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Games for the two membranes problem

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We find viscosity solutions to the two membranes problem (that is, a system with two obstacle-type equations) with two different p-Laplacian operators taking limits of value functions of a sequence of games. We analyze two-player zero-sum games that are played in two boards with different rules in each board. At each turn both players (one inside each board) have the choice of playing without changing board or changing to the other board (and then playing one round of the other game). We show that the value functions corresponding to this kind of game converge uniformly to a viscosity solution of the two membranes problem. If in addition the possibility of having the choice to change boards depends on a coin toss we show that we also have convergence of the value functions to the two membranes problem that is supplemented with an extra condition inside the coincidence set.

1. Introduction

The deep connection between partial differential equations and probability is a well-known and widely studied subject. For linear operators, such as the Laplacian, this relation turns out to rely on the validity of mean value formulas for the solutions in the PDE side and martingale identities in the probability side. In fact, there is a standard connection between the Laplacian and the Brownian motion or with the limit of random walks as the step size goes to zero; see, for example, [Doob 1954; 1971; 1984; Hunt 1957; 1958; Kac 1947; Kakutani 1944; Knapp 1965; Williams 1991]. Recently, starting with [Peres et al. 2009], some of these connections were extended to cover nonlinear equations. For a probabilistic approximation of the infinity Laplacian there is a game (called tug-of-war in the literature), introduced in [Peres et al. 2009], whose value functions approximate solutions to the PDE as a parameter that controls the size of the steps in the game goes to zero. In [Peres and Sheffield 2008] — see also [Manfredi et al. 2012a; 2012b] — the authors introduce a modification of the game (called tug-of-war with noise) that is related to the normalized p-Laplacian. Approximation of solutions to linear and nonlinear PDEs using game theory is now a classical subject. The previously mentioned results were extended to cover very different equations (such as Pucci operators, the Monge–Ampère equation, the obstacle problem, etc); see the books [Blanc and Rossi 2019] and [Lewicka 2020]. However, much less is known concerning the relation between systems of PDEs and games. As a recent reference for a cooperative system we quote [Mitake and Tran 2017]. One of the systems that attracted the attention of the PDE community is the two membrane problem. This problem models the behavior of two elastic membranes that are clamped at the boundary of a prescribed domain

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(they are assumed to be ordered, one membrane above the other) and they are subject to different external forces (the membrane that is on top is pushed down and the one that is below is pushed up). The main assumption here is that the two membranes do not penetrate each other (they are assumed to be ordered in the whole domain). This situation can be modeled by a two obstacle problem; the lower membrane acts as an obstacle from below for the free elastic equation that describes the location of the upper membrane, while, conversely, the upper membrane is an obstacle from above for the equation for the lower membrane. When the equations that obey the two membranes have a variational structure this problem can be tackled using calculus of variations (one aims to minimize the sum of the two energies subject to the constraint that the functions that describe the position of the membranes are always ordered inside the domain; one is bigger than or equal to the other); see [Vergara Caffarelli 1971]. However, when the involved equations are not variational the analysis relies on monotonicity arguments (using the maximum principle). Once existence of a solution (in an appropriate sense) is obtained a lot of interesting questions arise, such as uniqueness, regularity of the involved functions, a description of the contact set, the regularity of the contact set, etc. See [Caffarelli et al. 2017; 2018; Silvestre 2005] and the dissertation [Vivas 2018].

Our main goal in this paper is to analyze games whose value functions approximate solutions to the two membranes problem with two different normalized p-Laplacians (these are fully nonlinear nonvariational equations; see below).

1.1. ***The normalized p-Laplacian and game theory.*** To begin, we introduce the normalized p-Laplacian and give the relation between this operator and the game called tug-of-war with noise in the literature; we refer to [Manfredi et al. 2012b] and the recent books [Blanc and Rossi 2019] and [Lewicka 2020] for details. This kind of game has been extensively studied.

Consider the classical p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $2 \le p < \infty$. Expanding the divergence we can (formally) write this operator as a combination of the Laplacian operator $\Delta u = \sum_{n=1}^{N} u_{x_n x_n}$ and the 1-homogeneous infinity Laplacian $\Delta_\infty^1 u = \big\langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\big\rangle = |\nabla u|^{-2} \sum_{1\le i,j\le N} u_{x_i} u_{x_i x_j} u_{x_j}$:

$$\Delta_p u = |\nabla u|^{p-2}((p-2)\Delta_\infty^1 u + \Delta u). \tag{1-1}$$

Now we want to recall the mean value formula associated to this operator obtained in [Manfredi et al. 2010] (see also [Arroyo and Llorente 2016] and [Lewicka and Manfredi 2017]). Given $0 < \alpha < 1$, let us consider $u : \Omega \to \mathbb{R}$ such that

$$u(x) = \alpha\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} u(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} u(y)\Big) + (1-\alpha) \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2) \tag{1-2}$$

as $\varepsilon \to 0$. It turns out that u satisfies this asymptotic mean value formula if and only if u is a viscosity solution to $\Delta_p u = 0$; see [Manfredi et al. 2010]. For general references on mean value formulas for solutions to nonlinear PDEs, we refer to [Arroyo and Llorente 2016; Blanc et al. 2021; Ishiwata et al. 2017; Kawohl et al. 2012; Lewicka and Manfredi 2017]. In fact, if we assume that u is smooth, using a simple Taylor expansion we have

$$\fint_{B_\varepsilon(x)} u(y)\,dy - u(x) = \frac{\varepsilon^2}{2(N+2)}\Delta u(x) + o(\varepsilon^2), \tag{1-3}$$

and if $|\nabla u(x)| \neq 0$, again using a simple Taylor expansion, we obtain

$$\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} u(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} u(y)\Big) - u(x)$$
$$\sim \left(\frac{1}{2}u\left(x+\varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) + \frac{1}{2}u\left(x-\varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right)\right) = \frac{1}{2}\varepsilon^2\Delta^1_\infty u(x) + o(\varepsilon^2). \quad (1\text{-}4)$$

Then, if we come back to (1-2), divide by ε^2, and take $\varepsilon \to 0$, we get

$$0 = \frac{\alpha}{2}\Delta^1_\infty u(x) + \frac{(1-\alpha)}{2(N+2)}\Delta u(x).$$

Thus, from (1-1), we get that the function u is a solution to the equation

$$-\Delta_p u(x) = 0$$

for $p > 2$ such that

$$\frac{\alpha}{1-\alpha} = \frac{p-2}{N+2}.$$

These computations can be made rigorous using viscosity theory; we refer to [Manfredi et al. 2010].

Now, let us introduce the normalized p-Laplacian (also called the game p-Laplacian).

Definition 1. Given $p > 2$, let the normalized p-Laplacian be defined as

$$\Delta^1_p u(x) = \frac{\alpha}{2}\Delta^1_\infty u(x) + \frac{1-\alpha}{2(N+2)}\Delta u(x),$$

with

$$\frac{\alpha}{1-\alpha} = \frac{p-2}{N+2}.$$

Note that this is a nonlinear elliptic 1-homogeneous operator that is a linear combination between the classical Laplacian and the ∞-Laplacian.

There is a game theoretical approximation to these operators. The connection between the Laplacian and the Brownian motion or with the limit of random walks as the step size goes to zero is well known; see [Kac 1947; Kakutani 1944; Knapp 1965]. For a probabilistic approximation of the infinity Laplacian there is a game (called tug-of-war in the literature) that was introduced in [Peres et al. 2009]. In [Peres and Sheffield 2008] — see also [Manfredi et al. 2012a; 2012b] — the authors introduce a two-player zero-sum game called tug-of-war with noise that is related to the normalized p-Laplacian. This is a two-player zero-sum game (two players, Player I and Player II, play one against the other and the total earnings of one player are exactly the total losses of the other). The rules of the game are as follows: In a bounded smooth domain Ω (what we need here is that $\partial\Omega$ satisfies an exterior sphere condition) given an initial position $x \in \Omega$, with probability α, Player I and Player II play tug-of-war (the players toss a fair coin and the winner chooses the next position of the token in $B_\varepsilon(x)$), and with probability $(1-\alpha)$, they move at random (the next position of the token is chosen at random in $B_\varepsilon(x)$). They continue playing with these rules until the game position leaves the domain Ω. At this stopping time, Player II pays Player I the amount determined by a payoff function defined outside Ω. The value of the game (defined as the best value that both players may expect to obtain) satisfies a mean value formula, called the dynamic programming principle (DPP), which in this case is given by

$$u^\varepsilon(x)=\alpha\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)}u^\varepsilon(y)+\frac{1}{2}\inf_{y\in B_\varepsilon(x)}u^\varepsilon(y)\Big)+(1-\alpha)\fint_{B_\varepsilon(x)}u^\varepsilon(y)\,dy.$$

This is exactly the same formula as (1-2) but without the error term $o(\varepsilon^2)$. Notice that the value function of this game depends on ε, the parameter that controls the size of the possible movements. Note that this equation can be written as

$$0=\alpha\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)}u^\varepsilon(y)+\frac{1}{2}\inf_{y\in B_\varepsilon(x)}u^\varepsilon(y)-u^\varepsilon(x)\Big)+(1-\alpha)\fint_{B_\varepsilon(x)}(u^\varepsilon(y)-u^\varepsilon(x))\,dy.$$

Using as a main tool the asymptotic formulas (1-3) and (1-4), in [Blanc et al. 2017] and [Manfredi et al. 2012b] the authors show that there is a uniform limit as $\varepsilon\to 0$,

$$u^\varepsilon\rightrightarrows u,$$

and that this limit u is the unique solution (in a viscosity sense) to the Dirichlet problem

$$\begin{cases}-\Delta_p^1 u(x)=0, & x\in\Omega,\\ u(x)=F(x), & x\in\partial\Omega.\end{cases}$$

When one wants to deal with a nonhomogeneous equation like $-\Delta_p^1 u(x)=h(x)$ one can add a running payoff to the game, that is, at every play Player I pays to Player II the amount $\varepsilon^2 h(x)$.

1.2. ***The two membranes problem.*** As we already mentioned, the two membranes problem describes the equilibrium position of two elastic membranes in contact with each other that are not allowed to cross. Hence, one of the membranes acts as an obstacle (from above or below) for the other. Given two differential operators $F(x,u,\nabla u,D^2u)$ and $G(x,v,\nabla v,D^2v)$ the mathematical formulation of the two membranes problem is the following:

$$\begin{cases}\min\big\{F\big(x,u(x),\nabla u(x),D^2u(x)\big),\,(u-v)(x)\big\}=0, & x\in\Omega,\\ \max\big\{G\big(x,v(x),\nabla v(x),D^2v(x)\big),\,(v-u)(x)\big\}=0, & x\in\Omega,\\ u(x)=f(x), & x\in\partial\Omega,\\ v(x)=g(x), & x\in\partial\Omega.\end{cases}$$

In general there is no uniqueness for the two membranes problem. For example, take u the solution to the first operator $F(u)=0$ with $u|_{\partial\Omega}=f$ and v the solution to the obstacle problem from above for $G(v)$ and boundary datum g. This pair (u,v) is a solution to the two membranes problem (v is a solution to the obstacle problem with u as upper obstacle and u is a solution to the obstacle problem with v as lower obstacle; in fact u is a solution in the whole domain and is above v). Analogously, one can consider $\tilde v$ as the solution to $G(\tilde v)=0$ with $v|_{\partial\Omega}=g$ and $\tilde u$ the solution to the obstacle problem from above for $F(\tilde u)$ and boundary datum f, to obtain a pair $(\tilde u,\tilde v)$ that is a solution to the two membranes problem. In general, it holds that $(u,v)\neq(\tilde u,\tilde v)$.

The two membranes problem for the Laplacian with a right-hand side, that is, for $F(D^2u) = -\Delta u + h_1$ and $F(D^2v) = -\Delta v - h_2$, was first considered in [Vergara Caffarelli 1971] using variational arguments. Later, in [Caffarelli et al. 2017] the authors solve the two membranes problem for two different fractional Laplacians of different order (two linear nonlocal operators defined by two different kernels). Notice that in this case the problem is still variational. In these cases an extra condition appears, namely, the sum of the two operators vanishes,

$$G(u) + F(v) = 0, \tag{1-5}$$

inside Ω. Moreover, this extra condition together with the variational structure is used to prove a $C^{1,\gamma}$ regularity result for the solution.

The two membranes problem for a nonlinear operator was studied in [Caffarelli et al. 2017; 2018; Silvestre 2005]. In particular, in [Caffarelli et al. 2018] the authors consider a version of the two membranes problem for two different fully nonlinear operators, $F(D^2u)$ and $G(D^2u)$. Assuming that F is convex and that

$$G(X) = -F(-X), \tag{1-6}$$

they prove that solutions are $C^{1,1}$ smooth.

We also mention that a more general version of the two membranes problem involving more than two membranes was considered by several authors (see, for example, [Azevedo et al. 2005; Carillo et al. 2005; Chipot and Vergara-Caffarelli 1985]).

1.3. ***Description of the main results.*** In this paper we use the previously described tug-of-war with noise game to obtain games whose value functions approximate solutions (in a viscosity sense) to a system with two obstacle-type equations (a two membrane problem).

1.3.1. *First game.* Let us describe the first game that we are going to study in more detail. Again, it is a two-player zero-sum game. The game is played in two boards, which we call board 1 and board 2, that are two copies of a fixed smooth bounded domain $\Omega \subset \mathbb{R}^N$. We fix two final payoff functions $f, g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ that are uniformly Lipschitz functions with $f \geq g$, and two running payoff functions $h_1, h_2 : \Omega \to \mathbb{R}$ (we also assume that they are uniformly Lipschitz functions), corresponding to the first and second board respectively. Take a positive parameter ε. Let us use two games with different rules for the first and second board respectively associated to two different p-Laplacian operators. To this end, let us fix two numbers $0 < \alpha_i < 1$ for $i = 1, 2$. Playing in the first board the rules are the following: with α_1 probability we play with tug-of-war rules, which means a fair coin is tossed and the player who wins the coin toss chooses the next position inside the ball $B_\varepsilon(x)$, and with $(1 - \alpha_1)$ probability we play with a random walk rule, meaning that the next position is chosen at random in $B_\varepsilon(x)$ with uniform probability. Playing in the first board we add a running payoff of amount $-\varepsilon^2 h_1(x_0)$ (Player I gets $-\varepsilon^2 h_1(x_0)$ and Player II $\varepsilon^2 h_1(x_0)$). We call this game the J_1 game. Analogously, in the second board we use α_2 to encode the probability that we play tug-of-war and $(1 - \alpha_2)$ for the probability to move at random, this time with a running payoff of amount $\varepsilon^2 h_2(x_0)$. We call this game J_2.

To the rules that we described in the two boards J_1 and J_2 we add the following ways of changing boards: in the first board, Player I decides to play with J_1 rules (and the game position remains at the first board) or to change boards and the new position of the token is chosen playing the J_2 game rule in the second board. In the second board the rule is just the opposite: in this case, Player II decides to play with J_2 game rules (and remains at the second board) or to change boards and play in the first board with the J_1 game rules.

The game starts with a token at an initial position $x_0 \in \Omega$ in one of the two boards. After the first play the game continues with the same rules (each player decides to change or continue in one board plus the rules for the two different tug-of-war with noise game at each board) until the tokes leaves the domain Ω (at this time the game ends). This gives a random sequence of points (positions of the token) and a stopping time τ (the first time that the position of the token is outside Ω in any of the two boards). The sequence of positions will be denoted by

$$\{(x_0, j_0), (x_1, j_1), \dots, (x_\tau, j_\tau)\},$$

where $x_k \in \Omega$ (and $x_\tau \notin \Omega$) and the second variable, $j_k \in \{1, 2\}$, is just an index that indicates in which board we are playing, with $j_k = 1$ if the position of the token is in the first board and $j_k = 2$ if we are in the second board. As we mentioned, the game ends when the token leaves Ω at some point (x_τ, j_τ). In this case the final payoff (the amount that Player I gets and Player II pays) is given by $f(x_\tau)$ if $j_\tau = 1$ (the token leaves the domain in the first board) and $g(x_\tau)$ if $j_\tau = 2$ (the token leaves in the second board). Hence, taking into account the running payoff and the final payoff, the total payoff of a particular occurrence of the game is given by

$$\text{total payoff} := f(x_\tau)\chi_{\{1\}}(j_\tau) + g(x_\tau)\chi_{\{2\}}(j_\tau) - \varepsilon^2 \sum_{k=0}^{\tau-1} \big(h_1(x_k)\chi_{\{1\}}(j_{k+1}) - h_2(x_k)\chi_{\{j=2\}}(j_{k+1})\big).$$

Notice that the total payoff is the sum of the final payoff (given by $f(x_\tau)$ or by $g(x_\tau)$ according to the board at which the position leaves the domain) and the running payoff that is given by $-\varepsilon^2 h_1(x_k)$ and $\varepsilon^2 h_2(x_k)$ corresponding to the board in which we play at step $k+1$.

Now, the players fix two strategies, S_{I} for Player I and S_{II} for Player II. That is, both players decide to play or to change boards in the respective board, and in each board they select the point to go provided the coin toss of the tug-of-war game is favorable. Notice that the decision on the board where the game takes place is made by the players at each turn (according to the board at which the position is, one of the players makes the choice). Therefore, when the strategies of both players are fixed, the board in which the game occurs at each turn is given (and it is not random). Then, once we fix the strategies S_{I} and S_{II}, everything depends only on the underlying probability: the coin toss that decides when to play tug-of-war and when to move at random (note that this probability is given by α_1 or α_2 and it is different in the two boards) and the coin toss $\big($with probability $\frac{1}{2}$-$\frac{1}{2}\big)$ that decides who chooses the next position of the game if the tug-of-war game is played. With respect to this underlying probability, with fixed strategies S_{I} and S_{II}, we can compute the expected final payoff starting at (x, j)

(recall that $j = 1, 2$ indicates the board at which is the position of the game),

$$\mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x,j)}[\text{total payoff}].$$

The game is said to have a value if

$$w^\varepsilon(x, j) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x,j)}[\text{total payoff}] = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x,j)}[\text{total payoff}]. \tag{1-7}$$

Notice that this value w^ε is the best possible expected outcome that Player I and Player II may expect to obtain playing their best. Here we will prove that this game has a value. The value of the game, w^ε, is composed in fact of two functions, the first one defined in the first board,

$$u^\varepsilon(x) := w^\varepsilon(x, 1),$$

which is the expected outcome of the game if the initial position is at the first board (and the players play their best) and

$$v^\varepsilon(x) := w^\varepsilon(x, 2),$$

which is the expected outcome of the game when the initial position is in the second board. It turns out that these two functions u^ε, v^ε satisfy a system of equations that is called the dynamic programming principle (DPP) in the literature. In our case, the corresponding DPP for the game is given by

$$\begin{cases} u^\varepsilon(x) = \max\big\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\big\}, & x \in \Omega, \\ v^\varepsilon(x) = \min\big\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\big\}, & x \in \Omega, \\ u^\varepsilon(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^\varepsilon(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1-8}$$

where

$$J_1(w)(x) = \alpha_1\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} w(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} w(y)\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x)} w(y)\,dy - \varepsilon^2 h_1(x)$$

and

$$J_2(w)(x) = \alpha_2\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} w(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} w(y)\Big) + (1-\alpha_2)\fint_{B_\varepsilon(x)} w(y)\,dy + \varepsilon^2 h_2(x).$$

Remark 2. From the DPP and the condition $f \ge g$ it is clear that the value functions of the game are ordered. We have

$$u^\varepsilon(y) \ge v^\varepsilon(y)$$

for all $y \in \mathbb{R}^N$.

Remark 3. Observe that the DPP reflects the rules for the game described above. That is, the J_1 rule says that with α_1 probability we play with tug-of-war game and with $(1-\alpha_1)$ probability we play the random walk game with a running payoff that involves h_1. Analogously, in the J_2 game the probability is given by α_2 and the running payoff involves h_2. Also notice that the max and min that are arise in the DPP correspond to the choices of the players to change board (or not). In the first board the first player (who aims to maximize the expected outcome) is the one who decides while in the second board the second player (who wants to minimize) decides.

Our first result says that the value functions of the game converge uniformly as $\varepsilon \to 0$ to a pair of continuous functions (u, v) that is a viscosity solution to a system of partial differential equations in which two equations of obstacle type appear.

Theorem 4. *There exists a sequence $\varepsilon_j \to 0$ such that $(u^{\varepsilon_j}, v^{\varepsilon_j})$ converges to a pair of continuous functions (u, v), that is,*

$$u^{\varepsilon_j} \rightrightarrows u, \quad v^{\varepsilon_j} \rightrightarrows v$$

uniformly in $\overline{\Omega}$. The limit pair is a viscosity solution to the two membrane system with two different p-Laplacians, that is,

$$\begin{cases} u(x) \geq v(x), & & x \in \Omega, \\ -\Delta_p^1 u(x) + h_1(x) \geq 0, & -\Delta_q^1 v(x) - h_2(x) \leq 0, & x \in \Omega, \\ -\Delta_p^1 u(x) + h_1(x) = 0, & -\Delta_q^1 v(x) - h_2(x) = 0, & x \in \{u > v\} \cap \Omega, \\ u(x) = f(x), & & x \in \partial\Omega, \\ v(x) = g(x), & & x \in \partial\Omega. \end{cases} \tag{1-9}$$

Here p and q are given by

$$\frac{\alpha_1}{1-\alpha_1} = \frac{p-2}{N+2} \quad \textit{and} \quad \frac{\alpha_2}{1-\alpha_2} = \frac{q-2}{N+2}. \tag{1-10}$$

Remark 5. Using that $u^\varepsilon \rightrightarrows u$, $v^\varepsilon \rightrightarrows v$ and that $u^\varepsilon \geq v^\varepsilon$ we immediately obtain

$$u(y) \geq v(y) \quad \text{for all } \; y \in \mathbb{R}^N.$$

Remark 6. We can write the system (1-9) as

$$\begin{cases} \min\{-\Delta_p^1 u(x) + h_1(x), (u-v)(x)\} = 0, & x \in \Omega, \\ \max\{-\Delta_q^1 v(x) - h_2(x), (v-u)(x)\} = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \\ v(x) = g(x), & x \in \partial\Omega. \end{cases}$$

Here the first equation says that u is a solution to the obstacle problem for the p-Laplacian with v as a obstacle and boundary datum f, and the second equation says that v is a solution to the obstacle problem for the q-Laplacian with u as a obstacle from above and boundary datum g.

This formulation corresponds to a two membrane problem in which the membranes are clamped on the boundary of the domain and each membrane acts as an obstacle for the other.

Remark 7. Since in general there is no uniqueness for the two membranes problem we can only show convergence taking a sequence $\varepsilon_j \to 0$ using a compactness argument.

Let us briefly comment on the main difficulties that appear in the proof of this result. To show that the DPP has a solution we argue using monotonicity arguments in the spirit of Perron's method (a solution is obtained as the supremum of subsolutions). Once we proved existence of a solution to the DPP we use this solution to construct quasioptimal strategies for the players and show that the game has a value that coincides with a solution to the DPP (this fact implies uniqueness for solutions to the DPP). At this point we want to mention the cruciality of the rule that forces one round of play of the game when one of the players decides to change boards. If one changes boards without playing a round in the other board the game may never end (and even if we penalize games that never end it is not clear that the game has a value). See [Peres et al. 2009] for an example of a tug-of-war game that does not have a value. After proving existence and uniqueness for the DPP and the existence of a value for the game we study its behavior as $\varepsilon \to 0$. Uniform convergence will follow from a variant of the Arzelà–Ascoli lemma; see Lemma 20 (this idea was used before to obtain convergence of value functions of games; see several examples in [Blanc and Rossi 2019]). To this end we need that when the game starts close to the boundary in any of the two boards any of the two players has a strategy that forces the game to end close to the starting point in a controlled number of plays with large probability. For example, starting in the first board the first player may choose the strategy to never change boards and to point to a boundary point when the tug-of-war game is played. One can show that this strategy gives the desired one-sided estimate. However, starting in the first board, to find a strategy for Player II that achieves similar bounds is trickier since the player who may decide to change boards is Player I. To obtain such bounds for the terminal position and the expected number of plays in this case is one of the main difficulties that we deal with. Once we proved uniform convergence of the value functions we use the DPP to obtain, using the usual viscosity approach, that the limit pair is a solution to the two membranes problem.

1.3.2. *Second game.* Let us consider a variant of the previous game in which the possibility of the players to change boards also depends on a coin toss.

This new game has the following rules: If the position of the game is at $(x_k, 1)$ the players toss a fair coin $\big($probability $\frac{1}{2}$-$\frac{1}{2}\big)$, and if Player I wins, he decides to play the J_1 game in the first board or to play the J_2 game in the second board. On the other hand, if the winner is Player II the only option is to play J_1 in the first board. If the position is in the second board, say at $(x_k, 2)$, the situation is analogous but with the roles of the players reversed: the players toss a fair coin again, and if Player II wins, she decides between playing J_2 in the second board or jumping to the first board and playing J_1, while if Player I wins the only option is to play J_2 in the second board. Here the rules of J_1 and J_2 are exactly as before, the only thing that we changed is that the decision to change boards or not is also dependent on a fair coin toss.

This game has associated to it the following DPP:

$$
\begin{cases}
u^{\varepsilon}(x) = \frac{1}{2}\max\big\{J_1(u^{\varepsilon})(x),\, J_2(v^{\varepsilon})(x)\big\} + \frac{1}{2}J_1(u^{\varepsilon})(x), & x \in \Omega,\\
v^{\varepsilon}(x) = \frac{1}{2}\min\big\{J_1(u^{\varepsilon})(x),\, J_2(v^{\varepsilon})(x)\big\} + \frac{1}{2}J_2(v^{\varepsilon})(x), & x \in \Omega,\\
u^{\varepsilon}(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega,\\
v^{\varepsilon}(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
$$

This DPP also reflects the rules of the game. For instance, the first equation says that with probability $\frac{1}{2}$ the first player decides to play J_1 or to change boards and play J_2 (thus the term $\max\{J_1(u^{\varepsilon})(x),\, J_2(v^{\varepsilon})(x)\}$ appears) and with probability $\frac{1}{2}$ the position stays in the first board (they just play J_1).

Remark 8. Also in this case, from the DPP and the condition $f \geq g$ it is immediate that

$$u^{\varepsilon}(y) \geq v^{\varepsilon}(y)$$

holds for all $y \in \mathbb{R}^N$.

In this case the pair $(u^{\varepsilon}, v^{\varepsilon})$ also converges uniformly along a subsequence $\varepsilon_j \to 0$ to a continuous pair (u, v), and this limit pair is also a viscosity solution to the two membrane problem with an extra condition in the contact set.

Theorem 9. *There exists a sequence $\varepsilon_j \to 0$ such that $(u^{\varepsilon_j}, v^{\varepsilon_j})$ converges to a pair of continuous functions (u, v), that is,*

$$u^{\varepsilon_j} \rightrightarrows u, \quad v^{\varepsilon_j} \rightrightarrows v$$

uniformly in $\overline{\Omega}$. The limit pair is a viscosity solution to the two membrane system with the two different p-Laplacians (1-9), with p and q given by (1-10). Moreover, the extra condition

$$(-\Delta_p^1 u(x) + h_1(x)) + (-\Delta_q^1 v(x) - h_2(x)) = 0, \quad x \in \Omega, \tag{1-11}$$

holds.

Remark 10. Let us observe that the extra condition (1-11) trivially holds in $\{u > v\}$ (since u and v solve $-\Delta_p^1 u(x) + h_1(x) = 0$ and $-\Delta_q^1 v(x) - h_2(x) = 0$ respectively). Then this extra condition gives us new information in the contact set $\{u = v\}$. Notice that this extra condition (the sum of the two equations is equal to zero) is similar to the one that appears in [Caffarelli et al. 2017] — compare (1-5) — but it is not the same as the one assumed in [Caffarelli et al. 2018] to obtain regularity of the solutions — see (1-6) — since for the normalized p-Laplacian it is not true that $-\Delta_p^1 u(x) = \Delta_q^1(-u)(x)$ unless $p = q$.

The paper is organized as follows: In Section 2 we analyze the first game; in Section 2.1 we prove that the game has a value and that this value is the unique solution to the DPP. The proof of Theorem 4 is divided across Sections 2.2 and 2.3. In the first one we prove uniform convergence along a subsequence and in the second we show that the uniform limit is a viscosity solution to the PDE system (1-9).

In Section 3 we include a brief description of the analysis for the second game (the arguments used to show uniform convergence are quite similar). Here we focus on the details needed to show that we obtain an extra condition in the contact set.

Finally, in Section 4 we include some remarks and comment on possible extensions of our results.

2. Analysis for the first game

2.1. ***Existence and uniqueness for the DPP.*** In this section we first prove that there is a solution to the DPP (1-8), next we show that the existence of a solution to the DPP implies that the game has a value (it allows us to find quasioptimal strategies for the players), and at the end we obtain the uniqueness of solutions to the DPP.

To show existence of a solution to the DPP we use a variant of Perron's method (that is, a solution can be obtained as supremum of subsolutions).

Let us consider the set of functions

$$\mathscr{A} = \{(\underline{u}^\varepsilon, \underline{v}^\varepsilon) : \underline{u}^\varepsilon \text{ and } \underline{v}^\varepsilon \text{are bounded functions such that (2-1) holds}\}$$

with

$$\begin{cases} u^\varepsilon(x) \le \max\bigl\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\bigr\}, & x \in \Omega, \\ v^\varepsilon(x) \le \min\bigl\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\bigr\}, & x \in \Omega, \\ u^\varepsilon(x) \le f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^\varepsilon(x) \le g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2-1}$$

Notice that (2-1) is just the DPP with inequalities that say that $(\underline{u}^\varepsilon, \underline{v}^\varepsilon)$ is a subsolution to the DPP (1-8). For the precise definition of sub- and supersolutions to DPP systems we refer to [Miranda and Rossi 2020; 2023].

Let us begin proving that $\mathscr{A}$ is nonempty. To this end we introduce an auxiliary function. As $\Omega \subset \mathbb{R}^N$ is bounded there exists $R > 0$ such that $\Omega \Subset B_R(0) \setminus \{0\}$ (without loss of generality we may assume that $0 \notin \Omega$). Consider the function

$$z_0(x) = \begin{cases} 2K(|x|^2 - R) - M & \text{if } x \in \overline{B_R(0)}, \\ \quad -M & \text{if } x \in \mathbb{R}^N \setminus B_R(0). \end{cases}$$

This function has the following properties: the function z_0 is $C^2(\Omega)$, and for $x \in B_R(0) \setminus \{0\}$,

$$\Delta z_0(x) = (z_0)_{rr} + \frac{N-1}{r}(z_0)_r = 4K + \frac{N-1}{r} 4Kr = 4K + 4K(N-1) \quad \text{and} \quad \Delta^1_\infty z_0(x) = (z_0)_{rr}(x) = 4K.$$

Notice that when we compute the infinity Laplace operator of z_0 we have to pay special care at the origin (where the gradient of z_0 vanishes), since the operator is not well defined there. In doing this we use that z_0 is a radial function and compute the infinity Laplacian in the classical sense at points in $B_R(0) \setminus \{0\}$ (where the gradient does not vanish).

Then we get

$$\Delta^1_p z_0(x) = \frac{1}{2}\alpha_1(4K) + \frac{(1-\alpha_1)}{2(N+2)}(4K + 4K(N-1)) \ge 4K,$$

$$\Delta^1_q z_0(x) = \frac{1}{2}\alpha_2(4K) + \frac{(1-\alpha_2)}{2(N+2)}(4K + 4K(N-1)) \ge 4K.$$

We are ready to prove the first lemma.

Lemma 11. *For ε small enough, $\mathscr{A} \neq \varnothing$.*

Proof. We consider z_0 with the constants

$$K = \max\{\|h_1\|_\infty, \|h_2\|_\infty\} + 1 \quad \text{and} \quad M = \max\{\|f\|_\infty, \|g\|_\infty\} + 1,$$

and claim that

$$(z_0, z_0) \in \mathscr{A}.$$

Let us prove this claim. First, we observe that the inequality (2-1) holds for $x \in \mathbb{R}^N \setminus \Omega$. Then we are left to prove that for $x \in \Omega$,

$$z_0(x) \le \min\bigl\{J_1(z_0)(x),\, J_2(z_0)(x)\bigr\}.$$

That is, we aim to show that

$$0 \le \min\bigl\{J_1(z_0)(x) - z_0(x),\, J_2(z_0)(x) - z_0(x)\bigr\}. \tag{2-2}$$

Using Taylor expansions we obtain

$$\begin{aligned} J_1(z_0)(x) - z_0(x) &= \alpha_1\Bigl(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x)}(z_0(y)-z_0(x)) + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x)}(z_0(y)-z_0(x))\Bigr) \\ &\qquad\qquad + (1-\alpha_1)\fint_{B_\varepsilon(x)}(z_0(y)-z_0(x))\,dy - \varepsilon^2 h_1(x) \\ &= \Bigl(\frac{1}{2}\alpha_1\Delta^1_\infty z_0(x) + \frac{(1-\alpha_1)}{2(N+2)}\Delta z_0(x)\Bigr)\varepsilon^2 - \varepsilon^2 h_1(x) + o(\varepsilon^2). \end{aligned}$$

Analogously,

$$J_2(z_0)(x) - z_0(x) = \Bigl(\frac{1}{2}\alpha_2\Delta^1_\infty z_0(x) + \frac{(1-\alpha_2)}{2(N+2)}\Delta z_0(x)\Bigr)\varepsilon^2 + \varepsilon^2 h_2(x) + o(\varepsilon^2).$$

If we come back to (2-2) and we divide by ε^2 we obtain

$$0 \le \min\bigl\{\Delta^1_p z_0(x) - h_1(x),\, \Delta^1_q z_0(x) + h_2(x)\bigr\}. \tag{2-3}$$

Using the properties of z_0 we have

$$\Delta^1_p z_0(x) - h_1(x) \ge 3K \quad \text{and} \quad \Delta^1_q z_0(x) + h_2(x) \ge 3K.$$

Thus, the inequality (2-3) holds for ε small enough. This ends the proof. □

Remark 12. We can define a different auxiliary function z^ε as the solution to the following problem:

$$\begin{cases} z^\varepsilon(x) = \min\limits_{i\in\{1,2\}}\Bigl\{\alpha_i\Bigl(\frac{1}{2}\sup\limits_{y\in B_\varepsilon(x)} z^\varepsilon(y) + \frac{1}{2}\inf\limits_{y\in B_\varepsilon(x)} z^\varepsilon(y)\Bigr) + (1-\alpha_i)\fint_{B_\varepsilon(x)} z^\varepsilon(y)\,dy\Bigr\} - \varepsilon^2 K, & x \in \Omega, \\ z^\varepsilon(x) = -M, & x \notin \Omega. \end{cases} \tag{2-4}$$

The existence of this function is given in [Blanc et al. 2017, Theorem 1.5]. In fact, (2-4) is the DPP that corresponds to a game in which one player (the one that wants to minimize the expected payoff) chooses the coin that decides the game to play between tug-of-war and a random walk.

If we argue as before we can prove that $(z^\varepsilon, z^\varepsilon) \in \mathscr{A}$.

Now our goal is to show that the functions $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ are uniformly bounded. To prove this fact we will need some lemmas. Let us consider the function $w_0 = -z_0$. This function has the following properties:

$$\Delta w_0(x) = -\Delta z_0 = -4K - 4K(N-1) \quad \text{and} \quad \Delta^1_\infty w_0(x) = -\Delta^1_\infty w_0(x) = -4K.$$

Then we have

$$\Delta_p^1 w_0(x) = \frac{1}{2}\alpha_1(-4K) + \frac{(1-\alpha_1)}{2(N+2)}(-4K - 4K(N-1)) \leq -4K,$$
$$\Delta_q^1 w_0(x) = \frac{1}{2}\alpha_2(-4K) + \frac{(1-\alpha_2)}{2(N+2)}(-4K - 4K(N-1)) \leq -4K.$$

Let us prove a technical lemma.

Lemma 13. *Given* $K = \max\{\|h_1\|_\infty, \|h_1\|_\infty\} + 1$ *and* $M = \max\{\|f\|_\infty, \|g\|_\infty\} + 1$, *there exists* $\varepsilon_0 > 0$ *such that the function* w_0 *satisfies*

$$\begin{cases} w_0(x) \geq \max\{J_1(w_0)(x), J_2(w_0)(x)\} + K\varepsilon^2, & x \in \Omega, \\ w_0(x) \geq M, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2-5}$$

for every $\varepsilon < \varepsilon_0$.

Proof. First, let us observe that the inequality $w^\varepsilon(x) \geq M$ holds for $x \in \mathbb{R}^N \setminus \Omega$ when $\widetilde{M}$ is large enough. Then we are left to prove that for $x \in \Omega$,

$$w_0(x) \geq \max\{J_1(w_0)(x), J_2(w_0)(x)\} + K\varepsilon^2.$$

That is,

$$0 \geq \max\{J_1(w_0)(x) - w_0(x), J_2(w_0)(x) - w_0(x)\} + K\varepsilon^2. \tag{2-6}$$

Using Taylor expansions we obtain

$$\begin{aligned} J_1(w_0)(x) - w_0(x) &= \alpha_1\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)}(w_0(y) - w_0(x)) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)}(w_0(y) - w_0(x))\Big) \\ &\qquad + (1-\alpha_1)\fint_{B_\varepsilon(x)}(w_0(y) - w_0(x))\,dy + \varepsilon^2 h_1(x) \\ &= \Big(\frac{\alpha_1}{2}\Delta_\infty^1 w_0(x) + \frac{(1-\alpha_1)}{2(N+2)}\Delta w_0(x)\Big)\varepsilon^2 - \varepsilon^2 h_1(x) + o(\varepsilon^2). \end{aligned}$$

Analogously,

$$J_2(w_0)(x) - w_0(x) = \Big(\frac{\alpha_2}{2}\Delta_\infty^1 w_0(x) + \frac{(1-\alpha_2)}{2(N+2)}\Delta w_0(x)\Big)\varepsilon^2 - \varepsilon^2 h_2(x) + o(\varepsilon^2).$$

If we come back to (2-6) and we divide by ε^2 we get

$$0 \geq \max\{\Delta_p^1 w_0(x) - h_1(x), \Delta_q^1 w_0(x) + h_2(x)\} + K. \tag{2-7}$$

Using the properties of w_0 we arrive at

$$\Delta_p^1 w_0(x) - h_1(x) \leq -4K \quad \text{and} \quad \Delta_q^1 w_0(x) + h_2(x) \leq -3K.$$

Thus, the inequality (2-7) holds for ε small enough. This ends the proof. □

Our next result says that the subsolutions to the DPP (pairs $(\underline{u}, \underline{v}) \in \mathscr{A}$) are indeed bounded by w_0. This shows that functions in $\mathscr{A}$ are uniformly bounded. From the proof of the following result one can obtain a comparison principle for the DPP.

Lemma 14. *Let* $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ (bounded subsolutions to the DPP (1-8)), *and let* w^ε *be a function that satisfies* (2-5), *that is,*

$$\begin{cases} w^\varepsilon(x) \geq \max\big\{J_1(w^\varepsilon)(x),\, J_2(w^\varepsilon)(x)\big\} + K\varepsilon^2, & x \in \Omega, \\ w^\varepsilon(x) \geq M, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then it holds that

$$\underline{u}^\varepsilon(x) \leq w^\varepsilon(x) \quad \textit{and} \quad \underline{v}^\varepsilon(x) \leq w^\varepsilon(x), \qquad x \in \mathbb{R}^N.$$

Proof. We argue by contradiction. Assume that

$$\max\big\{\sup(\underline{u}^\varepsilon - w^\varepsilon),\, \sup(\underline{v}^\varepsilon - w^\varepsilon)\big\} = \theta > 0.$$

It is clear that

$$\underline{u}^\varepsilon(x) \leq M \leq w^\varepsilon(x) \quad \text{and} \quad \underline{v}^\varepsilon(x) \leq M \leq w^\varepsilon(x)$$

for $x \notin \Omega$. Thus, we have to concentrate on what happens inside Ω. We divide the proof into two cases.

Case 1: Assume that

$$\sup(\underline{v}^\varepsilon - w^\varepsilon) = \theta.$$

Given $n \in \mathbb{N}$ let $x_n \in \Omega$ be such that

$$\theta - \frac{1}{n} < (\underline{v}^\varepsilon - w^\varepsilon)(x_n).$$

We use the inequalities satisfied by the involved functions to obtain

$$\begin{aligned} \theta - \frac{1}{n} &< (\underline{v}^\varepsilon - w^\varepsilon)(x_n) \\ &\leq J_2(\underline{v}^\varepsilon)(x_n) - J_2(w^\varepsilon)(x_n) \\ &= \alpha_2\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) - \frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} w^\varepsilon(y) - \frac{1}{2}\inf_{y\in B_\varepsilon(x_n)} w^\varepsilon(y)\Big) \\ &\qquad\qquad + (1-\alpha_2) \fint_{B_\varepsilon(x_n)} (\underline{v}^\varepsilon - w^\varepsilon)(y)\,dy + \varepsilon^2 h_2(x_n) - \varepsilon^2 K \\ &\leq \alpha_2\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) - \frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} w^\varepsilon(y) + \frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} (\underline{v}^\varepsilon - w^\varepsilon)(y)\Big) \\ &\qquad\qquad + (1-\alpha_2) \fint_{B_\varepsilon(x_n)} (\underline{v}^\varepsilon - w^\varepsilon)(y)\,dy + \varepsilon^2 h_2(x_n) - \varepsilon^2 K. \end{aligned}$$

Now we use that

$$\sup_{y\in B_\varepsilon(x_n)} (\underline{v}^\varepsilon - w^\varepsilon)(y) \leq \theta \quad \text{and} \quad \fint_{B_\varepsilon(x_n)} (\underline{v} - w)(y)\,dy \leq \theta$$

to arrive at

$$\theta - \frac{2}{n\alpha_2} < \sup_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) - \sup_{y\in B_\varepsilon(x_n)} w^\varepsilon(y) + (2h_2(x_n) - 2K)\frac{\varepsilon^2}{\alpha_2}.$$

Take $y_n \in B_\varepsilon(x_n)$ such that

$$\sup_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) - \frac{1}{n} < \underline{v}^\varepsilon(y_n).$$

Then we get

$$\theta - \frac{2}{n\alpha_2} < \underline{v}^\varepsilon(y_n) + \frac{1}{n} - \sup_{y\in B_\varepsilon(x_\varepsilon)} w^\varepsilon(y)(2h_2(x_n) - 2K\varepsilon^2) \le \underline{v}^\varepsilon(y_n) + \frac{1}{n} - w^\varepsilon(y_n) - \frac{\varepsilon^2}{\alpha_2}.$$

Here we use that $h_2(x) - K \le -1$. Hence

$$\theta - \frac{2-\alpha_2}{n\alpha_2} + \frac{\varepsilon^2}{\alpha_2} < (\underline{v}^\varepsilon - w^\varepsilon)(y_n) \le \theta,$$

which leads to a contradiction if $n \in \mathbb{N}$ is large enough that

$$-\frac{2-\alpha_2}{n\alpha_2} + \frac{\varepsilon^2}{\alpha_2} > 0,$$

since in this case we obtain

$$\theta < \theta - \frac{2-\alpha_2}{n\alpha_2} + \frac{\varepsilon^2}{\alpha_2} < (\underline{u}^\varepsilon - w^\varepsilon)(y_n) \le \theta.$$

This ends the proof in the first case.

Case 2: Assume that

$$\sup(\underline{u}^\varepsilon - w^\varepsilon) = \theta.$$

In this case we take again a sequence $x_n \in \Omega$ such that

$$\theta - \frac{1}{n} < (\underline{u}^\varepsilon - w^\varepsilon)(x_n).$$

Let us assume first that

$$\max\left\{J_1(\underline{u}^\varepsilon)(x_n),\, J_2(\underline{v}^\varepsilon)(x_n)\right\} = J_2(\underline{v}^\varepsilon)(x_n),$$

and then we obtain

$$(\underline{u}^\varepsilon - w^\varepsilon)(x_n) \le J_2(\underline{v}^\varepsilon)(x_n) - J_2(w^\varepsilon)(x_n).$$

We are again in the first case and we arrive at a contradiction arguing as before.

Finally, let us assume that

$$\max\left\{J_1(\underline{u}^\varepsilon)(x_n),\, J_2(\underline{v}^\varepsilon)(x_n)\right\} = J_1(\underline{u}^\varepsilon)(x_n),$$

and then we obtain

$$(\underline{u}^\varepsilon - w^\varepsilon)(x_n) \le J_1(\underline{u}^\varepsilon)(x_n) - J_1(w^\varepsilon)(x_n).$$

If we argue as in the first case we arrive at a contradiction. This ends the proof. □

Now, using that w_0 is continuous $\mathbb{R}^N$ and hence bounded in the ball $\overline{B}_R$, we can deduce that there exists a constant $\Lambda > 0$ that depends on the data f, g, h and the domain Ω such that $w_0(x) \le \Lambda$. Then, using the previous lemmas, we obtain a uniform bound for functions in $\mathscr{A}$.

Theorem 15. *There exists a constant $\Lambda > 0$ that depends on f, g, h and Ω such that for every $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ it holds that*

$$\underline{u}^\varepsilon(x) \le \Lambda \quad \textit{and} \quad \underline{v}^\varepsilon(x) \le \Lambda,$$

for every $x \in \mathbb{R}^N$ and every $\varepsilon \le \varepsilon_0$ (here ε_0 is given by Lemma 13).

Proof. It follows from Lemmas 13 and 14 and the boundedness of w_0. □

With this result at hand we can define for $x \in \mathbb{R}^N$,

$$u^\varepsilon(x) = \sup_{(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}} \underline{u}^\varepsilon(x) \quad \text{and} \quad v^\varepsilon(x) = \sup_{(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}} \underline{v}^\varepsilon(x).$$

The previous result, Theorem 15, gives that these two functions u^ε and v^ε are well defined and bounded. It turns out that they are a solution to the DPP.

Theorem 16. *The pair of functions $(u^\varepsilon, v^\varepsilon)$ is a solution to the DPP (1-8).*

Proof. First, let us show that $(u^\varepsilon, v^\varepsilon) \in \mathscr{A}$. Given $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ and $x \in \Omega$ we have that

$$\underline{u}^\varepsilon(x) \le \max\big\{ J_1(\underline{u}^\varepsilon)(x), J_2(\underline{v}^\varepsilon)(x) \big\}.$$

Taking the supremum over $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ we obtain

$$\underline{u}^\varepsilon(x) \le \max\big\{ J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x) \big\},$$

and hence (taking the supremum in the left-hand side) we conclude that

$$u^\varepsilon(x) \le \max\big\{ J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x) \big\}.$$

An analogous computation for the second equation shows that v^ε satisfies

$$v^\varepsilon(x) \le \min\big\{ J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x) \big\}$$

for $x \in \Omega$. Finally, as $\underline{u}^\varepsilon(x) \le f(x)$ and $\underline{v}^\varepsilon(x) \le g(x)$ for $x \in \mathbb{R}^N \setminus \Omega$, taking the supremum we obtain $u^\varepsilon(x) \le f(x)$ and $v^\varepsilon(x) \le g(x)$ for $x \in \mathbb{R}^N \setminus \Omega$, and we conclude that $(u^\varepsilon, v^\varepsilon) \in \mathscr{A}$.

We have a set of inequalities for the pair $(u^\varepsilon, v^\varepsilon) \in \mathscr{A}$. To show that the pair is indeed a solution to the DPP we have to show that they are in fact equalities. To prove this fact we argue by contradiction. Assume that we have a strict inequality for some $x_0 \in \mathbb{R}^N$. If $x_0 \in \mathbb{R}^N \setminus \Omega$ and we have $u^\varepsilon(x_0) < f(x_0)$ we will reach a contradiction considering

$$u_0^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & \text{if } x \ne x_0, \\ u^\varepsilon(x_0) + \delta & \text{if } x = x_0, \end{cases}$$

with $\delta > 0$ small enough such that $u^\varepsilon(x_0) + \delta < f(x_0)$. Indeed, one can check that the pair $(u_0^\varepsilon, v^\varepsilon)$ belongs to $\mathscr{A}$ but at x_0 we have $u_0^\varepsilon(x_0) = u^\varepsilon(x_0) + \delta > u^\varepsilon(x_0) = \sup_{\mathscr{A}} \underline{u}^\varepsilon(x_0)$, a contradiction. A similar argument can be used when $x_0 \in \mathbb{R}^N \setminus \Omega$, and we have $v^\varepsilon(x_0) < g(x_0)$. We conclude that $u^\varepsilon(x) = f(x)$ and $v^\varepsilon(x) = g(x)$ for every $x \in \mathbb{R}^N \setminus \Omega$.

Now, let us assume that the point at which we have a strict inequality is inside Ω, $x_0 \in \Omega$. First, assume that we have

$$u^\varepsilon(x_0) < \max\{J_1(u^\varepsilon)(x_0), J_2(v^\varepsilon)(x_0)\}.$$

Let us consider

$$\max\{J_1(u^\varepsilon)(x_0), J_2(v^\varepsilon)(x_0)\} - u^\varepsilon(x_0) = \delta > 0,$$

and, as before, the function

$$u_0^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & \text{if } x \neq x_0, \\ u^\varepsilon(x_0) + \frac{1}{2}\delta & \text{if } x = x_0. \end{cases}$$

Then we have

$$u_0^\varepsilon(x_0) = u^\varepsilon(x_0) + \tfrac{1}{2}\delta < \max\{J_1(u^\varepsilon)(x_0), J_2(v^\varepsilon)(x_0)\}.$$

At other points $x \in \Omega$ we also have

$$u_0^\varepsilon(x) \le \max\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\} \le \max\{J_1(u_0^\varepsilon)(x), J_2(v^\varepsilon)(x)\}.$$

Finally, concerning v^ε we get (at any point $x \in \Omega$),

$$v^\varepsilon(x) \le \min\{J_1(u_0^\varepsilon)(x), J_2(v^\varepsilon)(x)\}.$$

Hence, we have that the pair $(u_0^\varepsilon, v^\varepsilon)$ belongs to $\mathscr{A}$, getting a contradiction as before, since $u_0^\varepsilon(x_0) > u^\varepsilon(x_0)$.

Analogously, one can deal with the case in which $x_0 \in \Omega$ and

$$v^\varepsilon(x_0) < \min\{J_1(u^\varepsilon)(x_0), J_2(v^\varepsilon)(x_0)\}. \qquad \square$$

Corollary 17. *There exists a constant* $\Lambda > 0$ *such that*

$$|u^\varepsilon(x)| < \Lambda \quad \textit{and} \quad |v^\varepsilon(x)| < \Lambda$$

for all $x \in \mathbb{R}^N$.

Proof. Every solution to the DPP belongs to $\mathcal{A}$. Hence the result follows from Theorem 15. $\square$

Now, for completeness, we include the precise statement of the optional stopping theorem (a key tool from probability theory that we will use in what follows).

Optional stopping theorem. We briefly recall (see [Williams 1991]) that a sequence of random variables $\{M_k\}_{k\ge1}$ is a supermartingale (respectively, submartingale) if

$$\mathbb{E}[M_{k+1} \mid M_0, M_1, \dots, M_k] \le M_k \quad \text{(respectively, } \ge\text{)}.$$

Suppose that τ is a stopping time such that one of the following conditions holds:

(a) The stopping time τ is bounded almost surely.

(b) $\mathbb{E}[\tau] < \infty$ and there exists a constant $c > 0$ such that

$$\mathbb{E}[M_{k+1} - M_k \mid M_0, \ldots, M_k] \leq c.$$

(c) There exists a constant $c > 0$ such that $|M_{\min\{\tau,k\}}| \leq c$ almost surely for every k.

For such a τ, the optional stopping theorem (OST) states that

$$\mathbb{E}[M_\tau] \leq \mathbb{E}[M_0] \quad \text{(respectively, } \geq \text{)}$$

if $\{M_k\}_{k\geq 0}$ is a supermartingale (respectively, submartingale). For the proof of this classical result, see [Doob 1971; Williams 1991].

Let us finish this section proving the following theorem.

Theorem 18. *The functions u^ε and v^ε that satisfy the DPP (1-8) are the functions that give the value of the game in (1-7). This means that the function*

$$w^\varepsilon(x, j) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{(x,j)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}] = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x,j)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}]$$

satisfies

$$w^\varepsilon(x, 1) = u^\varepsilon(x) \quad \textit{and} \quad w^\varepsilon(x, 2) = v^\varepsilon(x)$$

for any pair $(u^\varepsilon, v^\varepsilon)$ that solves the DPP, that is, for any pair that satisfies

$$\begin{cases} u^\varepsilon(x) = \max\big\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\big\}, & x \in \Omega, \\ u^\varepsilon(x) = \min\big\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\big\}, & x \in \Omega, \\ u^\varepsilon(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^\varepsilon(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Proof. Fix $\delta > 0$. Assume that we start at a point in the first board, $(x_0, 1)$. Then we choose a strategy S_{I}^* for Player I using the solution to the DPP (1-8) as follows: Whenever $j_k = 1$, Player I decides to stay in the first board if

$$\max\big\{J_1(u^\varepsilon)(x_k), J_2(v^\varepsilon)(x_k)\big\} = J_1(u^\varepsilon)(x_k),$$

and in this case Player I chooses a point

$$x^{\mathrm{I}}_{k+1} = S_{\mathrm{I}}^*((x_0, j_0), \ldots, (x_k, j_k)) \quad \text{such that} \quad \sup_{y\in B_\varepsilon(x_k)} u^\varepsilon(y) - \frac{\delta}{2^{k+1}} \leq u^\varepsilon(x^{\mathrm{I}}_{k+1}).$$

On the other hand, Player I decides to jump to the second board if

$$\max\big\{J_1(u^\varepsilon)(x_k), J_2(v^\varepsilon)(x_k)\big\} = J_2(v^\varepsilon)(x_k),$$

and in this case Player I chooses a point

$$x^{\mathrm{I}}_{k+1} = S_{\mathrm{I}}^*((x_0, j_0), \ldots, (x_k, j_k)) \quad \text{such that} \quad \sup_{y\in B_\varepsilon(x_k)} v^\varepsilon(y) - \frac{\delta}{2^{k+1}} \leq v^\varepsilon(x^{\mathrm{I}}_{k+1}).$$

Given this strategy for Player I and any strategy for Player II, we consider the sequence of random variables

$$M_k = w^\varepsilon(x_k, j_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k},$$

where $w^\varepsilon(x_k, 1) = u^\varepsilon(x_k),\ w^\varepsilon(x_k, 2) = v^\varepsilon(x_k)$ and

$$\chi_{\{j=i\}}(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Let us see that $(M_k)_{k\geq 0}$ is a *submartingale*. To this end, we need to estimate

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[M_{k+1} \mid M_0, \dots, M_k].$$

Let us consider several cases.

Case 1: Suppose that $j_k = 1$ and $j_{k+1} = 1$ (that is, we stay in the first board). Then

$$\begin{aligned}
&\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[M_{k+1} \mid M_0, \dots, M_k] \\
&= \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}\Big[u^\varepsilon(x_{k+1}) - \varepsilon^2 \sum_{l=0}^{k}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0, \dots, M_k\Big] \\
&= \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}\Big[u^\varepsilon(x_{k+1}) - \varepsilon^2 h_1(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0, \dots, M_k\Big] \\
&= \alpha_1\Big(\frac{1}{2}u^\varepsilon(x^{\mathrm{I}}_{k+1}) + \frac{1}{2}u^\varepsilon(x^{\mathrm{II}}_{k+1})\Big) + (1-\alpha_1) \fint_{B_\varepsilon(x_k)} u^\varepsilon(y)\,dy - \varepsilon^2 h_1(x_k) \\
&\qquad - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}}.
\end{aligned}$$

In the second equality, we used that $j_{k+1} = 1$. We obtain

$$\begin{aligned}
&\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[M_{k+1} \mid M_0, \dots, M_k] \\
&\geq \alpha_1\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_k)} u^\varepsilon(y) - \frac{\delta}{2^{k+1}} + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_k)} u^\varepsilon(y)\Big) + (1-\alpha_1) \fint_{B_\varepsilon(x_k)} u^\varepsilon(y)\,dy - \varepsilon^2 h_1(x_k) \\
&\qquad - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \\
&\geq J_1(u^\varepsilon)(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} \\
&= \max\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} \\
&= u^\varepsilon(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} = M_k.
\end{aligned}$$

Case 2: Suppose that $j_k = 1$ and $j_{k+1} = 2$ (that is, we jump to the second board). Then

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}[M_{k+1} \,|\, M_0,\dots,M_k] \\
&\quad = \mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}\Big[v^\varepsilon(x_{k+1}) - \varepsilon^2 \sum_{l=0}^{k} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0,\dots,M_k \Big] \\
&\quad = \mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}\Big[v^\varepsilon(x_{k+1}) + \varepsilon^2 h_2(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0,\dots,M_k \Big].
\end{aligned}$$

In the second equality, we used that $j_{k+1} = 2$. We then obtain

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}[M_{k+1} \mid M_0,\dots,M_k] \\
&\quad \geq \alpha_2\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_k)} v^\varepsilon(y) - \frac{\delta}{2^{k+1}} + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_k)} v^\varepsilon(y)\Big) + (1-\alpha_2) \fint_{B_\varepsilon(x_k)} v^\varepsilon(y)\,dy + \varepsilon^2 h_2(x_k) \\
&\qquad\qquad - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \\
&\quad = J_2(v^\varepsilon)(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} \\
&\quad = \max\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} \\
&\quad = u^\varepsilon(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} = M_k.
\end{aligned}$$

Case 3: Suppose that $j_k = 2$ and $j_{k+1} = 2$ (that is, we stay in the second board). Then

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}[M_{k+1} \,|\, M_0,\dots,M_k] \\
&\quad = \mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}\Big[v^\varepsilon(x_{k+1}) - \varepsilon^2 \sum_{l=0}^{k} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0,\dots,M_k \Big] \\
&\quad = \mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}\Big[v^\varepsilon(x_{k+1}) + \varepsilon^2 h_2(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0,\dots,M_k \Big].
\end{aligned}$$

In the second equality, we used that $j_{k+1} = 2$. Therefore, we have

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}[M_{k+1} \mid M_0,\dots,M_k] \\
&\qquad \geq \alpha_2\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_k)} v^\varepsilon(y) - \frac{\delta}{2^{k+1}} + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_k)} v^\varepsilon(y)\Big) + (1-\alpha_2) \fint_{B_\varepsilon(x_k)} v^\varepsilon(y)\,dy + \varepsilon^2 h_2(x_k) \\
&\qquad\qquad - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \\
&\qquad \geq J_2(v^\varepsilon)(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k}
\end{aligned}$$

$$\geq \min\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k}$$
$$= v^\varepsilon(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} = M_k.$$

Case 4: Suppose that $j_k = 2$ and $j_{k+1} = 1$ (that is, we jump to the first board). Then

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{k+1}\,|\,M_0,\dots,M_k]$$
$$= \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}\Big[u^\varepsilon(x_{k+1}) - \varepsilon^2 \sum_{l=0}^{k}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}}\,\Big|\, M_0,\dots,M_k\Big]$$
$$= \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}\Big[u^\varepsilon(x_{k+1}) - \varepsilon^2 h_1(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}}\,\Big|\, M_0,\dots,M_k\Big].$$

In the second equality, we used that $j_{k+1} = 1$. Hence,

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{k+1}\mid M_0,\dots,M_k]$$
$$\geq \alpha_1\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_k)} u^\varepsilon(y) - \frac{\delta}{2^{k+1}} + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_k)} u^\varepsilon(y)\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x_k)} u^\varepsilon(y)\,dy - \varepsilon^2 h_1(x_k)$$
$$- \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}}$$
$$= J_1(u^\varepsilon)(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k}$$
$$\geq \min\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k}$$
$$= v^\varepsilon(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} = M_k.$$

Thus, gathering the four cases, we conclude that M_k is a submartingale.

Using the OST we obtain

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{\tau\wedge k}] \geq M_0.$$

Taking the limit as $k\to\infty$ we get

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_\tau] \geq M_0.$$

If we take $\inf_{S_{\mathrm{II}}}$ and then $\sup_{S_{\mathrm{I}}}$ we arrive at

$$\sup\nolimits_{S_{\mathrm{I}}}\inf\nolimits_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[M_\tau] \geq M_0.$$

This inequality says that

$$\sup\nolimits_{S_{\mathrm{I}}}\inf\nolimits_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}] \geq u(x_0) - \delta.$$

To prove an inequality in the opposite direction we fix a strategy for Player II as follows: Whenever $j_k = 1$ Player II decides to stay in the second board if

$$\min\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} = J_2(v^\varepsilon)(x_k),$$

and Player II decides to jump to the first board when

$$\min\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} = J_1(v^\varepsilon)(x_k).$$

If we play tug-of-war (in both boards) Player II chooses

$$x_{k+1}^{\mathrm{II}} = S_{\mathrm{II}}^*((x_0, j_0), \dots, (x_k, j_k)) \quad \text{such that} \quad \inf_{y\in B_\varepsilon(x_k)} w^\varepsilon(y, j_{k+1}) + \frac{\delta}{2^{k+1}} \geq w^\varepsilon(x_{k+1}^{\mathrm{II}}, j_{k+1}).$$

Given this strategy for Player II and any strategy for Player I, using computations similar to the ones we made before, we can prove that the sequence of random variables

$$N_k = w^\varepsilon(x_k, j_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) + \frac{\delta}{2^k}$$

is a supermartingale. Finally, using the OST we arrive at

$$\inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] \leq u^\varepsilon(x_0) + \delta.$$

Then we have obtained

$$u^\varepsilon(x_0) - \delta \leq \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] \leq \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] \leq u^\varepsilon(x_0) + \delta$$

for any positive δ.

Analogously, we can prove that

$$v^\varepsilon(x_0) - \delta \leq \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,2)}[\text{total payoff}] \leq \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,2)}[\text{total payoff}] \leq v^\varepsilon(x_0) + \delta.$$

Since δ is arbitrary, this proves that the game has a value,

$$\begin{aligned}\sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] = w(x_0, 1),\\ \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,2)}[\text{total payoff}] = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,2)}[\text{total payoff}] = w(x_0, 2),\end{aligned}$$

and that these functions coincide with the solution to the DPP,

$$w(x_0, 1) = u^\varepsilon(x_0) \quad \text{and} \quad w(x_0, 2) = v^\varepsilon(x_0),$$

as we wanted to show. □

Since solutions to the DPP coincide with the value of the game and this is unique, we obtain uniqueness of solutions to the DPP.

Corollary 19. *There exists a unique solution to the DPP (1-8).*

Proof. Existence follows from Theorem 16 and uniqueness from the fact that in Theorem 18 we proved that any solution to the DPP coincides with the value function of the game, that is, it satisfies

$$u^\varepsilon(x) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{(x,1)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}] = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x,1)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}],$$
$$v^\varepsilon(x) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{(x,2)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}] = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x,2)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}].$$ □

2.2. *Uniform convergence as $\varepsilon \to 0$.* To obtain a convergent subsequence of the values of the game u^ε and v^ε we will use the following Arzelà–Ascoli type lemma. For its proof, see [Manfredi et al. 2012b, Lemma 4.2].

Lemma 20. *Let*

$$\{u^\varepsilon : \overline{\Omega} \to \mathbb{R}\}_{\varepsilon>0}$$

be a set of functions such that

(1) *there exists $C > 0$ such that $|u^\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*
(2) *given $\delta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x-y| < r_0$,*

$$|u^\varepsilon(x) - u^\varepsilon(y)| < \delta.$$

Then there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence still denoted by $\{u^\varepsilon\}$ such that

$$u^\varepsilon \to u \quad \textit{uniformly in } \overline{\Omega} \textit{ as } \varepsilon \to 0.$$

So our task now is to show that u^ε and v^ε both satisfy the hypotheses of the previous lemma. First, we observe that we already proved that they are uniformly bounded (see Corollary 17).

To obtain the second hypothesis of Lemma 20 we will need to prove some technical lemmas. This part of the paper is delicate and involves the choice of particular strategies for the players.

First of all, we need to find an upper bound for the expectation of the total number of plays,

$$\mathbb{E}[\tau].$$

To this end we define an auxiliary game as follows: In the next lemma we play a tug-of-war or random walk game in an annulus and one of the players uses the strategy of pointing to the center of the annulus when they play tug-of-war. Then, no matter if we play tug-of-war or random walk at each turn and no matter the strategy used by the other player, we can obtain a precise bound (in terms of the configuration of the annulus and the distance of the initial position to the inner boundary) for the expected number of plays until one reaches the ball inside the annulus.

The key point here is that if one of the players pulls towards 0 each time that they play tug-of-war then the expected number of plays is bounded above by a precise expression that scales as ε^{-2} independently of the game that is played at every round (tug-of-war or random walk). This upper bound translates to our game (starting at any of the two boards) since the result implies that if one of the players chooses to pulls towards 0 then, independently of the choice of the other player and independently of the board at which we play — that is, independently of the coin toss that selects the game that is played (tug-of-war or random walk) — the game ends in an expected number of times that satisfies the obtained upper bound. See Remark 22 below.

Lemma 21. *Given $0 < \delta < R$, let us consider the annular domain $B_R(0) \setminus B_\delta(0)$. In this domain we consider the following game: given $x \in B_R(0) \setminus B_\delta(0)$ the next position of the token can be chosen using the game tug-of-war or a random walk. When tug-of-war is played, one of the players pulls towards 0. In all cases the next position is assumed to be in $B_\varepsilon(x) \cap B_R(0)$. The game ends when the token reaches $B_\delta(0)$. Then, if τ^* is the exit time, we have the estimate*

$$\varepsilon^2 \mathbb{E}^{x_0}[\tau^*] \le C_1(R/\delta)\,\mathrm{dist}(\partial B_\delta(0), x_0) + o(1),$$

where $o(1) \to 0$ if $\varepsilon \to 0$.

Proof. Without loss of generality we can suppose that S_{I}^* is to pull towards 0. Let us call

$$E_\varepsilon(x) = \mathbb{E}^x_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[\tau^*].$$

Notice that E is radial and increasing in $r = |x|$. Since our aim is to obtain a bound that is independent of the game (tug-of-war or random walk) that is played at each round, if we try to maximize the expectation for the exit time, we have that the function E satisfies

$$E_\varepsilon(x) \le \max\left\{\left(\frac{1}{2}\sup_{y\in B_\varepsilon(x)\cap B_R(0)} E_\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)\cap B_R(0)} E_\varepsilon(y), \fint_{B_\varepsilon(x)\cap B_R(0)} E_\varepsilon(y)\,dy\right)\right\} + 1.$$

Hence, let us consider the DPP

$$\widetilde{E}_\varepsilon(x) = \max\left\{\left(\frac{1}{2}\sup_{y\in B_\varepsilon(x)\cap B_R(0)} \widetilde{E}_\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)\cap B_R(0)} \widetilde{E}_\varepsilon(y), \fint_{B_\varepsilon(x)\cap B_R(0)} \widetilde{E}_\varepsilon(y)\,dy\right)\right\} + 1.$$

Writing $F_\varepsilon(x) = \varepsilon^2 \widetilde{E}_\varepsilon(x)$, we then obtain

$$F_\varepsilon(x) = \max\left\{\left(\frac{1}{2}\sup_{y\in B_\varepsilon(x)\cap B_R(0)} F_\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)\cap B_R(0)} F_\varepsilon(y), \fint_{B_\varepsilon(x)\cap B_R(0)} F_\varepsilon(y)\,dy\right)\right\} + \varepsilon^2.$$

This induces us to look for a function F such that

$$F(x) \ge \fint_{B_\varepsilon(x)} F(y)\,dy + \varepsilon^2 \quad\text{and}\quad F(x) \ge \frac{1}{2}\sup_{y\in B_\varepsilon(x)} F(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} F(y) + \varepsilon^2. \tag{2-8}$$

We arrived at a sort of discrete version of the inequalities

$$\begin{cases} \Delta F(x) \le -2(N+2), & x \in B_{R+\varepsilon}(0) \setminus \overline{B}_{\delta-\varepsilon}(0), \\ \Delta^1_\infty F(x) \le -2, & x \in B_{R+\varepsilon}(0) \setminus \overline{B}_{\delta-\varepsilon}(0). \end{cases} \tag{2-9}$$

If we assume that F is radial and increasing if $r = |x|$ we get

$$\Delta^1_\infty F = \partial_{rr} F \le \partial_{rr} F + \frac{N-1}{r}\partial_r F = \Delta F.$$

Hence, to find a solution of (2-9), we can consider the problem

$$\begin{cases} \Delta F(x) = -2(N+2), & x \in B_{R+\varepsilon}(0) \setminus \bar{B}_\delta(0), \\ F(x) = 0, & x \in \partial B_\delta(0), \\ \frac{\partial F}{\partial \nu}(x) = 0, & \text{for } x \in \partial B_{R+\varepsilon}(0), \end{cases} \tag{2-10}$$

where $\partial F/\partial \nu$ refers to the outward normal derivative. The solution to this problem takes the form

$$F(r) = -ar^2 - br^{2-N} + c \quad \text{for } N > 2 \qquad \text{and} \qquad F(r) = -ar^2 - b\log(r) + c \quad \text{for } N = 2,$$

with $a, b, c \in \mathbb{R}$ that depends of δ, R, ε, N. For example, for $N > 2$, we obtain that a, b and c are given by the solution to the following equations:

$$\begin{gathered} \Delta F = -2aN = -2(N+2), \\ \partial_r F(R+\varepsilon) = -2a(R+\varepsilon) - b(2-N)(R+\varepsilon)^{1-N} = 0, \\ F(\delta) = -a\delta^2 - b\delta^{2-N} + c = 0. \end{gathered}$$

Observe that the resulting function $F(r)$ is increasing.

In this way we find F that satisfies the inequalities (2-9). The classical calculation using Taylor expansions shows that F satisfies (2-8) for each $B_\varepsilon(x) \subset B_{R+\varepsilon} \setminus \bar{B}_{\delta-\varepsilon}(0)$. Moreover, since F is increasing in r, it holds that for each $x \in B_R(0) \setminus B_\delta(0)$,

$$⨍_{B_\varepsilon(x)\cap B_R(0)} F \le ⨍_{B_\varepsilon(x)} F \le F(x) - \varepsilon^2$$

and

$$\tfrac{1}{2}\sup_{y\in B_\varepsilon(x)\cap B_R(0)} F + \tfrac{1}{2}\inf_{y\in B_\varepsilon(x)\cap B_R(0)} F \le \tfrac{1}{2}\sup_{y\in B_\varepsilon(x)} F + \tfrac{1}{2}\inf_{y\in B_\varepsilon(x)} F \le F(x) - \varepsilon^2.$$

Consider the sequence of random variables $(M_k)_{k\ge1}$ given by

$$M_k = F(x_k) + k\varepsilon^2.$$

Let us prove that $(M_k)_{k\ge0}$ is a *supermartingale*. Indeed, we have

$$\begin{aligned} \mathbb{E}[M_{k+1} \mid M_0, \dots, M_k] &= \mathbb{E}[F(x_{k+1}) + (k+1)\varepsilon^2 \mid M_0, \dots, M_k] \\ &\le \max\left\{\frac{1}{2}\sup_{y\in B_\varepsilon(x)\cap B_R(0)} F(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)\cap B_R(0)} F(y), ⨍_{B_\varepsilon(x)\cap B_R(0)} F(y)\,dy\right\} \\ &\le F(x_k) + k\varepsilon^2 \end{aligned}$$

if $x_k \in B_R(0) \setminus \overline{B_\delta(0)}$. Thus, M_k is a supermartingale. Using the OST we obtain

$$\mathbb{E}[M_{\tau^*\wedge k}] \le M_0.$$

This means

$$\mathbb{E}^{x_0}[F(x_{\tau^*\wedge k}) + (\tau^* \wedge k)\varepsilon^2] \le F(x_0).$$

Using that $x_{\tau^*} \in \overline{B_\delta(0)} \setminus \overline{B_{\delta-\varepsilon}(0)}$ we get

$$0 \le -\mathbb{E}^{x_0}[F(x_{\tau^*})] \le o(1).$$

Furthermore, the estimate

$$0 \le F(x_0) \le C(R/\delta)\,\mathrm{dist}(\partial B_\delta, x_0)$$

holds for the solution of (2-10). Then, taking the limit as $k \to \infty$, we obtain

$$\varepsilon^2 \mathbb{E}[\tau^*] \le F(x_0) - \mathbb{E}[F(x_{\tau^*})] \le C(R/\delta)\,\mathrm{dist}(\partial B_\delta(0), x_0) + o(1).$$

This completes the proof. □

Remark 22. Suppose that we are playing the previous game in two boards with Ω inside the annulus. Let us recall that Ω satisfies an exterior sphere condition: there exists $\delta > 0$ such that given $y \in \partial\Omega$ there exists $z_y \in \mathbb{R}^N$ such that $B_\delta(z_y) \subset \mathbb{R}^N \setminus \Omega$ and $\overline{B_\delta(z_y)} \cap \overline{\Omega} = \{y\}$. Note that if we have some δ_0 that satisfies the exterior sphere property for ball of that radius, then the exterior sphere property is also satisfied for balls with radius δ for every $\delta < \delta_0$. Then we can consider simultaneously the game defined in Lemma 21 in the annular domain $B_R(0) \setminus B_\delta(0)$ with $\Omega \subset B_R(0) \setminus B_\delta(0)$, and with the same strategies for Player I and Player II, so that no mater which game, J_1 or J_2, is played in any of the two boards we obtain the bound for the expected number of plays given in Lemma 21. That is, if in the two boards game we start for example in $(x_0, 1)$ and Player I decides to stay in the first board and play J_1, in the one board game with the annular domain the third player decides to play tug-of-war with α_1 probability, or random walk with $1-\alpha_1$ probability, but if the player decides to jump to the second board and play J_2, then in the one board game the third player decides to play tug-of-war with α_2 probability and random walk with $1-\alpha_2$ probability. Thus, using that $\Omega \subset B_R(0) \setminus B_\delta(0)$ we deduce that in the two boards game the exit time τ is smaller than or equal to the exit time τ^* corresponding to the one board game considered in the previous lemma. This means that we have

$$\mathbb{E}[\tau] \le \mathbb{E}[\tau^*].$$

Next we derive an estimate for the asymptotic uniform continuity of the so-called nonhomogeneous p-Laplacian functions.

Lemma 23. *Let be Ω as above, $h : \Omega \to \mathbb{R}$ and $F : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ two Lipschitz functions. For $0 < \beta < 1$ let $\mu^\varepsilon : \mathbb{R}^N \to \mathbb{R}$ be a function that satisfies the following DPP*:

$$\begin{cases} \mu^\varepsilon(x) = \beta\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} \mu^\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} \mu^\varepsilon(y)\Big) + (1-\beta) \fint_{B_\varepsilon(x)} \mu^\varepsilon(y)\,dy + \varepsilon^2 h(x), & x \in \Omega, \\ \mu^\varepsilon(x) = F(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then, given $\eta > 0$, there exists $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$|\mu^\varepsilon(x) - \mu^\varepsilon(y)| < \eta$$

if $|x-y| < r_0$ and $\varepsilon < \varepsilon_0$.

Proof. We have several cases:

Case 1: If $x, y \in \mathbb{R}^N \setminus \Omega$ we have

$$|\mu^\varepsilon(x) - \mu^\varepsilon(y)| = |F(x) - F(y)| \le L(F)|x - y| < \eta$$

if $r_0 < \eta/L(F)$.

Case 2: Suppose $x \in \Omega$ and $y \in \partial\Omega$. Without loss of generality we can suppose that $\Omega \subset B_R(0) \setminus B_\delta(0)$ and $y \in \partial B_\delta(0)$. Let us call $x_0 = x$ the first position in the game. In the first case suppose that Player I uses the strategy of pulling towards 0, denoted by S_{I}^*. Let us consider the sequence of random variables

$$M_k = |x_k| - C\varepsilon^2 k.$$

If $C > 0$ is large enough, M_k is a supermartingale. Indeed

$$\mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}\big[|x_{k+1}| \mid x_0, \dots, x_k\big] \le \beta\Big(\frac{1}{2}(|x_k| + \varepsilon) + \frac{1}{2}(|x_k| - \varepsilon)\Big) + (1 - \beta) \fint_{B_\varepsilon(x_k)} |z|\, dz \le |x_k| + C\varepsilon^2.$$

The first inequality follows from the choice of the strategy, and the second from the estimate

$$\fint_{B_\varepsilon(x)} |z|\, dz \le |x| + C\varepsilon^2.$$

Using the OST we obtain

$$\mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}[|x_\tau|] \le |x_0| + C\varepsilon^2 \mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[\tau].$$

Now, Lemma 21 and Remark 22 give us the estimate

$$\varepsilon^2 \mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}[\tau] \le \varepsilon^2 \mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}[\tau^*] \le C_1(R/\delta)\operatorname{dist}(\partial B_\delta(0), x_0) + o(1).$$

Then

$$\mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}[|x_\tau|] \le |x_0 - y| + \delta + C_2(R/\delta)|x_0 - y| + o(1).$$

Here $C_2(R/\delta) = C_1(R/\delta)$. If we rewrite this inequality we obtain

$$\mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}[|x_\tau|] \le \delta + C_3(R/\delta)|x_0 - y| + o(1)$$

with $C_3(R/\delta) = C_2(R/\delta) + 1$.

Using that F is a Lipschitz function we have

$$|F(x_\tau) - F(0)| \le L(F)|x_\tau|.$$

Hence, we get

$$\begin{aligned}
\mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}[F(x_\tau)] &\ge F(0) - L(F)\mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}[|x_\tau|] \\
&\ge F(y) - L(F)\delta - L(F)C_3(R/\delta)|x_0 - y| + o(1) \\
&\ge F(y) - L(F)\delta - L(F)C_3(R/\delta)r_0 - o(1).
\end{aligned}$$

Then

$$\mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}^{x_0}\Bigg[F(x_\tau) + \varepsilon^2 \sum_{j=0}^{\tau-1} h(x_j)\Bigg] \ge F(y) - L(F)\delta - L(F)C_3 r_0 - \|h\|_\infty C r_0 - o(1).$$

Thus, taking $\inf_{S_{\mathrm{II}}}$ and then $\sup_{S_{\mathrm{I}}}$ we get

$$\mu^{\varepsilon}(x_0) > F(y) - L(F)\delta - L(F)C_3 r_0 - \|h\|_\infty C r_0 - o(1) > F(y) - \eta.$$

Here we take $\delta > 0$ such that $L(F)\delta < \frac{1}{3}\eta$, and then take $r_0 > 0$ such that $(L(F)C_3 + \|h\|_\infty C)r_0 < \frac{1}{3}\eta$ and $o(1) < \frac{1}{3}\eta$.

Analogously, we can obtain the estimate

$$\mu^{\varepsilon}(x_0) < F(y) + \eta$$

if Player II use the strategy that pulls towards 0. This ends the proof in this case.

Case 3: Now, given two points x and y inside Ω with $|x-y| < r_0$, we couple the game starting at $x_0 = x$ with the game starting at $y_0 = y$ making the same movements. This coupling generates two sequences of positions x_i and y_i such that $|x_i - y_i| < r_0$ and $j_i = k_i$. This continues until one of the games exits the domain (say at $y_\tau \notin \Omega$). At this point for the game starting at x_0 we have that its position x_τ is close to the exterior point $y_\tau \notin \Omega$ (since we have $|x_\tau - y_\tau| < r_0$) and hence we can use our previous estimates for points close to the boundary to conclude that

$$|\mu^{\varepsilon}(x_0) - \mu^{\varepsilon}(y_0)| < \eta.$$

□

Now we are ready to prove the second condition of the Arzelà–Ascoli type result, Lemma 20.

Lemma 24. *Let $(u^\varepsilon, v^\varepsilon)$ be a pair of functions that is a solution to the DPP (1-8) given by*

$$\begin{cases} u^\varepsilon(x) = \max\big\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\big\}, & x \in \Omega, \\ v^\varepsilon(x) = \min\big\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\big\}, & x \in \Omega, \\ u^\varepsilon(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^\varepsilon(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Given $\eta > 0$, there exists $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$|u^\varepsilon(x) - u^\varepsilon(y)| < \eta \quad \textit{and} \quad |v^\varepsilon(x) - v^\varepsilon(y)| < \eta$$

if $|x-y| < r_0$ and $\varepsilon < \varepsilon_0$.

Proof. We will proceed by repeating the ideas used in Lemma 23.

We consider again several cases.

Case 1: Suppose that $x, y \in \mathbb{R}^N \setminus \Omega$. Then we have

$$|u^\varepsilon(x) - u^\varepsilon(y)| = |f(x) - f(y)| \le L(f)|x-y| < \eta \quad \text{and} \quad |v^\varepsilon(x) - v^\varepsilon(y)| = |g(x) - g(y)| \le L(g)|x-y| < \eta$$

if $\max\{L(f), L(g)\} r_0 < \eta$.

Case 2: Let us begin with the estimate of u^ε. Suppose now that $x \in \Omega$ and $y \in \partial\Omega$ in the first board (we write $(x,1)$ and $(y,1)$). Without loss of generality we suppose again that $\Omega \subset B_R(0) \setminus B_\delta(0)$ and $y \in \partial B_\delta(0)$. Let us call $x_0 = x$ the first position in the game. Player I uses the following strategy, called S_{I}^*: the token always stay in the first board (Player I decides not to change boards), and pulls towards 0 when tug-of-war is played. In this case we have that u^ε is a supersolution to the DPP that appears in Lemma 23 (with $\beta = \alpha_1$). Notice that the game is always played in the first board. As Player I wants to maximize the expected value we get that the first component for our system, u^ε, satisfies

$$u^\varepsilon(x) \geq \mu^\varepsilon(x)$$

(the value function when the player that wants to maximize is allowed to choose to change boards is bigger than or equal to the value function of a game where the player does not have the possibility of making this choice). From this bound and Lemma 23, a lower bound for u^ε close to the boundary follows. That is, from the estimate obtained in that lemma we get

$$u^\varepsilon(x) > f(y) - \eta.$$

Let us be more precise and consider the sequence of random variables

$$M_k = |x_k| - C\varepsilon^2 k.$$

We obtain arguing as before that M_k is a supermartingale for $C > 0$ large enough. If we repeat the reasoning of the Lemma 23 (this can be done because we stay in the first board) we arrive at

$$u^\varepsilon(x) > f(y) - \eta$$

if $|x - y| < r_0$ and $\varepsilon < \varepsilon_0$ for some r_0 and ε_0.

Now, the next estimate requires a particular strategy for Player II, called S^*_{II}: when play the tug-of-war game, Player II pulls towards 0 (in both boards) and if in some step Player I decides to jump to the second board, then Player II decides to stay always in this board and the position never comes back to the first board. Let us consider

$$M_k = |x_k| - C\varepsilon^2 k.$$

We want to estimate

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}, S^*_{\mathrm{II}}}\big[|x_{k+1}| \mid x_0, \ldots, x_k\big].$$

Now, Lemma 23 says that

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}, S^*_{\mathrm{II}}}\big[|x_{k+1}| \mid x_0, \ldots, x_k\big] \leq |x_k| + C\varepsilon^2$$

for all possible combinations of j_k and j_{k+1}. Using the OST we obtain

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}, S^*_{\mathrm{II}}}[|x_\tau|] \leq |x_0| + C\varepsilon^2 \mathbb{E}_{S_{\mathrm{I}}, S^*_{\mathrm{II}}}[\tau].$$

Let us suppose that $j_\tau = 1$. This means that $j_k = 1$ for all $0 \leq k \leq \tau$. If we proceed as in Lemma 23, we obtain

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}, S^*_{\mathrm{II}}}[\text{final payoff}] \leq f(y) + L\delta + LCr_0 + \|h_1\|_\infty Cr_0 + o(1).$$

On the other hand, if $j_\tau = 2$, we have

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}, S^*_{\mathrm{II}}}[g(x_\tau)] \leq g(y) + L\delta + LC_3 r_0 + o(1) \leq f(y) + L\delta + LC_3 r_0 + o(1).$$

Thus, we get

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}},S^*_{\mathrm{II}}}\left[g(x_\tau)+\sum_{l=0}^{\tau-1}\big(h_1(x_l)\chi_{\{l=1\}}(l)+h_2(x_l)\chi_{\{l=2\}}(l)\big)\right]\le f(y)+L\delta+LCr_0+(\|h_1\|+\|h_2\|_\infty)Cr_0+o(1).$$

In both cases, taking $\sup_{S_{\mathrm{I}}}$ and then $\inf_{S_{\mathrm{II}}}$ we arrive at

$$u^\varepsilon(x_0)\le f(y)+\eta,$$

taking $\delta>0,\ r_0>0$ and $\varepsilon>0$ small enough.

Analogously we can obtain the estimates for v^ε and complete the proof. □

As a corollary we obtain uniform convergence along a sequence $\varepsilon_j\to 0$.

Corollary 25. *There exists a sequence $\varepsilon_j\to 0$ and a pair of functions (u,v) that are continuous in $\overline{\Omega}$ such that*

$$u^{\varepsilon_j}\rightrightarrows u,\qquad v^{\varepsilon_j}\rightrightarrows v$$

uniformly in $\overline{\Omega}$.

Proof. The result follows from Lemma 20. □

2.3. ***The limit is a viscosity solution to the PDE system.*** Our main goal in this section is to prove that the limit pair (u,v) is a viscosity solution to (1-9).

First, let us state the precise definition of what we understand as a viscosity solution for the system (1-9). We refer to [Crandall et al. 1992] for a general reference to viscosity theory.

Viscosity solutions. We begin with the definition of a viscosity solution to a fully nonlinear second-order elliptic PDE. Fix a function

$$P:\Omega\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{S}^N\to\mathbb{R},$$

where $\mathbb{S}^N$ denotes the set of symmetric $N\times N$ matrices, and consider the PDE

$$P\big(x,u(x),Du(x),D^2u(x)\big)=0,\qquad x\in\Omega. \tag{2-11}$$

We will assume that P is degenerate elliptic, that is, P satisfies a monotonicity property with respect to the matrix variable, that is,

$$X\le Y\ \text{in}\ \mathbb{S}^N\quad\Longrightarrow\quad P(x,r,p,X)\ge P(x,r,p,Y)$$

for all $(x,r,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^N$.

Definition 26. A lower semicontinuous function u is a viscosity supersolution of (2-11) if for every $\phi\in C^2$ such that ϕ touches u at $x\in\Omega$ strictly from below (that is, $u-\phi$ has a strict minimum at x with $u(x)=\phi(x)$), we have

$$P\big(x,\phi(x),D\phi(x),D^2\phi(x)\big)\ge 0.$$

An upper semicontinuous function u is a subsolution of (2-11) if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above (that is, $u - \psi$ has a strict maximum at x with $u(x) = \psi(x)$), we have

$$P\big(x, \phi(x), D\phi(x), D^2\phi(x)\big) \le 0.$$

Finally, u is a viscosity solution of (2-11) if it is both a super- and subsolution.

When P is not continuous one has to consider the upper and lower semicontinuous envelopes of P, which we denote by P^* and P_* respectively, and consider

$$P^*(x, \phi(x), D\phi(x), D^2\phi(x)) \ge 0 \quad \text{and} \quad P_*(x, \phi(x), D\phi(x), D^2\phi(x)) \le 0$$

when defining super- and subsolutions.

In our system (1-9) we have two equations given by the functions

$$F_1(x, u, v, p, X) = \min\left\{ -\frac{\alpha_1}{2}\left\langle X\frac{p}{|p|}, \frac{p}{|p|}\right\rangle - \frac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x), (u-v)(x)\right\},$$
$$F_2(x, u, v, q, Y) = \max\left\{ -\frac{\alpha_2}{2}\left\langle Y\frac{q}{|q|}, \frac{q}{|q|}\right\rangle - \frac{(1-\alpha_2)}{2(N+2)}\operatorname{trace}(Y) - h_2(x), (v-u)(x)\right\}.$$

These functions F_1 and F_2 are not continuous (they are not even well defined for $p = 0$ and for $q = 0$ respectively). The upper semicontinuous envelope of F_1 is given by

$$(F_1)^*(x, u, v, p, X) = \begin{cases} \min\left\{ -\dfrac{\alpha_1}{2}\left\langle X\dfrac{p}{|p|}, \dfrac{p}{|p|}\right\rangle - \dfrac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x), (u-v)(x)\right\}, & p \ne 0, \\ \min\left\{ -\dfrac{\alpha_1}{2}\lambda_1(X) - \dfrac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x), (u-v)(x)\right\}, & p = 0. \end{cases}$$

Here $\lambda_1(X) = \min\{\lambda : \lambda \text{ is an eigenvalue of } X\}$. While the lower semicontinuous envelope is

$$(F_1)_*(x, u, v, p, X) = \begin{cases} \min\left\{ -\dfrac{\alpha_1}{2}\left\langle X\dfrac{p}{|p|}, \dfrac{p}{|p|}\right\rangle - \dfrac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x), (u-v)(x)\right\}, & p \ne 0, \\ \min\left\{ -\dfrac{\alpha_1}{2}\lambda_N(X) - \dfrac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x), (u-v)(x)\right\}, & p = 0. \end{cases}$$

Here $\lambda_N(X) = \max\{\lambda : \lambda \text{ is an eigenvalue of } X\}$.

Analogous formulas hold for $(F_2)^*$ and $(F_2)_*$, changing α_1 by α_2.

Then the definition of a viscosity solution for the system (1-9) that we will use here is the following.

Definition 27. A pair of continuous functions $u, v : \overline{\Omega} \mapsto \mathbb{R}$ is a viscosity solution to (1-9) if

(1) $u(x) \ge v(x)$ for $x \in \overline{\Omega}$,

(2) $u|_{\partial\Omega} = f$ and $v|_{\partial\Omega} = g$,

(3) u is a viscosity solution to $F_1(x, u, v(x), \nabla u, D^2u) = 0$ in $\{x : u(x) > v(x)\}$ and u is a viscosity supersolution to $F_1(x, u, v(x), \nabla u, D^2u) = 0$ in Ω,

(4) v is a viscosity solution to $F_2(x, u(x), v, \nabla v, D^2v) = 0$ in $\{x : u(x) > v(x)\}$ and v is a viscosity solution to $F_2(x, u(x), v, \nabla v, D^2v) = 0$ in Ω.

Remark 28. The meaning of Definition 27 is that we understand a solution to (1-9) as a pair of continuous up to the boundary functions that satisfy the boundary conditions pointwise and such that u is a viscosity solution to the obstacle problem (from below) for the first equation in the system (with v as a fixed continuous function of x as obstacle from below) and v solves the obstacle problem (from above) for the second equation in the system (regarding u as a fixed function of x as obstacle from above).

With this definition at hand we are ready to show that any uniform limit of the value functions of our game is a viscosity solution to the two membranes problem with the two different p-Laplacians.

Theorem 29. *Let (u, v) be continuous functions that are a uniform limit of a sequence of values of the game, that is,*

$$u^{\varepsilon_j} \rightrightarrows u, \quad v^{\varepsilon_j} \rightrightarrows v$$

uniformly in $\overline{\Omega}$ as $\varepsilon_j \to 0$. Then the limit pair (u, v) is a viscosity solution to (1-9).

Proof. We divide the proof into several steps.

(1) <u>u and v are ordered</u>: From the fact that

$$u^{\varepsilon_j} \geq v^{\varepsilon_j}$$

in $\overline{\Omega}$ and the uniform convergence we immediately get

$$u \geq v$$

in $\overline{\Omega}$.

(2) <u>The boundary conditions</u>: As we have that

$$u^{\varepsilon_j} = f, \quad v^{\varepsilon_j} = g,$$

in $\mathbb{R}^N \setminus \Omega$ we get

$$u|_{\partial\Omega} = f, \quad v|_{\partial\Omega} = g.$$

(3) <u>The equation for u</u>: First, let us show that u is a viscosity supersolution to

$$-\Delta_p^1 u(x) + h_1(x) = 0$$

for $x \in \Omega$. To this end, consider a point $x_0 \in \Omega$ and a smooth function $\varphi \in C^2(\Omega)$ such that $(u-\varphi)(x_0) = 0$ is a strict minimum of $(u - \varphi)$. Then from the uniform convergence there exists a sequence of points, which we will denote by $(x_\varepsilon)_{\varepsilon>0}$, such that $x_\varepsilon \to x_0$ and

$$(u^\varepsilon - \varphi)(x_\varepsilon) \leq (u^\varepsilon - \varphi)(y) + o(\varepsilon^2),$$

that is,

$$u^\varepsilon(y) - u^\varepsilon(x_\varepsilon) \geq \varphi(y) - \varphi(x_\varepsilon) - o(\varepsilon^2). \tag{2-12}$$

From the DPP (1-8) we have

$$0 = \max\big\{J_1(u^\varepsilon)(x_\varepsilon) - u(x_\varepsilon),\, J_2(v^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon)\big\} \geq J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon). \tag{2-13}$$

Writing $J_1(u^\varepsilon)(x_\varepsilon) - u(x_\varepsilon)$ we obtain

$$J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon) = \alpha_1\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y)-u^\varepsilon(x_\varepsilon)) + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y)-u^\varepsilon(x_\varepsilon))\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y)-u^\varepsilon(x_\varepsilon))\,dy - \varepsilon^2 h_1(x_\varepsilon),$$

and then, using (2-12), we get

$$J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon) \geq \alpha_1 \underbrace{\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_\varepsilon)}(\varphi(y)-\varphi(x_\varepsilon)) + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_\varepsilon)}(\varphi(y)-\varphi(x_\varepsilon))\Big)}_{\mathrm{I}} + \beta_1 \underbrace{\fint_{B_\varepsilon(x_\varepsilon)}(\varphi(y)-\varphi(x_\varepsilon))\,dy}_{\mathrm{II}} - \varepsilon^2 h_1(x_\varepsilon) + o(\varepsilon^2).$$

Let us analyze I and II. We begin with I: Assume that $\nabla\varphi(x_0)\neq 0$. Let $z_\varepsilon \in B_1(0)$ be such that

$$\max_{y\in \bar{B}_\varepsilon(x_\varepsilon)} \varphi(y) = \varphi(x_\varepsilon + \varepsilon z_\varepsilon).$$

Then we have

$$I = \tfrac{1}{2}\big(\varphi(x_\varepsilon+\varepsilon z_\varepsilon)-\varphi(x_\varepsilon)\big) + \tfrac{1}{2}\big(\varphi(x_\varepsilon-\varepsilon z_\varepsilon)-\varphi(x_\varepsilon)\big) + o(\varepsilon^2).$$

From a simple Taylor expansion we conclude that

$$\tfrac{1}{2}\big(\varphi(x_\varepsilon+\varepsilon z_\varepsilon)-\varphi(x_\varepsilon)\big) + \tfrac{1}{2}\big(\varphi(x_\varepsilon-\varepsilon z_\varepsilon)-\varphi(x_\varepsilon)\big) + o(\varepsilon^2) = \tfrac{1}{2}\varepsilon^2\langle D^2\varphi(x_\varepsilon)z_\varepsilon, z_\varepsilon\rangle + o(\varepsilon^2).$$

Dividing the first inequality by ε^2 and taking the limit as $\varepsilon \to 0$ (see [Miranda and Rossi 2020]) we arrive at

$$I \to \tfrac{1}{2}\Delta^1_\infty \varphi(x_0).$$

When $\nabla\varphi = 0$, arguing again using Taylor expansions, we get

$$\limsup I \geq \tfrac{1}{2}\lambda_1(D^2\varphi(x_0)).$$

See [Blanc and Rossi 2019] for more details.

Now, we look at II: Using again Taylor expansions we obtain

$$\fint_{B_\varepsilon(x_\varepsilon)}(\varphi(y)-\varphi(x_\varepsilon))\,dy = \frac{\varepsilon^2}{2(N+1)}\Delta\varphi(x_\varepsilon) + o(\varepsilon^2).$$

Dividing by ε^2 and taking limits as $\varepsilon \to 0$ we get

$$\mathrm{II} \to \frac{1}{2(N+2)}\Delta\varphi(x_0).$$

Therefore, if we come back to (2-13), dividing by ε^2 and taking the limit as $\varepsilon \to 0$ we obtain

$$0 \geq \frac{\alpha_1}{2}\Delta^1_\infty\varphi(x_0) + \frac{1-\alpha_1}{2(N+1)}\Delta\varphi(x_0) - h_1(x_0)$$

when $\nabla\varphi(x_0)\neq 0$, and

$$0\geq \frac{\alpha_1}{2}\lambda_1(D^2\varphi(x_0))+\frac{1-\alpha_1}{2(N+1)}\Delta\varphi(x_0)-h_1(x_0)$$

when $\nabla\varphi(x_0)=0$.

Using the definition of the normalized p-Laplacian we have arrived at

$$-\Delta_p^1\varphi(x_0)+h_1(x_0)\geq 0,$$

in the sense of Definition 26.

Now we are going to prove that u is a viscosity solution to

$$-\Delta_p^1 u(x)+h_1(x)=0 \tag{2-14}$$

in the set $\Omega\cap\{u>v\}$. Let us consider $x_0\in\Omega\cap\{u>v\}$. Let $\eta>0$ be such that

$$u(x_0)\geq v(x_0)+3\eta.$$

Then, using that u and v are continuous functions, there exists $\delta>0$ such that

$$u(y)\geq v(y)+2\eta \quad \text{for all } \ y\in B_\delta(x_0),$$

and, using that $u^\varepsilon\rightrightarrows u$ and $v^\varepsilon\rightrightarrows v$, we have

$$u^\varepsilon(y)\geq v^\varepsilon(y)+\eta \quad \text{for all } \ y\in B_\delta(x_0)$$

for $0<\varepsilon<\varepsilon_0$ for some $\varepsilon_0>0$. Given $z\in B_{\delta/2}(x_0)$ and $\varepsilon<\min\{\varepsilon_0,\delta/2\}$, we obtain

$$B_\varepsilon(z)\subset B_\delta(x_0).$$

Using that $u^\varepsilon\rightrightarrows u$ we obtain the following limits:

$$\sup\nolimits_{y\in B_\varepsilon(z)} u^\varepsilon(y)\to u(z) \quad \text{as } \ \varepsilon\to 0. \tag{2-15}$$

In fact, from our previous estimates we have that

$$|\sup\nolimits_{y\in B_\varepsilon(z)} u^\varepsilon(y)-u(z)|\leq \sup\nolimits_{y\in B_\varepsilon(z)}|u^\varepsilon(y)-u(y)|+\sup\nolimits_{y\in B_\varepsilon(z)}|u(y)-u(z)|.$$

Using that $u^\varepsilon\rightrightarrows u$, there exists $\varepsilon_1>0$ such that if $\varepsilon<\varepsilon_1$,

$$|(u^\varepsilon-u)(x)|<\tfrac12\theta \quad \text{for all } \ x\in\Omega.$$

Now, using that u is continuous, there exists $\varepsilon_2>0$ such that

$$|u(y)-u(z)|<\tfrac12\theta \quad \text{if } \ |y-z|<\varepsilon_2,$$

and thus, if we take $\varepsilon<\min\{\varepsilon_0,\varepsilon_1,\varepsilon_2,\delta/2\}$ we obtain

$$|\sup\nolimits_{y\in B_\varepsilon(z)} u^\varepsilon(y)-u(z)|<\theta.$$

This proves (2-15).

Also, with a similar argument, we have

$$\inf_{y\in B_\varepsilon(z)} u^\varepsilon(y) \xrightarrow{\varepsilon\to 0} u(z). \tag{2-16}$$

Finally, we get

$$\fint_{B_\varepsilon(z)} u^\varepsilon(y)\,dy \xrightarrow{\varepsilon\to 0} u(z). \tag{2-17}$$

In fact, let us compute

$$\left|\fint_{B_\varepsilon(z)} u^\varepsilon(y)\,dy - u(z)\right| \le \fint_{B_\varepsilon(z)} |u^\varepsilon(y)-u(y)|\,dy + \fint_{B_\varepsilon(z)} |u(y)-u(z)|\,dz.$$

Now we use again that $u^\varepsilon \rightrightarrows u$ and that u is a continuous function to obtain

$$\fint_{B_\varepsilon(z)} |u^\varepsilon(y)-u(y)|\,dy < \frac{1}{2}\theta \quad \text{and} \quad \fint_{B_\varepsilon(z)} |u(y)-u(z)|\,dz < \frac{1}{2}\theta$$

for $\varepsilon > 0$ small enough. Thus we obtain

$$\left|\fint_{B_\varepsilon(z)} u^\varepsilon(y)\,dy - u(z)\right| < \theta.$$

Using the previous limits, (2-15), (2-16) and (2-17), we obtain

$$J_1(u^\varepsilon)(z) \to u(z) \quad \text{as } \varepsilon \to 0.$$

Analogously, we can prove that

$$J_2(v^\varepsilon)(z) \to v(z) \quad \text{as } \varepsilon \to 0.$$

Now, if we recall that $u(z) \ge v(z) + 2\eta$, we obtain

$$J_1(u^\varepsilon)(z) \ge J_2(v^\varepsilon)(z) + \eta$$

if $\varepsilon > 0$ is small enough. Then, using the DPP, we obtain

$$u^\varepsilon(z) = \max\{J_1(u^\varepsilon)(z),\, J_2(v^\varepsilon)(z)\} = J_1(u^\varepsilon)(z)$$

for all $z \in B_{\delta/2}(x_0)$ and every $\varepsilon > 0$ small enough. Let us prove that u is viscosity subsolution of (2-14). Given now $\varphi \in \mathscr{C}^2(\Omega)$ such that $(u-\varphi)(x_0) = 0$ is maximum of $u-\varphi$, then, from the uniform convergence, there exists a sequence of points $(x_\varepsilon)_{\varepsilon>0} \subset B_{\delta/2}(x_0)$ such that $x_\varepsilon \to x_0$ and

$$(u^\varepsilon - \varphi)(x_\varepsilon) \ge (u^\varepsilon - \varphi)(y) + o(\varepsilon^2),$$

that is,

$$u^\varepsilon(y) - u^\varepsilon(x_\varepsilon) \le \varphi(y) - \varphi(x_\varepsilon) - o(\varepsilon^2). \tag{2-18}$$

From the DPP (1-8) we have

$$0 = \max\{J_1(u^\varepsilon)(x_\varepsilon) - u(x_\varepsilon),\, J_2(v^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon)\} = J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon).$$

Writing $J_1(u^\varepsilon)(x_\varepsilon)-u(x_\varepsilon)$ we obtain

$$J_1(u^\varepsilon)(x_\varepsilon)-u^\varepsilon(x_\varepsilon)=\alpha_1\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y)-u^\varepsilon(x_\varepsilon))+\frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y)-u^\varepsilon(x_\varepsilon))\Big)\\ +(1-\alpha_1)\fint_{B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y)-u^\varepsilon(x_\varepsilon))\,dy-\varepsilon^2h_1(x_\varepsilon),$$

and then, using (2-18), we get

$$J_1(u^\varepsilon)(x_\varepsilon)-u^\varepsilon(x_\varepsilon)\le\alpha_1\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_\varepsilon)}(\varphi(y)-\varphi(x_\varepsilon))+\frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_\varepsilon)}(\varphi(y)-\varphi(x_\varepsilon))\Big)\\ +(1-\alpha_1)\fint_{B_\varepsilon(x_\varepsilon)}(\varphi(y)-\varphi(x_\varepsilon))\,dy-\varepsilon^2h_1(x_\varepsilon)+o(\varepsilon^2).$$

Passing to the limit as before we obtain

$$0\le\frac{\alpha_1}{2}\Delta^1_\infty\varphi(x_0)+\frac{(1-\alpha_1)}{2(N+1)}\Delta\varphi(x_0)-h_1(x_0)$$

when $\nabla\varphi(x_0)\neq 0$, and

$$0\le\frac{\alpha_1}{2}\lambda_N(D^2\varphi(x_0))+\frac{(1-\alpha_1)}{2(N+1)}\Delta\varphi(x_0)-h_1(x_0)$$

if $\nabla\varphi(x_0)=0$. Hence we arrived at

$$-\Delta^1_p\varphi(x_0)+h_1(x_0)\le 0,$$

according to Definition 26. This proves that u is a viscosity subsolution of (2-14) inside the open set $\{u>v\}$.

As we have that u is a viscosity supersolution in the whole Ω, we conclude that u is a viscosity solution to

$$-\Delta^1_p u(x_0)+h_1(x_0)=0$$

in the set $\{u>v\}$.

(4) The equation for v: The case that v is a viscosity subsolution to

$$-\Delta^1_q v(x)+h_2(x)=0$$

is analogous. Here we use that

$$0=\min\big\{J_2(v^\varepsilon)(x_\varepsilon)-v(x_\varepsilon),\,J_1(u^\varepsilon)(x_\varepsilon)-v^\varepsilon(x_\varepsilon)\big\}\le J_2(v^\varepsilon)(x_\varepsilon)-v^\varepsilon(x_\varepsilon).$$

To show that v is a viscosity solution

$$-\Delta^1_q v(x_0)-h_2(x_0)=0$$

if $x_0\in\Omega\cap\{u>v\}$, we proceed as before. □

3. A game that gives an extra condition on the contact set

In this section we will study the value functions of the second game. In this case, they are given by a pair of functions $(u^\varepsilon, v^\varepsilon)$ that satisfies the DPP

$$\begin{cases} u^\varepsilon(x) = \frac{1}{2}\max\bigl\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\bigr\} + \frac{1}{2}J_1(u^\varepsilon)(x), & x \in \Omega, \\ v^\varepsilon(x) = \frac{1}{2}\min\bigl\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\bigr\} + \frac{1}{2}J_2(v^\varepsilon)(x), & x \in \Omega, \\ u^\varepsilon(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^\varepsilon(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3-1}$$

It is clear from the DPP that

$$u^\varepsilon \geq v^\varepsilon.$$

We aim to show that these functions converge (along subsequences $\varepsilon_j \to 0$) to a pair of functions (u, v) that is a viscosity solution to the system

$$\begin{cases} u(x) \geq v(x), & x \in \Omega, \\ -\Delta_p^1 u(x) + h_1(x) \geq 0, \; -\Delta_q^1 v(x) - h_2(x) \leq 0, & x \in \Omega, \\ -\Delta_p^1 u(x) + h_1(x) = 0, \; -\Delta_q^1 v(x) - h_2(x) = 0, & x \in \{u > v\} \cap \Omega, \\ (-\Delta_p^1 u(x) + h_1(x)) + (-\Delta_q^1 v(x) - h_2(x)) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \\ v(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{3-2}$$

Notice that this is the classical formulation of the two membranes problem, but with the extra condition

$$(-\Delta_p^1 u(x) + h_1(x)) + (-\Delta_q^1 v(x) - h_2(x)) = 0,$$

which is meaningful for $x \in \{u(x) = v(x)\}$.

The existence and uniqueness of the pair of functions $(u^\varepsilon, v^\varepsilon)$ can be proved as before. In fact, we can reproduce the arguments of Perron's method to obtain existence of a solution. Next, we show that given a solution to the DPP we can build quasioptimal strategies and show that the game has a value and that this value coincides with the solution to the DPP, from where uniqueness of solutions to the DPP follows.

Uniform convergence also follows with the same arguments used before using the Arzelà–Ascoli type result together with the estimates close to the boundary proved in the previous section. Notice that here we can prescribe the same strategies as the ones used before. For example, Player I may decide to stay in the first board (if the coin toss allows a choice) and to point to a prescribed point when tug-of-war is played. Also note that the crucial bound on the expected number of plays given in Lemma 21 can also be used here to obtain a bound for the total number of plays in the variant of the game.

Passing to the limit in the viscosity sense is also analogous. One only has to pay special attention to the extra condition. Therefore, let us prove now that the extra condition in (3-2),

$$(-\Delta_p^1 u(x) + h_1(x)) + (-\Delta_q^1 v(x) - h_2(x)) = 0,$$

holds in the viscosity sense in the set $\{x : u(x) = v(x)\}$.

Let us start proving the subsolution case. Given $x_0 \in \{u = v\}$ and $\varphi \in \mathscr{C}^2(\Omega)$ such that $(u-\varphi)(x_0) = 0$ is a maximum of $u-\varphi$, notice that since $v(x_0) = u(x_0)$ and $v \le u$ in Ω we also have that $(v-\varphi)(x_0) = 0$ is a maximum of $v-\varphi$. Then, by uniform convergence, there exists a sequence of points $(x_\varepsilon)_{\varepsilon>0} \subset B_{\delta/2}(x_0)$, such that $x_\varepsilon \to x_0$ and

$$(u^\varepsilon - \varphi)(x_\varepsilon) \ge (u^\varepsilon - \varphi)(y) + o(\varepsilon^2). \tag{3-3}$$

Case 1: Suppose that $u^\varepsilon(x_{\varepsilon_j}) > v^\varepsilon(x_{\varepsilon_j})$ for a subsequence such that $\varepsilon_j \to 0$. Let us observe that, if

$$J_1(u^\varepsilon)(z) < J_2(v^\varepsilon)(z),$$

we have that

$$u^\varepsilon(z) = \tfrac{1}{2} J_1(u^\varepsilon)(z) + \tfrac{1}{2} J_2(v^\varepsilon)(z) \quad \text{and} \quad v^\varepsilon(z) = \tfrac{1}{2} J_1(u^\varepsilon)(z) + \tfrac{1}{2} J_2(v^\varepsilon)(z),$$

and then we get

$$u^\varepsilon(z) = v^\varepsilon(z)$$

in this case.

This remark implies that when $u^\varepsilon(x_{\varepsilon_j}) > v^\varepsilon(x_{\varepsilon_j})$ we have

$$J_1(u^\varepsilon)(x_{\varepsilon_j}) \ge J_2(v^\varepsilon)(x_{\varepsilon_j}).$$

If we use the DPP (3-1) we get

$$\begin{aligned} 0 &= \tfrac{1}{2}\big(J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j})\big) + \tfrac{1}{2} \max\big\{J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j}),\, J_2(v^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j})\big\} \\ &= \tfrac{1}{2}\big(J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j})\big) + \tfrac{1}{2} J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j}) \\ &= J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j}), \end{aligned}$$

and using (3-3) we obtain

$$0 = J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j}) \le J_1(\varphi)(x_{\varepsilon_j}) - \varphi(x_{\varepsilon_j}),$$

and taking the limit as $\varepsilon_j \to 0$ as before we get

$$-\Delta_p^1 \varphi(x_0) + h_1(x_0) \le 0. \tag{3-4}$$

We proved before that v is a subsolution to

$$-\Delta_q^1 v(x) - h_2(x) = 0$$

in the whole Ω. Therefore, as $(v-\varphi)(x_0) = 0$ is a maximum of $v-\varphi$ we get

$$-\Delta_q^1 \varphi(x_0) - h_2(x_0) \le 0. \tag{3-5}$$

Thus, from (3-4) and (3-5) we conclude that

$$(-\Delta_p^1 \varphi(x_0) + h_1(x_0)) + (-\Delta_q^1 \varphi(x_0) - h_2(x_0)) \le 0.$$

Case 2: If $u^\varepsilon(x_\varepsilon) = v^\varepsilon(x_\varepsilon)$ for $\varepsilon < \varepsilon_0$. Using the DPP (3-1) we have

$$u^\varepsilon(x_\varepsilon) = \tfrac{1}{2} J_1(u^\varepsilon)(x_\varepsilon) + \tfrac{1}{2} J_2(v^\varepsilon)(x_\varepsilon) \quad \text{and} \quad v^\varepsilon(x_\varepsilon) = \tfrac{1}{2} J_1(u^\varepsilon)(x_\varepsilon) + \tfrac{1}{2} J_2(v^\varepsilon)(x_\varepsilon),$$

and then we get

$$\max\big\{J_1(u^\varepsilon)(x_\varepsilon),\, J_2(v^\varepsilon)(x_\varepsilon)\big\} = J_2(v^\varepsilon)(x_\varepsilon) \quad \text{and} \quad \min\big\{J_1(u^\varepsilon)(x_\varepsilon),\, J_2(v^\varepsilon)(x_\varepsilon)\big\} = J_1(u^\varepsilon)(x_\varepsilon).$$

If we use again (3-3) we get

$$\varphi(y) - \varphi(x_\varepsilon) \geq u^\varepsilon(y) - u^\varepsilon(x_\varepsilon) + o(\varepsilon^2) \geq v^\varepsilon(y) - v^\varepsilon(x_\varepsilon) + o(\varepsilon^2),$$

where we used that $u^\varepsilon \geq v^\varepsilon$ and $u^\varepsilon(x_\varepsilon) = v^\varepsilon(x_\varepsilon)$. Thus

$$0 = \tfrac{1}{2}\big(J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon)\big) + \tfrac{1}{2}\big(J_2(v^\varepsilon)(x_\varepsilon) - v^\varepsilon(x_\varepsilon)\big) \leq \tfrac{1}{2}\big(J_1(\varphi)(x_\varepsilon) - \varphi(x_\varepsilon)\big) + \tfrac{1}{2}\big(J_2(\varphi)(x_\varepsilon) - \varphi(x_\varepsilon)\big).$$

Taking the limit $\varepsilon \to 0$ we obtain

$$(-\Delta_p^1 \varphi(x_0) + h_1(x_0)) + (-\Delta_q^1 \varphi(x_0) - h_2(x_0)) \leq 0,$$

in the viscosity sense (taking care of the semicontinuous envelopes when the gradient of φ vanishes). We have just proved that the extra condition is satisfied with an inequality when we touch u and v from above at some point x_0 with a smooth test function.

The proof that the other inequality holds when we touch u and v from below is analogous and hence we omit the details.

4. Final remarks

Below we gather some brief comments on possible extensions of our results.

4.1. ***n membranes.*** We can extend our results to the case in which we have n membranes. For the PDE problem we refer to [Azevedo et al. 2005; Carillo et al. 2005; Chipot and Vergara-Caffarelli 1985].

We can generalize the game to an n-dimensional system. Let us suppose that we have, for $1 \leq k \leq n$,

$$J_k(w)(x) = \alpha_k\Big(\frac{1}{2}\sup_{y \in B_\varepsilon(x)} w(y) + \frac{1}{2}\inf_{y \in B_\varepsilon(x)} w(y)\Big) + (1-\alpha_k) \fint_{B_\varepsilon(x)} w(y)\, dy - \varepsilon^2 h_k(x).$$

These games have associated to them the operators

$$L_k(w) = -\Delta_{p_k}^1 w + h_k.$$

Given $f_1 \geq f_2 \geq \cdots \geq f_n$ defined outside Ω, we can consider the DPP

$$\begin{cases} u_k^\varepsilon(x) = \frac{1}{2}\max_{i \geq k}\{J_i(u_i^\varepsilon)\} + \frac{1}{2}\min_{l \leq k}\{J_l(u_l^\varepsilon)\}, & x \in \Omega, \\ u_k^\varepsilon(x) = f_k(x), & x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

for $1 \leq k \leq n$.

This DPP is associated to a game that is played in n boards. In board k a fair coin is tossed and the winner is allowed to change boards but Player I can only choose to change to a board with index bigger than or equal to k while Player II may choose a board with index smaller than or equal to k.

The functions $(u_1^\varepsilon, \cdots, u_n^\varepsilon)$ converge uniformly as $\varepsilon \to 0$ (along a subsequence) to continuous functions $\{u_k\}_{1\le k\le n}$ that are viscosity solutions to the n membranes problem

$$\begin{cases} u_k(x) \ge u_{k+1}(x), & x \in \Omega, \\ L_k(u_k) \ge 0, \quad L_{k+l}(u_{k+l}) \le 0 & x \in \{u_{k-1} > u_k \equiv u_{k+1} \equiv \cdots \equiv u_{k+l} > u_{k+l+1}\} \cap \Omega, \\ L_k(u_k) + L_{k+l}(u_{k+l}) = 0, & x \in \{u_{k-1} > u_k \equiv u_{k+1} \equiv \cdots \equiv u_{k+l} > u_{k+l+1}\} \cap \Omega, \\ L_k(u_k) = 0, & x \in \{u_{k-1} > u_k > u_{k+1}\} \cap \Omega, \\ u_k(x) = f_k(x), & x \in \partial\Omega \end{cases}$$

for $1 \le k \le n$.

Notice that here the extra condition

$$L_k(u_k) + L_{k+l}(u_{k+l}) = 0, \quad x \in \{u_{k-1} > u_k \equiv u_{k+1} \equiv \cdots \equiv u_{k+l} > u_{k+l+1}\} \cap \Omega$$

appears.

4.2. ***Other operators.*** Our results can also be extended to the two membranes problem with different operators as soon as there are games J_1 and J_2 whose value functions approximate the solutions to the corresponding PDEs and for which the key estimates of Section 2 can be proved. Namely, we need that starting close to the boundary each player has a strategy that forces the game to end close to the initial position in the same board with large probability and in a controlled expected number of plays regardless the choices of the other player.

For instance, our results can be extended to deal with the two membranes problem for Pucci operators (for a game related to Pucci operators we refer to [Blanc et al. 2019]). Pucci operators are uniformly elliptic and are given in terms of two positive constants, λ and Λ, by the formulas

$$M^+_{\lambda,\Lambda}(D^2u) = \sup_{A\in L_{\lambda,\Lambda}} \operatorname{trace}(AD^2u) \quad \text{and} \quad M^-_{\lambda,\Lambda}(D^2u) = \inf_{A\in L_{\lambda,\Lambda}} \operatorname{trace}(AD^2u)$$

with

$$L_{\lambda,\Lambda} = \{A \in \mathbb{S}^n : \lambda \operatorname{Id} \le A \le \Lambda \operatorname{Id}\}.$$

Notice that the extra condition that we obtain with the second game reads as

$$M^+_{\lambda_1,\Lambda_1}(D^2u(x)) + M^-_{\lambda_2,\Lambda_2}(D^2v(x)) = h_2(x) - h_1(x)$$

if we play with a game associated to the equation $M^+_{\lambda_1,\Lambda_1}(D^2u(x)) + h_1(x) = 0$ in the first board and with a game associated to $M^-_{\lambda_2,\Lambda_2}(D^2u(x)) + h_1(x) = 0$ in the second board.

We leave the details to the reader.

4.3. ***Playing with an unfair coin modifies the extra condition.*** One can also deal with the game in which the coin toss that is used to determine if the player can make the choice to change boards or not is not a fair coin. Assume that a coin is tossed in the first board with probabilities γ and $(1-\gamma)$ and in the second board with reverse probabilities, $(1-\gamma)$ and γ. In this case the equations that are involved in the DPP read as

$$\begin{cases} u^\varepsilon(x) = \gamma \max\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\} + (1-\gamma) J_1(u^\varepsilon)(x), & x \in \Omega, \\ v^\varepsilon(x) = (1-\gamma) \min\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\} + \gamma J_2(v^\varepsilon)(x), & x \in \Omega. \end{cases}$$

In this case, the extra condition that we obtain is given by

$$\gamma(-\Delta_p^1 u(x) + h_1(x)) + (1-\gamma)(-\Delta_q^1 v(x) - h_2(x)) = 0, \quad x \in \Omega,$$

Notice that there are two extreme cases, $\gamma = 0$ and $\gamma = 1$. When $\gamma = 1$, the second player cannot decide to change boards but the first player has this possibility (with probability one) in the first board. In this case, in the limit problem the second component, v, is a solution to $-\Delta_q^1 v(x) - h_2(x) = 0$ in the whole Ω and u is the solution to the obstacle problem (with v as obstacle from below). On the other hand, if $\gamma = 0$, it is the first player who cannot decide to change and the second player has the command in the second board and in this case in the limit u is the component that is a solution to the equation $-\Delta_p^1 u(x) + h_1(x) = 0$, and v the one that solves the obstacle problem (with u as obstacle from above).

Note that the value functions are increasing with respect to γ, that is, $u^\varepsilon_{\gamma_1}(x) \le u^\varepsilon_{\gamma_2}(x)$ and $v^\varepsilon_{\gamma_1}(x) \le v^\varepsilon_{\gamma_2}(x)$ for $\gamma_1 \le \gamma_2$. Therefore, passing to the limit as $\varepsilon \to 0$ we obtain a family of solutions to the two membranes problem that is increasing with γ,

$$u_0(x) \le u_{\gamma_1}(x) \le u_{\gamma_2}(x) \le u_1(x) \quad \text{and} \quad v_0(x) \le v_{\gamma_1}(x) \le v_{\gamma_2}(x) \le v_1(x)$$

for $\gamma_1 \le \gamma_2$.

The pair (u_0, v_0) is the minimal solution to the two membranes problem in the sense that $u_0 \le u$ and $v_0 \le v$ for any other solution (u, v). In fact, since u is a supersolution and u_0 is a solution to $-\Delta_p^1 u(x) + h_1(x) = 0$ from the comparison principle we obtain $u_0 \le u$. Then we obtain that $v_0 \le v$ from the fact that they are solutions to the obstacle problem from above with obstacles u_0 and u respectively.

Analogously, the pair (u_1, v_1) is the maximal solution to the two membranes problem.

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