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Counting compatible indexing systems for C_{p^n}

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We count the number of compatible pairs of indexing systems for the cyclic group C_{p^n}. Building on work of Balchin, Barnes, and Roitzheim, we show that this sequence of natural numbers is another family of Fuss–Catalan numbers. We count this two different ways: showing how the conditions of compatibility give natural recursive formulas for the number of admissible sets and using an enumeration of ways to extend indexing systems by conceptually simpler pieces.

1. Introduction

Recent work in equivariant algebra has studied the beautiful variety of different multiplicative structures that can arise when we mix in the action of a finite group. Blumberg and Hill [2015] showed that this is a fundamentally combinatorial structure. The various multiplicative norms or additive transfers are entirely governed by certain particularly well-behaved subcategories of finite G-sets.

This classification was reformulated in independent work of Balchin, Barnes, and Roitzheim [Balchin et al. 2021] and of Rubin [2021], in which they further underscored the combinatorial structure by showing the norms and transfers are encoded in "transfer systems", certain refinements of the poset of subgroups of G under inclusion.

Definition 1.1 [Rubin 2021, Definition 3.4; Balchin et al. 2021, Definition 7]. A transfer system for a finite group G is a partial order $\to$ relation on the set $\mathrm{Sub}(G)$ of subsets of G such that

(1) if $K \to H$, then $K \leq H$ (so this is a "weak subposet"),

(2) if $K \to H$ and $g \in G$, then $gKg^{-1} \to gHg^{-1}$, and

(3) if $K \to H$ and $J \leq H$, then $K \cap J \to J$.

Let $\mathrm{Tran}(G)$ denote the set of transfer systems for G.

The set of all transfer systems for G itself has a partial order: we say $\mathcal{O} \leq \mathcal{O}'$ if the identity map is order-preserving.

Example 1.2. For $G = C_{p^n}$, the subgroup lattice is order-isomorphic to the linear order

$$\{1 \leq 2 \cdots \leq n+1\} = [n+1].$$

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The conjugation condition is always satisfied, and the restriction condition here can be rephrased as saying that if $C_{p^i} \to C_{p^j}$ and $i \le k \le j$, then $C_{p^i} \to C_{p^k}$.

Balchin, Barnes, and Roitzheim [Balchin et al. 2021, Theorem 25] showed that the poset of transfer systems for C_{p^n} is order-isomorphic to the Tamari lattice.

In this paper, we will focus on the groups C_{p^n}. Example 1.2 stresses that we could equivalently look at "transfer systems for the poset $[n+1]$" in the sense of Franchere, Ormsby, Osorno, Qin, and Waugh.

Definition 1.3 [Franchere et al. 2022, Definition 4.1]. A transfer system on

$$[n] = \{1 \le \cdots \le n\}$$

is a weak subposet of $[n]$ with partial order $\to$ that contains all the elements and which satisfies the "restriction condition": if $i \to j$ and $i \le k \le j$, then $i \to k$.

It is helpful to view these as a graded set in which we allow n to vary, as this will help encode certain natural operations.

Definition 1.4. For each $n \in \mathbb{N}$, let

$$T_n = \begin{cases} \mathrm{Tran}([n]) & n \ge 1, \\ \{\varnothing\} & n = 0. \end{cases}$$

Balchin, Barnes, and Roitzheim counted the number of transfer systems for C_{p^n}:

Theorem 1.5 [Balchin et al. 2021, Theorem 20]. *For each $n \in \mathbb{N}$,*

$$|T_n| = \frac{1}{2n+1}\binom{2n+1}{n} = \mathrm{Cat}(n).$$

The numbers $\mathrm{Cat}(n)$ are the Catalan numbers. These are ubiquitous in combinatorics, parametrizing structures from binary rooted trees to Dyck paths. This sequence fits into a bivariant family of sequences.

Definition 1.6. The *Fuss–Catalan numbers* are defined by

$$A_n(p, r) = \frac{r}{np+r}\binom{np+r}{n}$$

for nonnegative n and positive p and r.

The Catalan numbers arise here:

$$\mathrm{Cat}(n) = A_n(2, 1).$$

In this paper, we show the next term of the sequence is also related to equivariant algebra.

Blumberg and Hill [2022] studied what kind of compatibility conditions arise if we allow both the additive transfers and multiplicative norms to each be structured by various transfer systems. As a slogan, "the presence of certain multiplicative norms forces some additive transfers". These conditions were simplified by Chan [2022], who gave a definition internal to transfer systems. When the conditions are satisfied, we say that $(\mathcal{O}_a, \mathcal{O}_m)$ is compatible. Counting these for C_{p^n} is the main result of this paper.

Theorem 1.7. *For $C_{p^{n-1}}$, there are*

$$A_n(3, 1)$$

compatible pairs of transfer systems.

We present two proofs of this theorem in Sections 4 and 5, each of which underscores a different combinatorial feature of the number of compatible pairs. The mathematics in this paper arose from an REU project in Summer 2021. The two junior coauthors each came up with a distinct solution to this counting problem, so we include here both of their solutions.

A third proof by Henry Ma will appear separately. This proof uses a direct bijection between compatible pairs of transfer systems and Kreweras pairs. Recent work of Balchin, MacBrough, and Ormsby [Balchin et al. 2022] shows that Kreweras pairs are equinumerous with composition-closed premodel structures on the poset $[n]$. We do not see a direct connection between compatible pairs and composition-closed premodel structures, however.

2. Operations on transfer systems

Concatenation. A key piece of structure on transfer systems for $[n]$ is the ability to "concatenate" a transfer system for $[k]$ and one for $[n-k]$ to build one for $[n]$. This makes computations with the entire graded set easier to understand.

Definition 2.1. Let $\mathcal{O}_L$ be a transfer system for $[k]$ and $\mathcal{O}_R$ be a transfer system for $[n-k]$. Then we define a relation $\mathcal{O} = \mathcal{O}_L \oplus \mathcal{O}_R$ on $[n]$ by saying $i \xrightarrow{\mathcal{O}} j$ if and only if

(1) $j \leq k$ and $i \xrightarrow{\mathcal{O}_L} j$ or

(2) $i \geq (k+1)$ and $(i-k) \xrightarrow{\mathcal{O}_R} (j-k)$.

We call $\mathcal{O}$ the *direct sum* or *concatenation* of $\mathcal{O}_L$ and $\mathcal{O}_R$.

Remark 2.2. As a poset, this is just the disjoint union of the two posets $\mathcal{O}_L$ and $\mathcal{O}_R$. The additional data is the map to $[n]$ in this case.

Proposition 2.3. *The concatenation of transfer systems is a transfer system.*

Proof. The definition of $\mathcal{O}$ is that of the disjoint union of posets, so $\mathcal{O}$ is a poset, and it visibly maps to the usual inclusions.

We need only check the restriction condition, and here, it suffices to check the restriction of $i+k \to j+k$ along $m \leq j+k$ for some $m \leq k$. In this case, both the source and target are m. □

The direct sum is a graded operation here:

$$T_n \times T_m \xrightarrow{\oplus} T_{n+m}.$$

We extend this to T_0 by declaring

$$\varnothing \oplus \mathcal{O} = \mathcal{O} \oplus \varnothing = \mathcal{O}$$

for any transfer system $\mathcal{O}$ or the empty set.

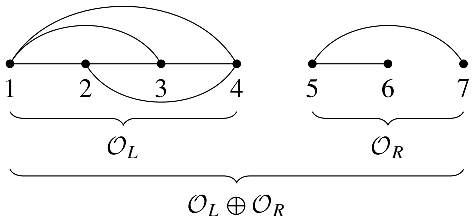

Figure 1. Example of concatenation of $\mathcal{O}_L$ and $\mathcal{O}_R$.

Restriction. Given a transfer system for $[n]$, we have two natural ways to build transfer systems for smaller natural numbers. These both arise from the inclusion of subposets.

Definition 2.4. If $k \leq n$, then let

$$\iota : [k] \to [n]$$

be the map sending $i \to i$, and let

$$\phi : [n-k] \to [n]$$

be the map sending i to $i+k$.

We can restrict a transfer system on $[n]$ along either of these inclusions.

Definition 2.5. Let $\mathcal{O}$ be a transfer system for $[n]$, and let $k \leq n$.

Let $i_k^*\mathcal{O}$ be defined by saying for all $i \leq j \leq k$,

$$i \xrightarrow{i_k^*\mathcal{O}} j \quad \text{if and only if} \quad i \xrightarrow{\mathcal{O}} j.$$

Let $\Phi^k\mathcal{O}$ be defined by saying for all $i \leq j \leq (n-k)$,

$$i \xrightarrow{\Phi^k\mathcal{O}} j \quad \text{if and only if} \quad (i+k) \xrightarrow{\mathcal{O}} (j+k).$$

Proposition 2.6. *For a transfer system on $\mathcal{O}$, $i_k^*\mathcal{O}$ is a transfer system on $[k]$ and $\Phi^k\mathcal{O}$ is a transfer system on $[n-k]$.*

Proof. The construction gives wide, weak subposets of $[k]$ and $[n-k]$ respectively, by observation. Since the two inclusions ι and ϕ are interval inclusions, the restriction condition is easily checked. $\square$

The two restrictions can be visualized as simply throwing away any transfers that start or end outside of the given range. An example picture of this is shown in Figure 2.

Remark 2.7. The maps i_k^* and Φ^k have topological meaning. Recall that the subgroup lattice of $C_{p^{n-1}}$ is isomorphic to $[n]$ via the map that sends a subgroup H to $\log_p(|H|)+1$. The inclusion of the subgroups of $C_{p^{k-1}}$ into those of $C_{p^{n-1}}$ corresponds via this identification to the inclusion $\iota : [k] \hookrightarrow [n]$. Similarly, the subgroups of $C_{p^{n-1}}$ which contain C_{p^k} correspond to the inclusion $\phi : [n-k] \hookrightarrow [n]$.

Using these, the map i_k^* on transfer systems is just the ordinary restriction of transfer systems for $C_{p^{n-1}}$ to transfer systems for the subgroup $C_{p^{k-1}}$. The map Φ^k also has a direct topological meaning. Blumberg and Hill [2015, Lemma B.1] showed that for any N_∞-operad $\mathcal{O}$ for G and for any normal subgroup N

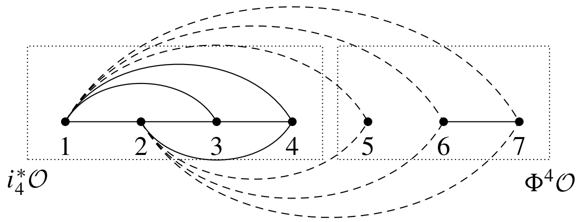

Figure 2. Example of $i_k^*\mathcal{O}$ and $\Phi^k\mathcal{O}$. The dashed transfers are forgotten to form $i_4^*\mathcal{O}$ and $\Phi^4\mathcal{O}$.

of G, the N-fixed points of $\mathcal{O}$ are an N_∞-operad for G/N. The corresponding operation on transfer systems is Φ^k.

Wrapped and saturated systems. We single out two families of transfer systems useful for our counts.

Definition 2.8. A transfer system $\mathcal{O}$ for $[n]$ is *wrapped* if $1 \xrightarrow{\mathcal{O}} n$.

Example 2.9. The transfer systems $\mathcal{O}_L$ and $\mathcal{O}_R$ from Figure 1 are wrapped; the transfer system $\mathcal{O}_L \oplus \mathcal{O}_R$ is not.

Example 2.10. The transfer system $\mathcal{O}$ in Figure 2 is wrapped, as is $i_4^*\mathcal{O}$. The transfer system $\Phi^4\mathcal{O}$ is not.

By the restriction condition, in a wrapped transfer system, we have $1 \to j$ for all j, and we think of the largest transfer as "wrapping" the rest of the transfer system. Additionally, we can always "wrap" any given transfer system, using the circle-dot product of [Balchin et al. 2021]. We use a single case of their construction here.

Definition 2.11. If $\mathcal{O} \in T_n$, let $w(\mathcal{O})$ be the transfer system for $[n+1]$ with

(1) for all j, $1 \to j$ in $w(\mathcal{O})$, and

(2) $\Phi^1(w(\mathcal{O})) = \mathcal{O}$.

This is implicit in Balchin, Barnes, and Roitzheim's "circle-dot product" and their decomposition theorem.

Theorem 2.12 [Balchin et al. 2021, Corollary 21]. *Any transfer system for* $[n]$ *with* $n \geq 1$ *decomposes uniquely as*

$$\mathcal{O} \odot \mathcal{O}' = \mathcal{O} \oplus w(\mathcal{O}')$$

for some transfer systems $\mathcal{O} \in T_m$ *and* $\mathcal{O}' \in T_{n-m-1}$.

Corollary 2.13. *Any transfer system* $\mathcal{O}$ *for* $[n]$ *can be written uniquely as*

$$\mathcal{O} = w(\mathcal{O}_1) \oplus \cdots \oplus w(\mathcal{O}_k)$$

for some transfer systems $\mathcal{O}_1, \ldots, \mathcal{O}_k$.

A special case of wrapped transfer systems is given by complete ones.

Definition 2.14. The *complete* transfer system for $[n]$ is the one for which $i \to j$ for all $i \le j$. Denote it by $\mathcal{O}_n^{\mathrm{cpt}}$.

Remark 2.15. The complete transfer system for $[n]$ is one for which the partial order is just the inclusion. This means that complete transfer systems are maximal elements in the poset of transfer systems.

These complete transfer systems will be especially useful building blocks for us.

Definition 2.16 [Rubin 2021, Definition 3.4]. A transfer system $\mathcal{O}$ is *saturated* if whenever $i \to j$ and $i \le k \le j$, then $k \to j$.

Remark 2.17. The saturation condition is equivalent to a 2-out-of-3 property for the transfer system, since $i \to j$ with $i \le k \le j$ always implies $i \to k$ by the restriction condition. This formulation has been used by Hafeez, Marcus, Ormsby, and Osorno [Hafeez et al. 2022, Definition 2.5] in their study of the saturation conjecture.

Proposition 2.18. *A saturated transfer system on* $[n]$ *is a direct sum of complete ones.*

Proof. Complete transfer systems and direct sums of these are visibly saturated. For the other direction, write $\mathcal{O}$ as a direct sum of wrapped transfer systems:

$$\mathcal{O} = \mathcal{O}_{n_1} \oplus \cdots \oplus \mathcal{O}_{n_k}.$$

The transfer system $\mathcal{O}$ is saturated if and only if each of the pieces is, since there are no transfers connecting the individual summands. It suffices to then show that a wrapped saturated transfer system is complete. This however follows by downward induction on n. □

Given any transfer system on $[n]$, there is a minimal saturated transfer system that contains it. This is immediate from the observation that the intersection of two saturated transfer systems is again a saturated transfer system. For concreteness, we spell this out directly here.

Proposition 2.19. *If* $\mathcal{O}$ *is any transfer system, then there is a minimal saturated transfer system that contains* $\mathcal{O}$.

Proof. Write $\mathcal{O}$ as a direct sum of wrapped transfer systems $\mathcal{O} = \mathcal{O}_{n_1} \oplus \cdots \oplus \mathcal{O}_{n_r}$, with $\mathcal{O}_{n_i} \in T_{n_i}$. Then the minimal saturated transfer system containing it is simply

$$\mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_r}^{\mathrm{cpt}},$$

where we replace each summand with the complete one of that size. □

Definition 2.20. If $\mathcal{O}$ is a transfer system for $[n]$, let $\mathrm{hull}(\mathcal{O})$ denote the minimal saturated transfer system that contains it. This is the *saturated hull*.

For transfer systems for $[n]$, there is a kind of dual notion of a maximal saturated transfer systems inside any given transfer system.

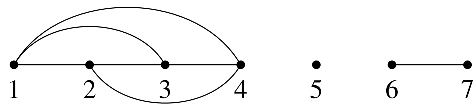

Figure 3. Core of $\mathcal{O}$ from Figure 2.

Complete transfer systems have two useful properties:

(1) Whenever $i \to j$, we also have $k \to j$ for all $i \leq k \leq j$.

(2) They are generated as a poset as the transitive closure of the relation $i \to i+1$ for all $1 \leq i \leq n-1$.

These two properties give us two different ways to repackage the condition of being saturated.

Proposition 2.21. *A transfer system $\mathcal{O}$ for n is saturated if and only if whenever $i \to j$ with $i < j$, we have $(j-1) \to j$.*

Proof. If $\mathcal{O}$ is saturated, then, by definition of the direct sum, if $i \to j$, then i and j correspond to subgroups from the same direct summand. By completeness, we therefore have all intermediate transfers.

Using the decomposition of a transfer system into wrapped ones (Corollary 2.13), we see that it suffices to show that if a wrapped transfer system that has the property that $i \to j$ implies $(j-1) \to j$, then the transfer system is complete. This follows from downward induction on n, using that $1 \to n$ by the wrapped assumption, and hence $1 \to j$ for all j by restriction. □

Corollary 2.22. *A transfer system $\mathcal{O}$ is saturated if and only if it is generated as a partial order by relations $i \to (i+1)$ for some collection of positive i at most $n-1$.*

The possibly surprising part here is that we only need the partial order: the other parts of being a transfer system come along for free in this case, since we are generating by a covering condition. This gives us a second kind of structural result.

Definition 2.23. If $\mathcal{O}$ is a transfer system, let the *core* of $\mathcal{O}$, denoted by $\operatorname{core}\mathcal{O}$, be the partial order generated by $i \to (i+1)$, where i ranges over the integers from 1 to $n-1$ such that $i \to (i+1)$.

Example 2.24. In Figure 3, we see the core of the transfer system $\mathcal{O}$ from Figure 2.

3. Compatible pairs

Our main object of study is the notion of compatible pairs. Blumberg and Hill [2022] defined these to describe "compatibility" between equivariant norms and transfers in an abstract, categorical way. Chan [2022, Theorem 4.10] reformulated this in the language of transfer systems, giving a purely combinatorial formulation. We use that here for the special case of C_{p^n}.

Notation 3.1. Given a pair of transfer systems $(\mathcal{O}_a, \mathcal{O}_m)$, let $i \xrightarrow{a} j$ be $i \xrightarrow{\mathcal{O}_a} j$, and similarly for $\xrightarrow{m}$.

Definition 3.2 [Chan 2022, Definition 4.6]. A pair of transfer systems $(\mathcal{O}_a, \mathcal{O}_m)$ for $[n]$ is compatible if for whenever $i \xrightarrow{m} j$, we have $k \xrightarrow{a} j$ for all $i \leq k \leq j$.

Note here that the conditions are asymmetrical: arrows in $\mathcal{O}_m$ force those in $\mathcal{O}_a$. Moreover, this is a kind of relative saturation condition, with an arrow in $\mathcal{O}_m$ actually forcing $\mathcal{O}_a$ to look saturated in a range. This gives us two equivalent forms.

Proposition 3.3. *A pair $(\mathcal{O}_a, \mathcal{O}_m)$ is compatible if and only if the following equivalent comparisons hold*:

(1) $\mathcal{O}_m \leq \mathrm{core}(\mathcal{O}_a)$,

(2) $\mathrm{hull}(\mathcal{O}_m) \leq \mathcal{O}_a$, *and*

(3) $\mathrm{hull}(\mathcal{O}_m) \leq \mathrm{core}(\mathcal{O}_a)$.

Proof. The conditions of Definition 3.2 are a restatement of the condition that the saturated hull of $\mathcal{O}_m$ is less than or equal to $\mathcal{O}_a$. For the equivalence of the three conditions, we use that the core of $\mathcal{O}$ is the largest saturated transfer system less than or equal to $\mathcal{O}$ and the hull is the smallest saturated transfer system greater than or equal to $\mathcal{O}$. □

Corollary 3.4. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ be a compatible pair of transfer systems for $[n]$. If for some $1 \leq k \leq n-1$, we have $k \overset{a}{\nrightarrow} (k+1)$, then, for all $j \leq k$ and $\ell \geq (k+1)$, we must have $j \overset{m}{\nrightarrow} \ell$.*

Put another way, we see that $\mathcal{O}_m$ must break apart at k, and this must be compatible with $\mathcal{O}_a$.

Corollary 3.5. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ be a compatible pair of transfer systems for $[n]$. If for some $1 \leq k \leq (n-1)$, we have $k \overset{a}{\nrightarrow} (k+1)$, then*

(1) $\mathcal{O}_m = i_k^*\mathcal{O}_m \oplus \Phi^k\mathcal{O}_m$, *and*

(2) *the pairs*
$$(i_k^*\mathcal{O}_a, i_k^*\mathcal{O}_m) \quad \text{and} \quad (\Phi^k\mathcal{O}_a, \Phi^k\mathcal{O}_m)$$
are compatible.

Definition 3.6. Let
$$\mathcal{D}_n^{(0)} = \{(\mathcal{O}_a, \mathcal{O}_m) \mid \text{compatible}\} \subset T_n \times T_n.$$

We have projection maps
$$p_a, p_m : \mathcal{D}_n^{(0)} \to T_n$$
which take a pair $(\mathcal{O}_a, \mathcal{O}_m)$ to $\mathcal{O}_a$ or $\mathcal{O}_m$ respectively. Our main goal is to find the cardinality of $\mathcal{D}_n^{(0)}$ for all n. We solve this in several different ways using different aspects of Proposition 3.3.

4. Solving the recurrence relations

Decomposition and recurrence relation. The wrapping map w defines a natural filtration on the collection of transfer systems. We can use this to build a recursive relation describing compatible pairs.

Definition 4.1. For each $i \geq 0$, let
$$(\mathcal{F}^i T)_n = \mathrm{Im}(w^{\circ i}) \subset T_n,$$
viewed as a graded subset.

Definition 4.2. Let
$$\mathcal{D}_n^{(i)} = p_a^{-1}((\mathcal{F}^i T)_n)$$
be the set of composable pairs with $\mathcal{O}_a \in \mathcal{F}^i T$.

Let $d(n,i) = |\mathcal{D}_n^{(i)}|$ be the corresponding cardinality.

We deduce our recursive formulae from the Balchin, Barnes, and Roitzheim decomposition theorem (Theorem 2.12). We restate the result here to set up our decomposition.

Proposition 4.3. *If $\mathcal{O}$ is a transfer system in $(\mathcal{F}^i T)_n$, then there is*

(1) *a unique natural number $1 \le j \le n-i$,*

(2) *a unique wrapped transfer system $w\mathcal{O}_R$ in T_j, and*

(3) *a unique transfer system $\mathcal{O}_L$ in T_{n-i-j}*

such that
$$\mathcal{O} = w^i(\mathcal{O}_L \oplus w\mathcal{O}_R).$$

Notation 4.4. Let $(\mathcal{F}^i T)_{n,j}$ be the set of transfer systems in $(\mathcal{F}^i T)_n$ which decompose as
$$\mathcal{O} = w^i(\mathcal{O}_L \oplus w\mathcal{O}_R)$$
with $w\mathcal{O}_R \in T_j$ a wrapped transfer system.

Proposition 4.5. *The map*
$$\left(\coprod_{j=1}^{n-i-1} \mathcal{D}_{n-j}^{(i)} \times \mathcal{D}_j^{(1)}\right) \sqcup \mathcal{D}_n^{(i+1)} \to \mathcal{D}_n^{(i)}$$
given on $\mathcal{D}_{n-j}^{(i)} \times \mathcal{D}_j^{(1)}$ by
$$\big((w^i\mathcal{O}'_a, \mathcal{O}'_m), (w\mathcal{O}''_a, \mathcal{O}''_m)\big) \mapsto \big(w^i(\mathcal{O}'_a \oplus w\mathcal{O}''_a), \mathcal{O}'_m \oplus \mathcal{O}''_m\big),$$
and on the last summand by the natural inclusion, is a bijection.

Proof. We use Proposition 4.3 to further break up $\mathcal{D}_n^{(i)}$, since the decomposition here gives a disjoint union decomposition
$$(\mathcal{F}^i T)_n = \coprod_{j=1}^{n-i} (\mathcal{F}^i T)_{n,j}.$$
This decomposition induces a decomposition of $\mathcal{D}_n^{(i)}$:
$$\mathcal{D}_{n,j}^{(i)} = p_1^{-1}(\mathcal{F}^i T)_{n,j}.$$
Since $T_0 = \{\varnothing\}$, the unit for $\oplus$, we have
$$\mathcal{D}_{n,n-i}^{(i)} = \mathcal{D}_n^{(i+1)},$$
given by the usual inclusion. Now let $1 \le j < n-i$, and consider an element $(\mathcal{O}_a, \mathcal{O}_m)$ in $\mathcal{D}_{n,j}^{(i)}$. By definition, we have
$$\mathcal{O}_a = w^i(\mathcal{O}_{a,L} \oplus w\mathcal{O}_{a,R}),$$

with $\mathcal{O}_{a,L} \neq \varnothing$ and $w\mathcal{O}_{a,R}$ wrapped, and hence we are missing the transfer

$$(i+n-j) \to (i+n-j+1)$$

in $\mathcal{O}_a$. This means that $\mathcal{O}_m$ breaks up into a direct sum

$$\mathcal{O}'_m \oplus \mathcal{O}''_m,$$

where $\mathcal{O}'_m \in T_{i+n-j}$ and $\mathcal{O}''_m \in T_j$, by Corollary 3.5. Moreover, we know that the pairs

$$(i^*_{i+n-j}\mathcal{O}_a, \mathcal{O}'_m) \quad \text{and} \quad (\Phi^{i+n-j}\mathcal{O}_a, \mathcal{O}''_m)$$

are compatible. The result follows, since

$$i^*_{i+n-j}\mathcal{O}_a = w^i\mathcal{O}_{a,L} \quad \text{and} \quad \Phi^{i+n-j}\mathcal{O}_a = w\mathcal{O}_{a,R}. \qquad \square$$

Corollary 4.6. *We have a recursive formula*

$$d(n,i) = d(n,i+1) + \sum_{j=1}^{n-i-1} d(n-j,i)d(j,1).$$

The base case here is actually $\mathcal{D}_n^{(n)}$, which is $p_a^{-1}(\mathcal{O}_n^{\mathrm{cpt}})$. Every transfer system is compatible with the complete additive transfer system. Note also that we have an important edge case:

$$(\mathcal{F}^{n-1}T)_n = (\mathcal{F}^nT)_n,$$

since both correspond to the unique complete transfer system on $[n]$.

Proposition 4.7 [Balchin et al. 2021, Theorem 20]. *For each n, we have*

$$d(n,n) = d(n,n-1) = \mathrm{Cat}(n).$$

Rewriting the recurrence relation. We now can solve the recurrence relation, giving our first proof of the main theorem. Recall the definition of the Fuss–Catalan numbers:

$$A_n(p,r) = \frac{r}{np+r}\binom{np+r}{n}.$$

We begin with some helpful properties of the Fuss–Catalan number.

Proposition 4.8. *For any positive integer p and positive n, r, the following properties hold*:

(1) $A_n(3,r) = \sum_{j=0}^{n} A_j(2,1)A_{n-j}(3, j+r-1)$.

(2) $A_n(p,r) = A_n(p,r-1) + A_{n-1}(p, p+r-1)$.

(3) $A_{n+1}(p,1) = A_n(p,p)$.

(4) $A_n(p,s+r) = \sum_{j=0}^{n} A_j(p,r)A_{n-j}(p,s)$.

(5) $A_n(3,2) = \sum_{j=1}^{n} A_{j+1}(2,1)A_{n-j}(3,j)$.

Proof. Four of these are from work of Młotkowski. Formula (1) is a special case of Proposition 2.1 in [Młotkowski 2010]. Formula (2) is Equation 2.2 there, (3) is Equation 2.3, and (4) is Equation 2.4.

We will only prove (5). It can be done by induction on n. The base case $n = 1$ is a straightforward check. Take $n \geq 2$.

Write

$$s = \sum_{j=1}^{n} A_{j+1}(2, 1) A_{n-j}(3, j).$$

Replacing j with $j+1$, we can rewrite this as

$$s = \sum_{j=2}^{n+1} A_j(2, 1) A_{n-j+1}(3, j-1).$$

Applying (2) twice, we have

$$\begin{aligned} A_{n-j+1}(3, j+1) &= A_{n-j+1}(3, j) + A_{n-j}(3, j+3) \\ &= \big(A_{n-j+1}(3, j-1) + A_{n-j}(3, j+2)\big) + A_{n-j}(3, j+3). \end{aligned}$$

With the help of (1), we can expand $A_{n+1}(3, 2)$ to

$$A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 2) + \sum_{j=2}^{n+1} A_{n-j+1}(3, j+1) A_j(2, 1),$$

and substituting in for $A_{n-j+1}(3, j+1)$, we can rewrite this as

$$A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 2) + s + \sum_{j=2}^{n} A_{n-j}(3, j+2) A_j(2, 1) + \sum_{j=2}^{n} A_{n-j}(3, j+3) A_j(2, 1).$$

The last two sums also show up in the expansions of $A_n(3, 3)$ and $A_n(3, 4)$, respectively, by (1):

$$\begin{aligned} \sum_{j=2}^{n} A_{n-j}(3, j+2) A_j(2, 1) &= A_n(3, 3) - A_n(3, 2) - A_{n-1}(3, 3), \\ \sum_{j=2}^{n} A_{n-j}(3, j+3) A_j(2, 1) &= A_n(3, 4) - A_n(3, 3) - A_{n-1}(3, 4). \end{aligned}$$

This gives an equality

$$\begin{aligned} A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 2) + s &+ \big(A_n(3, 3) - A_n(3, 2) - A_{n-1}(3, 3)\big) \\ &+ \big(A_n(3, 4) - A_n(3, 3) - A_{n-1}(3, 4)\big). \end{aligned} \tag{4.9}$$

We can use (3) to rewrite (4.9) as

$$A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 4) + \big(s - A_n(3, 1) - A_{n-1}(3, 4)\big).$$

However, applying (2) to $A_{n+1}(3, 2)$, we get exactly

$$A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 4),$$

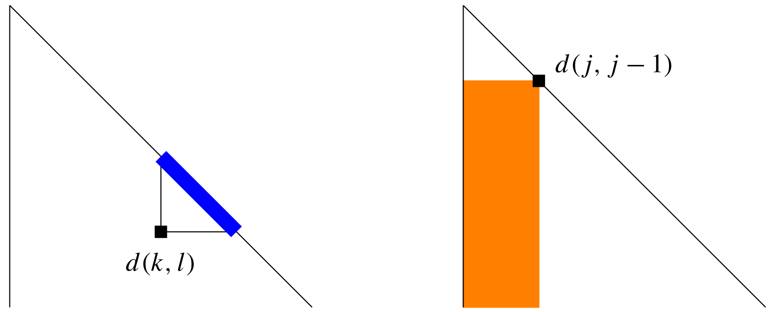

Figure 4. Contributions to each term: $d(k, l)$ is a linear combination of $d(j, j-1)$ from the blue shaded region, and a given $d(j, j-1)$ contributes only in the orange region.

so we deduce

$$s = A_n(3, 1) + A_{n-1}(3, 4).$$

By (2) again, we get $s = A_n(3, 2)$. □

Theorem 4.10. *For any $n \in \mathbb{N}$, $A_{n-1}(3, 2)$ is $d(n, 1)$.*

Before proving the theorem, we first discuss the strategy of the proof. Instead of focusing on the recurrence steps for a specific n, we put all $d(k, j)$ together, forming a triangle as follows:

$$\begin{array}{llll} d(2,1) & & & \\ d(3,1) & d(3,2) & & \\ d(4,1) & d(4,2) & d(4,3) & \\ d(5,1) & d(5,2) & d(5,3) & d(5,4) \\ \quad\vdots & & & \end{array}$$

Recall Corollary 4.6. For $i \in \mathbb{N}_+$, the elements in the set $\{d(j+i, j) : j \geq 1\}$ share the same recursive formula. In fact, we can generate the above triangle hypotenuse-by-hypotenuse from outside to inside. To begin with, we generate the second outermost hypotenuse $d(3, 1)$, $d(4, 2)$, $d(5, 3)$, ... by the outermost hypotenuse $d(2, 1)$, $d(3, 2)$, $d(4, 3)$ ... according to the recursive formula for $d(n, n-2)$. In other words, $d(n, n-2)$ is a linear combination of $d(j, j-1)$ with coefficients 1 or $d(1, 1)$. Similarly, we can generate the third outermost hypotenuse by the outermost and the second outermost hypotenuse according to the recursive formula for $d(n, n-3)$. Now the coefficients are from the space spanned by $d(2, 1)$, $d(1, 1)$ and 1.

As the whole triangle can be generated hypotenuse-by-hypotenuse, we can conclude that any $d(k, l)$ in the triangle can be written as a linear combination of $d(j, j-1)$ where $l+1 \leq j \leq k$ with coefficients from the space spanned by 1 and $d(i, 1)$ where $1 \leq i \leq k-l-1$. Graphically, $d(k, l)$ is a linear combination of $d(j, j-1)$ in the blue area of Figure 4.

Proof of Theorem 4.10. This is easy to check for $d(2, 1)$. Then, by induction, it suffices to show the theorem is true for $d(n, 1)$ if the theorem holds for $d(i, 1)$ for any $2 \leq i \leq n-1$.

Fix j such that $2 \le j \le n$. For our convenience, we denote the coefficient of $d(j, j-1)$ for $d(k,l)$ as $c(k,l)$. Our goal is to determine $c(n,1)$. We will achieve this by induction.

First, we figure out the coefficients of $d(j, j-1)$ for the vertical line $d(j+l, j-1)$ where $0 \le l \le n-j$. We know $c(j, j-1) = 1 = A_0(3,1)$ and for $0 \le l \le n-j$,

$$c(j+l, j-1) = \sum_{i=1}^{l} d(i,1)c(j+l-i, j-1).$$

As $d(i,1)$ is $A_{i-1}(3,2)$ for $2 \le i \le n-1$, we get $c(j+l, j-1) = A_l(3,1)$ by (4) of Proposition 4.8 and the equality $A_n(3,3) = A_{n+1}(3,1)$ for $0 \le l \le n-j$ inductively.

In fact, we claim that for $1 \le i \le j-1$, $c(j+l, j-i) = A_l(3,i)$ for $0 \le l \le n-j$. We have shown the case for $i = 1$. Now it is sufficient to show the case $i = k$ provided the claim holds for $1 \le i \le k-1$. Similarly, we know $c(j, j-k) = 1 = A_0(3,k)$ and for $0 \le l \le n-j$,

$$c(j+l, j-k) = c(j+l, j-(k-1)) + \sum_{i=1}^{l} d(i,1)c(j+l-i, j-k).$$

As $d(i,1)$ is $A_{i-1}(3,2)$ for $2 \le i \le n-1$, we get $c(j+l, j-k) = A_l(3,k)$ by (2) and (4) of Proposition 4.8 for $0 \le l \le n-j$ inductively. Thus, we get $c(n,1) = A_{n-j}(3, j-1)$. As $d(n,1)$ depends only on $d(j, j-1)$ for $2 \le j \le n$,

$$d(n,1) = \sum_{j=2}^{n} A_{n-j}(3, j-1)d(j, j-1) = \sum_{j=2}^{n} A_{n-j}(3, j-1)A_j(2,1),$$

where the last equality is Proposition 4.7. By (5) of Proposition 4.8, we conclude $d(n,1) = A_{n-1}(3,2)$. □

Theorem 4.11. *There are $A_n(3,1)$ compatible transfer systems for $[n]$.*

Proof. We will prove this by induction. It is easy to check the case for $[1]$. Assume the theorem is true for $[i]$ where $1 \le i < n$. We want to show it is true for $[n]$.

Corollary 4.6 shows that the number of compatible systems for $[n]$ is

$$d(n,0) = d(n,1) + \sum_{j=1}^{n-1} d(j,0)d(n-j,1).$$

Theorem 4.10 and the inductive hypothesis let us rewrite this as

$$d(n,0) = A_{n-1}(3,2) + \sum_{j=1}^{n-1} A_j(3,1)A_{n-j-1}(3,2) = \sum_{j=0}^{n-1} A_j(3,1)A_{n-j-1}(3,2).$$

By (4) of Proposition 4.8, we deduce

$$d(n,0) = A_{n-1}(3,3),$$

which, by (3) of Proposition 4.8, equals $A_n(3,1)$. □

5. Extensions of saturated systems

Counting using additive cores. Instead of using the filtration by powers of w on the additive indexing system, we can use the first part of Proposition 3.3. This says that compatibility of $(\mathcal{O}_a, \mathcal{O}_m)$ is the same question as compatibility of $(\operatorname{core}\mathcal{O}_a, \mathcal{O}_m)$, since both reduce to the comparison

$$\mathcal{O}_m \leq \operatorname{core}\mathcal{O}_a.$$

Since the core breaks up as a direct some of complete transfer systems, this last condition is really the same as asking that if

$$\operatorname{core}\mathcal{O}_a = \mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_m}^{\mathrm{cpt}},$$

then we have a direct sum decomposition

$$\mathcal{O}_m = \mathcal{O}'_{n_1} \oplus \cdots \oplus \mathcal{O}'_{n_m},$$

where $\mathcal{O}'_{n_i} \in T_{n_i}$ for all i. This is the only condition here, so we deduce the following proposition.

Proposition 5.1. *If $\mathcal{O}_a$ is a transfer system with*

$$\operatorname{core}\mathcal{O}_a = \mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_m}^{\mathrm{cpt}},$$

then there are

$$\prod_{j=1}^{m} \operatorname{Cat}(n_j)$$

transfer systems $\mathcal{O}_m$ such that the pair $(\mathcal{O}_a, \mathcal{O}_m)$ is compatible.

This reduces our question of the number of compatible pairs to two parts:

(1) Enumerate all of the transfer systems with a fixed core.

(2) Then evaluate the corresponding sum.

Notation 5.2. Let $\vec{k} = (k_1, \ldots, k_m)$ be a sequence of positive integers. For each $1 \leq s \leq m$, let

$$K_s = k_1 + \cdots + k_s.$$

Let

$$\mathcal{O}_{\vec{k}}^{\mathrm{sat}} = \mathcal{O}_{k_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{k_m}^{\mathrm{cpt}}.$$

Definition 5.3. For a sequence $\vec{k} = (k_1, \ldots, k_m)$ with $n = k_1 + \cdots + k_m$, let

$$\mathcal{E}_{\vec{k}} = \{\mathcal{O} \in T_k \mid \operatorname{core}\mathcal{O} = \mathcal{O}_{\vec{k}}^{\mathrm{sat}}\},$$

and let

$$e_{\vec{k}} = |\mathcal{E}_{\vec{k}}|.$$

Proposition 5.4. *The number of admissible pairs of transfer systems for $[n]$ is*

$$\sum_{\vec{k}} e_{\vec{k}} \prod_{j} \operatorname{Cat}(k_j),$$

where $\vec{k}$ in the first sum ranges over the partitions of n.

Proof. We partition the set of transfer systems by their core:

$$T_n = \bigsqcup_{\vec{k}} \mathcal{E}_{\vec{k}},$$

where $\vec{k}$ ranges over the partitions of n. This induces a partition of the set of compatible pairs:

$$\mathcal{D}_n^{(0)} = \bigsqcup_{\vec{k}} \{(\mathcal{O}_a, \mathcal{O}_m) \mid \text{compatible}, \mathcal{O}_a \in \mathcal{E}_{\vec{k}}\}.$$

The first part of Proposition 3.3 shows that

$$\{\mathcal{O}_m \mid (\mathcal{O}_a, \mathcal{O}_m) \text{ compatible}\}$$

is constant for $\mathcal{O}_a \in \mathcal{E}_{\vec{k}}$, and Proposition 5.1 identifies the cardinality as

$$\prod_j \mathrm{Cat}(k_j).$$

Putting this all together, we find there are

$$\sum_{\vec{k}} e_{\vec{k}} \prod_j \mathrm{Cat}(k_j)$$

compatible pairs of transfer systems. □

Catalan tuples. The enumeration and exact sum were analyzed by de Jong, Hock, and Wulkenhaar [de Jong et al. 2022] in a slightly different guise. They consider certain sequences which they call Catalan tuples.

Definition 5.5 [de Jong et al. 2022, Definition 3.1]. For each positive integer n, a *Catalan tuple of length n* is a sequence of nonnegative integers

$$\vec{s} = (s_0, \ldots, s_r)$$

with three properties:

(1) For all j, we have
$$\sum_{i=0}^{j} s_i > j.$$

(2) At the end of the sequence,
$$\sum_{i=0}^{r} s_i = n.$$

(3) If $n > 0$, then $s_r > 0$.

Let $\mathcal{S}_n$ be the set of all Catalan tuples of length n.

Remark 5.6. We have slightly modified the definition here to ignore trailing zeros. This removes our ability to predict the length of the string, but it will better connect with the extensions.

We can restate the conditions in Definition 5.5 slightly to start connecting with extensions.

Definition 5.7. The *excess* of a Catalan tuple $\vec{s} = (s_1, \ldots, s_r)$ of length n is

$$e(\vec{s}) := n - r - 1.$$

Remark 5.8. Note that the edge condition in Definition 5.5 of $j = r$ implies the inequality $n > r$. This means the excess is always nonnegative.

Proposition 5.9. *Let $\vec{s}$ be a Catalan tuple of length n and excess $e = n - r - 1$. Then, for any $k > 0$, the sequence*

$$\vec{s}_k(\ell) = (s_0, \ldots, s_r, \underbrace{0, \ldots, 0}_{\ell}, k)$$

is a Catalan tuple if and only if

$$0 \leq \ell \leq e.$$

Proof. For $0 \leq j \leq r$, the Catalan tuple condition holds since it does for $\vec{s}$. If $\ell = 0$, then we have

$$\sum_{i=0}^{r} s_i + k = n + k > r + 1,$$

since $n > r$ and $k \geq 1$. Assume now that $\ell > 0$, and consider a $1 \leq j \leq \ell$. We have

$$\sum_{i=0}^{r+j} s_i = n = r + e + 1.$$

On the other hand, this is greater than $r + j$ if and only if $j < e + 1$. This gives the bounds on ℓ. Finally, note that the analysis for the case $\ell = 0$ now also implies the case $j = \ell + 1$. □

Proposition 5.10. *Let $\vec{s}$ be a Catalan tuple of length n and excess $e = n - r - 1$. Then, for any $k > 0$ and $0 \leq \ell \leq e$, the excess of the Catalan tuple*

$$\vec{s}_k(\ell) = (s_0, \ldots, s_r, \underbrace{0, \ldots, 0}_{\ell}, k)$$

is

$$e(\vec{s}_k(\ell)) = k - 1 + e - \ell = n + k - (r + \ell + 2).$$

Proof. The Catalan tuple given has length $n + k$. The sequence $\vec{s}$ has length $(r + 1)$, and we added $\ell + 1$ new terms to form $\vec{s}_k(\ell)$. □

This lets us rewrite Catalan tuples using only the nonzero entries.

Definition 5.11. If $\vec{s}$ is a Catalan tuple, then let $\mathrm{core}(\vec{s})$ be the subsequence of nonzero entries of $\vec{s}$. Given a partition $\vec{k}$ of k, let $\mathcal{S}_{\vec{k}}$ be the subset of $\mathcal{S}_k$ of Catalan tuples with core $\vec{k}$:

$$\mathcal{S}_{\vec{k}} := \{\vec{s} \in \mathcal{S}_k \mid \mathrm{core}\, \vec{s} = \vec{k}\}.$$

Corollary 5.12. *Catalan tuples with core $\vec{k}$ are those sequences of the form*

$$k_1,\ \underbrace{0,\ \ldots,\ 0}_{\ell_1},\ k_2,\ \underbrace{0,\ \ldots,\ 0}_{\ell_2},\ k_3,\ \ldots,\ k_{n-1},\ \underbrace{0,\ \ldots,\ 0}_{\ell_{n-1}},\ k_n$$

such that for all $1 \leq j \leq n-1$, *we have*

$$\sum_{i=1}^{j}(k_i - \ell_i) \geq j.$$

The excess of such a sequence is

$$(k_n - 1) + \sum_{i=1}^{n-1}(k_i - 1 - \ell_i).$$

In their work, de Jong, Hock, and Wulkenhaar consider certain collections of Catalan tuples.

Definition 5.13 [de Jong et al. 2022, Definition 4.1]. A *nested Catalan tuple of length n* is a sequence of Catalan tuples $(\vec{s}_{i_1}, \ldots, \vec{s}_{i_r})$ such that $\vec{s}_{i_j} \in \mathcal{S}_{i_j}$ and the sequence

$$(i_1 + 1, i_2, \ldots, i_r)$$

is a Catalan tuple of length n.

A key result in [de Jong et al. 2022] is the cardinality of the number of nested Catalan tuple that begin with (0). For this, we need a straightforward lemma.

Lemma 5.14. *The map* $\Sigma : \mathcal{S}_n \to \mathcal{S}_{n+1}$ *given by*

$$(s_0, \ldots, s_r) \mapsto (1, s_0, \ldots, s_r)$$

is an injection with image those sequences which begin with 1.

Proposition 5.15 [de Jong et al. 2022, Corollary 4.6]. *The number of nested Catalan tuples of length* $(n+1)$ *with first term* (0) *is*

$$\sum_{\vec{s}\in\mathcal{S}_n}\prod_{i}\mathrm{Cat}(s_i) = \frac{1}{2n+1}\binom{3n}{n} = \frac{1}{3n+1}\binom{3n+1}{n} = A_n(3, 1).$$

We will produce an explicit bijection

$$\sigma : T_n \to \mathcal{S}_n$$

by building bijections between $\mathcal{E}_{\vec{k}}$ and $\mathcal{S}_{\vec{k}}$.

Enumerating extensions by a complete transfer system. It is helpful to think of elements of $\mathcal{E}_{\vec{k}}$ also as various "extensions" of the complete transfer systems $\mathcal{O}^{\mathrm{cpt}}_{k_1}, \ldots, \mathcal{O}^{\mathrm{cpt}}_{k_n}$. For our count, it is easier to instead consider a more general class.

Definition 5.16. An *extension of* $\mathcal{O}' \in T_m$ *by* $\mathcal{O}'' \in T_k$ is a transfer system $\mathcal{O} \in T_{m+k}$ such that

(1) $i_m^*\mathcal{O} = \mathcal{O}'$,

(2) $\Phi^m\mathcal{O} = \mathcal{O}''$,

and an extension $\mathcal{O}$ of $\mathcal{O}'$ by $\mathcal{O}''$ is *core-preserving* if, moreover,

$$\operatorname{core}\mathcal{O} = \operatorname{core}\mathcal{O}' \oplus \operatorname{core}\mathcal{O}''.$$

Note that since by assumption we have specified i_m^* and Φ^m in an extension, we need only determine the transfers with source $i \le m$ and target $j > m$.

Definition 5.17. Any transfer $i \to j$ with $i \le m$ and $j > m$ in an extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}''$ is a *crossing transfer*.

Proposition 5.18. *Let $\mathcal{O}$ be an extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}'' \in T_k$. Then following are equivalent*:

(1) *The extension is core-preserving.*

(2) *If there is a transfer $m \to j$, then $j = m$.*

Proof. Note that the existence of a nontrivial transfer $m \to j$ is equivalent to the existence of a transfer $m \to (m+1)$, by the restriction axiom. If we have

$$\operatorname{core}\mathcal{O}' = \mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_j}^{\mathrm{cpt}} \quad \text{and} \quad \operatorname{core}\mathcal{O}'' = \mathcal{O}_{m_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{m_i}^{\mathrm{cpt}},$$

then, by construction of the core, the existence of the transfer $m \to (m+1)$ is equivalent to $\operatorname{core}\mathcal{O}$ satisfying

$$\operatorname{core}\mathcal{O} = \mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_{j-1}}^{\mathrm{cpt}} \oplus \mathcal{O}_{n_j+m_1}^{\mathrm{cpt}} \oplus \mathcal{O}_{m_2} \oplus \cdots \oplus \mathcal{O}_{m_i}.$$

The result follows. □

Because we want to enumerate transfer systems with a fixed core, we now restrict attention to core-preserving extensions of $\mathcal{O}$ by a complete transfer system $\mathcal{O}'$. This significantly simplifies our combinatorics.

Lemma 5.19. *Let $\mathcal{O}$ be a core-preserving extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}_k^{\mathrm{cpt}}$. Then, for each $i \le m$, the following are equivalent*:

(1) *We have a crossing transfer $i \to m + j$ for some $j > 0$.*

(2) *We have crossing transfers $i \to m + j$ for all $0 \le j \le k$.*

Proof. One direction is immediate. For the other, if we have a transfer $i \to m + j$, then, by the restriction axiom, we have transfers $i \to m$ and $i \to m+1$. Since $\mathcal{O}_k^{\mathrm{cpt}}$ is complete, in our extension, we have transfers $m+1 \to m+j$ for any $1 \le j \le k$, which gives the second result. □

Remark 5.20. We singled out the transfer $i \to m$ since this constrains the number of possible sources for a transfer from $[m]$ up to $[k]$. Any crossing transfer has source an element of $[m]$ that transfers up to m in $\mathcal{O}'$.

Definition 5.21. If $\mathcal{O} \in T_m$, then let

$$\tau(\mathcal{O}) = \{d \in [m] \mid d \to m\}$$

be the set of elements which transfer up to m in $\mathcal{O}$.

The element m always transfers to m, so $\tau(\mathcal{O})$ is always nonempty.

Definition 5.22. If $\mathcal{O} \in T_m$, then let

$$e(\mathcal{O}) = |\tau(\mathcal{O})| - 1$$

be the number of elements of $\{1, \ldots, m-1\}$ which transfer up to m in $\mathcal{O}$.

As a subset of the totally ordered set $[m]$, the set $\tau(\mathcal{O})$ inherits a total order:

$$\tau(\mathcal{O}) = \{d_1, d_2, \ldots, d_{e(\mathcal{O})+1}\},$$

where by assumption, we always will have that if $i < j$, then $d_i < d_j$. The crossing transfers have a kind of "nondecreasing" property for this order.

Lemma 5.23. *Let $\mathcal{O}$ be a core-preserving extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}_k^{\mathrm{cpt}}$. In $\mathcal{O}$, if we have a transfer*

$$d \to m+k,$$

for some $d \in \tau(\mathcal{O}')$, then, for all $d' \in \tau(\mathcal{O}')$ such that $d' \leq d$, we have transfers

$$d' \to m+k.$$

Proof. By the restriction axiom, whenever $d' \leq d$, we have a transfer $d' \to d$. The result follows from transitivity. □

Definition 5.24. Let $\mathcal{O}' \in T_m$, let $e = e(\mathcal{O}')$, and write

$$\tau(\mathcal{O}') = \{d_1, \ldots, d_{e+1}\}.$$

For each $k > 0$ and for each $0 \leq \ell \leq e$, define a relation $\to_\ell$ on $[m+k]$ that refines the partial order $\leq$ by saying that

(1) if $i \leq j \leq m$, then $i \to_\ell j$ if and only if $i \to j$ in $\mathcal{O}'$,

(2) if $m < i \leq j$, then $i \to_\ell j$, and

(3) if $i \leq m < j$, then $i \to_\ell j$ if and only if $i = d_s$ for some $1 \leq s \leq r - \ell$.

The following proposition is a straightforward application of the definition of a transfer system.

Proposition 5.25. *The relation $\to_\ell$ is a transfer system on $[n+k]$ that is a core-preserving extension of $\mathcal{O}'$ by $\mathcal{O}_k^{\mathrm{cpt}}$.*

Definition 5.26. Let $\mathcal{O}_{\mathcal{O}',k}(\ell)$ denote the transfer system $\to_\ell$ on $[n+k]$.

Remark 5.27. There is a special case of the extensions: $\ell = e$. In this case, we have the direct sum $\mathcal{O}' \oplus \mathcal{O}_k^{\mathrm{cpt}}$.

There is a crucial observation about the number of transfers to $[m+k]$ here.

Proposition 5.28. *Let $\mathcal{O}'$ be a transfer system on $[m]$ with $e(\mathcal{O}') = e$, and let $0 \leq \ell \leq e$. Then*

$$e(\mathcal{O}_{\mathcal{O}',k}(\ell)) = (k-1) + (e - \ell).$$

Proof. All of the k elements of $[k]$ transfer up, and by construction, the $r-\ell$ elements $d_1, \dots, d_{r-\ell}$ are the only elements from $[m]$ which also transfer up to $m+k$. This gives

$$\tau(\mathcal{O}_{\mathcal{O}',k}(\ell)) = \{d_1, \dots, d_{r-\ell}, m+1, \dots, m+k\},$$

and the result follows. □

Putting these together gives a complete classification of the core-preserving extensions.

Theorem 5.29. *Let $\mathcal{O}' \in T_m$ be a transfer system, and let $e = e(\mathcal{O}')$. Then there are $(e+1)$ core-preserving extensions of $\mathcal{O}'$ by $\mathcal{O}_k^{\mathrm{cpt}}$ given by $\mathcal{O}_{\mathcal{O}',k}(\ell)$ for $0 \le \ell \le e$.*

Proof. Lemmas 5.19 and 5.23 show that any core-preserving extension has this form. The converse is the content of Proposition 5.25. □

Enumerating transfer systems with a fixed core. Now let $\mathcal{O}$ be a transfer system with

$$\operatorname{core}\mathcal{O} = \mathcal{O}_{\vec{k}}^{\mathrm{sat}}.$$

Write $\vec{k} = (k_1, \dots, k_n)$. We can immediately identify $\mathcal{O}$ inductively as a type considered in the last section.

Proposition 5.30. *The transfer system $\mathcal{O}$ is a core-preserving extension of $i_{K_{n-1}}^*\mathcal{O}$ by $\mathcal{O}_{k_n}^{\mathrm{cpt}}$.*

This turns our problem into an inductive one, working down on the number of summands in the partition of k. We can now build our bijection.

Definition 5.31. Let

$$\sigma : \mathcal{E}_{\vec{k}} \to \mathcal{S}_{\vec{k}}$$

be defined inductively by the following procedure. If $\vec{k} = (k)$, then $\mathcal{O} = \mathcal{O}_k^{\mathrm{cpt}}$, and we define

$$\sigma(\mathcal{O}_k^{\mathrm{cpt}}) = k.$$

For a general $\vec{k} = (k_1, \dots, k_n)$ with $n > 1$ and $\mathcal{O} \in \mathcal{E}_{\vec{k}}$, let ℓ_{n-1} be the unique number $0 \le \ell_{n-1}$ such that

$$\mathcal{O} = \mathcal{O}_{\mathcal{O}',k_n}(\ell_{n-1}),$$

where $\mathcal{O}' = i_K^*\mathcal{O}$, and define

$$\sigma(\mathcal{O}) = (\sigma(\mathcal{O}'), \underbrace{0, \dots, 0}_{\ell_{n-1}}, k_n).$$

Example 5.32. For $\mathcal{O}$ the transfer system in Figure 2, we have

$$\sigma(\mathcal{O}) = (4, 0, 1, 2).$$

We need to verify that σ actually lands in the set $\mathcal{S}_{\vec{k}}$.

Proposition 5.33. *For any $\mathcal{O}$ with* $\operatorname{core}\mathcal{O} = \vec{k}$,

(1) *$\sigma(\mathcal{O}) \in \mathcal{S}_{\vec{k}}$, and*

(2) *$e(\sigma(\mathcal{O})) = e(\mathcal{O})$, that is, the excess of $\sigma(\mathcal{O})$ is the number of elements j between 1 and $K_n - 1$ that transfer up to K_n in $\mathcal{O}$.*

Proof. We show this by induction on n. The base case of $n = 1$ is immediate by the definitions of σ and the excess, so assume this is true for partitions with fewer than n terms.

Let $\mathcal{O}' = i^*_{K_{n-1}}\mathcal{O}$, and let r be the number of $j < K_{n-1}$ which transfer up to K_{n-1} in $\mathcal{O}'$. By the inductive hypothesis, $\sigma(\mathcal{O}')$ is a Catalan tuple with core $(k_1, \dots, k_{n-1})$ and we also have

$$e(\sigma(\mathcal{O}')) = r.$$

Now if $0 \le \ell_n \le r$ is such that $\mathcal{O} = \mathcal{O}_{\mathcal{O}',k_n}(\ell_{n-1})$, then, since $\ell_{n-1} \le e(\sigma(\mathcal{O}'))$, the first claim is Proposition 5.9. For the second part, Proposition 5.10 shows that the excess of $\sigma(\mathcal{O})$ is

$$e(\sigma(\mathcal{O})) = e - \ell_{n-1} + (k_n - 1).$$

Proposition 5.28 shows this is exactly the number of elements smaller than K_n which transfer up to K_n. $\square$

This gives us the final piece for our argument.

Corollary 5.34. *The map σ is a bijection $\mathcal{E}_{\vec{k}} \to \mathcal{S}_{\vec{k}}$.*

Proof. By induction on n, we see that there are exactly as many extensions of $i^*_{K_{n-1}}\mathcal{O}$ by $\mathcal{O}_{k_n}$ as there are extensions of the Catalan tuple $\sigma(i^*_{K_{n-1}}\mathcal{O})$ to a Catalan tuple ending with k_n, and the map σ gives a bijection between these. $\square$

We close by completing a proof of Theorem 1.7 using these extensions.

Second proof of Theorem 1.7. Proposition 5.15 says that we have

$$A_n(3, 1) = \sum_{\vec{s} \in \mathcal{S}_n} \prod_i \mathrm{Cat}(s_i).$$

The set S_n of Catalan tuples of length n is partitioned by the cores:

$$S_n = \coprod_{\vec{k}} S_{\vec{k}}.$$

So we can rewrite the sum from Proposition 5.15 as

$$\sum_{\vec{s} \in \mathcal{S}_n} \prod_i \mathrm{Cat}(s_i) = \sum_{\vec{k}} \sum_{\vec{s} \in S_{\vec{k}}} \prod_i \mathrm{Cat}(s_i).$$

Since $\mathrm{Cat}(0) = 1$, only terms in the core of $\vec{s}$ contribute in a nontrivial way to the product:

$$\prod_i \mathrm{Cat}(s_i) = \prod_j \mathrm{Cat}(k_j),$$

where in the second product, we are running over the entries of $\vec{k} = \operatorname{core} \vec{s}$. In particular, we can again rewrite the sum from Proposition 5.15:

$$\sum_{\vec{k}} \sum_{\vec{s} \in S_{\vec{k}}} \prod_i \mathrm{Cat}(s_i) = \sum_{\vec{k}} |S_{\vec{k}}| \prod_j \mathrm{Cat}(k_j),$$

since all Catalan tuples with the same core contribute the same product to the sum. Corollary 5.34 shows

$$|S_{\vec{k}}| = |\mathcal{E}_{\vec{k}}| = e_{\vec{k}},$$

the number of transfer systems with core $\mathcal{O}^{\mathrm{sat}}_{\vec{k}}$. Proposition 5.4 identifies this last sum with the number of compatible pairs, completing the proof. □

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