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We establish the existence of positive segregated solutions for competitive nonlinear Schrödinger systems in the presence of an external trapping potential, which have the property that each component is obtained from the previous one by a rotation, and we study their behavior as the forces of interaction become very small or very large.

As a consequence, we obtain optimal partitions for the Schrödinger equation by sets that are linearly isometric to each other.

1. Introduction

Consider the nonlinear Schrödinger system

$$-\Delta u_i+V_i(x)u_i=|u_i|^{2p-2}u_i+\sum_{\substack{j=1\\ j\neq i}}^{\ell}\beta_{ij}|u_j|^p|u_i|^{p-2}u_i,\quad u_i\in H^1(\mathbb{R}^N),\quad u_i>0,\qquad i=1,\dots,\ell,\tag{1-1}$$

where $N\geq 2,\ p>1$ and $p<N/(N-2)$ if $N\geq 3,\ \beta_{ij}=\beta_{ji}\in\mathbb{R}$, and $V_i\in\mathcal{C}^0(\mathbb{R}^N)$.

For the cubic nonlinearity ($p=2$) in dimensions $N=2,3$ this system arises in the study of Bose–Einstein condensation for a mixture of ℓ different states which overlap in space. It has been widely studied in the last two decades. Most work has been done in the autonomous case (i.e., for constant V_i). We refer the reader to the recent paper [Li et al. 2022], where the authors provide an exhaustive list of references. The nonautonomous case turns out to be much more difficult. Some results have been recently obtained by Peng and Wang [2013], Pistoia and Vaira [2022], and Li, Wei and Wu [Li et al. 2022].

The system (1-1) for a more general subcritical nonlinearity in higher dimensions has been much less studied. Even if it does not have an immediate physical motivation, finding a solution in this general setting is a quite interesting and challenging problem from a mathematical point of view. Besides, Schrödinger equations in higher dimensions have been widely studied in applications; see for instance [Dong 2011]. To our knowledge, the only result for the system (1-1) in higher dimensions is that by Gao and Guo [2020], who proved the existence of infinitely many solutions for only two equations ($\ell=2$) when the coupling parameter β_{12} is negative, and both equations have a common potential $V_1=V_2$ which does not enjoy any symmetry properties, but satisfies suitable decay assumptions at infinity. However, nothing is said about the sign of the solutions.

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Here we study (1-1) in a fully symmetric setting, namely we consider the nonlinear Schrödinger system

$$-\Delta u_i+V(x)u_i=|u_i|^{2p-2}u_i+\beta\sum_{\substack{j=1\\ j\neq i}}^{\ell}|u_j|^p|u_i|^{p-2}u_i,\quad u_i\in H^1(\mathbb{R}^N),\quad u_i>0,\qquad i=1,\dots,\ell,\tag{1-2}$$

where $N\geq 4$, $1<p<N/(N-2)$, $\beta<0$, and $V\in\mathcal{C}^0(\mathbb{R}^N)$ satisfies the following assumptions for some $n\in\mathbb{N}$:

(V_1) V is radial.

(V_2) $0<\inf_{x\in\mathbb{R}^N}V(x)$ and $V(x)\to V_\infty>0$ as $|x|\to\infty$.

(V_3^n) There exist C_0, $R_0>0$ and $\lambda\in\big(0,2\sin\frac{\pi}{\ell n}\big)$ such that

$$V(x)\leq V_\infty-C_0\mathrm{e}^{-\lambda\sqrt{V_\infty}|x|}\quad\text{for every } x\in\mathbb{R}^N \text{ with } |x|\geq R_0.$$

We look for *fully nontrivial* solutions to (1-2), i.e., solutions with all components u_i different from zero. Set

$$\|u\|_V^2:=\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2).$$

We prove the following results.

Theorem 1.1. *Let $n\in\mathbb{N}$ and assume that V satisfies (V_1), (V_2) and (V_3^n). Then the system (1-2) has a fully nontrivial solution $\boldsymbol{u}=(u_1,\dots,u_\ell)$ satisfying, for every $(z,y)\in\mathbb{C}\times\mathbb{R}^{N-2}\equiv\mathbb{R}^N$,*

$$\begin{cases} u_1(\mathrm{e}^{2\pi\mathrm{i}/n}z,\theta y)=u_1(z,y) & \text{for every } \theta\in O(N-2),\\ u_{j+1}(z,y)=u_1(\mathrm{e}^{2\pi\mathrm{i}j/\ell n}z,y) & \text{for every } j=1,\dots,\ell-1.\end{cases}\tag{1-3}$$

This solution has least energy among all nontrivial solutions satisfying (1-3). Furthermore, the energy of each component satisfies

$$\frac{p-1}{2p}\|u_i\|_V^2<n\mathfrak{c}_\infty,$$

where $\mathfrak{c}_\infty$ is the ground state energy of the Schrödinger equation

$$-\Delta u+V_\infty u=|u|^{2p-2}u,\quad u\in H^1(\mathbb{R}^N).\tag{1-4}$$

As usual, $O(N-2)$ denotes the group of linear isometries of $\mathbb{R}^{N-2}$. The symmetries (1-3) of the solutions given by Theorem 1.1 suggests calling them *pinwheel solutions*. For an autonomous system of two equations in dimensions 2 and 3, solutions of this kind were found by Wei and Weth [2007].

Since the potential V is assumed to be radial, using the compactness of the embedding of the subspace of radial functions in $H^1(\mathbb{R}^N)$ into $L^{2p}(\mathbb{R}^N)$ and following the argument given in [Clapp and Szulkin 2019, Theorem 1.1], it is easy to see that the system (1-2) has a solution all of whose components are radial. Note however that if $\boldsymbol{u}=(u_1,\dots,u_\ell)$ satisfies (1-2) and (1-3) and some component u_i is radial, then $u_1=\cdots=u_\ell=:u$ and u is a nontrivial solution of the equation

$$-\Delta u+V(x)u=(1+(\ell-1)\beta)|u|^{2p-2}u,\quad u\in H^1(\mathbb{R}^N).$$

Therefore, if $1+(\ell-1)\beta \leq 0$, a nontrivial solution to the system (1-2) satisfying (1-3) cannot be radial. In fact, more can be said. The following result, combined with Theorem 1.1, yields multiple positive nonradial solutions when the assumption (V_3^n) is satisfied for large enough n.

Proposition 1.2. *Let $\beta \leq -1/(\ell-1)$, and for some $m, q \in \mathbb{N}$, let $\boldsymbol{u}_m$ and $\boldsymbol{u}_q$ be solutions to (1-2) satisfying (1-3) with $n=\ell^m$ and $n=\ell^q$ respectively. If $m \neq q$, then $\boldsymbol{u}_m \neq \boldsymbol{u}_q$.*

One may wonder if the solution given by Theorem 1.1 for $\beta \in (-1/(\ell-1), 0)$ is radial or not. The following result gives a partial answer in terms of the nonautonomous Schrödinger equation (1-5). Namely, if the least energy solutions to this equation that satisfy (1-6) are nonradial, then the solutions to the system (1-2) satisfying (1-3) are nonradial for β close enough to 0.

Theorem 1.3. *Let $n \in \mathbb{N}$ and assume that V satisfies (V_1), (V_2) and (V_3^n). Let $\boldsymbol{u}_k = (u_{k,1}, \dots, u_{k,\ell})$ be a least energy fully nontrivial solution to (1-2) and (1-3) with $\beta = \beta_k$. Assume that $\beta_k < 0$ and $\beta_k \to 0$ as $k \to \infty$. Then, after passing to a subsequence, $u_{k,j} \to u_{0,j}$ strongly in $H^1(\mathbb{R}^N)$, $u_{0,j} \geq 0$, $\boldsymbol{u}_0 = (u_{0,1}, \dots, u_{0,\ell})$ satisfies (1-3), $u_{0,j}$ is a nontrivial solution to the equation*

$$-\Delta u + V(x)u = |u|^{2p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{1-5}$$

and $u_{0,j}$ has least energy among all solutions to (1-5) satisfying

$$u(\mathrm{e}^{2\pi \mathrm{i}/n} z, \theta y) = u(z, y) \quad \text{for every } \theta \in O(N-2),\ (z, y) \in \mathbb{R}^N. \tag{1-6}$$

Furthermore,

$$\frac{p-1}{2p} \|u_{0,j}\|_V^2 < n\mathfrak{c}_\infty.$$

Next, we describe the behavior of the solutions given by Theorem 1.1 as $\beta \to -\infty$. As shown by Conti, Terracini and Verzini [Conti et al. 2002; 2005] and Chang, Lin, Lin and Lin [Chang et al. 2004], there is a connection between variational elliptic systems with strong competitive interaction and optimal partition problems.

We shall call an ℓ-tuple $(\Omega_1, \dots, \Omega_\ell)$ of nonempty open subsets of $\mathbb{R}^N$ an *(n, ℓ)-pinwheel partition* of $\mathbb{R}^N$ if $\Omega_i \cap \Omega_j = \varnothing$ whenever $i \neq j$ and it satisfies following two symmetry conditions:

(S_1) $\Omega_{j+1} = \{(z, y) \in \mathbb{C} \times \mathbb{R}^{N-2} : (\mathrm{e}^{2\pi \mathrm{i} j/\ell n} z, y) \in \Omega_1\}$ for each $j = 1, \dots, \ell-1$.

(S_2) If $(z, y) \in \Omega_1$, then $(\mathrm{e}^{2\pi \mathrm{i}/n} z, \theta y) \in \Omega_1$ for every $\theta \in O(N-2)$.

We denote the set of all (n, ℓ)-pinwheel partitions by $\mathcal{P}_\ell^n$. If Ω is an open subset of $\mathbb{R}^N$ satisfying (S_2), a minimizer for

$$\inf_{u \in \mathcal{M}_\Omega} \frac{p-1}{2p} \|u\|_V^2 =: \mathfrak{c}_\Omega$$

on the Nehari manifold

$$\mathcal{M}_\Omega := \Big\{ u \in H_0^1(\Omega) : u \neq 0,\ \|u\|_V^2 = \int_{\mathbb{R}^N} |u|^{2p}, \text{ and } u(\mathrm{e}^{2\pi \mathrm{i}/n} z, \theta y) = u(z, y) \text{ for all } \theta \in O(N-2) \text{ and } (z, y) \in \Omega \Big\} \tag{1-7}$$

is a *least energy solution* to the problem

$$\begin{cases} -\Delta u + V(x)u = |u|^{2p-2}u, & u \in H_0^1(\Omega), \\ u(e^{2\pi i/n}z, \theta y) = u(z, y) & \text{for every } \theta \in O(N-2),\ (z, y) \in \Omega. \end{cases} \tag{1-8}$$

We say that $(\Omega_1, \dots, \Omega_\ell)$ is an *optimal (n, ℓ)-pinwheel partition* for (1-5) if $\mathfrak{c}_{\Omega_j}$ is attained on $\mathcal{M}_{\Omega_j}$ and

$$\sum_{j=1}^{\ell} \mathfrak{c}_{\Omega_j} = \inf_{(\Theta_1,\dots,\Theta_\ell)\in\mathcal{P}_\ell^n} \sum_{j=1}^{\ell} \mathfrak{c}_{\Theta_j}.$$

Theorem 1.4. *Let $n \in \mathbb{N}$ and assume that V satisfies (V_1), (V_2) and (V_3^n). Let $\boldsymbol{u}_k = (u_{k,1}, \dots, u_{k,\ell})$ be a least energy fully nontrivial solution to* (1-2) *and* (1-3) *with $\beta = \beta_k$. Assume that $\beta_k \to -\infty$ as $k \to \infty$. Then, after passing to a subsequence:*

(i) *$u_{k,j} \to u_{\infty,j}$ strongly in $H^1(\mathbb{R}^N)$, $u_{\infty,j} \geq 0$, $u_{\infty,j} \neq 0$, $u_{\infty,i}u_{\infty,j} = 0$ if $i \neq j$, $\boldsymbol{u}_\infty = (u_{\infty,1}, \dots, u_{\infty,\ell})$ satisfies* (1-3), *and*

$$\int_{\mathbb{R}^N} \beta_k u_{k,j}^p u_{k,i}^p \to 0 \quad \text{as } k \to \infty \qquad \text{whenever } i \neq j.$$

(ii) *$u_{\infty,j} \in \mathcal{C}^0(\mathbb{R}^N)$, the restriction of $u_{\infty,j}$ to the open set $\Omega_j := \{x \in \mathbb{R}^N : u_{\infty,j}(x) > 0\}$ is a least energy solution to the problem* (1-8) *in Ω_j, and $(\Omega_1, \dots, \Omega_\ell)$ is an optimal (n, ℓ)-pinwheel partition for* (1-5).

(iii) *$\mathbb{R}^N \smallsetminus \bigcup_{j=1}^{\ell} \Omega_j = \mathscr{R} \cup \mathscr{S}$, where $\mathscr{R} \cap \mathscr{S} = \varnothing$, $\mathscr{R}$ is an $(m-1)$-dimensional $\mathcal{C}^{1,\alpha}$-submanifold of $\mathbb{R}^N$ and $\mathscr{S}$ is a closed subset of $\mathbb{R}^N$ with Hausdorff measure $\leq m-2$. Furthermore, if $\xi \in \mathscr{R}$, there exist i, j such that*

$$\lim_{x\to\xi^+} |\nabla u_i(x)| = \lim_{x\to\xi^-} |\nabla u_j(x)| \neq 0,$$

where $x \to \xi^\pm$ are the limits taken from opposite sides of $\mathscr{R}$, and if $\xi \in \mathscr{S}$, then

$$\lim_{x\to\xi} |\nabla u_j(x)| = 0 \quad \text{for every } j = 1, \dots, \ell.$$

(iv) *If $\ell = 2$, then $u_{\infty,1} - u_{\infty,2}$ is a sign-changing solution to* (1-5) *satisfying* (1-6).

Note that (iii) implies that the partition exhausts $\mathbb{R}^N$, i.e., $\mathbb{R}^N = \bigcup_{j=1}^{\ell} \overline{\Omega}_j$. Thus, every Ω_j is unbounded.

The regularity properties of optimal partitions have been established, in different settings, for instance, in [Caffarelli and Lin 2008; Clapp et al. 2021b; Noris et al. 2010; Soave et al. 2016; Tavares and Terracini 2012].

Theorem 1.4 establishes the existence of optimal partitions having an additional property: each set of the partition is obtained from any other by means of a linear isometry. Pinwheel partitions are an example of this type of partition, but others are conceivable. In Section 2 we present a general symmetric variational setting for the system (1-2) that produces other examples.

The existence of sign-changing solutions to (1-5) having the additional property that their negative part is obtained from the positive one by means of a linear isometry and a change of sign has been established in [Clapp and Salazar 2012]. This includes those given by Theorem 1.4(iv). The tool for producing this type of solution is a homomorphism from some group of linear isometries of $\mathbb{R}^N$ onto the group with two elements. As shown in Section 2 this tool also serves to get positive solutions of the system (1-2) for $\ell = 2$ with the property that each component is obtained from the other by composition with a linear isometry. The general tool for obtaining a similar result for the system (1-2) of ℓ equations is a homomorphism into the group of permutations of a set of ℓ elements.

Rather than search for results in the general setting of Section 2, we decided, for the sake of clarity, to look at pinwheel solutions only. The solutions found in [Peng and Wang 2013; Pistoia and Vaira 2022] for $N = 2, 3$ and $p = 2$ were of this type. Peng and Wang [2013] focused on the case where the potential V is greater than its limit at infinity, and for a system of two equations, they established the existence of pinwheel solutions for β sufficiently negative. Pistoia and Vaira [2022] raised the question of whether solutions exist when V is below its limit at infinity and showed in that case that the system (1-2) has a solution satisfying (1-3) for β close enough to 0. The energy of each component approaches $n\mathfrak{c}_\infty$ as $\beta \to 0$.

Our results can be easily extended to dimension $N = 2$. In contrast, the dimension $N = 3$ requires a more delicate analysis because compactness can also be lost by the presence of solutions to the autonomous system (with $V = V_\infty$) that travel to infinity; see Remark 3.3.

In Section 2 we present the general variational framework and in Section 3 we study the behavior of minimizing sequences of pinwheel solutions for the system (1-2). In Section 4 we prove Theorem 1.1 and Proposition 1.2. Section 5 is devoted to the proofs of Theorems 1.3 and 1.4.

2. The symmetric variational setting

Let G be a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$, and for $\ell \geq 2$, let S_ℓ be the group of permutations of the set $\{1, \dots, \ell\}$ acting on $\mathbb{R}^\ell$ in the obvious way, i.e.,

$$\sigma(u_1, \dots, u_\ell) = (u_{\sigma(1)}, \dots, u_{\sigma(\ell)}) \quad \text{for every } \sigma \in S_\ell,\ (u_1, \dots, u_\ell) \in \mathbb{R}^\ell.$$

Let $\phi : G \to S_\ell$ be a continuous homomorphism of groups. A function $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^\ell$ will be called *ϕ-equivariant* if

$$\boldsymbol{u}(gx) = \phi(g)\boldsymbol{u}(x) \quad \text{for all } g \in G,\ x \in \mathbb{R}^N. \tag{2-1}$$

Note that if $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^\ell$ is ϕ-equivariant, then $\boldsymbol{u}$ is K_ϕ-invariant, where $K_\phi := \ker(\phi)$.

These data define a G-action on $\mathcal{H} := (H^1(\mathbb{R}^N))^\ell$ as follows:

$$(g\boldsymbol{u})(x) := \phi(g)\boldsymbol{u}(g^{-1}x) \quad \text{for every } g \in G,\ \boldsymbol{u} = (u_1, \dots, u_\ell) \in \mathcal{H}.$$

For $u, v \in H_0^1(\mathbb{R}^N)$ we set

$$\langle u, v\rangle_V := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) \quad \text{and} \quad \|u\|_V := \sqrt{\langle u, u\rangle_V}.$$

The solutions to the system (1-2) are the positive critical points of the functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ given by

$$\mathcal{J}(\boldsymbol{u}) := \frac{1}{2}\sum_{i=1}^{\ell} \|u_i\|_V^2 - \frac{1}{2p}\sum_{i=1}^{\ell}\int_{\mathbb{R}^N} |u_i|^{2p} - \frac{\beta}{2p}\sum_{\substack{i,j=1\\ i\neq j}}^{\ell}\int_{\mathbb{R}^N} |u_i|^p|u_j|^p,$$

which is of class $\mathcal{C}^1$. Its i-th partial derivative is

$$\partial_i \mathcal{J}(\boldsymbol{u})v = \langle u_i, v\rangle_V - \int_{\mathbb{R}^N} |u_i|^{2p-2}u_i v - \beta \sum_{\substack{j=1\\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} |u_j|^p |u_i|^{p-2} u_i v$$

for any $\boldsymbol{u} \in \mathcal{H}$, $v \in H^1(\mathbb{R}^N)$. The functional $\mathcal{J}$ is G-invariant, i.e.,

$$\mathcal{J}(g\boldsymbol{u}) = \mathcal{J}(\boldsymbol{u}) \quad \text{for every } g \in G,\ \boldsymbol{u} = (u_1, \dots, u_\ell) \in \mathcal{H}.$$

So, by the principle of symmetric criticality [Willem 1996, Theorem 1.28], the critical points of the restriction of $\mathcal{J}$ to the G-fixed point space of $\mathcal{H}$,

$$\mathcal{H}^\phi := \{\boldsymbol{u} \in \mathcal{H} : g\boldsymbol{u} = \boldsymbol{u} \text{ for all } g \in G\} = \{\boldsymbol{u} \in \mathcal{H} : \boldsymbol{u} \text{ is } \phi\text{-equivariant}\},$$

are critical points of $\mathcal{J}$, i.e., they are the solutions to the system (1-2) satisfying (2-1). We denote by $\mathcal{J}^\phi$ the restriction of $\mathcal{J}$ to $\mathcal{H}^\phi$. Note that

$$(\mathcal{J}^\phi)'(\boldsymbol{u})\boldsymbol{v} = \mathcal{J}'(\boldsymbol{u})\boldsymbol{v} = \sum_{i=1}^{\ell} \partial_i \mathcal{J}(\boldsymbol{u}) v_i \quad \text{for any } \boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^\phi.$$

The fully nontrivial critical points of $\mathcal{J}^\phi$ belong to the set

$$\mathcal{N}^\phi := \{\boldsymbol{u} \in \mathcal{H}^\phi : u_i \neq 0 \text{ and } \partial_i \mathcal{J}(\boldsymbol{u}) u_i = 0 \text{ for all } i = 1, \dots, \ell\}.$$

Observe that

$$\mathcal{J}^\phi(\boldsymbol{u}) = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_i\|_V^2 \quad \text{if } \boldsymbol{u} \in \mathcal{N}^\phi.$$

Set

$$c^\phi := \inf_{\boldsymbol{u} \in \mathcal{N}^\phi} \mathcal{J}^\phi(\boldsymbol{u}).$$

We consider also the single equation

$$-\Delta u + V(x) u = |u|^{2p-2}u, \quad u \in H^1(\mathbb{R}^N)^G, \tag{2-2}$$

where $H^1(\mathbb{R}^N)^G := \{u \in H^1(\mathbb{R}^N) : u \text{ is } G\text{-invariant}\}$, and we denote by $J : H^1(\mathbb{R}^N)^G \to \mathbb{R}$ and $\mathcal{M}^G$ the energy functional and the Nehari manifold associated to it, i.e.,

$$J(u) := \frac{1}{2}\|u\|_V^2 - \frac{1}{2p} \int_{\mathbb{R}^N} |u|^{2p} \tag{2-3}$$

and

$$\mathcal{M}^G := \left\{ u \in H^1(\mathbb{R}^N)^G : u \neq 0,\ \|u\|_V^2 = \int_{\mathbb{R}^N} |u|^{2p} \right\}.$$

Similarly, we denote by $J_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ and $\mathcal{M}_\infty$ the energy functional and the Nehari manifold associated to (1-4). Set

$$\mathfrak{c}_\infty := \inf_{u \in \mathcal{M}_\infty} J_\infty(u) \quad \text{and} \quad \mathfrak{c}^G := \inf_{u \in \mathcal{M}^G} J(u). \tag{2-4}$$

We shall focus our attention on the following example.

Example 2.1. Let $\mathbb{Z}_m := \{\mathrm{e}^{2\pi\mathrm{i}j/m} : j = 0, \dots, m-1\}$ act on $\mathbb{C}$ by complex multiplication, and let $G_m := \mathbb{Z}_m \times O(N-2)$ act on $\mathbb{R}^N$ as

$$\begin{aligned} \alpha x &:= (\alpha z, y) \quad \text{for all } \alpha \in \mathbb{Z}_m, \\ \theta x &:= (z, \theta y) \quad \text{for all } \theta \in O(N-2) \text{ and } x = (z, y) \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N. \end{aligned}$$

Let $\sigma_1 \in S_\ell$ be the cyclic permutation $\sigma_1(i) := i + 1 \bmod \ell$, and let $\phi_n : G_{\ell n} \to S_\ell$ be the homomorphism given by $\phi_n(\mathrm{e}^{2\pi\mathrm{i}/\ell n}, \theta) := \sigma_1$ for any $\theta \in O(N-2)$. Then $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^\ell$ is ϕ_n-equivariant if and only if

$$\big(u_1(\mathrm{e}^{2\pi\mathrm{i}/\ell n} z, \theta y), \dots, u_\ell(\mathrm{e}^{2\pi\mathrm{i}/\ell n} z, \theta y)\big) = \big(u_2(z, y), \dots, u_\ell(z, y), u_1(z, y)\big)$$

for every $(z, y) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $\theta \in O(N-2)$, i.e., if and only if (1-3) holds. Note that every u_j is G_n-invariant.

3. The behavior of minimizing sequences

From now on, we fix n, and we take G_n and $\phi_n : G_{\ell n} \to S_\ell$ as in Example 2.1. Then, for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^{\phi_n}$,

$$(\mathcal{J}^{\phi_n})'(\boldsymbol{u})\boldsymbol{v} = \sum_{i=1}^{\ell} \partial_i \mathcal{J}(\boldsymbol{u}) v_i = \ell\, \partial_j \mathcal{J}(\boldsymbol{u}) v_j \quad \text{for any } j = 1, \dots, \ell, \tag{3-1}$$

and the set $\mathcal{N}^{\phi_n}$ is the usual Nehari manifold associated to the functional $\mathcal{J}^{\phi_n} : \mathcal{H}^{\phi_n} \to \mathbb{R}$, i.e.,

$$\mathcal{N}^{\phi_n} = \{\boldsymbol{u} \in \mathcal{H}^{\phi_n} : \boldsymbol{u} \neq 0,\ (\mathcal{J}^{\phi_n})'(\boldsymbol{u})\boldsymbol{u} = 0\}.$$

It has the following properties.

Proposition 3.1. (a) $\mathcal{N}^{\phi_n} \neq \varnothing$.

(b) $c^{\phi_n} \geq \ell \mathfrak{c}^{G_n} > 0$.

(c) *$\mathcal{N}^{\phi_n}$ is a closed $\mathcal{C}^1$-submanifold of codimension 1 of $\mathcal{H}^{\phi_n}$, and a natural constraint for $\mathcal{J}^{\phi_n}$.*

(d) *If $\boldsymbol{u} \in \mathcal{H}^{\phi_n}$ is such that, for each $i = 1, \dots, \ell$,*

$$\int_{\mathbb{R}^N} |u_i|^{2p} + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \beta \int_{\mathbb{R}^N} |u_i|^p |u_j|^p > 0,$$

then there exists a unique $s_{\boldsymbol{u}} \in (0, \infty)$ such that $s_{\boldsymbol{u}} \boldsymbol{u} \in \mathcal{N}^{\phi_n}$. Furthermore,

$$\mathcal{J}^{\phi_n}(s_{\boldsymbol{u}} \boldsymbol{u}) = \max_{s \in (0,\infty)} \mathcal{J}^{\phi_n}(s\boldsymbol{u}).$$

(e) $c^{\phi_n} \leq \ell n \mathfrak{c}_\infty$.

Proof. The proof is easy. We give the details for the sake of completeness.

(a) Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be a nontrivial radial function with $\|\varphi\|_V^2 = \int_{\mathbb{R}^N} |\varphi|^{2p}$. Set $\xi_{i,j} := (\mathrm{e}^{2\pi\mathrm{i}(i+\ell j)/\ell n}, 0) \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N$ and define

$$u_{R,i+1}(x) := \sum_{j=0}^{n-1} \varphi(x - R\xi_{i,j}), \quad i = 0, \dots, n-1,$$

where $R > 0$ is taken large enough that $u_{R,i}$ and $u_{R,j}$ have disjoint supports for every $i \neq j$. Then $\boldsymbol{u}_R := (u_{R,1}, \ldots, u_{R,1}) \in \mathcal{N}^{\phi_n}$.

(b) Let $\boldsymbol{u} = (u_1, \ldots, u_\ell) \in \mathcal{N}^{\phi_n}$. As $\beta < 0$, we have

$$0 < \|u_i\|_V^2 = \|u_1\|_V^2 \leq \int_{\mathbb{R}^N} |u_1|^{2p} = \int_{\mathbb{R}^N} |u_i|^{2p} \quad \text{for all } i = 2, \ldots, \ell.$$

Hence, there exists $s \in (0,1]$ such that $su_i \in \mathcal{M}^{G_n}$ for every $i = 1, \ldots, \ell$. Therefore,

$$\ell \mathfrak{c}^{G_n} \leq \sum_{i=1}^{\ell} J(s_i u_i) = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|s_i u_i\|_V^2 \leq \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_i\|_V^2 = \mathcal{J}^{\phi}(\boldsymbol{u}).$$

It follows that $\ell \mathfrak{c}^{G_n} \leq c^{\phi_n}$.

(c) The function $\Psi : \mathcal{H}^{\phi_n} \smallsetminus \{0\} \to \mathbb{R}$ given by $\Psi(\boldsymbol{u}) := (\mathcal{J}^{\phi_n})'(\boldsymbol{u})\boldsymbol{u}$ is of class $\mathcal{C}^1$, and $\mathcal{N}^{\phi_n} = \Psi^{-1}(0)$. It follows from (b) that $\mathcal{N}^{\phi_n}$ is a closed subset of $\mathcal{H}^{\phi_n}$. As

$$\Psi'(\boldsymbol{u})\boldsymbol{u} = (2-2p)\ell \|u_1\|_V^2 \neq 0,$$

we have that 0 is a regular value of Ψ. This shows that $\mathcal{N}^{\phi_n}$ is a $\mathcal{C}^1$-submanifold of codimension 1 of $\mathcal{H}^{\phi_n}$. It also shows that $\boldsymbol{u} \notin \ker \Psi'(\boldsymbol{u}) =: T_{\boldsymbol{u}}\mathcal{N}^{\phi_n}$, the tangent space of $\mathcal{N}^{\phi_n}$ at $\boldsymbol{u}$. Hence,

$$\mathcal{H}^{\phi_n} = T_{\boldsymbol{u}}\mathcal{N}^{\phi_n} \oplus \mathbb{R}\boldsymbol{u}.$$

Since, by definition, $(\mathcal{J}^{\phi_n})'(\boldsymbol{u})\boldsymbol{u} = 0$ for every $\boldsymbol{u} \in \mathcal{N}^{\phi_n}$, we infer that a critical point of the restriction of $\mathcal{J}^{\phi_n}$ to $\mathcal{N}^{\phi_n}$ is a critical point of $\mathcal{J}^{\phi_n}$.

(d) The proof is straightforward. The number $s_{\boldsymbol{u}}$ is

$$s_{\boldsymbol{u}} = \left(\frac{\|u_1\|_V^2}{\int_{\mathbb{R}^N} |u_1|^{2p} + \sum\limits_{j=2}^{\ell} \beta \int_{\mathbb{R}^N} |u_1|^p |u_j|^p} \right)^{1/(2p-2)}.$$

(e) Let ω be the least energy positive radial solution to (1-4). Set $\xi_{i,j} = (\mathrm{e}^{2\pi \mathrm{i}(i+\ell j)/\ell n}, 0) \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N$. Define

$$w_{R,i+1}(x) := \sum_{j=0}^{n-1} \omega(x - R\xi_{i,j}), \quad i = 0, \ldots, n-1.$$

Then $\boldsymbol{w}_R = (w_{R,1}, \ldots, w_{R,\ell}) \in \mathcal{H}^{\phi_n}$. If R is sufficiently large, statement (d) yields $s_R \in (0,\infty)$ such that $s_R \boldsymbol{w}_R \in \mathcal{N}^{\phi_n}$ and $s_R \to 1$ as $R \to \infty$. Using assumption (V_2) we obtain

$$c^{\phi_n} \leq \mathcal{J}^{\phi_n}(s_R \boldsymbol{w}_R) = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|s_{R,i} w_{R,i}\|_V^2 \to \ell n \mathfrak{c}_\infty \quad \text{as } R \to \infty.$$

This shows that $c^{\phi_n} \leq \ell n \mathfrak{c}_\infty$, as claimed. $\square$

Lemma 3.2. *Let (x_k) be a sequence in $\mathbb{R}^N$, where $N \geq 4$. After passing to a subsequence, there exists a sequence (ξ_k) in $\mathbb{R}^N$ and a constant $C_0 > 0$ such that*

$$|x_k - \xi_k| \leq C_0 \quad \text{for all } k \in \mathbb{N},$$

and one of the following statements holds true:

- $\xi_k = 0$ *for all k, or*
- $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ *and* $|\zeta_k| \to \infty$, *or*
- *for each $m \in \mathbb{N}$ there exist $\gamma_1, \ldots, \gamma_m \in O(N-2)$ such that $|\gamma_i \xi_k - \gamma_j \xi_k| \to \infty$ if $i \neq j$.*

Proof. See [Clapp et al. 2021a, Lemma 3.1]. □

Remark 3.3. This lemma is not true in dimension $N = 3$, because $O(1) = \{1, -1\}$.

Theorem 3.4. *Let $\boldsymbol{u}_k = (u_{k,1}, \ldots, u_{k,\ell}) \in \mathcal{N}^{\phi_n}$ be such that $\mathcal{J}^{\phi_n}(\boldsymbol{u}_k) \to c^{\phi_n}$ and $u_{k,i} \geq 0$. Then, after passing to a subsequence, either $\boldsymbol{u}_k \to \boldsymbol{u}$ strongly in $\mathcal{H}^{\phi_n}$ with $u_i \geq 0$, or there are points $(z_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N$ such that $|z_k| \to \infty$,*

$$\lim_{k\to\infty} \left\| u_{k,1} - \sum_{j=0}^{n} \omega(\,\cdot - (\mathrm{e}^{2\pi \mathrm{i} j/n} z_k, 0)) \right\| = 0,$$

and $c^{\phi_n} = \ell n \mathfrak{c}_\infty$, where ω is the least energy positive radial solution to (1-4).

Proof. Invoking Ekeland's variational principle [Willem 1996, Theorem 8.5] we may assume that $(\mathcal{J}^{\phi_n})'(\boldsymbol{u}_k) \to 0$ in $(\mathcal{H}^{\phi_n})'$.

Since $\beta < 0$, Proposition 3.1(b) yields $c_0 > 0$ such that

$$\int_{\mathbb{R}^N} |u_{k,1}|^{2p} > c_0 \quad \text{for all } k \in \mathbb{N}.$$

By Lions' lemma [Willem 1996, Lemma 1.21] there exist $\delta > 0$ and $x_k \in \mathbb{R}^N$ such that, after passing to a subsequence,

$$\int_{B_1(x_k)} |u_{k,1}|^{2p} > \delta \quad \text{for all } k \in \mathbb{N}.$$

For (x_k) we fix a sequence (ξ_k) and a constant $C_0 > 0$ such that $|x_k - \xi_k| \leq C_0$ for all $k \in \mathbb{N}$, satisfying one of the alternatives stated in Lemma 3.2. Then

$$\int_{B_{C_0+1}(\xi_k)} |u_{k,1}|^{2p} \geq \int_{B_1(x_k)} |u_{k,1}|^{2p} > \delta \quad \text{for all } k \in \mathbb{N}. \tag{3-2}$$

It follows that either $\xi_k = 0$, or $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $\zeta_k \to \infty$. Otherwise, by Lemma 3.2, for each $m \in \mathbb{N}$ there would exist $\gamma_1, \ldots, \gamma_m \in O(N-2)$ such that $|\gamma_i \xi_k - \gamma_j \xi_k| \geq 2(C_0 + 1)$ if $i \neq j$ for large enough $k \in \mathbb{N}$, and as $u_{k,1}$ is G_n-invariant, we would have that

$$\int_{\mathbb{R}^N} |u_{k,1}|^{2p} \geq \sum_{i=1}^{m} \int_{B_{C_0+1}(\gamma_i \xi_k)} |u_{k,1}|^{2p} = m \int_{B_{C_0+1}(\xi_k)} |u_{k,1}|^{2p} > m\delta$$

for all $m \in \mathbb{N}$. This is impossible because $(u_{k,1})$ is bounded in $L^{2p}(\mathbb{R}^N)$.

Next, we distinguish two cases.

Case 1: $\xi_k = 0$ for all $k \in \mathbb{N}$.

Since the sequence $(u_{k,1})$ is bounded in $H^1(\mathbb{R}^N)$, passing to a subsequence, we have that $u_{k,1} \rightharpoonup u_1$ weakly in $H^1(\mathbb{R}^N)$, $u_{k,1} \to u_1$ in $L^{2p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_{k,1} \to u_1$ a.e. in $\mathbb{R}^N$. Hence, $u_1 \geq 0$ and it follows from (3-2) that $u_1 \neq 0$. Note that, as $u_{k,1} \in H^1(\mathbb{R}^N)^{G_n}$, $u_1 \in H^1(\mathbb{R}^N)^{G_n}$. Set $u_{j+1}(z, y) := u_1(\mathrm{e}^{2\pi \mathrm{i} j/\ell n} z, y)$ for $(z, y) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $j = 1, \dots, \ell - 1$, and set $\boldsymbol{u} = (u_1, \dots, u_\ell)$. Then $u_{k,j+1} \rightharpoonup u_{j+1}$ weakly in $H^1(\mathbb{R}^N)$, and as $(\mathcal{J}^{\phi_n})'(\boldsymbol{u}_k) \to 0$ in $(\mathcal{H}^{\phi_n})'$, we derive from (3-1) that

$$0 = \lim_{k\to\infty} \partial_1 \mathcal{J}(\boldsymbol{u}_k)\varphi = \partial_1 \mathcal{J}(\boldsymbol{u})\varphi \quad \text{for every } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^N)^{G_n}.$$

Hence, $\boldsymbol{u} \in \mathcal{N}^{\phi_n}$ and

$$c^{\phi_n} \leq \mathcal{J}^{\phi_n}(\boldsymbol{u}) = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_i\|_V^2 \leq \liminf_{k\to\infty} \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_{k,i}\|_V^2 = \lim_{k\to\infty} \mathcal{J}^{\phi_n}(\boldsymbol{u}_k) = c^{\phi_n}.$$

Therefore, $\boldsymbol{u}_k \to \boldsymbol{u}$ strongly in $\mathcal{H}^{\phi_n}$. This shows that, in Case 1, the first alternative stated in Theorem 3.4 holds true.

Case 2: $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $\zeta_k \to \infty$.

Set

$$w_{k,i}(x) := u_{k,i}(x + \xi_k), \quad i = 1, \dots, \ell.$$

Note that $w_{k,i}$ is $O(N-2)$-invariant. Since the sequence $(w_{k,i})$ is bounded in $H^1(\mathbb{R}^N)$, a subsequence satisfies $w_{k,i} \rightharpoonup w_i$ weakly in $H^1(\mathbb{R}^N)^{O(N-2)}$, $w_{k,i} \to w_i$ in $L^{2p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $w_{k,i} \to w_i$ a.e. in $\mathbb{R}^N$. Hence, $w_i \geq 0$. To simplify notation, set $\alpha := \mathrm{e}^{2\pi\mathrm{i}/n}$. Note that, as $|\alpha^j \xi_k - \alpha^m \xi_k| \to \infty$ if $j \neq m$, we have that

$$w_{k,i} \circ \alpha^{-m} - \sum_{j=m+1}^{n-1} (w_i \circ \alpha^{-m})(\,\cdot - \alpha^j \xi_k + \alpha^m \xi_k) \rightharpoonup w_i \circ \alpha^{-m}$$

weakly in $H^1(\mathbb{R}^N)$. Hence, setting $V_k(x) := V(x + \xi_k)$, Lemma A.1 gives

$$\|w_i \circ \alpha^{-m}\|^2_{V_\infty} = \Big\| w_{k,i} \circ \alpha^{-m} - \sum_{j=m+1}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k + \alpha^m \xi_k) \Big\|^2_{V_k} - \Big\| w_{k,i} \circ \alpha^{-m} - \sum_{j=m}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k + \alpha^m \xi_k) \Big\|^2_{V_k} + o(1).$$

Since $u_{k,i}$ is G_n-invariant, the change of variable $y = z - \alpha^m \xi_k$ yields

$$\Big\| u_{k,i} - \sum_{j=m+1}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k) \Big\|^2_V = \Big\| u_{k,i} - \sum_{j=m}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k) \Big\|^2_V + \|w_i\|^2_{V_\infty} + o(1),$$

and iterating this identity we obtain

$$\|u_{k,i}\|^2_V = \Big\| u_{k,i} - \sum_{j=0}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k) \Big\|^2_V + n\|w_i\|^2_{V_\infty} + o(1). \tag{3-3}$$

On the other hand, for any given $v \in H^1(\mathbb{R}^N)^{O(N-2)}$ set $v_k(y) := v(y-\xi_k)$ and

$$\hat{v}_k(y) := \sum_{j=0}^{n-1} v_k(\alpha^j y).$$

Recalling that $u_{k,i}$ is G_n-invariant and performing the translation $y = x+\xi_k$, we obtain

$$\begin{aligned}\partial_i \mathcal{J}(\boldsymbol{u}_k)\hat{v}_k &= \sum_{j=0}^{n-1} \partial_i \mathcal{J}(\boldsymbol{u}_k)(v_k \circ \alpha^j) = n\partial_i \mathcal{J}(\boldsymbol{u}_k)v_k \\ &= n\bigg(\int_{\mathbb{R}^N}(\nabla w_{k,i}\cdot\nabla v + V_k(x)w_{k,i}v) - \int_{\mathbb{R}^N}|w_{k,i}|^{2p-2}w_{k,i}v - \beta\sum_{\substack{j=1\\ j\neq i}}^{\ell}\int_{\mathbb{R}^N}|w_{k,j}|^p|w_{k,i}|^{p-2}w_{k,i}v\bigg).\end{aligned}$$

Note that $\hat{v}_k$ is G_n-invariant. As $(\mathcal{J}^{\phi_n})'(\boldsymbol{u}_k) \to 0$, invoking (3-1) and assumption (V_2), and passing to the limit as $k\to\infty$, we get

$$0 = \int_{\mathbb{R}^N}(\nabla w_i\cdot\nabla v + V_\infty w_i v) - \int_{\mathbb{R}^N}|w_i|^{2p-2}w_i v - \beta\sum_{\substack{j=1\\ j\neq i}}^{\ell}\int_{\mathbb{R}^N}|w_j|^p|w_i|^{p-2}w_i v \tag{3-4}$$

for every $v \in H^1(\mathbb{R}^N)^{O(N-2)}$ and $i = 1, \dots, \ell$. Since, by (3-2),

$$\int_{B_{C_0+1}(0)}|w_{k,1}|^{2p} \geq \int_{B_{C_0+1}(\xi_k)}|u_{k,1}|^{2p} \geq \delta > 0,$$

we see that $w_1 \neq 0$. Furthermore, (3-4) implies that

$$\|w_1\|_{V_\infty}^2 = \int_{\mathbb{R}^N}|w_1|^{2p} + \beta\sum_{j=2}^{\ell}\int_{\mathbb{R}^N}|w_j|^p|w_1|^p \leq \int_{\mathbb{R}^N}|w_1|^{2p}, \tag{3-5}$$

so there exists $t\in(0,1]$ such that $\|tw_1\|_{V_\infty}^2 = \int_{\mathbb{R}^N}|tw_1|^{2p}$. It follows that $tw_1 \in \mathcal{M}_\infty$, and from (3-3) and Proposition 3.1(e) we derive

$$n\mathfrak{c}_\infty \leq \frac{p-1}{2p}n\|tw_1\|_{V_\infty}^2 \leq \frac{p-1}{2p}n\|w_1\|_{V_\infty}^2 \leq \lim_{k\to\infty}\frac{p-1}{2p}\|u_{k,1}\|_V^2 = \frac{1}{\ell}c^{\phi_n} \leq n\mathfrak{c}_\infty.$$

Therefore, $t=1$, $w_1\in\mathcal{M}_\infty$ and $J_\infty(w_1) = ((p-1)/(2p))\|w_1\|_{V_\infty}^2 = \mathfrak{c}_\infty$, i.e., w_1 is a least energy solution of (1-4). Moreover, from (3-3) we get that

$$\lim_{k\to\infty}\bigg\|u_{k,1} - \sum_{j=0}^{n-1}(w_1\circ\alpha^{-j})(\,\cdot - \alpha^j\xi_k)\bigg\|_V^2 = 0.$$

Since the positive least energy solution to (1-4) is unique up to translation and w_1 is $O(N-2)$-invariant, there exists $\xi = (\zeta, 0) \in \mathbb{C}\times\mathbb{R}^{N-2}$ such that $w_1(x) = \omega(x+\xi)$. Hence, $(w_1\circ\alpha^{-j})(x-\alpha^j\xi_k) = \omega(\alpha^{-j}x - \xi_k - \xi) = \omega(x - \alpha^j(\xi_k+\xi))$. So, setting $z_k := \zeta_k + \zeta$, we obtain

$$\lim_{k\to\infty}\bigg\|u_{k,1} - \sum_{j=0}^{n}\omega(\,\cdot - (\mathrm{e}^{2\pi \mathrm{i} j/n}z_k, 0))\bigg\| = 0.$$

This shows that, in Case 2, the second alternative stated in Theorem 3.4 holds true. ∎

Corollary 3.5. *If $c^{\phi_n} < \ell n \mathfrak{c}_\infty$, the system (1-2) has a least energy fully nontrivial solution satisfying (1-3).*

4. Existence of a solution

We define the set of *weak (n,ℓ)-pinwheel partitions* as

$$\mathcal{W}_\ell^n := \{(u_1,\dots,u_\ell)\in\mathcal{H}^{\phi_n} : u_i\neq 0,\ \|u_i\|_V^2 = |u_i|_{2p}^{2p},\ u_iu_j=0 \text{ in } \mathbb{R}^N \text{ if } i\neq j\},$$

and set

$$\hat{c}^{\phi_n} := \inf_{(u_1,\dots,u_\ell)\in\mathcal{W}_\ell^n} \frac{p-1}{2p}\sum_{i=1}^{\ell}\|u_i\|_V^2.$$

Our next goal is to give an upper estimate for $\hat{c}^{\phi_n}$. To this end, we choose $\varepsilon\in\big(0,(d_{\ell n}-\lambda)/(d_{\ell n}+\lambda)\big)$ and a radial function $\chi\in\mathcal{C}^\infty(\mathbb{R}^N)$ satisfying $0\le\chi\le 1$, $\chi(x)=1$ if $|x|\le 1-\varepsilon$ and $\chi(x)=0$ if $|x|\ge 1$. Let ω be the positive least energy radial solution to (1-4). For each $r>0$ define

$$\omega_r(x) := \chi\Big(\frac{x}{r}\Big)\omega(x).$$

Lemma 4.1. *As $r\to\infty$,*

$$\big|\|\omega\|^2-\|\omega_r\|^2\big| = O(\mathrm{e}^{-2(1-\varepsilon)\sqrt{V_\infty}r}) \quad \textit{and} \quad \big||\omega|_{2p}^{2p}-|\omega_r|_{2p}^{2p}\big| = O(\mathrm{e}^{-2p(1-\varepsilon)\sqrt{V_\infty}r}),$$

where $|\cdot|_{2p}$ denotes the norm in $L^{2p}(\mathbb{R}^N)$.

Proof. These statements follow easily from the well-known estimates $|\omega(x)| = O(|x|^{-\frac{1}{2}(N-1)}\mathrm{e}^{-\sqrt{V_\infty}|x|})$ and $|\nabla\omega(x)| = O(|x|^{-\frac{1}{2}(N-1)}\mathrm{e}^{-\sqrt{V_\infty}|x|})$, as in [Clapp and Weth 2004, Lemma 2]. □

Set $\varrho := \frac{1}{4}(d_{\ell n}+\lambda)$, and for $R>1$ define

$$\hat{w}_{1,R}(x) := \sum_{j=0}^{n-1}\omega_{\varrho R}(x-R(\mathrm{e}^{2\pi\mathrm{i}j/n},0)) \quad \text{and} \quad w_{1,R} := t_R\hat{w}_{1,R},$$

where $t_R\in(0,\infty)$ is such that $\|w_{1,R}\|_V^2 = |w_{1,R}|_{2p}^{2p}$. Note that $t_R\to 1$ as $R\to\infty$, $w_{1,R}$ is G_n-invariant and

$$\operatorname{supp}\big(\omega_{\varrho R}(\cdot - R(\mathrm{e}^{2\pi\mathrm{i}j/\ell n},0))\big)\subset\overline{B_{\varrho R}(R(\mathrm{e}^{2\pi\mathrm{i}j/\ell n},0))}.$$

Set $w_{j+1,R}(\mathrm{e}^{2\pi\mathrm{i}j/\ell n}z,y) := w_{1,R}(z,y)$ for $(z,y)\in\mathbb{C}\times\mathbb{R}^{N-2}$ and $j=1,\dots,\ell-1$. Since $\varrho<\frac{1}{2}d_{\ell n}$ we have that $\operatorname{supp}(w_{i,R})\cap\operatorname{supp}(w_{j,R})=\varnothing$ if $i\neq j$. Hence, $\boldsymbol{w}_R=(w_{1,R},\dots,w_{\ell,R})\in\mathcal{W}_\ell^n$.

Lemma 4.2. *There exist $C_1, R_1>0$ such that*

$$\frac{p-1}{2p}\sum_{i=1}^{\ell}\|w_{i,R}\|_V^2 = \mathcal{J}^{\phi_n}(\boldsymbol{w}_R)\le \ell n\mathfrak{c}_\infty - C_1\mathrm{e}^{-\lambda\sqrt{V_\infty}R} \quad \textit{for all } R\ge R_1.$$

Proof. Because $\boldsymbol{w}_R=(w_{1,R},\dots,w_{\ell,R})\in\mathcal{W}_\ell^n$, the equality holds true. To prove the inequality note that $t_R \in \left[\frac{1}{2}, 2\right]$ for R large enough. Assumption (V_3^n) yields

$$\begin{aligned}\int_{\mathbb{R}^N} (V(x) - V_\infty)\left|t_R\omega_{\varrho R}(x - R(1,0))\right|^2 \mathrm{d}x &= \int_{|x|\le\varrho R} \big(V(x + R(1,0)) - V_\infty\big)|t_R\omega_{\varrho R}(x)|^2 \mathrm{d}x \\ &= -\frac{1}{4}C_0 \int_{|x|\le\varrho R} \mathrm{e}^{-\lambda\sqrt{V_\infty}|x+R(1,0)|}|\omega(x)|^2 \mathrm{d}x \\ &\le -\frac{1}{4}C_0\bigg(\int_{\mathbb{R}^N} \mathrm{e}^{-\lambda\sqrt{V_\infty}|x|}|\omega(x)|^2 \mathrm{d}x\bigg)\mathrm{e}^{-\lambda\sqrt{V_\infty}R} \\ &=: -2C\mathrm{e}^{-\lambda\sqrt{V_\infty}R}.\end{aligned}$$

Using Lemma 4.1, for R large enough we get

$$\begin{aligned}\mathcal{J}^{\phi_n}(\boldsymbol{w}_R) &= \frac{1}{2}\sum_{i=1}^{\ell}\|w_{i,R}\|_V^2 - \frac{1}{2p}\sum_{i=1}^{\ell}\int_{\mathbb{R}^N}|w_{i,R}|^{2p} - \frac{\beta}{2p}\sum_{\substack{i,j=1\\ i\neq j}}^{\ell}\int_{\mathbb{R}^N}|w_{i,R}|^p|w_{j,R}|^p \\ &= \ell n\bigg(\frac{1}{2}\|t_R\omega_{\varrho R}(\,\cdot - R(1,0))\|_V^2 - \frac{1}{2p}\left|t_R\omega_{\varrho R}(\,\cdot - R(1,0))\right|_{2p}^{2p}\bigg) \\ &= \ell n\bigg(\frac{1}{2}\|t_R\omega_{\varrho R}\|_{V_\infty}^2 + \frac{1}{2}\int_{\mathbb{R}^N}(V - V_\infty)\left|t_R\omega_{\varrho R}(\,\cdot - R(1,0))\right|^2 - \frac{1}{2p}|t_R\omega_{\varrho R}|_{2p}^{2p}\bigg) \\ &= \ell n\bigg(\frac{1}{2}\|t_R\omega\|_{V_\infty}^2 - C\mathrm{e}^{-\lambda\sqrt{V_\infty}R} - \frac{1}{2p}|t_R\omega|_{2p}^{2p} + O(\mathrm{e}^{-2(1-\varepsilon)\sqrt{V_\infty}\varrho R})\bigg) \\ &\le \ell n\mathfrak{c}_\infty - C_1\mathrm{e}^{-\lambda\sqrt{V_\infty}R},\end{aligned}$$

because $2(1-\varepsilon)\varrho > \frac{1}{2}(d_{\ell n} + \lambda)\big(1 - (d_{\ell n} - \lambda)/(d_{\ell n} + \lambda)\big) = \lambda$. $\square$

Proof of Theorem 1.1. Note that $\mathcal{W}_\ell^n \subset \mathcal{N}^{\phi_n}$. Hence, from Lemma 4.2 we get

$$c^{\phi_n} \le \hat{c}^{\phi_n} < \ell n\mathfrak{c}_\infty,$$

and Corollary 3.5 yields the result. $\square$

Proof of Proposition 1.2. Arguing by contradiction, assume that $\boldsymbol{u}$ is a solution to (1-2) satisfying (1-3) with $n = \ell^m$ and with $n = \ell^q$, respectively, and that $1 \le m < q$. Then, for $k = \ell^{q-m-1}j$ with $j = 1, \dots, \ell - 1$,

$$u_1(x) = u_1(\mathrm{e}^{2\pi\mathrm{i}k/\ell^q}x) = u_1(\mathrm{e}^{2\pi\mathrm{i}j/\ell^m\ell}x) = u_{j+1}(x),$$

and as $1 + \beta(\ell - 1) \le 0$, we obtain

$$\|u_1\|_V^2 = \int_{\mathbb{R}^N}|u_1|^{2p} + \beta\sum_{j=1}^{\ell-1}|u_{j+1}|^p|u_1|^p = (1 + \beta(\ell - 1))\int_{\mathbb{R}^N}|u_1|^{2p} \le 0,$$

a contradiction. $\square$

5. The limit profiles of the solutions

We start with the case $\beta \to 0$.

Proof of Theorem 1.3. We write $\mathcal{J}_k^{\phi_n}$ and $\mathcal{N}_k^{\phi_n}$ for the functional and the Nehari set associated to the system (1-2) with $\beta=\beta_k$, and we define

$$c_k^{\phi_n} := \inf_{\mathcal{N}_k^{\phi_n}} \mathcal{J}_k^{\phi_n}.$$

As $\mathcal{W}_\ell^n \subset \mathcal{N}_k^{\phi_n}$ for every $k\in\mathbb{N}$, invoking Lemma 4.2 we see that

$$\frac{p-1}{2p}\sum_{i=1}^{\ell}\|u_{k,i}\|_V^2 = c_k^{\phi_n} \le \hat{c}^{\phi_n} < \ell n \mathfrak{c}_\infty \quad \text{for all } k\in\mathbb{N}. \tag{5-1}$$

After passing to a subsequence, we have that $u_{k,i} \rightharpoonup u_{0,i}$ weakly in $H^1(\mathbb{R}^N)$, $u_{k,i}\to u_{0,i}$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_{k,i}\to u_{0,i}$ a.e. in $\mathbb{R}^N$, for each $i=1,\dots,\ell$. Hence, $u_{0,i}\ge 0$ and $\boldsymbol{u}_0=(u_{0,1},\dots,u_{0,\ell})\in\mathcal{H}^{\phi_n}$.

We claim that

$$u_{0,i}\neq 0 \quad \text{for all } i=1,\dots,\ell.$$

To prove this claim assume, arguing by contradiction, that $u_{0,i}=0$. Following the argument in the proof of Theorem 3.4 we see that, after passing to a subsequence, there exist $\xi_k\in\mathbb{R}^N$, $C_0>0$ and $\delta>0$ such that

$$\int_{B_{C_0+1}(\xi_k)} |u_{k,i}|^{2p} > \delta > 0 \quad \text{for all } k\in\mathbb{N}, \tag{5-2}$$

where either $\xi_k=0$, or $\xi_k=(\zeta_k,0)\in\mathbb{C}\times\mathbb{R}^{N-2}$ and $\zeta_k\to\infty$. Since $u_{k,i}\to 0$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, (5-2) implies that $\xi_k\neq 0$. Now, as in Case 2 of Theorem 3.4, we set

$$w_{k,i}(x) := u_{k,i}(x+\xi_k), \quad i=1,\dots,\ell,$$

and we take a subsequence satisfying $w_{k,i}\rightharpoonup w_i$ weakly in $H^1(\mathbb{R}^N)$, $w_{k,i}\to w_i$ in $L^{2p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $w_{k,i}\to w_i$ a.e. in $\mathbb{R}^N$. Hence, $w_i\in H^1(\mathbb{R}^N)^{O(N-2)}$, $w_i\ge 0$ and following the proof of (3-3) we obtain

$$\|u_{k,i}\|_V^2 = \Big\|u_{k,i} - \sum_{j=0}^{n-1}(w_i\circ\alpha^{-j})(\cdot-\alpha^j\xi_k)\Big\|_V^2 + n\|w_i\|_{V_\infty}^2 + o(1). \tag{5-3}$$

Furthermore, following the proof of (3-4) we derive

$$\int_{\mathbb{R}^N}(\nabla w_i\cdot\nabla v + V_\infty w_i v) = \int_{\mathbb{R}^N}|w_i|^{2p-2}w_i v + \beta_k\sum_{\substack{j=1\\ j\neq i}}^{\ell}\int_{\mathbb{R}^N}|w_j|^p|w_i|^{p-2}w_i v$$

for every $v\in H^1(\mathbb{R}^N)^{O(N-2)}$, and taking $v=w_i$ we get

$$\|w_i\|_{V_\infty}^2 = \int_{\mathbb{R}^N}|w_i|^{2p} + \beta_k\sum_{\substack{j=1\\ j\neq i}}^{\ell}\int_{\mathbb{R}^N}|w_j|^p|w_i|^p \le \int_{\mathbb{R}^N}|w_i|^{2p}.$$

Since, by (5-2),

$$\int_{B_{C_0+1}(0)}|w_{k,i}|^{2p} \ge \int_{B_{C_0+1}(\xi_k)}|u_{k,i}|^{2p} \ge \delta > 0,$$

we see that $w_i \neq 0$. Hence, there exists $t \in (0,1]$ such that $\|tw_i\|^2_{V_\infty} = \int_{\mathbb{R}^N} |tw_i|^{2p}$, and (5-3) yields

$$n\mathfrak{c}_\infty \leq n\,\frac{p-1}{2p}\|tw_i\|^2_{V_\infty} \leq n\,\frac{p-1}{2p}\|w_i\|^2_{V_\infty} \leq \frac{p-1}{2p}\|u_{k,i}\|^2_V.$$

As a consequence,

$$\ell n\mathfrak{c}_\infty \leq \frac{p-1}{2p}\sum_{i=1}^{\ell}\|u_{k,i}\|^2_V,$$

contradicting (5-1). This shows that $u_{0,i} \neq 0$, as claimed.

As $(\mathcal{J}_k^{\phi_n})'(\boldsymbol{u}_k) = 0,\ u_{k,i} \geq 0,\ u_{0,i} \geq 0$ and $\beta_k < 0$, we have that

$$\begin{aligned}\langle u_{k,i}, u_{0,i}\rangle_V &= \int_{\mathbb{R}^N} |u_{k,i}|^{2p-2}u_{k,i}u_{0,i} + \beta_k \sum_{\substack{j=1\\ j\neq i}}^{\ell}\int_{\mathbb{R}^N}|u_{k,j}|^p|u_{k,i}|^{p-2}u_{k,i}u_{0,i}\\ &\leq \int_{\mathbb{R}^N}|u_{k,i}|^{2p-2}u_{k,i}u_{0,i},\end{aligned}$$

and passing to the limit we obtain $\|u_{0,i}\|^2_V \leq |u_{0,i}|^{2p}_{2p}$. Hence, there exists $s \in (0,1]$ such that $\|su_{0,i}\|^2_V = |su_{0,i}|^{2p}_{2p}$ and we have that

$$\mathfrak{c}^{G_n} \leq \frac{p-1}{2p}\|su_{0,i}\|^2_V \leq \frac{p-1}{2p}\|u_{0,i}\|^2_V \leq \liminf_{k\to\infty}\frac{p-1}{2p}\|u_{k,i}\|^2_V, \tag{5-4}$$

with $\mathfrak{c}^{G_n}$ as in (2-4). We claim that these are equalities.

To prove this claim, let $v_k \in \mathcal{M}^{G_n}$ be such that $J(v_k) = ((p-1)/(2p))\|v_k\|^2_V \to \mathfrak{c}^{G_n}$. Set $u_{k,1} := v_k$ and define $u_{k,j+1}$ as in (1-3) for $j = 1, \dots, \ell-1$. Set $\boldsymbol{u}_k = (u_{k,1}, \dots, u_{k,\ell})$. Because (v_k) is bounded in $H^1(\mathbb{R}^N)$ and $\beta_k \to 0$, we have that

$$\lim_{k\to\infty}\beta_k\int_{\mathbb{R}^N}|u_{k,j}|^p|u_{k,i}|^p = 0 \quad \text{for every } i, j,$$

so, by Proposition 3.1(d), for k large enough there exists $s_k \in (0,\infty)$ such that $s_k\boldsymbol{u}_k \in \mathcal{N}_k^{\phi_n}$ and $s_k \to 1$ as $k \to \infty$. Thus,

$$c_k^{\phi_n} \leq \mathcal{J}^{\phi_n}(s_k\boldsymbol{u}_k) = \frac{p-1}{2p}\sum_{i=1}^{\ell}\|s_ku_{k,i}\|^2_V = \frac{p-1}{2p}\ell s_k^2\|v_k\|^2_V \longrightarrow \ell\mathfrak{c}^{G_n}. \tag{5-5}$$

Combining (5-4) and (5-5) we see that $s = 1$, thus $u_{0,i} \in \mathcal{M}^{G_n}$, that $u_{k,i} \to u_{0,i}$ strongly in $H^1(\mathbb{R}^N)$ and that

$$J(u_{0,i}) = \mathfrak{c}^{G_n} = \frac{1}{\ell}c_k^{\phi_n} < n\mathfrak{c}_\infty.$$

This completes the proof. □

Now we turn to the case $\beta \to -\infty$. For the proof of Theorem 1.4 we need the following result.

Lemma 5.1. *Let $\beta_k < 0$ and $(u_{k,1}, \dots, u_{k,\ell})$ be a solution to (1-2) with $\beta = \beta_k$ such that $u_{k,i} \to u_{\infty,i}$ strongly in $H^1(\mathbb{R}^N)$ for every $i = 1, \dots, \ell$. Then $(u_{k,i})$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$.*

Proof. Let $s \geq 0$ and assume that $u_{k,i} \in L^{2(s+1)}(\mathbb{R}^N)$ for every $k \in \mathbb{N}$. Fix $L > 0$ and define $w_{k,i} := u_{k,i} \min\{u_{k,i}^s, L\}$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w_{k,i}|^2 &\leq (1+s) \int_{\mathbb{R}^N} \nabla u_{k,i} \cdot \nabla(u_{k,i} \min\{u_{k,i}^{2s}, L^2\}) \\ &= (1+s)\Big(\int_{\mathbb{R}^N} |u_{k,i}|^{2p-2} w_{k,i}^2 + \beta \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} |u_{k,j}|^p |u_{k,i}|^{p-2} w_{k,i}^2 - \int_{\mathbb{R}^N} V w_{k,i}^2 \Big) \\ &\leq (1+s) \int_{\mathbb{R}^N} |u_{k,i}|^{2p-2} w_{k,i}^2. \end{aligned} \tag{5-6}$$

On the other hand, for any $K > 0$ we have that

$$\begin{aligned}\int_{\mathbb{R}^N} |u_{k,i}|^{2p-2} w_{k,i}^2 &\leq \int_{\mathbb{R}^N} (|u_{k,i}|^{2p-2} - |u_{\infty,i}|^{2p-2}) w_{k,i}^2 + \int_{|u_{\infty,i}|^{2p-2} \geq K} |u_{\infty,i}|^{2p-2} w_{k,i}^2 + K \int_{\mathbb{R}^N} w_{k,i}^2 \\ &\leq \big| |u_{k,i}|^{2p-2} - |u_{\infty,i}|^{2p-2} \big|_{2p}^{2p-2} |w_{k,i}|_{2p}^2 + \Big(\int_{|u_{\infty,i}|^{2p-2} \geq K} |u_{\infty,i}|^{2p} \Big)^{(p-1)/p} |w_{k,i}|_{2p}^2 \\ &\qquad + K |w_{k,i}|_2^2. \end{aligned}$$

As $u_{k,i} \to u_{\infty,i}$ strongly in $H^1(\mathbb{R}^N)$, choosing $k_0 > 0$ and K sufficiently large, we get that

$$\int_{\mathbb{R}^N} |u_{k,i}|^{2p-2} w_{k,i}^2 \leq \tfrac{1}{2} |w_{k,i}|_{2p}^2 + K |w_{k,i}|_2^2 \quad \text{for every } k \geq k_0. \tag{5-7}$$

Because $H^1(\mathbb{R}^N)$ is continuously embedded into $L^{2p}(\mathbb{R}^N)$, we derive from (5-6) and (5-7) that, for every $k \in \mathbb{N}$,

$$|w_{k,i}|_{2p}^2 \leq K_s |w_{k,i}|_2^2,$$

for some constant K_s independent of L, and letting $L \to \infty$ we get

$$|u_{k,i}|_{2p(s+1)}^{2(s+1)} = |u_{k,i}^{s+1}|_{2p}^2 \leq K_s |u_{k,i}^{s+1}|_2^2 = K_s |u_{k,i}|_{2(s+1)}^{2(s+1)}.$$

As $(u_{k,i})$ is uniformly bounded in $L^2(\mathbb{R}^N)$, iterating this inequality starting with $s = 0$ and using interpolation, we conclude that $(u_{k,i})$ is uniformly bounded in $L^q(\mathbb{R}^N)$ for any $q \in [2, \infty)$ and each $i = 1, \dots, \ell$. This implies that

$$f_{k,i} := |u_{k,i}|^{2p-2} u_{k,i} + \beta \sum_{j \neq i} |u_{k,j}|^p |u_{k,i}|^{p-2} u_{k,i}$$

is uniformly bounded in $L^q(\mathbb{R}^N)$ for any $q \in [2, \infty)$. Then, by the Calderón–Zygmund inequality, $(u_{k,i})$ is uniformly bounded in $W^{2,q}(\mathbb{R}^N)$ for every $q \in [2, \infty)$, and choosing q large enough, we derive from the Sobolev embedding theorem that $(u_{k,i})$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$, as claimed. □

Proof of Theorem 1.4. (i) As before, we write $\mathcal{J}_k^{\phi_n}$ and $\mathcal{N}_k^{\phi_n}$ for the functional and the Nehari set associated to the system (1-2) with $\beta = \beta_k$, and set

$$c_k^{\phi_n} := \inf_{\mathcal{N}_k^{\phi_n}} \mathcal{J}_k^{\phi_n}.$$

Arguing as in the proof of Theorem 1.3, we see that, after passing to a subsequence, $u_{k,i} \rightharpoonup u_{\infty,i}$ weakly in $H^1(\mathbb{R}^N)$, $u_{k,i} \to u_{\infty,i}$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_{k,i} \to u_{\infty,i}$ a.e. in $\mathbb{R}^N$, for each $i = 1, \dots, \ell$. Hence, $u_{\infty,i} \geq 0$ and $\boldsymbol{u}_\infty = (u_{\infty,1}, \dots, u_{\infty,\ell}) \in \mathcal{H}^{\phi_n}$, so $\boldsymbol{u}_\infty$ satisfies (1-3). We also get that

$$u_{\infty,i} \neq 0 \quad \text{and} \quad \|u_{\infty,i}\|_V^2 \leq |u_{\infty,i}|_{2p}^{2p} \qquad \text{for all } \ i = 1, \dots, \ell.$$

Furthermore, as $(\mathcal{J}_k^{\phi_n})'(\boldsymbol{u}_k) = 0$, we have that, for every $j \neq i$,

$$0 \leq \int_{\mathbb{R}^N} |u_{k,j}|^p |u_{k,i}|^p \leq \frac{|u_{k,i}|_{2p}^{2p}}{-\beta_k} \leq \frac{C}{-\beta_k}.$$

As $\beta_k \to -\infty$, passing to the limit and using Fatou's lemma, we obtain

$$0 \leq \int_{\mathbb{R}^N} |u_{\infty,j}|^p |u_{\infty,i}|^p \leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} |u_{k,j}|^p |k_{n,i}|^p = 0.$$

This implies that $u_{\infty,j} u_{\infty,i} = 0$ a.e. in $\mathbb{R}^N$ whenever $i \neq j$.

Let $s \in (0, 1]$ be such that $\|su_{\infty,i}\|_V^2 = |su_{\infty,i}|_{2p}^{2p}$. Then $s\boldsymbol{u}_\infty \in \mathcal{W}_\ell^n$, and using (5-1) we get

$$\hat{c}^{\phi_n} \leq \frac{p-1}{2p} \sum_{i=1}^{\ell} \|su_{\infty,i}\|_V^2 \leq \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_{\infty,i}\|_V^2 \leq \frac{p-1}{2p} \sum_{i=1}^{\ell} \liminf_{k\to\infty} \|u_{k,i}\|_V^2 \leq \hat{c}^{\phi_n}.$$

This proves that $s = 1$, $\boldsymbol{u}_\infty \in \mathcal{W}_\ell^n$, $u_{k,i} \to u_{\infty,i}$ strongly in $H^1(\mathbb{R}^N)$ and

$$\hat{c}^{\phi_n} = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_{\infty,i}\|_V^2. \tag{5-8}$$

Finally, as $\lim_{k\to\infty} \|u_{k,i}\|_V^2 = \|u_{\infty,i}\|_V^2 = |u_{\infty,i}|_{2p}^{2p} = \lim_{k\to\infty} |u_{k,i}|_{2p}^{2p}$, from

$$\lim_{k\to\infty} \|u_{k,i}\|_V^2 = \lim_{k\to\infty} |u_{k,i}|_{2p}^{2p} + \lim_{k\to\infty} \beta_k \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} |u_{k,j}|^p |u_{k,i}|^p,$$

we obtain

$$\int_{\mathbb{R}^N} \beta_k u_{k,j}^p u_{k,i}^p \to 0 \quad \text{as } k \to \infty \qquad \text{whenever } i \neq j.$$

(ii) It follows from Lemma 5.1 and [Clapp et al. 2021b, Theorem B.2] that $(u_{k,i})$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(K)$ for each compact subset K of $\mathbb{R}^N$ and $\alpha \in (0, 1)$. So from the Arzelà-Ascoli theorem we get that $u_{\infty,i} \in \mathcal{C}^0(\mathbb{R}^N)$. Therefore $\Omega_i := \{x \in \mathbb{R}^N : u_{\infty,i}(x) > 0\}$ is open. Because $u_{\infty,i} u_{\infty,j} = 0$ if $i \neq j$ and $\boldsymbol{u}_\infty$ satisfies (1-3), we have that $\Omega_i \cap \Omega_j = \varnothing$ if $i \neq j$ and the ℓ-tuple $(\Omega_1, \dots, \Omega_\ell)$ satisfies (S_1) and (S_2). Thus, it is an (n, ℓ)-pinwheel partition.

Since $\boldsymbol{u}_\infty \in \mathcal{W}_\ell^n$, we have that $u_{\infty,i}$ belongs to the Nehari manifold $\mathcal{M}_{\Omega_i}$ defined in (1-7). Therefore, $((p-1)/(2p))\|u_{\infty,i}\|_V^2 \geq \mathfrak{c}_{\Omega_i}$. Equality must hold true as, otherwise, there would exist $v_1 \in \mathcal{M}_{\Omega_1}$ such that $((p-1)/(2p))\|u_{\infty,1}\|_V^2 > ((p-1)/(2p))\|v_1\|_V^2 \geq \mathfrak{c}_{\Omega_i}$, and defining v_{j+1} as in (1-3), we would have that $(v_1, \dots, v_\ell) \in \mathcal{W}_\ell^n$ and

$$\frac{p-1}{2p}\sum_{i=1}^{\ell}\|v_i\|_V^2 < \frac{p-1}{2p}\sum_{i=1}^{\ell}\|u_{\infty,i}\|_V^2 = \hat{c}^{\phi_n}$$

by (5-8), which is a contradiction. This shows that $u_{\infty,i}$ is a least energy solution of (1-8) in Ω_i. Now, since $((p-1)/(2p))\|u_{\infty,i}\|_V^2 = \mathfrak{c}_{\Omega_i}$, we get

$$\inf_{(\Theta_1,\dots,\Theta_\ell)\in\mathcal{P}_\ell^n}\sum_{j=1}^{\ell}\mathfrak{c}_{\Theta_j} \leq \sum_{j=1}^{\ell}\mathfrak{c}_{\Omega_j} = \hat{c}^{\phi_n} \leq \inf_{(\Theta_1,\dots,\Theta_\ell)\in\mathcal{P}_\ell^n}\sum_{j=1}^{\ell}\mathfrak{c}_{\Theta_j}.$$

This shows that $(u_{\infty,1}, \dots, u_{\infty,\ell})$ is an optimal (n,ℓ)-pinwheel partition.

(iii) This is a local statement. Recall that $(u_{k,i})$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(\Omega)$ for each open subset Ω compactly contained in $\mathbb{R}^N$ and $\alpha \in (0,1)$. So, from the Arzelà-Ascoli theorem, we get that $u_{k,i} \to u_{\infty,i}$ in $\mathcal{C}^{0,\alpha}(\Omega)$. Thus, all hypotheses of [Clapp et al. 2021b, Theorem C.1] are satisfied and (iii) follows.

(iv) Let G_{2n} be the group defined in Example 2.1 with $\ell = 2$, and let $\tau_n : G_{2n} \to \mathbb{Z}_2 := \{1, -1\}$ be the homomorphism given by $\tau_n(\mathrm{e}^{2\pi\mathrm{i}/2n}) = -1$ and $\tau_n(\theta) = 1$ for every $\theta \in O(N-2)$. A solution to the Schrödinger equation (1-5) satisfying

$$u(gx) = \tau_n(g)u(x) \quad \text{for all } g \in G_{2n},\ x \in \mathbb{R}^N \tag{5-9}$$

is a critical point of the functional $J : H^1(\mathbb{R}^N)^{\tau_n} \to \mathbb{R}$ defined by (2-3) on the space

$$H^1(\mathbb{R}^N)^{\tau_n} := \{u \in H^1(\mathbb{R}^N) : u \text{ satisfies (5-9)}\}.$$

The nontrivial ones belong to the Nehari manifold

$$\mathcal{M}^{\tau_n} := \{u \in H^1(\mathbb{R}^N)^{\tau_n} : u \neq 0,\ \|u\|_V^2 = |u|_{2p}^{2p}\},$$

which is a natural constraint for J. Note that every nontrivial function satisfying (5-9) is nonradial and changes sign.

There is a one-to-one correspondence

$$\mathcal{W}_2^n \to \mathcal{M}^{\tau_n}, \qquad (u_1, u_2) \mapsto u_1 - u_2,$$

whose inverse is $u \mapsto (u^+, -u^-)$, with $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$, satisfying

$$\frac{p-1}{2p}(\|u_1\|_V^2 + \|u_2\|_V^2) = J(u_1 - u_2).$$

Therefore,

$$J(u_{\infty,1} - u_{\infty,2}) = \frac{p-1}{2p}(\|u_{\infty,1}\|_V^2 + \|u_{\infty,2}\|_V^2) = \inf_{(u_1,u_2)\in\mathcal{W}_2^n}\frac{p-1}{2p}(\|u_1\|_V^2 + \|u_2\|_V^2) = \inf_{u\in\mathcal{M}^{\tau_n}} J(u).$$

This shows that $u_{\infty,1} - u_{\infty,2}$ is a least energy solution to (1-5) and (5-9). □

Appendix: An auxiliary result

Lemma A.1. *Assume $v_k \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, $\xi_k \in \mathbb{R}^N$ satisfies $|\xi_k| \to \infty$ and $V \in \mathcal{C}^0(\mathbb{R}^N)$ satisfies assumption (V_2). Set $V_k(x) := V(x+\xi_k)$. Then*

$$\lim_{k\to\infty} \|v_k\|_{V_k}^2 - \lim_{k\to\infty} \|v_k - v\|_{V_k}^2 = \|v\|_{V_\infty}^2.$$

Proof. As $v_k \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, one has

$$\begin{aligned}\|v\|_{V_\infty}^2 + o(1) &= \|v_k\|_{V_\infty}^2 - \|v_k - v\|_{V_\infty}^2 \\ &= \|v_k\|_{V_k}^2 - \|v_k - v\|_{V_k}^2 + 2\int_{\mathbb{R}^N} (V_\infty - V_k) v_k v - \int_{\mathbb{R}^N} (V_\infty - V_k) v^2.\end{aligned}$$

Given $\varepsilon > 0$, choose $R > 0$ large enough that

$$\int_{\mathbb{R}^N \smallsetminus B_R} |V_\infty - V_k||v|^2 \le 2 \sup_{x\in\mathbb{R}^N} V(x) \int_{\mathbb{R}^N \smallsetminus B_R} |v|^2 < \tfrac{1}{2}\varepsilon.$$

Now take k_0 such that

$$|V_\infty - V(x+\xi_k)| < \frac{\varepsilon}{2|v|_2^2} =: \delta \quad \text{for every } x \in B_R \text{ and } k \ge k_0.$$

Then, for $k \ge k_0$,

$$\int_{\mathbb{R}^N} |V_\infty - V_k||v|^2 \le \int_{B_R} |V_\infty - V_k||v|^2 + \int_{\mathbb{R}^N \smallsetminus B_R} |V_\infty - V_k||v|^2 < \varepsilon$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} |(V_\infty - V_k) v_k v| &\le \Big(\int_{\mathbb{R}^N} |V_\infty - V_k||v_k|^2\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |V_\infty - V_k||v|^2\Big)^{\frac{1}{2}} \\ &\le C\sqrt{\varepsilon}.\end{aligned}$$

This completes the proof. □

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ORBITA MATHEMATICAE

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Subspaces fixed by a nilpotent matrix 1
Marvin Anas Hahn, Gabriele Nebe, Mima Stanojkovski and Bernd Sturmfels

Pinwheel solutions to Schrödinger systems 17
Mónica Clapp and Angela Pistoia

Counting compatible indexing systems for C_{p^n} 37
Michael A. Hill, Jiayun Meng and Nan Li

Games for the two membranes problem 59
Alfredo Miranda and Julio D. Rossi

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y el Caribe