL} = \mathsf{s}_n\mathbf{MGL} = \Sigma^{n+(n)}\mathbf{M}\mathbb{L}_n$ with the Eilenberg–MacLane spectrum given by the degree-n part of the Lazard ring, at least after inverting the exponential characteristic of S.*

Proof. See [60]. □

For example, $\mathsf{s}_2\mathbf{MGL} = \Sigma^{2+(2)}(\mathbf{M}\mathbb{Z}\{x_1^2\} \vee \mathbf{M}\mathbb{Z}\{x_2\})$.

Let $\mathbf{KGL}$ denote the motivic spectrum representing (homotopy) algebraic K-theory over S in the sense that

$$\pi_{s+(w)}\mathbf{KGL} \cong K^Q_{s-w}(S),$$

where the superscript "Q" stands for "Quillen". It is a motivic ring spectrum in a natural and unique way. The difference of the tautological and the trivial line bundle on the projective line $\boldsymbol{P}^1$ define an element $\beta \in \pi_{1+(1)}\mathbf{KGL}$ such that multiplication with it yields an equivalence $\Sigma^{1+(1)}\mathbf{KGL} \simeq \mathbf{KGL}$ often called "Bott periodicity".

Theorem 4.9 (Levine, Voevodsky). *Let S be the spectrum of a field. The unit* $\mathbf{1} \to \mathbf{KGL}$ *induces an isomorphism on zero slices. Bott periodicity thus describes the graded slice as*

$$\mathsf{s}_*\mathbf{KGL} = \mathsf{s}_0\mathbf{KGL}[\beta^{\pm}] = \mathbf{M}\mathbb{Z}[\beta^{\pm}],$$

where $\deg(\beta) = 1 + (1)$. *Also*, $\mathbf{KGL}$ *is slice complete*: $\operatorname{holim} \mathsf{f}_q\mathbf{KGL} \simeq *$.

Proof. Levine's proof via the homotopy coniveau tower works over any field. Since it is going to be used anyhow, instead a presentation of $\mathbf{kgl} := \mathsf{f}_0\mathbf{KGL}$, the effective cover of $\mathbf{KGL}$, will be used. The projective bundle formula in K-theory supplies a canonical map $\mathbf{MGL} \to \mathbf{KGL}$ over any base scheme, thanks to the universal property of $\mathbf{MGL}$. It factors over $\mathbf{MGL}/(x_2, x_3, \dots)$, and the latter coincides with $\mathbf{kgl}$ up to inverting the exponential characteristic. A description of $\mathsf{s}_0\mathbf{KGL} = \mathsf{s}_0\mathbf{kgl} = \mathsf{s}_0\mathbf{MGL}$ follows. Bott periodicity provides an identification

$$\mathsf{f}_q\mathbf{KGL} \simeq \mathsf{f}_q(\Sigma^{q+(q)}\mathbf{KGL}) \simeq \Sigma^{q+(q)}(\mathsf{f}_0\mathbf{KGL}) = \Sigma^{q+(q)}\mathbf{kgl}$$

and similarly for the slices. This identification also shows that $\mathsf{f}_q\mathbf{KGL} = \mathsf{vf}_q\mathbf{KGL}$ for all q. Since $\operatorname{holim}_q \mathsf{vf}_q\mathsf{E} \simeq *$ for all E, convergence as stated follows. Additionally the columns of the first page of the slice spectral sequence of $\mathbf{KGL}$ are finite. □

Remark 4.10. Theorem 4.9 implies that the slice spectral sequence for $\mathbf{KGL}$ converges strongly. Recent work of Bachmann [11] shows that Theorem 4.9 holds over any Dedekind domain. Even more recent work implies the same over any quasicompact quasiseparated scheme.

Example 4.11. Let S be the spectrum of a field. For all $w \geq 0$, $\pi_{0+(-w)}\mathbf{kgl} \cong \mathbf{K}^{\mathsf{M}}_w$. The canonical map $\mathbf{kgl} \to \mathbf{KGL}$ induces the canonical map from Milnor to Quillen K-theory. Given that the first slice differential $\mathsf{s}_0\mathbf{kgl} \to \Sigma^1\mathsf{s}_1\mathbf{kgl}$ coincides with the Steenrod operation

$$\mathbf{M}\mathbb{Z} \xrightarrow{\mathrm{pr}_2^\infty} \mathbf{M}\mathbb{Z}/2 \xrightarrow{\mathsf{Sq}^2} \Sigma^{1+(1)}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\partial^2_\infty} \Sigma^{2+(1)}\mathbf{M}\mathbb{Z}$$

(which will follow from the corresponding differential for $\mathbf{kq}$ stated in Theorem 5.2 using notation introduced immediately before), there results an exact sequence

$$H^{w-2,w} \xrightarrow{\partial^2_\infty\mathsf{Sq}^2\mathrm{pr}^\infty_2} H^{w+1,w+1} \to \pi_{1-(w)}\mathbf{kgl} \to H^{w-1,w} \to 0$$

in which the surjection $\pi_{1-(w)}\mathbf{kgl} \to H^{w-1,w}$ usually does not split. In the case $S = \operatorname{Spec}(\mathbb{Q})$ this sequence has the form

$$H^{0,2}(\mathbb{Q}) \xrightarrow{0} H^{3,3}(\mathbb{Q}) = \mathbb{Z}/2 \to \mathbb{Z}/48 \to H^{1,2}(\mathbb{Q}) \to 0$$

for $w = -2$, as [36] implies.

Convergence of the slice spectral sequence is often a subtle issue, as the following example will show.

Definition 4.12. Let S be a scheme containing $\frac{1}{2}$. Then $\mathbf{KQ} \in \mathbf{SH}(S)$ denotes the motivic spectrum representing hermitian K-theory. Let $\mathbf{kq} := \mathsf{vf}_0\mathbf{KQ}$ denote its very effective cover.

A few of the homotopy groups or sheaves of $\mathbf{KQ}$ are known explicitly. For example, if S is the spectrum of a field or a discrete valuation ring in which 2 is invertible, $\pi_{0+(0)}\mathbf{KQ} \cong \pi_{0+(0)}\mathbf{kq} \cong \mathbf{GW}(S)$ and $\pi_{0+(w)}\mathbf{KQ} \cong \pi_{0+(w)}\mathbf{kq} \cong \boldsymbol{W}(S)$ for $w > 0$.

Proposition 4.13. *Over a field of characteristic $\neq 2$, multiplication with the Hopf map η induces cofiber sequences*

$$\Sigma^{(1)}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{\text{forget}} \mathbf{kgl} \to \Sigma^{1+(1)}\mathbf{kq}, \tag{4-3}$$

$$\Sigma^{(1)}\mathbf{KQ} \xrightarrow{\eta} \mathbf{KQ} \xrightarrow{\text{forget}} \mathbf{KGL} \xrightarrow{\Sigma^{1+(1)}\text{hyper}\circ\beta} \Sigma^{1+(1)}\mathbf{KQ}. \tag{4-4}$$

Here forget *and* hyper *are induced by the forgetful and hyperbolic maps between algebraic and hermitian K-theory, respectively.*

Proof. See [54; 57] □

There exists a periodicity element $\alpha \in \pi_{4+(4)}\mathbf{KQ}$ such that multiplication with α induces an equivalence $\Sigma^{4+(4)}\mathbf{KQ} \simeq \mathbf{KQ}$. One has $\text{forget}(\alpha) = \beta^4 \in \pi_{4+(4)}\mathbf{KGL}$, relating the two periodicities.

Remark 4.14. As a consequence of Proposition 4.13 and the slice completeness of $\mathbf{kgl}$, which in turn follows from Theorem 4.9, the slice completion of $\mathbf{kq}$ coincides with its η-completion:

$$\mathsf{sc}(\mathbf{kq}) \simeq \mathbf{kq}^{\wedge}_{\eta}. \tag{4-5}$$

This identification is helpful, because for any motivic spectrum E the canonical η-arithmetic square

$$\begin{array}{ccc} \mathsf{E} & \longrightarrow & \mathsf{E}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \mathsf{E}^{\wedge}_{\eta} & \longrightarrow & \mathsf{E}^{\wedge}_{\eta}[\eta^{-1}] \end{array} \tag{4-6}$$

is a homotopy pullback square.[2]

Another consequence of Proposition 4.13 is a determination of the slices of $\mathbf{kq}$ and $\mathbf{KQ}$. For comparing with the sphere spectrum, it suffices to focus on $\mathbf{kq}$. Note that the periodicity element $\alpha \in \pi_{4+(4)}\mathbf{KQ}$ induces an equivalence $\mathsf{s}_q\mathbf{KQ} \simeq$

[2]There is nothing special about η; this holds for any endomorphism of the sphere spectrum.

$\Sigma^{4+(4)}\mathsf{s}_{q-4}\mathbf{KQ}$ of period 4. Funnily enough, on the level of slices a periodicity of period 2 holds, induced by an element $\sqrt{\alpha}$ of degree $2+(2)$.

Theorem 4.15. *Over a field of characteristic not two the nonnegative slices of* $\mathbf{kq}$ *are given as*

$$\mathsf{s}_q\mathbf{kq} = \begin{cases} \Sigma^{(2n)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(2n)}\mathbf{M}\mathbb{Z}/2 \vee \\ \qquad \cdots \vee \Sigma^{2n-2+(2n)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+(2n)}\mathbf{M}\mathbb{Z}, & q=2n, \\ \Sigma^{(2n+1)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(2n+1)}\mathbf{M}\mathbb{Z}/2 \vee \\ \qquad \cdots \vee \Sigma^{2n+(2n+1)}\mathbf{M}\mathbb{Z}/2, & q=2n+1, \end{cases}$$

and in closed form

$$\mathsf{s}_*\mathbf{kq} = \mathbf{M}\mathbb{Z}[\eta, \sqrt{\alpha}]/(2\eta = 0,\, \eta^2 \xrightarrow{\partial^2_\infty} \sqrt{\alpha}),$$

where η *of degree* (1) *is induced by the Hopf map. The negative slices of* $\mathbf{kq}$ *are zero. The canonical map* $\mathbf{kq} \to \mathbf{KQ}$ *induces a natural inclusion as a direct summand on slices, and respects the multiplicative structure. The multiplicative relation* $\eta^2 \xrightarrow{\partial^2_\infty} \sqrt{\alpha}$ *says that the product*

$$\mathsf{s}_1\mathbf{kq} \wedge_{\mathsf{s}_0\mathbf{kq}} \mathsf{s}_1\mathbf{kq} \cong \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{1+(2)}\mathbf{M}\mathbb{Z}/2 \\ \to \mathsf{s}_2\mathbf{kq} \cong \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2\{\eta^2\} \vee \Sigma^{2+(2)}\mathbf{M}\mathbb{Z}\{\sqrt{\alpha}\}$$

maps via the (*unique*) *nontrivial map to the integral summand.*

Proof. Since $\mathbf{kq} = \mathsf{f}_0(\mathbf{KQ}_{\geq 0})$ is (very) effective, its negative slices are zero. Applying the slice functor to (4-3) yields a cofiber sequence. The natural isomorphism $\mathsf{s}_q \circ \Sigma^{(1)} \cong \Sigma^{(1)} \circ \mathsf{s}_{q-1}$ of [54, Lemma 2.1] shows the forgetful map forget : $\mathbf{kq} \to \mathbf{kgl}$ induces, on zero slices, an isomorphism

$$\mathsf{s}_0\mathbf{kq} \xrightarrow{\cong} \mathsf{s}_0\mathbf{kgl},$$

and likewise for the unit map $\mathbf{1} \to \mathbf{kq}$. For the 1-slices (4-3) induces a cofiber sequence

$$\Sigma^{(1)}\mathsf{s}_0\mathbf{kq} = \Sigma^{(1)}\mathbf{M}\mathbb{Z} \xrightarrow{\eta} \mathsf{s}_1\mathbf{kq} \\ \xrightarrow{\mathsf{s}_1\mathrm{forget}} \mathsf{s}_1\mathbf{kgl} = \Sigma^{1+(1)}\mathbf{M}\mathbb{Z} \to \Sigma^{1+(1)}\mathsf{s}_0\mathbf{kq} = \Sigma^{1+(1)}\mathbf{M}\mathbb{Z}.$$

Hence $\mathsf{s}_1$hyper can be identified with an integer $n \in \mathbb{Z}$. Note that as an endomorphism of $\mathbf{KGL}$, the composition forget $\circ$ hyper $= 1 + \psi^{-1}$ is the sum of the identity $\mathrm{id}_{\mathbf{KGL}}$ and the map induced by sending a vector bundle to its dual. Note further that as an endomorphism of $\mathbf{KQ}$, the composition hyper $\circ$ forget coincides with multiplication by h. It follows that $n = 2$, so that $\mathsf{s}_1\mathbf{kq} = \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2$. For the 2-slices then (4-3) induces a cofiber sequence

$$\Sigma^{(1)}\mathsf{s}_1\mathbf{kq} = \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\eta} \mathsf{s}_2\mathbf{kq} \\ \xrightarrow{\mathsf{s}_2\mathrm{forget}} \mathsf{s}_2\mathbf{kgl} = \Sigma^{2+(2)}\mathbf{M}\mathbb{Z} \to \Sigma^{1+(1)}\mathsf{s}_1\mathbf{kq} = \Sigma^{1+(2)}\mathbf{M}\mathbb{Z}/2.$$

Hence the last map is zero for simplicial degree reasons, the cofiber sequence splits, and we get $\mathsf{s}_2\mathbf{kq} = \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(2)}\mathbf{M}\mathbb{Z}$. Also, $\mathsf{s}_2$forget is the projection map onto $\Sigma^{2+(2)}\mathbf{M}\mathbb{Z}$. On 3-slices (4-3) induces a cofiber sequence

$$\Sigma^{(1)}\mathsf{s}_2\mathbf{kq} = \Sigma^{(3)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(3)}\mathbf{M}\mathbb{Z} \xrightarrow{\eta} \mathsf{s}_3\mathbf{kq} \xrightarrow{\mathsf{s}_3\text{forget}} \mathsf{s}_3\mathbf{kgl} = \Sigma^{3+(3)}\mathbf{M}\mathbb{Z} \to \Sigma^{1+(1)}\mathsf{s}_2\mathbf{kq}.$$

Here the last map lands trivially in $\Sigma^{1+(3)}\mathbf{M}\mathbb{Z}/2$ for simplicial degree reasons, while its component mapping to $\Sigma^{3+(3)}\mathbf{M}\mathbb{Z}$ can be identified with an integer $n \in \mathbb{Z}$. We deduce $n = 2$ by comparison with the hyperbolic map $\mathbf{KGL} \to \mathbf{KQ}$ in [54, §4.3]. Hence we obtain $\mathsf{s}_3\mathbf{kq} \cong \Sigma^{(3)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(3)}\mathbf{M}\mathbb{Z}/2$. Iterating these arguments produces the claimed additive calculation. The statement regarding $\mathbf{kq} \to \mathbf{KQ}$ follows from applying the main result of [30]. The "polynomial part" of the multiplicative structure follows from the periodicity of $\mathbf{KQ}$ and $\mathbf{KQ}[\eta^{-1}]$. The relation between η^2 and $\sqrt{\alpha}$ follows from the commutative diagram

$$\begin{array}{ccc} \mathsf{s}_1\mathbf{kq} \wedge_{\mathsf{s}_0\mathbf{kq}} \mathsf{s}_1\mathbf{kq} & \longrightarrow & \mathsf{s}_2\mathbf{kq} \\ \big\downarrow{\scriptstyle \mathsf{s}_1\text{forget} \wedge \mathsf{s}_1\text{forget}} & & \big\downarrow{\scriptstyle \mathsf{s}_2\text{forget}} \\ \mathsf{s}_1\mathbf{kgl} \wedge_{\mathsf{s}_0\mathbf{kgl}} \mathsf{s}_1\mathbf{kgl} & \xrightarrow{\simeq} & \mathsf{s}_2\mathbf{kgl} \end{array}$$

and the above identification of the vertical maps, using the bottom horizontal equivalence given by Theorem 4.9. □

Remark 4.16. Contrary to the calculation of the slices of $\mathbf{KQ}$ in [54] there is no "mysterious summand" appearing in Theorem 4.15, thanks to the connectivity of $\mathbf{kq}$. Each slice of $\mathbf{kq}$ is a finite sum of motivic Eilenberg–MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$. The odd slices of $\mathbf{kq}$ are cellular of finite type for every F [55, §3.3], and likewise for all the slices when $\mathsf{char}(F) = 0$.

Corollary 4.17. *When* $\mathsf{char}(F) \neq 2$ *the slices of* $\mathbf{kq}[\frac{1}{\eta}] = \mathbf{KW}_{\geq 0}$ *are given by*

$$\mathsf{s}_q(\mathbf{KW}_{\geq 0}) = \Sigma^{(q)}(\mathbf{M}\mathbb{Z}/2 \vee \Sigma^2\mathbf{M}\mathbb{Z}/2 \vee \Sigma^4\mathbf{M}\mathbb{Z}/2 \vee \cdots),$$

and

$$\mathsf{s}_*(\mathbf{KW}_{\geq 0}) \cong \mathbf{M}\mathbb{Z}[\eta^{\pm 1}, \sqrt{\alpha}]/(2\eta = 2\sqrt{\alpha} = 0, \eta^2 \xrightarrow{\mathsf{Sq}^1} \sqrt{\alpha}).$$

The canonical map $\mathbf{KW}_{\geq 0} \to \mathbf{KW}$ *induces the natural inclusion on slices, and respects the multiplicative structure.*

As in the case of $\mathsf{s}_*\mathbf{kq}$, the multiplicative structure is not quite polynomial and, because of the multiplicative relation involving Sq^1 (which is similar to the multiplicative relation in $\mathsf{s}_*\mathbf{kq}$ from Theorem 4.15), not $\mathbf{M}\mathbb{Z}/2$-linear.

Lemma 4.18. *Let $n \in \mathbb{Z}$. The sequence*

$$\mathbf{K}^{\mathsf{M}}_{n+1} \xrightarrow{\mathsf{h}} \mathbf{K}^{\mathsf{MW}}_{n+1} \xrightarrow{\eta} \mathbf{K}^{\mathsf{MW}}_{n} \to \mathbf{K}^{\mathsf{M}}_{n} \to 0$$

is exact.

Proof. Multiplication with $\mathsf{h} = 2 + \eta[-1] \in \mathbf{K}^{\mathsf{MW}}_0$ on $\mathbf{K}^{\mathsf{MW}}_{n+1}$ factors over the projection $\mathbf{K}^{\mathsf{MW}}_{n+1} \to \mathbf{K}^{\mathsf{M}}_{n+1} = \mathbf{K}^{\mathsf{MW}}_{n+1}/\eta\mathbf{K}^{\mathsf{MW}}_{n+2}$, because $\eta\mathsf{h} = 0$. This explains the choice of the first homomorphism in the sequence above. It remains to prove that an element in the kernel of $\eta : \mathbf{K}^{\mathsf{MW}}_{n+1} \to \mathbf{K}^{\mathsf{MW}}_{n}$ is a multiple of h. The exercise at the end of Section 3 says that multiplication with η is an isomorphism for $n < -1$. For $n = -1$, multiplication with η corresponds to the canonical map from the Grothendieck–Witt ring to the Witt ring. By definition, its kernel is a copy of the integers generated by the hyperbolic plane h, thus giving exactness for $n = -1$.

Define Witt K-theory as the quotient $\mathbf{K}^{\mathsf{W}}_n := \mathbf{K}^{\mathsf{MW}}_n/\mathsf{h}\mathbf{K}^{\mathsf{MW}}_n \cong \mathbf{K}^{\mathsf{MW}}_n/\mathsf{h}\mathbf{K}^{\mathsf{M}}_n$, where the last identification follows from the relation $\eta\mathsf{h} = 0$. This relation also implies that the surjection $\mathbf{K}^{\mathsf{MW}}_n \to \mathbf{K}^{\mathsf{W}}_n$ is injective as well for $n < 0$. Let $\boldsymbol{I} \subset \boldsymbol{W}$ denote the fundamental ideal, that is, the kernel of the mod 2 rank homomorphism $\boldsymbol{W} \to \mathbb{Z}/2$, or, equivalently, the kernel of the rank homomorphism $\mathbf{GW} \to \mathbb{Z}$. Sending the symbol for a unit $[u]$ to the additive inverse (in $\boldsymbol{I}$) of the class of the quadratic form $\langle 1, -u\rangle$ of rank 2 and η to $\langle 1\rangle$ defines a graded homomorphism $\mathbf{K}^{\mathsf{MW}}_n \to \boldsymbol{I}^n$ for all n where $\boldsymbol{I}^n = \boldsymbol{W}$ for $n \leq 0$. As it sends h to zero, there results a homomorphism $\mathbf{K}^{\mathsf{W}}_n \to \boldsymbol{I}^n$ which is an isomorphism by [44] (see also [28, Theorem 3.8]). As explained in [28, Theorem 5.4], one may then conclude that the canonical homomorphism $\mathbf{K}^{\mathsf{MW}}_{n+1}/\mathsf{h}\mathbf{K}^{\mathsf{M}}_{n+1} \to \mathbf{K}^{\mathsf{MW}}_n$ is an injection whose image coincides with $\boldsymbol{I}^{n+1}$. ☐

Theorem 4.19. *Let $n \in \mathbb{Z}$. The canonical map $\mathbf{K}^{\mathsf{MW}}_n \to \pi_{(-n)}\mathbf{kq}$ is an isomorphism.*

Proof. The proof is by induction on n. If $n \leq 0$, then $\pi_{(-n)}\mathbf{kq} = \pi_{(-n)}\mathbf{KQ}$ by definition of $\mathbf{kq} \to \mathbf{KQ}$. Since the canonical map $\mathbf{K}^{\mathsf{MW}}_n \to \pi_{(-n)}\mathbf{KQ}$ is an isomorphism for $n \leq 0$, the induction may start at $n = 0$. Suppose $n \geq 0$ is such that the statement is true for n. For the induction step consider the long exact sequence of homotopy sheaves induced by the cofiber sequence

$$\Sigma^{(1)}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{\text{forget}} \mathbf{kgl} \xrightarrow{\text{hyper}} \Sigma^{1+(1)}\mathbf{kq}.$$

It fits into a commutative diagram

$$\begin{array}{ccccccccccc}
 & & \mathbf{K}^{\mathsf{M}}_{n+1} & \xrightarrow{\mathsf{h}} & \mathbf{K}^{\mathsf{MW}}_{n+1} & \xrightarrow{\eta} & \mathbf{K}^{\mathsf{MW}}_{n} & \longrightarrow & \mathbf{K}^{\mathsf{M}}_{n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \\
\cdots & \xrightarrow{\text{forget}} & \pi_{1-(n)}\mathbf{kgl} & \xrightarrow{\text{hyper}} & \pi_{-(n+1)}\mathbf{kq} & \xrightarrow{\eta} & \pi_{-(n)}\mathbf{kq} & \xrightarrow{\text{forget}} & \pi_{-(n)}\mathbf{kgl} & \longrightarrow & 0
\end{array}$$

of exact sequences (using Lemma 4.18 and Proposition 4.13) in which the leftmost vertical map is the composition

$$\mathbf{K}^{\mathsf{M}}_{n+1} \cong \pi_{1-(n+1)}\mathsf{s}_1\mathbf{kgl} \xleftarrow{\cong} \mathsf{f}_1\mathbf{kgl} \to \pi_{1-(n+1)}\mathbf{kgl}. \tag{4-7}$$

Let $z \in \mathbf{K}^{\mathsf{MW}}_{n+1}$ such that $\phi(z) = 0$. Then by the induction assumption, $z\eta = 0$. Lemma 4.18 implies that there is an element $y \in \mathbf{K}^{\mathsf{M}}_{n+1}$ with $z = y\mathsf{h}$. Since $\psi(y) \in \pi_{1-(n)}\mathbf{kgl}$ is such that $\mathsf{h}(\psi(y)) = \phi(y\mathsf{h}) = 0$, there exists $x \in \pi_{1-(n)}\mathbf{kq}$ such that $\operatorname{forget}(x) = \psi(y)$. The forgetful map $\mathbf{kq} \to \mathbf{kgl}$ induces an isomorphism on zero slices. Hence the element $x \in \pi_{1-(n)}\mathbf{kq}$ lifts to an element $w \in \pi_{1-(n)}\mathsf{f}_1\mathbf{kq}$. The element $\mathsf{f}_1(\operatorname{forget})(w) \in \pi_{1-(n)}\mathsf{f}_1\mathbf{kgl}$ may not coincide with y, but their images in $\pi_{1-(n)}\mathbf{kgl}$ do, being $\operatorname{forget}(x)$ and $\psi(y)$, respectively. Hence there exists an element $t \in \pi_{2-(n)}\mathsf{s}_0\mathbf{kgl} \cong H^{n-2,n}$ which is mapped to the difference $y - \mathsf{f}_1(\operatorname{forget})(w)$. Since the forgetful map induces an isomorphism of zero slices, subtracting the image of t from w gives w' such that $\mathsf{f}_1(\operatorname{forget})(w') = y$. As the map $\pi_{1-(n)}\mathsf{f}_1\mathbf{kgl} \to \pi_{1-(n)}\mathsf{s}_1\mathbf{kgl} \cong \mathbf{K}^{\mathsf{M}}_{n+1}$ is an isomorphism and $\mathsf{s}_1(\operatorname{forget})$ factors over the map $\partial^2_\infty : \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2 \to \Sigma^{1+(1)}\mathbf{M}\mathbb{Z}$, the element y is in the image of ∂^2_∞. Hence, $y = y' \cdot \{-1\}$ for some $y' \in \mathbf{K}^{\mathsf{M}}_n$. Thus $z = y\mathsf{h} = y' \cdot \{-1\} \cdot \mathsf{h} = y' \cdot 0 = 0$, whence ϕ is injective.

Surjectivity of ϕ can be proven as follows. Let $z \in \pi_{-(n+1)}\mathbf{kq}$. By the induction assumption, there exists $y \in \mathbf{K}^{\mathsf{MW}}_{n+1}$ with $\eta(\phi(y) - z) = 0$. The forgetful map induces an isomorphism on zero slices, whence the canonical diagram

$$\begin{array}{ccc} \mathsf{f}_1\mathbf{kq} & \xrightarrow{\mathsf{f}_1(\text{forget})} & \mathsf{f}_1\mathbf{kgl} \\ \downarrow & & \downarrow \\ \mathbf{kq} & \xrightarrow{\mathsf{f}_1(\text{forget})} & \mathbf{kgl} \end{array}$$

is a homotopy pushout diagram. In particular, the homotopy cofiber of $\mathsf{f}_1(\operatorname{forget})$ is $\Sigma^{1+(1)}\mathbf{kq}$, whose $\pi_{1-(n)}$ contains $\phi(y) - z$. Since $\eta(\phi(y) - z) = 0$, there exists $x \in \pi_{1-(n)}\mathsf{f}_1\mathbf{kgl}$ with $\operatorname{hyper}(x) = \phi(y) - z$. Viewing $x \in \mathbf{K}^{\mathsf{M}}_{n+1}$ via the isomorphism (4-7), the equation $z = \phi(y - \mathsf{h} \cdot x)$, and thus surjectivity of ϕ, follows. □

As a consequence of Theorem 4.19, the canonical map $\mathbf{K}^{\mathsf{MW}} \to \pi_{0+(\star)}\mathbf{1}$ is injective, thereby proving part of Theorem 3.10. Its surjectivity requires further arguments which will not be discussed here.

5. The slice filtration on the sphere spectrum

Corollary 4.8 implies a description of all slices of the sphere spectrum, as suggested by Voevodsky in [65], provided in [38] and, in slightly different form and with multiplicative structure, in [55].

Theorem 5.1. *Suppose S is ind-smooth over a field or a Dedekind domain of mixed characteristic. Let P denote the set of positive residue characteristics in S. Then* *the slices of the P-inverted sphere spectrum over S are*

$$\mathsf{s}_q(\mathbf{1}[P^{-1}]) \cong \bigvee_{p\geq 0} \Sigma^{2q-p,q}\mathbf{M}(\mathrm{Ext}^{p,2q}_{\mathbf{MU}_*\mathbf{MU}}(\mathbf{MU}_*, \mathbf{MU}_*)[P^{-1}]).$$

Sketch of proof. The cosimplicial **MGL**-based Adams resolution of the sphere spectrum

$$\mathbf{1} \longrightarrow \mathbf{MGL} \rightrightarrows \mathbf{MGL}\wedge\mathbf{MGL} \Rrightarrow \cdots$$

induces natural equivalences

$$\mathsf{s}_q(\mathbf{1}_\Lambda) \xrightarrow{\cong} \operatorname{holim}_{\Delta} \mathsf{s}_q(\mathbf{MGL}_\Lambda^{\wedge\bullet})$$

for every q by downward induction and the 1-effectivity of the cofiber of $\mathbf{1}\to\mathbf{MGL}$, the latter holding over every base scheme (exercise). It remains to identify the latter with the corresponding motivic Eilenberg–MacLane spectra associated to $\mathrm{Ext}^{p,2q}_{\mathbf{MU}_*\mathbf{MU}}(\mathbf{MU}_*, \mathbf{MU}_*)$, the E^2-page of the Adams–Novikov spectral sequence for the topological sphere spectrum. This uses Corollary 4.8 and the fact that a perfect chain complex of modules over a principal ideal domain is quasi-isomorphic to its homology. □

Although only a finite portion of $\mathrm{Ext}^{p,2q}_{\mathbf{MU}_*\mathbf{MU}}(\mathbf{MU}_*, \mathbf{MU}_*)$, the E^2-page of the Adams–Novikov spectral sequence for the topological sphere spectrum, is known explicitly, certain infinite families are well understood, as explained very well in [50]. Funnily, some properties of the so-called α-family and their powers can be discovered by comparison with the slices of **kq** given in Theorem 4.15 via Lemma 5.10 below. More concretely, over a field of exponential characteristic $e\neq 2$, one has $s_1\mathbf{1}[e^{-1}]\simeq \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2\{\alpha_1\}$ and $s_3\mathbf{1}[e^{-1}]\simeq \Sigma^{(3)}\mathbf{M}\mathbb{Z}/2\{\alpha_1^3\}\vee\Sigma^{2+(3)}\mathbf{M}\mathbb{Z}/2\{\alpha_3\}$. If also $e\neq 3$, then $s_2\mathbf{1}[e^{-1}]\simeq \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2\{\alpha_1^2\}\vee\Sigma^{1+(2)}\mathbf{M}\mathbb{Z}/12\{\alpha_{2/2}\}$.

In order to use the information given by the form of the slices, a description of the first slice differential often helps. If E is a motivic spectrum, its *first slice differential* (as a map of motivic spectra) in weight q is the composition

$$\mathsf{d}^1_{\mathsf{E}}(q) : \mathsf{s}_q\mathsf{E} \to \Sigma\mathsf{f}_{q+1}\mathsf{E} \to \Sigma\mathsf{s}_{q+1}\mathsf{E} \tag{5-1}$$

involving canonical maps from the slice filtration on E. The induced homomorphisms on the first page of the slice spectral sequence are denoted

$$d^1_{\mathsf{E}} : E^1_{s+(w),q}(\mathsf{E}) := \pi_{s+(w)}\mathsf{s}_q\mathsf{E} \to \pi_{s-1+(w)}\mathsf{s}_{q+1}\mathsf{E} = E^1_{s-1+(w),q+1}(\mathsf{E}). \tag{5-2}$$

Higher pages and differential homomorphisms $d^r_{\mathsf{E}} : E^r_{s+(w),q}(\mathsf{E}) \to E^r_{s-1+(w),q+r}(\mathsf{E})$ are then produced as usual [15]. The advantage of using the first slice differential as a map of motivic spectra instead of just the induced homomorphisms is that the possibilities are much more restricted. For example, $\mathsf{d}^1_{\mathbf{kq}}(q) : \mathsf{s}_q\mathbf{kq} \to \Sigma\mathsf{s}_{q+1}\mathbf{kq}$, the first slice differential as a map of motivic spectra for **kq**, is a map between finite sums of motivic Eilenberg–MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$ by Theorem 4.15. Thus $\mathsf{d}^1_{\mathbf{kq}}(q)$ can be described via its restriction $\mathsf{d}^1_{\mathbf{kq}}(q,i)$ to the summand corresponding to the unique suspension $\Sigma^{i+(q)}$. Furthermore, $\mathsf{d}^1_{\mathbf{kq}}(q,i)$ splits into at most three nontrivial components. Voevodsky's work on the motivic Steenrod algebra [66] implies in particular that nonzero cohomology operations increasing the weight by one can only increase the simplicial degree by numbers in $\{0, 1, 2, 3, 4\}$, which limits the possible components.

To describe these, let $\tau \in h^{0,1} \cong \mu_2(F)$ and $\rho \in h^{1,1} \cong F^\times/2$ denote the classes represented by $-1 \in F$; $h^{p,q}$ is shorthand for the mod-2 motivic cohomology group of F in degree p and weight q. There are canonical maps $\mathrm{pr} = \mathrm{pr}^\infty_2 : \mathbf{M}\mathbb{Z} \to \mathbf{M}\mathbb{Z}/2$ and $\partial = \partial^2_\infty : \mathbf{M}\mathbb{Z}/2 \to \Sigma^{1,0}\mathbf{M}\mathbb{Z}$ such that the first motivic Steenrod square Sq^1 equals $\mathrm{pr}^\infty_2 \circ \partial^2_\infty$.

Theorem 5.2. *When* $\mathsf{char}(F) \neq 2$ *the* d^1*-differential in the slice spectral sequence for* $\mathbf{kq}$ *is given by*

$$\mathsf{d}^1_{\mathbf{kq}}(q,i) = \begin{cases} (0, \mathsf{Sq}^2, \mathsf{Sq}^3\mathsf{Sq}^1), & q-1 > i \equiv 0 \bmod 4, \\ (\tau, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \mathsf{Sq}^3\mathsf{Sq}^1), & q-1 > i \equiv 2 \bmod 4, \end{cases}$$

$$\mathsf{d}^1_{\mathbf{kq}}(q,q) = \begin{cases} (0, \mathsf{Sq}^2 \circ \mathrm{pr}, 0), & q \equiv 0 \bmod 4, \\ (\tau \circ \mathrm{pr}, \mathsf{Sq}^2 \circ \mathrm{pr}), & q \equiv 2 \bmod 4, \end{cases}$$

$$\mathsf{d}^1_{\mathbf{kq}}(q,q-1) = \begin{cases} (0, \mathsf{Sq}^2, \partial\mathsf{Sq}^2\mathsf{Sq}^1), & q \equiv 1 \bmod 4, \\ (\tau, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \partial\mathsf{Sq}^2\mathsf{Sq}^1), & q \equiv 3 \bmod 4. \end{cases}$$

Proof. Whenever it is nontrivial, the first slice differential for $\mathbf{kgl}/2$ coincides with the first slice differential for $\mathbf{KGL}/2$. The latter is $1+(1)$-periodic, and hence amounts to a map

$$\mathsf{d}^1_{\mathbf{KGL}/2} : \mathbf{M}\mathbb{Z}/2 \to \Sigma^{2+(1)}\mathbf{M}\mathbb{Z}/2,$$

which squares to zero. Voevodsky's computation of the motivic Steenrod algebra at the prime two [67] implies that $\mathsf{d}^1_{\mathbf{KGL}/2} \in \{0, \mathsf{Sq}^2\mathsf{Sq}^1 + \mathsf{Sq}^1\mathsf{Sq}^2\}$. The convergence result for the slice filtration on $\mathbf{KGL}$, the computation of $K^Q_*(\mathbb{R}; \mathbb{Z}/2)$ due to Suslin, and base change imply that $\mathsf{d}^1_{\mathbf{KGL}/2} = \mathsf{Sq}^1\mathsf{Sq}^2 + \mathsf{Sq}^2\mathsf{Sq}^1$. Naturality of slices with respect to the composition forget : $\mathbf{kq} \to \mathbf{kgl} \to \mathbf{kgl}/2$ implies that

$$\mathsf{Sq}^2 \circ \mathrm{pr} = \mathsf{d}^1_{\mathbf{kq}} : \mathsf{s}_0\mathbf{kq} \to \Sigma\mathsf{s}_1\mathbf{kq}.$$

Periodicity with respect to η then implies occurrences of Sq^2 on the slice summands generated by powers of η. The Adem relations $\mathsf{Sq}^2\mathsf{Sq}^2 = \tau\mathsf{Sq}^3\mathsf{Sq}^1$ and $\mathsf{Sq}^3\mathsf{Sq}^1\tau = (\mathsf{Sq}^2 + \rho\mathsf{Sq}^1)(\mathsf{Sq}^2 + \rho\mathsf{Sq}^1)$ then basically imply the rest (at least in the region of the slices where η acts invertibly). In this region the slices inject into the slices of $\mathbf{KW}_{\geq 0}$ given by Corollary 4.17; compare with the identification of $\mathsf{d}^1_{\mathbf{KW}}$ recorded in [54, Theorem 5.3]. □

Theorem 5.3. *When* $\mathsf{char}(F) \neq 2$ *the* d^1*-differential in the slice spectral sequence for* $\mathbf{KW}_{\geq 0}$ *is given by*

$$\mathsf{d}^1_{\mathbf{KW}_{\geq 0}}(q,i) = \begin{cases} (0, \mathsf{Sq}^2, \mathsf{Sq}^3\mathsf{Sq}^1), & i \equiv 0 \bmod 4, \\ (\tau, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \mathsf{Sq}^3\mathsf{Sq}^1), & i \equiv 2 \bmod 4. \end{cases}$$

Proof. This follows essentially from Theorem 5.2. □

Before getting back to the topic of this section, the slice filtration of the sphere spectrum, the case of $\mathbf{kq}$ will be elaborated further, in order to demonstrate the applicability of the slice spectral sequence.

Lemma 5.4. *The groups* $E^1_{0-(n),m}(\mathbf{kq})$ *and* $E^1_{1-(n),m}(\mathbf{kq})$ *consist of permanent cycles.*

Proof. As an exercise, one should compute the second page of the slice spectral sequence for $\mathbf{KW}_{\geq 0}$ using Theorem 5.3. It is concentrated in columns divisible by 4. Hence all higher slice differential homomorphisms are zero, whence $E^2(\mathbf{KW}_{\geq 0}) = E^\infty(\mathbf{KW}_{\geq 0})$. Since $E^1_{0-(n),m}(\mathbf{kq}) \cong E^1_{0-(n),m}(\mathbf{KW}_{\geq 0})$ naturally for $m > 0$, the result follows. □

Remark 5.5. The computation of the abutment of the slice spectral sequence for $\mathbf{KW}_{\geq 0}$ from the proof of Lemma 5.4 is compatible with Milnor's conjecture on quadratic forms. More precisely, the known coefficients

$$\pi_{s+(w)}\mathbf{KW}_{\geq 0} \cong \begin{cases} W(F), & s \equiv 0 \bmod 4, \\ 0, & \text{else}, \end{cases}$$

and passage to an algebraic closure of the base field imply that

$$\pi_{s+(w)}\mathsf{f}_1\mathbf{KW}_{\geq 0} \cong \begin{cases} \boldsymbol{I}(F), & s \equiv 0 \bmod 4, \\ 0, & \text{else}, \end{cases}$$

where $\boldsymbol{I}(F) \subset \boldsymbol{W}(F)$ is the fundamental ideal, that is, the kernel of the rank homomorphism. The multiplicativity of the slice filtration and the form of the abutment then implies that the slice filtration on $\pi_{0+(\star)}\mathbf{KW}_{\geq 0}$ coincides with the fundamental ideal filtration on the Witt ring. Moreover, Milnor's conjecture on quadratic forms holds for any field of characteristic not two. Details can be found in [54]. Since the fundamental ideal filtration on the Witt ring is separated by the main result of [3], the slice filtration for $\mathbf{KW}_{\geq 0}$ converges. Nevertheless, $\mathbf{KW}_{\geq 0}$ is not slice complete in general. The canonical map $\mathbf{KW}_{\geq 0} \to \mathsf{sc}(\mathbf{KW}_{\geq 0})$ induces the canonical homomorphism

$$\pi_{0+(0)}\mathbf{KW}_{\geq 0} \cong \boldsymbol{W}(F) \to \boldsymbol{W}(F)^\wedge_{\boldsymbol{I}} \cong \pi_{0+(0)}\mathsf{sc}(\mathbf{KW}_{\geq 0})$$

from the Witt ring to its completion at the fundamental ideal. This homomorphism is always injective. However, it is surjective if and only if the fundamental ideal filtration is finite, which is equivalent to the 2-cohomological dimension of F being finite. Every formally real field has infinite 2-cohomological dimension.

Lemma 5.6. *The possibly nontrivial groups in the first column* $E^2_{1-(n),m}(\mathbf{kq})$ *are*

$$\begin{aligned} E^2_{1-(n),0}(\mathbf{kq}) &\cong H^{n-1,n}, \\ E^2_{1-(n),1}(\mathbf{kq}) &\cong h^{n,n+1}/\mathsf{Sq}^2\mathrm{pr}H^{n-2,n}, \\ E^2_{1-(n),2}(\mathbf{kq}) &\cong h^{n+1,n+2}/\mathsf{Sq}^2h^{n-1,n+1}. \end{aligned}$$

Proof. To prove that the group $E^2_{1-(n),m}(\mathbf{kq})$ is trivial for $m \geq 3$, one has to show that the d^1-differential entering $E^1_{1-(n),m}(\mathbf{kq}) = h^{n+m-1,n+m}$ is surjective. For $m \geq 4$ the differential is given by

$$E^1_{2-(n),m-1}(\mathbf{kq}) = h^{n+m-3,n+m-1} \oplus h^{n+m-1,n+m-1} \to h^{n+m-1,n+m} = E^1_{1-(n),m}(\mathbf{kq}),$$
$$(b_2, b_0) \mapsto \mathsf{Sq}^2 b_2 + \tau b_0,$$

as stated in Theorem 5.2. The claim follows since multiplying with the map $\tau : h^{n+m-1,n+m-1} \to h^{n+m-1,n+m}$ is surjective. For $m = 3$,

$$E^1_{-n+2,2,-n}(\mathbf{kq}) = h^{n,n+2} \oplus H^{n+2,n+2} \to h^{n+2,n+3} = E^1_{-n+1,3,-n}(\mathbf{kq}),$$
$$(b_2, B_0) \mapsto \mathsf{Sq}^2 b_2 + \tau \mathrm{pr} B_0,$$

is surjective, since τ and $\mathrm{pr}^\infty_2 : H^{n+2,n+2} \to h^{n+2,n+2}$ are both surjective maps. The remaining identifications follow from the determination of the slice d^1-differential in Theorem 5.2. □

Remark 5.7. Lemma 5.6 implies that the slice spectral sequence for $\mathbf{kq}$ admits at most one further nonzero differential to the first column, but Lemma 5.8 below shows that it is trivial. Furthermore, Lemma 5.6 indicates that a triple iteration of η on $\pi_1\mathsf{sc}(\mathbf{kq})$ is zero, the reason being on the one hand that the slice spectral sequence for E computes the homotopy groups of $\mathsf{sc}(\mathsf{E})$, and on the other hand that the Hopf map $\Sigma^{(1)}\mathsf{E} \to \mathsf{E}$ induces maps

$$\mathsf{s}_q(\eta \wedge \mathsf{E}) : \mathsf{s}_q\Sigma^{(1)}\mathsf{E} \simeq \Sigma^{(1)}\mathsf{s}_{q-1}\mathsf{E} \to \mathsf{s}_q\mathsf{E}. \tag{5-3}$$

In particular, $\pi_1\mathsf{sc}(\mathbf{kq})[\eta^{-1}] = 0$. Similarly, one can show $\pi_2\mathsf{sc}(\mathbf{kq})[\eta^{-1}] = 0$ by proving $E^2_{2-(n),m}(\mathbf{kq}) = 0$ for $m > 3$ (exercise involving Theorem 5.2), thereby implying $\pi_1\mathbf{kq} = \pi_1\mathsf{sc}(\mathbf{kq})$. Hence the first column of the slice spectral sequence for $\mathbf{kq}$ computes $\pi_1\mathbf{kq}$, not just $\pi_1\mathsf{sc}(\mathbf{kq})$.

The next statement, which was employed in the previous proof, uses the element $\eta_{\mathsf{top}} \in \pi_{1+(0)}\mathbf{1}_{\mathbb{Z}}$ which is the image of the topological Hopf map associated with the Hopf construction on the topological group S^1 under the "constant sheaf" map $\pi_1\mathbb{S} \to \pi_{1+(0)}\mathbf{1}_{\mathbb{Z}}$. It is necessary to distinguish it from the element $\eta \in \pi_{0+(1)}\mathbf{1}_{\mathbb{Z}}$, which can be obtained as the Hopf construction on the algebraic group $\boldsymbol{G}_{\mathsf{m}}$. The Hopf construction on the algebraic group $\mathrm{SL}_2 \simeq \boldsymbol{A}^2 \smallsetminus \{0\} \simeq \Sigma^{1+(2)}$ gives an unstable representative of ν over the integers which complex-realizes to the topological second Hopf map $\nu_{\mathsf{top}} \in \pi_3\mathbb{S}$, and real-realizes to the topological first Hopf map[3] $\eta_{\mathsf{top}} \in \pi_1\mathbb{S}$.

Lemma 5.8. *The second differential*

$$E^2_{2-(n),0}(\mathbf{kq}) \to E^2_{1-(n),2}(\mathbf{kq})$$

in the slice spectral sequence for $\mathbf{kq}$ *is trivial.*

Proof. The cofiber sequence

$$\mathsf{f}_1\mathbf{kq} \to \mathbf{kq} \to \mathsf{s}_0\mathbf{kq} \to \Sigma\mathsf{f}_1\mathbf{kq}$$

induces a long exact sequence of homotopy modules

$$\cdots \to \pi_2\mathsf{s}_0\mathbf{kq} \to \pi_1\mathsf{f}_1\mathbf{kq} \to \pi_1\mathbf{kq} \to \pi_1\mathsf{s}_0\mathbf{kq} \to 0.$$

Since η acts trivially on $\mathsf{s}_0\mathbf{kq} \cong \mathbf{M}\mathbb{Z}$, the homotopy module $\pi_2\mathsf{s}_0\mathbf{kq}$ is a $\mathbf{K}^{\mathsf{M}}$-module. Hence the $\mathbf{K}^{\mathsf{MW}}$-module homomorphism $\pi_2\mathsf{s}_0\mathbf{kq} \to \pi_1\mathsf{f}_1\mathbf{kq}$ factors as

$$\pi_2\mathsf{s}_0\mathbf{kq} \to {}_{\eta}(\pi_1\mathsf{f}_1\mathbf{kq}) \to \pi_1\mathsf{f}_1\mathbf{kq},$$

where the first map is a $\mathbf{K}^{\mathsf{M}}$-module homomorphism to the η-torsion in $\pi_1\mathsf{f}_1\mathbf{kq}$. Its target is identified as a consequence of the $\mathbf{K}^{\mathsf{MW}}$-module presentation of $\pi_1\mathsf{f}_1\mathbf{kq}$ given in Lemma 5.9 below. Set $\mathbf{k}^{\mathsf{M}} := \mathbf{K}^{\mathsf{M}}/2 \cong \pi_0\mathbf{M}\mathbb{Z}/2$ to be Milnor K-theory modulo 2, and more generally $\mathbf{k}^{\mathsf{M}}(w) \cong \pi_0\Sigma^{(w)}\mathbf{M}\mathbb{Z}/2$ its w-fold weight shift, where $w \in \mathbb{Z}$. The image of the $\mathbf{K}^{\mathsf{MW}}$-module

$${}_{\eta}(\pi_1\mathsf{f}_1\mathbf{kq}) \cong \mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1) \oplus \rho^2(\mathbf{k}^{\mathsf{M}}(-2))$$

in $\pi_1\mathsf{s}_1\mathbf{kq}$ coincides with $\rho^2(\mathbf{k}^{\mathsf{M}}(-2))$ generated by the element $[-1]^2\eta_{\mathsf{top}}$, because $\eta\eta_{\mathsf{top}}$ maps trivially to $\pi_{1+(1)}\mathsf{s}_1\mathbf{kq}$. The image of the map $\pi_2\mathbf{M}\mathbb{Z}/2 \to \pi_1\mathsf{s}_1\mathbf{kq}$ given by the first slice differential $\mathsf{Sq}^2\mathrm{pr}^{\infty}_2 : \mathsf{s}_0\mathbf{kq} \to \Sigma\mathsf{s}_1\mathbf{kq}$ is (strictly) contained in $\rho^2(\mathbf{k}^{\mathsf{M}}(-2))$. It remains to observe that $\pi_2\mathbf{M}\mathbb{Z} \to \mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1)$ maps trivially to the first summand of ${}_{\eta}(\pi_1\mathsf{f}_1\mathbf{kq})$. Since the target is 2-torsion by Lemma 5.9 below, ψ factors over $(\pi_2\mathbf{M}\mathbb{Z})/2$. This occurs in the $\mathbf{K}^{\mathsf{MW}}$-module short exact sequence

$$0 \to (\pi_2\mathbf{M}\mathbb{Z})/2 \to \pi_2\mathbf{M}\mathbb{Z}/2 \to {}_2\pi_1\mathbf{M}\mathbb{Z} \to 0,$$

where the first map is induced by $\mathrm{pr}^{\infty}_2 : \mathbf{M}\mathbb{Z} \to \mathbf{M}\mathbb{Z}/2$. It coincides with the map $\mathsf{s}_0(\mathbf{kq} \xrightarrow{\text{canonical}} \mathbf{kq}/2)$ up to equivalence. Since $\pi_1\mathsf{f}_1\mathbf{kq}$ is 2-torsion, the map $\pi_1\mathsf{f}_1\mathbf{kq} \to \mathsf{f}_1\mathbf{kq}/2$ is injective. Owing to the commutative diagram

[3] Pardon the slight abuse of notation.

$$\begin{array}{ccccc} \pi_2\mathbf{M}\mathbb{Z} & \longrightarrow & (\pi_2\mathbf{M}\mathbb{Z})/2 & \longrightarrow & \pi_2\mathbf{M}\mathbb{Z}/2 \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1\mathsf{f}_1\mathbf{kq} & \xrightarrow{\text{id}} & \pi_1\mathsf{f}_1\mathbf{kq} & \longrightarrow & \pi_1\mathsf{f}_1\mathbf{kq}/2 \end{array}$$

it suffices to prove $\pi_2\mathbf{M}\mathbb{Z}/2 \cong \mathbf{k}^{\mathsf{M}}(-2) \to \pi_1\mathsf{f}_1\mathbf{kq}/2$ maps trivially to the summand $\mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1)$. The latter is determined by the image of the (unique) $\mathbf{K}^{\mathsf{MW}}$-module generator $g = g_F \in \pi_{2+(-2)}\mathbf{M}\mathbb{Z}/2$. The generator g_F is the image of the generator $g_{F_0} \in \pi_{2+(-2)}\mathbf{M}\mathbb{Z}/2$, where F_0 is the prime field of F. The commutative diagram

$$\begin{array}{ccc} \pi_{2+(-2)}(\mathbf{M}\mathbb{Z}/2)_{F_0} & \xrightarrow{\text{id}} & \pi_{2+(-2)}(\mathbf{M}\mathbb{Z}/2)_F \\ \downarrow & & \downarrow \\ \mathbf{k}^{\mathsf{M}}_3(F_0)/\rho^2\mathbf{k}^{\mathsf{M}}_1(F_0) & \longrightarrow & \mathbf{k}^{\mathsf{M}}_3(F)/\rho^2\mathbf{k}^{\mathsf{M}}_1(F) \end{array}$$

implies the right-hand side vertical map is zero, because $\mathbf{k}^{\mathsf{M}}_3(F_0)/\rho^2\mathbf{k}^{\mathsf{M}}_1(F_0) = 0$ due to [42, Example 1.5, Appendix]. □

Lemma 5.9. *The $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1\mathsf{f}_1\mathbf{kq} \cong \mathbf{K}^{\mathsf{MW}}/(2, \eta^2)$ is generated by the image of the topological Hopf map $\eta_{\mathsf{top}} \in \pi_{1+(0)}\mathbf{1}$ in $\pi_{1+(0)}\mathsf{f}_1\mathbf{kq}$. The image of $\pi_1\mathsf{f}_2\mathbf{kq} \to \pi_1\mathsf{f}_1\mathbf{kq}$ is the submodule generated by $\eta\eta_{\mathsf{top}}$ and it is isomorphic to $\mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1)$.*

Proof. The column on the second page of the slice spectral sequence for $\mathsf{f}_1\mathbf{kq}$ computing π_1 is concentrated in s_1 and s_2. The d^1-differential $\mathsf{Sq}^2 : \mathsf{s}_1\mathbf{kq} \to \Sigma\mathsf{f}_2\mathbf{kq} \to \Sigma\mathsf{s}_2\mathbf{kq}$ induces $\mathsf{Sq}^2 : \mathbf{k}^{\mathsf{M}}(-1) \cong \pi_2\mathsf{s}_1\mathbf{kq} \to \pi_1\mathsf{f}_2\mathbf{kq} \cong \pi_1\mathsf{s}_2\mathbf{kq} \cong \mathbf{k}^{\mathsf{M}}(1)$, whence the image in $\pi_1\mathsf{f}_2\mathbf{kq}$ coincides with $\rho^2\mathbf{k}^{\mathsf{M}}(-1)$. There is no room for higher differentials. Thus the long exact sequence

$$\cdots \to \pi_2\mathsf{s}_1\mathbf{kq} \to \pi_1\mathsf{f}_2\mathbf{kq} \to \pi_1\mathsf{f}_1\mathbf{kq} \to \pi_1\mathsf{s}_1\mathbf{kq} \to 0$$

induced by the cofiber sequence

$$\mathsf{f}_2\mathbf{kq} \to \mathsf{f}_1\mathbf{kq} \to \mathsf{s}_1\mathbf{kq} \to \Sigma\mathsf{f}_2\mathbf{kq}$$

yields a $\mathbf{K}^{\mathsf{MW}}$-module short exact sequence

$$0 \to \mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1) \to \pi_1\mathsf{f}_1\mathbf{kq} \to \pi_1\mathsf{s}_1\mathbf{kq} \cong \mathbf{k}^{\mathsf{M}} \to 0. \tag{5-4}$$

Étale or complex realization implies the rightmost term in (5-4) is generated by the image of the topological Hopf map η_{top} in $\pi_{1+(0)}\mathsf{s}_1\mathbf{kq}$. The multiplicative structure of the slices shows the image of $\eta\eta_{\mathsf{top}}$ in $\pi_{1+(1)}\mathsf{f}_1\mathbf{kq}$ is nontrivial, hence it coincides with the (unique) generator of $\mathbf{k}^{\mathsf{M}}(1)/\rho^2\mathbf{k}^{\mathsf{M}}(-1)$. In particular, the map $\mathbf{K}^{\mathsf{MW}} \to \pi_1\mathsf{f}_1\mathbf{kq}$ sending 1 to the image of η_{top} is surjective. As in the proof of [52, Lemma 2.3], one concludes the desired isomorphisms. □

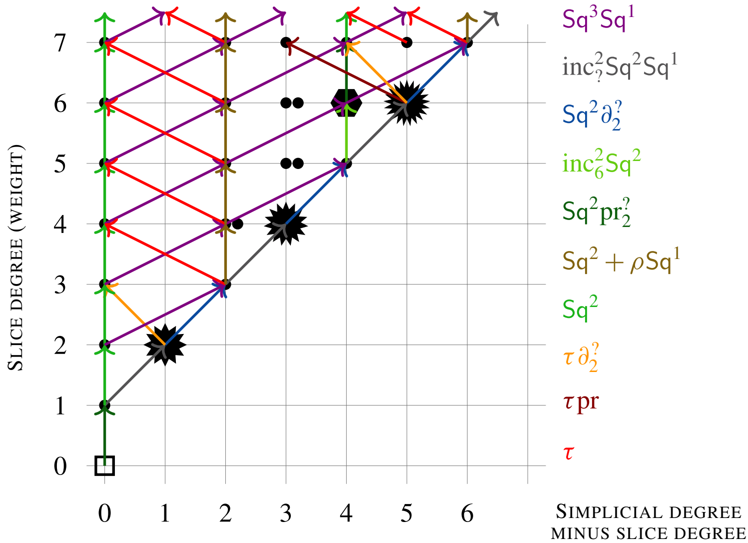

Figure 2. The first slice differential for **1**.

Lemma 5.8 implies the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1\mathbf{kq}$ is given by the short exact sequence

$$0 \to \pi_1\mathsf{f}_1\mathbf{kq}/\mathsf{Sq}^2\mathrm{pr}_2^{\infty}\pi_2\mathbf{M}\mathbb{Z} \to \pi_1\mathbf{kq} \to \pi_1\mathbf{M}\mathbb{Z} \to 0, \tag{5-5}$$

which does not split in general, neither as an extension of $\mathbf{K}^{\mathsf{MW}}$-modules nor degreewise as an extension of abelian groups. Since η^2 acts as zero on $\pi_1\mathsf{f}_1\mathbf{kq}$ by Lemma 5.9, η^3 acts as zero on $\pi_1\mathbf{kq}$.

To return to the topic at hand, substantial information on the first slice differential for **1**, at least over fields of characteristic not two, can be deduced from the first slice differential on **kq** given in Theorem 5.2 along the unit map $\mathbf{1}\to\mathbf{kq}$. Instead of a proper "theorem", the result will be displayed as Figure 2. It is based on Lemma 5.10 about the behavior of the unit map $\mathbf{1}\to\mathbf{kq}$ on slices. A more complete description can be found in [55, Lemmas 4.1 and 4.2]. In Figure 2 each slice is displayed along a horizontal line indexed by its weight. Every direct summand of a slice is placed on the vertical line indexed by its simplicial suspension degree. An open square refers to a motivic Eilenberg–MacLane spectrum with $\mathbb{Z}$-coefficients and solid dots to $\mathbb{Z}/2$-coefficients. The solid polygons indicate coefficients in $\mathbb{Z}/12$, $\mathbb{Z}/240$, $\mathbb{Z}/6$, and $\mathbb{Z}/504$, respectively. The colors of the d^1-differentials are split according to the respective direct sum decomposition and refer to elements of the motivic Steenrod algebra ordered by simplicial degree. Note that over a field of odd characteristic p, every occurrence of "$\mathbb{Z}$" should be replaced with "$\mathbb{Z}[\frac{1}{p}]$".

Lemma 5.10. *Let F be a field of characteristic not two. On 0-slices and 1-slices the unit map $\mathbf{1}\to\mathbf{kq}$ induces the identity, on 2-slices a rather canonical map, and* *for $q \geq 3$ the induced map $\mathsf{s}_q(\mathbf{1} \to \mathbf{kq})$ is the identity on the summands generated by α_1^q and $\alpha_3\alpha_1^{q-3}$. In particular, the unit map $\mathbf{1} \to \mathbf{kq}$ induces the identity on 3-slices. In other words,*

$$\begin{aligned}
&\mathsf{s}_0(\mathbf{1}) = \mathbf{M}\mathbb{Z} \xrightarrow{1} \mathbf{M}\mathbb{Z} = \mathsf{s}_0(\mathbf{kq}),\\
&\mathsf{s}_1(\mathbf{1}) = \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2\{\alpha_1\} \xrightarrow{1} \Sigma^{(1)}\mathbf{M}\mathbb{Z}/2 = \mathsf{s}_1(\mathbf{kq}),\\
&\mathsf{s}_2(\mathbf{1}) = \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2\{\alpha_1^2\} \vee \Sigma^{1+(2)}\mathbf{M}\mathbb{Z}/12\{\alpha_{2/2}\}\\
&\qquad\qquad \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & \partial_\infty^{12}\end{pmatrix}} \Sigma^{(2)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(2)}\mathbf{M}\mathbb{Z} = \mathsf{s}_2(\mathbf{kq}),\\
&s_q(\mathbf{1}) \hookleftarrow \Sigma^{(q)}\mathbf{M}\mathbb{Z}/2\{\alpha_1^q\} \vee \Sigma^{2+(q)}\mathbf{M}\mathbb{Z}/2\{\alpha_3\alpha_1^{q-3}\}\\
&\qquad\qquad \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} \Sigma^{(q)}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2+(q)}\mathbf{M}\mathbb{Z}/2 \hookrightarrow \mathsf{s}_q\mathbf{kq}.
\end{aligned}$$

Proof. The zero slice functor preserves the ring structure; see also [55, Lemma 2.29]. More generally, the graded slice functor preserves the ring structure, which implies the statements on the summands generated by α_1^q and $\alpha_1^{q-3}\alpha_3$. By [55, Lemma 2.30] the second diagonal entry for $\mathsf{s}_2(\mathbf{1} \to \mathbf{kq})$ has the form $n \cdot \partial_\infty^{12}$ for n an odd integer. The commutative diagram of motivic ring maps

$$\begin{array}{ccc}
\mathbf{1} & \longrightarrow & \mathbf{MGL}\\
\downarrow & & \downarrow\\
\mathbf{KQ} & \longrightarrow & \mathbf{KGL}
\end{array}$$

implies that n can be identified from the map on 2-slices induced by the unit map $\mathbf{1} \to \mathbf{MGL}$. This computation can be derived from the proof of Theorem 5.1 and shows that n is not divisible by 3. The result follows by applying a suitable isomorphism to $\mathsf{s}_2(\mathbf{1})$. □

Lemma 5.11. *For all $n \in \mathbb{Z}$ the unit map $\mathbf{1} \to \mathbf{kq}$ induces a surjection on $E^1_{1-(n),2}$ and $E^1_{2-(n),4}$, and for all $m, k \in \mathbb{Z}$ an isomorphism on*

$$\begin{aligned}
&E^1_{0-(n),m},\\
&E^1_{1-(n),m}, \quad m \neq 2,\\
&E^1_{2-(n),m}, \quad m \neq 2, 4,\\
&E^1_{k-(n),m}, \quad m \leq 1.
\end{aligned}$$

Proof. This follows from Lemma 5.10. While each slice may contribute many homotopy classes, these occur always on the same row and to the right of the respective slice. □

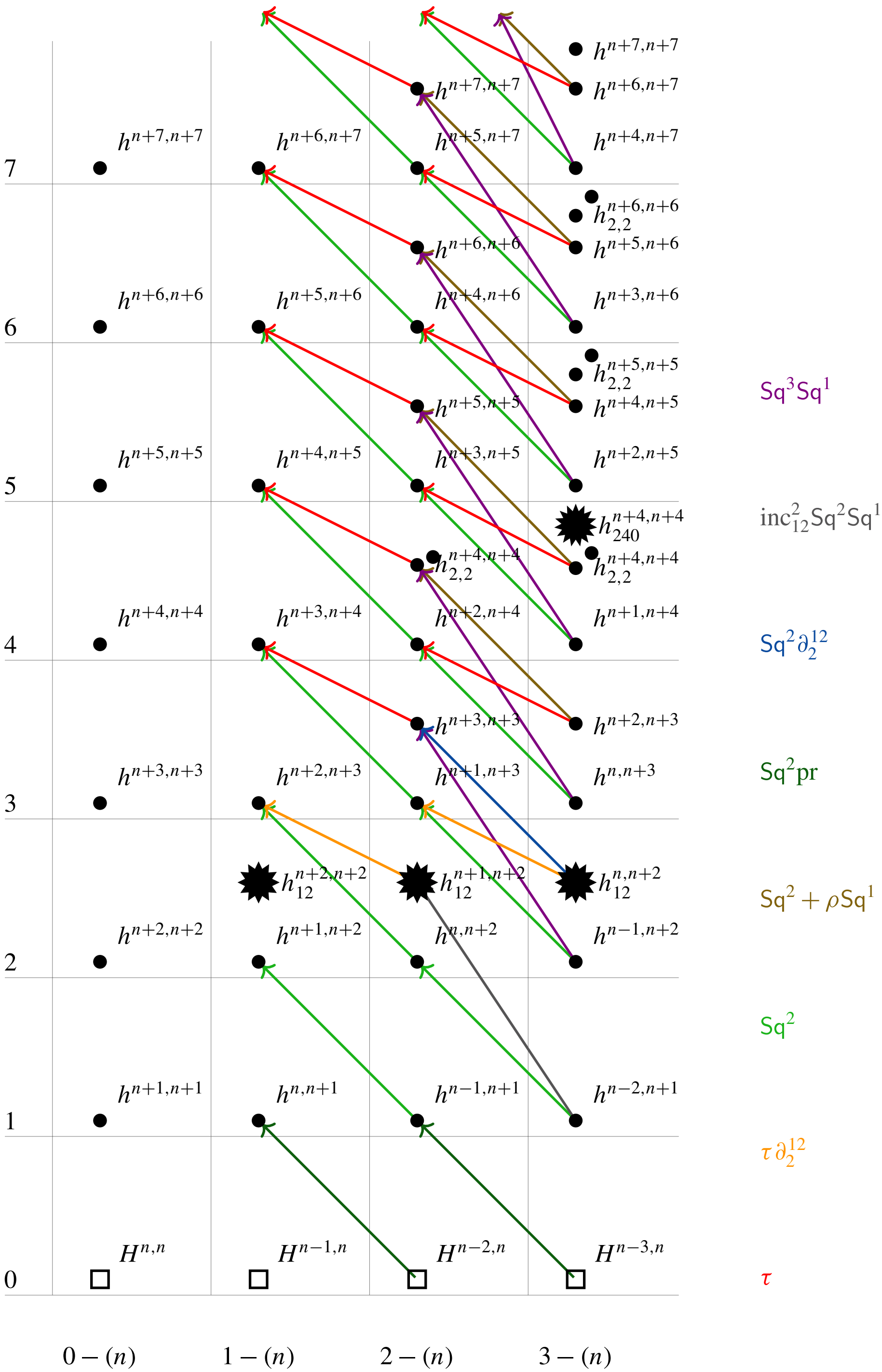

Figure 3. E^1-page of the weight $-n$-th slice spectral sequence for $\mathbf{1}_\Lambda$.

The description of the first four nontrivial columns of the first page of the slice spectral sequence of the sphere spectrum — together with the first slice differential — in Figure 3 allows a computation of the first three nontrivial columns of the second page.

Proposition 5.12. *Let F be a field of characteristic not two, and $n \in \mathbb{Z}$. The second page of the slice spectral sequence for the sphere spectrum in weight $-n$ contains the following groups*:

q	$E^2_{0-(n),q}(\mathbf{1})$	$E^2_{1-(n),q}(\mathbf{1})$	$E^2_{2-(n),q}(\mathbf{1})$
5	$h^{n+5,n+5}$	0	0
4	$h^{n+4,n+4}$	0	$h^{n+4,n+4}$
3	$h^{n+3,n+3}$	$h^{n+2,n+3}/\tau\partial^{12}_2 h^{n+1,n+2}_{12}$	$h^{n+1,n+3}/\mathsf{Sq}^2 h^{n-1,n+2}$
2	$h^{n+2,n+2}$	$h^{n+2,n+2}_{12} \oplus h^{n+1,n+2}/\mathsf{Sq}^2 h^{n-1,n+1}$	$\ker(h^{n+1,n+2}_{12} \xrightarrow{\partial^{12}_2} h^{n+2,n+2}) \oplus h^{n,n+2}/\mathsf{Sq}^2 h^{n-2,n+1}$
1	$h^{n+1,n+1}$	$h^{n,n+1}/\mathsf{Sq}^2\mathrm{pr}^{\infty}_2 H^{n-2,n}$	$\ker(h^{n-1,n+1} \xrightarrow{\mathsf{Sq}^2} h^{n+1,n+2})$
0	$H^{n,n}$	$H^{n-1,n}$	$\ker(H^{n-2,n} \xrightarrow{\mathsf{Sq}^2\mathrm{pr}^{\infty}_2} h^{n,n+1})$

In particular, $E^2_{1-(n),q}(\mathbf{1}) = 0$ *for* $q > 3$ *and* $E^2_{2-(n),q}(\mathbf{1}) = 0$ *for* $q > 4$.

Proof. As mentioned right before the statement, this is a direct consequence of Figure 3. □

In order to draw consequences for actual homotopy groups, a short interlude on convergence is required. By construction, the slice spectral sequence of E converges to the slice completion $\mathsf{sc}(\mathsf{E})$. Recall from Remark 5.5 that $\mathbf{KW}_{\geq 0}$ is not slice complete in general, which implies the same for $\mathbf{kq}$: the slice filtration on $\pi_0\mathbf{kq} \cong \mathbf{K}^{\mathsf{MW}}$ coincides with the fundamental ideal filtration, and if this filtration is infinite, $\mathbf{K}^{\mathsf{MW}} \to (\mathbf{K}^{\mathsf{MW}})^{\wedge}_{\boldsymbol{I}}$ is not surjective. Since $\pi_0\mathbf{1} \to \pi_0\mathbf{kq}$ is an isomorphism as a consequence of Theorems 3.10 and 4.19, the same is true for $\mathbf{1}$. On the other hand, Remark 4.14 identifies $\mathsf{sc}(\mathbf{kq}) \simeq \mathbf{kq}^{\wedge}_{\eta}$, which was used in the identification $\pi_1\mathbf{kq} \cong \pi_1\mathsf{sc}(\mathbf{kq})$. The same works for the sphere spectrum, with more effort.

Theorem 5.13. *Let F be a field of exponential characteristic e. Then* $\mathsf{sc}(\mathbf{1}[e^{-1}])$ *and* $\mathbf{1}[e^{-1}]^{\wedge}_{\eta}$ *are naturally equivalent.*

Proof. This is [55, Theorem 3.50], and its proof occupies most of [55, Section 3]. The basic idea, modeled on [34], is to prove the statement for motivic spectra which are cellular of finite type. It is then necessary to see that slices of such are again cellular of finite type. Theorem 5.1 implies that it suffices to prove this for motivic Eilenberg–MacLane spectra for $\mathbb{Z}$ and for finite abelian groups. This in turn follows from Theorem 4.7, giving two reasons for inverting the exponential characteristic.

Extending cellularity of finite type to slice completions requires the connectivity to increase with the slice. Theorem 5.1 shows that it does not hold for the sphere spectrum $\mathbf{1}$: every positive slice $\mathsf{s}_q(\mathbf{1})$ contains $\Sigma^{(q)}\mathbf{M}\mathbb{Z}/2$ as a direct summand, which satisfies $\pi_0\Sigma^{(q)}\mathbf{M}\mathbb{Z}/2 \cong h^{q,q}$, a nontrivial group for many fields, such as formally real fields. Nevertheless, these summands are generated by powers of α_1, the element detecting the Hopf map. The main result of [2] implies that this pattern persists: summands affecting increasing connectivity of slices of the sphere spectrum are always generated by a product of an element in the α-family with a suitable power of α_1. As a consequence, the connectivity of $\mathsf{s}_q(\mathbf{1}[e^{-1}]/\eta)$ tends to ∞ with q, and the same is then true for $\mathsf{s}_q(\mathsf{E}[e^{-1}]/\eta^\ell)$ for all ℓ and for all E which are cellular of finite type. □

Remark 5.14. In case F is a field of finite cohomological dimension in the sense that the cohomology of its absolute Galois cohomology is nonzero only in finitely many degrees, Levine showed in [37] that $\mathbf{1}[e^{-1}]$ is slice complete. Here e is the exponential characteristic of F. In fact, he proved the rather amazing and much more general statement that for every compact motivic spectrum E over a field of finite cohomological dimension the canonical map $\mathsf{E}[e^{-1}] \to \mathsf{sc}\mathsf{E}[e^{-1}]$ is an equivalence. See [58] for details on Galois cohomology and examples of such fields.

Recall from Remark 4.14 that the canonical square

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{1}[\eta^{-1}] \\ \downarrow & & \downarrow \\ \mathbf{1}^\wedge_\eta & \longrightarrow & \mathbf{1}^\wedge_\eta[\eta^{-1}] \end{array}$$

is a homotopy pullback square. Theorem 5.13 implies that the slice spectral sequence for $\mathbf{1}[e^{-1}]$ converges to the homotopy groups of $\mathbf{1}[e^{-1}]^\wedge_\eta$. The finiteness of the first and second column of the second page of the slice spectral sequence stated in Proposition 5.12 yields the vanishing

$$\pi_1(\mathbf{1}[e^{-1}])^\wedge_\eta[\eta^{-1}] = \pi_2(\mathbf{1}[e^{-1}])^\wedge_\eta[\eta^{-1}] = 0. \tag{5-6}$$

On the one hand, this implies the aforementioned injectivity of $\pi_0\mathbf{1} \to \pi_0\mathbf{1}^\wedge_\eta$. On the other hand, there results an isomorphism

$$\pi_1\mathbf{1}[e^{-1}] \xrightarrow{\cong} \pi_1(\mathbf{1}[e^{-1}])^\wedge_\eta \oplus \pi_1\mathbf{1}[e^{-1}, \eta^{-1}]$$

of homotopy groups from the long exact sequence associated to (4-6) for $\mathsf{E} = \mathbf{1}[e^{-1}]$. The second summand has been identified in [51].

Theorem 5.15. *Let F be a field of characteristic not two. Then $\pi_1\mathbf{1}[\eta^{-1}] = \pi_2\mathbf{1}[\eta^{-1}] = 0$.*

Proof. To sketch the argument, Morel's Theorem 3.10 implies that the canonical map $\mathbf{1}[\eta^{-1}] \to \mathbf{KW}_{\geq 0}$ induces an isomorphism on π_s in degrees $s < 1$; π_0 being isomorphic to $W(F)[\eta, \eta^{-1}]$. The vanishing $\pi_1\mathbf{KW}_{\geq 0} = \pi_2\mathbf{KW}_{\geq 0} = 0$ is known and can also be deduced from the slice spectral sequence, as in Remark 5.5. Hence the canonical map $\mathbf{1}[\eta^{-1}] \to \mathbf{KW}_{\geq 0}$ is a 1-connected map in the sense that $\pi_s \mathsf{C} = 0$ for $s \leq 1$, where C is its homotopy cofiber. Then $\pi_s \mathsf{C} \wedge \mathsf{C} = 0$ for $s \leq 3$, whence $\mathsf{C} \to \mathbf{KW}_{\geq 0} \wedge \mathsf{C}$ is 3-connected. A rather involved comparison of $\mathbf{KW}_{\geq 0} \to \mathbf{KW}_{\geq 0} \wedge \mathbf{KW}_{\geq 0}$ with the corresponding map $\mathrm{ko} \to \mathrm{ko} \wedge \mathrm{ko}$ for connective real topological K-theory ko implies that this map, and hence its cofiber $\mathbf{KW}_{\geq 0} \wedge \mathsf{C}$, is 3-connected. It follows that C is even 2-connected, whence $\mathsf{C} \wedge \mathsf{C}$ is 5-connected, and thus C is indeed 3-connected. □

A lot more is known about the homotopy of $\mathbf{1}[\eta^{-1}]$; see [29] over the complex numbers, [69] over the rational numbers, [49] over various "small" fields, and [10] over fields of characteristic not two. In any case, Theorem 5.15 suffices to conclude the following.

Corollary 5.16. *Let F be a field of exponential characteristic $e \neq 2$. Then the map $\pi_1\mathbf{1}[e^{-1}] \to \pi_1 \mathsf{sc}(\mathbf{1}[e^{-1}])$ is an isomorphism.*

Proof. As explained above, this follows from Theorem 5.15 and the vanishing in (5-6). □

Theorem 5.17. *Let F be a field of exponential characteristic $e \neq 2$. The unit map $\mathbf{1} \to \mathbf{kq}$ induces a surjection $\pi_{1+(\star)}\mathbf{1} \to \pi_{1+(\star)}\mathbf{kq}$ of $\mathbf{K}^{\mathsf{MW}}$-modules, whose kernel coincides with $\mathbf{K}^{\mathsf{M}}_{2-\star}/24$ after inverting e. In particular, since $\pi_{1+(n)}\mathbf{kq} = 0$ for $n \geq 2$, there are isomorphisms $\pi_{1+(2)}\mathbf{1}[e^{-1}] \cong \mathbb{Z}/24[e^{-1}]$ and $\pi_{1+(n)}\mathbf{1}[e^{-1}] = 0$ for $n \geq 3$. Also, $\mathbf{K}^{\mathsf{M}}_{2-\star}/24$ is generated by the second Hopf map $\nu \in \pi_{1+(2)}\mathbf{1}$. The relations $\eta\nu = 0 \in \pi_{1+(3)}\mathbf{1}$ and $\eta^2\eta_{\mathsf{top}} = 12\nu \in \pi_{1+(2)}\mathbf{1}$ hold.*

Proof. Lemma 5.9 implies that the unit map induces a surjection $\pi_1\mathsf{f}_1\mathbf{1} \to \pi_1\mathsf{f}_1\mathbf{kq}$, since the target is generated as a $\mathbf{K}^{\mathsf{MW}}$-module by the image of η_{top}. This element naturally lives in $\pi_{1+(0)}\mathbf{1}$ and lifts uniquely to $\pi_{1+(0)}\mathsf{f}_1\mathbf{1}$ because $\pi_{2+(0)}\mathsf{s}_0\mathbf{1} = \pi_{1+(0)}\mathsf{s}_0\mathbf{1} = 0$. Since $\mathsf{s}_0\mathbf{1} \xrightarrow{\cong} \mathsf{s}_0\mathbf{kq}$ is an equivalence by Lemma 5.10, surjectivity for $\pi_1\mathbf{1} \to \pi_1\mathbf{kq}$ follows as soon as the connecting map $\pi_1\mathsf{s}_0\mathbf{1} \to \pi_0\mathsf{f}_1\mathbf{1}$ is zero. As explained in Remark 5.5 for $\mathbf{KW}_{\geq 0}$, the group $\pi_0\mathsf{f}_1\mathbf{1}$ injects into $\pi_0\mathbf{1}$; see also Lemma 5.4.

In order to identify the kernel of $\pi_1\mathbf{1}[e^{-1}] \to \pi_1\mathbf{kq}[e^{-1}]$, observe that both $\mathbf{K}^{\mathsf{MW}}$-modules are computed by the respective slice spectral sequence, thanks to Remark 5.7 and Corollary 5.16. The column of the second page responsible for $\pi_1\mathbf{1}[e^{-1}]$ has been determined in Proposition 5.12. The major problem is to show that all further differentials ending in this column are zero. Basic techniques are shrinking the base field, a short exact sequence from [48, Theorem 3.2], real realization, and — for differentials originating in the zero slice — passing to suitable quotients of the sphere spectrum.

Possibly nonzero targets for these differentials are the terms $E^2_{1-(n),q}(\mathbf{1}[e^{-1}])$ for $q \in \{2,3\}$. As a warm-up, consider the second differential $E^2_{2-(n),1}(\mathbf{1}[e^{-1}]) \to E^2_{1-(n),3}(\mathbf{1}[e^{-1}])$. The connecting homomorphism

$$\mathbf{k}^{\mathsf{M}}(-1) \cong \pi_2 \mathsf{s}_1 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{f}_2 \mathbf{1}[e^{-1}] \tag{5-7}$$

is defined on a $\mathbf{K}^{\mathsf{MW}}$-module having a (unique) generator, here denoted g, in $\mathbf{k}^{\mathsf{M}}(-1)_1 = \mathbf{k}^{\mathsf{M}}_0$. To describe the image of g, observe that the long exact sequence

$$\cdots \to \pi_1 \mathsf{f}_4 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{f}_3 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{s}_3 \mathbf{1}[e^{-1}] \to \pi_0 \mathsf{f}_4 \mathbf{1}[e^{-1}] \to \cdots$$

induces an isomorphism $\pi_1 \mathsf{f}_3 \mathbf{1}[e^{-1}] \cong \pi_1 \mathsf{s}_3 \mathbf{1}[e^{-1}] \cong \mathbf{k}^{\mathsf{M}}(2)$. This $\mathbf{K}^{\mathsf{MW}}$-module has a (unique) generator given by the element $\eta^2 \eta_{\mathsf{top}} \in \pi_{1+(2)} \mathbf{1}$, which naturally lifts to $\pi_1 \mathsf{f}_3 \mathbf{1}$. The short exact sequence

$$0 \to \pi_1 \mathsf{f}_3 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{f}_2 \mathbf{1}[e^{-1}] \to \pi_1 \mathsf{s}_2 \mathbf{1}[e^{-1}] \to 0$$

of $\mathbf{K}^{\mathsf{MW}}$-modules can be determined as follows. Ignoring the possible 3-primary component of $\pi_1 \mathsf{s}_2 \mathbf{1}[e^{-1}]$, the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1 \mathsf{f}_2 \mathbf{1}[e^{-1}]$ is classified by an element in the group

$$\operatorname{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{k}^{\mathsf{M}}(1) \oplus \mathbf{K}^{\mathsf{M}}/4(2), \mathbf{k}^{\mathsf{M}}(2)) \cong \operatorname{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{k}^{\mathsf{M}}, \mathbf{k}^{\mathsf{M}}(1)) \oplus \operatorname{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}/4, \mathbf{k}^{\mathsf{M}}).$$

The multiplicative structure on the slices of $\mathbf{1}[e^{-1}]$ (or a suitable étale or complex realization) identifies the first component of this element as the unique nonzero element in $\mathbf{k}^{\mathsf{M}}(1)_{-1} = \mathbf{k}^{\mathsf{M}}_0$ represented by $\eta\eta_{\mathsf{top}}$, because $\eta \cdot \eta\eta_{\mathsf{top}} = \eta^2\eta_{\mathsf{top}}$. For the same reason, the second component of this element is the unique nonzero element in $\mathbf{k}^{\mathsf{M}}_0$, because $\nu \in \pi_{1+(2)}\mathbf{1}$ naturally lifts to $\pi_{1+(2)}\mathsf{f}_2\mathbf{1}$ and satisfies $4\nu \neq 0$. Thus $\pi_1 \mathsf{f}_2 \mathbf{1}[e^{-1}]$ is generated, as a $\mathbf{K}^{\mathsf{MW}}$-module, by the elements $\eta\eta_{\mathsf{top}}$ and ν, subject to the relations $2\eta\eta_{\mathsf{top}} = 0 = \eta\nu$ and $12\nu = \eta^2\eta_{\mathsf{top}}$; see also [22]. The image of the generator g under the connecting homomorphism (5-7) is thus of the form $x\eta\eta_{\mathsf{top}} + y\nu$, where $x \in \mathbf{K}^{\mathsf{MW}}_2$ and $y \in \mathbf{K}^{\mathsf{MW}}_3$. The form of the first differential originating in the 1-slice given in Figure 2 then implies $x = \rho^2$ and $y = 0$. In particular, over fields where $\rho^2 = 0$ (such as fields of positive characteristic), the homomorphism (5-7) is zero, and hence so is the second differential induced by it. Suppose now that F is a field (of characteristic zero) in which $\rho^2 \neq 0$. Being a pure symbol of degree 2, [48, Theorem 3.2] shows that every element in

$$E^2_{2-(\star),1}(\mathbf{1}) = {}_{\rho^2}\mathbf{k}^{\mathsf{M}}(-1)$$

is in the image of the transfer map for an étale field extension $f : F \hookrightarrow E$ such that $\rho^2 = 0 \in \mathbf{k}^{\mathsf{M}}_2(E)$. By [31, Theorem 1.9] this transfer map is induced by a map $f_\sharp \mathbf{1}_E \to \mathbf{1}_F$ of motivic spectra. Since the slice spectral sequence is natural with respect to maps of motivic spectra, and the second differential homomorphism in question is zero for E, it is also zero for F. At this stage, one can deduce that the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1\mathsf{f}_1\mathbf{1}[e^{-1}]$ is generated by $\eta_{\mathsf{top}} \in \pi_{1+(0)}\mathsf{f}_1\mathbf{1}$ and $\nu \in \pi_{1+(2)}\mathsf{f}_1\mathbf{1}$, subject to the relations $2\eta_{\mathsf{top}} = 0 = \eta\nu$ and $12\nu = \eta^2\eta_{\mathsf{top}}$; see also [52, Theorem 2.5].

One cannot argue similarly for the second differential $E^2_{2-(n),0}(\mathbf{1}) \to E^2_{1-(n),2}(\mathbf{1})$, because no manageable set of $\mathbf{K}^{\mathsf{M}}$-generators for $\pi_2\mathsf{s}_0\mathbf{1}$ in small degrees is known. This is resolved by passage to quotient spectra $\mathbf{1} \to \mathbf{1}/\beta$ for suitable $\beta \in \mathbf{K}^{\mathsf{MW}}$; here "suitable" implies that $\mathbf{1} \to \mathbf{1}/\beta$ induces an injection on the target group for the differential in question. To illustrate this, observe that the second differential $E^2_{2-(n),0}(\mathbf{1}[e^{-1}]) \to E^2_{1-(n),2}(\mathbf{1}[e^{-1}])$ lands in a direct sum. The component ending in $h^{n+1,n+2}/\mathsf{Sq}^2 h^{n-1,n+1}$ is zero by comparison with $\mathbf{kq}$, as Lemma 5.8 shows.

The component ending in $h^{n+2,n+2}_{12}$ can be split further into the direct sum of the 3-primary component $h^{n+2,n+2}_3$ and the 2-primary component $h^{n+2,n+2}_4$. As the 3-primary component of $\pi_1\mathsf{s}_2(\mathbf{1}[3^{-1}])$ is zero, without loss of generality the base field F has characteristic different from 3. The canonical map $\mathbf{1} \to \mathbf{1}/3$ then induces an injection on the 3-primary component of $\pi_1\mathsf{s}_2$. Hence it suffices to prove that the second differential

$$
\begin{aligned}
&E^2_{2-(n),0}(\mathbf{1}/3)\\
&\quad = \pi_{2-(n)}\mathsf{s}_0\mathbf{1}/3 = \pi_{2-(n)}\mathbf{M}\mathbb{Z}/3 \to \pi_{1-(n)}\mathsf{s}_2\mathbf{1}/3 = E^1_{1-(n),2}(\mathbf{1}/3) = E^2_{1-(n),2}(\mathbf{1}/3)
\end{aligned}
$$

is zero. If F contains a primitive third root of unity ω, $\pi_{2-(n)}\mathbf{M}\mathbb{Z}/3$ is generated by $\omega \in h^{0,2}_3(F)$ as a $\mathbf{K}^{\mathsf{M}}(F)$-module. The second differential maps this generator to the group $h^{4,4}_3(F)$. To conclude that the image of this generator is zero, consider the subfield $G := F_0(\omega)$, where $F_0 \subset F$ is the prime field of F. Functoriality with respect to field extensions shows that the diagram

$$
\begin{array}{ccc}
E^2_{2-(n),0}(\mathbf{1}_G/3) & \longrightarrow & E^2_{1-(n),2}(\mathbf{1}_G/3) = 0 \\
\downarrow & & \downarrow \\
E^2_{2-(n),0}(\mathbf{1}_F/3) & \longrightarrow & E^2_{1-(n),2}(\mathbf{1}_F/3)
\end{array}
$$

commutes. The group $h^{4,4}_3(G) = 0$ is zero, because the cohomological dimension of G is at most 2. Hence the image of the generator, which lifts to the initial corner, is zero. If F does not contain a primitive third root of unity, transfer with respect to the quadratic field extension $F \hookrightarrow F(\omega)$ induces an injection on the relevant 3-primary component, whence this case follows from the previous one.

To analyze the component ending in $h^{n+2,n+2}_4$ requires a different quotient spectrum. The projection $\mathbf{1} \to \mathbf{1}/4\mathsf{h}$ works; consider the proof of [55, Lemma 4.15] for details. One interesting feature is that the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_2\mathsf{s}_0(\mathbf{1}/4\mathsf{h}) \cong \pi_2\mathbf{M}\mathbb{Z}/8$ is generated by a single element in $\pi_{2-(2)}\mathbf{M}\mathbb{Z}/8 = h^{0,2}_8$, which lifts to the prime field. Another interesting feature is that, after eliminating the easy case of odd characteristic, shrinking to the prime field $\mathbb{Q}$ allows the usage of real realization. The aforementioned generator hits an element in $h_4^{4,4}(\mathbb{Q})$, a group whose unique nonzero element real-realizes to the nonzero element $\eta_{\mathsf{top}} \in \pi_1\mathbb{S}$, and hence cannot be hit. Thus also this second differential is zero.

With all second differentials ending in the first column being zero, there remains a single third differential

$$E^3_{2-(n),0}(\mathbf{1}[e^{-1}]) = E^2_{2-(n),0}(\mathbf{1}[e^{-1}]) \to E^2_{1-(n),3}(\mathbf{1}[e^{-1}]) = E^3_{1-(n),3}(\mathbf{1}[e^{-1}])$$

to consider. It is treated as before (exercise; which $\beta \in \mathbf{K}^{\mathsf{MW}}$ will work?). Hence the kernel A_n of the unit map $\pi_{1+(n)}\mathbf{1}[e^{-1}] \to \pi_{1+(n)}\mathbf{kq}[e^{-1}]$ admits a filtration whose associated graded consists of the two terms listed in Proposition 5.12 which map trivially to the E^∞-page for $\mathbf{kq}$. In other words, the slice filtration induces a short exact sequence

$$0 \to h^{n+2,n+2}/\partial_2^{12} h_{12}^{n+1,n+2} \to A_{-n} \to h_{12}^{n+2,n+2}[e^{-1}] \to 0. \tag{5-8}$$

Removing τ from the corresponding entry in Proposition 5.12 is justified, because multiplication with τ is an isomorphism on $h^{*,\star}$ for $\star \geq 0$. Assembling all weights in (5-8) together, there results a short exact sequence

$$0 \to \mathbf{k}^{\mathsf{M}}(2)/\partial_2^{12}\pi_1\Sigma^{(2)}\mathbf{M}\mathbb{Z}/12 \to A_{-\star} \to \mathbf{K}^{\mathsf{M}}(2)/12[e^{-1}] \to 0 \tag{5-9}$$

of $\mathbf{K}^{\mathsf{MW}}$-modules, whence $A_{-\star}$ is classified by an element in

$$\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}(2)/12[e^{-1}], \mathbf{k}^{\mathsf{M}}(2)/\partial_2^{12}\pi_1\Sigma^{(2)}\mathbf{M}\mathbb{Z}/12) \cong \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}/12, \mathbf{k}^{\mathsf{M}}/\partial_2^{12}\pi_1\mathbf{M}\mathbb{Z}/12).$$

The outer terms in (5-9) are in fact $\mathbf{K}^{\mathsf{M}}$-modules, with canonical generators in degree $\star = -2$. It follows that also $A_{-\star}$ is a $\mathbf{K}^{\mathsf{M}}$-module, because the action of η on $A_{-\star}$ is determined by its effect on a lift $\tilde{g} \in A_2$ of a generator $g \in \mathbf{K}_0^{\mathsf{M}}/12[e^{-1}]$. However, $\eta\tilde{g} \in A_3 = 0$ necessarily lands in the trivial group. Hence $A_{-\star}$ is classified by an element in

$$\mathrm{Ext}^1_{\mathbf{K}^{\mathsf{MW}}}(\mathbf{K}^{\mathsf{M}}/12, \mathbf{k}^{\mathsf{M}}/\partial_2^{12}\pi_1\mathbf{M}\mathbb{Z}/12) \cong \mathrm{Ext}^1_{\mathbf{K}^{\mathsf{M}}}(\mathbf{K}^{\mathsf{M}}/12, \mathbf{k}^{\mathsf{M}}/\partial_2^{12}\pi_1\mathbf{M}\mathbb{Z}/12) \cong \mathbf{k}_0^{\mathsf{M}} = \mathbb{Z}/2$$

and is thus the split extension or the unique nontrivial extension $\mathbf{K}^{\mathsf{M}}(2)/24[e^{-1}]$. That $A_{-\star}$ is the latter follows either by contemplating the multiplicative structure of the slice spectral sequence, or by complex or étale realization $\pi_{1+(2)}\mathbf{1}[e^{-1}] \to \pi_3\mathbb{S}[e^{-1}]$, which hits an element of order 8. Viewed as an element in $\pi_{1+(2)}\mathbf{1}$, it thus generates the kernel of $\pi_1\mathbf{1}[e^{-1}] \to \pi_1\mathbf{kq}[e^{-1}]$ as a $\mathbf{K}^{\mathsf{MW}}$-module. The equation $\eta\nu = 0$ follows from the above argument, but is already known by [22, Theorem

1.4]. The element $\eta^2\eta_{\mathsf{top}}$ is a nontrivial element of order 2, and since e is odd by assumption, has to coincide with 12ν already in $\pi_{1+(2)}\mathbf{1}$. □

Remark 5.18. With a bit more effort, the second Milnor–Witt stem $\pi_{2+(\star)}\mathbf{1}_F$ over a field F of characteristic $p \neq 2$ can be determined in the following sense: the kernel of the (nonsurjective!) unit map

$$\pi_{2+(\star)}\mathbf{1} \to \pi_{2+(\star)}\mathbf{kq}$$

is isomorphic to the $\mathbf{K}^{\mathsf{M}}$-module $\mathbf{k}^{\mathsf{M}}(4) \oplus \pi_1\Sigma^{(2)}\mathbf{M}\mathbb{Z}/24$ after inverting p if $p > 0$. The (unique) generator for $\mathbf{k}^{\mathsf{M}}(4)$ is $\nu^2 \in \pi_{2+(4)}\mathbf{1}$. See [56] for details and the proof. Also in this case the first slice differential is nontrivial, but the higher ones are zero. This is different for the third Milnor–Witt stem $\pi_{3+(\star)}\mathbf{1}_F$, as [23] implies in the case $F = \mathbb{R}$.

6. Applications

While Morel's identification $\pi_{0+(0)}\mathbf{1}_F \cong \mathbf{GW}(F)$ has spawned the area of refined enumerative geometry, pioneered by Levine [40] and Wickelgren [68]; see also [14], the computation of $\pi_1\mathbf{1}$ gave rise to several applications in geometry and K-theory.

In [5], Asok and Fasel determine the unstable $\boldsymbol{A}^1$-homotopy sheaf $\underline{\pi}_3^{\boldsymbol{A}^1}\boldsymbol{A}^3 \smallsetminus \{0\}$ over infinite fields of characteristic not two. Their statement involves Voevodsky's contraction, an operation sending a sheaf A of groups on $\mathbf{Sm}_S$ to the sheaf defined as the kernel of the map $A(X \times_S \boldsymbol{G}_{\mathsf{m}}) \to A(X)$ given by restriction along the morphism $X \xrightarrow{X\times_S 1} X \times_X \boldsymbol{G}_{\mathsf{m}}$ at every $X \in \mathbf{Sm}_S$.

Theorem 6.1 (Asok–Fasel). *Let F be an infinite perfect field of characteristic not two. There is a canonical "Suslin matrix" map inducing a surjection in a short exact sequence*

$$0 \to \boldsymbol{F}_5 \to \underline{\pi}_3^{\boldsymbol{A}^1}\boldsymbol{A}^3 \smallsetminus \{0\} \to \underline{\pi}_{1-(3)}\mathbf{KQ} \to 0.$$

The fifth contraction of the kernel $\boldsymbol{F}_5$ is isomorphic to the pullback $\mathbf{K}_0^{\mathsf{M}}/24 \times_{\mathbf{K}_0^{\mathsf{M}}/2} \boldsymbol{W}$ as a sheaf.

While [5] says more about $\boldsymbol{F}_5$ and in particular suggests that it is generated by an appropriate suspension of the unstable Hopf map $\nu : S^{3+(4)} \to S^{2+(2)}$, Theorem 6.1 already indicates a close relation to the stable computation in Theorem 5.17. The geometric consequence is a proof of a case of a conjecture of Murthy's.

Corollary 6.2. *Let X be a smooth affine variety of dimension four over an algebraically closed field of characteristic not two. If $V \to X$ is a vector bundle of rank* 3 *over X, then E splits off a trivial rank-*1 *summand if and only if $c_3(E) = 0$ in $CH^3(X) = H^{6,3}(X, \mathbb{Z})$.*

The advantage of the computation in Theorem 6.1 is that it paves a way to prove Murthy's conjecture in general, provided the following unstable computation of a homotopy sheaf holds.

Conjecture 6.3 (Asok–Fasel). *Let F be an infinite perfect field of characteristic not two. There is a canonical "Suslin matrix" map inducing an exact sequence*

$$\mathbf{K}^{\mathsf{M}}_{2+n}/24 \to \underline{\pi}_n^{\boldsymbol{A}^1} \boldsymbol{A}^n \smallsetminus \{0\} \to \underline{\pi}_{1-(n)}\mathbf{KQ}$$

of Nisnevich sheaves which becomes short exact after n-fold contraction.

This conjecture was stated in [4] and implies Murthy's conjecture in any dimension. However, it is in fact much stronger, as it gives the complete secondary obstruction to splitting a free rank-1 summand of a vector bundle on a smooth affine scheme over an infinite perfect field. The unstable "Suslin matrix" map stabilizes to the $\boldsymbol{P}^1$-stable unit map $\mathbf{1} \to \mathbf{KQ}$, whose factorization over $\mathbf{1} \to \mathbf{kq}$ figures in Theorem 5.17. However, while the unstable Conjecture 6.3 implies a $\boldsymbol{P}^1$-stable exact sequence

$$\mathbf{K}^{\mathsf{M}}_{2+n}/24 \to \pi_{1-(n)}\mathbf{1} \to \pi_{1-(n)}\mathbf{KQ}$$

over any infinite perfect field, Theorem 5.17 does not imply Conjecture 6.3 even after inverting the exponential characteristic, due to a lack of a $\boldsymbol{P}^1$-Freudenthal suspension theorem. Nevertheless, it is possible to derive unstable information from Theorem 5.17 and reprove the part of Theorem 6.1 which is relevant for Murthy's conjecture. This part turns out to be useful for a proper application of Theorem 5.17 provided in [6].

Theorem 6.4 (Asok–Fasel–Williams). *Let F be an infinite field of characteristic different from two and three, and A an essentially smooth local F-algebra. The image of the Suslin–Hurewicz homomorphism $K_5^{\mathrm{Quillen}}(A) \to \mathbf{K}^{\mathsf{M}}_5(A)$ coincides with $24\mathbf{K}^{\mathsf{M}}_5(A)$.*

Suslin described the homomorphism $K_n^{\mathrm{Quillen}}(F) \to \mathbf{K}^{\mathsf{M}}_n(F)$ for infinite fields in [63] and showed that the image for $n = 3$ coincides with $2\mathbf{K}^{\mathsf{M}}_3(F)$ if and only if the Milnor conjectures on Galois cohomology and on quadratic forms hold in degree 3, which he settled with Merkurjev in [41]. The case of $n = 4$ was recently proven in [53].

A final application, provided in [1], concerns the question whether vector bundles, perhaps equipped with a special linear or symplectic structure, induce Thom isomorphisms for generalized cohomology theories of algebraic varieties. One particularly interesting cohomology theory is the universal stable $\boldsymbol{A}^1$-derived cohomology $\mathsf{H}_{\boldsymbol{A}^1}\mathbb{Z}$, obtained by linearizing the sphere spectrum. It resides naturally as the unit object in the symmetric monoidal stable $\boldsymbol{A}^1$-derived category of S, obtained from the category of chain complexes of Nisnevich sheaves of abelian groups on $\mathbf{Sm}_S$ by inverting the affine line and stabilizing with respect to the projective line. See [16, Chapter 5] for a detailed definition, and [9] for the étale version. The latter admits Thom isomorphisms, whereas the former does not, by the following theorem.

Theorem 6.5 (Ananyevskiy). *Let F be a field. Then there is a short exact sequence*

$$\pi_0\mathbf{1} \xrightarrow{\eta_{\mathsf{top}}} \pi_1\mathbf{1} \to \pi_1\mathsf{H}_{\boldsymbol{A}^1}\mathbb{Z} \to 0$$

of $\mathbf{K}^{\mathsf{MW}}$*-modules.*

In particular, if $\mathsf{char}(F) \neq 2$, the $\mathbf{K}^{\mathsf{MW}}$-module $\pi_1\mathsf{H}_{\boldsymbol{A}^1}\mathbb{Z}$ contains an element of order 4 in weight (2), given by the image of the second Hopf map ν, as Theorem 5.17 implies. Étale realization proves that this image is nonzero of order 3 if $\mathsf{char}(F) = 2$. As a consequence of this property, $\mathsf{H}_{\boldsymbol{A}^1}\mathbb{Z}$ cannot be equipped with Thom isomorphisms associated with special linear or symplectic vector bundles. An equivalent formulation is that $\mathsf{H}_{\boldsymbol{A}^1}\mathbb{Z}$ is neither a module over **MSL** nor over **MSp**, special linear or symplectic algebraic bordism.

As another application of Theorem 5.17, Morel's Theorem 3.10 will be partially extended to other bases than fields. Due to the length of the arguments involved, a new section is in order.

7. The sphere spectrum over discrete valuation rings

Recall from Theorem 2.7 that, given a closed embedding $i : Z \hookrightarrow S$ of base schemes, with open complement $j : S \smallsetminus Z \hookrightarrow S$, the natural maps define a localization homotopy cofiber sequence

$$j_\sharp j^*\mathsf{E} \to \mathsf{E} \to i_*i^*\mathsf{E} \to \Sigma j_\sharp j^*\mathsf{E}$$

in $\mathbf{SH}(S)$ for any motivic spectrum E over S. As an exercise, one may check that i_* commutes with infinite direct sums, which implies it admits a right adjoint $i^! : \mathbf{SH}(S) \to \mathbf{SH}(Z)$. Taking right adjoints in the localization homotopy cofiber sequence above yields another distinguished localization homotopy cofiber sequence

$$i_*i^!\mathsf{E} \to \mathsf{E} \to j_*j^*\mathsf{E} \to \Sigma i_*i^!\mathsf{E}, \tag{7-1}$$

which in turn induces a long exact sequence

$$\cdots \to \pi_s i_*i^!\mathsf{E} \to \pi_s\mathsf{E} \to \pi_s j_*j^*\mathsf{E} \to \pi_{s-1}i_*i^!\mathsf{E} \to \cdots \tag{7-2}$$

of homotopy modules in $\mathbf{SH}(S)$. By adjointness, $\pi_{s+(w)}f_*\mathsf{E} \cong \pi_{s+(w)}\mathsf{E}$ for any morphism $f : S \to R$ of base schemes, where the first homotopy group is computed in $\mathbf{SH}(R)$, and the second in $\mathbf{SH}(S)$. In favorable cases (such as algebraic bordism **MGL** and its variants, homotopy algebraic K-theory **KGL**, and the sphere spectrum $\mathbf{1}$), there exists a version E_S over any base scheme S, such that E_S is canonically isomorphic to $f^*\mathsf{E}_R$ for any morphism $f : S \to R$ of base schemes. One calls such a collection an *absolute* motivic spectrum.

Example 7.1. Any motivic spectrum over $\mathrm{Spec}(\mathbb{Z})$ defines an absolute motivic spectrum. In particular, Spitzweck's motivic Eilenberg–MacLane spectrum $\mathbf{M}\mathbb{Z}$, which is constructed in [61] over Dedekind rings, defines an absolute motivic spectrum. Part of the construction is a verification that its pullback along a morphism from the spectrum of a field to the spectrum of the Dedekind ring coincides with Voevodsky's motivic Eilenberg–MacLane spectrum, as cited in Theorem 2.6. As indicated already, $\mathbf{1}$, $\mathbf{MSp}$, $\mathbf{MSL}$, $\mathbf{MGL}$, $\mathbf{KGL}$ are absolute motivic spectra. Hermitian K-theory may be represented by an absolute motivic spectrum; it certainly is over base schemes in which 2 is invertible.

Hence for an absolute motivic spectrum $\{\mathsf{E}_S\}$, the long exact sequence (7-2) contains the homotopy groups of E_S, the homotopy groups of E_U, and the homotopy groups of $i^!\mathsf{E}_S$, a motivic spectrum over Z. In even more favorable cases, this spectrum can be described in terms of E_Z. One case is given in Theorem 2.8. If $i : Z \hookrightarrow S$ is a closed embedding of smooth R-schemes, then $\Sigma^\infty_{\boldsymbol{P}^1_R} S/U \simeq \mathsf{Th}(Ni)$. The target of this canonical equivalence is the motivic Thom spectrum of the normal bundle of $i : Z \hookrightarrow S$, and the source is the canonical cone of the counit $j_\sharp j^*\mathbf{1}_S \to \mathbf{1}_S$, which by Theorem 2.7 is equivalent to $i_*i^*\mathbf{1}_S$, both viewed as objects in $\mathbf{SH}(R)$. This equivalence induces an equivalence $i^*\mathsf{E} \simeq i^!\mathsf{E} \wedge \mathsf{Th}(Ni)$ in $\mathbf{SH}(Z)$ for every motivic spectrum E over S [7]. Since motivic Thom spectra are invertible with respect to the smash product, there results an identification of the functor $i^!$ as

$$i^!\mathsf{E} \simeq \underline{\mathrm{Hom}}_Z(\mathsf{Th}(Ni), i^*\mathsf{E}) \tag{7-3}$$

for every closed embedding of schemes which are smooth over a base scheme. As a consequence of the six-functor formalism alluded to in and right after Theorem 2.4, [19] generalizes this equivalence as follows.

Theorem 7.2 (Déglise–Jin–Khan). *For every separated morphism $f : R \to S$ which can be factored as a regular closed embedding, followed by a smooth morphism, there exists an element $a \in K^0(R)$ and a natural transformation*

$$\mathrm{pur}_f(\mathsf{E}) : f^*\mathsf{E} \to f^!\mathsf{E} \wedge \mathsf{Th}(a) \tag{7-4}$$

of functors $\mathbf{SH}(S) \to \mathbf{SH}(R)$ which is suitably natural.

The natural transformation (7-4) listed in Theorem 7.2 specializes to the homotopy purity equivalence from Theorem 2.8 in case f already is smooth; then a is the inverse of the class of the tangent bundle of f in $K^0(R)$. Hence the "interesting" case of Theorem 7.2 is where $f = i : Z \to S$ is a regular closed embedding. The obvious candidate for the K-theory class a is then the class of the normal bundle $Ni \to Z$. Before pursuing this, an example might be helpful.

Example 7.3. Let $i : Z \hookrightarrow S$ be the zero scheme of a single regular function $\phi \in \mathcal{O}_S$. Then ϕ generates the conormal ideal sheaf $(\phi)/(\phi)^2$ and hence defines a trivialization of the normal bundle of rank 1. The restriction of ϕ to the complement $U := S \smallsetminus i(Z)$ along the open embedding $j : U \hookrightarrow S$ is an invertible element in $\mathcal{O}_U$, and thus defines an element $j^*(\phi) \in \pi_{(-1)}\mathbf{1}_U$. Its image in $\pi_{-1-(1)}i^!\mathbf{1}_S$ under the connecting map in the localization sequence (7-2) can be viewed as an incarnation of the purity transformation

$$i^*\mathbf{1}_S = \mathbf{1}_Z \to \Sigma^{1+(1)}i^!\mathbf{1}_S \simeq i^!\mathbf{1}_S \wedge \mathsf{Th}(Ni)$$

evaluated at $\mathbf{1}_S$.

Definition 7.4. Let $i : Z \hookrightarrow S$ be a regular closed immersion. A motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ is *i-pure* if the purity transformation

$$\mathrm{pur}_i : i^*\mathsf{E} \to i^!\mathsf{E} \wedge \mathsf{Th}(Ni)$$

is an equivalence.

Example 7.5. The absolute motivic spectrum **KGL** is i-pure for every regular closed immersion i of regular schemes, and the same is true for the motivic Eilenberg–MacLane spectrum with rational coefficients [16]. Both motivic spectra are orientable, whence their smash product with a Thom spectrum for a K-theory class of rank r is equivalent to a suspension with $S^{r+(r)}$.

With integral coefficients, the situation is more complicated, as usual, and “absolute purity” is currently not known to hold. However, based on [61], the integral motivic Eilenberg–MacLane spectrum satisfies the following, by [27, Proposition A.3].

Theorem 7.6 (Spitzweck). *Let $S = \mathrm{Spec}(D)$ be the spectrum of a Dedekind ring and $i : Z \hookrightarrow S$ the inclusion of a closed point. Then the motivic Eilenberg–MacLane spectrum $\mathbf{M}\mathbb{Z}_S$ with integral coefficients is i-pure.*

Theorem 7.6, whose proof is too involved for these lecture notes, will be used as the starting point for the proof of a special case of the following conjecture from [18].

Conjecture 7.7 (Déglise). *The absolute motivic spectra* **1** *and* **MGL** *are i-pure for every regular closed immersion i of regular schemes.*

The rationalized versions of **1** and **MGL** are i-pure for every regular closed immersion of regular schemes, thus supplying the current evidence for Conjecture 7.7. Further evidence is provided by the following theorem.

Theorem 7.8. *Let D be a discrete valuation ring with residue field k of characteristic $p > 0$ and fraction field F of characteristic zero. Let $i : \mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(D)$ denote the inclusion of the closed point. Then the purity transformation*

$$\mathrm{pur}_i(\mathbf{1}_D[p^{-1}]) : \mathbf{1}_k[p^{-1}] \to \Sigma^{1+(1)} i^! \mathbf{1}_D[p^{-1}]$$

is an equivalence.

The proof of Theorem 7.8 requires some preparatory lemmata.

Lemma 7.9. *Let $i : Z \to S$ be a closed embedding. The class of i-pure motivic spectra over S is closed under homotopy colimits.*

Proof. By construction, i^* and $- \wedge \mathsf{Th}(Ni)$ commute with homotopy colimits. It is also true that $i^!$ preserves homotopy colimits [27, Lemma 7.8], thereby implying the result by naturality of the purity transformation. Here is the argument. Applying i^* to the localization homotopy cofiber sequence (7-1) and using the counit equivalence $i^* i_* \simeq \mathrm{id}$ provides a homotopy cofiber sequence

$$i^! \to i^* \to i^* j_* j^* \to \Sigma i^!$$

in which i^*, j^*, j_* all commute with homotopy colimits. Hence so does $i^!$. □

Dealing with homotopy limits requires more effort. For a base scheme X, let X_{Nis} denote the small Nisnevich site of X. Standard properties of the Nisnevich topology imply that the cohomological dimension of X_{Nis} coincides with the Krull dimension of X. Let Sp denote the classical category of spectra of pointed simplicial sets, and let $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$ denote the category of Nisnevich sheaves with values in spectra. Every motivic spectrum $\mathsf{E} \in \mathbf{SH}(S)$ restricts canonically to a Nisnevich sheaf $\mathsf{E}|_{X_{\mathrm{Nis}}} \in \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$ for every $X \in \mathbf{Sm}_S$ by viewing it as an S^1-$\boldsymbol{G}_{\mathrm{m}}$-bispectrum, taking the zeroth S^1-spectrum, and restricting the resulting Nisnevich sheaf of S^1-spectra on the big site $\mathbf{Sm}_S$ to the small site X_{Nis}. A closed inclusion $i : W \hookrightarrow X$ induces a base change functor $i^\diamond : \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp}) \to \mathrm{Sh}(W_{\mathrm{Nis}}, \mathrm{Sp})$ which commutes with homotopy colimits and preserves connected sheaves of spectra. It is unclear whether the diagram

$$\begin{array}{ccc} \mathbf{SH}(S) & \longrightarrow & \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp}) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle (i \times_S X)^\diamond} \\ \mathbf{SH}(Z) & \longrightarrow & \mathrm{Sh}((Z \times_S X)_{\mathrm{Nis}},) \end{array}$$

commutes. Nevertheless, commuting $i^\diamond$ with homotopy limits works on the small site X_{Nis} under suitable connectivity assumptions.

Lemma 7.10. *Let $i : W \hookrightarrow X$ be a closed inclusion of base schemes. If*

$$\cdots \to E_{n+1} \to E_n \to \cdots \to E_1 \to E_0$$

is a tower in the category $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$ *such that the connectivity of the homotopy fibers* $\mathrm{hofib}(E_n \to E_{n-1})$ *tends to* ∞ *as n does, then the canonical map*

$$i^\diamond \operatorname*{holim}_n E_n \to \operatorname*{holim}_n i^\diamond(E_n)$$

is an equivalence in $\mathrm{Sh}(W_{\mathrm{Nis}}, \mathrm{Sp})$.

Proof. Testing on schemes in X_{Nis} and using finite cohomological dimension of X_{Nis} it follows from the assumption on the tower and the $\lim^1$-exact sequence that the canonical map

$$[\Sigma^s Y_+, \operatorname*{holim}_{m>n} \mathrm{hofib}(E_m \to E_n)] \to \lim_{m>n} [\Sigma^s Y_+, \mathrm{hofib}(E_m \to E_n)]$$

is an isomorphism for every $s \in \mathbb{Z}$ and every $Y \in X_{\mathrm{Nis}}$; here "$[-,-]$" denotes the Hom groups in the homotopy category of $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$. In particular also the connectivity of

$$\operatorname*{holim}_{m>n} \mathrm{hofib}(E_m \to E_n)$$

tends to ∞ with n. Moreover $i^\diamond : \mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp}) \to \mathrm{Sh}(W_{\mathrm{Nis}}, \mathrm{Sp})$ preserves connected objects. For formal reasons, both holim and $i^\diamond$ preserve homotopy fiber sequences. Hence for every $s \in \mathbb{Z}$ and every $V \in W_{\mathrm{Nis}}$, there exists a natural number n such that both nonhorizontal maps in the diagram

$$\begin{array}{ccc} [\Sigma^s V_+, i^\diamond \operatorname{holim}_{m>n} E_m] & \longrightarrow & [\Sigma^s V_+, \operatorname{holim}_{m>n} i^\diamond E_m] \\ & \searrow \quad [\Sigma^s V_+, E_n] \quad \swarrow & \end{array}$$

are isomorphisms. Hence the horizontal map is an isomorphism for every $s \in \mathbb{Z}$ and every $V \in W_{\mathrm{Nis}}$, showing the claim. □

Given the right assumptions on a tower of motivic spectra, Lemma 7.10 suffices to commute i^* with the homotopy limit of the tower.

Lemma 7.11. *Let* $i : Z \hookrightarrow S$ *be a closed inclusion of base schemes. Let*

$$\cdots \to \mathsf{E}_{n+1} \to \mathsf{E}_n \to \cdots \to \mathsf{E}_1 \to \mathsf{E}_0$$

be a tower in $\mathbf{SH}(S)$ *with associated homotopy fibers* $\mathsf{D}_n := \mathrm{hofib}(\mathsf{E}_n \to \mathsf{E}_{n-1})$ *for every* $n > 0$. *Assume that*

(1) *for every* $w \in \mathbb{Z}$ *and* $X \in \mathbf{Sm}_S$ *the connectivity of the restriction of* $\Sigma^{(w)}\mathsf{D}_n$ *to* $\mathrm{Sh}(X_{\mathrm{Nis}}, \mathrm{Sp})$ *tends to* ∞ *with* n, *and*

(2) *for every* $w \in \mathbb{Z}$ *there is an* $N \in \mathbb{N}$ *such that for every* $X \in \mathbf{Sm}_S$ *and* $n > N$ *the natural map*

$$(i \times_S X)^\diamond((\Sigma^{(w)}\mathsf{D}_n)|_{X_{\mathrm{Nis}}}) \to (i^*\Sigma^{(w)}\mathsf{D}_n)|_{(Z\times_S X)_{\mathrm{Nis}}}$$

is an equivalence in $\mathrm{Sh}((Z\times_S X)_{\mathrm{Nis}},\mathrm{Sp})$.

Then the natural map

$$i^*\operatorname*{holim}_n \mathsf{E}_n \to \operatorname*{holim}_n i^*\mathsf{E}_n$$

is an equivalence in $\mathbf{SH}(Z)$.

Proof. For $w\in\mathbb{Z}$ and sufficiently large n, the restriction of

$$\Sigma^{(w)}\operatorname*{holim}_m \operatorname{hofib}(\mathsf{E}_m\to\mathsf{E}_n)$$

to $\mathrm{Sh}(X_{\mathrm{Nis}},\mathrm{Sp})$ and the restriction of

$$\Sigma^{(w)}\operatorname*{holim}_m i^*\operatorname{fib}(\mathsf{E}_m\to\mathsf{E}_n)$$

to $\mathrm{Sh}((Z\times_S X)_{\mathrm{Nis}},\mathrm{Sp})$ satisfy the assumptions of [61, Proposition 8.6]. The result then follows from Lemma 7.10 and [61, Proposition 8.6]. □

Remark 7.12. In order to apply Lemma 7.11 to the slice filtration, one has to verify the relevant assumptions on the tower. Suppose S is the spectrum of a Dedekind domain whose cohomological dimension is finite. If each D_n is of the form $\bigvee_{j\in J_n}\Sigma^{s_j+(w_j)}\mathsf{M}A_j$, with each A_j a finite abelian group and $s_j+w_j\geq\varphi(n)$, where $\varphi:\mathbb{N}_{>0}\to\mathbb{Z}$ is a function with $\lim_{n\to\infty}\varphi(n)=\infty$, then the assumptions of Lemma 7.11 are satisfied. The first condition follows from [61, Theorem 3.9], and the second condition can be deduced from the proof of [61, Proposition 8.7]. In the case of the slice filtration for $\mathsf{f}_1\mathbf{1}_D[p^{-1}]$, where D is a discrete valuation ring whose fraction field has characteristic zero and whose residue field has characteristic $p>0$, one may use $\varphi(n)=n-1$ by Theorem 5.1.

Proof of Theorem 7.8. The argument will use the slice filtration. In the situation at hand, Theorem 5.1 provides the slices of $\mathbf{1}_D[p^{-1}]$. In particular, every slice of $\mathbf{1}_D[p^{-1}]$ is a (finite) coproduct of suspensions of motivic Eilenberg–MacLane spectra with coefficients in (finitely generated) modules over $\mathbb{Z}[p^{-1}]$. Thus every slice of $\mathbf{1}_D[p^{-1}]$ is i-pure, because of Lemma 7.9 and the fact that $\mathbf{M}\mathbb{Z}_D$ is i-pure by Theorem 7.6. Furthermore, the n-th effective cover $\mathsf{f}_n\to\mathrm{id}$ sits in a natural homotopy cofiber sequence

$$\mathsf{f}_n\to\mathrm{id}\to\mathsf{f}^{n-1}\to\Sigma\mathsf{f}_n$$

such that $\mathsf{f}^0\mathbf{1}=\mathsf{s}_0\mathbf{1}$ (and more generally for all effective motivic spectra). One obtains natural homotopy cofiber sequences

$$\mathsf{s}_n\to\mathsf{f}^n\to\mathsf{f}^{n-1}\to\Sigma\mathsf{s}_n$$

for all n. Note that the slice completion defined in (4-2) identifies with $\mathsf{scE}=\operatorname{holim}_{n\to\infty}\mathsf{f}^{n-1}\mathsf{E}$.

Theorem 5.1 and Example 7.1 then imply that the canonical natural transformation $i^* \circ \mathsf{s}_n \to \mathsf{s}_n \circ i^*$ is an equivalence when evaluated at $\mathbf{1}_D[p^{-1}]$; hence the same is true for the natural transformation $i^* \circ \mathsf{f}^{n-1} \to \mathsf{f}^{n-1} \circ i^*$. Suppose now that the residue field k of D is perfect and has finite cohomological dimension. Then [37, Theorem 4] says that the composition

$$\mathbf{1}_k[p^{-1}] = i^*\mathbf{1}_D[p^{-1}] \to i^*\mathsf{sc}(\mathbf{1}_D[p^{-1}]) \to \mathsf{sc}(i^*\mathbf{1}_D[p^{-1}]) = \mathsf{sc}(\mathbf{1}_k[p^{-1}])$$

is an equivalence. However, at this stage it is unclear whether i^* commutes with sc; there is no reason to assume that i^* preserves homotopy limits.

If additionally the fraction field F of D has finite cohomological dimension (a condition which is automatic according to [59, Dimension cohomologique: Theorem 2.2] in the case D is Henselian), [37, Theorem 4] implies that $\mathbf{1}_F$ and $\mathbf{1}_F[p^{-1}]$ are slice convergent. Hence the canonical map $\mathbf{1}_D[p^{-1}] \to \mathsf{sc}(\mathbf{1}_D[p^{-1}])$ is such that its image under j^* is an equivalence. The functor j^*, having a left adjoint $j_\sharp$, commutes with homotopy limits. Thus the image of the canonical map $\mathbf{1}_D[p^{-1}] \to \mathsf{sc}(\mathbf{1}_D[p^{-1}])$ under j^* is, up to equivalence, the canonical map $\mathbf{1}_F[p^{-1}] \to \mathsf{sc}(\mathbf{1}_F[p^{-1}])$, and in particular an equivalence.

Lemma 7.11 and Remark 7.12 imply that i^* commutes with the homotopy limit over the slice filtration of $\mathbf{1}_D[p^{-1}]$. Hence the canonical map $\mathbf{1}_D[p^{-1}] \to \mathsf{sc}(\mathbf{1}_D[p^{-1}])$ is such that its image under i^* is an equivalence as well, just as for j^*, and also coincides with the canonical map $\mathbf{1}_k[p^{-1}] \to \mathsf{sc}(\mathbf{1}_k[p^{-1}])$ up to equivalence. Theorem 2.7 implies that $\mathbf{1}_D[p^{-1}]$ is slice convergent. More importantly for the discussion at hand, the purity transformation

$$\mathrm{pur}_i(\mathbf{1}_D[p^{-1}]) : i^*\mathbf{1}_D[p^{-1}] \to i^!\mathbf{1}_D[p^{-1}] \wedge \mathsf{Th}(Ni) \tag{7-5}$$

is then, up to natural equivalence, the homotopy limit of the purity transformations

$$\mathrm{pur}_i(\mathsf{f}^n\mathbf{1}_D[p^{-1}]) : i^*\mathsf{f}^n\mathbf{1}_D[p^{-1}] \to i^!\mathsf{f}^n\mathbf{1}_D[p^{-1}] \wedge \mathsf{Th}(Ni),$$

which are equivalences. Thus also (7-5) is an equivalence if D is a discrete valuation ring whose perfect residue and fraction field have finite cohomological dimension.

The condition on the fraction field is unnecessary for the following reason. The essentially étale passage $f : D \to D_{\mathrm{Hens}}$ from a discrete valuation ring D to its Henselization induces an equality on residue fields. Since f is essentially étale, properties of the six-functor formalism imply that $\mathbf{1}_D[p^{-1}]$ is i-pure if and only if $\mathbf{1}_{D_{\mathrm{Hens}}}[p^{-1}]$ is i_{Hens}-pure. Note that D_{Hens} has finite cohomological dimension if and only if its residue field k does, by the aforementioned [59, Dimension cohomologique: Theorem 2.2].

Furthermore, if $k \hookrightarrow \ell$ is a purely inseparable field extension of characteristic $p > 0$, it induces an equivalence $\mathbf{SH}(k)[p^{-1}] \xrightarrow{\simeq} \mathbf{SH}(\ell)[p^{-1}]$ by [26]. This holds in particular for the passage from a field of characteristic $p > 0$ to its perfection.

Hence the assumption on the residue field of D being perfect may be dropped (using that $\mathbf{1}_k[p^{-1}]$ is slice complete if and only if $\mathbf{1}_\ell[p^{-1}]$ is).

Finally, if D is a discrete valuation ring with residue field k of characteristic $p > 0$ and fraction field F of characteristic zero, express F as the filtered colimit of subfields $E \subset F$ having finite transcendence degree over $\mathbb{Q}$. For every such E, the intersection $D \cap E$ is again a discrete valuation ring, simply by restricting the valuation, with fraction field E. The residue field ℓ of $D \cap E$ necessarily has finite transcendence degree over $\mathbb{F}_p$. The reason is that any lift of a family of elements in ℓ generating a subfield of transcendence degree at least d to $D \cap E$ generates a subfield of E of transcendence degree at least d. Since ℓ then has finite cohomological dimension by [58], this concludes the proof by the case already completed. □

As a consequence of Theorem 7.8, Morel's computation of $\pi_0 \mathbf{1}_F$ for F a field can be extended to discrete valuation rings of mixed characteristic, subject to inverting the residue characteristic and restricting the weight.[4] Here are some details. Still remaining over the discrete valuation ring D of mixed characteristic with residue characteristic $p > 0$, Theorem 7.8 gives the localization long exact sequence (7-2) the following form for the p-inverted sphere spectrum:

$$\cdots \to \pi_{s+1+(\star+1)} \mathbf{1}_k[p^{-1}] \to \pi_{s+(\star)} \mathbf{1}_D[p^{-1}] \\ \to \pi_{s+(\star)} \mathbf{1}_F[p^{-1}] \to \pi_{s+(\star+1)} \mathbf{1}_k[p^{-1}] \to \cdots \quad (7\text{-}6)$$

This long exact sequence contains a homomorphism

$$\mathbf{K}^{\mathsf{MW}}(F)[p^{-1}] \cong \pi_{0+(\star)} \mathbf{1}_F[p^{-1}] \to \pi_{0+(\star+1)} \mathbf{1}_k[p^{-1}] \cong \mathbf{K}^{\mathsf{MW}}(k)[p^{-1}],$$

which commutes with multiplication by η and units in D. Moreover, since the purity transformation used implicitly can be made explicit by choosing a uniformizing element, as explained in Example 7.3, one can check that it coincides with a homomorphism constructed by Morel [46, Theorem 3.15].

The maps of spheres discussed in Section 3 give rise to a ring homomorphism from $\mathbf{K}^{\mathsf{MW}}(D)$ as discussed in [46, Theorem 3.22] and defined — by a verbatim copy of Definition 3.8 — in [28] to $\pi_0 \mathbf{1}_D$.[5] The statements at the end of Section 3 extend to D by [28, Theorem 5.4].

Theorem 7.13. *Let D be a discrete valuation ring whose residue field k has characteristic $p > 2$ and at least five elements. Suppose further that its fraction field F has characteristic zero. For all $s < 0$, the group $\pi_s \mathbf{1}_D[p^{-1}]$ equals 0. For all $w < 2$ the canonical map $\mathbf{K}^{\mathsf{MW}}_w(D) \to \pi_{0-(w)} \mathbf{1}_D$ is an isomorphism after inverting p.*

[4] In the case of a 2-regular number ring R, [12, Theorem 5.2] provides an identification of $\pi_{0+(0)} \mathbf{1}_{R[1/2]}$ in weight zero with the Grothendieck–Witt ring of $R[1/2]$ after localizing at two.

[5] Strictly speaking, [28] restricts attention to local rings in which 2 is invertible and whose residue field contains at least 5 elements.

Proof. Of course Morel's Theorem 3.10 is used here. The connectivity statement gives $\pi_s \mathbf{1}_F = \pi_s \mathbf{1}_k$ for $s < 0$, whence $\pi_s \mathbf{1}_D[p^{-1}] = 0$ for $s < -1$ by the long exact sequence (7-6). The homomorphism

$$\mathbf{K}^{\mathsf{MW}}(F) \cong \pi_{0+(\star)} \mathbf{1}_F \to \pi_{0+(\star+1)} \mathbf{1}_k \cong \mathbf{K}^{\mathsf{MW}}(k)$$

is surjective, because every unit in k lifts, as does the Hopf map. Hence it is also surjective after inverting p, which implies $\pi_{-1} \mathbf{1}_D[p^{-1}] = 0$ by the long exact sequence (7-6). The second statement concerns the surjectivity of the homomorphism

$$\delta_1 : \pi_{1+(\star)} \mathbf{1}_F[p^{-1}] \to \pi_{1+(\star+1)} \mathbf{1}_k[p^{-1}].$$

It fits into a commutative diagram

$$\begin{array}{ccccccc} \pi_{1-(w)} \mathsf{f}_1 \mathbf{1}_F[p^{-1}] & \longrightarrow & \pi_{1-(w)} \mathbf{1}_F[p^{-1}] & \longrightarrow & \pi_{1-(w)} \mathbf{M}\mathbb{Z}_F[p^{-1}] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_{1-(w-1)} \mathsf{f}_1 \mathbf{1}_k[p^{-1}] & \longrightarrow & \pi_{1-(w-1)} \mathbf{1}_k[p^{-1}] & \longrightarrow & \pi_{1-(w-1)} \mathbf{M}\mathbb{Z}_k[p^{-1}] & \longrightarrow & 0 \end{array}$$

whose horizontal sequence is exact by construction and [55, Corollary 5.6]. Note that $\mathsf{f}_1 \mathbf{1}_D[p^{-1}]$ is i-pure by Theorem 7.6, Theorem 7.8 and Lemma 7.9. As a $\mathbf{K}^{\mathsf{MW}}(k)[p^{-1}]$-module, $\pi_{1+(\star)} \mathsf{f}_1 \mathbf{1}[p^{-1}]$ is generated by η_{top} and ν [52, Theorem 2.5]. These elements are in the image of δ_1, namely $\delta_1([u]\eta_{\mathsf{top}}) = \eta_{\mathsf{top}}$ and $\delta_1([u]\nu) = \nu$, where u is a uniformizing element in D. Hence the middle vertical arrow in the diagram above will be surjective as soon as the one right next to it is. It remains to see that the connecting map

$$\delta_1^{\mathbf{M}\mathbb{Z}} : \pi_{1-(w)} \mathbf{M}\mathbb{Z}_F[p^{-1}] \to \pi_{1-(w-1)} \mathbf{M}\mathbb{Z}_k[p^{-1}]$$

is surjective. This is definitely the case for $w < 3$, because $\pi_{1-(w-1)} \mathbf{M}\mathbb{Z}_k = H^{w-2,w-1}(k, \mathbb{Z})$ vanishes for $w < 3$. Hence the sequence

$$0 \to \pi_{0-(w)} \mathbf{1}_D[p^{-1}] \to \pi_{0-(w)} \mathbf{1}_F[p^{-1}] \to \pi_{0-(w-1)} \mathbf{1}_k[p^{-1}] \to 0$$

is exact for $w < 3$. In weight $w < 0$, this sequence is obtained by inverting p on the Gersten sequence

$$0 \to \boldsymbol{W}(D) \to \boldsymbol{W}(F) \to \boldsymbol{W}(k) \to 0$$

for the Witt group, because it is exact by direct computation, or by [13]. This result extends to the Grothendieck–Witt group, giving the desired result in weight $w = 0$. With the help of [28, Theorem 5.4] and the exactness of the sequence

$$0 \to D^\times = \mathbf{K}_1^{\mathsf{M}}(D) \to F^\times = \mathbf{K}_1^{\mathsf{M}}(F) \xrightarrow{\delta} \mathbb{Z} = \mathbf{K}_0^{\mathsf{M}}(k) \to 0$$

the result follows in weight $w = 1$ as well. □

The methods of [17] allow an extension of Theorem 7.13 to all weights, identifying $\pi_0\mathbf{1}_D[p^{-1}]$ with the so-called unramified Milnor–Witt K-theory defined in [28, Section 6.2]. Also, [20] might lead to an identification of $\mathbf{K}_2^{\mathsf{MW}}(D)[p^{-1}]$ with $\pi_{0-(2)}\mathbf{1}_D[p^{-1}]$.

8. Notation

S, R, U, Z	base (Noetherian separated of finite Krull dimension) schemes
F, E, G	base fields
$\mathbf{Sm}_S$	category of smooth separated S-schemes
$\boldsymbol{A}^d_S$, $\boldsymbol{P}^d_S$	affine and projective space of dimension d over S
$\mathbf{Spc}_\bullet(S)$	pointed simplicial presheaves on $\mathbf{Sm}_S$
$\mathbf{SH}(S)$	$\boldsymbol{P}^1$-stable $\boldsymbol{A}^1$-homotopy category of motivic spectra over S
D, E	generic motivic spectra
$\Sigma^{s+(w)}$	sphere of dimension $s+w$ and weight w, suspension functor
$\pi_{s+(w)}\mathsf{E}$	homotopy group $[S^{s+(w)}, \mathsf{E}] = \mathrm{Hom}_{\mathbf{SH}(S)}(S^{s+(w)}, \mathsf{E})$
$\underline{\pi}_{s+(w)}\mathsf{E}$	Nisnevich sheaf associated to $X \mapsto [\Sigma^{s+(w)X_+}, \mathsf{E}]$
$\pi_s\mathsf{E} = \pi_{s+(\star)}\mathsf{E}$	homotopy module $\bigoplus_{w\in\mathbb{Z}} \pi_{s+(w)}\mathsf{E}$
$\mathbf{M}A$	motivic Eilenberg–MacLane-spectrum for the abelian group A
$\mathbf{1}_S$	motivic sphere spectrum over S
$\mathbf{MGL}, \mathbf{MSL}, \mathbf{MSp}$	algebraic bordism spectra of various flavors
$\mathbf{KGL}, \mathbf{kgl}$	algebraic K-theory spectrum and its (very) effective cover
$\mathbf{KQ}, \mathbf{kq}$	hermitian K-theory spectrum and its (very) effective cover
$\mathbf{KW}, \mathbf{KW}_{\geq 0}$	Witt theory spectrum and its connective cover
$\mathsf{f}_n, \mathsf{s}_n$	n-th effective cover and slice
$\mathsf{vf}_n, \mathsf{vs}_n$	n-th very effective cover and slice
$\mathbf{K}^{\mathsf{MW}}(F)$	Milnor–Witt K-theory of F
$\mathbf{K}^{\mathsf{M}}(F)$	Milnor K-theory of F
$\mathbf{GW}(F), \boldsymbol{W}(F)$	Grothendieck–Witt ring and Witt ring of F
$[u] : S^{0+(0)} \to S^{0+(1)}$	map of spheres associated with unit u
$\mathsf{h} = 2 + \eta[-1]$	hyperbolic plane; zeroth algebraic Hopf map
$\eta : S^{1+(2)} \to S^{1+(1)}$	first algebraic Hopf map
$\nu : S^{3+(4)} \to S^{2+(2)}$	second algebraic Hopf map
$\eta_{\mathsf{top}} : S^{3+(0)} \to S^{2+(0)}$	first topological Hopf map

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