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Subspaces fixed by a nilpotent matrix

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The linear spaces that are fixed by a given nilpotent $n\times n$ matrix form a subvariety of the Grassmannian. We classify these varieties for small n. Muthiah, Weekes and Yacobi conjectured that their radical ideals are generated by certain linear forms known as shuffle equations. We prove this conjecture for $n \le 7$, and we disprove it for $n = 8$. The question remains open for nilpotent matrices arising from the affine Grassmannian.

1. Introduction

For an arbitrary field K, the Grassmannian $\mathrm{Gr}(\ell, n)$ parametrizes ℓ-dimensional subspaces L of the vector space K^n. Given any matrix $T \in K^{n\times n}$, we write LT for the image of L under the map given by T. This right action is compatible with representing L as the row space of an $\ell\times n$ matrix $\boldsymbol{L}$. The Plücker embedding of $\mathrm{Gr}(\ell, n)$ into $\mathbb{P}^{\binom{n}{\ell}-1}$ arises by representing L with the vector of maximal minors $p_{i_1 i_2\cdots i_\ell}$ of $\boldsymbol{L}$. Its homogeneous prime ideal has a natural Gröbner basis of quadrics [Sturmfels 1993, Theorem 3.1.7]. These are known as the *Plücker quadrics*.

In this paper we assume that T is nilpotent, i.e., $T^n = 0$, and we study the subvariety

$$\mathrm{Gr}(\ell, n)^T = \{L \in \mathrm{Gr}(\ell, n) : LT \subseteq L\}.$$

We are interested in its homogeneous radical ideal in the Plücker coordinates $p_{i_1 i_2\cdots i_\ell}$.

Example 1 ($n = 4$, $\ell = 2$). Fix a nonzero scalar ϵ and consider the nilpotent 4×4 matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The fixed point locus $\mathrm{Gr}(2, 4)^T$ is a singular surface in the 4-dimensional Grassmannian $\mathrm{Gr}(2, 4) = V(p_{12}p_{34} - p_{13}p_{23} + p_{14}p_{23})$. It is the quadratic cone in $\mathbb{P}^3$ defined by the prime ideal

$$\langle p_{13},\, p_{14} + \epsilon p_{23},\, p_{12}p_{34} - \epsilon p_{23}^2\rangle = \langle p_{13},\, p_{14} + \epsilon p_{23}\rangle + \text{ideal of } \mathrm{Gr}(2, 4). \tag{1}$$

On an affine chart of $\mathrm{Gr}(2, 4)$, each plane L that is fixed by T is the row span of a matrix

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & 0 & \epsilon x \end{pmatrix} \qquad \text{or} \qquad \boldsymbol{L} = \begin{pmatrix} \epsilon z & w & 1 & 0 \\ 0 & z & 0 & 1 \end{pmatrix} \quad \text{after setting } x = \frac{1}{\epsilon z} \text{ and } y = -\frac{w}{\epsilon z^2}.$$

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Next consider the special case $\epsilon = 0$. The ideal (1) is still radical, but it now decomposes:

$$\langle p_{13}, p_{14}, p_{12}p_{34}\rangle = \langle p_{13}, p_{14}, p_{34}\rangle \cap \langle p_{12}, p_{13}, p_{14}\rangle. \tag{2}$$

The quadratic cone degenerates into two planes $\mathbb{P}^2$ in $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$. They are given by

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{L} = \begin{pmatrix} 0 & w & 1 & 0 \\ 0 & z & 0 & 1 \end{pmatrix}.$$

We conclude that $\mathrm{Gr}(2,4)^T$ can be singular or reducible. For all values of ϵ in K, its radical ideal is generated by two linear forms plus the Plücker quadric $p_{12}p_{34} - p_{13}p_{23} + p_{14}p_{23}$. ◇

We assume from now on that the nilpotent matrix T is in Jordan canonical form. The necessary change of basis in K^n works over an arbitrary field K because all the eigenvalues of T are zero. The matrix T in Example 1 is in Jordan canonical form when $\epsilon = 0$ or $\epsilon = 1$.

Kreiman, Lakshmibai, Magyar, and Weyman [Kreiman et al. 2007] identified a natural set of linear forms in Plücker coordinates that vanish on $\mathrm{Gr}(\ell, n)^T$. These are called *shuffle equations* and they generalize the two linear forms seen in (1). It was conjectured in [Kreiman et al. 2007] that the shuffle equations cut out certain models of the affine Grassmannian. Muthiah, Weekes and Yacobi [Muthiah et al. 2022] gave a reformulation of the shuffle equations, and they proved the main conjecture of [Kreiman et al. 2007]. We refer to [Muthiah et al. 2022, Section 6] for that proof and for a conceptual discussion of the shuffle equations. It was subsequently conjectured in [Muthiah et al. 2022, Section 7] that the shuffle equations plus the Plücker quadrics generate the radical ideal of $\mathrm{Gr}(\ell, n)^T$. The present paper settles that conjecture.

Our presentation is organized as follows. In Section 2 we review the shuffle equations and we show how to generate them in `Macaulay2` [Macaulay2]. The duality result in Theorem 5 allows us to swap ℓ and $n - \ell$ in these computations. In Section 3 we present the classification of all varieties $\mathrm{Gr}(\ell, n)^T$ for $n \leq 8$. We compute their dimensions, degrees, irreducible components, and defining equations. We disprove the conjecture of Muthiah, Weekes and Yacobi [Muthiah et al. 2022, Conjecture 7.6] for $n = 8$, and we show that it holds for $n \leq 7$. Section 4 is devoted to finite-dimensional models of the affine Grassmannian. Here T is the nilpotent matrix given by a partition of rectangular shape. We prove that $\mathrm{Gr}(\ell, n)^T$ is irreducible for such T, and we give a matrix parametrization. We believe that Conjecture 7.1 in [Muthiah et al. 2022] holds. This is equivalent to [Muthiah et al. 2022, Conjecture 7.6] for rectangular shapes. We offer supporting evidence.

2. Shuffle equations

Fix a nilpotent $n \times n$ matrix $T = T_\lambda$ in Jordan canonical form. Here $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s)$ is any partition of the integer n. Each entry of the matrix T_λ is either 0 or 1. The entries 1 are located in positions $(j, j+1)$, where $j \in \{1, 2, \ldots, n-1\} \setminus \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_{s-1}\}$. In other words, T_λ is the nilpotent matrix in Jordan canonical form where the sizes of the Jordan blocks are given by the parts λ_i of the partition λ. The rank of T_λ equals $n - s$. We regard $\ker(T_\lambda)$ as a linear subspace of dimension $s - 1$ in the projective space $\mathbb{P}^{n-1}$.

The shuffle relations are defined as follows. Consider the $n\times n$ matrix $\mathrm{Id}_n + zT$, where z is a parameter. For $z \in K$, this is an automorphism of the vector space K^n. A subspace L of K^n satisfies $LT \subseteq L$ if and only if $L(\mathrm{Id}_n + zT) = L$ for all z. Writing $P \in K^{\binom{n}{\ell}}$ for the row vector of Plücker coordinates of L, the last equation is equivalent to the identity

$$P \cdot \textstyle\bigwedge_\ell(\mathrm{Id}_n + zT) = P. \tag{3}$$

Here $\bigwedge_\ell(\mathrm{Id}_n + zT)$ is the ℓ-th exterior power of the $n\times n$ matrix $\mathrm{Id}_n + zT$. This is an $\binom{n}{\ell}\times\binom{n}{\ell}$ matrix whose entries are polynomials in $\mathbb{Z}[z]$ of degree $\leq \ell$. Equivalently, we can write

$$\textstyle\bigwedge_\ell(\mathrm{Id}_n + zT) = \bigwedge_\ell \mathrm{Id}_n + \sum_{i=1}^{\ell} \left[\bigwedge_\ell(\mathrm{Id}_n + zT)\right]_i z^i, \tag{4}$$

where the coefficient $\left[\bigwedge_\ell(\mathrm{Id}_n + zT)\right]_i$ of z^i is an integer matrix of format $\binom{n}{\ell}\times\binom{n}{\ell}$. From (3) we then obtain

$$P \cdot \left[\textstyle\bigwedge_\ell(\mathrm{Id}_n + zT)\right]_i = 0 \quad \text{for } i = 1, 2, \ldots, \ell. \tag{5}$$

This is a finite collection of linear forms in the $\binom{n}{\ell}$ Plücker coordinates $p_{i_1 i_2 \cdots i_\ell}$. These are the *shuffle equations* of T. The following was proved by Muthiah et al. [2022, Proposition 6.6].

Proposition 2. *The variety* $\mathrm{Gr}(\ell, n)^T$ *is the intersection of the Grassmannian* $\mathrm{Gr}(\ell, n)$ *with a linear subspace in* $\mathbb{P}^{\binom{n}{\ell}-1}$. *That linear subspace is defined by the shuffle equations.*

Example 3 ($n = 4$, $\ell = 2$). We compute the shuffle equations for the matrix T in Example 1. Write $P = (p_{12}, p_{13}, p_{23}, p_{14}, p_{24}, p_{34})$. With this ordering of the Plücker coordinates, we have

$$\textstyle\bigwedge_2(\mathrm{Id}_4 + zT) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & z & \epsilon z & \epsilon z^2 & 0 \\ 0 & 0 & 1 & 0 & \epsilon z & 0 \\ 0 & 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In (5), we find $P \cdot \left[\bigwedge_2(\mathrm{Id}_4 + zT)\right]_1 = (0, 0, p_{13}, \epsilon p_{13}, (\epsilon p_{23} + p_{14}), 0)$ and $P \cdot \left[\bigwedge_2(\mathrm{Id}_4 + zT)\right]_2 = (0, 0, 0, 0, \epsilon p_{13}, 0)$. The coordinates are the shuffle equations. We saw these in (1). ◇

In the next example we demonstrate how the shuffle equations can be computed and analyzed within the computer algebra system `Macaulay2` [Macaulay2]. All computations of the varieties $\mathrm{Gr}(\ell, n)^T$ in this paper were carried out by this code, with `n`, `l` and `U` $= \mathrm{Id}_n + zT$ adjusted.

Example 4. We examine the smallest instance where $\mathrm{Gr}(\ell, n)^T$ has three irreducible components, namely $n = 6$, $\ell = 3$ and $\lambda = (3, 1, 1, 1)$, as seen in Table 2 below. The following `Macaulay2` code outputs the ideal `J` generated by the shuffle equations and Plücker quadrics:

```
n=6; l=3;
R = ring Grassmannian(l-1,n-1,CoefficientRing => QQ);
```

```
P = matrix{gens R}; S = R[z];
U = matrix {{1,z,0,0,0,0},
           {0,1,z,0,0,0},
           {0,0,1,0,0,0},
           {0,0,0,1,0,0},
           {0,0,0,0,1,0},
           {0,0,0,0,0,1}};
M = (toList coefficients(P*exteriorPower(l,U)))_1;
rowws = toList(0..((# entries M)-2));
I = minors(1,submatrix(M,rowws,))
J = I+Grassmannian(l-1,n-1,R); toString mingens J
betti mingens J, (dim J)-2, degree J
J == radical(J), isPrime J
```

The output of this code shows that the ideal J is radical but not prime. It is minimally generated by 12 linear forms and 8 quadrics. Its variety $\mathrm{Gr}(3,6)^T$ has dimension 4 and degree 2 in the ambient space $\mathbb{P}^{19}$ of $\mathrm{Gr}(3,6)$. We next compute the prime decomposition:

```
DJ =  decompose J; #DJ, betti mingens radical J
apply(DJ, T -> {T,codim T, degree T, betti mingens T})
```

The fixed point locus $\mathrm{Gr}(3,6)^T$ has three irreducible components. The largest component is defined by a quadric in a subspace $\mathbb{P}^5$. In addition, there are two coordinate subspaces $\mathbb{P}^3$. ◇

We next come to a duality result which will aid our computations in Section 3.

Theorem 5. *The varieties* $\mathrm{Gr}(\ell,n)^T$ *and* $\mathrm{Gr}(n-\ell,n)^T$ *coincide after a linear change of coordinates in the ambient space* $\mathbb{P}^{\binom{n}{\ell}-1}$. *This holds for all* ℓ *and* n *and all nilpotent* $n\times n$ *matrices* T. *Under this coordinate change, which depends on* T, *the shuffle equations coincide.*

Proof. Let $\mathrm{B}_m = (b_{ij})$ denote the $m\times m$ matrix with 1's on the antidiagonal and 0's elsewhere, i.e., $b_{ij}=1$ if $i+j=m+1$ and $b_{ij}=0$ otherwise. Given a partition λ of n and its matrix $T=T_\lambda$, we define $B=B_\lambda$ to be the block-diagonal $n\times n$ matrix $B_\lambda = \mathrm{diag}(\mathrm{B}_{\lambda_1},\ldots,\mathrm{B}_{\lambda_s})$. Note that $B^2=\mathrm{Id}_n$ and, if T^{t} denotes the transpose T, that $TB = BT^{\mathrm{t}}$, i.e., T is a self-adjoint linear operator for the nondegenerate symmetric bilinear form defined by B on K^n.

We consider the nondegenerate inner product on K^n that is defined by the invertible symmetric matrix B. The orthogonal space of a given ℓ-dimensional subspace L with respect to this inner product is the $(n-\ell)$-dimensional subspace

$$L^\perp = \ker(\boldsymbol{L}B) = \{v\in K^n : uBv^t = 0 \text{ for all } u\in L\}.$$

Suppose L is T-fixed. We claim that $L^\perp$ is T-fixed. Indeed, suppose $v\in L^\perp$, i.e., $uBv^t=0$ for all $u\in L$. This implies $uB(vT)^t = uBT^tv^t = (uT)Bv^t = 0$ for all $u\in L$, and so $vT\in L^\perp$. This shows that passing to the orthogonal space defines the desired linear isomorphism,

$$\mathrm{Gr}(\ell,n)^T \to \mathrm{Gr}(n-\ell,n)^T, \quad L \mapsto L^\perp. \tag{6}$$

For $T = 0_n$, this is the familiar isomorphism between the Grassmannians $\mathrm{Gr}(\ell,n)$ and $\mathrm{Gr}(n-\ell,n)$. A subtle point is that duality is taken relative to the inner product given by B.

We shall explicitly describe the linear change of coordinates on $\mathbb{P}^{\binom{n}{\ell}-1}$ that induces the isomorphism (6). We start with the *Hodge star isomorphism* $P \mapsto P^*$ that takes the vector $P = (p_{i_1\cdots i_\ell})_{1\le i_1<\cdots<i_\ell\le n}$ to the vector $P^* = (p^*_{j_1\cdots j_{n-\ell}})_{1\le j_1<\cdots<j_{n-\ell}\le n}$. If I is an ordered ℓ-subset of $[n] = \{1,2,\ldots,n\}$ and $J = [n]\setminus I$ is the complementary ordered $(n-\ell)$-subset then

$$p^*_J := \mathrm{sign}(I,J)\cdot p_I.$$

Here $\mathrm{sign}(I,J)$ is the sign of the permutation of $[n]$ given by the ordered sequence (I,J).

To be completely explicit, here is an example. For $n=4$ the formula for the Hodge star is

$$\begin{aligned}P^* &= (p^*_1,p^*_2,p^*_3,p^*_4 \mid p^*_{12},p^*_{13},p^*_{23},p^*_{14},p^*_{24},p^*_{34} \mid p^*_{123},p^*_{124},p^*_{134},p^*_{234})\\ &= (p_{234},-p_{134},p_{124},-p_{123} \mid p_{34},-p_{24},p_{14},p_{23},-p_{13},p_{12} \mid p_4,-p_3,p_2,-p_1).\end{aligned} \tag{7}$$

The restriction of the Hodge star to the Grassmannian $\mathrm{Gr}(\ell,n)$ in $\mathbb{P}^{\binom{n}{\ell}-1}$ takes a linear space to its orthogonal space with respect to the standard inner product. To incorporate the quadratic form B, we consider the automorphism of $\mathbb{P}^{\binom{n}{\ell}-1}$ that takes P to $\big(P\cdot(\bigwedge_\ell B)\big)^*$. The restriction of this automorphism to the Grassmannian $\mathrm{Gr}(\ell,n)$ is the isomorphism (6).

It remains to show that the map $P\mapsto \big(P\cdot(\bigwedge_\ell B)\big)^*$ preserves the shuffle equations. To do this, let M_ℓ be the $\binom{n}{\ell}\times\binom{n}{\ell}$ matrix with entries in K such that $P^* = PM_\ell$. Note that M_ℓ^2 is the identity matrix. Via conjugation, the Hodge star operator extends to $K^{\binom{n}{\ell}\times\binom{n}{\ell}}$, i.e., via sending N to $N^* = M_\ell N M_\ell$. Assuming that $P\cdot\bigwedge_\ell(\mathrm{Id}_n+zT) = P$ for all $z\in K$, we rewrite

$$\begin{aligned}\big(P\cdot(\textstyle\bigwedge_\ell B)\big)^*\textstyle\bigwedge_{n-\ell}(\mathrm{Id}_n+zT) &= P\cdot\big(\textstyle\bigwedge_\ell B\big)M_\ell\textstyle\bigwedge_{n-\ell}(\mathrm{Id}_n+zT)\\ &= P\cdot\big(\textstyle\bigwedge_\ell B\big)\textstyle\bigwedge_\ell(\mathrm{Id}_n+zT^t)\big(\textstyle\bigwedge_\ell B\big)\big(\textstyle\bigwedge_\ell B\big)M_\ell\\ &= P\cdot\textstyle\bigwedge_\ell(\mathrm{Id}_n+zBT^tB)\big(\textstyle\bigwedge_\ell B\big)M_\ell\\ &= P\cdot\textstyle\bigwedge_\ell(\mathrm{Id}_n+zT)\big(\textstyle\bigwedge_\ell B\big)M_\ell\\ &= P\cdot\big(\textstyle\bigwedge_\ell B\big)M_\ell\\ &= \big(P\cdot(\textstyle\bigwedge_\ell B)\big)^*.\end{aligned}$$

This shows that the shuffle equations for $\mathrm{Gr}(\ell,n)^T$ are mapped to those of $\mathrm{Gr}(n-\ell,n)^T$ under our automorphism of $\mathbb{P}^{\binom{n}{\ell}-1}$. This was the claim, and the proof of Theorem 5 is complete. □

Example 6 ($n=4$). Fix $\epsilon = 0$ in Example 1. Then $T = T_\lambda$ for $\lambda = (2,1,1)$, and we have

$$B = B_\lambda = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Our map $P \mapsto \big(P\cdot(\bigwedge_\ell B)\big)^*$, written for $\ell = 1, 2, 3$, is the following signed permutation of (7):

$$P \mapsto (-p_{134}, p_{234}, p_{124}, -p_{123} \mid -p_{34}, p_{14}, -p_{24}, -p_{13}, p_{23}, p_{12} \mid -p_4, p_3, -p_1, p_2). \qquad (8)$$

For $\ell = 1$ the unique shuffle equation is p_1. This is mapped to $-p_{134}$, which is the unique shuffle equation for $\ell = 3$. Likewise, p_{134} is mapped to $-p_1$. This makes sense because $\mathrm{Gr}(1,4)^T = V(p_1) = \mathrm{span}(e_2, e_3, e_4) = \ker(T)$, whereas $\mathrm{Gr}(3,4)^T = V(p_{134})$ consists of all hyperplanes in K^4 that contain e_2. Both are projective planes $\mathbb{P}^2$. Our involution swaps them.

For $\ell = n - \ell = 2$, there are two shuffle equations, namely p_{13} and p_{14}, as seen in Example 3. These two Plücker coordinates are swapped (up to sign) in (8), so our involution fixes $\mathrm{Gr}(2,4)^T$. Moreover, this involution interchanges the two irreducible components in (2). We see this in the coordinate change (8) which sends $p_{12} \mapsto -p_{34}$ and $p_{34} \mapsto p_{12}$. ◇

3. Classification and counterexample

The main result in this article is the determination of all fixed point loci $\mathrm{Gr}(\ell,n)^T$ for $n \le 8$. From this computational result, we extract the following theorem about the shuffle equations.

Theorem 7. *Fix $1 \le \ell < n \le 7$ and let T be any nilpotent $n \times n$ matrix. Then the shuffle equations generate the radical ideal of the fixed point locus $\mathrm{Gr}(\ell,n)^T$. The same does not hold for $n = 8$: there is a unique partition, namely $\lambda = (4,2,2)$, and a unique dimension, namely $\ell = 4$, such that the radical ideal of $\mathrm{Gr}(\ell,n)^{T_\lambda}$ is not generated by the shuffle equations.*

Proof. The proof is carried out by exhaustive computation of all varieties $\mathrm{Gr}(\ell,n)^{T_\lambda}$, where λ is any partition of $n \le 8$. Here we use the `Macaulay2` code from Example 4 and Theorem 5.

The results are summarized in Tables 1, 2 and 3. For each instance (λ, ℓ), we report a triple $[\sigma, \delta, \gamma]$ or $[\sigma, \delta, \gamma]^\kappa$. Here σ is the number of linearly independent shuffle equations. The entries δ and γ are the dimension and degree of $\mathrm{Gr}(\ell,n)^{T_\lambda}$ in its Plücker embedding into $\mathbb{P}^{\binom{n}{\ell}-1}$. The upper index κ is the number of irreducible components of $\mathrm{Gr}(\ell,n)^T$, and this index is dropped if $\kappa = 1$. The columns

λ	$\ell=1$	$\ell=2$
(1,1,1,1)	[0,3,1]	[0,4,2]
(2,1,1)	[1,2,1]	$[2,2,2]^2$
(2,2)	[2,1,1]	[2,2,2]
(3,1)	[2,1,1]	[4,1,1]
(4)	[3,0,1]	[5,0,1]

λ	$\ell=1$	$\ell=2$
(1,1,1,1,1)	[0,4,1]	[0,6,5]
(2,1,1,1)	[1,3,1]	$[3,4,2]^2$
(2,2,1)	[2,2,1]	[4,3,3]
(3,1,1)	[2,2,1]	$[6,2,2]^2$
(3,2)	[3,1,1]	[6,2,2]
(4,1)	[3,1,1]	[8,1,1]
(5)	[4,0,1]	[9,0,1]

Table 1. Fixed point loci $\mathrm{Gr}(\ell,n)^T$ for $n=4$ and $n=5$.

λ	$\ell=1$	$\ell=2$	$\ell=3$
(1,1,1,1,1,1)	[0,5,1]	[0,8,14]	[0,9,42]
(2,1,1,1,1)	[1,4,1]	$[4,6,5]^2$	$[6,6,10]^2$
(2,2,1,1)	[2,3,1]	$[6,4,6]^2$	[8,5,10]
(3,1,1,1)	[2,3,1]	$[8,4,2]^2$	$[12,4,2]^3$
(2,2,2)	[3,2,1]	[6,4,6]	[11,4,6]
(3,2,1)	[3,2,1]	[9,3,3]	$[12,3,6]^2$
(4,1,1)	[3,2,1]	$[11,2,2]^2$	$[16,2,2]^2$
(3,3)	[4,1,1]	[11,2,2]	[12,3,6]
(4,2)	[4,1,1]	[11,2,2]	[16,2,2]
(5,1)	[4,1,1]	[13,1,1]	[18,1,1]
(6)	[5,0,1]	[14,0,1]	[19,0,1]

λ	$\ell=1$	$\ell=2$	$\ell=3$
(1,1,1,1,1,1,1)	[0,6,1]	[35,10,42]	[140,12,462]
(2,1,1,1,1,1)	[1,5,1]	$[5,8,14]^2$	$[10,9,42]^2$
(2,2,1,1,1)	[2,4,1]	$[8,6,5]^2$	$[14,7,35]^2$
(3,1,1,1,1)	[2,4,1]	$[10,6,5]^2$	$[20,6,10]^3$
(2,2,2,1)	[3,3,1]	[9,5,10]	[17,6,30]
(3,2,1,1)	[3,3,1]	$[12,4,6]^2$	$[21,5,10]^2$
(4,1,1,1)	[3,3,1]	$[14,4,2]^2$	$[27,4,2]^3$
(3,2,2)	[4,2,1]	[12,4,6]	$[23,4,12]^2$
(3,3,1)	[4,2,1]	[15,3,3]	[23,4,12]
(4,2,1)	[4,2,1]	[15,3,3]	$[27,3,6]^2$
(5,1,1)	[4,2,1]	$[17,2,2]^2$	$[31,2,2]^2$
(4,3)	[5,1,1]	[17,2,2]	[27,3,6]
(5,2)	[5,1,1]	[17,2,2]	[31,2,2]
(6,1)	[5,1,1]	[19,1,1]	[33,1,1]
(7)	[6,0,1]	[20,0,1]	[34,0,1]

Table 2. Fixed point loci $\mathrm{Gr}(\ell, n)^T$ for $n = 6$ and $n = 7$.

for $\ell > n/2$ are omitted because of Theorem 5. In any given row of one of our tables, the entry for $n-\ell$ would be identical to that for ℓ.

In each case, we computed the irreducible components of the shuffle ideal. We recorded the prime ideal for each component, and we determined degree, dimension, singularities, etc. The intersection of these primes is the radical ideal of $\mathrm{Gr}(\ell, n)^T$. In all cases but one, we found that the radical ideal is generated by the shuffle equations plus the Plücker quadrics. The unique exceptional case is $\lambda = (4, 2, 2)$ and $\ell = 4$, with the highlighted entry $\mathbf{[54,4,24]^3}$. This means that there are 54 linearly independent shuffle relations plus 4 additional Plücker quadrics. However, this ideal is not radical. To generate the radical, we need one more linear form. Further below, we shall examine the geometry of this counterexample in detail.

An easy `Macaulay2` proof for the failure of `J` to be radical is running the following line:

```
apply(first entries promote(P,S),p -> {p % J, p^2 % J})
```

This reveals that the variable p_{1468} is not in `J` but its square is in `J`. Note that this coordinate corresponds to $\mathtt{p}_{0357}$ in the zero-based indexing of `Macaulay2`. This concludes the proof. ☐

Example 8 ($n = 6$). Consider the lower right entry on the left in Table 2. Here $\ell = 3$ and $\lambda = (6)$, so T is the nilpotent matrix that maps $e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto e_5 \mapsto e_6 \mapsto 0$. The variety $\mathrm{Gr}(3, 6)^T$ consists of a

λ	$\ell=2$	$\ell=3$	$\ell=4$
(1,1,1,1,1,1,1,1)	[0,12,132]	[420,15,6006]	[721,16,24024]
(2,1,1,1,1,1,1)	$[6,10,42]^2$	$[15,12,462]^2$	$[20,12,924]^2$
(2,2,1,1,1,1)	$[10,8,14]^2$	$[22,9,168]^2$	$[28,10,420]^3$
(3,1,1,1,1,1)	$[12,8,14]^2$	$[30,9,42]^3$	$[40,9,42]^3$
(2,2,2,1,1)	$[12,6,20]^2$	[26,8,140]	$[34,8,280]^2$
(3,2,1,1,1)	$[15,6,5]^2$	$[33,7,35]^3$	$[42,7,70]^2$
(4,1,1,1,1)	$[17,6,5]^2$	$[41,6,10]^3$	$[54,6,10]^4$
(2,2,2,2)	[12,6,20]	[32,7,70]	[34,8,280]
(3,2,2,1)	[16,5,10]	$[35,6,30]^2$	$[46,6,60]^2$
(3,3,1,1)	$[19,4,6]^2$	$[38,5,30]^2$	[46,6,60]
(4,2,1,1)	$[19,4,6]^2$	$[42,5,10]^2$	$[54,5,10]^3$
(5,1,1,1)	$[21,4,2]^2$	$[48,4,2]^3$	$[62,4,2]^3$
(3,3,2)	[19,4,6]	[38,5,30]	[52,5,30]
(4,2,2)	[19,4,6]	$[44,4,12]^2$	$[\mathbf{54},\mathbf{4},\mathbf{24}]^{\mathbf{3}}$
(4,3,1)	[22,3,3]	[44,4,12]	$[54,4,24]^2$
(5,2,1)	[22,3,3]	$[48,3,6]^2$	$[62,3,6]^2$
(6,1,1)	$[24,2,2]^2$	$[52,2,2]^2$	$[66,2,2]^2$
(4,4)	[24,2,2]	[48,3,6]	[54,4,24]
(5,3)	[24,2,2]	[48,3,6]	[62,3,6]
(6,2)	[24,2,2]	[52,2,2]	[66,2,2]
(7,1)	[26,1,1]	[54,1,1]	[68,1,1]
(8)	[27,0,1]	[55,0,1]	[69,0,1]

Table 3. Fixed point loci $\mathrm{Gr}(\ell,n)^T$ for $n=8$.

single point e_{456}. It is instructive to revisit the construction of the shuffle equations for this case. The 20 coordinates of the row vector $P\cdot\bigwedge_3(\mathrm{Id}_6+zT)$ are

$$p_{123},\quad p_{123}z+p_{124},\quad p_{123}z^2+p_{124}z+p_{134},\quad p_{123}z^3+p_{124}z^2+p_{134}z+p_{234},\quad p_{124}z+p_{125},$$
$$p_{124}z^2+(p_{134}+p_{125})z+p_{135},\quad p_{124}z^3+(p_{134}+p_{125})z^2+(p_{234}+p_{135})z+p_{235},$$
$$p_{134}z^2+p_{135}z+p_{145},\quad p_{134}z^3+(p_{234}+p_{135})z^2+(p_{235}+p_{145})z+p_{245},$$
$$p_{234}z^3+p_{235}z^2+p_{245}z+p_{345},\quad p_{125}z+p_{126},\quad p_{125}z^2+(p_{135}+p_{126})z+p_{136},$$
$$p_{125}z^3+(p_{135}+p_{126})z^2+(p_{235}+p_{136})z+p_{236},\quad p_{135}z^2+(p_{145}+p_{136})z+p_{146},$$

$$p_{135}z^3 + (p_{235} + p_{145} + p_{136})z^2 + (p_{245} + p_{236} + p_{146})z + p_{246},$$
$$p_{235}z^3 + p_{245} + p_{236}z^2 + (p_{345} + p_{246})z + p_{346},$$
$$p_{145}z^2 + p_{146}z + p_{156}, \quad p_{145}z^3 + (p_{245} + p_{146})z^2 + (p_{246}+p_{156})z + p_{256},$$
$$p_{245}z^3 + (p_{345}+p_{246})z^2 + (p_{346}+p_{256})z + p_{356}, \quad p_{345}z^3+p_{346}z^2+p_{356}z+p_{456}.$$

The shuffle equations are the coefficients of z^3, z^2 and z. They span the ideal of all Plücker coordinates except p_{456}. This is the homogeneous maximal ideal of $\mathrm{Gr}(3,6)^T = \{e_{456}\}$. ◇

Example 9 ($n = 8$). The smallest instance of a variety $\mathrm{Gr}(\ell, n)^{T_\lambda}$ with four irreducible components occurs for $n = 8$, $\lambda = (4, 1, 1, 1, 1)$, and $\ell = 4$. There are 54 linearly independent shuffle equations, and 46 Plücker quadrics remain modulo these linear forms. The variety $\mathrm{Gr}(4, 8)^{T_\lambda}$ has dimension 6 and degree 10 in $\mathbb{P}^{69}$. It is the union of four irreducible components, two of dimension 6 and degree 5, and two linear spaces of dimension 4. ◇

We now present a detailed study of our counterexample to [Muthiah et al. 2022, Conjecture 7.6]. We have $n = 8$, $\ell = 4$, and the matrix $T = T_\lambda$ given by the partition $\lambda = (4, 2, 2)$, i.e., operating as

$$e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto 0, \quad e_5 \mapsto e_6 \mapsto 0, \quad e_7 \mapsto e_8 \mapsto 0.$$

We consider the scheme structure on $\mathrm{Gr}(4, 8)^T$ given by the shuffle ideal $\mathtt{J}$. There are three minimal primes, each of dimension 4 and degree 6. One component is nonreduced of multiplicity 2, so the degree of our scheme is $24 = 6 + 6 + 2 \cdot 6$. It has no embedded primes.

We begin with the two reduced components. Each of these is a Segre fourfold $\mathbb{P}^2 \times \mathbb{P}^2$ lying in a $\mathbb{P}^8$ inside a coordinate subspace $\mathbb{P}^{11}$. The two ambient coordinate subspaces are

$$\mathrm{span}\{e_{1234}, e_{1346}, e_{1348}, e_{2345}, e_{2346}, e_{2347}, e_{2348}, e_{3456}, e_{3458}, e_{3467}, e_{3468}, e_{3478}\},$$
$$\mathrm{span}\{e_{3456}, e_{3458}, e_{3467}, e_{3468}, e_{3478}, e_{3568}, e_{3678}, e_{4567}, e_{4568}, e_{4578}, e_{4678}, e_{5678}\}.$$

In suitable affine coordinates, the two reduced components are parametrized by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & a & b & c & d \\ 0 & 1 & 0 & 0 & 0 & a & 0 & c \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & a & b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 1 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The T-module structures on these subspaces L are given by the partitions (4) and (2, 2).

We now study the nonreduced component. It lies in a $\mathbb{P}^8$ inside the coordinate subspace

$$\mathrm{span}\{e_{3468}, e_{2346}, e_{2348}, e_{2468}, e_{3456}, e_{3458}, e_{3467}, e_{3478}, e_{4568}, e_{4678}\} \simeq \mathbb{P}^9.$$

Geometrically, it is a cone over a hyperplane slice of $\mathbb{P}^2 \times \mathbb{P}^2$. It has the matrix representation

$$\begin{pmatrix} 0 & a & 1 & 0 & b & 0 & c & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & e & 1 & f & 0 \\ 0 & g & 0 & 0 & h & 0 & i & 1 \end{pmatrix} \quad \text{where the } 3{\times}3 \text{ block } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ has trace 0 and rank} \leq 1.$$

The zero matrix gives a singular point on this component. There are moreover three distinct T-module structures on the subspaces L in this component, namely $(3, 1)$, $(2, 2)$ and $(2, 1, 1)$.

Remark 10. The first nontrivial entry in each table is $\lambda = (2, 1, 1, \ldots, 1)$. Each irreducible component of $\mathrm{Gr}(\ell, n)^{T_\lambda}$ is a Grassmannian. This is obvious for $\ell = 1$. We sketch a proof for $2 \le \ell \le \lceil n/2 \rceil$. The matrix T maps $e_1 \mapsto e_2$ and $e_i \mapsto 0$ for $i \ge 2$. Thus, $\ker(T)$ is a hyperplane in K^n and each $L \in \mathrm{Gr}(\ell, n)^T$ possesses a minimal subspace $\widetilde{L}$ satisfying $L = \widetilde{L} + \widetilde{L}T$. The space $\widetilde{L}$ might not be unique, but its dimension is, being either ℓ or $\ell - 1$. In the first case, $\widetilde{L} = L$ and L is a subspace of $\ker(T)$. In the second case, $\widetilde{L}$ is an $(\ell-1)$-dimensional subspace of K^n with $\widetilde{L} \not\subseteq \ker T$ and $\widetilde{L}T = \mathrm{span}(e_2)$. This implies that $\mathrm{Gr}(\ell, n)^T$ has two irreducible components, namely the Grassmannians $\mathrm{Gr}(\ell, n-1)$ and $\mathrm{Gr}(\ell-1, n)$.

4. The affine Grassmannian

The key player in the articles [Kreiman et al. 2007] and [Muthiah et al. 2022] is the affine Grassmannian, which is an infinite-dimensional variety. Our varieties $\mathrm{Gr}(\ell, n)^T$ serve as finite-dimensional models, when restricting to $T = T_\lambda$, where λ is a *rectangular partition*. By this we mean partitions $\lambda = (r, r, \ldots, r)$ with d parts, so that $dr = n$ and $d, r \ge 2$. This section revolves around the next two points.

Theorem 11. *If λ is a rectangular partition then the variety $\mathrm{Gr}(\ell, n)^{T_\lambda}$ is irreducible.*

Conjecture 12. Conjecture 7.6 in [Muthiah et al. 2022] holds for rectangular partitions λ. In other words, for rectangular partitions, the shuffle equations plus Plücker quadrics generate a prime ideal.

Remark 13. Tables 1, 2 and 3 show that Theorem 11 and Conjecture 12 are true for $n \le 8$. In that range, the only rectangular partitions λ are $(2, 2)$, $(2, 2, 2)$, $(3, 3)$, $(2, 2, 2, 2)$ and $(4, 4)$. We see that the shuffle ideals that cut out their varieties $\mathrm{Gr}(\ell, n)^{T_\lambda}$ are prime for all ℓ.

We shall derive Theorem 11 from known facts about Schubert varieties in affine Grassmannians. We aim to explain this approach in a manner that is as self-contained as possible. The section concludes with some further evidence in support of Conjecture 12.

Let $\mathbb{K} = K((t))$ be the field of Laurent series with coefficients in K. Its valuation ring $\mathcal{O}_\mathbb{K} = K[[t]]$ consists of formal power series with nonnegative integer exponents. The residue field is K. The $\mathbb{K}$-vector space $\mathbb{K}^d$ is a module over $\mathcal{O}_\mathbb{K}$. A *lattice* L is an $\mathcal{O}_\mathbb{K}$-submodule of $\mathbb{K}^d$ of maximal rank d. Two lattices L and L' are *equivalent* if $L' = t^a L$ for some $a \in \mathbb{Z}$. To parametrize all lattices, we consider the groups $\mathrm{GL}_d(\mathbb{K})$ and $\mathrm{GL}_d(\mathcal{O}_\mathbb{K})$ of invertible $d \times d$ matrices with entries in $\mathbb{K}$ and $\mathcal{O}_\mathbb{K}$ respectively. The *affine Grassmannian* is the coset space

$$\mathrm{GL}_d(\mathbb{K})/\mathrm{GL}_d(\mathcal{O}_\mathbb{K}). \tag{9}$$

Its points are the lattices L. Indeed, every L is the column span over $\mathcal{O}_\mathbb{K}$ of a matrix in $\mathrm{GL}_d(\mathbb{K})$. Two matrices define the same L if they differ via right multiplication by a matrix in $\mathrm{GL}_d(\mathcal{O}_\mathbb{K})$. To obtain finite-dimensional varieties we can study with a computer, we set

$$\mathcal{B}_r = \{L \text{ lattice} : t^r \mathcal{O}_\mathbb{K}^d \subseteq L \subseteq \mathcal{O}_\mathbb{K}^d\}. \tag{10}$$

We note that (9) modulo equivalence of lattices equals the *Bruhat–Tits building* for $\mathrm{GL}_d(\mathbb{K})$. The set $\mathcal{B}_r$ represents the ball of radius r around the standard lattice $\mathcal{O}_{\mathbb{K}}^d$ in that building.

Both $t^r\mathcal{O}_{\mathbb{K}}^d$ and $\mathcal{O}_{\mathbb{K}}^d$ are infinite-dimensional vector spaces over K. Their quotient is a finite-dimensional vector space over K. This space has dimension $n = dr$, and we make the identification

$$K^n = \mathcal{O}_{\mathbb{K}}^d / t^r \mathcal{O}_{\mathbb{K}}^d. \tag{11}$$

Writing $e_1, e_2, \ldots, e_d$ for the standard basis of K^d, we shall use the following basis for K^n:

$$e_1, te_1, \ldots, t^{r-1}e_1, \quad e_2, te_2, \ldots, t^{r-1}e_2, \quad \ldots, \quad e_d, te_d, \ldots, t^{r-1}e_d. \tag{12}$$

In this basis, multiplication with t is given by the nilpotent $n\times n$ matrix T_λ for $\lambda = (r, \ldots, r)$.

Every lattice $L \in \mathcal{B}_r$ is determined by its image in (11). We also write L for that image. Hence L is a subspace of K^n that satisfies $LT_\lambda \subseteq L$. Conversely, every subspace L of K^n satisfying $LT_\lambda \subseteq L$ comes from a unique lattice in $\mathcal{B}_r$. This establishes the following result.

Proposition 14. *The radius r ball in (10) is the following finite union of projective varieties*:

$$\mathcal{B}_r = \bigcup_{\ell=0}^{dr} \mathrm{Gr}(\ell, n)^{T_\lambda}, \qquad \textit{where } \lambda = (r, r, \ldots, r). \tag{13}$$

Example 15 ($d = r = 2$). Here $n = rd = 4$, $T = T_{(2,2)}$, and the disjoint union in (13) equals

$$\mathcal{B}_2 = \mathrm{Gr}(0,4)^T \cup \mathrm{Gr}(1,4)^T \cup \mathrm{Gr}(2,4)^T \cup \mathrm{Gr}(3,4)^T \cup \mathrm{Gr}(4,4)^T.$$

The first and last Grassmannian are the points that represent the lattices $\mathcal{O}_{\mathbb{K}}^2$ and $t^2\mathcal{O}_{\mathbb{K}}^2$. The second and fourth Grassmannian are projective lines $\mathbb{P}^1$. The middle Grassmannian is a quadratic cone in $\mathbb{P}^3$. We saw this in (1) for $\epsilon = 1$. Note the row $\lambda = (2, 2)$ in Table 1. ◇

Example 16 ($n = 8$). The two options are $d = 4$, $r = 2$ and $d = 2$, $r = 4$. These are the rows $\lambda = (2, 2, 2, 2)$ and $\lambda = (4, 4)$ of Table 3. In either case, $\mathcal{B}_r$ is the disjoint union of nine irreducible varieties, indexed by $\ell = 0, 1, 2, 3, 4, 5, 6, 7, 8$, and soon to be called Schubert varieties. Their dimensions are 0, 3, 6, 7, 8, 7, 6, 3, 0 and 0, 1, 2, 3, 4, 3, 2, 1, 0 respectively. ◇

We turn towards the proof of Theorem 11. We will give a polynomial parametrization for each variety $\mathrm{Gr}(\ell, n)^{T_\lambda}$ in (13). The elements of the group $\mathrm{GL}_d(\mathcal{O}_{\mathbb{K}})$ are $d\times d$ matrices $\boldsymbol{A} = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \cdots$, where each A_i is a $d\times d$ matrix with entries in K, and $\det(A_0) \neq 0$. This group acts naturally on (10) and on (11). The $d\times d$ matrix $\boldsymbol{A}$ with entries in $\mathcal{O}_{\mathbb{K}} \subset \mathbb{K}$ admits the following representation by an $n\times n$ matrix over the residue field K:

$$A = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{r-2} & A_{r-1} \\ 0 & A_0 & A_1 & \cdots & A_{r-3} & A_{r-2} \\ 0 & 0 & A_0 & \cdots & A_{r-4} & A_{r-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_0 & A_1 \\ 0 & 0 & 0 & \cdots & 0 & A_0 \end{pmatrix}. \tag{14}$$

To get this nice block form, the basis of K^n shown in (12) has to be reordered as follows:

$$e_1, e_2, \ldots, e_d, \quad te_1, te_2, \ldots, te_d, \quad \ldots, \quad t^{r-1}e_1, t^{r-1}e_2, \ldots, t^{r-1}e_d. \tag{15}$$

The matrices A act on each of the components in (13). We are interested in their orbits.

Let μ be a partition of the integer ℓ with at most d parts and largest part at most r. To be precise, we write $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d$, where $r \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_d \geq 0$ and $\sum_{i=1}^d \mu_i = \ell$. With this partition we associate the lattice $L_\mu = t^{r-\mu_1}\mathcal{O}_{\mathbb{K}}e_1 \oplus t^{r-\mu_2}\mathcal{O}_{\mathbb{K}}e_2 \oplus \cdots \oplus t^{r-\mu_d}\mathcal{O}_{\mathbb{K}}e_d$. The corresponding subspace of K^n is spanned by standard basis vectors:

$$L_\mu = K\{t^{r-i}e_j : 1 \leq i \leq \mu_j \text{ and } 1 \leq j \leq d\}.$$

By construction, we have $L_\mu \in \mathrm{Gr}(\ell, n)^T$. Since L_μ is a coordinate subspace, its Plücker coordinates are given by one of the basis points in $\mathbb{P}^{\binom{n}{\ell}-1}$, here denoted by e_μ for simplicity.

The orbit of L_μ under the above group action is a constructible subset of $\mathrm{Gr}(\ell, n)^T \subset \mathbb{P}^{\binom{n}{\ell}-1}$. It consists of all points $e_\mu \cdot \bigwedge_\ell A$ that represent the subspaces $L_\mu A$, where A runs over all matrices of the form (14). Let W_μ denote the Zariski closure of this orbit. In symbols,

$$W_\mu = \overline{\mathrm{GL}_d(\mathcal{O}_{\mathbb{K}}) \cdot L_\mu} \subseteq \mathrm{Gr}(\ell, n)^T.$$

The variety W_μ is called a *Schubert variety*. We immediately obtain the following lemma.

Remark 17. For each partition μ of ℓ, the Schubert variety W_μ is irreducible. It is given by an explicit polynomial parametrization, namely $A \mapsto e_\mu \cdot \bigwedge_\ell A$, which encodes $A \mapsto L_\mu A$.

Example 18 ($d = 3,\ r = 2,\ \ell = 3$). Let $\mu = (2, 1, 0)$ with the basis (15) of K^6. The subspace L_μ corresponds to the point $e_\mu = e_{145}$ in $\mathrm{Gr}(3, 6)^T \subset \mathbb{P}^{19}$. Its image under A is the row space of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot A = \begin{pmatrix} a_{011} & a_{012} & a_{013} & a_{111} & a_{112} & a_{113} \\ 0 & 0 & 0 & a_{011} & a_{012} & a_{013} \\ 0 & 0 & 0 & a_{021} & a_{022} & a_{023} \end{pmatrix}. \tag{16}$$

The action of the group $\mathrm{GL}_3(\mathcal{O}_{\mathbb{K}})$ on K^6 is given by the matrix in (14), here written as

$$A = \begin{pmatrix} A_0 & A_1 \\ 0 & A_0 \end{pmatrix} = \begin{pmatrix} a_{011} & a_{012} & a_{013} & a_{111} & a_{112} & a_{113} \\ a_{021} & a_{022} & a_{023} & a_{121} & a_{122} & a_{123} \\ a_{031} & a_{032} & a_{033} & a_{131} & a_{132} & a_{133} \\ 0 & 0 & 0 & a_{011} & a_{012} & a_{013} \\ 0 & 0 & 0 & a_{021} & a_{022} & a_{023} \\ 0 & 0 & 0 & a_{031} & a_{032} & a_{033} \end{pmatrix}.$$

The Schubert variety W_μ is parametrized by all matrices (16). As a subvariety of the Grassmannian $\mathrm{Gr}(3, 6)$, it is defined by the following 11 linear forms in the 20 Plücker coordinates:

$$p_{123},\ p_{124},\ p_{134},\ p_{234},\ p_{125},\ p_{135},\ p_{235},\ p_{126},\ p_{136},\ p_{236},\ p_{156} - p_{246} + p_{345}. \tag{17}$$

This subvariety has dimension 4 and degree 6, and we find that $\mathrm{Gr}(3, 6)^T = W_\mu$. It is the entry [11, 4, 6] for $\lambda = (2, 2, 2)$ of Table 2. The expressions (17) are the shuffle equations. $\diamond$

The duality of Theorem 5 acts on the Schubert varieties as follows. The complement to $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$ is the partition $\mu^c = (r-\mu_d, r-\mu_{d-1}, \ldots, r-\mu_1)$ of the integer $n-\ell$. Then the inclusion $W_{\mu^c} \subseteq \mathrm{Gr}(n-\ell, n)^T$ is isomorphic to the inclusion $W_\mu \subseteq \mathrm{Gr}(\ell, n)^T$.

We summarize the above discussion as follows: for any partition μ of ℓ with $\leq d$ parts of size $\leq r$, we have constructed an irreducible subvariety W_μ of $\mathrm{Gr}(\ell, n)^T$. Here $n = dr$ and $T = T_\lambda$ for $\lambda = (r, r, \ldots, r)$. The union of these varieties equals $\mathrm{Gr}(\ell, n)^T$ because every lattice in the ball $\mathcal{B}_r$ lies in the $\mathrm{GL}(\mathcal{O}_\mathbb{K})$-orbit of some lattice $L_\mu = t^{r-\mu_1}\mathcal{O}_\mathbb{K} e_1 \oplus \cdots \oplus t^{r-\mu_d}\mathcal{O}_\mathbb{K} e_d$.

To proceed further, we record the following fact about inclusions of Schubert varieties.

Lemma 19. *Let μ and ν be two partitions of ℓ with at most d parts and largest part at most r. Then the inclusion $W_\mu \subseteq W_\nu$ holds if and only if $\mu \leq \nu$ in the dominance order on partitions.*

Proof. This is well known in algebraic combinatorics; see, e.g., [Lakshmibai and Brown 2015, Remark 5.3.4]. □

Deriving the irreducibility of $\mathrm{Gr}(\ell, n)^T$ is now reduced to a combinatorial argument.

Proof of Theorem 11. Two partitions satisfy $\mu \leq \nu$ in dominance order if and only if $\mu_1 + \cdots + \mu_i \leq \nu_1 + \cdots + \nu_i$ for all i. Consider the set $\mathcal{P}$ of all partitions of ℓ with at most d parts whose largest part has size at most r. The restriction of dominance order to this set has a unique largest element $\mu_{\max}$. Namely, this largest partition equals $\mu_{\max} = (r, r, \ldots, r, b)$. The partition $\mu_{\max}$ has a blocks of size r, where $\ell = ar + b$ and $0 \leq b < r$. Lemma 19 implies

$$W_{\mu_{\max}} = \bigcup_{\mu \in \mathcal{P}} W_\mu = \mathrm{Gr}(\ell, n)^T. \tag{18}$$

In light of Remark 17, this proves the irreducibility of $\mathrm{Gr}(\ell, n)^T$, i.e., Theorem 11 holds. □

Corollary 20. *Fix $T = T_\lambda$, where $\lambda = (r, r, \ldots, r)$ and set $a = \lfloor \ell/r \rfloor$ and $b = \ell - ar$. The dimension of the irreducible variety $\mathrm{Gr}(\ell, n)^T$ is equal to $(d-a)\ell - (a+1)b$.*

Proof. We compute the dimension of W_μ for any $\mu \in \mathcal{P}$. For the action of $\mathrm{GL}_d(\mathcal{O}_\mathbb{K})$ by the group of matrices A, we determine the stabilizer of the distinguished point L_μ. Every matrix in this stabilizer has $A_1 = A_2 = \cdots = A_{r-1} = 0$. The matrix A_0 breaks into blocks according to various levels given by powers of t. This can be expressed conveniently by the partition $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_r^*)$ that is conjugate to $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$. Here μ_i^* is the number of indices j such that $\mu_j \geq i$. Note that μ^* is a partition of ℓ with at most r parts and largest part at most d. The desired stabilizer is the product of the matrix groups $\mathrm{GL}_{\mu_i^*}(K)$ for $i = 1, 2, \ldots, r$. In particular, the dimension of the stabilizer is $\sum_{i=1}^r (\mu_i^*)^2$. This implies

$$\dim(W_\mu) = d\ell - \sum_{i=1}^r (\mu_i^*)^2 = \sum_{1 \leq i \leq j \leq d} (\mu_i - \mu_j).$$

We learned the last identity from [Voll 2010, (26)]. We apply the middle formula to the maximal partition $\mu = \mu_{\max} = (r, r, \ldots, r, b)$. Its conjugate partition is $\mu^* = (a+1, \ldots, a+1, a, \ldots, a)$, with b blocks of

λ	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$
(3, 3, 3)	[27, 4, 6]*	[57, 6, 90]*	[99, 6, 90]*	[99, 6, 90]*
(5, 5)	[41, 2, 2]*	[112, 3, 6]*	[194, 4, 24]*	[220, 5, 120]*
(2, 2, 2, 2, 2)	[20, 8, 70]*	[70, 10, 1050]*	[110, 12, 23100]	[152, 12, 23100]
(6, 6)	[62, 2, 2]*	[212, 3, 6]*	[479, 4, 24]	[760, 5, ??]
(4, 4, 4)	[57, 4, 6]*	[193, 6, 90]*	[414, 8, 2520]	[711, 8, ??]
(3, 3, 3, 3)	[50, 6, 20]*	[156, 9, 1680]	[399, 10, 8400]	[648, 11, ??]
(2, 2, 2, 2, 2, 2)	[30, 10, 252]*	[130, 13, 18018]	[270, 16, ??]	[492, 17, ??]

Table 4. Fixed point loci for rectangular partitions of $n = 9, 10, 12$.

size $a+1$ and $r-b$ blocks of size a. The middle formula yields

$$\dim(W_\mu) = d\ell - b(a+1)^2 - (r-b)a^2 = (d-a)\ell - (a+1)b. \tag{19}$$

The assertion now follows from (18). □

We now present further evidence in favor of Conjecture 12, beginning with the computational results shown in Table 4. For each table entry we verified that the shuffle equations span the space of linear forms that vanish on $\mathrm{Gr}(\ell, n)^T$. For all entries marked with a star, the `Macaulay2` command `isPrime J` terminated and proved that the shuffle ideal is prime.

The extremal cases $\lambda = (1, 1, \ldots, 1)$ and $\lambda = (n)$ had been excluded from the definition of rectangular partition, but it is worthwhile to consider these now. Conjecture 12 holds for both of these cases. Indeed for $\lambda = (1, 1, \ldots, 1)$, we have $T = 0_n$, so there are no shuffle equations. The corresponding variety $\mathrm{Gr}(\ell, n)^{T_\lambda}$ agrees with the Grassmannian $\mathrm{Gr}(\ell, n)$. This is defined by the Plücker quadrics, which are well known to generate a prime ideal.

We conclude by addressing the case $\lambda = (n)$. This was studied for $n = 6$ in Example 8. We now generalize what we saw there, namely that Conjecture 12 holds for the one-part partition.

Proposition 21. *For $\lambda = (n)$ and any ℓ, the shuffle equations are all Plücker coordinates p_I except for $I = (n-\ell+1, \ldots, n)$. These generate the prime ideal of the point* $\mathrm{Gr}(\ell, n)^T = \{e_I\}$.

Proof. Consider any ordered ℓ-set I in $[n]$. If $n \notin I$ then p_I equals the coefficient of z^ℓ in the coordinate of the row vector $P \cdot \bigwedge_\ell(I+zT)$ that is indexed by $I+(1, 1, \ldots, 1, 1)$. If $I = (J, n)$ but $n-1 \notin J$ then, modulo the above Plücker coordinates, p_I equals the coefficient of $z^{\ell-1}$ of the coordinate in $P \cdot \bigwedge_\ell(I+zT)$ that is indexed by $I+(1, 1, \ldots, 1, 0)$. If $I = (J, n-1, n)$ but $n-2 \notin J$ then, modulo the above Plücker coordinates, p_I equals the coefficient of $z^{\ell-2}$ of the coordinate that is indexed by $I + (1, \ldots, 1, 0, 0)$, etc. Iterating this process yields all Plücker coordinates other than the last one, $I = (n-\ell+1, n-\ell+2, \ldots, n-1, n)$. For $n = 6$ and $\ell = 3$, our argument can be checked by looking at the 20 expressions in Example 8. □

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Pinwheel solutions to Schrödinger systems

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We establish the existence of positive segregated solutions for competitive nonlinear Schrödinger systems in the presence of an external trapping potential, which have the property that each component is obtained from the previous one by a rotation, and we study their behavior as the forces of interaction become very small or very large.

As a consequence, we obtain optimal partitions for the Schrödinger equation by sets that are linearly isometric to each other.

1. Introduction

Consider the nonlinear Schrödinger system

$$-\Delta u_i+V_i(x)u_i=|u_i|^{2p-2}u_i+\sum_{\substack{j=1\\ j\neq i}}^{\ell}\beta_{ij}|u_j|^p|u_i|^{p-2}u_i,\quad u_i\in H^1(\mathbb{R}^N),\quad u_i>0,\qquad i=1,\dots,\ell,\tag{1-1}$$

where $N\geq 2$, $p>1$ and $p<N/(N-2)$ if $N\geq 3$, $\beta_{ij}=\beta_{ji}\in\mathbb{R}$, and $V_i\in\mathcal{C}^0(\mathbb{R}^N)$.

For the cubic nonlinearity ($p=2$) in dimensions $N=2,3$ this system arises in the study of Bose–Einstein condensation for a mixture of ℓ different states which overlap in space. It has been widely studied in the last two decades. Most work has been done in the autonomous case (i.e., for constant V_i). We refer the reader to the recent paper [Li et al. 2022], where the authors provide an exhaustive list of references. The nonautonomous case turns out to be much more difficult. Some results have been recently obtained by Peng and Wang [2013], Pistoia and Vaira [2022], and Li, Wei and Wu [Li et al. 2022].

The system (1-1) for a more general subcritical nonlinearity in higher dimensions has been much less studied. Even if it does not have an immediate physical motivation, finding a solution in this general setting is a quite interesting and challenging problem from a mathematical point of view. Besides, Schrödinger equations in higher dimensions have been widely studied in applications; see for instance [Dong 2011]. To our knowledge, the only result for the system (1-1) in higher dimensions is that by Gao and Guo [2020], who proved the existence of infinitely many solutions for only two equations ($\ell=2$) when the coupling parameter β_{12} is negative, and both equations have a common potential $V_1=V_2$ which does not enjoy any symmetry properties, but satisfies suitable decay assumptions at infinity. However, nothing is said about the sign of the solutions.

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Here we study (1-1) in a fully symmetric setting, namely we consider the nonlinear Schrödinger system

$$-\Delta u_i+V(x)u_i=|u_i|^{2p-2}u_i+\beta\sum_{\substack{j=1\\ j\neq i}}^{\ell}|u_j|^p|u_i|^{p-2}u_i,\quad u_i\in H^1(\mathbb{R}^N),\quad u_i>0,\qquad i=1,\dots,\ell, \tag{1-2}$$

where $N\geq 4$, $1<p<N/(N-2)$, $\beta<0$, and $V\in\mathcal{C}^0(\mathbb{R}^N)$ satisfies the following assumptions for some $n\in\mathbb{N}$:

(V_1) V is radial.

(V_2) $0<\inf_{x\in\mathbb{R}^N}V(x)$ and $V(x)\to V_\infty>0$ as $|x|\to\infty$.

(V_3^n) There exist $C_0,\,R_0>0$ and $\lambda\in\big(0,2\sin\frac{\pi}{\ell n}\big)$ such that

$$V(x)\leq V_\infty-C_0\mathrm{e}^{-\lambda\sqrt{V_\infty}|x|}\quad\text{for every } x\in\mathbb{R}^N \text{ with } |x|\geq R_0.$$

We look for *fully nontrivial* solutions to (1-2), i.e., solutions with all components u_i different from zero. Set

$$\|u\|_V^2:=\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2).$$

We prove the following results.

Theorem 1.1. *Let $n\in\mathbb{N}$ and assume that V satisfies (V_1), (V_2) and (V_3^n). Then the system (1-2) has a fully nontrivial solution $\boldsymbol{u}=(u_1,\dots,u_\ell)$ satisfying, for every $(z,y)\in\mathbb{C}\times\mathbb{R}^{N-2}\equiv\mathbb{R}^N$,*

$$\begin{cases} u_1(\mathrm{e}^{2\pi\mathrm{i}/n}z,\theta y)=u_1(z,y) & \text{for every } \theta\in O(N-2),\\ u_{j+1}(z,y)=u_1(\mathrm{e}^{2\pi\mathrm{i}j/\ell n}z,y) & \text{for every } j=1,\dots,\ell-1.\end{cases}\tag{1-3}$$

This solution has least energy among all nontrivial solutions satisfying (1-3). Furthermore, the energy of each component satisfies

$$\frac{p-1}{2p}\|u_i\|_V^2<n\mathfrak{c}_\infty,$$

where $\mathfrak{c}_\infty$ is the ground state energy of the Schrödinger equation

$$-\Delta u+V_\infty u=|u|^{2p-2}u,\quad u\in H^1(\mathbb{R}^N).\tag{1-4}$$

As usual, $O(N-2)$ denotes the group of linear isometries of $\mathbb{R}^{N-2}$. The symmetries (1-3) of the solutions given by Theorem 1.1 suggests calling them *pinwheel solutions*. For an autonomous system of two equations in dimensions 2 and 3, solutions of this kind were found by Wei and Weth [2007].

Since the potential V is assumed to be radial, using the compactness of the embedding of the subspace of radial functions in $H^1(\mathbb{R}^N)$ into $L^{2p}(\mathbb{R}^N)$ and following the argument given in [Clapp and Szulkin 2019, Theorem 1.1], it is easy to see that the system (1-2) has a solution all of whose components are radial. Note however that if $\boldsymbol{u}=(u_1,\dots,u_\ell)$ satisfies (1-2) and (1-3) and some component u_i is radial, then $u_1=\cdots=u_\ell=:u$ and u is a nontrivial solution of the equation

$$-\Delta u+V(x)u=(1+(\ell-1)\beta)|u|^{2p-2}u,\quad u\in H^1(\mathbb{R}^N).$$

Therefore, if $1+(\ell-1)\beta\leq 0$, a nontrivial solution to the system (1-2) satisfying (1-3) cannot be radial. In fact, more can be said. The following result, combined with Theorem 1.1, yields multiple positive nonradial solutions when the assumption (V_3^n) is satisfied for large enough n.

Proposition 1.2. *Let $\beta\leq -1/(\ell-1)$, and for some $m,q\in\mathbb{N}$, let $\boldsymbol{u}_m$ and $\boldsymbol{u}_q$ be solutions to (1-2) satisfying (1-3) with $n=\ell^m$ and $n=\ell^q$ respectively. If $m\neq q$, then $\boldsymbol{u}_m\neq\boldsymbol{u}_q$.*

One may wonder if the solution given by Theorem 1.1 for $\beta\in(-1/(\ell-1),0)$ is radial or not. The following result gives a partial answer in terms of the nonautonomous Schrödinger equation (1-5). Namely, if the least energy solutions to this equation that satisfy (1-6) are nonradial, then the solutions to the system (1-2) satisfying (1-3) are nonradial for β close enough to 0.

Theorem 1.3. *Let $n\in\mathbb{N}$ and assume that V satisfies (V_1), (V_2) and (V_3^n). Let $\boldsymbol{u}_k=(u_{k,1},\dots,u_{k,\ell})$ be a least energy fully nontrivial solution to (1-2) and (1-3) with $\beta=\beta_k$. Assume that $\beta_k<0$ and $\beta_k\to 0$ as $k\to\infty$. Then, after passing to a subsequence, $u_{k,j}\to u_{0,j}$ strongly in $H^1(\mathbb{R}^N)$, $u_{0,j}\geq 0$, $\boldsymbol{u}_0=(u_{0,1},\dots,u_{0,\ell})$ satisfies (1-3), $u_{0,j}$ is a nontrivial solution to the equation*

$$-\Delta u+V(x)u=|u|^{2p-2}u,\quad u\in H^1(\mathbb{R}^N),\tag{1-5}$$

and $u_{0,j}$ has least energy among all solutions to (1-5) satisfying

$$u(e^{2\pi i/n}z,\theta y)=u(z,y)\quad\text{for every }\theta\in O(N-2),\ (z,y)\in\mathbb{R}^N.\tag{1-6}$$

Furthermore,

$$\frac{p-1}{2p}\|u_{0,j}\|_V^2<n\mathfrak{c}_\infty.$$

Next, we describe the behavior of the solutions given by Theorem 1.1 as $\beta\to-\infty$. As shown by Conti, Terracini and Verzini [Conti et al. 2002; 2005] and Chang, Lin, Lin and Lin [Chang et al. 2004], there is a connection between variational elliptic systems with strong competitive interaction and optimal partition problems.

We shall call an ℓ-tuple $(\Omega_1,\dots,\Omega_\ell)$ of nonempty open subsets of $\mathbb{R}^N$ an (n,ℓ)*-pinwheel partition* of $\mathbb{R}^N$ if $\Omega_i\cap\Omega_j=\varnothing$ whenever $i\neq j$ and it satisfies following two symmetry conditions:

(S_1) $\Omega_{j+1}=\{(z,y)\in\mathbb{C}\times\mathbb{R}^{N-2}:(e^{2\pi ij/\ell n}z,y)\in\Omega_1\}$ for each $j=1,\dots,\ell-1$.

(S_2) If $(z,y)\in\Omega_1$, then $(e^{2\pi i/n}z,\theta y)\in\Omega_1$ for every $\theta\in O(N-2)$.

We denote the set of all (n,ℓ)-pinwheel partitions by $\mathcal{P}_\ell^n$. If Ω is an open subset of $\mathbb{R}^N$ satisfying (S_2), a minimizer for

$$\inf_{u\in\mathcal{M}_\Omega}\frac{p-1}{2p}\|u\|_V^2=:\mathfrak{c}_\Omega$$

on the Nehari manifold

$$\mathcal{M}_\Omega:=\Big\{u\in H_0^1(\Omega):u\neq 0,\ \|u\|_V^2=\int_{\mathbb{R}^N}|u|^{2p},\ \text{and}\ u(e^{2\pi i/n}z,\theta y)=u(z,y)\ \text{for all}\ \theta\in O(N-2)\ \text{and}\ (z,y)\in\Omega\Big\}\tag{1-7}$$

is a *least energy solution* to the problem

$$\begin{cases} -\Delta u + V(x)u = |u|^{2p-2}u, & u \in H_0^1(\Omega), \\ u(e^{2\pi i/n}z, \theta y) = u(z, y) & \text{for every } \theta \in O(N-2),\ (z, y) \in \Omega. \end{cases} \tag{1-8}$$

We say that $(\Omega_1, \ldots, \Omega_\ell)$ is an *optimal* (n, ℓ)*-pinwheel partition* for (1-5) if $\mathfrak{c}_{\Omega_j}$ is attained on $\mathcal{M}_{\Omega_j}$ and

$$\sum_{j=1}^{\ell} \mathfrak{c}_{\Omega_j} = \inf_{(\Theta_1,\ldots,\Theta_\ell)\in\mathcal{P}_\ell^n} \sum_{j=1}^{\ell} \mathfrak{c}_{\Theta_j}.$$

Theorem 1.4. *Let $n \in \mathbb{N}$ and assume that V satisfies (V_1), (V_2) and (V_3^n). Let $\boldsymbol{u}_k = (u_{k,1}, \ldots, u_{k,\ell})$ be a least energy fully nontrivial solution to (1-2) and (1-3) with $\beta = \beta_k$. Assume that $\beta_k \to -\infty$ as $k \to \infty$. Then, after passing to a subsequence:*

(i) *$u_{k,j} \to u_{\infty,j}$ strongly in $H^1(\mathbb{R}^N)$, $u_{\infty,j} \geq 0$, $u_{\infty,j} \neq 0$, $u_{\infty,i}u_{\infty,j} = 0$ if $i \neq j$, $\boldsymbol{u}_\infty = (u_{\infty,1}, \ldots, u_{\infty,\ell})$ satisfies (1-3), and*

$$\int_{\mathbb{R}^N} \beta_k u_{k,j}^p u_{k,i}^p \to 0 \quad \text{as } k \to \infty \qquad \text{whenever } i \neq j.$$

(ii) *$u_{\infty,j} \in \mathcal{C}^0(\mathbb{R}^N)$, the restriction of $u_{\infty,j}$ to the open set $\Omega_j := \{x \in \mathbb{R}^N : u_{\infty,j}(x) > 0\}$ is a least energy solution to the problem (1-8) in Ω_j, and $(\Omega_1, \ldots, \Omega_\ell)$ is an optimal (n, ℓ)-pinwheel partition for (1-5).*

(iii) *$\mathbb{R}^N \smallsetminus \bigcup_{j=1}^{\ell} \Omega_j = \mathscr{R} \cup \mathscr{S}$, where $\mathscr{R} \cap \mathscr{S} = \varnothing$, $\mathscr{R}$ is an $(m-1)$-dimensional $\mathcal{C}^{1,\alpha}$-submanifold of $\mathbb{R}^N$ and $\mathscr{S}$ is a closed subset of $\mathbb{R}^N$ with Hausdorff measure $\leq m-2$. Furthermore, if $\xi \in \mathscr{R}$, there exist i, j such that*

$$\lim_{x\to\xi^+} |\nabla u_i(x)| = \lim_{x\to\xi^-} |\nabla u_j(x)| \neq 0,$$

where $x \to \xi^\pm$ are the limits taken from opposite sides of $\mathscr{R}$, and if $\xi \in \mathscr{S}$, then

$$\lim_{x\to\xi} |\nabla u_j(x)| = 0 \quad \text{for every } j = 1, \ldots, \ell.$$

(iv) *If $\ell = 2$, then $u_{\infty,1} - u_{\infty,2}$ is a sign-changing solution to (1-5) satisfying (1-6).*

Note that (iii) implies that the partition exhausts $\mathbb{R}^N$, i.e., $\mathbb{R}^N = \bigcup_{j=1}^{\ell} \overline{\Omega}_j$. Thus, every Ω_j is unbounded.

The regularity properties of optimal partitions have been established, in different settings, for instance, in [Caffarelli and Lin 2008; Clapp et al. 2021b; Noris et al. 2010; Soave et al. 2016; Tavares and Terracini 2012].

Theorem 1.4 establishes the existence of optimal partitions having an additional property: each set of the partition is obtained from any other by means of a linear isometry. Pinwheel partitions are an example of this type of partition, but others are conceivable. In Section 2 we present a general symmetric variational setting for the system (1-2) that produces other examples.

The existence of sign-changing solutions to (1-5) having the additional property that their negative part is obtained from the positive one by means of a linear isometry and a change of sign has been established in [Clapp and Salazar 2012]. This includes those given by Theorem 1.4(iv). The tool for producing this type of solution is a homomorphism from some group of linear isometries of $\mathbb{R}^N$ onto the group with two elements. As shown in Section 2 this tool also serves to get positive solutions of the system (1-2) for $\ell = 2$ with the property that each component is obtained from the other by composition with a linear isometry. The general tool for obtaining a similar result for the system (1-2) of ℓ equations is a homomorphism into the group of permutations of a set of ℓ elements.

Rather than search for results in the general setting of Section 2, we decided, for the sake of clarity, to look at pinwheel solutions only. The solutions found in [Peng and Wang 2013; Pistoia and Vaira 2022] for $N = 2, 3$ and $p = 2$ were of this type. Peng and Wang [2013] focused on the case where the potential V is greater than its limit at infinity, and for a system of two equations, they established the existence of pinwheel solutions for β sufficiently negative. Pistoia and Vaira [2022] raised the question of whether solutions exist when V is below its limit at infinity and showed in that case that the system (1-2) has a solution satisfying (1-3) for β close enough to 0. The energy of each component approaches $n\mathfrak{c}_\infty$ as $\beta \to 0$.

Our results can be easily extended to dimension $N = 2$. In contrast, the dimension $N = 3$ requires a more delicate analysis because compactness can also be lost by the presence of solutions to the autonomous system (with $V = V_\infty$) that travel to infinity; see Remark 3.3.

In Section 2 we present the general variational framework and in Section 3 we study the behavior of minimizing sequences of pinwheel solutions for the system (1-2). In Section 4 we prove Theorem 1.1 and Proposition 1.2. Section 5 is devoted to the proofs of Theorems 1.3 and 1.4.

2. The symmetric variational setting

Let G be a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$, and for $\ell \geq 2$, let S_ℓ be the group of permutations of the set $\{1, \dots, \ell\}$ acting on $\mathbb{R}^\ell$ in the obvious way, i.e.,

$$\sigma(u_1, \dots, u_\ell) = (u_{\sigma(1)}, \dots, u_{\sigma(\ell)}) \quad \text{for every } \sigma \in S_\ell,\ (u_1, \dots, u_\ell) \in \mathbb{R}^\ell.$$

Let $\phi : G \to S_\ell$ be a continuous homomorphism of groups. A function $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^\ell$ will be called *ϕ-equivariant* if

$$\boldsymbol{u}(gx) = \phi(g)\boldsymbol{u}(x) \quad \text{for all } g \in G,\ x \in \mathbb{R}^N. \tag{2-1}$$

Note that if $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^\ell$ is ϕ-equivariant, then $\boldsymbol{u}$ is K_ϕ-invariant, where $K_\phi := \ker(\phi)$.

These data define a G-action on $\mathcal{H} := (H^1(\mathbb{R}^N))^\ell$ as follows:

$$(g\boldsymbol{u})(x) := \phi(g)\boldsymbol{u}(g^{-1}x) \quad \text{for every } g \in G,\ \boldsymbol{u} = (u_1, \dots, u_\ell) \in \mathcal{H}.$$

For $u, v \in H_0^1(\mathbb{R}^N)$ we set

$$\langle u, v\rangle_V := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) \quad \text{and} \quad \|u\|_V := \sqrt{\langle u, u\rangle_V}.$$

The solutions to the system (1-2) are the positive critical points of the functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ given by

$$\mathcal{J}(\boldsymbol{u}) := \frac{1}{2}\sum_{i=1}^{\ell}\|u_i\|_V^2 - \frac{1}{2p}\sum_{i=1}^{\ell}\int_{\mathbb{R}^N}|u_i|^{2p} - \frac{\beta}{2p}\sum_{\substack{i,j=1\\ i\neq j}}^{\ell}\int_{\mathbb{R}^N}|u_i|^p|u_j|^p,$$

which is of class $\mathcal{C}^1$. Its i-th partial derivative is

$$\partial_i \mathcal{J}(\boldsymbol{u})v = \langle u_i, v\rangle_V - \int_{\mathbb{R}^N} |u_i|^{2p-2}u_i v - \beta \sum_{\substack{j=1\\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} |u_j|^p |u_i|^{p-2} u_i v$$

for any $\boldsymbol{u} \in \mathcal{H}$, $v \in H^1(\mathbb{R}^N)$. The functional $\mathcal{J}$ is G-invariant, i.e.,

$$\mathcal{J}(g\boldsymbol{u}) = \mathcal{J}(\boldsymbol{u}) \quad \text{for every } g \in G,\ \boldsymbol{u} = (u_1, \dots, u_\ell) \in \mathcal{H}.$$

So, by the principle of symmetric criticality [Willem 1996, Theorem 1.28], the critical points of the restriction of $\mathcal{J}$ to the G-fixed point space of $\mathcal{H}$,

$$\mathcal{H}^\phi := \{\boldsymbol{u} \in \mathcal{H} : g\boldsymbol{u} = \boldsymbol{u} \text{ for all } g \in G\} = \{\boldsymbol{u} \in \mathcal{H} : \boldsymbol{u} \text{ is } \phi\text{-equivariant}\},$$

are critical points of $\mathcal{J}$, i.e., they are the solutions to the system (1-2) satisfying (2-1). We denote by $\mathcal{J}^\phi$ the restriction of $\mathcal{J}$ to $\mathcal{H}^\phi$. Note that

$$(\mathcal{J}^\phi)'(\boldsymbol{u})\boldsymbol{v} = \mathcal{J}'(\boldsymbol{u})\boldsymbol{v} = \sum_{i=1}^{\ell} \partial_i \mathcal{J}(\boldsymbol{u}) v_i \quad \text{for any } \boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^\phi.$$

The fully nontrivial critical points of $\mathcal{J}^\phi$ belong to the set

$$\mathcal{N}^\phi := \{\boldsymbol{u} \in \mathcal{H}^\phi : u_i \neq 0 \text{ and } \partial_i \mathcal{J}(\boldsymbol{u}) u_i = 0 \text{ for all } i = 1, \dots, \ell\}.$$

Observe that

$$\mathcal{J}^\phi(\boldsymbol{u}) = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_i\|_V^2 \quad \text{if } \boldsymbol{u} \in \mathcal{N}^\phi.$$

Set

$$c^\phi := \inf_{\boldsymbol{u} \in \mathcal{N}^\phi} \mathcal{J}^\phi(\boldsymbol{u}).$$

We consider also the single equation

$$-\Delta u + V(x)u = |u|^{2p-2}u, \quad u \in H^1(\mathbb{R}^N)^G, \tag{2-2}$$

where $H^1(\mathbb{R}^N)^G := \{u \in H^1(\mathbb{R}^N) : u \text{ is } G\text{-invariant}\}$, and we denote by $J : H^1(\mathbb{R}^N)^G \to \mathbb{R}$ and $\mathcal{M}^G$ the energy functional and the Nehari manifold associated to it, i.e.,

$$J(u) := \frac{1}{2}\|u\|_V^2 - \frac{1}{2p} \int_{\mathbb{R}^N} |u|^{2p} \tag{2-3}$$

and

$$\mathcal{M}^G := \left\{ u \in H^1(\mathbb{R}^N)^G : u \neq 0,\ \|u\|_V^2 = \int_{\mathbb{R}^N} |u|^{2p} \right\}.$$

Similarly, we denote by $J_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ and $\mathcal{M}_\infty$ the energy functional and the Nehari manifold associated to (1-4). Set

$$\mathfrak{c}_\infty := \inf_{u \in \mathcal{M}_\infty} J_\infty(u) \quad \text{and} \quad \mathfrak{c}^G := \inf_{u \in \mathcal{M}^G} J(u). \tag{2-4}$$

We shall focus our attention on the following example.

Example 2.1. Let $\mathbb{Z}_m := \{e^{2\pi i j/m} : j = 0, \dots, m-1\}$ act on $\mathbb{C}$ by complex multiplication, and let $G_m := \mathbb{Z}_m \times O(N-2)$ act on $\mathbb{R}^N$ as

$$\alpha x := (\alpha z, y) \quad \text{for all } \alpha \in \mathbb{Z}_m,$$
$$\theta x := (z, \theta y) \quad \text{for all } \theta \in O(N-2) \text{ and } x = (z, y) \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N.$$

Let $\sigma_1 \in S_\ell$ be the cyclic permutation $\sigma_1(i) := i + 1 \mod \ell$, and let $\phi_n : G_{\ell n} \to S_\ell$ be the homomorphism given by $\phi_n(e^{2\pi i/\ell n}, \theta) := \sigma_1$ for any $\theta \in O(N-2)$. Then $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^\ell$ is ϕ_n-equivariant if and only if

$$\big(u_1(e^{2\pi i/\ell n} z, \theta y), \dots, u_\ell(e^{2\pi i/\ell n} z, \theta y)\big) = \big(u_2(z, y), \dots, u_\ell(z, y), u_1(z, y)\big)$$

for every $(z, y) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $\theta \in O(N-2)$, i.e., if and only if (1-3) holds. Note that every u_j is G_n-invariant.

3. The behavior of minimizing sequences

From now on, we fix n, and we take G_n and $\phi_n : G_{\ell n} \to S_\ell$ as in Example 2.1. Then, for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}^{\phi_n}$,

$$(\mathcal{J}^{\phi_n})'(\boldsymbol{u})\boldsymbol{v} = \sum_{i=1}^{\ell} \partial_i \mathcal{J}(\boldsymbol{u}) v_i = \ell\, \partial_j \mathcal{J}(\boldsymbol{u}) v_j \quad \text{for any } j = 1, \dots, \ell, \tag{3-1}$$

and the set $\mathcal{N}^{\phi_n}$ is the usual Nehari manifold associated to the functional $\mathcal{J}^{\phi_n} : \mathcal{H}^{\phi_n} \to \mathbb{R}$, i.e.,

$$\mathcal{N}^{\phi_n} = \{\boldsymbol{u} \in \mathcal{H}^{\phi_n} : \boldsymbol{u} \neq 0,\ (\mathcal{J}^{\phi_n})'(\boldsymbol{u})\boldsymbol{u} = 0\}.$$

It has the following properties.

Proposition 3.1. (a) $\mathcal{N}^{\phi_n} \neq \varnothing$.

(b) $c^{\phi_n} \geq \ell \mathfrak{c}^{G_n} > 0$.

(c) *$\mathcal{N}^{\phi_n}$ is a closed $\mathcal{C}^1$-submanifold of codimension 1 of $\mathcal{H}^{\phi_n}$, and a natural constraint for $\mathcal{J}^{\phi_n}$.*

(d) *If $\boldsymbol{u} \in \mathcal{H}^{\phi_n}$ is such that, for each $i = 1, \dots, \ell$,*

$$\int_{\mathbb{R}^N} |u_i|^{2p} + \sum_{\substack{j=1\\ j\neq i}}^{\ell} \beta \int_{\mathbb{R}^N} |u_i|^p |u_j|^p > 0,$$

then there exists a unique $s_{\boldsymbol{u}} \in (0, \infty)$ such that $s_{\boldsymbol{u}}\boldsymbol{u} \in \mathcal{N}^{\phi_n}$. Furthermore,

$$\mathcal{J}^{\phi_n}(s_{\boldsymbol{u}}\boldsymbol{u}) = \max_{s \in (0,\infty)} \mathcal{J}^{\phi_n}(s\boldsymbol{u}).$$

(e) $c^{\phi_n} \leq \ell n \mathfrak{c}_\infty$.

Proof. The proof is easy. We give the details for the sake of completeness.

(a) Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be a nontrivial radial function with $\|\varphi\|_V^2 = \int_{\mathbb{R}^N} |\varphi|^{2p}$. Set $\xi_{i,j} := (e^{2\pi i(i+\ell j)/\ell n}, 0) \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N$ and define

$$u_{R,i+1}(x) := \sum_{j=0}^{n-1} \varphi(x - R\xi_{i,j}), \quad i = 0, \dots, n-1,$$

where $R > 0$ is taken large enough that $u_{R,i}$ and $u_{R,j}$ have disjoint supports for every $i \neq j$. Then $\boldsymbol{u}_R := (u_{R,1}, \dots, u_{R,1}) \in \mathcal{N}^{\phi_n}$.

(b) Let $\boldsymbol{u} = (u_1, \dots, u_\ell) \in \mathcal{N}^{\phi_n}$. As $\beta < 0$, we have

$$0 < \|u_i\|_V^2 = \|u_1\|_V^2 \leq \int_{\mathbb{R}^N} |u_1|^{2p} = \int_{\mathbb{R}^N} |u_i|^{2p} \quad \text{for all } i = 2, \dots, \ell.$$

Hence, there exists $s \in (0, 1]$ such that $su_i \in \mathcal{M}^{G_n}$ for every $i = 1, \dots, \ell$. Therefore,

$$\ell \mathfrak{c}^{G_n} \leq \sum_{i=1}^{\ell} J(s_i u_i) = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|s_i u_i\|_V^2 \leq \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_i\|_V^2 = \mathcal{J}^{\phi}(\boldsymbol{u}).$$

It follows that $\ell \mathfrak{c}^{G_n} \leq c^{\phi_n}$.

(c) The function $\Psi : \mathcal{H}^{\phi_n} \smallsetminus \{0\} \to \mathbb{R}$ given by $\Psi(\boldsymbol{u}) := (\mathcal{J}^{\phi_n})'(\boldsymbol{u})\boldsymbol{u}$ is of class $\mathcal{C}^1$, and $\mathcal{N}^{\phi_n} = \Psi^{-1}(0)$. It follows from (b) that $\mathcal{N}^{\phi_n}$ is a closed subset of $\mathcal{H}^{\phi_n}$. As

$$\Psi'(\boldsymbol{u})\boldsymbol{u} = (2 - 2p)\ell \|u_1\|_V^2 \neq 0,$$

we have that 0 is a regular value of Ψ. This shows that $\mathcal{N}^{\phi_n}$ is a $\mathcal{C}^1$-submanifold of codimension 1 of $\mathcal{H}^{\phi_n}$. It also shows that $\boldsymbol{u} \notin \ker \Psi'(\boldsymbol{u}) =: T_{\boldsymbol{u}}\mathcal{N}^{\phi_n}$, the tangent space of $\mathcal{N}^{\phi_n}$ at $\boldsymbol{u}$. Hence,

$$\mathcal{H}^{\phi_n} = T_{\boldsymbol{u}}\mathcal{N}^{\phi_n} \oplus \mathbb{R}\boldsymbol{u}.$$

Since, by definition, $(\mathcal{J}^{\phi_n})'(\boldsymbol{u})\boldsymbol{u} = 0$ for every $\boldsymbol{u} \in \mathcal{N}^{\phi_n}$, we infer that a critical point of the restriction of $\mathcal{J}^{\phi_n}$ to $\mathcal{N}^{\phi_n}$ is a critical point of $\mathcal{J}^{\phi_n}$.

(d) The proof is straightforward. The number $s_{\boldsymbol{u}}$ is

$$s_{\boldsymbol{u}} = \left(\frac{\|u_1\|_V^2}{\int_{\mathbb{R}^N} |u_1|^{2p} + \sum\limits_{j=2}^{\ell} \beta \int_{\mathbb{R}^N} |u_1|^p |u_j|^p} \right)^{1/(2p-2)}.$$

(e) Let ω be the least energy positive radial solution to (1-4). Set $\xi_{i,j} = (\mathrm{e}^{2\pi \mathrm{i}(i+\ell j)/\ell n}, 0) \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N$. Define

$$w_{R,i+1}(x) := \sum_{j=0}^{n-1} \omega(x - R\xi_{i,j}), \quad i = 0, \dots, n-1.$$

Then $\boldsymbol{w}_R = (w_{R,1}, \dots, w_{R,\ell}) \in \mathcal{H}^{\phi_n}$. If R is sufficiently large, statement (d) yields $s_R \in (0, \infty)$ such that $s_R \boldsymbol{w}_R \in \mathcal{N}^{\phi_n}$ and $s_R \to 1$ as $R \to \infty$. Using assumption (V_2) we obtain

$$c^{\phi_n} \leq \mathcal{J}^{\phi_n}(s_R \boldsymbol{w}_R) = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|s_{R,i} w_{R,i}\|_V^2 \to \ell n \mathfrak{c}_\infty \quad \text{as } R \to \infty.$$

This shows that $c^{\phi_n} \leq \ell n \mathfrak{c}_\infty$, as claimed. ∎

Lemma 3.2. *Let (x_k) be a sequence in $\mathbb{R}^N$, where $N \geq 4$. After passing to a subsequence, there exists a sequence (ξ_k) in $\mathbb{R}^N$ and a constant $C_0 > 0$ such that*

$$|x_k - \xi_k| \leq C_0 \quad \textit{for all } k \in \mathbb{N},$$

and one of the following statements holds true:

- $\xi_k = 0$ *for all k, or*
- $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ *and* $|\zeta_k| \to \infty$, *or*
- *for each $m \in \mathbb{N}$ there exist $\gamma_1, \ldots, \gamma_m \in O(N-2)$ such that $|\gamma_i \xi_k - \gamma_j \xi_k| \to \infty$ if $i \neq j$.*

Proof. See [Clapp et al. 2021a, Lemma 3.1]. □

Remark 3.3. This lemma is not true in dimension $N = 3$, because $O(1) = \{1, -1\}$.

Theorem 3.4. *Let $\boldsymbol{u}_k = (u_{k,1}, \ldots, u_{k,\ell}) \in \mathcal{N}^{\phi_n}$ be such that $\mathcal{J}^{\phi_n}(\boldsymbol{u}_k) \to c^{\phi_n}$ and $u_{k,i} \geq 0$. Then, after passing to a subsequence, either $\boldsymbol{u}_k \to \boldsymbol{u}$ strongly in $\mathcal{H}^{\phi_n}$ with $u_i \geq 0$, or there are points $(z_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2} \equiv \mathbb{R}^N$ such that $|z_k| \to \infty$,*

$$\lim_{k\to\infty} \left\| u_{k,1} - \sum_{j=0}^{n} \omega(\,\cdot - (\mathrm{e}^{2\pi \mathrm{i} j/n} z_k, 0)) \right\| = 0,$$

and $c^{\phi_n} = \ell n \mathfrak{c}_\infty$, where ω is the least energy positive radial solution to (1-4).

Proof. Invoking Ekeland's variational principle [Willem 1996, Theorem 8.5] we may assume that $(\mathcal{J}^{\phi_n})'(\boldsymbol{u}_k) \to 0$ in $(\mathcal{H}^{\phi_n})'$.

Since $\beta < 0$, Proposition 3.1(b) yields $c_0 > 0$ such that

$$\int_{\mathbb{R}^N} |u_{k,1}|^{2p} > c_0 \quad \text{for all } k \in \mathbb{N}.$$

By Lions' lemma [Willem 1996, Lemma 1.21] there exist $\delta > 0$ and $x_k \in \mathbb{R}^N$ such that, after passing to a subsequence,

$$\int_{B_1(x_k)} |u_{k,1}|^{2p} > \delta \quad \text{for all } k \in \mathbb{N}.$$

For (x_k) we fix a sequence (ξ_k) and a constant $C_0 > 0$ such that $|x_k - \xi_k| \leq C_0$ for all $k \in \mathbb{N}$, satisfying one of the alternatives stated in Lemma 3.2. Then

$$\int_{B_{C_0+1}(\xi_k)} |u_{k,1}|^{2p} \geq \int_{B_1(x_k)} |u_{k,1}|^{2p} > \delta \quad \text{for all } k \in \mathbb{N}. \tag{3-2}$$

It follows that either $\xi_k = 0$, or $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $\zeta_k \to \infty$. Otherwise, by Lemma 3.2, for each $m \in \mathbb{N}$ there would exist $\gamma_1, \ldots, \gamma_m \in O(N-2)$ such that $|\gamma_i \xi_k - \gamma_j \xi_k| \geq 2(C_0 + 1)$ if $i \neq j$ for large enough $k \in \mathbb{N}$, and as $u_{k,1}$ is G_n-invariant, we would have that

$$\int_{\mathbb{R}^N} |u_{k,1}|^{2p} \geq \sum_{i=1}^{m} \int_{B_{C_0+1}(\gamma_i \xi_k)} |u_{k,1}|^{2p} = m \int_{B_{C_0+1}(\xi_k)} |u_{k,1}|^{2p} > m\delta$$

for all $m \in \mathbb{N}$. This is impossible because $(u_{k,1})$ is bounded in $L^{2p}(\mathbb{R}^N)$.

Next, we distinguish two cases.

Case 1: $\xi_k = 0$ for all $k \in \mathbb{N}$.

Since the sequence $(u_{k,1})$ is bounded in $H^1(\mathbb{R}^N)$, passing to a subsequence, we have that $u_{k,1} \rightharpoonup u_1$ weakly in $H^1(\mathbb{R}^N)$, $u_{k,1} \to u_1$ in $L^{2p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_{k,1} \to u_1$ a.e. in $\mathbb{R}^N$. Hence, $u_1 \geq 0$ and it follows from (3-2) that $u_1 \neq 0$. Note that, as $u_{k,1} \in H^1(\mathbb{R}^N)^{G_n}$, $u_1 \in H^1(\mathbb{R}^N)^{G_n}$. Set $u_{j+1}(z, y) := u_1(\mathrm{e}^{2\pi \mathrm{i} j/\ell n} z, y)$ for $(z, y) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $j = 1, \dots, \ell - 1$, and set $\boldsymbol{u} = (u_1, \dots, u_\ell)$. Then $u_{k,j+1} \rightharpoonup u_{j+1}$ weakly in $H^1(\mathbb{R}^N)$, and as $(\mathcal{J}^{\phi_n})'(\boldsymbol{u}_k) \to 0$ in $(\mathcal{H}^{\phi_n})'$, we derive from (3-1) that

$$0 = \lim_{k\to\infty} \partial_1 \mathcal{J}(\boldsymbol{u}_k)\varphi = \partial_1 \mathcal{J}(\boldsymbol{u})\varphi \quad \text{for every } \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^N)^{G_n}.$$

Hence, $\boldsymbol{u} \in \mathcal{N}^{\phi_n}$ and

$$c^{\phi_n} \leq \mathcal{J}^{\phi_n}(\boldsymbol{u}) = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_i\|_V^2 \leq \liminf_{k\to\infty} \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_{k,i}\|_V^2 = \lim_{k\to\infty} \mathcal{J}^{\phi_n}(\boldsymbol{u}_k) = c^{\phi_n}.$$

Therefore, $\boldsymbol{u}_k \to \boldsymbol{u}$ strongly in $\mathcal{H}^{\phi_n}$. This shows that, in Case 1, the first alternative stated in Theorem 3.4 holds true.

Case 2: $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $\zeta_k \to \infty$.

Set

$$w_{k,i}(x) := u_{k,i}(x + \xi_k), \quad i = 1, \dots, \ell.$$

Note that $w_{k,i}$ is $O(N-2)$-invariant. Since the sequence $(w_{k,i})$ is bounded in $H^1(\mathbb{R}^N)$, a subsequence satisfies $w_{k,i} \rightharpoonup w_i$ weakly in $H^1(\mathbb{R}^N)^{O(N-2)}$, $w_{k,i} \to w_i$ in $L^{2p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $w_{k,i} \to w_i$ a.e. in $\mathbb{R}^N$. Hence, $w_i \geq 0$. To simplify notation, set $\alpha := \mathrm{e}^{2\pi \mathrm{i}/n}$. Note that, as $|\alpha^j \xi_k - \alpha^m \xi_k| \to \infty$ if $j \neq m$, we have that

$$w_{k,i} \circ \alpha^{-m} - \sum_{j=m+1}^{n-1} (w_i \circ \alpha^{-m})(\,\cdot - \alpha^j \xi_k + \alpha^m \xi_k) \rightharpoonup w_i \circ \alpha^{-m}$$

weakly in $H^1(\mathbb{R}^N)$. Hence, setting $V_k(x) := V(x + \xi_k)$, Lemma A.1 gives

$$\begin{aligned}\|w_i \circ \alpha^{-m}\|^2_{V_\infty} = {} & \Big\| w_{k,i} \circ \alpha^{-m} - \sum_{j=m+1}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k + \alpha^m \xi_k) \Big\|^2_{V_k} \\ & - \Big\| w_{k,i} \circ \alpha^{-m} - \sum_{j=m}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k + \alpha^m \xi_k) \Big\|^2_{V_k} + o(1).\end{aligned}$$

Since $u_{k,i}$ is G_n-invariant, the change of variable $y = z - \alpha^m \xi_k$ yields

$$\Big\| u_{k,i} - \sum_{j=m+1}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k) \Big\|^2_V = \Big\| u_{k,i} - \sum_{j=m}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k) \Big\|^2_V + \|w_i\|^2_{V_\infty} + o(1),$$

and iterating this identity we obtain

$$\|u_{k,i}\|^2_V = \Big\| u_{k,i} - \sum_{j=0}^{n-1} (w_i \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k) \Big\|^2_V + n\|w_i\|^2_{V_\infty} + o(1). \tag{3-3}$$

On the other hand, for any given $v \in H^1(\mathbb{R}^N)^{O(N-2)}$ set $v_k(y) := v(y - \xi_k)$ and

$$\hat{v}_k(y) := \sum_{j=0}^{n-1} v_k(\alpha^j y).$$

Recalling that $u_{k,i}$ is G_n-invariant and performing the translation $y = x + \xi_k$, we obtain

$$\begin{aligned}\partial_i \mathcal{J}(\boldsymbol{u}_k)\hat{v}_k &= \sum_{j=0}^{n-1} \partial_i \mathcal{J}(\boldsymbol{u}_k)(v_k \circ \alpha^j) = n \partial_i \mathcal{J}(\boldsymbol{u}_k) v_k \\ &= n\biggl(\int_{\mathbb{R}^N} (\nabla w_{k,i} \cdot \nabla v + V_k(x) w_{k,i} v) - \int_{\mathbb{R}^N} |w_{k,i}|^{2p-2} w_{k,i} v - \beta \sum_{\substack{j=1\\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} |w_{k,j}|^p |w_{k,i}|^{p-2} w_{k,i} v \biggr).\end{aligned}$$

Note that $\hat{v}_k$ is G_n-invariant. As $(\mathcal{J}^{\phi_n})'(\boldsymbol{u}_k) \to 0$, invoking (3-1) and assumption (V_2), and passing to the limit as $k \to \infty$, we get

$$0 = \int_{\mathbb{R}^N} (\nabla w_i \cdot \nabla v + V_\infty w_i v) - \int_{\mathbb{R}^N} |w_i|^{2p-2} w_i v - \beta \sum_{\substack{j=1\\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} |w_j|^p |w_i|^{p-2} w_i v \tag{3-4}$$

for every $v \in H^1(\mathbb{R}^N)^{O(N-2)}$ and $i = 1, \dots, \ell$. Since, by (3-2),

$$\int_{B_{C_0+1}(0)} |w_{k,1}|^{2p} \geq \int_{B_{C_0+1}(\xi_k)} |u_{k,1}|^{2p} \geq \delta > 0,$$

we see that $w_1 \neq 0$. Furthermore, (3-4) implies that

$$\|w_1\|^2_{V_\infty} = \int_{\mathbb{R}^N} |w_1|^{2p} + \beta \sum_{j=2}^{\ell} \int_{\mathbb{R}^N} |w_j|^p |w_1|^p \leq \int_{\mathbb{R}^N} |w_1|^{2p}, \tag{3-5}$$

so there exists $t \in (0, 1]$ such that $\|t w_1\|^2_{V_\infty} = \int_{\mathbb{R}^N} |t w_1|^{2p}$. It follows that $t w_1 \in \mathcal{M}_\infty$, and from (3-3) and Proposition 3.1(e) we derive

$$n\mathfrak{c}_\infty \leq \frac{p-1}{2p} n \|t w_1\|^2_{V_\infty} \leq \frac{p-1}{2p} n \|w_1\|^2_{V_\infty} \leq \lim_{k\to\infty} \frac{p-1}{2p} \|u_{k,1}\|^2_V = \frac{1}{\ell} c^{\phi_n} \leq n\mathfrak{c}_\infty.$$

Therefore, $t = 1$, $w_1 \in \mathcal{M}_\infty$ and $J_\infty(w_1) = ((p-1)/(2p))\|w_1\|^2_{V_\infty} = \mathfrak{c}_\infty$, i.e., w_1 is a least energy solution of (1-4). Moreover, from (3-3) we get that

$$\lim_{k\to\infty} \biggl\| u_{k,1} - \sum_{j=0}^{n-1} (w_1 \circ \alpha^{-j})(\,\cdot - \alpha^j \xi_k) \biggr\|^2_V = 0.$$

Since the positive least energy solution to (1-4) is unique up to translation and w_1 is $O(N-2)$-invariant, there exists $\xi = (\zeta, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ such that $w_1(x) = \omega(x + \xi)$. Hence, $(w_1 \circ \alpha^{-j})(x - \alpha^j \xi_k) = \omega(\alpha^{-j} x - \xi_k - \xi) = \omega(x - \alpha^j(\xi_k + \xi))$. So, setting $z_k := \zeta_k + \zeta$, we obtain

$$\lim_{k\to\infty} \biggl\| u_{k,1} - \sum_{j=0}^{n} \omega(\,\cdot - (\mathrm{e}^{2\pi \mathrm{i} j/n} z_k, 0)) \biggr\| = 0.$$

This shows that, in Case 2, the second alternative stated in Theorem 3.4 holds true. □

Corollary 3.5. *If $c^{\phi_n} < \ell n \mathfrak{c}_\infty$, the system (1-2) has a least energy fully nontrivial solution satisfying (1-3).*

4. Existence of a solution

We define the set of *weak (n, ℓ)-pinwheel partitions* as

$$\mathcal{W}_\ell^n := \{(u_1, \dots, u_\ell) \in \mathcal{H}^{\phi_n} : u_i \neq 0,\ \|u_i\|_V^2 = |u_i|_{2p}^{2p},\ u_i u_j = 0 \text{ in } \mathbb{R}^N \text{ if } i \neq j\},$$

and set

$$\hat{c}^{\phi_n} := \inf_{(u_1,\dots,u_\ell)\in\mathcal{W}_\ell^n} \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_i\|_V^2.$$

Our next goal is to give an upper estimate for $\hat{c}^{\phi_n}$. To this end, we choose $\varepsilon \in \big(0, (d_{\ell n} - \lambda)/(d_{\ell n} + \lambda)\big)$ and a radial function $\chi \in \mathcal{C}^\infty(\mathbb{R}^N)$ satisfying $0 \le \chi \le 1$, $\chi(x) = 1$ if $|x| \le 1 - \varepsilon$ and $\chi(x) = 0$ if $|x| \ge 1$. Let ω be the positive least energy radial solution to (1-4). For each $r > 0$ define

$$\omega_r(x) := \chi\Big(\frac{x}{r}\Big)\omega(x).$$

Lemma 4.1. *As $r \to \infty$,*

$$\big|\|\omega\|^2 - \|\omega_r\|^2\big| = O(\mathrm{e}^{-2(1-\varepsilon)\sqrt{V_\infty} r}) \quad \textit{and} \quad \big||\omega|_{2p}^{2p} - |\omega_r|_{2p}^{2p}\big| = O(\mathrm{e}^{-2p(1-\varepsilon)\sqrt{V_\infty} r}),$$

where $|\cdot|_{2p}$ denotes the norm in $L^{2p}(\mathbb{R}^N)$.

Proof. These statements follow easily from the well-known estimates $|\omega(x)| = O(|x|^{-\frac{1}{2}(N-1)}\mathrm{e}^{-\sqrt{V_\infty}|x|})$ and $|\nabla\omega(x)| = O(|x|^{-\frac{1}{2}(N-1)}\mathrm{e}^{-\sqrt{V_\infty}|x|})$, as in [Clapp and Weth 2004, Lemma 2]. □

Set $\varrho := \frac{1}{4}(d_{\ell n} + \lambda)$, and for $R > 1$ define

$$\hat{w}_{1,R}(x) := \sum_{j=0}^{n-1} \omega_{\varrho R}(x - R(\mathrm{e}^{2\pi \mathrm{i} j/n}, 0)) \quad \text{and} \quad w_{1,R} := t_R \hat{w}_{1,R},$$

where $t_R \in (0, \infty)$ is such that $\|w_{1,R}\|_V^2 = |w_{1,R}|_{2p}^{2p}$. Note that $t_R \to 1$ as $R \to \infty$, $w_{1,R}$ is G_n-invariant and

$$\operatorname{supp}\big(\omega_{\varrho R}(\cdot - R(\mathrm{e}^{2\pi \mathrm{i} j/\ell n}, 0))\big) \subset \overline{B_{\varrho R}(R(\mathrm{e}^{2\pi \mathrm{i} j/\ell n}, 0))}.$$

Set $w_{j+1,R}(\mathrm{e}^{2\pi \mathrm{i} j/\ell n} z, y) := w_{1,R}(z, y)$ for $(z, y) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $j = 1, \dots, \ell - 1$. Since $\varrho < \frac{1}{2} d_{\ell n}$ we have that $\operatorname{supp}(w_{i,R}) \cap \operatorname{supp}(w_{j,R}) = \varnothing$ if $i \neq j$. Hence, $\boldsymbol{w}_R = (w_{1,R}, \dots, w_{\ell,R}) \in \mathcal{W}_\ell^n$.

Lemma 4.2. *There exist $C_1, R_1 > 0$ such that*

$$\frac{p-1}{2p} \sum_{i=1}^{\ell} \|w_{i,R}\|_V^2 = \mathcal{J}^{\phi_n}(\boldsymbol{w}_R) \le \ell n \mathfrak{c}_\infty - C_1 \mathrm{e}^{-\lambda\sqrt{V_\infty} R} \quad \textit{for all} \ \ R \ge R_1.$$

Proof. Because $\boldsymbol{w}_R = (w_{1,R}, \dots, w_{\ell,R}) \in \mathcal{W}_\ell^n$, the equality holds true. To prove the inequality note that $t_R \in \left[\frac{1}{2}, 2\right]$ for R large enough. Assumption (V_3^n) yields

$$\begin{aligned}
\int_{\mathbb{R}^N} (V(x) - V_\infty)\left|t_R\omega_{\varrho R}(x - R(1,0))\right|^2 \mathrm{d}x &= \int_{|x|\le\varrho R}\big(V(x + R(1,0)) - V_\infty\big)|t_R\omega_{\varrho R}(x)|^2 \mathrm{d}x \\
&= -\frac{1}{4}C_0\int_{|x|\le\varrho R} \mathrm{e}^{-\lambda\sqrt{V_\infty}\,|x+R(1,0)|}|\omega(x)|^2 \mathrm{d}x \\
&\le -\frac{1}{4}C_0\bigg(\int_{\mathbb{R}^N} \mathrm{e}^{-\lambda\sqrt{V_\infty}|x|}|\omega(x)|^2 \mathrm{d}x\bigg)\mathrm{e}^{-\lambda\sqrt{V_\infty}R} \\
&=: -2C\mathrm{e}^{-\lambda\sqrt{V_\infty}R}.
\end{aligned}$$

Using Lemma 4.1, for R large enough we get

$$\begin{aligned}
\mathcal{J}^{\phi_n}(\boldsymbol{w}_R) &= \frac{1}{2}\sum_{i=1}^{\ell}\|w_{i,R}\|_V^2 - \frac{1}{2p}\sum_{i=1}^{\ell}\int_{\mathbb{R}^N}|w_{i,R}|^{2p} - \frac{\beta}{2p}\sum_{\substack{i,j=1\\ i\neq j}}^{\ell}\int_{\mathbb{R}^N}|w_{i,R}|^p|w_{j,R}|^p \\
&= \ell n\bigg(\frac{1}{2}\|t_R\omega_{\varrho R}(\,\cdot - R(1,0))\|_V^2 - \frac{1}{2p}\left|t_R\omega_{\varrho R}(\,\cdot - R(1,0))\right|_{2p}^{2p}\bigg) \\
&= \ell n\bigg(\frac{1}{2}\|t_R\omega_{\varrho R}\|_{V_\infty}^2 + \frac{1}{2}\int_{\mathbb{R}^N}(V - V_\infty)\left|t_R\omega_{\varrho R}(\,\cdot - R(1,0))\right|^2 - \frac{1}{2p}|t_R\omega_{\varrho R}|_{2p}^{2p}\bigg) \\
&= \ell n\bigg(\frac{1}{2}\|t_R\omega\|_{V_\infty}^2 - C\mathrm{e}^{-\lambda\sqrt{V_\infty}R} - \frac{1}{2p}|t_R\omega|_{2p}^{2p} + O(\mathrm{e}^{-2(1-\varepsilon)\sqrt{V_\infty}\varrho R})\bigg) \\
&\le \ell n\mathfrak{c}_\infty - C_1\mathrm{e}^{-\lambda\sqrt{V_\infty}R},
\end{aligned}$$

because $2(1-\varepsilon)\varrho > \frac{1}{2}(d_{\ell n} + \lambda)\big(1 - (d_{\ell n} - \lambda)/(d_{\ell n} + \lambda)\big) = \lambda$. □

Proof of Theorem 1.1. Note that $\mathcal{W}_\ell^n \subset \mathcal{N}^{\phi_n}$. Hence, from Lemma 4.2 we get

$$c^{\phi_n} \le \hat{c}^{\phi_n} < \ell n\mathfrak{c}_\infty,$$

and Corollary 3.5 yields the result. □

Proof of Proposition 1.2. Arguing by contradiction, assume that $\boldsymbol{u}$ is a solution to (1-2) satisfying (1-3) with $n = \ell^m$ and with $n = \ell^q$, respectively, and that $1 \le m < q$. Then, for $k = \ell^{q-m-1}j$ with $j = 1, \dots, \ell - 1$,

$$u_1(x) = u_1(\mathrm{e}^{2\pi\mathrm{i}k/\ell^q}x) = u_1(\mathrm{e}^{2\pi\mathrm{i}j/\ell^m\ell}x) = u_{j+1}(x),$$

and as $1 + \beta(\ell - 1) \le 0$, we obtain

$$\|u_1\|_V^2 = \int_{\mathbb{R}^N}|u_1|^{2p} + \beta\sum_{j=1}^{\ell-1}|u_{j+1}|^p|u_1|^p = (1 + \beta(\ell - 1))\int_{\mathbb{R}^N}|u_1|^{2p} \le 0,$$

a contradiction. □

5. The limit profiles of the solutions

We start with the case $\beta \to 0$.

Proof of Theorem 1.3. We write $\mathcal{J}_k^{\phi_n}$ and $\mathcal{N}_k^{\phi_n}$ for the functional and the Nehari set associated to the system (1-2) with $\beta = \beta_k$, and we define

$$c_k^{\phi_n} := \inf_{\mathcal{N}_k^{\phi_n}} \mathcal{J}_k^{\phi_n}.$$

As $\mathcal{W}_\ell^n \subset \mathcal{N}_k^{\phi_n}$ for every $k \in \mathbb{N}$, invoking Lemma 4.2 we see that

$$\frac{p-1}{2p}\sum_{i=1}^{\ell}\|u_{k,i}\|_V^2 = c_k^{\phi_n} \le \hat{c}^{\phi_n} < \ell n \mathfrak{c}_\infty \quad \text{for all } k \in \mathbb{N}. \tag{5-1}$$

After passing to a subsequence, we have that $u_{k,i} \rightharpoonup u_{0,i}$ weakly in $H^1(\mathbb{R}^N)$, $u_{k,i} \to u_{0,i}$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_{k,i} \to u_{0,i}$ a.e. in $\mathbb{R}^N$, for each $i = 1, \dots, \ell$. Hence, $u_{0,i} \ge 0$ and $\boldsymbol{u}_0 = (u_{0,1}, \dots, u_{0,\ell}) \in \mathcal{H}^{\phi_n}$.

We claim that

$$u_{0,i} \neq 0 \quad \text{for all } i = 1, \dots, \ell.$$

To prove this claim assume, arguing by contradiction, that $u_{0,i} = 0$. Following the argument in the proof of Theorem 3.4 we see that, after passing to a subsequence, there exist $\xi_k \in \mathbb{R}^N$, $C_0 > 0$ and $\delta > 0$ such that

$$\int_{B_{C_0+1}(\xi_k)} |u_{k,i}|^{2p} > \delta > 0 \quad \text{for all } k \in \mathbb{N}, \tag{5-2}$$

where either $\xi_k = 0$, or $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $\zeta_k \to \infty$. Since $u_{k,i} \to 0$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, (5-2) implies that $\xi_k \neq 0$. Now, as in Case 2 of Theorem 3.4, we set

$$w_{k,i}(x) := u_{k,i}(x + \xi_k), \qquad i = 1, \dots, \ell,$$

and we take a subsequence satisfying $w_{k,i} \rightharpoonup w_i$ weakly in $H^1(\mathbb{R}^N)$, $w_{k,i} \to w_i$ in $L^{2p}_{\mathrm{loc}}(\mathbb{R}^N)$ and $w_{k,i} \to w_i$ a.e. in $\mathbb{R}^N$. Hence, $w_i \in H^1(\mathbb{R}^N)^{O(N-2)}$, $w_i \ge 0$ and following the proof of (3-3) we obtain

$$\|u_{k,i}\|_V^2 = \Big\|u_{k,i} - \sum_{j=0}^{n-1}(w_i \circ \alpha^{-j})(\cdot - \alpha^j \xi_k)\Big\|_V^2 + n\|w_i\|_{V_\infty}^2 + o(1). \tag{5-3}$$

Furthermore, following the proof of (3-4) we derive

$$\int_{\mathbb{R}^N}(\nabla w_i \cdot \nabla v + V_\infty w_i v) = \int_{\mathbb{R}^N}|w_i|^{2p-2}w_i v + \beta_k \sum_{\substack{j=1 \\ j\neq i}}^{\ell}\int_{\mathbb{R}^N}|w_j|^p|w_i|^{p-2}w_i v$$

for every $v \in H^1(\mathbb{R}^N)^{O(N-2)}$, and taking $v = w_i$ we get

$$\|w_i\|_{V_\infty}^2 = \int_{\mathbb{R}^N}|w_i|^{2p} + \beta_k \sum_{\substack{j=1 \\ j\neq i}}^{\ell}\int_{\mathbb{R}^N}|w_j|^p|w_i|^p \le \int_{\mathbb{R}^N}|w_i|^{2p}.$$

Since, by (5-2),

$$\int_{B_{C_0+1}(0)}|w_{k,i}|^{2p} \ge \int_{B_{C_0+1}(\xi_k)}|u_{k,i}|^{2p} \ge \delta > 0,$$

we see that $w_i \neq 0$. Hence, there exists $t \in (0,1]$ such that $\|tw_i\|^2_{V_\infty} = \int_{\mathbb{R}^N} |tw_i|^{2p}$, and (5-3) yields

$$n\mathfrak{c}_\infty \leq n\frac{p-1}{2p}\|tw_i\|^2_{V_\infty} \leq n\frac{p-1}{2p}\|w_i\|^2_{V_\infty} \leq \frac{p-1}{2p}\|u_{k,i}\|^2_V.$$

As a consequence,

$$\ell n\mathfrak{c}_\infty \leq \frac{p-1}{2p}\sum_{i=1}^{\ell}\|u_{k,i}\|^2_V,$$

contradicting (5-1). This shows that $u_{0,i} \neq 0$, as claimed.

As $(\mathcal{J}_k^{\phi_n})'(\boldsymbol{u}_k) = 0$, $u_{k,i} \geq 0$, $u_{0,i} \geq 0$ and $\beta_k < 0$, we have that

$$\begin{aligned}\langle u_{k,i}, u_{0,i}\rangle_V &= \int_{\mathbb{R}^N} |u_{k,i}|^{2p-2}u_{k,i}u_{0,i} + \beta_k \sum_{\substack{j=1\\ j\neq i}}^{\ell}\int_{\mathbb{R}^N}|u_{k,j}|^p|u_{k,i}|^{p-2}u_{k,i}u_{0,i}\\ &\leq \int_{\mathbb{R}^N}|u_{k,i}|^{2p-2}u_{k,i}u_{0,i},\end{aligned}$$

and passing to the limit we obtain $\|u_{0,i}\|^2_V \leq |u_{0,i}|^{2p}_{2p}$. Hence, there exists $s \in (0,1]$ such that $\|su_{0,i}\|^2_V = |su_{0,i}|^{2p}_{2p}$ and we have that

$$\mathfrak{c}^{G_n} \leq \frac{p-1}{2p}\|su_{0,i}\|^2_V \leq \frac{p-1}{2p}\|u_{0,i}\|^2_V \leq \liminf_{k\to\infty}\frac{p-1}{2p}\|u_{k,i}\|^2_V, \tag{5-4}$$

with $\mathfrak{c}^{G_n}$ as in (2-4). We claim that these are equalities.

To prove this claim, let $v_k \in \mathcal{M}^{G_n}$ be such that $J(v_k) = ((p-1)/(2p))\|v_k\|^2_V \to \mathfrak{c}^{G_n}$. Set $u_{k,1} := v_k$ and define $u_{k,j+1}$ as in (1-3) for $j = 1, \dots, \ell-1$. Set $\boldsymbol{u}_k = (u_{k,1}, \dots, u_{k,\ell})$. Because (v_k) is bounded in $H^1(\mathbb{R}^N)$ and $\beta_k \to 0$, we have that

$$\lim_{k\to\infty}\beta_k\int_{\mathbb{R}^N}|u_{k,j}|^p|u_{k,i}|^p = 0 \quad \text{for every } i, j,$$

so, by Proposition 3.1(d), for k large enough there exists $s_k \in (0,\infty)$ such that $s_k\boldsymbol{u}_k \in \mathcal{N}_k^{\phi_n}$ and $s_k \to 1$ as $k \to \infty$. Thus,

$$c_k^{\phi_n} \leq \mathcal{J}^{\phi_n}(s_k\boldsymbol{u}_k) = \frac{p-1}{2p}\sum_{i=1}^{\ell}\|s_ku_{k,i}\|^2_V = \frac{p-1}{2p}\ell s_k^2\|v_k\|^2_V \longrightarrow \ell\mathfrak{c}^{G_n}. \tag{5-5}$$

Combining (5-4) and (5-5) we see that $s = 1$, thus $u_{0,i} \in \mathcal{M}^{G_n}$, that $u_{k,i} \to u_{0,i}$ strongly in $H^1(\mathbb{R}^N)$ and that

$$J(u_{0,i}) = \mathfrak{c}^{G_n} = \frac{1}{\ell}c_k^{\phi_n} < n\mathfrak{c}_\infty.$$

This completes the proof. □

Now we turn to the case $\beta \to -\infty$. For the proof of Theorem 1.4 we need the following result.

Lemma 5.1. *Let $\beta_k < 0$ and $(u_{k,1}, \dots, u_{k,\ell})$ be a solution to (1-2) with $\beta = \beta_k$ such that $u_{k,i} \to u_{\infty,i}$ strongly in $H^1(\mathbb{R}^N)$ for every $i = 1, \dots, \ell$. Then $(u_{k,i})$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$.*

Proof. Let $s \geq 0$ and assume that $u_{k,i} \in L^{2(s+1)}(\mathbb{R}^N)$ for every $k \in \mathbb{N}$. Fix $L > 0$ and define $w_{k,i} := u_{k,i} \min\{u_{k,i}^s, L\}$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w_{k,i}|^2 &\leq (1+s) \int_{\mathbb{R}^N} \nabla u_{k,i} \cdot \nabla (u_{k,i} \min\{u_{k,i}^{2s}, L^2\}) \\ &= (1+s)\Bigg(\int_{\mathbb{R}^N} |u_{k,i}|^{2p-2} w_{k,i}^2 + \beta \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} |u_{k,j}|^p |u_{k,i}|^{p-2} w_{k,i}^2 - \int_{\mathbb{R}^N} V w_{k,i}^2 \Bigg) \\ &\leq (1+s) \int_{\mathbb{R}^N} |u_{k,i}|^{2p-2} w_{k,i}^2. \end{aligned} \tag{5-6}$$

On the other hand, for any $K > 0$ we have that

$$\begin{aligned}\int_{\mathbb{R}^N} |u_{k,i}|^{2p-2} w_{k,i}^2 &\leq \int_{\mathbb{R}^N} (|u_{k,i}|^{2p-2} - |u_{\infty,i}|^{2p-2}) w_{k,i}^2 + \int_{|u_{\infty,i}|^{2p-2} \geq K} |u_{\infty,i}|^{2p-2} w_{k,i}^2 + K \int_{\mathbb{R}^N} w_{k,i}^2 \\ &\leq \big| |u_{k,i}|^{2p-2} - |u_{\infty,i}|^{2p-2} \big|_{2p}^{2p-2} |w_{k,i}|_{2p}^2 + \Bigg(\int_{|u_{\infty,i}|^{2p-2} \geq K} |u_{\infty,i}|^{2p} \Bigg)^{(p-1)/p} |w_{k,i}|_{2p}^2 \\ &\qquad + K |w_{k,i}|_2^2. \end{aligned}$$

As $u_{k,i} \to u_{\infty,i}$ strongly in $H^1(\mathbb{R}^N)$, choosing $k_0 > 0$ and K sufficiently large, we get that

$$\int_{\mathbb{R}^N} |u_{k,i}|^{2p-2} w_{k,i}^2 \leq \tfrac{1}{2} |w_{k,i}|_{2p}^2 + K |w_{k,i}|_2^2 \quad \text{for every } k \geq k_0. \tag{5-7}$$

Because $H^1(\mathbb{R}^N)$ is continuously embedded into $L^{2p}(\mathbb{R}^N)$, we derive from (5-6) and (5-7) that, for every $k \in \mathbb{N}$,

$$|w_{k,i}|_{2p}^2 \leq K_s |w_{k,i}|_2^2,$$

for some constant K_s independent of L, and letting $L \to \infty$ we get

$$|u_{k,i}|_{2p(s+1)}^{2(s+1)} = |u_{k,i}^{s+1}|_{2p}^2 \leq K_s |u_{k,i}^{s+1}|_2^2 = K_s |u_{k,i}|_{2(s+1)}^{2(s+1)}.$$

As $(u_{k,i})$ is uniformly bounded in $L^2(\mathbb{R}^N)$, iterating this inequality starting with $s = 0$ and using interpolation, we conclude that $(u_{k,i})$ is uniformly bounded in $L^q(\mathbb{R}^N)$ for any $q \in [2, \infty)$ and each $i = 1, \ldots, \ell$. This implies that

$$f_{k,i} := |u_{k,i}|^{2p-2} u_{k,i} + \beta \sum_{j \neq i} |u_{k,j}|^p |u_{k,i}|^{p-2} u_{k,i}$$

is uniformly bounded in $L^q(\mathbb{R}^N)$ for any $q \in [2, \infty)$. Then, by the Calderón–Zygmund inequality, $(u_{k,i})$ is uniformly bounded in $W^{2,q}(\mathbb{R}^N)$ for every $q \in [2, \infty)$, and choosing q large enough, we derive from the Sobolev embedding theorem that $(u_{k,i})$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$, as claimed. □

Proof of Theorem 1.4. (i) As before, we write $\mathcal{J}_k^{\phi_n}$ and $\mathcal{N}_k^{\phi_n}$ for the functional and the Nehari set associated to the system (1-2) with $\beta = \beta_k$, and set

$$c_k^{\phi_n} := \inf_{\mathcal{N}_k^{\phi_n}} \mathcal{J}_k^{\phi_n}.$$

Arguing as in the proof of Theorem 1.3, we see that, after passing to a subsequence, $u_{k,i} \rightharpoonup u_{\infty,i}$ weakly in $H^1(\mathbb{R}^N)$, $u_{k,i} \to u_{\infty,i}$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_{k,i} \to u_{\infty,i}$ a.e. in $\mathbb{R}^N$, for each $i = 1, \dots, \ell$. Hence, $u_{\infty,i} \geq 0$ and $\boldsymbol{u}_\infty = (u_{\infty,1}, \dots, u_{\infty,\ell}) \in \mathcal{H}^{\phi_n}$, so $\boldsymbol{u}_\infty$ satisfies (1-3). We also get that

$$u_{\infty,i} \neq 0 \quad \text{and} \quad \|u_{\infty,i}\|_V^2 \leq |u_{\infty,i}|_{2p}^{2p} \qquad \text{for all } \ i = 1, \dots, \ell.$$

Furthermore, as $(\mathcal{J}_k^{\phi_n})'(\boldsymbol{u}_k) = 0$, we have that, for every $j \neq i$,

$$0 \leq \int_{\mathbb{R}^N} |u_{k,j}|^p |u_{k,i}|^p \leq \frac{|u_{k,i}|_{2p}^{2p}}{-\beta_k} \leq \frac{C}{-\beta_k}.$$

As $\beta_k \to -\infty$, passing to the limit and using Fatou's lemma, we obtain

$$0 \leq \int_{\mathbb{R}^N} |u_{\infty,j}|^p |u_{\infty,i}|^p \leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} |u_{k,j}|^p |k_{n,i}|^p = 0.$$

This implies that $u_{\infty,j} u_{\infty,i} = 0$ a.e. in $\mathbb{R}^N$ whenever $i \neq j$.

Let $s \in (0, 1]$ be such that $\|s u_{\infty,i}\|_V^2 = |s u_{\infty,i}|_{2p}^{2p}$. Then $s\boldsymbol{u}_\infty \in \mathcal{W}_\ell^n$, and using (5-1) we get

$$\hat{c}^{\phi_n} \leq \frac{p-1}{2p} \sum_{i=1}^{\ell} \|s u_{\infty,i}\|_V^2 \leq \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_{\infty,i}\|_V^2 \leq \frac{p-1}{2p} \sum_{i=1}^{\ell} \liminf_{k\to\infty} \|u_{k,i}\|_V^2 \leq \hat{c}^{\phi_n}.$$

This proves that $s = 1$, $\boldsymbol{u}_\infty \in \mathcal{W}_\ell^n$, $u_{k,i} \to u_{\infty,i}$ strongly in $H^1(\mathbb{R}^N)$ and

$$\hat{c}^{\phi_n} = \frac{p-1}{2p} \sum_{i=1}^{\ell} \|u_{\infty,i}\|_V^2. \tag{5-8}$$

Finally, as $\lim_{k\to\infty} \|u_{k,i}\|_V^2 = \|u_{\infty,i}\|_V^2 = |u_{\infty,i}|_{2p}^{2p} = \lim_{k\to\infty} |u_{k,i}|_{2p}^{2p}$, from

$$\lim_{k\to\infty} \|u_{k,i}\|_V^2 = \lim_{k\to\infty} |u_{k,i}|_{2p}^{2p} + \lim_{k\to\infty} \beta_k \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \int_{\mathbb{R}^N} |u_{k,j}|^p |u_{k,i}|^p,$$

we obtain

$$\int_{\mathbb{R}^N} \beta_k u_{k,j}^p u_{k,i}^p \to 0 \quad \text{as } k \to \infty \qquad \text{whenever } i \neq j.$$

(ii) It follows from Lemma 5.1 and [Clapp et al. 2021b, Theorem B.2] that $(u_{k,i})$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(K)$ for each compact subset K of $\mathbb{R}^N$ and $\alpha \in (0, 1)$. So from the Arzelà-Ascoli theorem we get that $u_{\infty,i} \in \mathcal{C}^0(\mathbb{R}^N)$. Therefore $\Omega_i := \{x \in \mathbb{R}^N : u_{\infty,i}(x) > 0\}$ is open. Because $u_{\infty,i} u_{\infty,j} = 0$ if $i \neq j$ and $\boldsymbol{u}_\infty$ satisfies (1-3), we have that $\Omega_i \cap \Omega_j = \varnothing$ if $i \neq j$ and the ℓ-tuple $(\Omega_1, \dots, \Omega_\ell)$ satisfies (S_1) and (S_2). Thus, it is an (n, ℓ)-pinwheel partition.

Since $\boldsymbol{u}_\infty \in \mathcal{W}_\ell^n$, we have that $u_{\infty,i}$ belongs to the Nehari manifold $\mathcal{M}_{\Omega_i}$ defined in (1-7). Therefore, $((p-1)/(2p))\|u_{\infty,i}\|_V^2 \geq \mathfrak{c}_{\Omega_i}$. Equality must hold true as, otherwise, there would exist $v_1 \in \mathcal{M}_{\Omega_1}$ such that $((p-1)/(2p))\|u_{\infty,1}\|_V^2 > ((p-1)/(2p))\|v_1\|_V^2 \geq \mathfrak{c}_{\Omega_i}$, and defining v_{j+1} as in (1-3), we would have that $(v_1, \dots, v_\ell) \in \mathcal{W}_\ell^n$ and

$$\frac{p-1}{2p}\sum_{i=1}^{\ell}\|v_i\|_V^2 < \frac{p-1}{2p}\sum_{i=1}^{\ell}\|u_{\infty,i}\|_V^2 = \hat{c}^{\phi_n}$$

by (5-8), which is a contradiction. This shows that $u_{\infty,i}$ is a least energy solution of (1-8) in Ω_i. Now, since $((p-1)/(2p))\|u_{\infty,i}\|_V^2 = \mathfrak{c}_{\Omega_i}$, we get

$$\inf_{(\Theta_1,\dots,\Theta_\ell)\in\mathcal{P}_\ell^n}\sum_{j=1}^{\ell}\mathfrak{c}_{\Theta_j} \leq \sum_{j=1}^{\ell}\mathfrak{c}_{\Omega_j} = \hat{c}^{\phi_n} \leq \inf_{(\Theta_1,\dots,\Theta_\ell)\in\mathcal{P}_\ell^n}\sum_{j=1}^{\ell}\mathfrak{c}_{\Theta_j}.$$

This shows that $(u_{\infty,1}, \dots, u_{\infty,\ell})$ is an optimal (n, ℓ)-pinwheel partition.

(iii) This is a local statement. Recall that $(u_{k,i})$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(\Omega)$ for each open subset Ω compactly contained in $\mathbb{R}^N$ and $\alpha \in (0, 1)$. So, from the Arzelà-Ascoli theorem, we get that $u_{k,i} \to u_{\infty,i}$ in $\mathcal{C}^{0,\alpha}(\Omega)$. Thus, all hypotheses of [Clapp et al. 2021b, Theorem C.1] are satisfied and (iii) follows.

(iv) Let G_{2n} be the group defined in Example 2.1 with $\ell = 2$, and let $\tau_n : G_{2n} \to \mathbb{Z}_2 := \{1, -1\}$ be the homomorphism given by $\tau_n(\mathrm{e}^{2\pi\mathrm{i}/2n}) = -1$ and $\tau_n(\theta) = 1$ for every $\theta \in O(N-2)$. A solution to the Schrödinger equation (1-5) satisfying

$$u(gx) = \tau_n(g)u(x) \quad \text{for all } g \in G_{2n},\ x \in \mathbb{R}^N \tag{5-9}$$

is a critical point of the functional $J : H^1(\mathbb{R}^N)^{\tau_n} \to \mathbb{R}$ defined by (2-3) on the space

$$H^1(\mathbb{R}^N)^{\tau_n} := \{u \in H^1(\mathbb{R}^N) : u \text{ satisfies (5-9)}\}.$$

The nontrivial ones belong to the Nehari manifold

$$\mathcal{M}^{\tau_n} := \{u \in H^1(\mathbb{R}^N)^{\tau_n} : u \neq 0,\ \|u\|_V^2 = |u|_{2p}^{2p}\},$$

which is a natural constraint for J. Note that every nontrivial function satisfying (5-9) is nonradial and changes sign.

There is a one-to-one correspondence

$$\mathcal{W}_2^n \to \mathcal{M}^{\tau_n}, \quad (u_1, u_2) \mapsto u_1 - u_2,$$

whose inverse is $u \mapsto (u^+, -u^-)$, with $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$, satisfying

$$\frac{p-1}{2p}(\|u_1\|_V^2 + \|u_2\|_V^2) = J(u_1 - u_2).$$

Therefore,

$$J(u_{\infty,1} - u_{\infty,2}) = \frac{p-1}{2p}(\|u_{\infty,1}\|_V^2 + \|u_{\infty,2}\|_V^2) = \inf_{(u_1,u_2)\in\mathcal{W}_2^n}\frac{p-1}{2p}(\|u_1\|_V^2 + \|u_2\|_V^2) = \inf_{u\in\mathcal{M}^{\tau_n}} J(u).$$

This shows that $u_{\infty,1} - u_{\infty,2}$ is a least energy solution to (1-5) and (5-9). ∎

Appendix: An auxiliary result

Lemma A.1. *Assume $v_k \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, $\xi_k \in \mathbb{R}^N$ satisfies $|\xi_k| \to \infty$ and $V \in \mathcal{C}^0(\mathbb{R}^N)$ satisfies assumption (V_2). Set $V_k(x) := V(x+\xi_k)$. Then*

$$\lim_{k\to\infty} \|v_k\|^2_{V_k} - \lim_{k\to\infty} \|v_k - v\|^2_{V_k} = \|v\|^2_{V_\infty}.$$

Proof. As $v_k \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, one has

$$\begin{aligned}\|v\|^2_{V_\infty} + o(1) &= \|v_k\|^2_{V_\infty} - \|v_k - v\|^2_{V_\infty} \\ &= \|v_k\|^2_{V_k} - \|v_k - v\|^2_{V_k} + 2\int_{\mathbb{R}^N} (V_\infty - V_k) v_k v - \int_{\mathbb{R}^N} (V_\infty - V_k) v^2.\end{aligned}$$

Given $\varepsilon > 0$, choose $R > 0$ large enough that

$$\int_{\mathbb{R}^N \smallsetminus B_R} |V_\infty - V_k||v|^2 \le 2 \sup_{x\in\mathbb{R}^N} V(x) \int_{\mathbb{R}^N \smallsetminus B_R} |v|^2 < \tfrac{1}{2}\varepsilon.$$

Now take k_0 such that

$$|V_\infty - V(x+\xi_k)| < \frac{\varepsilon}{2|v|_2^2} =: \delta \quad \text{for every } x \in B_R \text{ and } k \ge k_0.$$

Then, for $k \ge k_0$,

$$\int_{\mathbb{R}^N} |V_\infty - V_k||v|^2 \le \int_{B_R} |V_\infty - V_k||v|^2 + \int_{\mathbb{R}^N \smallsetminus B_R} |V_\infty - V_k||v|^2 < \varepsilon$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} |(V_\infty - V_k) v_k v| &\le \left(\int_{\mathbb{R}^N} |V_\infty - V_k||v_k|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |V_\infty - V_k||v|^2\right)^{\frac{1}{2}} \\ &\le C\sqrt{\varepsilon}.\end{aligned}$$

This completes the proof. □

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Counting compatible indexing systems for C_{p^n}

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We count the number of compatible pairs of indexing systems for the cyclic group C_{p^n}. Building on work of Balchin, Barnes, and Roitzheim, we show that this sequence of natural numbers is another family of Fuss–Catalan numbers. We count this two different ways: showing how the conditions of compatibility give natural recursive formulas for the number of admissible sets and using an enumeration of ways to extend indexing systems by conceptually simpler pieces.

1. Introduction

Recent work in equivariant algebra has studied the beautiful variety of different multiplicative structures that can arise when we mix in the action of a finite group. Blumberg and Hill [2015] showed that this is a fundamentally combinatorial structure. The various multiplicative norms or additive transfers are entirely governed by certain particularly well-behaved subcategories of finite G-sets.

This classification was reformulated in independent work of Balchin, Barnes, and Roitzheim [Balchin et al. 2021] and of Rubin [2021], in which they further underscored the combinatorial structure by showing the norms and transfers are encoded in "transfer systems", certain refinements of the poset of subgroups of G under inclusion.

Definition 1.1 [Rubin 2021, Definition 3.4; Balchin et al. 2021, Definition 7]. A transfer system for a finite group G is a partial order $\to$ relation on the set $\mathrm{Sub}(G)$ of subsets of G such that

(1) if $K \to H$, then $K \leq H$ (so this is a "weak subposet"),

(2) if $K \to H$ and $g \in G$, then $gKg^{-1} \to gHg^{-1}$, and

(3) if $K \to H$ and $J \leq H$, then $K \cap J \to J$.

Let $\mathrm{Tran}(G)$ denote the set of transfer systems for G.

The set of all transfer systems for G itself has a partial order: we say $\mathcal{O} \leq \mathcal{O}'$ if the identity map is order-preserving.

Example 1.2. For $G = C_{p^n}$, the subgroup lattice is order-isomorphic to the linear order

$$\{1 \leq 2 \cdots \leq n+1\} = [n+1].$$

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The conjugation condition is always satisfied, and the restriction condition here can be rephrased as saying that if $C_{p^i} \to C_{p^j}$ and $i \le k \le j$, then $C_{p^i} \to C_{p^k}$.

Balchin, Barnes, and Roitzheim [Balchin et al. 2021, Theorem 25] showed that the poset of transfer systems for C_{p^n} is order-isomorphic to the Tamari lattice.

In this paper, we will focus on the groups C_{p^n}. Example 1.2 stresses that we could equivalently look at "transfer systems for the poset $[n+1]$" in the sense of Franchere, Ormsby, Osorno, Qin, and Waugh.

Definition 1.3 [Franchere et al. 2022, Definition 4.1]. A transfer system on

$$[n] = \{1 \le \cdots \le n\}$$

is a weak subposet of $[n]$ with partial order $\to$ that contains all the elements and which satisfies the "restriction condition": if $i \to j$ and $i \le k \le j$, then $i \to k$.

It is helpful to view these as a graded set in which we allow n to vary, as this will help encode certain natural operations.

Definition 1.4. For each $n \in \mathbb{N}$, let

$$T_n = \begin{cases} \mathrm{Tran}([n]) & n \ge 1, \\ \{\varnothing\} & n = 0. \end{cases}$$

Balchin, Barnes, and Roitzheim counted the number of transfer systems for C_{p^n}:

Theorem 1.5 [Balchin et al. 2021, Theorem 20]. *For each $n \in \mathbb{N}$,*

$$|T_n| = \frac{1}{2n+1}\binom{2n+1}{n} = \mathrm{Cat}(n).$$

The numbers $\mathrm{Cat}(n)$ are the Catalan numbers. These are ubiquitous in combinatorics, parametrizing structures from binary rooted trees to Dyck paths. This sequence fits into a bivariant family of sequences.

Definition 1.6. The *Fuss–Catalan numbers* are defined by

$$A_n(p,r) = \frac{r}{np+r}\binom{np+r}{n}$$

for nonnegative n and positive p and r.

The Catalan numbers arise here:

$$\mathrm{Cat}(n) = A_n(2,1).$$

In this paper, we show the next term of the sequence is also related to equivariant algebra.

Blumberg and Hill [2022] studied what kind of compatibility conditions arise if we allow both the additive transfers and multiplicative norms to each be structured by various transfer systems. As a slogan, "the presence of certain multiplicative norms forces some additive transfers". These conditions were simplified by Chan [2022], who gave a definition internal to transfer systems. When the conditions are satisfied, we say that $(\mathcal{O}_a, \mathcal{O}_m)$ is compatible. Counting these for C_{p^n} is the main result of this paper.

Theorem 1.7. *For $C_{p^{n-1}}$, there are*

$$A_n(3,1)$$

compatible pairs of transfer systems.

We present two proofs of this theorem in Sections 4 and 5, each of which underscores a different combinatorial feature of the number of compatible pairs. The mathematics in this paper arose from an REU project in Summer 2021. The two junior coauthors each came up with a distinct solution to this counting problem, so we include here both of their solutions.

A third proof by Henry Ma will appear separately. This proof uses a direct bijection between compatible pairs of transfer systems and Kreweras pairs. Recent work of Balchin, MacBrough, and Ormsby [Balchin et al. 2022] shows that Kreweras pairs are equinumerous with composition-closed premodel structures on the poset $[n]$. We do not see a direct connection between compatible pairs and composition-closed premodel structures, however.

2. Operations on transfer systems

Concatenation. A key piece of structure on transfer systems for $[n]$ is the ability to "concatenate" a transfer system for $[k]$ and one for $[n-k]$ to build one for $[n]$. This makes computations with the entire graded set easier to understand.

Definition 2.1. Let $\mathcal{O}_L$ be a transfer system for $[k]$ and $\mathcal{O}_R$ be a transfer system for $[n-k]$. Then we define a relation $\mathcal{O} = \mathcal{O}_L \oplus \mathcal{O}_R$ on $[n]$ by saying $i \xrightarrow{\mathcal{O}} j$ if and only if

(1) $j \le k$ and $i \xrightarrow{\mathcal{O}_L} j$ or

(2) $i \ge (k+1)$ and $(i-k) \xrightarrow{\mathcal{O}_R} (j-k)$.

We call $\mathcal{O}$ the *direct sum* or *concatenation* of $\mathcal{O}_L$ and $\mathcal{O}_R$.

Remark 2.2. As a poset, this is just the disjoint union of the two posets $\mathcal{O}_L$ and $\mathcal{O}_R$. The additional data is the map to $[n]$ in this case.

Proposition 2.3. *The concatenation of transfer systems is a transfer system.*

Proof. The definition of $\mathcal{O}$ is that of the disjoint union of posets, so $\mathcal{O}$ is a poset, and it visibly maps to the usual inclusions.

We need only check the restriction condition, and here, it suffices to check the restriction of $i+k \to j+k$ along $m \le j+k$ for some $m \le k$. In this case, both the source and target are m. □

The direct sum is a graded operation here:

$$T_n \times T_m \xrightarrow{\oplus} T_{n+m}.$$

We extend this to T_0 by declaring

$$\varnothing \oplus \mathcal{O} = \mathcal{O} \oplus \varnothing = \mathcal{O}$$

for any transfer system $\mathcal{O}$ or the empty set.

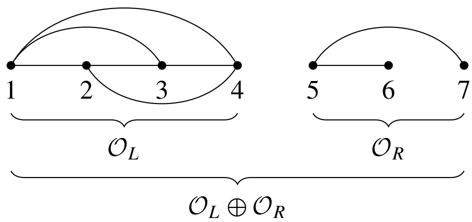

Figure 1. Example of concatenation of $\mathcal{O}_L$ and $\mathcal{O}_R$.

Restriction. Given a transfer system for $[n]$, we have two natural ways to build transfer systems for smaller natural numbers. These both arise from the inclusion of subposets.

Definition 2.4. If $k \leq n$, then let

$$\iota : [k] \to [n]$$

be the map sending $i \to i$, and let

$$\phi : [n-k] \to [n]$$

be the map sending i to $i+k$.

We can restrict a transfer system on $[n]$ along either of these inclusions.

Definition 2.5. Let $\mathcal{O}$ be a transfer system for $[n]$, and let $k \leq n$.

Let $i_k^*\mathcal{O}$ be defined by saying for all $i \leq j \leq k$,

$$i \xrightarrow{i_k^*\mathcal{O}} j \quad \text{if and only if} \quad i \xrightarrow{\mathcal{O}} j.$$

Let $\Phi^k\mathcal{O}$ be defined by saying for all $i \leq j \leq (n-k)$,

$$i \xrightarrow{\Phi^k\mathcal{O}} j \quad \text{if and only if} \quad (i+k) \xrightarrow{\mathcal{O}} (j+k).$$

Proposition 2.6. *For a transfer system on $\mathcal{O}$, $i_k^*\mathcal{O}$ is a transfer system on $[k]$ and $\Phi^k\mathcal{O}$ is a transfer system on $[n-k]$.*

Proof. The construction gives wide, weak subposets of $[k]$ and $[n-k]$ respectively, by observation. Since the two inclusions ι and ϕ are interval inclusions, the restriction condition is easily checked. $\square$

The two restrictions can be visualized as simply throwing away any transfers that start or end outside of the given range. An example picture of this is shown in Figure 2.

Remark 2.7. The maps i_k^* and Φ^k have topological meaning. Recall that the subgroup lattice of $C_{p^{n-1}}$ is isomorphic to $[n]$ via the map that sends a subgroup H to $\log_p(|H|)+1$. The inclusion of the subgroups of $C_{p^{k-1}}$ into those of $C_{p^{n-1}}$ corresponds via this identification to the inclusion $\iota : [k] \hookrightarrow [n]$. Similarly, the subgroups of $C_{p^{n-1}}$ which contain C_{p^k} correspond to the inclusion $\phi : [n-k] \hookrightarrow [n]$.

Using these, the map i_k^* on transfer systems is just the ordinary restriction of transfer systems for $C_{p^{n-1}}$ to transfer systems for the subgroup $C_{p^{k-1}}$. The map Φ^k also has a direct topological meaning. Blumberg and Hill [2015, Lemma B.1] showed that for any N_∞-operad $\mathcal{O}$ for G and for any normal subgroup N

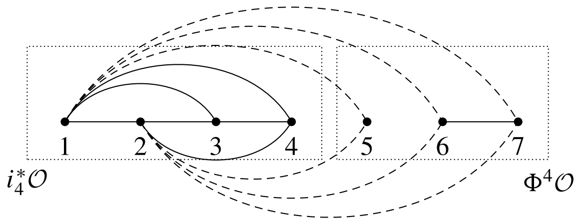

Figure 2. Example of $i_k^*\mathcal{O}$ and $\Phi^k\mathcal{O}$. The dashed transfers are forgotten to form $i_4^*\mathcal{O}$ and $\Phi^4\mathcal{O}$.

of G, the N-fixed points of $\mathcal{O}$ are an N_∞-operad for G/N. The corresponding operation on transfer systems is Φ^k.

Wrapped and saturated systems. We single out two families of transfer systems useful for our counts.

Definition 2.8. A transfer system $\mathcal{O}$ for $[n]$ is *wrapped* if $1 \xrightarrow{\mathcal{O}} n$.

Example 2.9. The transfer systems $\mathcal{O}_L$ and $\mathcal{O}_R$ from Figure 1 are wrapped; the transfer system $\mathcal{O}_L \oplus \mathcal{O}_R$ is not.

Example 2.10. The transfer system $\mathcal{O}$ in Figure 2 is wrapped, as is $i_4^*\mathcal{O}$. The transfer system $\Phi^4\mathcal{O}$ is not.

By the restriction condition, in a wrapped transfer system, we have $1 \to j$ for all j, and we think of the largest transfer as "wrapping" the rest of the transfer system. Additionally, we can always "wrap" any given transfer system, using the circle-dot product of [Balchin et al. 2021]. We use a single case of their construction here.

Definition 2.11. If $\mathcal{O} \in T_n$, let $w(\mathcal{O})$ be the transfer system for $[n+1]$ with

(1) for all j, $1 \to j$ in $w(\mathcal{O})$, and
(2) $\Phi^1(w(\mathcal{O})) = \mathcal{O}$.

This is implicit in Balchin, Barnes, and Roitzheim's "circle-dot product" and their decomposition theorem.

Theorem 2.12 [Balchin et al. 2021, Corollary 21]. *Any transfer system for* $[n]$ *with* $n \geq 1$ *decomposes uniquely as*

$$\mathcal{O} \odot \mathcal{O}' = \mathcal{O} \oplus w(\mathcal{O}')$$

for some transfer systems $\mathcal{O} \in T_m$ *and* $\mathcal{O}' \in T_{n-m-1}$.

Corollary 2.13. *Any transfer system* $\mathcal{O}$ *for* $[n]$ *can be written uniquely as*

$$\mathcal{O} = w(\mathcal{O}_1) \oplus \cdots \oplus w(\mathcal{O}_k)$$

for some transfer systems $\mathcal{O}_1, \ldots, \mathcal{O}_k$.

A special case of wrapped transfer systems is given by complete ones.

Definition 2.14. The *complete* transfer system for $[n]$ is the one for which $i \to j$ for all $i \leq j$. Denote it by $\mathcal{O}_n^{\mathrm{cpt}}$.

Remark 2.15. The complete transfer system for $[n]$ is one for which the partial order is just the inclusion. This means that complete transfer systems are maximal elements in the poset of transfer systems.

These complete transfer systems will be especially useful building blocks for us.

Definition 2.16 [Rubin 2021, Definition 3.4]. A transfer system $\mathcal{O}$ is *saturated* if whenever $i \to j$ and $i \leq k \leq j$, then $k \to j$.

Remark 2.17. The saturation condition is equivalent to a 2-out-of-3 property for the transfer system, since $i \to j$ with $i \leq k \leq j$ always implies $i \to k$ by the restriction condition. This formulation has been used by Hafeez, Marcus, Ormsby, and Osorno [Hafeez et al. 2022, Definition 2.5] in their study of the saturation conjecture.

Proposition 2.18. *A saturated transfer system on $[n]$ is a direct sum of complete ones.*

Proof. Complete transfer systems and direct sums of these are visibly saturated. For the other direction, write $\mathcal{O}$ as a direct sum of wrapped transfer systems:

$$\mathcal{O} = \mathcal{O}_{n_1} \oplus \cdots \oplus \mathcal{O}_{n_k}.$$

The transfer system $\mathcal{O}$ is saturated if and only if each of the pieces is, since there are no transfers connecting the individual summands. It suffices to then show that a wrapped saturated transfer system is complete. This however follows by downward induction on n. □

Given any transfer system on $[n]$, there is a minimal saturated transfer system that contains it. This is immediate from the observation that the intersection of two saturated transfer systems is again a saturated transfer system. For concreteness, we spell this out directly here.

Proposition 2.19. *If $\mathcal{O}$ is any transfer system, then there is a minimal saturated transfer system that contains $\mathcal{O}$.*

Proof. Write $\mathcal{O}$ as a direct sum of wrapped transfer systems $\mathcal{O} = \mathcal{O}_{n_1} \oplus \cdots \oplus \mathcal{O}_{n_r}$, with $\mathcal{O}_{n_i} \in T_{n_i}$. Then the minimal saturated transfer system containing it is simply

$$\mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_r}^{\mathrm{cpt}},$$

where we replace each summand with the complete one of that size. □

Definition 2.20. If $\mathcal{O}$ is a transfer system for $[n]$, let $\mathrm{hull}(\mathcal{O})$ denote the minimal saturated transfer system that contains it. This is the *saturated hull*.

For transfer systems for $[n]$, there is a kind of dual notion of a maximal saturated transfer systems inside any given transfer system.

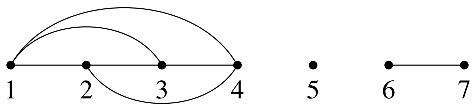

Figure 3. Core of $\mathcal{O}$ from Figure 2.

Complete transfer systems have two useful properties:

(1) Whenever $i \to j$, we also have $k \to j$ for all $i \leq k \leq j$.

(2) They are generated as a poset as the transitive closure of the relation $i \to i+1$ for all $1 \leq i \leq n-1$.

These two properties give us two different ways to repackage the condition of being saturated.

Proposition 2.21. *A transfer system $\mathcal{O}$ for n is saturated if and only if whenever $i \to j$ with $i < j$, we have $(j-1) \to j$.*

Proof. If $\mathcal{O}$ is saturated, then, by definition of the direct sum, if $i \to j$, then i and j correspond to subgroups from the same direct summand. By completeness, we therefore have all intermediate transfers.

Using the decomposition of a transfer system into wrapped ones (Corollary 2.13), we see that it suffices to show that if a wrapped transfer system that has the property that $i \to j$ implies $(j-1) \to j$, then the transfer system is complete. This follows from downward induction on n, using that $1 \to n$ by the wrapped assumption, and hence $1 \to j$ for all j by restriction. $\square$

Corollary 2.22. *A transfer system $\mathcal{O}$ is saturated if and only if it is generated as a partial order by relations $i \to (i+1)$ for some collection of positive i at most $n-1$.*

The possibly surprising part here is that we only need the partial order: the other parts of being a transfer system come along for free in this case, since we are generating by a covering condition. This gives us a second kind of structural result.

Definition 2.23. If $\mathcal{O}$ is a transfer system, let the *core* of $\mathcal{O}$, denoted by $\operatorname{core}\mathcal{O}$, be the partial order generated by $i \to (i+1)$, where i ranges over the integers from 1 to $n-1$ such that $i \to (i+1)$.

Example 2.24. In Figure 3, we see the core of the transfer system $\mathcal{O}$ from Figure 2.

3. Compatible pairs

Our main object of study is the notion of compatible pairs. Blumberg and Hill [2022] defined these to describe "compatibility" between equivariant norms and transfers in an abstract, categorical way. Chan [2022, Theorem 4.10] reformulated this in the language of transfer systems, giving a purely combinatorial formulation. We use that here for the special case of C_{p^n}.

Notation 3.1. Given a pair of transfer systems $(\mathcal{O}_a, \mathcal{O}_m)$, let $i \xrightarrow{a} j$ be $i \xrightarrow{\mathcal{O}_a} j$, and similarly for $\xrightarrow{m}$.

Definition 3.2 [Chan 2022, Definition 4.6]. A pair of transfer systems $(\mathcal{O}_a, \mathcal{O}_m)$ for $[n]$ is compatible if for whenever $i \xrightarrow{m} j$, we have $k \xrightarrow{a} j$ for all $i \leq k \leq j$.

Note here that the conditions are asymmetrical: arrows in $\mathcal{O}_m$ force those in $\mathcal{O}_a$. Moreover, this is a kind of relative saturation condition, with an arrow in $\mathcal{O}_m$ actually forcing $\mathcal{O}_a$ to look saturated in a range. This gives us two equivalent forms.

Proposition 3.3. *A pair $(\mathcal{O}_a, \mathcal{O}_m)$ is compatible if and only if the following equivalent comparisons hold*:

(1) $\mathcal{O}_m \leq \mathrm{core}(\mathcal{O}_a)$,

(2) $\mathrm{hull}(\mathcal{O}_m) \leq \mathcal{O}_a$, *and*

(3) $\mathrm{hull}(\mathcal{O}_m) \leq \mathrm{core}(\mathcal{O}_a)$.

Proof. The conditions of Definition 3.2 are a restatement of the condition that the saturated hull of $\mathcal{O}_m$ is less than or equal to $\mathcal{O}_a$. For the equivalence of the three conditions, we use that the core of $\mathcal{O}$ is the largest saturated transfer system less than or equal to $\mathcal{O}$ and the hull is the smallest saturated transfer system greater than or equal to $\mathcal{O}$. □

Corollary 3.4. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ be a compatible pair of transfer systems for $[n]$. If for some $1 \leq k \leq n-1$, we have $k \overset{a}{\nrightarrow} (k+1)$, then, for all $j \leq k$ and $\ell \geq (k+1)$, we must have $j \overset{m}{\nrightarrow} \ell$.*

Put another way, we see that $\mathcal{O}_m$ must break apart at k, and this must be compatible with $\mathcal{O}_a$.

Corollary 3.5. *Let $(\mathcal{O}_a, \mathcal{O}_m)$ be a compatible pair of transfer systems for $[n]$. If for some $1 \leq k \leq (n-1)$, we have $k \overset{a}{\nrightarrow} (k+1)$, then*

(1) $\mathcal{O}_m = i_k^* \mathcal{O}_m \oplus \Phi^k \mathcal{O}_m$, *and*

(2) *the pairs*

$$(i_k^*\mathcal{O}_a, i_k^*\mathcal{O}_m) \quad \text{and} \quad (\Phi^k\mathcal{O}_a, \Phi^k\mathcal{O}_m)$$

are compatible.

Definition 3.6. Let

$$\mathcal{D}_n^{(0)} = \{(\mathcal{O}_a, \mathcal{O}_m) \mid \text{compatible}\} \subset T_n \times T_n.$$

We have projection maps

$$p_a, p_m : \mathcal{D}_n^{(0)} \to T_n$$

which take a pair $(\mathcal{O}_a, \mathcal{O}_m)$ to $\mathcal{O}_a$ or $\mathcal{O}_m$ respectively. Our main goal is to find the cardinality of $\mathcal{D}_n^{(0)}$ for all n. We solve this in several different ways using different aspects of Proposition 3.3.

4. Solving the recurrence relations

Decomposition and recurrence relation. The wrapping map w defines a natural filtration on the collection of transfer systems. We can use this to build a recursive relation describing compatible pairs.

Definition 4.1. For each $i \geq 0$, let

$$(\mathcal{F}^i T)_n = \mathrm{Im}(w^{\circ i}) \subset T_n,$$

viewed as a graded subset.

Definition 4.2. Let
$$\mathcal{D}_n^{(i)} = p_a^{-1}((\mathcal{F}^i T)_n)$$
be the set of composable pairs with $\mathcal{O}_a \in \mathcal{F}^i T$.

Let $d(n,i) = |\mathcal{D}_n^{(i)}|$ be the corresponding cardinality.

We deduce our recursive formulae from the Balchin, Barnes, and Roitzheim decomposition theorem (Theorem 2.12). We restate the result here to set up our decomposition.

Proposition 4.3. *If $\mathcal{O}$ is a transfer system in $(\mathcal{F}^i T)_n$, then there is*

(1) *a unique natural number $1 \le j \le n-i$,*

(2) *a unique wrapped transfer system $w\mathcal{O}_R$ in T_j, and*

(3) *a unique transfer system $\mathcal{O}_L$ in T_{n-i-j}*

such that
$$\mathcal{O} = w^i(\mathcal{O}_L \oplus w\mathcal{O}_R).$$

Notation 4.4. Let $(\mathcal{F}^i T)_{n,j}$ be the set of transfer systems in $(\mathcal{F}^i T)_n$ which decompose as
$$\mathcal{O} = w^i(\mathcal{O}_L \oplus w\mathcal{O}_R)$$
with $w\mathcal{O}_R \in T_j$ a wrapped transfer system.

Proposition 4.5. *The map*
$$\left(\coprod_{j=1}^{n-i-1} \mathcal{D}_{n-j}^{(i)} \times \mathcal{D}_j^{(1)}\right) \sqcup \mathcal{D}_n^{(i+1)} \to \mathcal{D}_n^{(i)}$$
given on $\mathcal{D}_{n-j}^{(i)} \times \mathcal{D}_j^{(1)}$ by
$$\big((w^i\mathcal{O}'_a, \mathcal{O}'_m), (w\mathcal{O}''_a, \mathcal{O}''_m)\big) \mapsto \big(w^i(\mathcal{O}'_a \oplus w\mathcal{O}''_a), \mathcal{O}'_m \oplus \mathcal{O}''_m\big),$$
and on the last summand by the natural inclusion, is a bijection.

Proof. We use Proposition 4.3 to further break up $\mathcal{D}_n^{(i)}$, since the decomposition here gives a disjoint union decomposition
$$(\mathcal{F}^i T)_n = \coprod_{j=1}^{n-i} (\mathcal{F}^i T)_{n,j}.$$
This decomposition induces a decomposition of $\mathcal{D}_n^{(i)}$:
$$\mathcal{D}_{n,j}^{(i)} = p_1^{-1}(\mathcal{F}^i T)_{n,j}.$$
Since $T_0 = \{\varnothing\}$, the unit for $\oplus$, we have
$$\mathcal{D}_{n,n-i}^{(i)} = \mathcal{D}_n^{(i+1)},$$
given by the usual inclusion. Now let $1 \le j < n-i$, and consider an element $(\mathcal{O}_a, \mathcal{O}_m)$ in $\mathcal{D}_{n,j}^{(i)}$. By definition, we have
$$\mathcal{O}_a = w^i(\mathcal{O}_{a,L} \oplus w\mathcal{O}_{a,R}),$$

with $\mathcal{O}_{a,L} \neq \varnothing$ and $w\mathcal{O}_{a,R}$ wrapped, and hence we are missing the transfer

$$(i+n-j) \to (i+n-j+1)$$

in $\mathcal{O}_a$. This means that $\mathcal{O}_m$ breaks up into a direct sum

$$\mathcal{O}'_m \oplus \mathcal{O}''_m,$$

where $\mathcal{O}'_m \in T_{i+n-j}$ and $\mathcal{O}''_m \in T_j$, by Corollary 3.5. Moreover, we know that the pairs

$$(i^*_{i+n-j}\mathcal{O}_a, \mathcal{O}'_m) \quad \text{and} \quad (\Phi^{i+n-j}\mathcal{O}_a, \mathcal{O}''_m)$$

are compatible. The result follows, since

$$i^*_{i+n-j}\mathcal{O}_a = w^i\mathcal{O}_{a,L} \quad \text{and} \quad \Phi^{i+n-j}\mathcal{O}_a = w\mathcal{O}_{a,R}. \qquad \square$$

Corollary 4.6. *We have a recursive formula*

$$d(n,i) = d(n,i+1) + \sum_{j=1}^{n-i-1} d(n-j,i)d(j,1).$$

The base case here is actually $\mathcal{D}_n^{(n)}$, which is $p_a^{-1}(\mathcal{O}_n^{\text{cpt}})$. Every transfer system is compatible with the complete additive transfer system. Note also that we have an important edge case:

$$(\mathcal{F}^{n-1}T)_n = (\mathcal{F}^n T)_n,$$

since both correspond to the unique complete transfer system on $[n]$.

Proposition 4.7 [Balchin et al. 2021, Theorem 20]. *For each n, we have*

$$d(n,n) = d(n,n-1) = \text{Cat}(n).$$

Rewriting the recurrence relation. We now can solve the recurrence relation, giving our first proof of the main theorem. Recall the definition of the Fuss–Catalan numbers:

$$A_n(p,r) = \frac{r}{np+r}\binom{np+r}{n}.$$

We begin with some helpful properties of the Fuss–Catalan number.

Proposition 4.8. *For any positive integer p and positive n, r, the following properties hold*:

(1) $A_n(3,r) = \sum_{j=0}^{n} A_j(2,1)A_{n-j}(3,j+r-1)$.

(2) $A_n(p,r) = A_n(p,r-1) + A_{n-1}(p,p+r-1)$.

(3) $A_{n+1}(p,1) = A_n(p,p)$.

(4) $A_n(p,s+r) = \sum_{j=0}^{n} A_j(p,r)A_{n-j}(p,s)$.

(5) $A_n(3,2) = \sum_{j=1}^{n} A_{j+1}(2,1)A_{n-j}(3,j)$.

Proof. Four of these are from work of Młotkowski. Formula (1) is a special case of Proposition 2.1 in [Młotkowski 2010]. Formula (2) is Equation 2.2 there, (3) is Equation 2.3, and (4) is Equation 2.4.

We will only prove (5). It can be done by induction on n. The base case $n = 1$ is a straightforward check. Take $n \geq 2$.

Write

$$s = \sum_{j=1}^{n} A_{j+1}(2, 1) A_{n-j}(3, j).$$

Replacing j with $j + 1$, we can rewrite this as

$$s = \sum_{j=2}^{n+1} A_j(2, 1) A_{n-j+1}(3, j-1).$$

Applying (2) twice, we have

$$\begin{aligned} A_{n-j+1}(3, j+1) &= A_{n-j+1}(3, j) + A_{n-j}(3, j+3) \\ &= \big(A_{n-j+1}(3, j-1) + A_{n-j}(3, j+2)\big) + A_{n-j}(3, j+3). \end{aligned}$$

With the help of (1), we can expand $A_{n+1}(3, 2)$ to

$$A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 2) + \sum_{j=2}^{n+1} A_{n-j+1}(3, j+1) A_j(2, 1),$$

and substituting in for $A_{n-j+1}(3, j+1)$, we can rewrite this as

$$A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 2) + s + \sum_{j=2}^{n} A_{n-j}(3, j+2) A_j(2, 1) + \sum_{j=2}^{n} A_{n-j}(3, j+3) A_j(2, 1).$$

The last two sums also show up in the expansions of $A_n(3, 3)$ and $A_n(3, 4)$, respectively, by (1):

$$\begin{aligned} \sum_{j=2}^{n} A_{n-j}(3, j+2) A_j(2, 1) &= A_n(3, 3) - A_n(3, 2) - A_{n-1}(3, 3), \\ \sum_{j=2}^{n} A_{n-j}(3, j+3) A_j(2, 1) &= A_n(3, 4) - A_n(3, 3) - A_{n-1}(3, 4). \end{aligned}$$

This gives an equality

$$\begin{aligned} A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 2) + s + \big(A_n(3, 3) - A_n(3, 2) - A_{n-1}(3, 3)\big) \\ + \big(A_n(3, 4) - A_n(3, 3) - A_{n-1}(3, 4)\big). \end{aligned} \tag{4.9}$$

We can use (3) to rewrite (4.9) as

$$A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 4) + \big(s - A_n(3, 1) - A_{n-1}(3, 4)\big).$$

However, applying (2) to $A_{n+1}(3, 2)$, we get exactly

$$A_{n+1}(3, 2) = A_{n+1}(3, 1) + A_n(3, 4),$$

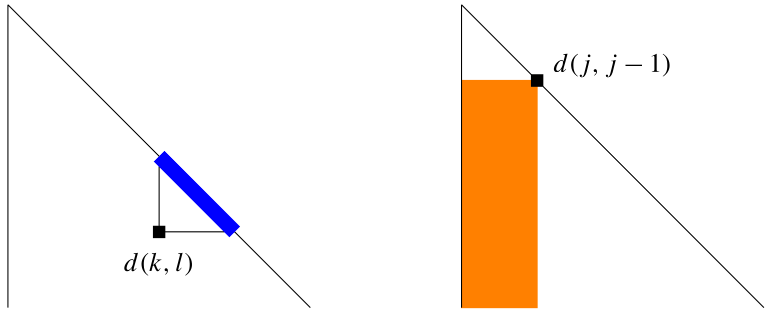

Figure 4. Contributions to each term: $d(k, l)$ is a linear combination of $d(j, j-1)$ from the blue shaded region, and a given $d(j, j-1)$ contributes only in the orange region.

so we deduce

$$s = A_n(3, 1) + A_{n-1}(3, 4).$$

By (2) again, we get $s = A_n(3, 2)$. □

Theorem 4.10. *For any* $n \in \mathbb{N}$, $A_{n-1}(3, 2)$ *is* $d(n, 1)$.

Before proving the theorem, we first discuss the strategy of the proof. Instead of focusing on the recurrence steps for a specific n, we put all $d(k, j)$ together, forming a triangle as follows:

$$\begin{array}{llll} d(2,1) & & & \\ d(3,1) & d(3,2) & & \\ d(4,1) & d(4,2) & d(4,3) & \\ d(5,1) & d(5,2) & d(5,3) & d(5,4) \\ \quad\vdots & & & \end{array}$$

Recall Corollary 4.6. For $i \in \mathbb{N}_+$, the elements in the set $\{d(j+i, j) : j \geq 1\}$ share the same recursive formula. In fact, we can generate the above triangle hypotenuse-by-hypotenuse from outside to inside. To begin with, we generate the second outermost hypotenuse $d(3, 1)$, $d(4, 2)$, $d(5, 3)$, ... by the outermost hypotenuse $d(2, 1)$, $d(3, 2)$, $d(4, 3)$ ... according to the recursive formula for $d(n, n-2)$. In other words, $d(n, n-2)$ is a linear combination of $d(j, j-1)$ with coefficients 1 or $d(1, 1)$. Similarly, we can generate the third outermost hypotenuse by the outermost and the second outermost hypotenuse according to the recursive formula for $d(n, n-3)$. Now the coefficients are from the space spanned by $d(2, 1)$, $d(1, 1)$ and 1.

As the whole triangle can be generated hypotenuse-by-hypotenuse, we can conclude that any $d(k, l)$ in the triangle can be written as a linear combination of $d(j, j-1)$ where $l+1 \leq j \leq k$ with coefficients from the space spanned by 1 and $d(i, 1)$ where $1 \leq i \leq k-l-1$. Graphically, $d(k, l)$ is a linear combination of $d(j, j-1)$ in the blue area of Figure 4.

Proof of Theorem 4.10. This is easy to check for $d(2, 1)$. Then, by induction, it suffices to show the theorem is true for $d(n, 1)$ if the theorem holds for $d(i, 1)$ for any $2 \leq i \leq n-1$.

Fix j such that $2 \leq j \leq n$. For our convenience, we denote the coefficient of $d(j, j-1)$ for $d(k,l)$ as $c(k,l)$. Our goal is to determine $c(n, 1)$. We will achieve this by induction.

First, we figure out the coefficients of $d(j, j-1)$ for the vertical line $d(j+l, j-1)$ where $0 \leq l \leq n-j$. We know $c(j, j-1) = 1 = A_0(3, 1)$ and for $0 \leq l \leq n-j$,

$$c(j+l, j-1) = \sum_{i=1}^{l} d(i, 1)c(j+l-i, j-1).$$

As $d(i, 1)$ is $A_{i-1}(3, 2)$ for $2 \leq i \leq n-1$, we get $c(j+l, j-1) = A_l(3, 1)$ by (4) of Proposition 4.8 and the equality $A_n(3, 3) = A_{n+1}(3, 1)$ for $0 \leq l \leq n-j$ inductively.

In fact, we claim that for $1 \leq i \leq j-1$, $c(j+l, j-i) = A_l(3, i)$ for $0 \leq l \leq n-j$. We have shown the case for $i = 1$. Now it is sufficient to show the case $i = k$ provided the claim holds for $1 \leq i \leq k-1$. Similarly, we know $c(j, j-k) = 1 = A_0(3, k)$ and for $0 \leq l \leq n-j$,

$$c(j+l, j-k) = c(j+l, j-(k-1)) + \sum_{i=1}^{l} d(i, 1)c(j+l-i, j-k).$$

As $d(i, 1)$ is $A_{i-1}(3, 2)$ for $2 \leq i \leq n-1$, we get $c(j+l, j-k) = A_l(3, k)$ by (2) and (4) of Proposition 4.8 for $0 \leq l \leq n-j$ inductively. Thus, we get $c(n, 1) = A_{n-j}(3, j-1)$. As $d(n, 1)$ depends only on $d(j, j-1)$ for $2 \leq j \leq n$,

$$d(n, 1) = \sum_{j=2}^{n} A_{n-j}(3, j-1)d(j, j-1) = \sum_{j=2}^{n} A_{n-j}(3, j-1)A_j(2, 1),$$

where the last equality is Proposition 4.7. By (5) of Proposition 4.8, we conclude $d(n, 1) = A_{n-1}(3, 2)$. □

Theorem 4.11. *There are $A_n(3, 1)$ compatible transfer systems for $[n]$.*

Proof. We will prove this by induction. It is easy to check the case for [1]. Assume the theorem is true for $[i]$ where $1 \leq i < n$. We want to show it is true for $[n]$.

Corollary 4.6 shows that the number of compatible systems for $[n]$ is

$$d(n, 0) = d(n, 1) + \sum_{j=1}^{n-1} d(j, 0)d(n-j, 1).$$

Theorem 4.10 and the inductive hypothesis let us rewrite this as

$$d(n, 0) = A_{n-1}(3, 2) + \sum_{j=1}^{n-1} A_j(3, 1)A_{n-j-1}(3, 2) = \sum_{j=0}^{n-1} A_j(3, 1)A_{n-j-1}(3, 2).$$

By (4) of Proposition 4.8, we deduce

$$d(n, 0) = A_{n-1}(3, 3),$$

which, by (3) of Proposition 4.8, equals $A_n(3, 1)$. □

5. Extensions of saturated systems

Counting using additive cores. Instead of using the filtration by powers of w on the additive indexing system, we can use the first part of Proposition 3.3. This says that compatibility of $(\mathcal{O}_a, \mathcal{O}_m)$ is the same question as compatibility of $(\operatorname{core}\mathcal{O}_a, \mathcal{O}_m)$, since both reduce to the comparison

$$\mathcal{O}_m \leq \operatorname{core}\mathcal{O}_a.$$

Since the core breaks up as a direct some of complete transfer systems, this last condition is really the same as asking that if

$$\operatorname{core}\mathcal{O}_a = \mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_m}^{\mathrm{cpt}},$$

then we have a direct sum decomposition

$$\mathcal{O}_m = \mathcal{O}'_{n_1} \oplus \cdots \oplus \mathcal{O}'_{n_m},$$

where $\mathcal{O}'_{n_i} \in T_{n_i}$ for all i. This is the only condition here, so we deduce the following proposition.

Proposition 5.1. *If $\mathcal{O}_a$ is a transfer system with*

$$\operatorname{core}\mathcal{O}_a = \mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_m}^{\mathrm{cpt}},$$

then there are

$$\prod_{j=1}^{m} \operatorname{Cat}(n_j)$$

transfer systems $\mathcal{O}_m$ such that the pair $(\mathcal{O}_a, \mathcal{O}_m)$ is compatible.

This reduces our question of the number of compatible pairs to two parts:

(1) Enumerate all of the transfer systems with a fixed core.

(2) Then evaluate the corresponding sum.

Notation 5.2. Let $\vec{k} = (k_1, \ldots, k_m)$ be a sequence of positive integers. For each $1 \leq s \leq m$, let

$$K_s = k_1 + \cdots + k_s.$$

Let

$$\mathcal{O}_{\vec{k}}^{\mathrm{sat}} = \mathcal{O}_{k_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{k_m}^{\mathrm{cpt}}.$$

Definition 5.3. For a sequence $\vec{k} = (k_1, \ldots, k_m)$ with $n = k_1 + \cdots + k_m$, let

$$\mathcal{E}_{\vec{k}} = \{\mathcal{O} \in T_k \mid \operatorname{core}\mathcal{O} = \mathcal{O}_{\vec{k}}^{\mathrm{sat}}\},$$

and let

$$e_{\vec{k}} = |\mathcal{E}_{\vec{k}}|.$$

Proposition 5.4. *The number of admissible pairs of transfer systems for $[n]$ is*

$$\sum_{\vec{k}} e_{\vec{k}} \prod_{j} \operatorname{Cat}(k_j),$$

where $\vec{k}$ in the first sum ranges over the partitions of n.

Proof. We partition the set of transfer systems by their core:

$$T_n = \bigsqcup_{\vec{k}} \mathcal{E}_{\vec{k}},$$

where $\vec{k}$ ranges over the partitions of n. This induces a partition of the set of compatible pairs:

$$\mathcal{D}_n^{(0)} = \bigsqcup_{\vec{k}} \{(\mathcal{O}_a, \mathcal{O}_m) \mid \text{compatible}, \mathcal{O}_a \in \mathcal{E}_{\vec{k}}\}.$$

The first part of Proposition 3.3 shows that

$$\{\mathcal{O}_m \mid (\mathcal{O}_a, \mathcal{O}_m) \text{ compatible}\}$$

is constant for $\mathcal{O}_a \in \mathcal{E}_{\vec{k}}$, and Proposition 5.1 identifies the cardinality as

$$\prod_j \mathrm{Cat}(k_j).$$

Putting this all together, we find there are

$$\sum_{\vec{k}} e_{\vec{k}} \prod_j \mathrm{Cat}(k_j)$$

compatible pairs of transfer systems. □

Catalan tuples. The enumeration and exact sum were analyzed by de Jong, Hock, and Wulkenhaar [de Jong et al. 2022] in a slightly different guise. They consider certain sequences which they call Catalan tuples.

Definition 5.5 [de Jong et al. 2022, Definition 3.1]. For each positive integer n, a *Catalan tuple of length n* is a sequence of nonnegative integers

$$\vec{s} = (s_0, \ldots, s_r)$$

with three properties:

(1) For all j, we have

$$\sum_{i=0}^{j} s_i > j.$$

(2) At the end of the sequence,

$$\sum_{i=0}^{r} s_i = n.$$

(3) If $n > 0$, then $s_r > 0$.

Let $\mathcal{S}_n$ be the set of all Catalan tuples of length n.

Remark 5.6. We have slightly modified the definition here to ignore trailing zeros. This removes our ability to predict the length of the string, but it will better connect with the extensions.

We can restate the conditions in Definition 5.5 slightly to start connecting with extensions.

Definition 5.7. The *excess* of a Catalan tuple $\vec{s} = (s_1, \dots, s_r)$ of length n is

$$e(\vec{s}) := n - r - 1.$$

Remark 5.8. Note that the edge condition in Definition 5.5 of $j = r$ implies the inequality $n > r$. This means the excess is always nonnegative.

Proposition 5.9. *Let $\vec{s}$ be a Catalan tuple of length n and excess $e = n - r - 1$. Then, for any $k > 0$, the sequence*

$$\vec{s}_k(\ell) = (s_0, \dots, s_r, \underbrace{0, \dots, 0}_{\ell}, k)$$

is a Catalan tuple if and only if

$$0 \leq \ell \leq e.$$

Proof. For $0 \leq j \leq r$, the Catalan tuple condition holds since it does for $\vec{s}$. If $\ell = 0$, then we have

$$\sum_{i=0}^{r} s_i + k = n + k > r + 1,$$

since $n > r$ and $k \geq 1$. Assume now that $\ell > 0$, and consider a $1 \leq j \leq \ell$. We have

$$\sum_{i=0}^{r+j} s_i = n = r + e + 1.$$

On the other hand, this is greater than $r + j$ if and only if $j < e + 1$. This gives the bounds on ℓ. Finally, note that the analysis for the case $\ell = 0$ now also implies the case $j = \ell + 1$. $\square$

Proposition 5.10. *Let $\vec{s}$ be a Catalan tuple of length n and excess $e = n - r - 1$. Then, for any $k > 0$ and $0 \leq \ell \leq e$, the excess of the Catalan tuple*

$$\vec{s}_k(\ell) = (s_0, \dots, s_r, \underbrace{0, \dots, 0}_{\ell}, k)$$

is

$$e(\vec{s}_k(\ell)) = k - 1 + e - \ell = n + k - (r + \ell + 2).$$

Proof. The Catalan tuple given has length $n + k$. The sequence $\vec{s}$ has length $(r + 1)$, and we added $\ell + 1$ new terms to form $\vec{s}_k(\ell)$. $\square$

This lets us rewrite Catalan tuples using only the nonzero entries.

Definition 5.11. If $\vec{s}$ is a Catalan tuple, then let $\mathrm{core}(\vec{s})$ be the subsequence of nonzero entries of $\vec{s}$. Given a partition $\vec{k}$ of k, let $\mathcal{S}_{\vec{k}}$ be the subset of $\mathcal{S}_k$ of Catalan tuples with core $\vec{k}$:

$$\mathcal{S}_{\vec{k}} := \{\vec{s} \in \mathcal{S}_k \mid \mathrm{core}\,\vec{s} = \vec{k}\}.$$

Corollary 5.12. *Catalan tuples with core $\vec{k}$ are those sequences of the form*

$$k_1,\ \underbrace{0,\ \dots,\ 0}_{\ell_1},\ k_2,\ \underbrace{0,\ \dots,\ 0}_{\ell_2},\ k_3,\ \dots,\ k_{n-1},\ \underbrace{0,\ \dots,\ 0}_{\ell_{n-1}},\ k_n$$

such that for all $1 \leq j \leq n-1$, we have

$$\sum_{i=1}^{j}(k_i - \ell_i) \geq j.$$

The excess of such a sequence is

$$(k_n - 1) + \sum_{i=1}^{n-1}(k_i - 1 - \ell_i).$$

In their work, de Jong, Hock, and Wulkenhaar consider certain collections of Catalan tuples.

Definition 5.13 [de Jong et al. 2022, Definition 4.1]. A *nested Catalan tuple of length n* is a sequence of Catalan tuples $(\vec{s}_{i_1}, \dots, \vec{s}_{i_r})$ such that $\vec{s}_{i_j} \in \mathcal{S}_{i_j}$ and the sequence

$$(i_1 + 1, i_2, \dots, i_r)$$

is a Catalan tuple of length n.

A key result in [de Jong et al. 2022] is the cardinality of the number of nested Catalan tuple that begin with (0). For this, we need a straightforward lemma.

Lemma 5.14. *The map $\Sigma : \mathcal{S}_n \to \mathcal{S}_{n+1}$ given by*

$$(s_0, \dots, s_r) \mapsto (1, s_0, \dots, s_r)$$

is an injection with image those sequences which begin with 1.

Proposition 5.15 [de Jong et al. 2022, Corollary 4.6]. *The number of nested Catalan tuples of length $(n+1)$ with first term (0) is*

$$\sum_{\vec{s} \in \mathcal{S}_n} \prod_i \operatorname{Cat}(s_i) = \frac{1}{2n+1}\binom{3n}{n} = \frac{1}{3n+1}\binom{3n+1}{n} = A_n(3, 1).$$

We will produce an explicit bijection

$$\sigma : T_n \to \mathcal{S}_n$$

by building bijections between $\mathcal{E}_{\vec{k}}$ and $\mathcal{S}_{\vec{k}}$.

Enumerating extensions by a complete transfer system. It is helpful to think of elements of $\mathcal{E}_{\vec{k}}$ also as various "extensions" of the complete transfer systems $\mathcal{O}^{\mathrm{cpt}}_{k_1}, \dots, \mathcal{O}^{\mathrm{cpt}}_{k_n}$. For our count, it is easier to instead consider a more general class.

Definition 5.16. An *extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}'' \in T_k$* is a transfer system $\mathcal{O} \in T_{m+k}$ such that

(1) $i_m^*\mathcal{O} = \mathcal{O}'$,

(2) $\Phi^m\mathcal{O} = \mathcal{O}''$,

and an extension $\mathcal{O}$ of $\mathcal{O}'$ by $\mathcal{O}''$ is *core-preserving* if, moreover,

$$\operatorname{core}\mathcal{O} = \operatorname{core}\mathcal{O}' \oplus \operatorname{core}\mathcal{O}''.$$

Note that since by assumption we have specified i_m^* and Φ^m in an extension, we need only determine the transfers with source $i \le m$ and target $j > m$.

Definition 5.17. Any transfer $i \to j$ with $i \le m$ and $j > m$ in an extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}''$ is a *crossing transfer*.

Proposition 5.18. *Let $\mathcal{O}$ be an extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}'' \in T_k$. Then following are equivalent*:

(1) *The extension is core-preserving.*

(2) *If there is a transfer $m \to j$, then $j = m$.*

Proof. Note that the existence of a nontrivial transfer $m \to j$ is equivalent to the existence of a transfer $m \to (m+1)$, by the restriction axiom. If we have

$$\operatorname{core}\mathcal{O}' = \mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_j}^{\mathrm{cpt}} \quad \text{and} \quad \operatorname{core}\mathcal{O}'' = \mathcal{O}_{m_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{m_i}^{\mathrm{cpt}},$$

then, by construction of the core, the existence of the transfer $m \to (m+1)$ is equivalent to core $\mathcal{O}$ satisfying

$$\operatorname{core}\mathcal{O} = \mathcal{O}_{n_1}^{\mathrm{cpt}} \oplus \cdots \oplus \mathcal{O}_{n_{j-1}}^{\mathrm{cpt}} \oplus \mathcal{O}_{n_j+m_1}^{\mathrm{cpt}} \oplus \mathcal{O}_{m_2} \oplus \cdots \oplus \mathcal{O}_{m_i}.$$

The result follows. □

Because we want to enumerate transfer systems with a fixed core, we now restrict attention to core-preserving extensions of $\mathcal{O}$ by a complete transfer system $\mathcal{O}'$. This significantly simplifies our combinatorics.

Lemma 5.19. *Let $\mathcal{O}$ be a core-preserving extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}_k^{\mathrm{cpt}}$. Then, for each $i \le m$, the following are equivalent*:

(1) *We have a crossing transfer $i \to m+j$ for some $j > 0$.*

(2) *We have crossing transfers $i \to m+j$ for all $0 \le j \le k$.*

Proof. One direction is immediate. For the other, if we have a transfer $i \to m+j$, then, by the restriction axiom, we have transfers $i \to m$ and $i \to m+1$. Since $\mathcal{O}_k^{\mathrm{cpt}}$ is complete, in our extension, we have transfers $m+1 \to m+j$ for any $1 \le j \le k$, which gives the second result. □

Remark 5.20. We singled out the transfer $i \to m$ since this constrains the number of possible sources for a transfer from $[m]$ up to $[k]$. Any crossing transfer has source an element of $[m]$ that transfers up to m in $\mathcal{O}'$.

Definition 5.21. If $\mathcal{O} \in T_m$, then let

$$\tau(\mathcal{O}) = \{d \in [m] \mid d \to m\}$$

be the set of elements which transfer up to m in $\mathcal{O}$.

The element m always transfers to m, so $\tau(\mathcal{O})$ is always nonempty.

Definition 5.22. If $\mathcal{O} \in T_m$, then let

$$e(\mathcal{O}) = |\tau(\mathcal{O})| - 1$$

be the number of elements of $\{1, \ldots, m-1\}$ which transfer up to m in $\mathcal{O}$.

As a subset of the totally ordered set $[m]$, the set $\tau(\mathcal{O})$ inherits a total order:

$$\tau(\mathcal{O}) = \{d_1, d_2, \ldots, d_{e(\mathcal{O})+1}\},$$

where by assumption, we always will have that if $i < j$, then $d_i < d_j$. The crossing transfers have a kind of "nondecreasing" property for this order.

Lemma 5.23. *Let $\mathcal{O}$ be a core-preserving extension of $\mathcal{O}' \in T_m$ by $\mathcal{O}_k^{\mathrm{cpt}}$. In $\mathcal{O}$, if we have a transfer*

$$d \to m + k,$$

for some $d \in \tau(\mathcal{O}')$, then, for all $d' \in \tau(\mathcal{O}')$ such that $d' \leq d$, we have transfers

$$d' \to m + k.$$

Proof. By the restriction axiom, whenever $d' \leq d$, we have a transfer $d' \to d$. The result follows from transitivity. $\square$

Definition 5.24. Let $\mathcal{O}' \in T_m$, let $e = e(\mathcal{O}')$, and write

$$\tau(\mathcal{O}') = \{d_1, \ldots, d_{e+1}\}.$$

For each $k > 0$ and for each $0 \leq \ell \leq e$, define a relation $\to_\ell$ on $[m+k]$ that refines the partial order $\leq$ by saying that

(1) if $i \leq j \leq m$, then $i \to_\ell j$ if and only if $i \to j$ in $\mathcal{O}'$,
(2) if $m < i \leq j$, then $i \to_\ell j$, and
(3) if $i \leq m < j$, then $i \to_\ell j$ if and only if $i = d_s$ for some $1 \leq s \leq r - \ell$.

The following proposition is a straightforward application of the definition of a transfer system.

Proposition 5.25. *The relation $\to_\ell$ is a transfer system on $[n+k]$ that is a core-preserving extension of $\mathcal{O}'$ by $\mathcal{O}_k^{\mathrm{cpt}}$.*

Definition 5.26. Let $\mathcal{O}_{\mathcal{O}',k}(\ell)$ denote the transfer system $\to_\ell$ on $[n+k]$.

Remark 5.27. There is a special case of the extensions: $\ell = e$. In this case, we have the direct sum $\mathcal{O}' \oplus \mathcal{O}_k^{\mathrm{cpt}}$.

There is a crucial observation about the number of transfers to $[m+k]$ here.

Proposition 5.28. *Let $\mathcal{O}'$ be a transfer system on $[m]$ with $e(\mathcal{O}') = e$, and let $0 \leq \ell \leq e$. Then*

$$e(\mathcal{O}_{\mathcal{O}',k}(\ell)) = (k-1) + (e - \ell).$$

Proof. All of the k elements of $[k]$ transfer up, and by construction, the $r-\ell$ elements $d_1,\dots,d_{r-\ell}$ are the only elements from $[m]$ which also transfer up to $m+k$. This gives

$$\tau(\mathcal{O}_{\mathcal{O}',k}(\ell)) = \{d_1,\dots,d_{r-\ell}, m+1,\dots,m+k\},$$

and the result follows. □

Putting these together gives a complete classification of the core-preserving extensions.

Theorem 5.29. *Let $\mathcal{O}' \in T_m$ be a transfer system, and let $e = e(\mathcal{O}')$. Then there are $(e+1)$ core-preserving extensions of $\mathcal{O}'$ by $\mathcal{O}_k^{\mathrm{cpt}}$ given by $\mathcal{O}_{\mathcal{O}',k}(\ell)$ for $0 \le \ell \le e$.*

Proof. Lemmas 5.19 and 5.23 show that any core-preserving extension has this form. The converse is the content of Proposition 5.25. □

Enumerating transfer systems with a fixed core. Now let $\mathcal{O}$ be a transfer system with

$$\operatorname{core}\mathcal{O} = \mathcal{O}_{\vec{k}}^{\mathrm{sat}}.$$

Write $\vec{k} = (k_1,\dots,k_n)$. We can immediately identify $\mathcal{O}$ inductively as a type considered in the last section.

Proposition 5.30. *The transfer system $\mathcal{O}$ is a core-preserving extension of $i_{K_{n-1}}^*\mathcal{O}$ by $\mathcal{O}_{k_n}^{\mathrm{cpt}}$.*

This turns our problem into an inductive one, working down on the number of summands in the partition of k. We can now build our bijection.

Definition 5.31. Let

$$\sigma : \mathcal{E}_{\vec{k}} \to \mathcal{S}_{\vec{k}}$$

be defined inductively by the following procedure. If $\vec{k} = (k)$, then $\mathcal{O} = \mathcal{O}_k^{\mathrm{cpt}}$, and we define

$$\sigma(\mathcal{O}_k^{\mathrm{cpt}}) = k.$$

For a general $\vec{k} = (k_1,\dots,k_n)$ with $n > 1$ and $\mathcal{O} \in \mathcal{E}_{\vec{k}}$, let ℓ_{n-1} be the unique number $0 \le \ell_{n-1}$ such that

$$\mathcal{O} = \mathcal{O}_{\mathcal{O}',k_n}(\ell_{n-1}),$$

where $\mathcal{O}' = i_K^*\mathcal{O}$, and define

$$\sigma(\mathcal{O}) = (\sigma(\mathcal{O}'), \underbrace{0,\dots,0}_{\ell_{n-1}}, k_n).$$

Example 5.32. For $\mathcal{O}$ the transfer system in Figure 2, we have

$$\sigma(\mathcal{O}) = (4,0,1,2).$$

We need to verify that σ actually lands in the set $\mathcal{S}_{\vec{k}}$.

Proposition 5.33. *For any $\mathcal{O}$ with* $\operatorname{core}\mathcal{O} = \vec{k}$,

1. *$\sigma(\mathcal{O}) \in \mathcal{S}_{\vec{k}}$, and*
2. *$e(\sigma(\mathcal{O})) = e(\mathcal{O})$, that is, the excess of $\sigma(\mathcal{O})$ is the number of elements j between 1 and $K_n - 1$ that transfer up to K_n in $\mathcal{O}$.*

Proof. We show this by induction on n. The base case of $n = 1$ is immediate by the definitions of σ and the excess, so assume this is true for partitions with fewer than n terms.

Let $\mathcal{O}' = i^*_{K_{n-1}}\mathcal{O}$, and let r be the number of $j < K_{n-1}$ which transfer up to K_{n-1} in $\mathcal{O}'$. By the inductive hypothesis, $\sigma(\mathcal{O}')$ is a Catalan tuple with core $(k_1, \dots, k_{n-1})$ and we also have

$$e(\sigma(\mathcal{O}')) = r.$$

Now if $0 \leq \ell_n \leq r$ is such that $\mathcal{O} = \mathcal{O}_{\mathcal{O}',k_n}(\ell_{n-1})$, then, since $\ell_{n-1} \leq e(\sigma(\mathcal{O}'))$, the first claim is Proposition 5.9. For the second part, Proposition 5.10 shows that the excess of $\sigma(\mathcal{O})$ is

$$e(\sigma(\mathcal{O})) = e - \ell_{n-1} + (k_n - 1).$$

Proposition 5.28 shows this is exactly the number of elements smaller than K_n which transfer up to K_n. □

This gives us the final piece for our argument.

Corollary 5.34. *The map σ is a bijection $\mathcal{E}_{\vec{k}} \to S_{\vec{k}}$.*

Proof. By induction on n, we see that there are exactly as many extensions of $i^*_{K_{n-1}}\mathcal{O}$ by $\mathcal{O}_{k_n}$ as there are extensions of the Catalan tuple $\sigma(i^*_{K_{n-1}}\mathcal{O})$ to a Catalan tuple ending with k_n, and the map σ gives a bijection between these. □

We close by completing a proof of Theorem 1.7 using these extensions.

Second proof of Theorem 1.7. Proposition 5.15 says that we have

$$A_n(3,1) = \sum_{\vec{s}\in\mathcal{S}_n} \prod_i \mathrm{Cat}(s_i).$$

The set S_n of Catalan tuples of length n is partitioned by the cores:

$$S_n = \coprod_{\vec{k}} S_{\vec{k}}.$$

So we can rewrite the sum from Proposition 5.15 as

$$\sum_{\vec{s}\in\mathcal{S}_n} \prod_i \mathrm{Cat}(s_i) = \sum_{\vec{k}} \sum_{\vec{s}\in S_{\vec{k}}} \prod_i \mathrm{Cat}(s_i).$$

Since $\mathrm{Cat}(0) = 1$, only terms in the core of $\vec{s}$ contribute in a nontrivial way to the product:

$$\prod_i \mathrm{Cat}(s_i) = \prod_j \mathrm{Cat}(k_j),$$

where in the second product, we are running over the entries of $\vec{k} = \mathrm{core}\,\vec{s}$. In particular, we can again rewrite the sum from Proposition 5.15:

$$\sum_{\vec{k}} \sum_{\vec{s}\in S_{\vec{k}}} \prod_i \mathrm{Cat}(s_i) = \sum_{\vec{k}} |S_{\vec{k}}| \prod_j \mathrm{Cat}(k_j),$$

since all Catalan tuples with the same core contribute the same product to the sum. Corollary 5.34 shows

$$|S_{\vec{k}}| = |\mathcal{E}_{\vec{k}}| = e_{\vec{k}},$$

the number of transfer systems with core $\mathcal{O}_{\vec{k}}^{\mathrm{sat}}$. Proposition 5.4 identifies this last sum with the number of compatible pairs, completing the proof. □

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Games for the two membranes problem

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We find viscosity solutions to the two membranes problem (that is, a system with two obstacle-type equations) with two different p-Laplacian operators taking limits of value functions of a sequence of games. We analyze two-player zero-sum games that are played in two boards with different rules in each board. At each turn both players (one inside each board) have the choice of playing without changing board or changing to the other board (and then playing one round of the other game). We show that the value functions corresponding to this kind of game converge uniformly to a viscosity solution of the two membranes problem. If in addition the possibility of having the choice to change boards depends on a coin toss we show that we also have convergence of the value functions to the two membranes problem that is supplemented with an extra condition inside the coincidence set.

1. Introduction

The deep connection between partial differential equations and probability is a well-known and widely studied subject. For linear operators, such as the Laplacian, this relation turns out to rely on the validity of mean value formulas for the solutions in the PDE side and martingale identities in the probability side. In fact, there is a standard connection between the Laplacian and the Brownian motion or with the limit of random walks as the step size goes to zero; see, for example, [Doob 1954; 1971; 1984; Hunt 1957; 1958; Kac 1947; Kakutani 1944; Knapp 1965; Williams 1991]. Recently, starting with [Peres et al. 2009], some of these connections were extended to cover nonlinear equations. For a probabilistic approximation of the infinity Laplacian there is a game (called tug-of-war in the literature), introduced in [Peres et al. 2009], whose value functions approximate solutions to the PDE as a parameter that controls the size of the steps in the game goes to zero. In [Peres and Sheffield 2008] — see also [Manfredi et al. 2012a; 2012b] — the authors introduce a modification of the game (called tug-of-war with noise) that is related to the normalized p-Laplacian. Approximation of solutions to linear and nonlinear PDEs using game theory is now a classical subject. The previously mentioned results were extended to cover very different equations (such as Pucci operators, the Monge–Ampère equation, the obstacle problem, etc); see the books [Blanc and Rossi 2019] and [Lewicka 2020]. However, much less is known concerning the relation between systems of PDEs and games. As a recent reference for a cooperative system we quote [Mitake and Tran 2017]. One of the systems that attracted the attention of the PDE community is the two membrane problem. This problem models the behavior of two elastic membranes that are clamped at the boundary of a prescribed domain

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(they are assumed to be ordered, one membrane above the other) and they are subject to different external forces (the membrane that is on top is pushed down and the one that is below is pushed up). The main assumption here is that the two membranes do not penetrate each other (they are assumed to be ordered in the whole domain). This situation can be modeled by a two obstacle problem; the lower membrane acts as an obstacle from below for the free elastic equation that describes the location of the upper membrane, while, conversely, the upper membrane is an obstacle from above for the equation for the lower membrane. When the equations that obey the two membranes have a variational structure this problem can be tackled using calculus of variations (one aims to minimize the sum of the two energies subject to the constraint that the functions that describe the position of the membranes are always ordered inside the domain; one is bigger than or equal to the other); see [Vergara Caffarelli 1971]. However, when the involved equations are not variational the analysis relies on monotonicity arguments (using the maximum principle). Once existence of a solution (in an appropriate sense) is obtained a lot of interesting questions arise, such as uniqueness, regularity of the involved functions, a description of the contact set, the regularity of the contact set, etc. See [Caffarelli et al. 2017; 2018; Silvestre 2005] and the dissertation [Vivas 2018].

Our main goal in this paper is to analyze games whose value functions approximate solutions to the two membranes problem with two different normalized p-Laplacians (these are fully nonlinear nonvariational equations; see below).

1.1. ***The normalized p-Laplacian and game theory.*** To begin, we introduce the normalized p-Laplacian and give the relation between this operator and the game called tug-of-war with noise in the literature; we refer to [Manfredi et al. 2012b] and the recent books [Blanc and Rossi 2019] and [Lewicka 2020] for details. This kind of game has been extensively studied.

Consider the classical p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $2 \le p < \infty$. Expanding the divergence we can (formally) write this operator as a combination of the Laplacian operator $\Delta u = \sum_{n=1}^{N} u_{x_n x_n}$ and the 1-homogeneous infinity Laplacian $\Delta_\infty^1 u = \left\langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right\rangle = |\nabla u|^{-2} \sum_{1\le i,j\le N} u_{x_i} u_{x_i x_j} u_{x_j}$:

$$\Delta_p u = |\nabla u|^{p-2}((p-2)\Delta_\infty^1 u + \Delta u). \tag{1-1}$$

Now we want to recall the mean value formula associated to this operator obtained in [Manfredi et al. 2010] (see also [Arroyo and Llorente 2016] and [Lewicka and Manfredi 2017]). Given $0 < \alpha < 1$, let us consider $u : \Omega \to \mathbb{R}$ such that

$$u(x) = \alpha\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} u(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} u(y)\Big) + (1-\alpha) \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2) \tag{1-2}$$

as $\varepsilon \to 0$. It turns out that u satisfies this asymptotic mean value formula if and only if u is a viscosity solution to $\Delta_p u = 0$; see [Manfredi et al. 2010]. For general references on mean value formulas for solutions to nonlinear PDEs, we refer to [Arroyo and Llorente 2016; Blanc et al. 2021; Ishiwata et al. 2017; Kawohl et al. 2012; Lewicka and Manfredi 2017]. In fact, if we assume that u is smooth, using a simple Taylor expansion we have

$$\fint_{B_\varepsilon(x)} u(y)\,dy - u(x) = \frac{\varepsilon^2}{2(N+2)}\Delta u(x) + o(\varepsilon^2), \tag{1-3}$$

and if $|\nabla u(x)| \neq 0$, again using a simple Taylor expansion, we obtain

$$\left(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} u(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} u(y)\right) - u(x) \\ \sim \left(\frac{1}{2}u\left(x+\varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) + \frac{1}{2}u\left(x-\varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right)\right) = \frac{1}{2}\varepsilon^2\Delta^1_\infty u(x) + o(\varepsilon^2). \quad (1\text{-}4)$$

Then, if we come back to (1-2), divide by ε^2, and take $\varepsilon \to 0$, we get

$$0 = \frac{\alpha}{2}\Delta^1_\infty u(x) + \frac{(1-\alpha)}{2(N+2)}\Delta u(x).$$

Thus, from (1-1), we get that the function u is a solution to the equation

$$-\Delta_p u(x) = 0$$

for $p > 2$ such that

$$\frac{\alpha}{1-\alpha} = \frac{p-2}{N+2}.$$

These computations can be made rigorous using viscosity theory; we refer to [Manfredi et al. 2010].

Now, let us introduce the normalized p-Laplacian (also called the game p-Laplacian).

Definition 1. Given $p > 2$, let the normalized p-Laplacian be defined as

$$\Delta^1_p u(x) = \frac{\alpha}{2}\Delta^1_\infty u(x) + \frac{1-\alpha}{2(N+2)}\Delta u(x),$$

with

$$\frac{\alpha}{1-\alpha} = \frac{p-2}{N+2}.$$

Note that this is a nonlinear elliptic 1-homogeneous operator that is a linear combination between the classical Laplacian and the ∞-Laplacian.

There is a game theoretical approximation to these operators. The connection between the Laplacian and the Brownian motion or with the limit of random walks as the step size goes to zero is well known; see [Kac 1947; Kakutani 1944; Knapp 1965]. For a probabilistic approximation of the infinity Laplacian there is a game (called tug-of-war in the literature) that was introduced in [Peres et al. 2009]. In [Peres and Sheffield 2008] — see also [Manfredi et al. 2012a; 2012b] — the authors introduce a two-player zero-sum game called tug-of-war with noise that is related to the normalized p-Laplacian. This is a two-player zero-sum game (two players, Player I and Player II, play one against the other and the total earnings of one player are exactly the total losses of the other). The rules of the game are as follows: In a bounded smooth domain Ω (what we need here is that $\partial\Omega$ satisfies an exterior sphere condition) given an initial position $x \in \Omega$, with probability α, Player I and Player II play tug-of-war (the players toss a fair coin and the winner chooses the next position of the token in $B_\varepsilon(x)$), and with probability $(1-\alpha)$, they move at random (the next position of the token is chosen at random in $B_\varepsilon(x)$). They continue playing with these rules until the game position leaves the domain Ω. At this stopping time, Player II pays Player I the amount determined by a payoff function defined outside Ω. The value of the game (defined as the best value that both players may expect to obtain) satisfies a mean value formula, called the dynamic programming principle (DPP), which in this case is given by

$$u^\varepsilon(x) = \alpha\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} u^\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} u^\varepsilon(y)\Big) + (1-\alpha)\fint_{B_\varepsilon(x)} u^\varepsilon(y)\,dy.$$

This is exactly the same formula as (1-2) but without the error term $o(\varepsilon^2)$. Notice that the value function of this game depends on ε, the parameter that controls the size of the possible movements. Note that this equation can be written as

$$0 = \alpha\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} u^\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} u^\varepsilon(y) - u^\varepsilon(x)\Big) + (1-\alpha)\fint_{B_\varepsilon(x)} (u^\varepsilon(y) - u^\varepsilon(x))\,dy.$$

Using as a main tool the asymptotic formulas (1-3) and (1-4), in [Blanc et al. 2017] and [Manfredi et al. 2012b] the authors show that there is a uniform limit as $\varepsilon \to 0$,

$$u^\varepsilon \rightrightarrows u,$$

and that this limit u is the unique solution (in a viscosity sense) to the Dirichlet problem

$$\begin{cases} -\Delta_p^1 u(x) = 0, & x \in \Omega, \\ u(x) = F(x), & x \in \partial\Omega. \end{cases}$$

When one wants to deal with a nonhomogeneous equation like $-\Delta_p^1 u(x) = h(x)$ one can add a running payoff to the game, that is, at every play Player I pays to Player II the amount $\varepsilon^2 h(x)$.

1.2. ***The two membranes problem.*** As we already mentioned, the two membranes problem describes the equilibrium position of two elastic membranes in contact with each other that are not allowed to cross. Hence, one of the membranes acts as an obstacle (from above or below) for the other. Given two differential operators $F(x, u, \nabla u, D^2 u)$ and $G(x, v, \nabla v, D^2 v)$ the mathematical formulation of the two membranes problem is the following:

$$\begin{cases} \min\big\{F\big(x, u(x), \nabla u(x), D^2 u(x)\big), (u-v)(x)\big\} = 0, & x \in \Omega, \\ \max\big\{G\big(x, v(x), \nabla v(x), D^2 v(x)\big), (v-u)(x)\big\} = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \\ v(x) = g(x), & x \in \partial\Omega. \end{cases}$$

In general there is no uniqueness for the two membranes problem. For example, take u the solution to the first operator $F(u) = 0$ with $u|_{\partial\Omega} = f$ and v the solution to the obstacle problem from above for $G(v)$ and boundary datum g. This pair (u, v) is a solution to the two membranes problem (v is a solution to the obstacle problem with u as upper obstacle and u is a solution to the obstacle problem with v as lower obstacle; in fact u is a solution in the whole domain and is above v). Analogously, one can consider $\tilde v$ as the solution to $G(\tilde v) = 0$ with $v|_{\partial\Omega} = g$ and $\tilde u$ the solution to the obstacle problem from above for $F(\tilde u)$ and boundary datum f, to obtain a pair $(\tilde u, \tilde v)$ that is a solution to the two membranes problem. In general, it holds that $(u, v) \neq (\tilde u, \tilde v)$.

The two membranes problem for the Laplacian with a right-hand side, that is, for $F(D^2u) = -\Delta u + h_1$ and $F(D^2v) = -\Delta v - h_2$, was first considered in [Vergara Caffarelli 1971] using variational arguments. Later, in [Caffarelli et al. 2017] the authors solve the two membranes problem for two different fractional Laplacians of different order (two linear nonlocal operators defined by two different kernels). Notice that in this case the problem is still variational. In these cases an extra condition appears, namely, the sum of the two operators vanishes,

$$G(u) + F(v) = 0, \tag{1-5}$$

inside Ω. Moreover, this extra condition together with the variational structure is used to prove a $C^{1,\gamma}$ regularity result for the solution.

The two membranes problem for a nonlinear operator was studied in [Caffarelli et al. 2017; 2018; Silvestre 2005]. In particular, in [Caffarelli et al. 2018] the authors consider a version of the two membranes problem for two different fully nonlinear operators, $F(D^2u)$ and $G(D^2u)$. Assuming that F is convex and that

$$G(X) = -F(-X), \tag{1-6}$$

they prove that solutions are $C^{1,1}$ smooth.

We also mention that a more general version of the two membranes problem involving more than two membranes was considered by several authors (see, for example, [Azevedo et al. 2005; Carillo et al. 2005; Chipot and Vergara-Caffarelli 1985]).

1.3. *Description of the main results.* In this paper we use the previously described tug-of-war with noise game to obtain games whose value functions approximate solutions (in a viscosity sense) to a system with two obstacle-type equations (a two membrane problem).

1.3.1. *First game.* Let us describe the first game that we are going to study in more detail. Again, it is a two-player zero-sum game. The game is played in two boards, which we call board 1 and board 2, that are two copies of a fixed smooth bounded domain $\Omega \subset \mathbb{R}^N$. We fix two final payoff functions $f, g : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ that are uniformly Lipschitz functions with $f \geq g$, and two running payoff functions $h_1, h_2 : \Omega \to \mathbb{R}$ (we also assume that they are uniformly Lipschitz functions), corresponding to the first and second board respectively. Take a positive parameter ε. Let us use two games with different rules for the first and second board respectively associated to two different p-Laplacian operators. To this end, let us fix two numbers $0 < \alpha_i < 1$ for $i = 1, 2$. Playing in the first board the rules are the following: with α_1 probability we play with tug-of-war rules, which means a fair coin is tossed and the player who wins the coin toss chooses the next position inside the ball $B_\varepsilon(x)$, and with $(1 - \alpha_1)$ probability we play with a random walk rule, meaning that the next position is chosen at random in $B_\varepsilon(x)$ with uniform probability. Playing in the first board we add a running payoff of amount $-\varepsilon^2 h_1(x_0)$ (Player I gets $-\varepsilon^2 h_1(x_0)$ and Player II $\varepsilon^2 h_1(x_0)$). We call this game the J_1 game. Analogously, in the second board we use α_2 to encode the probability that we play tug-of-war and $(1 - \alpha_2)$ for the probability to move at random, this time with a running payoff of amount $\varepsilon^2 h_2(x_0)$. We call this game J_2.

To the rules that we described in the two boards J_1 and J_2 we add the following ways of changing boards: in the first board, Player I decides to play with J_1 rules (and the game position remains at the first board) or to change boards and the new position of the token is chosen playing the J_2 game rule in the second board. In the second board the rule is just the opposite: in this case, Player II decides to play with J_2 game rules (and remains at the second board) or to change boards and play in the first board with the J_1 game rules.

The game starts with a token at an initial position $x_0 \in \Omega$ in one of the two boards. After the first play the game continues with the same rules (each player decides to change or continue in one board plus the rules for the two different tug-of-war with noise game at each board) until the tokes leaves the domain Ω (at this time the game ends). This gives a random sequence of points (positions of the token) and a stopping time τ (the first time that the position of the token is outside Ω in any of the two boards). The sequence of positions will be denoted by

$$\{(x_0, j_0), (x_1, j_1), \dots, (x_\tau, j_\tau)\},$$

where $x_k \in \Omega$ (and $x_\tau \notin \Omega$) and the second variable, $j_k \in \{1, 2\}$, is just an index that indicates in which board we are playing, with $j_k = 1$ if the position of the token is in the first board and $j_k = 2$ if we are in the second board. As we mentioned, the game ends when the token leaves Ω at some point (x_τ, j_τ). In this case the final payoff (the amount that Player I gets and Player II pays) is given by $f(x_\tau)$ if $j_\tau = 1$ (the token leaves the domain in the first board) and $g(x_\tau)$ if $j_\tau = 2$ (the token leaves in the second board). Hence, taking into account the running payoff and the final payoff, the total payoff of a particular occurrence of the game is given by

$$\text{total payoff} := f(x_\tau)\chi_{\{1\}}(j_\tau) + g(x_\tau)\chi_{\{2\}}(j_\tau) - \varepsilon^2 \sum_{k=0}^{\tau-1} \big(h_1(x_k)\chi_{\{1\}}(j_{k+1}) - h_2(x_k)\chi_{\{j=2\}}(j_{k+1})\big).$$

Notice that the total payoff is the sum of the final payoff (given by $f(x_\tau)$ or by $g(x_\tau)$ according to the board at which the position leaves the domain) and the running payoff that is given by $-\varepsilon^2 h_1(x_k)$ and $\varepsilon^2 h_2(x_k)$ corresponding to the board in which we play at step $k+1$.

Now, the players fix two strategies, S_{I} for Player I and S_{II} for Player II. That is, both players decide to play or to change boards in the respective board, and in each board they select the point to go provided the coin toss of the tug-of-war game is favorable. Notice that the decision on the board where the game takes place is made by the players at each turn (according to the board at which the position is, one of the players makes the choice). Therefore, when the strategies of both players are fixed, the board in which the game occurs at each turn is given (and it is not random). Then, once we fix the strategies S_{I} and S_{II}, everything depends only on the underlying probability: the coin toss that decides when to play tug-of-war and when to move at random (note that this probability is given by α_1 or α_2 and it is different in the two boards) and the coin toss $\big($with probability $\frac{1}{2}$-$\frac{1}{2}\big)$ that decides who chooses the next position of the game if the tug-of-war game is played. With respect to this underlying probability, with fixed strategies S_{I} and S_{II}, we can compute the expected final payoff starting at (x, j)

(recall that $j=1,2$ indicates the board at which is the position of the game),

$$\mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x,j)}[\text{total payoff}].$$

The game is said to have a value if

$$w^{\varepsilon}(x,j)=\sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}}\mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x,j)}[\text{total payoff}]=\inf_{S_{\mathrm{II}}}\sup_{S_{\mathrm{I}}}\mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x,j)}[\text{total payoff}]. \tag{1-7}$$

Notice that this value w^{ε} is the best possible expected outcome that Player I and Player II may expect to obtain playing their best. Here we will prove that this game has a value. The value of the game, w^{ε}, is composed in fact of two functions, the first one defined in the first board,

$$u^{\varepsilon}(x):=w^{\varepsilon}(x,1),$$

which is the expected outcome of the game if the initial position is at the first board (and the players play their best) and

$$v^{\varepsilon}(x):=w^{\varepsilon}(x,2),$$

which is the expected outcome of the game when the initial position is in the second board. It turns out that these two functions u^{ε}, v^{ε} satisfy a system of equations that is called the dynamic programming principle (DPP) in the literature. In our case, the corresponding DPP for the game is given by

$$\begin{cases} u^{\varepsilon}(x)=\max\left\{J_1(u^{\varepsilon})(x),\,J_2(v^{\varepsilon})(x)\right\}, & x\in\Omega,\\ v^{\varepsilon}(x)=\min\left\{J_1(u^{\varepsilon})(x),\,J_2(v^{\varepsilon})(x)\right\}, & x\in\Omega,\\ u^{\varepsilon}(x)=f(x), & x\in\mathbb{R}^N\setminus\Omega,\\ v^{\varepsilon}(x)=g(x), & x\in\mathbb{R}^N\setminus\Omega, \end{cases} \tag{1-8}$$

where

$$J_1(w)(x)=\alpha_1\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)}w(y)+\frac{1}{2}\inf_{y\in B_\varepsilon(x)}w(y)\Big)+(1-\alpha_1)\fint_{B_\varepsilon(x)}w(y)\,dy-\varepsilon^2h_1(x)$$

and

$$J_2(w)(x)=\alpha_2\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)}w(y)+\frac{1}{2}\inf_{y\in B_\varepsilon(x)}w(y)\Big)+(1-\alpha_2)\fint_{B_\varepsilon(x)}w(y)\,dy+\varepsilon^2h_2(x).$$

Remark 2. From the DPP and the condition $f\geq g$ it is clear that the value functions of the game are ordered. We have

$$u^{\varepsilon}(y)\geq v^{\varepsilon}(y)$$

for all $y\in\mathbb{R}^N$.

Remark 3. Observe that the DPP reflects the rules for the game described above. That is, the J_1 rule says that with α_1 probability we play with tug-of-war game and with $(1-\alpha_1)$ probability we play the random walk game with a running payoff that involves h_1. Analogously, in the J_2 game the probability is given by α_2 and the running payoff involves h_2. Also notice that the max and min that are arise in the DPP correspond to the choices of the players to change board (or not). In the first board the first player (who aims to maximize the expected outcome) is the one who decides while in the second board the second player (who wants to minimize) decides.

Our first result says that the value functions of the game converge uniformly as $\varepsilon \to 0$ to a pair of continuous functions (u, v) that is a viscosity solution to a system of partial differential equations in which two equations of obstacle type appear.

Theorem 4. *There exists a sequence $\varepsilon_j \to 0$ such that $(u^{\varepsilon_j}, v^{\varepsilon_j})$ converges to a pair of continuous functions (u, v), that is,*

$$u^{\varepsilon_j} \rightrightarrows u, \quad v^{\varepsilon_j} \rightrightarrows v$$

uniformly in $\overline{\Omega}$. The limit pair is a viscosity solution to the two membrane system with two different p-Laplacians, that is,

$$\begin{cases} u(x) \geq v(x), & & x \in \Omega, \\ -\Delta_p^1 u(x) + h_1(x) \geq 0, & -\Delta_q^1 v(x) - h_2(x) \leq 0, & x \in \Omega, \\ -\Delta_p^1 u(x) + h_1(x) = 0, & -\Delta_q^1 v(x) - h_2(x) = 0, & x \in \{u > v\} \cap \Omega, \\ u(x) = f(x), & & x \in \partial\Omega, \\ v(x) = g(x), & & x \in \partial\Omega. \end{cases} \tag{1-9}$$

Here p and q are given by

$$\frac{\alpha_1}{1-\alpha_1} = \frac{p-2}{N+2} \quad \text{and} \quad \frac{\alpha_2}{1-\alpha_2} = \frac{q-2}{N+2}. \tag{1-10}$$

Remark 5. Using that $u^{\varepsilon} \rightrightarrows u$, $v^{\varepsilon} \rightrightarrows v$ and that $u^{\varepsilon} \geq v^{\varepsilon}$ we immediately obtain

$$u(y) \geq v(y) \quad \text{for all } \ y \in \mathbb{R}^N.$$

Remark 6. We can write the system (1-9) as

$$\begin{cases} \min\big\{-\Delta_p^1 u(x) + h_1(x), (u-v)(x)\big\} = 0, & x \in \Omega, \\ \max\big\{-\Delta_q^1 v(x) - h_2(x), (v-u)(x)\big\} = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \\ v(x) = g(x), & x \in \partial\Omega. \end{cases}$$

Here the first equation says that u is a solution to the obstacle problem for the p-Laplacian with v as a obstacle and boundary datum f, and the second equation says that v is a solution to the obstacle problem for the q-Laplacian with u as a obstacle from above and boundary datum g.

This formulation corresponds to a two membrane problem in which the membranes are clamped on the boundary of the domain and each membrane acts as an obstacle for the other.

Remark 7. Since in general there is no uniqueness for the two membranes problem we can only show convergence taking a sequence $\varepsilon_j \to 0$ using a compactness argument.

Let us briefly comment on the main difficulties that appear in the proof of this result. To show that the DPP has a solution we argue using monotonicity arguments in the spirit of Perron's method (a solution is obtained as the supremum of subsolutions). Once we proved existence of a solution to the DPP we use this solution to construct quasioptimal strategies for the players and show that the game has a value that coincides with a solution to the DPP (this fact implies uniqueness for solutions to the DPP). At this point we want to mention the cruciality of the rule that forces one round of play of the game when one of the players decides to change boards. If one changes boards without playing a round in the other board the game may never end (and even if we penalize games that never end it is not clear that the game has a value). See [Peres et al. 2009] for an example of a tug-of-war game that does not have a value. After proving existence and uniqueness for the DPP and the existence of a value for the game we study its behavior as $\varepsilon \to 0$. Uniform convergence will follow from a variant of the Arzelà–Ascoli lemma; see Lemma 20 (this idea was used before to obtain convergence of value functions of games; see several examples in [Blanc and Rossi 2019]). To this end we need that when the game starts close to the boundary in any of the two boards any of the two players has a strategy that forces the game to end close to the starting point in a controlled number of plays with large probability. For example, starting in the first board the first player may choose the strategy to never change boards and to point to a boundary point when the tug-of-war game is played. One can show that this strategy gives the desired one-sided estimate. However, starting in the first board, to find a strategy for Player II that achieves similar bounds is trickier since the player who may decide to change boards is Player I. To obtain such bounds for the terminal position and the expected number of plays in this case is one of the main difficulties that we deal with. Once we proved uniform convergence of the value functions we use the DPP to obtain, using the usual viscosity approach, that the limit pair is a solution to the two membranes problem.

1.3.2. *Second game.* Let us consider a variant of the previous game in which the possibility of the players to change boards also depends on a coin toss.

This new game has the following rules: If the position of the game is at $(x_k, 1)$ the players toss a fair coin $\left(\text{probability } \frac{1}{2}\text{-}\frac{1}{2}\right)$, and if Player I wins, he decides to play the J_1 game in the first board or to play the J_2 game in the second board. On the other hand, if the winner is Player II the only option is to play J_1 in the first board. If the position is in the second board, say at $(x_k, 2)$, the situation is analogous but with the roles of the players reversed: the players toss a fair coin again, and if Player II wins, she decides between playing J_2 in the second board or jumping to the first board and playing J_1, while if Player I wins the only option is to play J_2 in the second board. Here the rules of J_1 and J_2 are exactly as before, the only thing that we changed is that the decision to change boards or not is also dependent on a fair coin toss.

This game has associated to it the following DPP:

$$\begin{cases} u^\varepsilon(x) = \frac{1}{2}\max\left\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\right\} + \frac{1}{2}J_1(u^\varepsilon)(x), & x \in \Omega, \\ v^\varepsilon(x) = \frac{1}{2}\min\left\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\right\} + \frac{1}{2}J_2(v^\varepsilon)(x), & x \in \Omega, \\ u^\varepsilon(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^\varepsilon(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

This DPP also reflects the rules of the game. For instance, the first equation says that with probability $\frac{1}{2}$ the first player decides to play J_1 or to change boards and play J_2 (thus the term $\max\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\}$ appears) and with probability $\frac{1}{2}$ the position stays in the first board (they just play J_1).

Remark 8. Also in this case, from the DPP and the condition $f \geq g$ it is immediate that

$$u^{\varepsilon}(y) \geq v^{\varepsilon}(y)$$

holds for all $y \in \mathbb{R}^N$.

In this case the pair $(u^{\varepsilon}, v^{\varepsilon})$ also converges uniformly along a subsequence $\varepsilon_j \to 0$ to a continuous pair (u, v), and this limit pair is also a viscosity solution to the two membrane problem with an extra condition in the contact set.

Theorem 9. *There exists a sequence* $\varepsilon_j \to 0$ *such that* $(u^{\varepsilon_j}, v^{\varepsilon_j})$ *converges to a pair of continuous functions* (u, v), *that is*,

$$u^{\varepsilon_j} \rightrightarrows u, \quad v^{\varepsilon_j} \rightrightarrows v$$

uniformly in $\overline{\Omega}$. *The limit pair is a viscosity solution to the two membrane system with the two different* p*-Laplacians* (1-9), *with* p *and* q *given by* (1-10). *Moreover, the extra condition*

$$(-\Delta_p^1 u(x) + h_1(x)) + (-\Delta_q^1 v(x) - h_2(x)) = 0, \quad x \in \Omega, \tag{1-11}$$

holds.

Remark 10. Let us observe that the extra condition (1-11) trivially holds in $\{u > v\}$ (since u and v solve $-\Delta_p^1 u(x) + h_1(x) = 0$ and $-\Delta_q^1 v(x) - h_2(x) = 0$ respectively). Then this extra condition gives us new information in the contact set $\{u = v\}$. Notice that this extra condition (the sum of the two equations is equal to zero) is similar to the one that appears in [Caffarelli et al. 2017] — compare (1-5) — but it is not the same as the one assumed in [Caffarelli et al. 2018] to obtain regularity of the solutions — see (1-6) — since for the normalized p-Laplacian it is not true that $-\Delta_p^1 u(x) = \Delta_q^1(-u)(x)$ unless $p = q$.

The paper is organized as follows: In Section 2 we analyze the first game; in Section 2.1 we prove that the game has a value and that this value is the unique solution to the DPP. The proof of Theorem 4 is divided across Sections 2.2 and 2.3. In the first one we prove uniform convergence along a subsequence and in the second we show that the uniform limit is a viscosity solution to the PDE system (1-9).

In Section 3 we include a brief description of the analysis for the second game (the arguments used to show uniform convergence are quite similar). Here we focus on the details needed to show that we obtain an extra condition in the contact set.

Finally, in Section 4 we include some remarks and comment on possible extensions of our results.

2. Analysis for the first game

2.1. ***Existence and uniqueness for the DPP.*** In this section we first prove that there is a solution to the DPP (1-8), next we show that the existence of a solution to the DPP implies that the game has a value (it allows us to find quasioptimal strategies for the players), and at the end we obtain the uniqueness of solutions to the DPP.

To show existence of a solution to the DPP we use a variant of Perron's method (that is, a solution can be obtained as supremum of subsolutions).

Let us consider the set of functions

$$\mathscr{A} = \{(\underline{u}^{\varepsilon}, \underline{v}^{\varepsilon}) : \underline{u}^{\varepsilon} \text{ and } \underline{v}^{\varepsilon} \text{are bounded functions such that (2-1) holds}\}$$

with

$$\begin{cases} u^{\varepsilon}(x) \le \max\big\{J_1(u^{\varepsilon})(x), J_2(v^{\varepsilon})(x)\big\}, & x \in \Omega, \\ v^{\varepsilon}(x) \le \min\big\{J_1(u^{\varepsilon})(x), J_2(v^{\varepsilon})(x)\big\}, & x \in \Omega, \\ u^{\varepsilon}(x) \le f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^{\varepsilon}(x) \le g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2-1}$$

Notice that (2-1) is just the DPP with inequalities that say that $(\underline{u}^{\varepsilon}, \underline{v}^{\varepsilon})$ is a subsolution to the DPP (1-8). For the precise definition of sub- and supersolutions to DPP systems we refer to [Miranda and Rossi 2020; 2023].

Let us begin proving that $\mathscr{A}$ is nonempty. To this end we introduce an auxiliary function. As $\Omega \subset \mathbb{R}^N$ is bounded there exists $R > 0$ such that $\Omega \Subset B_R(0) \setminus \{0\}$ (without loss of generality we may assume that $0 \notin \Omega$). Consider the function

$$z_0(x) = \begin{cases} 2K(|x|^2 - R) - M & \text{if } x \in \overline{B_R(0)}, \\ \quad -M & \text{if } x \in \mathbb{R}^N \setminus B_R(0). \end{cases}$$

This function has the following properties: the function z_0 is $C^2(\Omega)$, and for $x \in B_R(0) \setminus \{0\}$,

$$\Delta z_0(x) = (z_0)_{rr} + \frac{N-1}{r}(z_0)_r = 4K + \frac{N-1}{r} 4Kr = 4K + 4K(N-1) \quad \text{and} \quad \Delta^1_{\infty} z_0(x) = (z_0)_{rr}(x) = 4K.$$

Notice that when we compute the infinity Laplace operator of z_0 we have to pay special care at the origin (where the gradient of z_0 vanishes), since the operator is not well defined there. In doing this we use that z_0 is a radial function and compute the infinity Laplacian in the classical sense at points in $B_R(0) \setminus \{0\}$ (where the gradient does not vanish).

Then we get

$$\Delta^1_p z_0(x) = \frac{1}{2}\alpha_1(4K) + \frac{(1-\alpha_1)}{2(N+2)}(4K + 4K(N-1)) \ge 4K,$$
$$\Delta^1_q z_0(x) = \frac{1}{2}\alpha_2(4K) + \frac{(1-\alpha_2)}{2(N+2)}(4K + 4K(N-1)) \ge 4K.$$

We are ready to prove the first lemma.

Lemma 11. *For ε small enough, $\mathscr{A} \neq \varnothing$.*

Proof. We consider z_0 with the constants

$$K = \max\{\|h_1\|_\infty, \|h_2\|_\infty\} + 1 \quad \text{and} \quad M = \max\{\|f\|_\infty, \|g\|_\infty\} + 1,$$

and claim that

$$(z_0, z_0) \in \mathscr{A}.$$

Let us prove this claim. First, we observe that the inequality (2-1) holds for $x \in \mathbb{R}^N \setminus \Omega$. Then we are left to prove that for $x \in \Omega$,

$$z_0(x) \le \min\{J_1(z_0)(x),\, J_2(z_0)(x)\}.$$

That is, we aim to show that

$$0 \le \min\{J_1(z_0)(x) - z_0(x),\, J_2(z_0)(x) - z_0(x)\}. \tag{2-2}$$

Using Taylor expansions we obtain

$$\begin{aligned} J_1(z_0)(x) - z_0(x) &= \alpha_1\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x)}(z_0(y)-z_0(x)) + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x)}(z_0(y)-z_0(x))\Big) \\ &\qquad\qquad + (1-\alpha_1)\fint_{B_\varepsilon(x)}(z_0(y)-z_0(x))\,dy - \varepsilon^2 h_1(x) \\ &= \Big(\frac{1}{2}\alpha_1\Delta_\infty^1 z_0(x) + \frac{(1-\alpha_1)}{2(N+2)}\Delta z_0(x)\Big)\varepsilon^2 - \varepsilon^2 h_1(x) + o(\varepsilon^2). \end{aligned}$$

Analogously,

$$J_2(z_0)(x) - z_0(x) = \Big(\frac{1}{2}\alpha_2\Delta_\infty^1 z_0(x) + \frac{(1-\alpha_2)}{2(N+2)}\Delta z_0(x)\Big)\varepsilon^2 + \varepsilon^2 h_2(x) + o(\varepsilon^2).$$

If we come back to (2-2) and we divide by ε^2 we obtain

$$0 \le \min\{\Delta_p^1 z_0(x) - h_1(x),\, \Delta_q^1 z_0(x) + h_2(x)\}. \tag{2-3}$$

Using the properties of z_0 we have

$$\Delta_p^1 z_0(x) - h_1(x) \ge 3K \quad\text{and}\quad \Delta_q^1 z_0(x) + h_2(x) \ge 3K.$$

Thus, the inequality (2-3) holds for ε small enough. This ends the proof. □

Remark 12. We can define a different auxiliary function z^ε as the solution to the following problem:

$$\begin{cases} z^\varepsilon(x) = \displaystyle\min_{i\in\{1,2\}}\Big\{\alpha_i\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} z^\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} z^\varepsilon(y)\Big) + (1-\alpha_i)\fint_{B_\varepsilon(x)} z^\varepsilon(y)\,dy\Big\} - \varepsilon^2 K, & x\in\Omega, \\ z^\varepsilon(x) = -M, & x\notin\Omega. \end{cases} \tag{2-4}$$

The existence of this function is given in [Blanc et al. 2017, Theorem 1.5]. In fact, (2-4) is the DPP that corresponds to a game in which one player (the one that wants to minimize the expected payoff) chooses the coin that decides the game to play between tug-of-war and a random walk.

If we argue as before we can prove that $(z^\varepsilon, z^\varepsilon) \in \mathscr{A}$.

Now our goal is to show that the functions $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ are uniformly bounded. To prove this fact we will need some lemmas. Let us consider the function $w_0 = -z_0$. This function has the following properties:

$$\Delta w_0(x) = -\Delta z_0 = -4K - 4K(N-1) \quad\text{and}\quad \Delta_\infty^1 w_0(x) = -\Delta_\infty^1 w_0(x) = -4K.$$

Then we have

$$\Delta_p^1 w_0(x) = \frac{1}{2}\alpha_1(-4K) + \frac{(1-\alpha_1)}{2(N+2)}(-4K - 4K(N-1)) \le -4K,$$
$$\Delta_q^1 w_0(x) = \frac{1}{2}\alpha_2(-4K) + \frac{(1-\alpha_2)}{2(N+2)}(-4K - 4K(N-1)) \le -4K.$$

Let us prove a technical lemma.

Lemma 13. *Given* $K = \max\{\|h_1\|_\infty, \|h_1\|_\infty\} + 1$ *and* $M = \max\{\|f\|_\infty, \|g\|_\infty\} + 1$, *there exists* $\varepsilon_0 > 0$ *such that the function* w_0 *satisfies*

$$\begin{cases} w_0(x) \ge \max\big\{J_1(w_0)(x),\, J_2(w_0)(x)\big\} + K\varepsilon^2, & x \in \Omega, \\ w_0(x) \ge M, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2-5}$$

for every $\varepsilon < \varepsilon_0$.

Proof. First, let us observe that the inequality $w^\varepsilon(x) \ge M$ holds for $x \in \mathbb{R}^N \setminus \Omega$ when $\widetilde{M}$ is large enough. Then we are left to prove that for $x \in \Omega$,

$$w_0(x) \ge \max\big\{J_1(w_0)(x),\, J_2(w_0)(x)\big\} + K\varepsilon^2.$$

That is,

$$0 \ge \max\big\{J_1(w_0)(x) - w_0(x),\, J_2(w_0)(x) - w_0(x)\big\} + K\varepsilon^2. \tag{2-6}$$

Using Taylor expansions we obtain

$$\begin{aligned} J_1(w_0)(x) - w_0(x) &= \alpha_1\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x)}(w_0(y) - w_0(x)) + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x)}(w_0(y) - w_0(x))\Big) \\ &\qquad + (1-\alpha_1) \fint_{B_\varepsilon(x)} (w_0(y) - w_0(x))\,dy + \varepsilon^2 h_1(x) \\ &= \Big(\frac{\alpha_1}{2}\Delta_\infty^1 w_0(x) + \frac{(1-\alpha_1)}{2(N+2)}\Delta w_0(x)\Big)\varepsilon^2 - \varepsilon^2 h_1(x) + o(\varepsilon^2). \end{aligned}$$

Analogously,

$$J_2(w_0)(x) - w_0(x) = \Big(\frac{\alpha_2}{2}\Delta_\infty^1 w_0(x) + \frac{(1-\alpha_2)}{2(N+2)}\Delta w_0(x)\Big)\varepsilon^2 - \varepsilon^2 h_2(x) + o(\varepsilon^2).$$

If we come back to (2-6) and we divide by ε^2 we get

$$0 \ge \max\big\{\Delta_p^1 w_0(x) - h_1(x),\, \Delta_q^1 w_0(x) + h_2(x)\big\} + K. \tag{2-7}$$

Using the properties of w_0 we arrive at

$$\Delta_p^1 w_0(x) - h_1(x) \le -4K \quad \text{and} \quad \Delta_q^1 w_0(x) + h_2(x) \le -3K.$$

Thus, the inequality (2-7) holds for ε small enough. This ends the proof. □

Our next result says that the subsolutions to the DPP (pairs $(\underline{u}, \underline{v}) \in \mathscr{A}$) are indeed bounded by w_0. This shows that functions in $\mathscr{A}$ are uniformly bounded. From the proof of the following result one can obtain a comparison principle for the DPP.

Lemma 14. *Let* $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ (bounded subsolutions to the DPP (1-8)), *and let* w^ε *be a function that satisfies* (2-5), *that is,*

$$\begin{cases} w^\varepsilon(x) \geq \max\big\{J_1(w^\varepsilon)(x), J_2(w^\varepsilon)(x)\big\} + K\varepsilon^2, & x \in \Omega, \\ w^\varepsilon(x) \geq M, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then it holds that

$$\underline{u}^\varepsilon(x) \leq w^\varepsilon(x) \quad \textit{and} \quad \underline{v}^\varepsilon(x) \leq w^\varepsilon(x), \qquad x \in \mathbb{R}^N.$$

Proof. We argue by contradiction. Assume that

$$\max\big\{\sup(\underline{u}^\varepsilon - w^\varepsilon), \sup(\underline{v}^\varepsilon - w^\varepsilon)\big\} = \theta > 0.$$

It is clear that

$$\underline{u}^\varepsilon(x) \leq M \leq w^\varepsilon(x) \quad \text{and} \quad \underline{v}^\varepsilon(x) \leq M \leq w^\varepsilon(x)$$

for $x \notin \Omega$. Thus, we have to concentrate on what happens inside Ω. We divide the proof into two cases.

<u>Case 1</u>: Assume that

$$\sup(\underline{v}^\varepsilon - w^\varepsilon) = \theta.$$

Given $n \in \mathbb{N}$ let $x_n \in \Omega$ be such that

$$\theta - \frac{1}{n} < (\underline{v}^\varepsilon - w^\varepsilon)(x_n).$$

We use the inequalities satisfied by the involved functions to obtain

$$\begin{aligned} \theta - \frac{1}{n} &< (\underline{v}^\varepsilon - w^\varepsilon)(x_n) \\ &\leq J_2(\underline{v}^\varepsilon)(x_n) - J_2(w^\varepsilon)(x_n) \\ &= \alpha_2\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) - \frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} w^\varepsilon(y) - \frac{1}{2}\inf_{y\in B_\varepsilon(x_n)} w^\varepsilon(y)\Big) \\ &\qquad + (1-\alpha_2) \fint_{B_\varepsilon(x_n)} (\underline{v}^\varepsilon - w^\varepsilon)(y)\,dy + \varepsilon^2 h_2(x_n) - \varepsilon^2 K \\ &\leq \alpha_2\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) - \frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} w^\varepsilon(y) + \frac{1}{2}\sup_{y\in B_\varepsilon(x_n)} (\underline{v}^\varepsilon - w^\varepsilon)(y)\Big) \\ &\qquad + (1-\alpha_2) \fint_{B_\varepsilon(x_n)} (\underline{v}^\varepsilon - w^\varepsilon)(y)\,dy + \varepsilon^2 h_2(x_n) - \varepsilon^2 K. \end{aligned}$$

Now we use that

$$\sup_{y\in B_\varepsilon(x_n)} (\underline{v}^\varepsilon - w^\varepsilon)(y) \leq \theta \quad \text{and} \quad \fint_{B_\varepsilon(x_n)} (\underline{v} - w)(y)\,dy \leq \theta$$

to arrive at

$$\theta - \frac{2}{n\alpha_2} < \sup_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) - \sup_{y\in B_\varepsilon(x_n)} w^\varepsilon(y) + (2h_2(x_n) - 2K)\frac{\varepsilon^2}{\alpha_2}.$$

Take $y_n \in B_\varepsilon(x_n)$ such that

$$\sup_{y\in B_\varepsilon(x_n)} \underline{v}^\varepsilon(y) - \frac{1}{n} < \underline{v}^\varepsilon(y_n).$$

Then we get

$$\theta - \frac{2}{n\alpha_2} < \underline{v}^\varepsilon(y_n) + \frac{1}{n} - \sup_{y\in B_\varepsilon(x_\varepsilon)} w^\varepsilon(y)(2h_2(x_n) - 2K\varepsilon^2) \le \underline{v}^\varepsilon(y_n) + \frac{1}{n} - w^\varepsilon(y_n) - \frac{\varepsilon^2}{\alpha_2}.$$

Here we use that $h_2(x) - K \le -1$. Hence

$$\theta - \frac{2-\alpha_2}{n\alpha_2} + \frac{\varepsilon^2}{\alpha_2} < (\underline{v}^\varepsilon - w^\varepsilon)(y_n) \le \theta,$$

which leads to a contradiction if $n \in \mathbb{N}$ is large enough that

$$-\frac{2-\alpha_2}{n\alpha_2} + \frac{\varepsilon^2}{\alpha_2} > 0,$$

since in this case we obtain

$$\theta < \theta - \frac{2-\alpha_2}{n\alpha_2} + \frac{\varepsilon^2}{\alpha_2} < (\underline{u}^\varepsilon - w^\varepsilon)(y_n) \le \theta.$$

This ends the proof in the first case.

Case 2: Assume that

$$\sup(\underline{u}^\varepsilon - w^\varepsilon) = \theta.$$

In this case we take again a sequence $x_n \in \Omega$ such that

$$\theta - \frac{1}{n} < (\underline{u}^\varepsilon - w^\varepsilon)(x_n).$$

Let us assume first that

$$\max\left\{J_1(\underline{u}^\varepsilon)(x_n), J_2(\underline{v}^\varepsilon)(x_n)\right\} = J_2(\underline{v}^\varepsilon)(x_n),$$

and then we obtain

$$(\underline{u}^\varepsilon - w^\varepsilon)(x_n) \le J_2(\underline{v}^\varepsilon)(x_n) - J_2(w^\varepsilon)(x_n).$$

We are again in the first case and we arrive at a contradiction arguing as before.

Finally, let us assume that

$$\max\left\{J_1(\underline{u}^\varepsilon)(x_n), J_2(\underline{v}^\varepsilon)(x_n)\right\} = J_1(\underline{u}^\varepsilon)(x_n),$$

and then we obtain

$$(\underline{u}^\varepsilon - w^\varepsilon)(x_n) \le J_1(\underline{u}^\varepsilon)(x_n) - J_1(w^\varepsilon)(x_n).$$

If we argue as in the first case we arrive at a contradiction. This ends the proof. $\square$

Now, using that w_0 is continuous $\mathbb{R}^N$ and hence bounded in the ball $\overline{B}_R$, we can deduce that there exists a constant $\Lambda > 0$ that depends on the data f, g, h and the domain Ω such that $w_0(x) \le \Lambda$. Then, using the previous lemmas, we obtain a uniform bound for functions in $\mathscr{A}$.

Theorem 15. *There exists a constant $\Lambda > 0$ that depends on f, g, h and Ω such that for every $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ it holds that*

$$\underline{u}^\varepsilon(x) \le \Lambda \quad \textit{and} \quad \underline{v}^\varepsilon(x) \le \Lambda,$$

for every $x \in \mathbb{R}^N$ and every $\varepsilon \le \varepsilon_0$ (here ε_0 is given by Lemma 13).

Proof. It follows from Lemmas 13 and 14 and the boundedness of w_0. □

With this result at hand we can define for $x \in \mathbb{R}^N$,

$$u^\varepsilon(x) = \sup_{(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}} \underline{u}^\varepsilon(x) \quad \text{and} \quad v^\varepsilon(x) = \sup_{(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}} \underline{v}^\varepsilon(x).$$

The previous result, Theorem 15, gives that these two functions u^ε and v^ε are well defined and bounded. It turns out that they are a solution to the DPP.

Theorem 16. *The pair of functions $(u^\varepsilon, v^\varepsilon)$ is a solution to the DPP* (1-8).

Proof. First, let us show that $(u^\varepsilon, v^\varepsilon) \in \mathscr{A}$. Given $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ and $x \in \Omega$ we have that

$$\underline{u}^\varepsilon(x) \le \max\big\{J_1(\underline{u}^\varepsilon)(x), J_2(\underline{v}^\varepsilon)(x)\big\}.$$

Taking the supremum over $(\underline{u}^\varepsilon, \underline{v}^\varepsilon) \in \mathscr{A}$ we obtain

$$\underline{u}^\varepsilon(x) \le \max\big\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\big\},$$

and hence (taking the supremum in the left-hand side) we conclude that

$$u^\varepsilon(x) \le \max\big\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\big\}.$$

An analogous computation for the second equation shows that v^ε satisfies

$$v^\varepsilon(x) \le \min\big\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\big\}$$

for $x \in \Omega$. Finally, as $\underline{u}^\varepsilon(x) \le f(x)$ and $\underline{v}^\varepsilon(x) \le g(x)$ for $x \in \mathbb{R}^N \setminus \Omega$, taking the supremum we obtain $u^\varepsilon(x) \le f(x)$ and $v^\varepsilon(x) \le g(x)$ for $x \in \mathbb{R}^N \setminus \Omega$, and we conclude that $(u^\varepsilon, v^\varepsilon) \in \mathscr{A}$.

We have a set of inequalities for the pair $(u^\varepsilon, v^\varepsilon) \in \mathscr{A}$. To show that the pair is indeed a solution to the DPP we have to show that they are in fact equalities. To prove this fact we argue by contradiction. Assume that we have a strict inequality for some $x_0 \in \mathbb{R}^N$. If $x_0 \in \mathbb{R}^N \setminus \Omega$ and we have $u^\varepsilon(x_0) < f(x_0)$ we will reach a contradiction considering

$$u_0^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & \text{if } x \ne x_0, \\ u^\varepsilon(x_0) + \delta & \text{if } x = x_0, \end{cases}$$

with $\delta > 0$ small enough such that $u^\varepsilon(x_0) + \delta < f(x_0)$. Indeed, one can check that the pair $(u_0^\varepsilon, v^\varepsilon)$ belongs to $\mathscr{A}$ but at x_0 we have $u_0^\varepsilon(x_0) = u^\varepsilon(x_0) + \delta > u^\varepsilon(x_0) = \sup_{\mathscr{A}} \underline{u}^\varepsilon(x_0)$, a contradiction. A similar argument can be used when $x_0 \in \mathbb{R}^N \setminus \Omega$, and we have $v^\varepsilon(x_0) < g(x_0)$. We conclude that $u^\varepsilon(x) = f(x)$ and $v^\varepsilon(x) = g(x)$ for every $x \in \mathbb{R}^N \setminus \Omega$.

Now, let us assume that the point at which we have a strict inequality is inside Ω, $x_0 \in \Omega$. First, assume that we have

$$u^\varepsilon(x_0) < \max\bigl\{J_1(u^\varepsilon)(x_0),\, J_2(v^\varepsilon)(x_0)\bigr\}.$$

Let us consider

$$\max\bigl\{J_1(u^\varepsilon)(x_0),\, J_2(v^\varepsilon)(x_0)\bigr\} - u^\varepsilon(x_0) = \delta > 0,$$

and, as before, the function

$$u_0^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & \text{if } x \neq x_0, \\ u^\varepsilon(x_0) + \frac{1}{2}\delta & \text{if } x = x_0. \end{cases}$$

Then we have

$$u_0^\varepsilon(x_0) = u^\varepsilon(x_0) + \tfrac{1}{2}\delta < \max\bigl\{J_1(u^\varepsilon)(x_0),\, J_2(v^\varepsilon)(x_0)\bigr\}.$$

At other points $x \in \Omega$ we also have

$$u_0^\varepsilon(x) \le \max\bigl\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\bigr\} \le \max\bigl\{J_1(u_0^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\bigr\}.$$

Finally, concerning v^ε we get (at any point $x \in \Omega$),

$$v^\varepsilon(x) \le \min\bigl\{J_1(u_0^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\bigr\}.$$

Hence, we have that the pair $(u_0^\varepsilon, v^\varepsilon)$ belongs to $\mathscr{A}$, getting a contradiction as before, since $u_0^\varepsilon(x_0) > u^\varepsilon(x_0)$.

Analogously, one can deal with the case in which $x_0 \in \Omega$ and

$$v^\varepsilon(x_0) < \min\bigl\{J_1(u^\varepsilon)(x_0),\, J_2(v^\varepsilon)(x_0)\bigr\}. \qquad \square$$

Corollary 17. *There exists a constant* $\Lambda > 0$ *such that*

$$|u^\varepsilon(x)| < \Lambda \quad \textit{and} \quad |v^\varepsilon(x)| < \Lambda$$

for all $x \in \mathbb{R}^N$.

Proof. Every solution to the DPP belongs to $\mathcal{A}$. Hence the result follows from Theorem 15. $\square$

Now, for completeness, we include the precise statement of the optional stopping theorem (a key tool from probability theory that we will use in what follows).

Optional stopping theorem. We briefly recall (see [Williams 1991]) that a sequence of random variables $\{M_k\}_{k\ge1}$ is a supermartingale (respectively, submartingale) if

$$\mathbb{E}[M_{k+1} \mid M_0, M_1, \dots, M_k] \le M_k \quad \text{(respectively, } \ge).$$

Suppose that τ is a stopping time such that one of the following conditions holds:

(a) The stopping time τ is bounded almost surely.

(b) $\mathbb{E}[\tau] < \infty$ and there exists a constant $c > 0$ such that

$$\mathbb{E}[M_{k+1} - M_k \mid M_0, \dots, M_k] \le c.$$

(c) There exists a constant $c > 0$ such that $|M_{\min\{\tau,k\}}| \le c$ almost surely for every k.

For such a τ, the optional stopping theorem (OST) states that

$$\mathbb{E}[M_\tau] \le \mathbb{E}[M_0] \quad \text{(respectively, } \ge)$$

if $\{M_k\}_{k\ge 0}$ is a supermartingale (respectively, submartingale). For the proof of this classical result, see [Doob 1971; Williams 1991].

Let us finish this section proving the following theorem.

Theorem 18. *The functions u^ε and v^ε that satisfy the DPP (1-8) are the functions that give the value of the game in (1-7). This means that the function*

$$w^\varepsilon(x, j) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{(x,j)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}] = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x,j)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}]$$

satisfies

$$w^\varepsilon(x, 1) = u^\varepsilon(x) \quad \textit{and} \quad w^\varepsilon(x, 2) = v^\varepsilon(x)$$

for any pair $(u^\varepsilon, v^\varepsilon)$ that solves the DPP, that is, for any pair that satisfies

$$\begin{cases} u^\varepsilon(x) = \max\big\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\big\}, & x \in \Omega, \\ u^\varepsilon(x) = \min\big\{J_1(u^\varepsilon)(x), J_2(v^\varepsilon)(x)\big\}, & x \in \Omega, \\ u^\varepsilon(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^\varepsilon(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Proof. Fix $\delta > 0$. Assume that we start at a point in the first board, $(x_0, 1)$. Then we choose a strategy S_{I}^* for Player I using the solution to the DPP (1-8) as follows: Whenever $j_k = 1$, Player I decides to stay in the first board if

$$\max\big\{J_1(u^\varepsilon)(x_k), J_2(v^\varepsilon)(x_k)\big\} = J_1(u^\varepsilon)(x_k),$$

and in this case Player I chooses a point

$$x^{\mathrm{I}}_{k+1} = S^*_{\mathrm{I}}((x_0, j_0), \dots, (x_k, j_k)) \quad \text{such that} \quad \sup_{y\in B_\varepsilon(x_k)} u^\varepsilon(y) - \frac{\delta}{2^{k+1}} \le u^\varepsilon(x^{\mathrm{I}}_{k+1}).$$

On the other hand, Player I decides to jump to the second board if

$$\max\big\{J_1(u^\varepsilon)(x_k), J_2(v^\varepsilon)(x_k)\big\} = J_2(v^\varepsilon)(x_k),$$

and in this case Player I chooses a point

$$x^{\mathrm{I}}_{k+1} = S^*_{\mathrm{I}}((x_0, j_0), \dots, (x_k, j_k)) \quad \text{such that} \quad \sup_{y\in B_\varepsilon(x_k)} v^\varepsilon(y) - \frac{\delta}{2^{k+1}} \le v^\varepsilon(x^{\mathrm{I}}_{k+1}).$$

Given this strategy for Player I and any strategy for Player II, we consider the sequence of random variables

$$M_k = w^\varepsilon(x_k,\, j_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k},$$

where $w^\varepsilon(x_k, 1) = u^\varepsilon(x_k),\ w^\varepsilon(x_k, 2) = v^\varepsilon(x_k)$ and

$$\chi_{\{j=i\}}(j) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Let us see that $(M_k)_{k\geq 0}$ is a *submartingale*. To this end, we need to estimate

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{k+1} \mid M_0, \dots, M_k].$$

Let us consider several cases.

Case 1: Suppose that $j_k = 1$ and $j_{k+1} = 1$ (that is, we stay in the first board). Then

$$\begin{aligned}
&\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{k+1} \mid M_0, \dots, M_k] \\
&\quad= \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}\Big[u^\varepsilon(x_{k+1}) - \varepsilon^2 \sum_{l=0}^{k}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0, \dots, M_k\Big] \\
&\quad= \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}\Big[u^\varepsilon(x_{k+1}) - \varepsilon^2 h_1(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0, \dots, M_k\Big] \\
&\quad= \alpha_1\Big(\frac{1}{2}u^\varepsilon(x^{\mathrm{I}}_{k+1}) + \frac{1}{2}u^\varepsilon(x^{\mathrm{II}}_{k+1})\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x_k)} u^\varepsilon(y)\,dy - \varepsilon^2 h_1(x_k) \\
&\qquad\qquad - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}}.
\end{aligned}$$

In the second equality, we used that $j_{k+1} = 1$. We obtain

$$\begin{aligned}
&\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{k+1} \mid M_0, \dots, M_k] \\
&\quad\geq \alpha_1\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_k)} u^\varepsilon(y) - \frac{\delta}{2^{k+1}} + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_k)} u^\varepsilon(y)\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x_k)} u^\varepsilon(y)\,dy - \varepsilon^2 h_1(x_k) \\
&\qquad\qquad - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \\
&\quad\geq J_1(u^\varepsilon)(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} \\
&\quad= \max\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} \\
&\quad= u^\varepsilon(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} = M_k.
\end{aligned}$$

Case 2: Suppose that $j_k = 1$ and $j_{k+1} = 2$ (that is, we jump to the second board). Then

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}[M_{k+1} \,|\, M_0,\dots,M_k] \\
&\quad = \mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}\Big[v^\varepsilon(x_{k+1}) - \varepsilon^2 \sum_{l=0}^{k} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0,\dots,M_k \Big] \\
&\quad = \mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}\Big[v^\varepsilon(x_{k+1}) + \varepsilon^2 h_2(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0,\dots,M_k \Big].
\end{aligned}$$

In the second equality, we used that $j_{k+1} = 2$. We then obtain

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}[M_{k+1} \mid M_0,\dots,M_k] \\
&\quad \geq \alpha_2\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_k)} v^\varepsilon(y) - \frac{\delta}{2^{k+1}} + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_k)} v^\varepsilon(y)\Big) + (1-\alpha_2)\fint_{B_\varepsilon(x_k)} v^\varepsilon(y)\,dy + \varepsilon^2 h_2(x_k) \\
&\qquad\qquad - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \\
&\quad = J_2(v^\varepsilon)(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} \\
&\quad = \max\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} \\
&\quad = u^\varepsilon(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} = M_k.
\end{aligned}$$

Case 3: Suppose that $j_k = 2$ and $j_{k+1} = 2$ (that is, we stay in the second board). Then

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}[M_{k+1} \,|\, M_0,\dots,M_k] \\
&\quad = \mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}\Big[v^\varepsilon(x_{k+1}) - \varepsilon^2 \sum_{l=0}^{k} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0,\dots,M_k \Big] \\
&\quad = \mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}\Big[v^\varepsilon(x_{k+1}) + \varepsilon^2 h_2(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0,\dots,M_k \Big].
\end{aligned}$$

In the second equality, we used that $j_{k+1} = 2$. Therefore, we have

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}^{(x_0,1)}[M_{k+1} \mid M_0,\dots,M_k] \\
&\qquad \geq \alpha_2\Big(\frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x_k)} v^\varepsilon(y) - \frac{\delta}{2^{k+1}} + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x_k)} v^\varepsilon(y)\Big) + (1-\alpha_2)\fint_{B_\varepsilon(x_k)} v^\varepsilon(y)\,dy + \varepsilon^2 h_2(x_k) \\
&\qquad\qquad - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \\
&\qquad \geq J_2(v^\varepsilon)(x_k) - \varepsilon^2 \sum_{l=0}^{k-1} \big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k}
\end{aligned}$$

$$
\begin{aligned}
&\geq \min\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k}\\
&= v^\varepsilon(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} = M_k.
\end{aligned}
$$

Case 4: Suppose that $j_k = 2$ and $j_{k+1} = 1$ (that is, we jump to the first board). Then

$$
\begin{aligned}
&\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{k+1} \,|\, M_0, \dots, M_k]\\
&= \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}\Big[u^\varepsilon(x_{k+1}) - \varepsilon^2 \sum_{l=0}^{k}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0, \dots, M_k\Big]\\
&= \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}\Big[u^\varepsilon(x_{k+1}) - \varepsilon^2 h_1(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}} \,\Big|\, M_0, \dots, M_k\Big].
\end{aligned}
$$

In the second equality, we used that $j_{k+1} = 1$. Hence,

$$
\begin{aligned}
&\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{k+1} \mid M_0, \dots, M_k]\\
&\geq \alpha_1\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_k)} u^\varepsilon(y) - \frac{\delta}{2^{k+1}} + \frac{1}{2}\inf_{y\in B_\varepsilon(x_k)} u^\varepsilon(y)\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x_k)} u^\varepsilon(y)\,dy - \varepsilon^2 h_1(x_k)\\
&\qquad\qquad - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^{k+1}}\\
&= J_1(u^\varepsilon)(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k}\\
&\geq \min\big\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\big\} - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k}\\
&= v^\varepsilon(x_k) - \varepsilon^2 \sum_{l=0}^{k-1}\big(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\big) - \frac{\delta}{2^k} = M_k.
\end{aligned}
$$

Thus, gathering the four cases, we conclude that M_k is a submartingale.

Using the OST we obtain

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_{\tau\wedge k}] \geq M_0.$$

Taking the limit as $k \to \infty$ we get

$$\mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}}^*,S_{\mathrm{II}}}[M_\tau] \geq M_0.$$

If we take $\inf_{S_{\mathrm{II}}}$ and then $\sup_{S_{\mathrm{I}}}$ we arrive at

$$\sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[M_\tau] \geq M_0.$$

This inequality says that

$$\sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}} \mathbb{E}^{(x_0,1)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}] \geq u(x_0) - \delta.$$

To prove an inequality in the opposite direction we fix a strategy for Player II as follows: Whenever $j_k = 1$ Player II decides to stay in the second board if

$$\min\left\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\right\} = J_2(v^\varepsilon)(x_k),$$

and Player II decides to jump to the first board when

$$\min\left\{J_1(u^\varepsilon)(x_k),\, J_2(v^\varepsilon)(x_k)\right\} = J_1(v^\varepsilon)(x_k).$$

If we play tug-of-war (in both boards) Player II chooses

$$x_{k+1}^{\mathrm{II}} = S_{\mathrm{II}}^*((x_0, j_0), \ldots, (x_k, j_k)) \quad \text{such that} \quad \inf_{y\in B_\varepsilon(x_k)} w^\varepsilon(y, j_{k+1}) + \frac{\delta}{2^{k+1}} \geq w^\varepsilon(x_{k+1}^{\mathrm{II}}, j_{k+1}).$$

Given this strategy for Player II and any strategy for Player I, using computations similar to the ones we made before, we can prove that the sequence of random variables

$$N_k = w^\varepsilon(x_k, j_k) - \varepsilon^2 \sum_{l=0}^{k-1} \left(h_1(x_l)\chi_{\{j=1\}}(j_{l+1}) - h_2(x_l)\chi_{\{j=2\}}(j_{l+1})\right) + \frac{\delta}{2^k}$$

is a supermartingale. Finally, using the OST we arrive at

$$\inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] \leq u^\varepsilon(x_0) + \delta.$$

Then we have obtained

$$u^\varepsilon(x_0) - \delta \leq \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] \leq \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] \leq u^\varepsilon(x_0) + \delta$$

for any positive δ.

Analogously, we can prove that

$$v^\varepsilon(x_0) - \delta \leq \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,2)}[\text{total payoff}] \leq \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,2)}[\text{total payoff}] \leq v^\varepsilon(x_0) + \delta.$$

Since δ is arbitrary, this proves that the game has a value,

$$\sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,1)}[\text{total payoff}] = w(x_0, 1),$$
$$\sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,2)}[\text{total payoff}] = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{(x_0,2)}[\text{total payoff}] = w(x_0, 2),$$

and that these functions coincide with the solution to the DPP,

$$w(x_0, 1) = u^\varepsilon(x_0) \quad \text{and} \quad w(x_0, 2) = v^\varepsilon(x_0),$$

as we wanted to show. □

Since solutions to the DPP coincide with the value of the game and this is unique, we obtain uniqueness of solutions to the DPP.

Corollary 19. *There exists a unique solution to the DPP (1-8).*

Proof. Existence follows from Theorem 16 and uniqueness from the fact that in Theorem 18 we proved that any solution to the DPP coincides with the value function of the game, that is, it satisfies

$$u^\varepsilon(x) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{(x,1)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}] = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x,1)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}],$$
$$v^\varepsilon(x) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{(x,2)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}] = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{(x,2)}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\text{total payoff}]. \qquad \square$$

2.2. *Uniform convergence as $\varepsilon \to 0$.* To obtain a convergent subsequence of the values of the game u^ε and v^ε we will use the following Arzelà–Ascoli type lemma. For its proof, see [Manfredi et al. 2012b, Lemma 4.2].

Lemma 20. *Let*

$$\{u^\varepsilon : \overline{\Omega} \to \mathbb{R}\}_{\varepsilon>0}$$

be a set of functions such that

(1) *there exists $C > 0$ such that $|u^\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*

(2) *given $\delta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x-y| < r_0$,*

$$|u^\varepsilon(x) - u^\varepsilon(y)| < \delta.$$

Then there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence still denoted by $\{u^\varepsilon\}$ such that

$$u^\varepsilon \to u \quad \text{uniformly in } \overline{\Omega} \text{ as } \varepsilon \to 0.$$

So our task now is to show that u^ε and v^ε both satisfy the hypotheses of the previous lemma. First, we observe that we already proved that they are uniformly bounded (see Corollary 17).

To obtain the second hypothesis of Lemma 20 we will need to prove some technical lemmas. This part of the paper is delicate and involves the choice of particular strategies for the players.

First of all, we need to find an upper bound for the expectation of the total number of plays,

$$\mathbb{E}[\tau].$$

To this end we define an auxiliary game as follows: In the next lemma we play a tug-of-war or random walk game in an annulus and one of the players uses the strategy of pointing to the center of the annulus when they play tug-of-war. Then, no matter if we play tug-of-war or random walk at each turn and no matter the strategy used by the other player, we can obtain a precise bound (in terms of the configuration of the annulus and the distance of the initial position to the inner boundary) for the expected number of plays until one reaches the ball inside the annulus.

The key point here is that if one of the players pulls towards 0 each time that they play tug-of-war then the expected number of plays is bounded above by a precise expression that scales as ε^{-2} independently of the game that is played at every round (tug-of-war or random walk). This upper bound translates to our game (starting at any of the two boards) since the result implies that if one of the players chooses to pulls towards 0 then, independently of the choice of the other player and independently of the board at which we play — that is, independently of the coin toss that selects the game that is played (tug-of-war or random walk) — the game ends in an expected number of times that satisfies the obtained upper bound. See Remark 22 below.

Lemma 21. *Given $0 < \delta < R$, let us consider the annular domain $B_R(0) \setminus B_\delta(0)$. In this domain we consider the following game: given $x \in B_R(0) \setminus B_\delta(0)$ the next position of the token can be chosen using the game tug-of-war or a random walk. When tug-of-war is played, one of the players pulls towards 0. In all cases the next position is assumed to be in $B_\varepsilon(x) \cap B_R(0)$. The game ends when the token reaches $B_\delta(0)$. Then, if τ^* is the exit time, we have the estimate*

$$\varepsilon^2 \mathbb{E}^{x_0}[\tau^*] \le C_1(R/\delta)\,\mathrm{dist}(\partial B_\delta(0), x_0) + o(1),$$

where $o(1) \to 0$ if $\varepsilon \to 0$.

Proof. Without loss of generality we can suppose that S_{I}^* is to pull towards 0. Let us call

$$E_\varepsilon(x) = \mathbb{E}^x_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[\tau^*].$$

Notice that E is radial and increasing in $r = |x|$. Since our aim is to obtain a bound that is independent of the game (tug-of-war or random walk) that is played at each round, if we try to maximize the expectation for the exit time, we have that the function E satisfies

$$E_\varepsilon(x) \le \max\left\{\left(\frac{1}{2}\sup\nolimits_{y \in B_\varepsilon(x)\cap B_R(0)} E_\varepsilon(y) + \frac{1}{2}\inf\nolimits_{y \in B_\varepsilon(x)\cap B_R(0)} E_\varepsilon(y), \fint_{B_\varepsilon(x)\cap B_R(0)} E_\varepsilon(y)\,dy\right)\right\} + 1.$$

Hence, let us consider the DPP

$$\widetilde{E}_\varepsilon(x) = \max\left\{\left(\frac{1}{2}\sup\nolimits_{y \in B_\varepsilon(x)\cap B_R(0)} \widetilde{E}_\varepsilon(y) + \frac{1}{2}\inf\nolimits_{y \in B_\varepsilon(x)\cap B_R(0)} \widetilde{E}_\varepsilon(y), \fint_{B_\varepsilon(x)\cap B_R(0)} \widetilde{E}_\varepsilon(y)\,dy\right)\right\} + 1.$$

Writing $F_\varepsilon(x) = \varepsilon^2 \widetilde{E}_\varepsilon(x)$, we then obtain

$$F_\varepsilon(x) = \max\left\{\left(\frac{1}{2}\sup\nolimits_{y \in B_\varepsilon(x)\cap B_R(0)} F_\varepsilon(y) + \frac{1}{2}\inf\nolimits_{y \in B_\varepsilon(x)\cap B_R(0)} F_\varepsilon(y), \fint_{B_\varepsilon(x)\cap B_R(0)} F_\varepsilon(y)\,dy\right)\right\} + \varepsilon^2.$$

This induces us to look for a function F such that

$$F(x) \ge \fint_{B_\varepsilon(x)} F(y)\,dy + \varepsilon^2 \quad \text{and} \quad F(x) \ge \frac{1}{2}\sup\nolimits_{y\in B_\varepsilon(x)} F(y) + \frac{1}{2}\inf\nolimits_{y\in B_\varepsilon(x)} F(y) + \varepsilon^2. \tag{2-8}$$

We arrived at a sort of discrete version of the inequalities

$$\begin{cases} \Delta F(x) \le -2(N+2), & x \in B_{R+\varepsilon}(0) \setminus \bar{B}_{\delta-\varepsilon}(0), \\ \Delta^1_\infty F(x) \le -2, & x \in B_{R+\varepsilon}(0) \setminus \bar{B}_{\delta-\varepsilon}(0). \end{cases} \tag{2-9}$$

If we assume that F is radial and increasing if $r = |x|$ we get

$$\Delta^1_\infty F = \partial_{rr} F \le \partial_{rr} F + \frac{N-1}{r}\partial_r F = \Delta F.$$

Hence, to find a solution of (2-9), we can consider the problem

$$\begin{cases} \Delta F(x) = -2(N+2), & x \in B_{R+\varepsilon}(0) \setminus \bar{B}_\delta(0), \\ F(x) = 0, & x \in \partial B_\delta(0), \\ \frac{\partial F}{\partial \nu}(x) = 0, & \text{for } x \in \partial B_{R+\varepsilon}(0), \end{cases} \tag{2-10}$$

where $\partial F/\partial \nu$ refers to the outward normal derivative. The solution to this problem takes the form

$$F(r) = -ar^2 - br^{2-N} + c \quad \text{for } N > 2 \qquad \text{and} \qquad F(r) = -ar^2 - b\log(r) + c \quad \text{for } N = 2,$$

with $a, b, c \in \mathbb{R}$ that depends of $\delta,\ R,\ \varepsilon,\ N$. For example, for $N > 2$, we obtain that a, b and c are given by the solution to the following equations:

$$\Delta F = -2aN = -2(N+2),$$
$$\partial_r F(R+\varepsilon) = -2a(R+\varepsilon) - b(2-N)(R+\varepsilon)^{1-N} = 0,$$
$$F(\delta) = -a\delta^2 - b\delta^{2-N} + c = 0.$$

Observe that the resulting function $F(r)$ is increasing.

In this way we find F that satisfies the inequalities (2-9). The classical calculation using Taylor expansions shows that F satisfies (2-8) for each $B_\varepsilon(x) \subset B_{R+\varepsilon} \setminus \bar{B}_{\delta-\varepsilon}(0)$. Moreover, since F is increasing in r, it holds that for each $x \in B_R(0) \setminus B_\delta(0)$,

$$⨍_{B_\varepsilon(x)\cap B_R(0)} F \le ⨍_{B_\varepsilon(x)} F \le F(x) - \varepsilon^2$$

and

$$\tfrac{1}{2}\sup_{y\in B_\varepsilon(x)\cap B_R(0)} F + \tfrac{1}{2}\inf_{y\in B_\varepsilon(x)\cap B_R(0)} F \le \tfrac{1}{2}\sup_{y\in B_\varepsilon(x)} F + \tfrac{1}{2}\inf_{y\in B_\varepsilon(x)} F \le F(x) - \varepsilon^2.$$

Consider the sequence of random variables $(M_k)_{k\ge 1}$ given by

$$M_k = F(x_k) + k\varepsilon^2.$$

Let us prove that $(M_k)_{k\ge 0}$ is a *supermartingale*. Indeed, we have

$$\begin{aligned} \mathbb{E}[M_{k+1} \mid M_0, \dots, M_k] &= \mathbb{E}[F(x_{k+1}) + (k+1)\varepsilon^2 \mid M_0, \dots, M_k] \\ &\le \max\left\{ \frac{1}{2}\sup_{y\in B_\varepsilon(x)\cap B_R(0)} F(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)\cap B_R(0)} F(y), ⨍_{B_\varepsilon(x)\cap B_R(0)} F(y)\,dy \right\} \\ &\le F(x_k) + k\varepsilon^2 \end{aligned}$$

if $x_k \in B_R(0) \setminus \overline{B_\delta(0)}$. Thus, M_k is a supermartingale. Using the OST we obtain

$$\mathbb{E}[M_{\tau^*\wedge k}] \le M_0.$$

This means

$$\mathbb{E}^{x_0}[F(x_{\tau^*\wedge k}) + (\tau^*\wedge k)\varepsilon^2] \le F(x_0).$$

Using that $x_{\tau^*} \in \overline{B_\delta(0)} \setminus \overline{B_{\delta-\varepsilon}(0)}$ we get

$$0 \le -\mathbb{E}^{x_0}[F(x_{\tau^*})] \le o(1).$$

Furthermore, the estimate

$$0 \le F(x_0) \le C(R/\delta)\,\mathrm{dist}(\partial B_\delta, x_0)$$

holds for the solution of (2-10). Then, taking the limit as $k \to \infty$, we obtain

$$\varepsilon^2 \mathbb{E}[\tau^*] \le F(x_0) - \mathbb{E}[F(x_{\tau^*})] \le C(R/\delta)\,\mathrm{dist}(\partial B_\delta(0), x_0) + o(1).$$

This completes the proof. □

Remark 22. Suppose that we are playing the previous game in two boards with Ω inside the annulus. Let us recall that Ω satisfies an exterior sphere condition: there exists $\delta > 0$ such that given $y \in \partial\Omega$ there exists $z_y \in \mathbb{R}^N$ such that $B_\delta(z_y) \subset \mathbb{R}^N \setminus \Omega$ and $\overline{B_\delta(z_y)} \cap \overline{\Omega} = \{y\}$. Note that if we have some δ_0 that satisfies the exterior sphere property for ball of that radius, then the exterior sphere property is also satisfied for balls with radius δ for every $\delta < \delta_0$. Then we can consider simultaneously the game defined in Lemma 21 in the annular domain $B_R(0) \setminus B_\delta(0)$ with $\Omega \subset B_R(0) \setminus B_\delta(0)$, and with the same strategies for Player I and Player II, so that no mater which game, J_1 or J_2, is played in any of the two boards we obtain the bound for the expected number of plays given in Lemma 21. That is, if in the two boards game we start for example in $(x_0, 1)$ and Player I decides to stay in the first board and play J_1, in the one board game with the annular domain the third player decides to play tug-of-war with α_1 probability, or random walk with $1-\alpha_1$ probability, but if the player decides to jump to the second board and play J_2, then in the one board game the third player decides to play tug-of-war with α_2 probability and random walk with $1-\alpha_2$ probability. Thus, using that $\Omega \subset B_R(0) \setminus B_\delta(0)$ we deduce that in the two boards game the exit time τ is smaller than or equal to the exit time τ^* corresponding to the one board game considered in the previous lemma. This means that we have

$$\mathbb{E}[\tau] \le \mathbb{E}[\tau^*].$$

Next we derive an estimate for the asymptotic uniform continuity of the so-called nonhomogeneous p-Laplacian functions.

Lemma 23. *Let be Ω as above, $h : \Omega \to \mathbb{R}$ and $F : \mathbb{R}^N \setminus \Omega \to \mathbb{R}$ two Lipschitz functions. For $0 < \beta < 1$ let $\mu^\varepsilon : \mathbb{R}^N \to \mathbb{R}$ be a function that satisfies the following DPP*:

$$\begin{cases} \mu^\varepsilon(x) = \beta\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} \mu^\varepsilon(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} \mu^\varepsilon(y)\Big) + (1-\beta)\displaystyle\fint_{B_\varepsilon(x)} \mu^\varepsilon(y)\,dy + \varepsilon^2 h(x), & x \in \Omega, \\ \mu^\varepsilon(x) = F(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then, given $\eta > 0$, there exists $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$|\mu^\varepsilon(x) - \mu^\varepsilon(y)| < \eta$$

if $|x-y| < r_0$ and $\varepsilon < \varepsilon_0$.

Proof. We have several cases:

Case 1: If $x, y \in \mathbb{R}^N \setminus \Omega$ we have

$$|\mu^\varepsilon(x) - \mu^\varepsilon(y)| = |F(x) - F(y)| \leq L(F)|x - y| < \eta$$

if $r_0 < \eta / L(F)$.

Case 2: Suppose $x \in \Omega$ and $y \in \partial\Omega$. Without loss of generality we can suppose that $\Omega \subset B_R(0) \setminus B_\delta(0)$ and $y \in \partial B_\delta(0)$. Let us call $x_0 = x$ the first position in the game. In the first case suppose that Player I uses the strategy of pulling towards 0, denoted by S_{I}^*. Let us consider the sequence of random variables

$$M_k = |x_k| - C\varepsilon^2 k.$$

If $C > 0$ is large enough, M_k is a supermartingale. Indeed

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}\big[|x_{k+1}| \mid x_0, \dots, x_k\big] \leq \beta\Big(\frac{1}{2}(|x_k| + \varepsilon) + \frac{1}{2}(|x_k| - \varepsilon)\Big) + (1 - \beta) \fint_{B_\varepsilon(x_k)} |z|\, dz \leq |x_k| + C\varepsilon^2.$$

The first inequality follows from the choice of the strategy, and the second from the estimate

$$\fint_{B_\varepsilon(x)} |z|\, dz \leq |x| + C\varepsilon^2.$$

Using the OST we obtain

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[|x_\tau|] \leq |x_0| + C\varepsilon^2 \mathbb{E}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[\tau].$$

Now, Lemma 21 and Remark 22 give us the estimate

$$\varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[\tau] \leq \varepsilon^2 \mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[\tau^*] \leq C_1(R/\delta) \operatorname{dist}(\partial B_\delta(0), x_0) + o(1).$$

Then

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[|x_\tau|] \leq |x_0 - y| + \delta + C_2(R/\delta)|x_0 - y| + o(1).$$

Here $C_2(R/\delta) = C_1(R/\delta)$. If we rewrite this inequality we obtain

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[|x_\tau|] \leq \delta + C_3(R/\delta)|x_0 - y| + o(1)$$

with $C_3(R/\delta) = C_2(R/\delta) + 1$.

Using that F is a Lipschitz function we have

$$|F(x_\tau) - F(0)| \leq L(F)|x_\tau|.$$

Hence, we get

$$\begin{aligned}
\mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[F(x_\tau)] &\geq F(0) - L(F)\mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}[|x_\tau|] \\
&\geq F(y) - L(F)\delta - L(F)C_3(R/\delta)|x_0 - y| + o(1) \\
&\geq F(y) - L(F)\delta - L(F)C_3(R/\delta)r_0 - o(1).
\end{aligned}$$

Then

$$\mathbb{E}^{x_0}_{S_{\mathrm{I}}^*, S_{\mathrm{II}}}\left[F(x_\tau) + \varepsilon^2 \sum_{j=0}^{\tau-1} h(x_j)\right] \geq F(y) - L(F)\delta - L(F)C_3 r_0 - \|h\|_\infty C r_0 - o(1).$$

Thus, taking $\inf_{S_{\rm II}}$ and then $\sup_{S_{\rm I}}$ we get

$$\mu^{\varepsilon}(x_0) > F(y) - L(F)\delta - L(F)C_3 r_0 - \|h\|_\infty C r_0 - o(1) > F(y) - \eta.$$

Here we take $\delta > 0$ such that $L(F)\delta < \frac{1}{3}\eta$, and then take $r_0 > 0$ such that $(L(F)C_3 + \|h\|_\infty C)r_0 < \frac{1}{3}\eta$ and $o(1) < \frac{1}{3}\eta$.

Analogously, we can obtain the estimate

$$\mu^{\varepsilon}(x_0) < F(y) + \eta$$

if Player II use the strategy that pulls towards 0. This ends the proof in this case.

Case 3: Now, given two points x and y inside Ω with $|x-y| < r_0$, we couple the game starting at $x_0 = x$ with the game starting at $y_0 = y$ making the same movements. This coupling generates two sequences of positions x_i and y_i such that $|x_i - y_i| < r_0$ and $j_i = k_i$. This continues until one of the games exits the domain (say at $y_\tau \notin \Omega$). At this point for the game starting at x_0 we have that its position x_τ is close to the exterior point $y_\tau \notin \Omega$ (since we have $|x_\tau - y_\tau| < r_0$) and hence we can use our previous estimates for points close to the boundary to conclude that

$$|\mu^{\varepsilon}(x_0) - \mu^{\varepsilon}(y_0)| < \eta. \qquad \square$$

Now we are ready to prove the second condition of the Arzelà–Ascoli type result, Lemma 20.

Lemma 24. *Let $(u^{\varepsilon}, v^{\varepsilon})$ be a pair of functions that is a solution to the DPP (1-8) given by*

$$\begin{cases} u^{\varepsilon}(x) = \max\big\{J_1(u^{\varepsilon})(x),\, J_2(v^{\varepsilon})(x)\big\}, & x \in \Omega, \\ v^{\varepsilon}(x) = \min\big\{J_1(u^{\varepsilon})(x),\, J_2(v^{\varepsilon})(x)\big\}, & x \in \Omega, \\ u^{\varepsilon}(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^{\varepsilon}(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Given $\eta > 0$, there exists $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$|u^{\varepsilon}(x) - u^{\varepsilon}(y)| < \eta \quad \text{and} \quad |v^{\varepsilon}(x) - v^{\varepsilon}(y)| < \eta$$

if $|x-y| < r_0$ and $\varepsilon < \varepsilon_0$.

Proof. We will proceed by repeating the ideas used in Lemma 23.

We consider again several cases.

Case 1: Suppose that $x, y \in \mathbb{R}^N \setminus \Omega$. Then we have

$$|u^{\varepsilon}(x) - u^{\varepsilon}(y)| = |f(x) - f(y)| \le L(f)|x-y| < \eta \quad \text{and} \quad |v^{\varepsilon}(x) - v^{\varepsilon}(y)| = |g(x) - g(y)| \le L(g)|x-y| < \eta$$

if $\max\{L(f), L(g)\} r_0 < \eta$.

Case 2: Let us begin with the estimate of u^{ε}. Suppose now that $x \in \Omega$ and $y \in \partial\Omega$ in the first board (we write $(x, 1)$ and $(y, 1)$). Without loss of generality we suppose again that $\Omega \subset B_R(0) \setminus B_\delta(0)$ and $y \in \partial B_\delta(0)$. Let us call $x_0 = x$ the first position in the game. Player I uses the following strategy, called $S_{\rm I}^*$: the token always stay in the first board (Player I decides not to change boards), and pulls towards 0 when tug-of-war is played. In this case we have that u^ε is a supersolution to the DPP that appears in Lemma 23 (with $\beta = \alpha_1$). Notice that the game is always played in the first board. As Player I wants to maximize the expected value we get that the first component for our system, u^ε, satisfies

$$u^\varepsilon(x) \geq \mu^\varepsilon(x)$$

(the value function when the player that wants to maximize is allowed to choose to change boards is bigger than or equal to the value function of a game where the player does not have the possibility of making this choice). From this bound and Lemma 23, a lower bound for u^ε close to the boundary follows. That is, from the estimate obtained in that lemma we get

$$u^\varepsilon(x) > f(y) - \eta.$$

Let us be more precise and consider the sequence of random variables

$$M_k = |x_k| - C\varepsilon^2 k.$$

We obtain arguing as before that M_k is a supermartingale for $C > 0$ large enough. If we repeat the reasoning of the Lemma 23 (this can be done because we stay in the first board) we arrive at

$$u^\varepsilon(x) > f(y) - \eta$$

if $|x-y| < r_0$ and $\varepsilon < \varepsilon_0$ for some r_0 and ε_0.

Now, the next estimate requires a particular strategy for Player II, called S_{II}^*: when play the tug-of-war game, Player II pulls towards 0 (in both boards) and if in some step Player I decides to jump to the second board, then Player II decides to stay always in this board and the position never comes back to the first board. Let us consider

$$M_k = |x_k| - C\varepsilon^2 k.$$

We want to estimate

$$\mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^*}^{(x_0,1)}\big[|x_{k+1}| \mid x_0, \dots, x_k\big].$$

Now, Lemma 23 says that

$$\mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^*}^{(x_0,1)}\big[|x_{k+1}| \mid x_0, \dots, x_k\big] \leq |x_k| + C\varepsilon^2$$

for all possible combinations of j_k and j_{k+1}. Using the OST we obtain

$$\mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^*}^{(x_0,1)}[|x_\tau|] \leq |x_0| + C\varepsilon^2 \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^*}[\tau].$$

Let us suppose that $j_\tau = 1$. This means that $j_k = 1$ for all $0 \leq k \leq \tau$. If we proceed as in Lemma 23, we obtain

$$\mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^*}^{(x_0,1)}[\text{final payoff}] \leq f(y) + L\delta + LCr_0 + \|h_1\|_\infty Cr_0 + o(1).$$

On the other hand, if $j_\tau = 2$, we have

$$\mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^*}^{(x_0,1)}[g(x_\tau)] \leq g(y) + L\delta + LC_3 r_0 + o(1) \leq f(y) + L\delta + LC_3 r_0 + o(1).$$

Thus, we get

$$\mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^*}^{(x_0,1)}\left[g(x_\tau)+\sum_{l=0}^{\tau-1}\big(h_1(x_l)\chi_{\{l=1\}}(l)+h_2(x_l)\chi_{\{l=2\}}(l)\big)\right]\le f(y)+L\delta+LCr_0+(\|h_1\|+\|h_2\|_\infty)Cr_0+o(1).$$

In both cases, taking $\sup_{S_{\mathrm{I}}}$ and then $\inf_{S_{\mathrm{II}}}$ we arrive at

$$u^\varepsilon(x_0)\le f(y)+\eta,$$

taking $\delta>0,\ r_0>0$ and $\varepsilon>0$ small enough.

Analogously we can obtain the estimates for v^ε and complete the proof. □

As a corollary we obtain uniform convergence along a sequence $\varepsilon_j\to 0$.

Corollary 25. *There exists a sequence $\varepsilon_j\to 0$ and a pair of functions (u,v) that are continuous in $\overline{\Omega}$ such that*

$$u^{\varepsilon_j}\rightrightarrows u,\qquad v^{\varepsilon_j}\rightrightarrows v$$

uniformly in $\overline{\Omega}$.

Proof. The result follows from Lemma 20. □

2.3. ***The limit is a viscosity solution to the PDE system.*** Our main goal in this section is to prove that the limit pair (u,v) is a viscosity solution to (1-9).

First, let us state the precise definition of what we understand as a viscosity solution for the system (1-9). We refer to [Crandall et al. 1992] for a general reference to viscosity theory.

Viscosity solutions. We begin with the definition of a viscosity solution to a fully nonlinear second-order elliptic PDE. Fix a function

$$P:\Omega\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{S}^N\to\mathbb{R},$$

where $\mathbb{S}^N$ denotes the set of symmetric $N\times N$ matrices, and consider the PDE

$$P\big(x,u(x),Du(x),D^2u(x)\big)=0,\qquad x\in\Omega. \tag{2-11}$$

We will assume that P is degenerate elliptic, that is, P satisfies a monotonicity property with respect to the matrix variable, that is,

$$X\le Y\ \text{in}\ \mathbb{S}^N\quad\Longrightarrow\quad P(x,r,p,X)\ge P(x,r,p,Y)$$

for all $(x,r,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^N$.

Definition 26. A lower semicontinuous function u is a viscosity supersolution of (2-11) if for every $\phi\in C^2$ such that ϕ touches u at $x\in\Omega$ strictly from below (that is, $u-\phi$ has a strict minimum at x with $u(x)=\phi(x)$), we have

$$P\big(x,\phi(x),D\phi(x),D^2\phi(x)\big)\ge 0.$$

An upper semicontinuous function u is a subsolution of (2-11) if for every $\psi \in C^2$ such that ψ touches u at $x \in \Omega$ strictly from above (that is, $u - \psi$ has a strict maximum at x with $u(x) = \psi(x)$), we have

$$P\big(x, \phi(x), D\phi(x), D^2\phi(x)\big) \le 0.$$

Finally, u is a viscosity solution of (2-11) if it is both a super- and subsolution.

When P is not continuous one has to consider the upper and lower semicontinuous envelopes of P, which we denote by P^* and P_* respectively, and consider

$$P^*(x, \phi(x), D\phi(x), D^2\phi(x)) \ge 0 \quad \text{and} \quad P_*(x, \phi(x), D\phi(x), D^2\phi(x)) \le 0$$

when defining super- and subsolutions.

In our system (1-9) we have two equations given by the functions

$$F_1(x, u, v, p, X) = \min\left\{ -\frac{\alpha_1}{2}\left\langle X\frac{p}{|p|}, \frac{p}{|p|}\right\rangle - \frac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x),\, (u-v)(x)\right\},$$

$$F_2(x, u, v, q, Y) = \max\left\{ -\frac{\alpha_2}{2}\left\langle Y\frac{q}{|q|}, \frac{q}{|q|}\right\rangle - \frac{(1-\alpha_2)}{2(N+2)}\operatorname{trace}(Y) - h_2(x),\, (v-u)(x)\right\}.$$

These functions F_1 and F_2 are not continuous (they are not even well defined for $p = 0$ and for $q = 0$ respectively). The upper semicontinuous envelope of F_1 is given by

$$(F_1)^*(x, u, v, p, X) = \begin{cases} \min\left\{ -\frac{\alpha_1}{2}\left\langle X\frac{p}{|p|}, \frac{p}{|p|}\right\rangle - \frac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x),\, (u-v)(x)\right\}, & p \neq 0, \\ \min\left\{ -\frac{\alpha_1}{2}\lambda_1(X) - \frac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x),\, (u-v)(x)\right\}, & p = 0. \end{cases}$$

Here $\lambda_1(X) = \min\{\lambda : \lambda \text{ is an eigenvalue of } X\}$. While the lower semicontinuous envelope is

$$(F_1)_*(x, u, v, p, X) = \begin{cases} \min\left\{ -\frac{\alpha_1}{2}\left\langle X\frac{p}{|p|}, \frac{p}{|p|}\right\rangle - \frac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x),\, (u-v)(x)\right\}, & p \neq 0, \\ \min\left\{ -\frac{\alpha_1}{2}\lambda_N(X) - \frac{(1-\alpha_1)}{2(N+2)}\operatorname{trace}(X) + h_1(x),\, (u-v)(x)\right\}, & p = 0. \end{cases}$$

Here $\lambda_N(X) = \max\{\lambda : \lambda \text{ is an eigenvalue of } X\}$.

Analogous formulas hold for $(F_2)^*$ and $(F_2)_*$, changing α_1 by α_2.

Then the definition of a viscosity solution for the system (1-9) that we will use here is the following.

Definition 27. A pair of continuous functions $u, v : \overline{\Omega} \mapsto \mathbb{R}$ is a viscosity solution to (1-9) if

(1) $u(x) \ge v(x)$ for $x \in \overline{\Omega}$,

(2) $u|_{\partial\Omega} = f$ and $v|_{\partial\Omega} = g$,

(3) u is a viscosity solution to $F_1(x, u, v(x), \nabla u, D^2u) = 0$ in $\{x : u(x) > v(x)\}$ and u is a viscosity supersolution to $F_1(x, u, v(x), \nabla u, D^2u) = 0$ in Ω,

(4) v is a viscosity solution to $F_2(x, u(x), v, \nabla v, D^2v) = 0$ in $\{x : u(x) > v(x)\}$ and v is a viscosity solution to $F_2(x, u(x), v, \nabla v, D^2v) = 0$ in Ω.

Remark 28. The meaning of Definition 27 is that we understand a solution to (1-9) as a pair of continuous up to the boundary functions that satisfy the boundary conditions pointwise and such that u is a viscosity solution to the obstacle problem (from below) for the first equation in the system (with v as a fixed continuous function of x as obstacle from below) and v solves the obstacle problem (from above) for the second equation in the system (regarding u as a fixed function of x as obstacle from above).

With this definition at hand we are ready to show that any uniform limit of the value functions of our game is a viscosity solution to the two membranes problem with the two different p-Laplacians.

Theorem 29. *Let (u, v) be continuous functions that are a uniform limit of a sequence of values of the game, that is,*

$$u^{\varepsilon_j} \rightrightarrows u, \qquad v^{\varepsilon_j} \rightrightarrows v$$

uniformly in $\overline{\Omega}$ as $\varepsilon_j \to 0$. Then the limit pair (u, v) is a viscosity solution to (1-9).

Proof. We divide the proof into several steps.

(1) <u>u and v are ordered</u>: From the fact that

$$u^{\varepsilon_j} \geq v^{\varepsilon_j}$$

in $\overline{\Omega}$ and the uniform convergence we immediately get

$$u \geq v$$

in $\overline{\Omega}$.

(2) <u>The boundary conditions</u>: As we have that

$$u^{\varepsilon_j} = f, \qquad v^{\varepsilon_j} = g,$$

in $\mathbb{R}^N \setminus \Omega$ we get

$$u|_{\partial\Omega} = f, \qquad v|_{\partial\Omega} = g.$$

(3) <u>The equation for u</u>: First, let us show that u is a viscosity supersolution to

$$-\Delta_p^1 u(x) + h_1(x) = 0$$

for $x \in \Omega$. To this end, consider a point $x_0 \in \Omega$ and a smooth function $\varphi \in C^2(\Omega)$ such that $(u-\varphi)(x_0) = 0$ is a strict minimum of $(u - \varphi)$. Then from the uniform convergence there exists a sequence of points, which we will denote by $(x_\varepsilon)_{\varepsilon>0}$, such that $x_\varepsilon \to x_0$ and

$$(u^\varepsilon - \varphi)(x_\varepsilon) \leq (u^\varepsilon - \varphi)(y) + o(\varepsilon^2),$$

that is,

$$u^\varepsilon(y) - u^\varepsilon(x_\varepsilon) \geq \varphi(y) - \varphi(x_\varepsilon) - o(\varepsilon^2). \tag{2-12}$$

From the DPP (1-8) we have

$$0 = \max\big\{J_1(u^\varepsilon)(x_\varepsilon) - u(x_\varepsilon),\, J_2(v^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon)\big\} \geq J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon). \tag{2-13}$$

Writing $J_1(u^\varepsilon)(x_\varepsilon) - u(x_\varepsilon)$ we obtain

$$J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon) = \alpha_1\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y) - u^\varepsilon(x_\varepsilon)) + \frac{1}{2}\inf_{y\in B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y) - u^\varepsilon(x_\varepsilon))\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y) - u^\varepsilon(x_\varepsilon))\,dy - \varepsilon^2 h_1(x_\varepsilon),$$

and then, using (2-12), we get

$$J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon) \geq \alpha_1 \underbrace{\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_\varepsilon)}(\varphi(y) - \varphi(x_\varepsilon)) + \frac{1}{2}\inf_{y\in B_\varepsilon(x_\varepsilon)}(\varphi(y) - \varphi(x_\varepsilon))\Big)}_{\mathrm{I}} + \beta_1 \underbrace{\fint_{B_\varepsilon(x_\varepsilon)}(\varphi(y) - \varphi(x_\varepsilon))\,dy}_{\mathrm{II}} - \varepsilon^2 h_1(x_\varepsilon) + o(\varepsilon^2).$$

Let us analyze I and II. We begin with I: Assume that $\nabla\varphi(x_0) \neq 0$. Let $z_\varepsilon \in B_1(0)$ be such that

$$\max_{y\in \bar{B}_\varepsilon(x_\varepsilon)} \varphi(y) = \varphi(x_\varepsilon + \varepsilon z_\varepsilon).$$

Then we have

$$I = \tfrac{1}{2}\big(\varphi(x_\varepsilon + \varepsilon z_\varepsilon) - \varphi(x_\varepsilon)\big) + \tfrac{1}{2}\big(\varphi(x_\varepsilon - \varepsilon z_\varepsilon) - \varphi(x_\varepsilon)\big) + o(\varepsilon^2).$$

From a simple Taylor expansion we conclude that

$$\tfrac{1}{2}\big(\varphi(x_\varepsilon + \varepsilon z_\varepsilon) - \varphi(x_\varepsilon)\big) + \tfrac{1}{2}\big(\varphi(x_\varepsilon - \varepsilon z_\varepsilon) - \varphi(x_\varepsilon)\big) + o(\varepsilon^2) = \tfrac{1}{2}\varepsilon^2\langle D^2\varphi(x_\varepsilon)z_\varepsilon, z_\varepsilon\rangle + o(\varepsilon^2).$$

Dividing the first inequality by ε^2 and taking the limit as $\varepsilon \to 0$ (see [Miranda and Rossi 2020]) we arrive at

$$I \to \tfrac{1}{2}\Delta^1_\infty\varphi(x_0).$$

When $\nabla\varphi = 0$, arguing again using Taylor expansions, we get

$$\limsup I \geq \tfrac{1}{2}\lambda_1(D^2\varphi(x_0)).$$

See [Blanc and Rossi 2019] for more details.

Now, we look at II: Using again Taylor expansions we obtain

$$\fint_{B_\varepsilon(x_\varepsilon)}(\varphi(y) - \varphi(x_\varepsilon))\,dy = \frac{\varepsilon^2}{2(N+1)}\Delta\varphi(x_\varepsilon) + o(\varepsilon^2).$$

Dividing by ε^2 and taking limits as $\varepsilon \to 0$ we get

$$\mathrm{II} \to \frac{1}{2(N+2)}\Delta\varphi(x_0).$$

Therefore, if we come back to (2-13), dividing by ε^2 and taking the limit as $\varepsilon \to 0$ we obtain

$$0 \geq \frac{\alpha_1}{2}\Delta^1_\infty\varphi(x_0) + \frac{1-\alpha_1}{2(N+1)}\Delta\varphi(x_0) - h_1(x_0)$$

when $\nabla\varphi(x_0) \neq 0$, and

$$0 \geq \frac{\alpha_1}{2}\lambda_1(D^2\varphi(x_0)) + \frac{1-\alpha_1}{2(N+1)}\Delta\varphi(x_0) - h_1(x_0)$$

when $\nabla\varphi(x_0) = 0$.

Using the definition of the normalized p-Laplacian we have arrived at

$$-\Delta_p^1\varphi(x_0) + h_1(x_0) \geq 0,$$

in the sense of Definition 26.

Now we are going to prove that u is a viscosity solution to

$$-\Delta_p^1 u(x) + h_1(x) = 0 \tag{2-14}$$

in the set $\Omega \cap \{u > v\}$. Let us consider $x_0 \in \Omega \cap \{u > v\}$. Let $\eta > 0$ be such that

$$u(x_0) \geq v(x_0) + 3\eta.$$

Then, using that u and v are continuous functions, there exists $\delta > 0$ such that

$$u(y) \geq v(y) + 2\eta \quad \text{for all } \ y \in B_\delta(x_0),$$

and, using that $u^\varepsilon \rightrightarrows u$ and $v^\varepsilon \rightrightarrows v$, we have

$$u^\varepsilon(y) \geq v^\varepsilon(y) + \eta \quad \text{for all } \ y \in B_\delta(x_0)$$

for $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$. Given $z \in B_{\delta/2}(x_0)$ and $\varepsilon < \min\{\varepsilon_0, \delta/2\}$, we obtain

$$B_\varepsilon(z) \subset B_\delta(x_0).$$

Using that $u^\varepsilon \rightrightarrows u$ we obtain the following limits:

$$\sup_{y\in B_\varepsilon(z)} u^\varepsilon(y) \to u(z) \quad \text{as } \ \varepsilon \to 0. \tag{2-15}$$

In fact, from our previous estimates we have that

$$|\sup_{y\in B_\varepsilon(z)} u^\varepsilon(y) - u(z)| \leq \sup_{y\in B_\varepsilon(z)}|u^\varepsilon(y) - u(y)| + \sup_{y\in B_\varepsilon(z)}|u(y) - u(z)|.$$

Using that $u^\varepsilon \rightrightarrows u$, there exists $\varepsilon_1 > 0$ such that if $\varepsilon < \varepsilon_1$,

$$|(u^\varepsilon - u)(x)| < \tfrac{1}{2}\theta \quad \text{for all } \ x \in \Omega.$$

Now, using that u is continuous, there exists $\varepsilon_2 > 0$ such that

$$|u(y) - u(z)| < \tfrac{1}{2}\theta \quad \text{if } \ |y - z| < \varepsilon_2,$$

and thus, if we take $\varepsilon < \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \delta/2\}$ we obtain

$$|\sup_{y\in B_\varepsilon(z)} u^\varepsilon(y) - u(z)| < \theta.$$

This proves (2-15).

Also, with a similar argument, we have

$$\inf_{y\in B_\varepsilon(z)} u^\varepsilon(y) \xrightarrow{\varepsilon\to 0} u(z). \tag{2-16}$$

Finally, we get

$$\fint_{B_\varepsilon(z)} u^\varepsilon(y)\,dy \xrightarrow{\varepsilon\to 0} u(z). \tag{2-17}$$

In fact, let us compute

$$\left|\fint_{B_\varepsilon(z)} u^\varepsilon(y)\,dy - u(z)\right| \le \fint_{B_\varepsilon(z)} |u^\varepsilon(y)-u(y)|\,dy + \fint_{B_\varepsilon(z)} |u(y)-u(z)|\,dz.$$

Now we use again that $u^\varepsilon \rightrightarrows u$ and that u is a continuous function to obtain

$$\fint_{B_\varepsilon(z)} |u^\varepsilon(y)-u(y)|\,dy < \frac{1}{2}\theta \quad \text{and} \quad \fint_{B_\varepsilon(z)} |u(y)-u(z)|\,dz < \frac{1}{2}\theta$$

for $\varepsilon > 0$ small enough. Thus we obtain

$$\left|\fint_{B_\varepsilon(z)} u^\varepsilon(y)\,dy - u(z)\right| < \theta.$$

Using the previous limits, (2-15), (2-16) and (2-17), we obtain

$$J_1(u^\varepsilon)(z) \to u(z) \quad \text{as } \varepsilon \to 0.$$

Analogously, we can prove that

$$J_2(v^\varepsilon)(z) \to v(z) \quad \text{as } \varepsilon \to 0.$$

Now, if we recall that $u(z) \ge v(z) + 2\eta$, we obtain

$$J_1(u^\varepsilon)(z) \ge J_2(v^\varepsilon)(z) + \eta$$

if $\varepsilon > 0$ is small enough. Then, using the DPP, we obtain

$$u^\varepsilon(z) = \max\bigl\{J_1(u^\varepsilon)(z),\, J_2(v^\varepsilon)(z)\bigr\} = J_1(u^\varepsilon)(z)$$

for all $z \in B_{\delta/2}(x_0)$ and every $\varepsilon > 0$ small enough. Let us prove that u is viscosity subsolution of (2-14). Given now $\varphi \in \mathscr{C}^2(\Omega)$ such that $(u-\varphi)(x_0) = 0$ is maximum of $u-\varphi$, then, from the uniform convergence, there exists a sequence of points $(x_\varepsilon)_{\varepsilon>0} \subset B_{\delta/2}(x_0)$ such that $x_\varepsilon \to x_0$ and

$$(u^\varepsilon - \varphi)(x_\varepsilon) \ge (u^\varepsilon - \varphi)(y) + o(\varepsilon^2),$$

that is,

$$u^\varepsilon(y) - u^\varepsilon(x_\varepsilon) \le \varphi(y) - \varphi(x_\varepsilon) - o(\varepsilon^2). \tag{2-18}$$

From the DPP (1-8) we have

$$0 = \max\bigl\{J_1(u^\varepsilon)(x_\varepsilon) - u(x_\varepsilon),\, J_2(v^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon)\bigr\} = J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon).$$

Writing $J_1(u^\varepsilon)(x_\varepsilon) - u(x_\varepsilon)$ we obtain

$$J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon) = \alpha_1\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y) - u^\varepsilon(x_\varepsilon)) + \frac{1}{2}\inf_{y\in B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y) - u^\varepsilon(x_\varepsilon))\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x_\varepsilon)}(u^\varepsilon(y) - u^\varepsilon(x_\varepsilon))\,dy - \varepsilon^2 h_1(x_\varepsilon),$$

and then, using (2-18), we get

$$J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon) \le \alpha_1\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x_\varepsilon)}(\varphi(y) - \varphi(x_\varepsilon)) + \frac{1}{2}\inf_{y\in B_\varepsilon(x_\varepsilon)}(\varphi(y) - \varphi(x_\varepsilon))\Big) + (1-\alpha_1)\fint_{B_\varepsilon(x_\varepsilon)}(\varphi(y) - \varphi(x_\varepsilon))\,dy - \varepsilon^2 h_1(x_\varepsilon) + o(\varepsilon^2).$$

Passing to the limit as before we obtain

$$0 \le \frac{\alpha_1}{2}\Delta^1_\infty\varphi(x_0) + \frac{(1-\alpha_1)}{2(N+1)}\Delta\varphi(x_0) - h_1(x_0)$$

when $\nabla\varphi(x_0) \neq 0$, and

$$0 \le \frac{\alpha_1}{2}\lambda_N(D^2\varphi(x_0)) + \frac{(1-\alpha_1)}{2(N+1)}\Delta\varphi(x_0) - h_1(x_0)$$

if $\nabla\varphi(x_0) = 0$. Hence we arrived at

$$-\Delta^1_p\varphi(x_0) + h_1(x_0) \le 0,$$

according to Definition 26. This proves that u is a viscosity subsolution of (2-14) inside the open set $\{u > v\}$.

As we have that u is a viscosity supersolution in the whole Ω, we conclude that u is a viscosity solution to

$$-\Delta^1_p u(x_0) + h_1(x_0) = 0$$

in the set $\{u > v\}$.

(4) The equation for v: The case that v is a viscosity subsolution to

$$-\Delta^1_q v(x) + h_2(x) = 0$$

is analogous. Here we use that

$$0 = \min\big\{J_2(v^\varepsilon)(x_\varepsilon) - v(x_\varepsilon),\, J_1(u^\varepsilon)(x_\varepsilon) - v^\varepsilon(x_\varepsilon)\big\} \le J_2(v^\varepsilon)(x_\varepsilon) - v^\varepsilon(x_\varepsilon).$$

To show that v is a viscosity solution

$$-\Delta^1_q v(x_0) - h_2(x_0) = 0$$

if $x_0 \in \Omega \cap \{u > v\}$, we proceed as before. □

3. A game that gives an extra condition on the contact set

In this section we will study the value functions of the second game. In this case, they are given by a pair of functions $(u^\varepsilon, v^\varepsilon)$ that satisfies the DPP

$$\begin{cases} u^\varepsilon(x) = \frac{1}{2}\max\big\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\big\} + \frac{1}{2}J_1(u^\varepsilon)(x), & x \in \Omega, \\ v^\varepsilon(x) = \frac{1}{2}\min\big\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\big\} + \frac{1}{2}J_2(v^\varepsilon)(x), & x \in \Omega, \\ u^\varepsilon(x) = f(x), & x \in \mathbb{R}^N \setminus \Omega, \\ v^\varepsilon(x) = g(x), & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3-1}$$

It is clear from the DPP that

$$u^\varepsilon \geq v^\varepsilon.$$

We aim to show that these functions converge (along subsequences $\varepsilon_j \to 0$) to a pair of functions (u, v) that is a viscosity solution to the system

$$\begin{cases} u(x) \geq v(x), & x \in \Omega, \\ -\Delta_p^1 u(x) + h_1(x) \geq 0,\ -\Delta_q^1 v(x) - h_2(x) \leq 0, & x \in \Omega, \\ -\Delta_p^1 u(x) + h_1(x) = 0,\ -\Delta_q^1 v(x) - h_2(x) = 0, & x \in \{u > v\} \cap \Omega, \\ (-\Delta_p^1 u(x) + h_1(x)) + (-\Delta_q^1 v(x) - h_2(x)) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \\ v(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{3-2}$$

Notice that this is the classical formulation of the two membranes problem, but with the extra condition

$$(-\Delta_p^1 u(x) + h_1(x)) + (-\Delta_q^1 v(x) - h_2(x)) = 0,$$

which is meaningful for $x \in \{u(x) = v(x)\}$.

The existence and uniqueness of the pair of functions $(u^\varepsilon, v^\varepsilon)$ can be proved as before. In fact, we can reproduce the arguments of Perron's method to obtain existence of a solution. Next, we show that given a solution to the DPP we can build quasioptimal strategies and show that the game has a value and that this value coincides with the solution to the DPP, from where uniqueness of solutions to the DPP follows.

Uniform convergence also follows with the same arguments used before using the Arzelà–Ascoli type result together with the estimates close to the boundary proved in the previous section. Notice that here we can prescribe the same strategies as the ones used before. For example, Player I may decide to stay in the first board (if the coin toss allows a choice) and to point to a prescribed point when tug-of-war is played. Also note that the crucial bound on the expected number of plays given in Lemma 21 can also be used here to obtain a bound for the total number of plays in the variant of the game.

Passing to the limit in the viscosity sense is also analogous. One only has to pay special attention to the extra condition. Therefore, let us prove now that the extra condition in (3-2),

$$(-\Delta_p^1 u(x) + h_1(x)) + (-\Delta_q^1 v(x) - h_2(x)) = 0,$$

holds in the viscosity sense in the set $\{x : u(x) = v(x)\}$.

Let us start proving the subsolution case. Given $x_0 \in \{u = v\}$ and $\varphi \in \mathscr{C}^2(\Omega)$ such that $(u-\varphi)(x_0) = 0$ is a maximum of $u-\varphi$, notice that since $v(x_0) = u(x_0)$ and $v \le u$ in Ω we also have that $(v-\varphi)(x_0) = 0$ is a maximum of $v-\varphi$. Then, by uniform convergence, there exists a sequence of points $(x_\varepsilon)_{\varepsilon>0} \subset B_{\delta/2}(x_0)$, such that $x_\varepsilon \to x_0$ and

$$(u^\varepsilon - \varphi)(x_\varepsilon) \ge (u^\varepsilon - \varphi)(y) + o(\varepsilon^2). \tag{3-3}$$

Case 1: Suppose that $u^\varepsilon(x_{\varepsilon_j}) > v^\varepsilon(x_{\varepsilon_j})$ for a subsequence such that $\varepsilon_j \to 0$. Let us observe that, if

$$J_1(u^\varepsilon)(z) < J_2(v^\varepsilon)(z),$$

we have that

$$u^\varepsilon(z) = \tfrac{1}{2} J_1(u^\varepsilon)(z) + \tfrac{1}{2} J_2(v^\varepsilon)(z) \quad \text{and} \quad v^\varepsilon(z) = \tfrac{1}{2} J_1(u^\varepsilon)(z) + \tfrac{1}{2} J_2(v^\varepsilon)(z),$$

and then we get

$$u^\varepsilon(z) = v^\varepsilon(z)$$

in this case.

This remark implies that when $u^\varepsilon(x_{\varepsilon_j}) > v^\varepsilon(x_{\varepsilon_j})$ we have

$$J_1(u^\varepsilon)(x_{\varepsilon_j}) \ge J_2(v^\varepsilon)(x_{\varepsilon_j}).$$

If we use the DPP (3-1) we get

$$\begin{aligned} 0 &= \tfrac{1}{2}\big(J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j})\big) + \tfrac{1}{2}\max\big\{J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j}),\, J_2(v^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j})\big\} \\ &= \tfrac{1}{2}\big(J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j})\big) + \tfrac{1}{2} J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j}) \\ &= J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j}), \end{aligned}$$

and using (3-3) we obtain

$$0 = J_1(u^\varepsilon)(x_{\varepsilon_j}) - u^\varepsilon(x_{\varepsilon_j}) \le J_1(\varphi)(x_{\varepsilon_j}) - \varphi(x_{\varepsilon_j}),$$

and taking the limit as $\varepsilon_j \to 0$ as before we get

$$-\Delta_p^1 \varphi(x_0) + h_1(x_0) \le 0. \tag{3-4}$$

We proved before that v is a subsolution to

$$-\Delta_q^1 v(x) - h_2(x) = 0$$

in the whole Ω. Therefore, as $(v-\varphi)(x_0) = 0$ is a maximum of $v-\varphi$ we get

$$-\Delta_q^1 \varphi(x_0) - h_2(x_0) \le 0. \tag{3-5}$$

Thus, from (3-4) and (3-5) we conclude that

$$(-\Delta_p^1 \varphi(x_0) + h_1(x_0)) + (-\Delta_q^1 \varphi(x_0) - h_2(x_0)) \le 0.$$

Case 2: If $u^\varepsilon(x_\varepsilon) = v^\varepsilon(x_\varepsilon)$ for $\varepsilon < \varepsilon_0$. Using the DPP (3-1) we have

$$u^\varepsilon(x_\varepsilon) = \tfrac{1}{2} J_1(u^\varepsilon)(x_\varepsilon) + \tfrac{1}{2} J_2(v^\varepsilon)(x_\varepsilon) \quad \text{and} \quad v^\varepsilon(x_\varepsilon) = \tfrac{1}{2} J_1(u^\varepsilon)(x_\varepsilon) + \tfrac{1}{2} J_2(v^\varepsilon)(x_\varepsilon),$$

and then we get

$$\max\big\{J_1(u^\varepsilon)(x_\varepsilon),\, J_2(v^\varepsilon)(x_\varepsilon)\big\} = J_2(v^\varepsilon)(x_\varepsilon) \quad \text{and} \quad \min\big\{J_1(u^\varepsilon)(x_\varepsilon),\, J_2(v^\varepsilon)(x_\varepsilon)\big\} = J_1(u^\varepsilon)(x_\varepsilon).$$

If we use again (3-3) we get

$$\varphi(y) - \varphi(x_\varepsilon) \geq u^\varepsilon(y) - u^\varepsilon(x_\varepsilon) + o(\varepsilon^2) \geq v^\varepsilon(y) - v^\varepsilon(x_\varepsilon) + o(\varepsilon^2),$$

where we used that $u^\varepsilon \geq v^\varepsilon$ and $u^\varepsilon(x_\varepsilon) = v^\varepsilon(x_\varepsilon)$. Thus

$$0 = \tfrac{1}{2}\big(J_1(u^\varepsilon)(x_\varepsilon) - u^\varepsilon(x_\varepsilon)\big) + \tfrac{1}{2}\big(J_2(v^\varepsilon)(x_\varepsilon) - v^\varepsilon(x_\varepsilon)\big) \leq \tfrac{1}{2}\big(J_1(\varphi)(x_\varepsilon) - \varphi(x_\varepsilon)\big) + \tfrac{1}{2}\big(J_2(\varphi)(x_\varepsilon) - \varphi(x_\varepsilon)\big).$$

Taking the limit $\varepsilon \to 0$ we obtain

$$(-\Delta_p^1 \varphi(x_0) + h_1(x_0)) + (-\Delta_q^1 \varphi(x_0) - h_2(x_0)) \leq 0,$$

in the viscosity sense (taking care of the semicontinuous envelopes when the gradient of φ vanishes). We have just proved that the extra condition is satisfied with an inequality when we touch u and v from above at some point x_0 with a smooth test function.

The proof that the other inequality holds when we touch u and v from below is analogous and hence we omit the details.

4. Final remarks

Below we gather some brief comments on possible extensions of our results.

4.1. ***n membranes.*** We can extend our results to the case in which we have n membranes. For the PDE problem we refer to [Azevedo et al. 2005; Carillo et al. 2005; Chipot and Vergara-Caffarelli 1985].

We can generalize the game to an n-dimensional system. Let us suppose that we have, for $1 \leq k \leq n$,

$$J_k(w)(x) = \alpha_k\Big(\frac{1}{2}\sup_{y\in B_\varepsilon(x)} w(y) + \frac{1}{2}\inf_{y\in B_\varepsilon(x)} w(y)\Big) + (1-\alpha_k) \fint_{B_\varepsilon(x)} w(y)\,dy - \varepsilon^2 h_k(x).$$

These games have associated to them the operators

$$L_k(w) = -\Delta_{p_k}^1 w + h_k.$$

Given $f_1 \geq f_2 \geq \cdots \geq f_n$ defined outside Ω, we can consider the DPP

$$\begin{cases} u_k^\varepsilon(x) = \frac{1}{2}\max_{i\geq k}\{J_i(u_i^\varepsilon)\} + \frac{1}{2}\min_{l\leq k}\{J_l(u_l^\varepsilon)\}, & x \in \Omega, \\ u_k^\varepsilon(x) = f_k(x), & x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

for $1 \leq k \leq n$.

This DPP is associated to a game that is played in n boards. In board k a fair coin is tossed and the winner is allowed to change boards but Player I can only choose to change to a board with index bigger than or equal to k while Player II may choose a board with index smaller than or equal to k.

The functions $(u_1^\varepsilon, \cdots, u_n^\varepsilon)$ converge uniformly as $\varepsilon \to 0$ (along a subsequence) to continuous functions $\{u_k\}_{1\le k\le n}$ that are viscosity solutions to the n membranes problem

$$\begin{cases} u_k(x) \ge u_{k+1}(x), & x \in \Omega, \\ L_k(u_k) \ge 0, \quad L_{k+l}(u_{k+l}) \le 0 & x \in \{u_{k-1} > u_k \equiv u_{k+1} \equiv \cdots \equiv u_{k+l} > u_{k+l+1}\} \cap \Omega, \\ L_k(u_k) + L_{k+l}(u_{k+l}) = 0, & x \in \{u_{k-1} > u_k \equiv u_{k+1} \equiv \cdots \equiv u_{k+l} > u_{k+l+1}\} \cap \Omega, \\ L_k(u_k) = 0, & x \in \{u_{k-1} > u_k > u_{k+1}\} \cap \Omega, \\ u_k(x) = f_k(x), & x \in \partial\Omega \end{cases}$$

for $1 \le k \le n$.

Notice that here the extra condition

$$L_k(u_k) + L_{k+l}(u_{k+l}) = 0, \quad x \in \{u_{k-1} > u_k \equiv u_{k+1} \equiv \cdots \equiv u_{k+l} > u_{k+l+1}\} \cap \Omega$$

appears.

4.2. ***Other operators.*** Our results can also be extended to the two membranes problem with different operators as soon as there are games J_1 and J_2 whose value functions approximate the solutions to the corresponding PDEs and for which the key estimates of Section 2 can be proved. Namely, we need that starting close to the boundary each player has a strategy that forces the game to end close to the initial position in the same board with large probability and in a controlled expected number of plays regardless the choices of the other player.

For instance, our results can be extended to deal with the two membranes problem for Pucci operators (for a game related to Pucci operators we refer to [Blanc et al. 2019]). Pucci operators are uniformly elliptic and are given in terms of two positive constants, λ and Λ, by the formulas

$$M^+_{\lambda,\Lambda}(D^2u) = \sup\nolimits_{A\in L_{\lambda,\Lambda}} \operatorname{trace}(AD^2u) \quad \text{and} \quad M^-_{\lambda,\Lambda}(D^2u) = \inf\nolimits_{A\in L_{\lambda,\Lambda}} \operatorname{trace}(AD^2u)$$

with

$$L_{\lambda,\Lambda} = \{A \in \mathbb{S}^n : \lambda \operatorname{Id} \le A \le \Lambda \operatorname{Id}\}.$$

Notice that the extra condition that we obtain with the second game reads as

$$M^+_{\lambda_1,\Lambda_1}(D^2u(x)) + M^-_{\lambda_2,\Lambda_2}(D^2v(x)) = h_2(x) - h_1(x)$$

if we play with a game associated to the equation $M^+_{\lambda_1,\Lambda_1}(D^2u(x)) + h_1(x) = 0$ in the first board and with a game associated to $M^-_{\lambda_2,\Lambda_2}(D^2u(x)) + h_1(x) = 0$ in the second board.

We leave the details to the reader.

4.3. ***Playing with an unfair coin modifies the extra condition.*** One can also deal with the game in which the coin toss that is used to determine if the player can make the choice to change boards or not is not a fair coin. Assume that a coin is tossed in the first board with probabilities γ and $(1-\gamma)$ and in the second board with reverse probabilities, $(1-\gamma)$ and γ. In this case the equations that are involved in the DPP read as

$$\begin{cases} u^\varepsilon(x) = \gamma \max\big\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\big\} + (1-\gamma) J_1(u^\varepsilon)(x), & x \in \Omega, \\ v^\varepsilon(x) = (1-\gamma) \min\big\{J_1(u^\varepsilon)(x),\, J_2(v^\varepsilon)(x)\big\} + \gamma J_2(v^\varepsilon)(x), & x \in \Omega. \end{cases}$$

In this case, the extra condition that we obtain is given by

$$\gamma(-\Delta_p^1 u(x) + h_1(x)) + (1-\gamma)(-\Delta_q^1 v(x) - h_2(x)) = 0, \quad x \in \Omega,$$

Notice that there are two extreme cases, $\gamma = 0$ and $\gamma = 1$. When $\gamma = 1$, the second player cannot decide to change boards but the first player has this possibility (with probability one) in the first board. In this case, in the limit problem the second component, v, is a solution to $-\Delta_q^1 v(x) - h_2(x) = 0$ in the whole Ω and u is the solution to the obstacle problem (with v as obstacle from below). On the other hand, if $\gamma = 0$, it is the first player who cannot decide to change and the second player has the command in the second board and in this case in the limit u is the component that is a solution to the equation $-\Delta_p^1 u(x) + h_1(x) = 0$, and v the one that solves the obstacle problem (with u as obstacle from above).

Note that the value functions are increasing with respect to γ, that is, $u^\varepsilon_{\gamma_1}(x) \le u^\varepsilon_{\gamma_2}(x)$ and $v^\varepsilon_{\gamma_1}(x) \le v^\varepsilon_{\gamma_2}(x)$ for $\gamma_1 \le \gamma_2$. Therefore, passing to the limit as $\varepsilon \to 0$ we obtain a family of solutions to the two membranes problem that is increasing with γ,

$$u_0(x) \le u_{\gamma_1}(x) \le u_{\gamma_2}(x) \le u_1(x) \quad \text{and} \quad v_0(x) \le v_{\gamma_1}(x) \le v_{\gamma_2}(x) \le v_1(x)$$

for $\gamma_1 \le \gamma_2$.

The pair (u_0, v_0) is the minimal solution to the two membranes problem in the sense that $u_0 \le u$ and $v_0 \le v$ for any other solution (u, v). In fact, since u is a supersolution and u_0 is a solution to $-\Delta_p^1 u(x) + h_1(x) = 0$ from the comparison principle we obtain $u_0 \le u$. Then we obtain that $v_0 \le v$ from the fact that they are solutions to the obstacle problem from above with obstacles u_0 and u respectively.

Analogously, the pair (u_1, v_1) is the maximal solution to the two membranes problem.

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