JOURNAL OF **UMALCA**
Unión Matemática
de América Latina y el Caribe

ORBITA
MATHEMATICAE

KK- and *E*-theory via homotopy theory

Ulrich Bunke

msp

vol. 1 no. 2 2024

ORBITA MATHEMATICAE
Vol. 1, No. 2, 2024
https://doi.org/10.2140/om.2024.1.103

KK- and E-theory via homotopy theory

Ulrich Bunke

We provide a homotopy theorist's point of view on *KK*- and E-theory for C^*-algebras. We construct stable ∞-categories representing these theories through a sequence of Dwyer–Kan localizations of the category of C^*-algebras. Thereby we will reveal the homotopic-theoretic meaning of various classical constructions from C^*-algebra theory, in particular of Cuntz' q-construction. We will also discuss operator algebra K-theory in this framework.

1. Introduction

We describe a construction of stable ∞-categories representing *KK*- and E-theory for C^*-algebras through sequences of localizations of the category $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ of separable C^*-algebras followed by a left-Kan extension along the inclusion of separable C^*-algebras into all C^*-algebras. In contrast to the previous constructions of such an ∞-category [Land and Nikolaus 2018], particularly in the case of *KK*-theory [Bunke et al. 2021], the description presented here is independent of the classical group-valued *KK*-theory

---

These notes present an expanded version of the lecture notes of a course on noncommutative homotopy theory taught at the University of Regensburg in the winter term 2022/23. The author thanks the participants for suspicious attention and various critical remarks.

MSC2020: 19K35.
Keywords: *KK*-theory, E-theory, C^*-algebra K-theory, stable infinity category, Dwyer–Kan localization.

introduced in [Kasparov 1988; Cuntz 1987] which is described, e.g., in the textbooks [Blackadar 1998; Higson 1990b]. A stable ∞-category representing E-theory has not been considered so far.

The main goal of this note is to give a complete account of the basic categorical and functorial properties of KK- and E-theory using only the basic elements of C^*-algebra theory. In this way we hope to make these theories more accessible to readers with a homotopy theory background. The approach to KK- and E-theory described here can easily be generalized to the case of G-C^*-algebras for discrete groups G (see, for example, [Bunke and Duenzinger 2024] for E-theory) or to $C_0(X)$-algebras. With more modifications it should be possible to develop a similar approach to the algebraic version of KK-theory [Cortiñas and Thom 2007; Garkusha 2014; 2016; Ellis 2014]. It is also an interesting task to provide a homotopy-theoretic interpretation of the constructions from [Cuntz 1998] in the spirit of the present paper.

The starting point of our construction is the characterization of the stable ∞-category version of KK-theory through a universal property.

Definition 1.1. The functor $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ is initial for functors from $C^*\mathbf{Alg}^{\mathrm{nu}}$ to cocomplete stable ∞-categories which are homotopy invariant, stable, semiexact and s-finitary.

This means that kk has these properties, described in detail in Definition 2.4, and that for any cocomplete stable ∞-category $\mathbf{D}$ the restriction along kk induces an equivalence

$$\mathrm{kk}^* : \mathbf{Fun}^{\mathtt{colim}}(\mathrm{KK}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,se,sfin}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}).$$

Here the superscripts `colim` and h, s, se, $sfin$ stand for colimit-preserving and the corresponding properties listed in Definition 1.1.

The characterization of KK-theory by Definition 1.1 was given in [Bunke et al. 2021] following [Land and Nikolaus 2018]. A similar characterization of the group-valued KK-functor through universal properties has been known for a long time [Higson 1988].

The characterization of the stable ∞-category representing E-theory is similar and obtained by replacing in Definition 1.1 the condition of semiexactness by exactness. The motivation comes from the universal property of the classical E-theory stated in [Higson 1990a, Theorem 3.6].

Definition 1.2. The functor $\mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}$ is initial for functors from $C^*\mathbf{Alg}^{\mathrm{nu}}$ to cocomplete stable ∞-categories which are homotopy invariant, stable, exact and s-finitary.

In this case we have an equivalence

$$\mathrm{e}^* : \mathbf{Fun}^{\mathtt{colim}}(\mathrm{E}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,ex,sfin}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}).$$

Our construction proceeds with the following steps which are designed to force the universal properties stated above:

(1) $L_h : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is a Dwyer–Kan localization which inverts the homotopy equivalences. The resulting ∞-category $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is left-exact (see Section 3) and the functor L_h is Schochet-exact in the sense that it sends Schochet fibrant cartesian squares to cartesian squares.

(2) $L_K : C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is a smashing left-Bousfield localization which inverts the left-upper-corner inclusions and produces a semiadditive and left-exact ∞-category (see Section 4).

(3) We restrict to the full subcategory of separable algebras and form a left-exact Dwyer–Kan localization $L_{\mathrm{sep},!} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}$ for ! in {splt, se, ex} which forces split exact, semisplit exact or exact sequences to induce fiber sequences (see Section 5).

(4) For ! in {se, ex} the two-fold loop functor $\Omega^2_{\mathrm{sep},!} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}$ (where $\mathbf{C}^{\mathrm{group}}$ denotes the full subcategory of group objects in a semiadditive ∞-category $\mathbf{C}$) turns out to be the right-adjoint of a right-Bousfield localization and has a stable target. The composition of the localizations above gives functors

$$\mathrm{kk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}} := L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,\mathrm{se}}{}^{\mathrm{group}} \quad \text{and} \quad \mathrm{e}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{E}_{\mathrm{sep}} := L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,\mathrm{ex}}{}^{\mathrm{group}}.$$

(See Section 7.)

(5) We define the presentable stable ∞-categories KK and E as the `Ind`-completions of the stable ∞-categories $\mathrm{KK}_{\mathrm{sep}}$ and $\mathrm{E}_{\mathrm{sep}}$ and the functors

$$\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK} \quad \text{and} \quad \mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}$$

by left-Kan extending the compositions

$$C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \xrightarrow{\mathrm{kk}_{\mathrm{sep}}} \mathrm{KK}_{\mathrm{sep}} \xrightarrow{y} \mathrm{KK} \quad \text{and} \quad C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \xrightarrow{\mathrm{e}_{\mathrm{sep}}} \mathrm{E}_{\mathrm{sep}} \xrightarrow{y} \mathrm{E}$$

along the inclusion of separable C^*-algebras into all C^*-algebras (see Section 8). Theorem 8.5 states that the functors constructed by this procedure indeed satisfy the conditions of Definitions 1.1 and 1.2.

One interesting consequence of the constructions is that the functors

$$\mathrm{kk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}} \quad \text{and} \quad \mathrm{e}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{E}_{\mathrm{sep}}$$

are Dwyer–Kan localizations (see Proposition 7.5).

Our construction of *KK*- and *E*-theory for separable C^*-algebras via a sequence of localizations is analogous to the construction of an additive category representing *E*-theory in [Higson 1990a]. The idea of left-Kan extending *KK*-theory from separable C^*-algebras to all C^*-algebras also appears in [Skandalis 1988].

The category $C^*\mathbf{Alg}^{\mathrm{nu}}$ has symmetric monoidal structures $\otimes_{\max}$ and $\otimes_{\min}$. The kk- and e-theory functors have symmetric monoidal refinements which are characterized by symmetric monoidal versions of Definitions 1.1 and 1.2. We will discuss the universal properties of the symmetric monoidal refinements in the main body of the present paper.

The categories KK and E whose construction is sketched above are stable ∞-categories. For any two C^*-algebras A and B we therefore have mapping spectra

$$\mathrm{KK}(A, B) := \mathtt{map}_{\mathrm{KK}}(\mathrm{kk}(A), \mathrm{kk}(B)), \quad \mathrm{E}(A, B) := \mathtt{map}_{\mathrm{E}}(\mathrm{e}(A), \mathrm{e}(B)).$$

Taking homotopy groups we get $\mathbb{Z}$-graded KK- and E-theory groups

$$\mathrm{KK}_*(A,B) := \pi_*\mathrm{KK}(A,B), \quad \mathrm{E}_*(A,B) := \pi_*\mathrm{E}(A.B).$$

The approach to KK- and E-theory taken in the present note turns the classical constructions of these group-valued bifunctors into calculations. Our homotopy-theoretic construction of KK- and E-theory is straightforward once one knows which universal property one would like to enforce. Composition, homotopy invariance, stability and the respective exactness properties come for free. Also Bott periodicity is just a property which holds because of the existence of the Toeplitz extension. The real problem in our approach is to calculate the homotopy groups of the mapping spaces in order to see that they coincide with the classical groups. The latter are defined in terms of Kasparov modules (see [Cuntz 1987; 1998; Thomsen 1999; Dadarlat et al. 2018] for alternatives) in the case of KK-theory, or in case of E-theory, by the one-categorical localization procedure as in [Higson 1990a] or asymptotic morphisms [Connes and Higson 1990]. The comparison of the homotopy groups of the mapping spectra of the categories constructed in the present note with the classical groups is not obvious at all just from the construction. But it is crucial if one wants to use the models proposed in this note as a homotopy-theoretic replacement of the classical analytic constructions.

In the case of KK-theory one could argue by a comparison of the universal properties that the functors $\mathrm{kk}_{\mathrm{sep}}$ for separable algebras constructed in the present paper and in [Land and Nikolaus 2018] are canonically equivalent. Moreover, in [Bunke et al. 2021] we have shown that the composition

$$C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \xrightarrow{\mathrm{kk}_{\mathrm{sep}}} \mathrm{KK}_{\mathrm{sep}} \xrightarrow{\mathrm{ho}} \mathrm{hoKK}_{\mathrm{sep}}$$

is equivalent to the triangulated-category-valued KK-theory of [Meyer and Nest 2006], and that the KK-groups $\mathrm{KK}_0(A,B)$ for separable C^*-algebras A, B are canonically isomorphic to the KK-groups introduced in [Kasparov 1988]. But this argument has a draw back. Though the classical definition of KK-groups in terms of equivalence classes of Kasparov modules is not very complicated, this method of comparison also relies on the construction of the composition (i.e., the Kasparov product) and the verification of semiexactness in the classical theory which are deep theorems. It is therefore one of the guiding challenges of the present paper to give an independent complete proof for the comparison.

From the perspective of the present notes it is natural to compare the KK- and E-theory functors of the present paper with the classical ones by comparing their universal properties. This can be done in a model-independent way by defining the classical functors

$$\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}}, \quad \mathrm{e}^{\mathrm{class}}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{E}^{\mathrm{class}}_{\mathrm{sep}}$$

as the universal homotopy invariant, stable and split exact or half-exact functors, respectively, to an additive category in the sense of [Higson 1990a, Theorems 3.4 and 3.6]. These can directly be compared with the compositions

$$\mathrm{hokk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \xrightarrow{\mathrm{kk}_{\mathrm{sep}}} \mathrm{KK}_{\mathrm{sep}} \xrightarrow{\mathrm{ho}} \mathrm{hoKK}_{\mathrm{sep}}, \quad \mathrm{hoe}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \xrightarrow{\mathrm{e}_{\mathrm{sep}}} \mathrm{E}_{\mathrm{sep}} \xrightarrow{\mathrm{ho}} \mathrm{hoE}_{\mathrm{sep}}.$$

The following is a consequence of Theorem 13.16 and the Theorem 12.1 (which allows one to replace $\mathrm{kk}_{\mathrm{sep},q}$ appearing in Theorem 13.16 by $\mathrm{kk}_{\mathrm{sep}}$).

Corollary 1.3. *We have commutative squares*

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[r, "\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}"] \arrow[d, "\mathrm{kk}_{\mathrm{sep}}"] & \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}} \\ \mathrm{KK}_{\mathrm{sep}} \arrow[r, "\mathrm{ho}"] & \mathrm{hoKK}_{\mathrm{sep}} \arrow[u, dashed, "\simeq"] \end{tikzcd} \qquad \begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[r, "\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}"] \arrow[d, "\mathrm{e}_{\mathrm{sep}}"] & \mathrm{E}^{\mathrm{class}}_{\mathrm{sep}} \\ \mathrm{E}_{\mathrm{sep}} \arrow[r, "\mathrm{ho}"] & \mathrm{hoE}_{\mathrm{sep}} \arrow[u, dashed, "\simeq"] \end{tikzcd}$$

where the dashed arrows are equivalences of additive categories.

One could argue that in the case of *KK*-theory the proof of Corollary 1.3 has a similar problem as the argument mentioned above since we must know that our preferred model of $\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$ has the universal property stated in [Higson 1990a, Theorem 3.4]. We therefore will provide another, completely independent comparison with Cuntz' treatment of *KK* by showing (1-4) below. One could then read the arguments also in a different direction as showing that the Cuntz model indeed has the universal property [Higson 1990a, Theorem 3.4].

In our approach the enrichment of *KK*- and *E*-theory in spectra is a natural consequence of the stability of the ∞-categories KK or E. But point-set level constructions of spectral enrichments of *KK*-theory have previously been considered in [Joachim and Stolz 2009; Mitchener 2002].

As can be seen from the description above our approach to a stable ∞-category representing *KK*- and *E*- theory is different from other attempts to produce such stable ∞-categories which were guided by the methods of motivic homotopy theory [Østvær 2010; Mahanta 2015]. There the idea was to start from the category of presheaves $\mathbf{Fun}((C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}})^{\mathrm{op}}, \mathbf{Spc})$, to perform a series of left-Bousfield localizations forcing homotopy invariance, stability and the desired version of exactness ! in {splt, se, ex}, and finally to apply $-\otimes \mathbf{Sp}$ in presentable ∞-categories in order to stabilize. Let us denote the resulting presentable stable ∞-category by $\mathcal{KK}_{\mathrm{sep},!}$. It comes with a functor $kk_{\mathrm{sep},!} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathcal{KK}_{\mathrm{sep},!}$ which by construction has the universal property that

$$kk^*_{\mathrm{sep},!} : \mathbf{Fun}^{\mathtt{colim}}(\mathcal{KK}_{\mathrm{sep},!}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D})$$

for any presentable stable ∞-category $\mathbf{D}$. The main nontrivial question is then to understand the relation between $\pi_*\mathtt{map}_{\mathcal{KK}_{\mathrm{sep},!}}(kk_{\mathrm{sep},!}(A), kk_{\mathrm{sep},!}(B))$ and the classical *KK*-groups $\mathrm{KK}^{\mathrm{class}}_{\mathrm{sep},*}(A, B)$ (for $! = \mathrm{se}$) or *E*-theory groups $\mathrm{E}^{\mathrm{class}}_{\mathrm{sep},*}(A, B)$ (for $! = \mathrm{ex}$). We will not pursue this direction.

As said above the advantage of the constructions in the present note is that they do not require previous knowledge of *KK*- or *E*-theory. In contrast, in [Land and Nikolaus 2018; Bunke et al. 2021] the basic idea was to construct the category $\mathrm{KK}_{\mathrm{sep}}$ as a Dwyer–Kan localization of $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ at the kk-equivalences. The latter notion was imported from the classical theory. In the present paper we do not have to know from the beginning what a kk-equivalence is. The notion of a kk-equivalence comes out at the end as a morphism which is sent to an equivalence by the functor $\mathrm{kk}_{\mathrm{sep}}$. The input for the construction of $\mathrm{KK}_{\mathrm{sep}}$ in the present paper are only simple C^*-algebraic notions such as homotopy of homomorphisms, compact operators and semisplit exact sequences.

The construction of the ∞-categories KK and E via localizations and $\mathtt{Ind}$-completions is very suitable for understanding functors out of these categories. This will be employed in some subsequent papers. On the other hand, it is notoriously difficult to understand the homotopy types of the mapping spaces in a Dwyer–Kan localization just from the definition. In Sections 9 and 10 we will, with some effort, calculate the mapping spectra $\mathrm{E}(A, B)$ for $A \cong \mathbb{C}$ or $A \cong S(\mathbb{C}) \cong C_0(\mathbb{R})$ explicitly.

We first define the commutative ring spectrum $\mathrm{KU} := \mathrm{E}(\mathbb{C}, \mathbb{C})$. We justify this notation by providing a ring isomorphism $\pi_*\mathrm{KU} \cong \mathbb{Z}[b, b^{-1}]$ with $\deg(b) = -2$ and comparing $\Omega^\infty\mathrm{KU}$ with the classical constructions of an infinite loop space with the same name. Since $\mathrm{e}(\mathbb{C})$ is the tensor unit of E this category has a canonical enrichment over the category $\mathbf{Mod}(\mathrm{KU})$ of KU-module spectra. In Definition 9.3 we then define the lax symmetric monoidal $\mathbf{Mod}(\mathrm{KU})$-valued K-theory functor for C^*-algebras simply as

$$K(-) := \mathrm{E}(\mathbb{C}, -) : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Mod}(\mathrm{KU}). \tag{1-1}$$

This gives an effortless construction of a highly structured version of a K-theory functor for C^*-algebras. For previous constructions of spectrum-valued K-theory functors, see, for example, [Bunke et al. 2003; Dell'Ambrogio et al. 2011; Joachim 2003; Dadarlat and Pennig 2015].

Recall that the classical constructions of C^*-algebra K-theory groups as described, for instance, in [Blackadar 1998] employ equivalence classes of projections or components of unitary groups. In order to connect our definition (1-1) with the classical ones and in order to show that it gives the correct group-valued functors after taking homotopy groups, we relate the infinite loop-space valued functor $\Omega^\infty K$ with spaces of projections or unitaries. Thereby we take care of the natural commutative monoid or groups structures.

Using that $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is semiadditive we can define the commutative monoid (see Example 4.8)

$$\mathcal{P}\mathrm{roj}^s(B) := \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(\mathbb{C}, B)$$

of stable projections and the commutative group (see Example 4.9)

$$\mathcal{U}^s(B) := \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(S(\mathbb{C}), B)$$

of stable unitaries in B. The following result combines Proposition 9.4 and Corollary 10.8.

Corollary 1.4. (1) *If B is unital, then there is a canonical morphism $\mathcal{P}\mathrm{roj}^s(B) \to \Omega^\infty K(B)$ in* $\mathbf{CMon}(\mathbf{Spc})$ *which presents its target as the group completion.*

(2) *We have a canonical equivalence $\mathcal{U}^s(B) \simeq \Omega^{\infty-1}K(B)$ in* $\mathbf{CGroups}(\mathbf{Spc})$.

The canonical morphisms in Corollary 1.4 are induced by the steps (3)–(5) of the above sequence of localizations. The standard modification of Corollary 1.4(1) for nonunital C^*-algebras will be stated as Theorem 10.7.

If one goes over to connected components in (1) or homotopy groups in (2), and if one interprets $\pi_*K(B)$ as the classical version of K-theory of C^*-algebras, then the assertions of Corollary 1.4 are well-known. The main point of Corollary 1.4 is that $K(B)$ is not given by the classical definitions but is defined through mapping spectra of the category E which is constructed by a formal homotopy-theoretic procedure of Dwyer–Kan localizations. It is only by Corollary 1.4 that we know that these mapping spectra have the correct homotopy types to represent the classical K-theory of C^*-algebras.

An advantage of the definition of the K-theory functor for C^*-algebras in (1-1) is that it is homotopy invariant, stable, exact, and s-finitary by construction. In addition, in Corollary 9.7 we show, using the equivalence from Corollary 1.4(2), that it preserves filtered colimits. Of course, all these properties are well-known propositions about the classical definition.

Following [Rosenberg and Schochet 1987] we define the UCT-class in KK as the localizing subcategory generated by the tensor unit $\mathrm{kk}(\mathbb{C})$. Using Corollary 1.4(2) we will see in Corollary 9.16 that the natural map $\mathrm{KK}(B,-) \to \mathrm{E}(B,-)$ is an equivalence if B belongs to the UCT-class. Essentially by definition, the K-theory functor induces a symmetric monoidal equivalence between the UCT-class and the stable ∞-category $\mathbf{Mod}(\mathrm{KU})$. This leads to a simple picture of the universal coefficient theorem and the Künneth formula stated in Corollary 9.16.

Cuntz' treatment [1987] of KK-theory is based on the q-construction which involves a functor and a natural transformation

$$q : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}, \quad \iota : q \to \mathtt{id}_{C^*\mathbf{Alg}^{\mathrm{nu}}}.$$

The goal of Section 11 is to study the homotopical features of the q-construction. This whole section is essentially a translation of [Cuntz 1987] from abelian-group-valued functors to functors having values in semiadditive or additive ∞-categories. The main insight derived in this section is that inverting the image of the set $\{\iota_A : qA \to A \mid A \in C^*\mathbf{Alg}^{\mathrm{nu}}\}$ in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ (see step (2) above) yields the universal homotopy invariant, stable, split exact and Schochet-exact functor

$$L_{h,K,q} : C^*\mathbf{Alg}^{\mathrm{nu}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$$

with values in a left-exact additive ∞-category. By Proposition 11.6 it is a Dwyer–Kan localization. Using the deep result [Cuntz 1987, Theorem 1.6] (reproduced in these notes as Theorem 11.13) we will see in the separable case that the Dwyer–Kan localization $L_{\mathrm{sep},q} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ is actually a right-Bousfield localization, and we obtain the very simple formula

$$\ell\,\underline{\mathtt{Hom}}(qA, K \otimes B) \simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(A, B) \tag{1-2}$$

for the mapping space between two *separable* C^*-algebras A and B in this localization. The left-hand side of this equivalence is the space associated to the topological space of homomorphisms from qA to $K \otimes B$.

By [Cuntz 1987] it is known that for two separable C^*-algebras A and B there is an isomorphism

$$\pi_0\underline{\mathtt{Hom}}(qA, K \otimes B) \cong \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}}(A, B). \tag{1-3}$$

It is probably the deepest challenge of these notes to provide an accessible proof of the analogue

$$\pi_0\underline{\mathtt{Hom}}(qA, K \otimes B) \cong \mathrm{KK}_{\mathrm{sep},0}(A, B) \tag{1-4}$$

of this formula with classical KK-theory replaced by the homotopy-theoretic version constructed in the present paper. Note that in contrast to [Cuntz 1987], where (1-3) is essentially the definition of the right-hand side, in our situation the group on the right-hand side of (1-4) is defined as the group of components of the mapping space in a certain Dwyer–Kan localization. By (1-2) and the possibility to replace B by suspensions $S^i(B)$ for i in $\mathbb{N}$ we see that (1-4) is equivalent to the fact that the functor $L_{\mathrm{sep},h,K,q} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ is equivalent to the functor $\mathrm{kk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}}$. An equivalent formulation of this latter fact is the automatic semiexactness theorem, Theorem 12.4, stating that for any additive left-exact ∞-category $\mathbf{D}$, the natural inclusion

$$\mathbf{Fun}^{h,s,se+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \to \mathbf{Fun}^{h,s,splt+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D})$$

from homotopy invariant, stable, Schochet-exact and semiexact functors to homotopy invariant, stable, Schochet-exact and split exact functors is an equivalence. The proof of the automatic semiexactness theorem will be discussed in detail in Section 12.

Note that semiexactness of an exact sequence of C^*-algebras is defined in terms of the existence of a completely positive contractive (cpc) split. Since this is an analytic condition which somehow has to be exploited it is not surprising that the proof of the automatic semiexactness theorem in Section 12 is not purely homotopy theoretic in nature but contains various analytic arguments. But since we will avoid using Kasparov products or other deep results from the classical theory it might be quite accessible to homotopy theorists. In particular note that our proof does not depend on (1-2), whose proof in Theorem 11.13 involves Pedersen's derivation lifting.

But using (1-2) and the automatic semiexactness theorem together in Corollary 12.3 we can show in the context of the present notes that $\mathrm{KK}_{\mathrm{sep}}$ admits countable colimits and is therefore idempotent complete. In [Bunke et al. 2021, Lemma 2.19] this fact has been shown by using [Kasparov 1988, Theorem 2.9].[1]

The classical construction of KK-theory is based on the notion of Kasparov modules. Equivalence classes of Kasparov modules are interpreted as elements of $\mathrm{KK}_0(A, B)$. Kasparov modules in a certain standard form can be captured by Cuntz' work in terms of the q-construction. Essentially by definition, the left-hand side of (1-2) can be interpreted as the space of Kasparov (A, B)-modules. So the mapping spaces in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ are expressed in terms of spaces of Kasparov modules via (1-2), while the automatic semiexactness theorem implies that these are also the mapping spaces in $\mathrm{KK}_{\mathrm{sep}}$. We will not discuss the alternative models for the group-valued KK-theory based on asymptotic morphisms [Thomsen 1999] or localization algebras [Dadarlat et al. 2018].

Recall that the classical concrete model of E-theory [Connes and Higson 1990] involves asymptotic morphisms. In Section 14 we will show that asymptotic morphisms give rise to elements in $\mathrm{E}_0(A, B)$ in a way which is compatible with compositions.

We finally stress that these notes concentrate on the homotopy-theoretic and categorical aspects of KK- and E-theory. The full power of KK-theory to applications, such as the classification programs

[1] Using the results from [Bunke and Duenzinger 2024] one can show that $\mathrm{E}_{\mathrm{sep}}$ admits countable coproducts.

for C^*-algebras, only reveals itself if one employs the equivalence of different cycle-by-relation models based on Kasparov modules. This aspect will not be considered at all in these notes. Other applications, for instance, to the Baum–Connes or Novikov conjecture, require the ability to control the composition of morphisms in *KK*-theory explicitly. If one uses the model based on Kasparov modules, then there are well-developed methods serving this purpose, such as using connections. In the model given in the present paper it is quite tricky to calculate compositions of morphisms which do not simply come from morphisms between the C^*-algebras. We only give one nontrivial example of such a composition, which is Proposition 12.12 and is already complicated enough. But this calculation is absolutely crucial since it provides the last cornerstone for Theorem 12.1 which helps prove the comparison result Corollary 1.3.

2. C^*-algebras

We collect the basic facts from C^*-algebra theory which we will use. The material can be found in the introductory chapters of [Dixmier 1977; Pedersen 1979; Brown and Ozawa 2008; Williams 2007; Pisier 2020].

In order to fix set-theoretic size issues we choose three Grothendieck universes called the small, large, and very large sets.

In Definition 2.4 we will introduce the notions appearing in Definitions 1.1 and 1.2.

We let $C^*\mathbf{Alg}^{\mathrm{nu}}$ denote the large, but locally small category of small C^*-algebras and homomorphisms. By $C^*\mathbf{Alg}$ we denote its subcategory of unital C^*-algebras and unit-preserving homomorphisms. As we are interested in the categorical properties of the categories of C^*-algebras we will follow the approach in [Bunke 2020]. We consider $C^*\mathbf{Alg}^{\mathrm{nu}}$ as a full subcategory of the large locally small category of small $*$-algebras $^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ over $\mathbb{C}$. The latter is the category of small (possibly nonunital) algebras over $\mathbb{C}$ with an antilinear involution $*$ and structure-preserving maps.

A C^**-seminorm* on a $*$-algebra A is a submultiplicative seminorm satisfying the C^*-equality $\|a^*a\| = \|a\|^2$. For a in A we define the *maximal seminorm* of a by $\|a\|_{\max} := \sup_{\|-\|}\|a\|$, where the supremum runs over all C^*-seminorms on A.

We say that A is a *pre-*C^**-algebra* if all its elements have a finite maximal seminorm. The inclusion $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \to {}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ of the category of pre-C^*-algebras into the category of all $*$-algebras is the left-adjoint of a right-Bousfield localization whose right-adjoint is the *bounded elements functor* Bd^∞.

A C^**-algebra* is a pre-C^*-algebra A with the property that $(A, \|-\|_{\max})$ is a Banach space. The inclusion $C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$ of the category of C^*-algebras into the category of pre-C^*-algebras is the right-adjoint of a left-Bousfield localization whose left-adjoint is the *completion functor* compl.

In view of its algebraic description the category ${}^*\mathbf{Alg}^{\mathrm{nu}}_{\mathbb{C}}$ is clearly complete and cocomplete in the sense that it admits all small limits and colimits. As a consequence of the above characterization of C^*-algebras the category $C^*\mathbf{Alg}^{\mathrm{nu}}$ is complete and cocomplete, too. We obtain an explicit description of limits and colimits in terms of their algebraic counterparts indicated by a superscript alg. If $A : I \to C^*\mathbf{Alg}^{\mathrm{nu}}$ is an I-diagram of C^*-algebras for some small index category I, then

$$\lim_I A \cong \mathrm{Bd}^\infty(\lim_I{}^{\mathrm{alg}} A), \quad \mathrm{colim}_I A \cong \mathrm{compl}(\mathrm{colim}_I{}^{\mathrm{alg}} A). \tag{2-1}$$

In particular, the coproduct of the C^*-algebras A_0 and A_1 is represented by the free product of C^*-algebras $A_0 * A_1 := \mathrm{compl}(A_0 *^{\mathrm{alg}} A_1)$ together with the canonical morphisms $\iota_i : A_i \to A_0 * A_1$. Similarly, the product of A_0 and A_1 is represented by the (algebraic) sum $A_0 \oplus A_1$ together with the canonical projections $\mathtt{pr}_i : A_0 \oplus A_1 \to A_i$. If $(A_i)_{i\in I}$ is a small infinite family of C^*-algebras, then (2-1) says that $\prod_{i\in I} A_i \cong \mathrm{Bd}^\infty\bigl(\prod_{i\in I}^{\mathrm{alg}} A_i\bigr)$ is the subalgebra of the algebraic product of families $(a_i)_{i\in I}$ of elements a_i in A_i with $\sup_{i\in I} \|a_i\|_{A_i} < \infty$.

From now on we will suppress the size adjectives large and small as much as possible.

The category $C^*\mathbf{Alg}^{\mathrm{nu}}$ is pointed by the zero algebra 0.

The category $C^*\mathbf{Alg}^{\mathrm{nu}}$ has two canonical symmetric monoidal structures $\otimes_{\max}$ and $\otimes_{\min}$. For C^*-algebras A, B the *maximal tensor product* is defined by $A \otimes_{\max} B := \mathrm{compl}(A \otimes^{\mathrm{alg}} B)$, where we use that the $*$-algebra $A \otimes^{\mathrm{alg}} B$ is actually a pre-C^*-algebra.

In order to define the *minimal tensor product* (also called the spatial tensor product) we equip $A \otimes^{\mathrm{alg}} B$ with the minimal C^*-norm (not seminorm!) and form the closure. This minimal norm can alternatively be characterized as the norm induced by the representation $A \otimes B \to B(H \otimes L)$ induced by any two faithful representations $A \to B(L)$ and $B \to B(H)$ for Hilbert spaces L and H.

We always have a canonical morphism $A \otimes_{\max} B \to A \otimes_{\min} B$, and A is called *nuclear* if this morphism is an isomorphism for all B. Its is known that commutative C^*-algebras and the C^*-algebra K of compact operators on a separable Hilbert space are nuclear. If one of the tensor factors is nuclear we can safely write $\otimes$ and omit the subscript specifying the choice.

Example 2.1. The commutative algebra objects $\mathbf{CAlg}(C^*\mathbf{Alg}^{\mathrm{nu}})$ (say, with respect to $\otimes_{\max}$) are precisely the unital commutative C^*-algebras. □

If X is a compact topological space, then by $C(X)$ we denote the commutative C^*-algebra of continuous $\mathbb{C}$-valued functions on X. For C^*-algebras A and B we let $\underline{\mathtt{Hom}}(A, B)$ denote the compactly generated topological space characterized by the property that for every *compact* space X we have a natural bijection

$$\mathtt{Hom}_{\mathbf{Top}}(X, \underline{\mathtt{Hom}}(A, B)) \cong \mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, C(X) \otimes B). \tag{2-2}$$

The topology on $\underline{\mathtt{Hom}}(A, B)$ is equivalent to the maximal compactly generated topology containing the point-norm topology on $\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, B)$. In this way $C^*\mathbf{Alg}^{\mathrm{nu}}$ becomes a *category enriched in topological spaces*.

A homomorphism $f : B \to C$ between C^*-algebras is a *homotopy equivalence* if there exists a homomorphism $g : C \to B$, called a *homotopy inverse*, such that $f \circ g$ is homotopic to $\mathtt{id}_C$ in $\underline{\mathtt{Hom}}(C, C)$ and $g \circ f$ is homotopic to $\mathtt{id}_B$ in $\underline{\mathtt{Hom}}(B, B)$. Equivalently, one could require that the induced map

$$\underline{\mathtt{Hom}}(A, f) : \underline{\mathtt{Hom}}(A, B) \to \underline{\mathtt{Hom}}(A, C)$$

is a homotopy equivalence of topological spaces for all C^*-algebras A.

A *left-upper-corner inclusion* $A \to A \otimes K$ is a homomorphism of the form $a \mapsto a \otimes e$ where e is a minimal nonzero projection in K.

Remark 2.2. If one interprets K and $A \otimes K$ as algebras of $\mathbb{N}$-indexed matrices with entries in $\mathbb{C}$ or A, respectively, then we can write this map as

$$a \mapsto \begin{pmatrix} a & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This picture explains the name *left-upper-corner* inclusion. □

An exact sequence

$$0 \to I \to B \xrightarrow{\pi} Q \to 0$$

of C^*-algebras is called *semisplit exact* (or *split exact*), if π admits a completely positive contractive (cpc) right-inverse (or a right-inverse homomorphism, respectively). It is known that the functor $A \otimes_{\max} -$ preserves exact sequences and the condition of being semisplit exact or split exact. The functor $A \otimes_{\min} -$ preserves semisplit exact sequences and split exact sequences. A cartesian square in $C^*\mathbf{Alg}^{\mathrm{nu}}$

$$\begin{tikzcd} E \arrow[r] \arrow[d] & B \arrow[d] \\ D \arrow[r] & C \end{tikzcd}$$

is called *exact* (*semisplit*) if the vertical maps are surjective (admit a cpc split). The functor $A \otimes_{\max} -$ preserves exact cartesian squares and also semisplit cartesian squares, and $A \otimes_{\min} -$ preserves semisplit ones. Note, the fact that $B \to C$ is surjective or admits a cpc split implies that $E \to D$ has the same property.

A C^*-algebra is called *separable* if it contains a countable dense subset. We let $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ denote the full subcategory of separable C^*-algebras. Note that $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ is essentially small. For a C^*-algebra A we let $A' \subseteq_{\mathrm{sep}} A$ denote the poset of separable subalgebras of A. Then we have a canonical isomorphism

$$\operatorname*{\mathtt{colim}}_{A' \subseteq A} A' \cong A. \tag{2-3}$$

Example 2.3. The algebra of compact operators $K(H)$ on a separable Hilbert space H is separable. If $\dim(H) = \infty$, then the algebra of bounded operators $B(H)$ is not separable. If X is a separable metric space, then $C_0(X)$ is a separable C^*-algebra. If X is not compact, then the C^*-algebra of bounded continuous functions $C_b(X)$ on X is not separable. □

Let F be a functor defined on $C^*\mathbf{Alg}^{\mathrm{nu}}$ or $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$.

Definition 2.4. (1) F is *homotopy invariant* if F sends homotopy equivalences to equivalences.

(2) F is *stable* if it sends left-upper-corner inclusions to equivalences.

(3) F is *reduced* if $F(0)$ is a zero object.

(4) F is *exact* (*semisplit exact* or *split exact*) if F is reduced and F sends exact (semisplit exact or split exact) sequences to fiber sequences.

(5) If F is defined on $C^*\mathbf{Alg}^{\mathrm{nu}}$, then we say that F is *s-finitary* if for every C^*-algebra A the canonical morphism $\mathtt{colim}_{A'\subseteq_{\mathrm{sep}}A} F(A') \to F(A)$ is an equivalence.

In (3) and (4) we implicitly assume that the target of F is pointed. In (5) we further assume that the colimit exists.

Remark 2.5. In order to check that F is homotopy invariant it suffices to check $F(A) \to F(C([0,1])\otimes A)$ is an equivalence for every C^*-algebra A, where the map is induced by $(\mathbb{C} \to C([0,1])) \otimes \mathtt{id}_A$.

The functor F is s-finitary if and only if it represents the left-Kan extension of the restriction $F_{|C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}}$ along the inclusion $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$. By (2-3), a filtered colimit preserving functor is s-finitary. □

Remark 2.6. Many constructions in the present paper done for $C^*\mathbf{Alg}^{\mathrm{nu}}$ have a version for separable algebras. We will indicate this in the notation by adding subscripts sep to the categories or functors. If everything goes through for separable algebras word by word, then we will simply state that we *have a separable version.* At some places separability matters, and then we will be explicit. □

3. Inverting homotopy equivalences

We study the Dwyer–Kan localization of the category $C^*\mathbf{Alg}^{\mathrm{nu}}$ at the set of homotopy equivalences. We will show that the resulting ∞-category $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is presented by the topological enriched version of $C^*\mathbf{Alg}^{\mathrm{nu}}$ so that we understand the mapping spaces in $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ explicitly. It will turn out that $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is a pointed left-exact ∞-category.

We start with recalling the ∞-categorical background on Dwyer–Kan localizations. Let $\mathbf{C}$ be a ∞-category and W be a set of morphisms in $\mathbf{C}$. Then we can form the *Dywer–Kan localization*

$$L : \mathbf{C} \to \mathbf{C}[W^{-1}]$$

of $\mathbf{C}$ at W. It is characterized by the universal property that

$$L^* : \mathbf{Fun}(\mathbf{C}[W^{-1}], \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^W(\mathbf{C}, \mathbf{D}) \tag{3-1}$$

is an equivalence for every ∞-category $\mathbf{D}$, where the superscript W on the right indicates the full subcategory of functors which send the elements of W to equivalences [Lurie 2017, Definition 1.3.4.1 & Remark 1.3.4.2]. We will apply this to ∞-categories $\mathbf{C}$, $\mathbf{D}$ in the universe of large sets so that $\mathbf{C}[W^{-1}]$ also belongs to this universe.

Remark 3.1. The functor L is essentially surjective and in order to make formulas more readable we will usually denote the image $L(C)$ or $L(f)$ in $\mathbf{C}[W^{-1}]$ of an object or morphism in $\mathbf{C}$ simply by C or f. This convention in particular applies when we insert them into functors defined on $\mathbf{C}[W^{-1}]$. But sometimes we need the longer, more precise notation in order to avoid confusion. □

If $\mathbf{C}$ is symmetric monoidal, then we say that the localization L admits a symmetric monoidal refinement, if $\mathbf{C}[W^{-1}]$ has a symmetric monoidal structure, L has a symmetric monoidal refinement, and we have an equivalence

$$L^* : \mathbf{Fun}_{\otimes/\mathrm{lax}}(\mathbf{C}[W^{-1}], \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^W_{\otimes/\mathrm{lax}}(\mathbf{C}, \mathbf{D}) \tag{3-2}$$

for every symmetric monoidal ∞-category $\mathbf{D}$, where the notation $\otimes/\mathrm{lax}$ indicates two separate formulas, one for symmetric monoidal functors and one for lax symmetric monoidal functors.

In order to check that L has a symmetric monoidal refinement, by [Hinich 2016, Proposition 3.3.2], it suffices to check that the functor $C \otimes -$ preserves W for every object C of $\mathbf{C}$.

Definition 3.2. We let

$$L_h : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}_h \tag{3-3}$$

be the Dwyer–Kan localization of the category $C^*\mathbf{Alg}^{\mathrm{nu}}$ at the homotopy equivalences.

By definition it is characterized by the universal property that pull-back along L_h induces for any ∞-category $\mathbf{D}$ an equivalence

$$L_h^* : \mathbf{Fun}(C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^h(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}), \tag{3-4}$$

where the superscript h indicates the full subcategory of $\mathbf{Fun}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$ of homotopy invariant functors (see Definition 2.4(1)).

We consider the tensor product $\otimes_?$ on $C^*\mathbf{Alg}^{\mathrm{nu}}$ for ? in $\{\max, \min\}$.

Lemma 3.3. *For ? in $\{\max, \min\}$ the localization L_h has a symmetric monoidal refinement.*

Proof. It follows from the functoriality and associativity of $\otimes_?$ and (2-2) that for every C^*-algebra A the functor $A \otimes_? - : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ is continuous for the topological enrichment and therefore preserves homotopy equivalences. This implies that L_h has a symmetric monoidal refinement. □

Thus for every symmetric monoidal ∞-category $\mathbf{D}$ we have an equivalence

$$L_h^* : \mathbf{Fun}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^h_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{3-5}$$

Remark 3.4. Note that on $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ we have two symmetric monoidal structures $\otimes_?$, one for $? = \max$ and one for $? = \min$ which will be discussed in a parallel manner. In particular, (3-5) actually has two versions. □

In contrast to general Dwyer–Kan localizations, in the present case we can understand the mapping spaces in $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ explicitly. In fact, we will see that the topologically enriched category $C^*\mathbf{Alg}^{\mathrm{nu}}$ directly presents the localization. To this end we apply the singular complex functor `sing` to the $\underline{\mathrm{Hom}}$-spaces in order to get a Kan-complex enriched category. Further applying the homotopy coherent nerve we get an ∞-category $C^*\mathbf{Alg}^{\mathrm{nu}}_\infty$ together with a functor $C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}_\infty$ given by the inclusion of the zero skeleton of the mapping spaces.

Proposition 3.5. *The functor $C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}_\infty$ presents the Dwyer–Kan localization of $C^*\mathbf{Alg}^{\mathrm{nu}}$ at the homotopy equivalences.*

Proof. For every C^*-algebra B we define the *path algebra*

$$PB := C(\Delta^1) \otimes B. \tag{3-6}$$

By (2-2), defining a map of simplicial sets $[n] \to \mathtt{sign}(\underline{\mathtt{Hom}}(A, B))$ is equivalent to specifying an element in $\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, C(\Delta^n) \otimes B)$. We let $h_B : [1] \to \mathtt{sign}(\underline{\mathtt{Hom}}(PB, B))$ correspond to the identity of PB. One then checks that for any C^*-algebra A the canonically induced map

$$\mathtt{Hom}_{\mathbf{sSet}}([n], \mathtt{sing}(\underline{\mathtt{Hom}}(A, PB))) \to \mathtt{Hom}_{\mathbf{sSet}}([1] \times [n], \mathtt{sing}(\underline{\mathtt{Hom}}(A, B)))$$

is a bijection. The assertion of Proposition 3.5 now follows from [Lurie 2017, Theorem 1.3.4.7]. □

Corollary 3.6. *The ∞-category $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is locally small.*

Remark 3.7. At various places in this note we will use that small topological spaces present objects in the large ∞-category of small spaces[2] $\mathbf{Spc}$. This is achieved by the functor

$$\ell : \mathbf{Top} \to \mathbf{Spc}, \tag{3-7}$$

which presents the ∞-category $\mathbf{Spc}$ as the Dwyer–Kan localization of $\mathbf{Top}$ at the set of weak homotopy equivalences. One of the fundamental principles, called *Grothendieck's homotopy hypothesis*, states that the ∞-category $\mathbf{Spc}$ defined in this way is equivalent to the ∞-category of ∞-groupoids in which the mapping spaces of locally small ∞-categories naturally live. For a general large ∞-category they belong to the very large ∞-category of large spaces which we will denote by $\mathbf{SPC}$.

We will use that ℓ preserves coproducts, products and sends Serre fibrant cartesian squares to cartesian squares, where a cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ Z \arrow[r] & U \end{tikzcd}$$

in $\mathbf{Top}$ is called *Serre fibrant* if f is a Serre fibration. □

As an immediate corollary of Proposition 3.5 we get an explicit description of the mapping spaces in $C^*\mathbf{Alg}^{\mathrm{nu}}_h$.

Corollary 3.8. *For any two C^*-algebras A, B we have a natural equivalence of spaces*

$$\mathtt{Map}_{C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B) \simeq \ell\,\underline{\mathtt{Hom}}(A, B). \tag{3-8}$$

In the formula above we adopted the conventions from Remark 3.1.

We now discuss limits and colimits in $C^*\mathbf{Alg}^{\mathrm{nu}}_h$.

Proposition 3.9. *The category $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ admits finite products and arbitrary small coproducts, and the localization L_h preserves them.*

[2]This name is changed to *anima* in recent literature.

Proof. We start with finite products. Let $(B_i)_{i\in I}$ be a finite family of C^*-algebras and A be any C^*-algebra. Then we must show that the map

$$\mathtt{Map}_{C^*\mathbf{Alg}_h^{\mathrm{nu}}}\Big(A, \prod_i B_i\Big) \to \prod_{i\in I} \mathtt{Map}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, B_i)$$

induced by the family of projections $\big(\prod_{i\in I} B_i \to B_j\big)_{j\in I}$ is an equivalence. This follows from the fact that

$$\underline{\mathtt{Hom}}\Big(A, \prod_i B_i\Big) \to \prod_{i\in I} \underline{\mathtt{Hom}}(A, B_i) \tag{3-9}$$

is actually a homeomorphism.

We now consider coproducts. Let $(A_i)_{i\in I}$ be a small family of C^*-algebras and B be any C^*-algebra. Then we must show that the map

$$\mathtt{Map}_{C^*\mathbf{Alg}_h^{\mathrm{nu}}}\Big(\coprod_{i\in I} A_i, B\Big) \to \prod_{i\in I} \mathtt{Map}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A_i, B)$$

induced by the family of inclusions $\big(A_j \to \coprod_{i\in I} A_i\big)_{j\in I}$ is an equivalence. This follows from the fact that

$$\underline{\mathtt{Hom}}\Big(\coprod_{i\in I} A_i, B\Big) \to \prod_{i\in I} \underline{\mathtt{Hom}}(A_i, B)$$

is actually a homeomorphism. □

Remark 3.10. In the case of products we assume that the index set I is finite. If it is not finite, then the map (3-9) is no longer a homeomorphism. Let X be a compact topological space. Then the image under (3-9) of $\mathtt{Hom}_{\mathbf{Top}}\big(X, \underline{\mathtt{Hom}}(A, \prod_i B_i)\big)$ in

$$\mathtt{Hom}_{\mathbf{Top}}\Big(X, \prod_{i\in I} \underline{\mathtt{Hom}}(A, B_i)\Big) \cong \prod_{i\in I} \mathtt{Hom}_{\mathbf{Top}}(X, \underline{\mathtt{Hom}}(A, B_i))$$

consists of the families of maps $(\phi_i : X \to \underline{\mathtt{Hom}}(A, B_i))_{i\in I}$ such that the family $(\phi_i(a) : X \to B_i)_{i\in I}$ is equicontinuous for every a in A. □

Lemma 3.11. *The functor L_h is reduced and $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is pointed.*

Proof. The zero algebra represents the initial and the final object of $C^*\mathbf{Alg}_h^{\mathrm{nu}}$. □

Example 3.12. Let A be any C^*-algebra. Then $C_0([0, \infty)) \otimes A$ represents the zero object in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$. □

A morphism $f : B \to C$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$ is called a *Schochet fibration* if the map $f_* : \underline{\mathtt{Hom}}(A, B) \to \underline{\mathtt{Hom}}(A, C)$ is a Serre fibration of topological spaces for every C^*-algebra A [Schochet 1984].

Example 3.13. If $i : Y \to X$ is a map of compact spaces which has the homotopy extension property, then the restriction map $i^* : C(X) \to C(Y)$ is a Schochet fibration which in addition admits a cpc split. □

A cartesian square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ D \arrow[r] & C \end{tikzcd}$$

is called *Schochet fibrant* if f is a Schochet fibration. Note that a Schochet fibration is automatically surjective. If $D = 0$, then we say that

$$0 \to A \to B \to C \to 0$$

is a *Schochet-exact* sequence.

Definition 3.14. A functor $C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{C}$ will be called *Schochet-exact* if it sends Schochet fibrant cartesian squares to cartesian squares.

We will indicate Schochet-exact functors by a superscript as in $\mathbf{Fun}^{Sch}$.

Remark 3.15. In contrast to the other notions of exactness introduced in Definition 2.4(4), the notion of Schochet exactness is formulated in terms of squares instead of exact sequences.

If $\mathbf{C}$ is pointed, then a reduced Schochet-exact functor sends Schochet exact sequences to fiber sequences. If $\mathbf{C}$ is stable, then it is easy to see that the converse is also true. A functor which sends Schochet-exact sequences to fiber sequences is reduced and Schochet-exact; see [Bunke et al. 2021, Lemma 2.14] for analogous statements for semiexact functors and squares. $\square$

Remark 3.16. For the proof of Proposition 3.17(5) below we need the mapping cylinder construction. The *mapping cylinder* of a map $f : B \to C$ of C^*-algebras is defined by the Schochet fibrant and semisplit cartesian square

$$\begin{tikzcd} Z(f) \arrow[r] \arrow[d, "h_f"] & PC \arrow[d, "\mathtt{ev}_0"] \\ B \arrow[r, "f"] & C \end{tikzcd} \tag{3-10}$$

where PC is the path algebra as in (3-6). The maps $h_f : Z(f) \to B$ and $\mathtt{ev}_0$ are homotopy equivalences. We write elements in $Z(f)$ as pairs (b, γ) with b in B and γ in PC such that $\gamma(0) = f(b)$. The map $\tilde{f} : Z(f) \to C$ given by $(b, \gamma) \mapsto \gamma(1)$ is a Schochet fibration and also admits a cpc split $c \mapsto (0, \gamma_c)$ with $\gamma_c(t) := tc$. We further define the *mapping cone* of f by $C(f) := \mathtt{ker}(\tilde{f})$. An element of $C(f)$ is thus a pair (a, γ) of an element of A and a path in C with $f(a) = \gamma(0)$ and $\gamma(1) = 0$.

The sequence

$$0 \to C(f) \xrightarrow{\iota_f} Z(f) \xrightarrow{\tilde{f}} C \to 0 \tag{3-11}$$

is Schochet- and semisplit exact. $\square$

Recall that an ∞-category is called *left-exact* if it admits all finite limits. A functor between left-exact ∞-categories is called *left-exact* if it preserves finite limits. We use the notation $\mathbf{Fun}^{\mathrm{lex}}$ in order to denote the full subcategory of left-exact functors.

Proposition 3.17. (1) *The ∞-category $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is left-exact.*

(2) *The functor L_h sends Schochet fibrant cartesian squares to cartesian squares.*

(3) *The pull-back along L_h induces for every left-exact ∞-category $\mathbf{D}$ an equivalence*

$$L_h^* : \mathbf{Fun}^{\mathrm{lex}}(C^*\mathbf{Alg}_h^{\mathrm{nu}}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{3-12}$$

(4) *The pull-back along the symmetric monoidal refinement of L_h induces for every symmetric monoidal left-exact ∞-category $\mathbf{D}$ an equivalence*

$$L_h^* : \mathbf{Fun}^{\mathrm{lex}}_{\otimes,\mathrm{lax}}(C^*\mathbf{Alg}_h^{\mathrm{nu}}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,Sch}_{\otimes,\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}).$$

(5) *For $?$ in $\{\min, \max\}$ the functor $- \otimes_? - : C^*\mathbf{Alg}_h^{\mathrm{nu}} \times C^*\mathbf{Alg}_h^{\mathrm{nu}} \to C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is bileft-exact.*

Proof. We let W be the subcategory of homotopy equivalences in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and F be the subcategory of Schochet fibrations. Then $(C^*\mathbf{Alg}^{\mathrm{nu}}, W, F)$ is a category of fibrant objects in the sense of [Cisinski 2019, Definitions 7.4.12 and 7.5.7]. The corresponding verifications are due to [Uuye 2013, Theorem 2.19]. The main point is to see that the pull-back of a Schochet fibration or of a homotopy equivalence is again a Schochet fibration or a homotopy equivalence.

The assertions (1) and (2) then follow from [Cisinski 2019, Proposition 7.5.6]. For (2), in view of Corollary 3.8 one could argue alternatively that $\underline{\mathtt{Hom}}(A, -)$ sends Schochet fibrant cartesian squares to Serre fibrant cartesian squares, and that ℓ sends Serre fibrant cartesian squares to cartesian squares.

We now show (3). By (2) it is clear that L_h^* in (3-12) sends left-exact functors to Schochet-exact functors. Since it is the restriction of the equivalence in (3-4) it is fully faithful. We argue that it also essentially surjective. Let F be in $\mathbf{Fun}^{h,Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$. Then by (3-4) there exists $\hat{F}$ in $\mathbf{Fun}(C^*\mathbf{Alg}_h^{\mathrm{nu}}, \mathbf{D})$ such that $L_h^*\hat{F} \simeq F$. We must show that $\hat{F}$ is left-exact. Since it is reduced it suffices to show that it preserves cartesian squares.

In view of Corollary 3.8 any diagram of the shape

$$\begin{tikzcd} & B \arrow[d, "f"] \\ D \arrow[r] & C \end{tikzcd}$$

in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is equivalent to the image under L_h of a diagram of this shape in $C^*\mathbf{Alg}^{\mathrm{nu}}$. We can replace f by $\tilde{f} : Z(f) \to C$ without changing the image of the diagram under L_h up to equivalence. We now complete the diagram to a cartesian square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f'"] & Z(f) \arrow[d, "\tilde{f}"] \\ D \arrow[r] & C \end{tikzcd} \tag{3-13}$$

in $C^*\mathbf{Alg}^{\mathrm{nu}}$. It is Schochet fibrant and semisplit. Its image under L_h is then a cartesian square in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$, and every cartesian square in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is equivalent to one of this form. The image under $\hat{F}$ of L_h of the latter square is equivalent to the image under F of the original Schochet fibrant cartesian square and hence is a cartesian square.

Assertion (4) follows from a combination of (3) and the equivalence (3-5).

In the following, $\otimes$ stands for $\otimes_{\max}$ or $\otimes_{\min}$. Let A be a C^*-algebra. It follows from Proposition 3.9 that the endofunctor $A \otimes - : C^*\mathbf{Alg}^{\mathrm{nu}}_h \to C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is reduced and preserves finite products. So it suffices to show that it preserves cartesian squares. As seen above every cartesian square in $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is equivalent to the image under L_h of a square of the form (3-13).

Using the exactness of $A \otimes_{\max} -$ or the semiexactness of $A \otimes_{\min} -$ we see that

$$\begin{tikzcd} A \otimes P \arrow[r] \arrow[d] & A \otimes Z(f) \arrow[d, "\mathtt{id}_A \otimes \tilde{f}"] \\ A \otimes D \arrow[r] & A \otimes C \end{tikzcd} \tag{3-14}$$

is again cartesian. By analyzing the application of $A \otimes -$ to (3-10) we can obtain an isomorphism

$$\begin{tikzcd} A \otimes Z(f) \arrow[r, "\cong"] \arrow[d, "\mathtt{id}_A \otimes \tilde{f}"] & Z(\mathtt{id}_A \otimes f) \arrow[d, "\widetilde{\mathtt{id}_A \otimes f}"] \\ A \otimes C \arrow[r, equal] & A \otimes C \end{tikzcd}$$

We conclude that the square (3-14) is again a Schochet fibrant cartesian square, and that its image under L_h is a cartesian square in $C^*\mathbf{Alg}^{\mathrm{nu}}_h$. □

Example 3.18. The functor L_h sends the sequence (3-11) to a fiber sequence. Since the square

$$\begin{tikzcd} Z(f) \arrow[r, "\tilde{f}"] \arrow[d, "h_f"] & C \arrow[d, equal] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

commutes up to a preferred homotopy it provides an equivalence between $L_h(\tilde{f})$ and $L_h(f)$. In particular, the mapping cone $C(f)$ represents the fiber of $L_h(f)$. □

Example 3.19. A pointed left-exact ∞-category $\mathbf{C}$ has a loop endofunctor $\Omega : \mathbf{C} \to \mathbf{C}$. For an object C of $\mathbf{C}$, the object ΩC is determined by the pull-back

$$\begin{tikzcd} \Omega C \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd} \tag{3-15}$$

The category $C^*\mathbf{Alg}^{\mathrm{nu}}$ has the suspension endofunctor

$$S := C_0(\mathbb{R}) \otimes - : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}.$$

The square of restriction maps

$$\begin{tikzcd}
C_0(\mathbb{R}) \arrow[r] \arrow[d] & C_0((-\infty,0]) \arrow[d, "\mathtt{ev}_0"] \\
C_0([0,\infty)) \arrow[r, "\mathtt{ev}_0"] & \mathbb{C}
\end{tikzcd}$$

is a Schochet fibrant semisplit cartesian square by Example 3.13. Applying $L_h(-\otimes A)$ we get a cartesian square whose lower-left and upper-right corners are zero objects by Example 3.12. We therefore obtain

$$L_h \circ S \simeq \Omega \circ L_h : C^*\mathbf{Alg}_h^{\mathrm{nu}} \to C^*\mathbf{Alg}_h^{\mathrm{nu}}, \tag{3-16}$$

which is an equivalence of endofunctors. □

Example 3.20. We consider the Puppe sequence associated to a morphism $f : A \to B$. The latter gives rise to the mapping cone sequence (3-11). Since $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is left-exact we can form the diagram

$$\begin{tikzcd}
\Omega^2 L_h(B)) \arrow[r, "\Omega L_h(\partial_f)"] \arrow[d] & \Omega L_h(C(f)) \arrow[r] \arrow[d, "\Omega L_h(i_f)"] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega L_h(A) \arrow[r, "\Omega L_h(f)"] \arrow[d] & \Omega L_h(B) \arrow[r] \arrow[d, "\partial_f"] & 0 \arrow[d] \\
 & 0 \arrow[r] & L_h(C(f)) \arrow[r, "L_h(i_f)"] \arrow[d] & L_h(A) \arrow[d, "L_h(f)"] \\
 & & 0 \arrow[r] & L_h(B)
\end{tikzcd}$$

of pull-back squares in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$. The lower square is cartesian by Example 3.18. We further use the homotopy invariance of L_h applied to the homotopy equivalence h_f in order to replace $L_h(Z(f))$ by $L_h(A)$, and we use (3-15) in order to identify the corners with iterated loops of the objects. For instance, for the corner $\Omega L_h(A)$ just observe that the horizontal composition of the two middle squares is again cartesian. By (3-16) we can express looping in terms of suspension. The sequence

$$\cdots \to L_h(S^2(B)) \to L_h(S(C(f))) \to L_h(S(A)) \to L_h(S(B)) \to L_h(C(f)) \to L_h(A) \to L_h(B) \tag{3-17}$$

in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is called the *Puppe sequence* associated to f. Each consecutive pair of morphisms is a part of a fiber sequence in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$. □

Remark 3.21. If $\mathbf{C}$ is some ∞-category with a set of morphisms $W_{\mathbf{C}}$, $\mathbf{D}$ is a full subcategory, and if we set $\mathbf{W_D} := W \cap \mathbf{D}$, then we have a commutative square

$$\begin{tikzcd}
\mathbf{D} \arrow[r] \arrow[d] & \mathbf{C} \arrow[d] \\
\mathbf{D}[W_{\mathbf{D}}^{-1}] \arrow[r] & \mathbf{C}[W_{\mathbf{C}}^{-1}]
\end{tikzcd}$$

where the vertical functors are Dwyer–Kan localizations. In general the lower-horizontal map is not fully faithful. But this is true if we specialize to the case where $\mathbf{C} = C^*\mathbf{Alg}^{\mathrm{nu}}$, $W_{\mathbf{C}}$ are the homotopy equivalences, and $\mathbf{D} = C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$. □

We write $L_{\mathrm{sep},h} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ for the corresponding Dwyer–Kan localization. Thus for any ∞-category $\mathbf{D}$ pull-back along $L_{\mathrm{sep},h}$ induces an equivalence

$$L^*_{\mathrm{sep},h} : \mathbf{Fun}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^h(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}).$$

Note that $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ is essentially small. Using Corollary 3.8, Proposition 3.17 and its separable version and the fact that the tensor products preserve separable algebras we get the following statements.

Corollary 3.22. (1) *We have a commutative square*

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[r] \arrow[d, "L_{\mathrm{sep},h}"] & C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[d, "L_h"] \\ C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \arrow[r] & C^*\mathbf{Alg}^{\mathrm{nu}}_h \end{tikzcd}$$

whose vertical arrows are Dwyer–Kan localizations and whose horizontal arrows are fully faithful.

(2) *The square in* (1) *has a refinement to a diagram of symmetric monoidal categories and symmetric monoidal functors for* $\otimes_?$ *with* $? \in \{\min, \max\}$ *such that* $L_{\mathrm{sep},h}$ *becomes a symmetric monoidal localization.*

(3) $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ *is pointed and* $L_{\mathrm{sep},h}$ *is reduced.*

(4) $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ *admits finite products and countable coproducts, and* $L_{\mathrm{sep},h}$ *preserves them.*

(5) $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ *is left-exact and* $L_{\mathrm{sep},h}$ *sends Schochet fibrant cartesian squares of separable algebras to cartesian squares.*

(6) *The pull-back along* $L_{\mathrm{sep},h}$ *induces for every left-exact* ∞*-category* $\mathbf{D}$ *an equivalence*

$$L^*_{\mathrm{sep},h} : \mathbf{Fun}^{\mathrm{lex}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}). \tag{3-18}$$

(7) *The pull-back along the symmetric monoidal refinement of* $L_{\mathrm{sep},h}$ *induces for every symmetric monoidal left-exact* ∞*-category* $\mathbf{D}$ *an equivalence*

$$L^*_{\mathrm{sep},h} : \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}).$$

(8) *For* $? \in \{\min, \max\}$ *the functor* $-\otimes- : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \times C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ *is bileft-exact.*

In order to ensure separability of the coproducts, in Corollary 3.22(4) we must restrict to countable families.

4. Stabilization

We consider the Dwyer–Kan localization of the ∞-category $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ from (3-3) at the set of left-upper-corner inclusions $A \to K \otimes A$ for all C^*-algebras A, where K denotes the algebra of compact operators on a separable Hilbert space. It turns out that this localization is a left-Bousfield localization generated by the tensor idempotent K. This fact makes it easy to understand the resulting category $L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$. Its main new feature is semiadditivity.

We start with recalling the ∞-categorical background. Let $\mathbf{C}$ be an ∞-category with an endofunctor $L : \mathbf{C} \to \mathbf{C}$ and a natural transformation $\alpha : \mathtt{id}_{\mathbf{C}} \to L$. If for every object C the morphisms

$$\alpha_{L(C)}, L(\alpha_C) : L(C) \to L(L(C))$$

are equivalences, then $L : C \to L(C)$ is the left-adjoint of a *left-Bousfield localization* with unit α (see [Lurie 2009, Proposition 5.2.7.4]). It is also a Dwyer–Kan localization at the set of morphisms $W_L := \{\alpha_C \mid C \in \mathbf{C}\}$.

Let $\mathbf{C}$ be a left-exact ∞-category with a set of morphisms W. We say that the Dwyer–Kan localization $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ is *left-exact*, if $\mathbf{C}[W^{-1}]$ is left-exact and the functor L is left-exact. In this case, in addition to (3-1), we have an equivalence

$$L^* : \mathbf{Fun}^{\mathrm{lex}}(\mathbf{C}[W^{-1}], \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{\mathrm{lex},W}(\mathbf{C}, \mathbf{D})$$

for any left-exact ∞-category $\mathbf{D}$.

In the present section we encounter a smashing left-Bousfield localization which is generated by an idempotent object [Lurie 2017, Section 4.8.2]. An *idempotent object* in a symmetric monoidal ∞-category $\mathbf{C}$ with tensor unit $\mathbb{1}$ is an object $(\epsilon : \mathbb{1} \to A)$ in the slice $\mathbf{C}_{\mathbb{1}/}$ such that the map $\epsilon \otimes \mathtt{id}_A : A \simeq \mathbb{1} \otimes A \to A \otimes A$ is an equivalence. The inverse of this map is the multiplication of an essentially unique refinement of this object to a commutative algebra object in $\mathbf{C}$.

The functor $L_A := A \otimes - : \mathbf{C} \to L_A\mathbf{C}$ together with the unit $\mathtt{id} \xrightarrow{\epsilon \otimes \mathtt{id}_-} L_A$ satisfies the conditions above ensuring that $L_A : \mathbf{C} \to L_A\mathbf{C}$ is the left-adjoint of a left-Bousfield localization. The localization $L_A : \mathbf{C} \to L_A\mathbf{C}$ is also the Dwyer–Kan localization inverting the set W_A of morphisms $B \xrightarrow{\epsilon \otimes \mathtt{id}_B} A \otimes B$ for all B. By associativity of the tensor product the set W_A is preserved by the functor $- \otimes C$ for any object C of $\mathbf{C}$. It follows that the localization L_A has a symmetric monoidal refinement.

If $\mathbf{C}$ has arbitrary coproducts, then by general properties of a left-Bousfield localization so does $L_A\mathbf{C}$. Given a family $(B_i)_{i \in I}$ in $\mathbf{C}$ we have

$$\coprod_{i \in I}^{L_A\mathbf{C}} L_A(B_i) \simeq L_A\Big(\coprod_{i \in I}^{\mathbf{C}} B_i\Big).$$

Finally, if $\mathbf{C}$ is left-exact and $\otimes$ is bileft-exact, then L_A is a left-exact localization, and the induced tensor product $- \otimes - : L_A\mathbf{C} \times L_A\mathbf{C} \to L_A\mathbf{C}$ is bileft-exact.

We will apply the above constructions to the left-exact symmetric monoidal ∞-category $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ with one of the bileft-exact structures $\otimes_{\max}$ or $\otimes_{\min}$ and the tensor unit $\mathbb{C}$.

Recall that K is the algebra of compact operators on a separable Hilbert space H. Let e be a minimal nonzero projection in K and $\epsilon : \mathbb{C} \to K$ be the homomorphism $\lambda \mapsto \lambda e$.

Lemma 4.1. *$(\epsilon : \mathbb{C} \to K)$ is an idempotent object in $C^*\mathbf{Alg}^{\mathrm{nu}}_h$.*

Proof. For completeness of the presentation we add an argument for this well-known fact from C^*-algebra theory. For t in $[0, 1]$ we define the isometry $U_t : L^2((-\infty, 0]) \to L^2((-\infty, 1])$ by

$$U_t(f)(x) := \begin{cases} f(x), & x \in (-\infty, -t), \\ \frac{1}{\sqrt{2}} f\left(\frac{x-t}{2}\right), & x \in (-t, t), \\ 0, & x \in [t, 1]. \end{cases}$$

Then $t \mapsto U_t$ is strongly continuous and U_1 is unitary.

We now construct a square

$$\begin{tikzcd} H \arrow[r, "e\otimes\mathtt{id}_H"] \arrow[d, "v"] & H \otimes H \arrow[d, "w"] \\ L^2((-\infty, 0]) \arrow[r, "U_0"] & L^2((-\infty, 1]) \end{tikzcd}$$

of isometric maps between Hilbert spaces. For v we choose any unitary isomorphism. In order to construct w we choose an isomorphism $\mathtt{im}(e) \cong \mathbb{C}$ and a unitary isomorphism $v' : \mathtt{im}(e)^\perp \otimes H \to L^2([0, 1])$. Then we get the isomorphism $H \otimes H \cong \mathtt{im}(e) \otimes H \oplus \mathtt{im}(e)^\perp \otimes H \cong H \oplus \mathtt{im}(e)^\perp \otimes H$. We then define $w_{|H} := U_0 \circ v$ and $w_{|\mathtt{im}(e)^\perp \otimes H} := v'$.

Then $t \mapsto \phi_t := w^* U_t v(-) v^* U_t^* w : K \to K \otimes K$ is a point-norm continuous homotopy from $\epsilon \otimes \mathtt{id}_K$ to an isomorphism. $\square$

For every C^*-algebra A the map

$$\kappa_A : A \simeq \mathbb{C} \otimes A \xrightarrow{\epsilon\otimes\mathtt{id}_A} K \otimes A \tag{4-1}$$

is a *left-upper-corner inclusion.*

We let $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ denote the image of the functor $K \otimes -$. Since we have an isomorphism $K \otimes K \cong K$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$, it consists precisely of the objects which are represented by *K-stable C^*-algebras*, i.e., algebras A satisfying $A \cong K \otimes A$ (note that this isomorphism is not related with the left-upper-corner inclusion).

Definition 4.2. We define the functor

$$L_K : C^*\mathbf{Alg}^{\mathrm{nu}}_k \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \quad A \mapsto K \otimes A.$$

and the natural transformation $\kappa : \mathtt{id} \to L_K$ with components $(\kappa_A)_A$.

Note that $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is locally small by Corollary 3.6.

We define the functor

$$L_{h,K} : C^*\mathbf{Alg}^{\mathrm{nu}} \xrightarrow{L_h} C^*\mathbf{Alg}^{\mathrm{nu}}_h \xrightarrow{L_K} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h . \tag{4-2}$$

Corollary 4.3. (1) *The functor L_K is the left-adjoint of a left-Bousfield localization*

$$L_K : C^*\mathbf{Alg}^{\mathrm{nu}}_h \leftrightarrows L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h : \mathrm{incl}$$

unit κ.

(2) *The localization L_K is left-exact.*

(3) *$L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ admits arbitrary small coproducts and L_K preserves them.*

(4) *For $\otimes_?$ with $? \in \{\min, \max\}$ the localization L_K has a symmetric monoidal refinement.*

(5) *The functor $- \otimes_? - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \times L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is bileft-exact.*

For (3) we also used Proposition 3.9. The fact that L_K is a left-Bousfield localization yields the following formula for the mapping spaces in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$. For A, B in $C^*\mathbf{Alg}^{\mathrm{nu}}$ we have

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B) \simeq \mathtt{Map}_{C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, K \otimes B) \overset{(3\text{-}8)}{\simeq} \ell\underline{\mathtt{Hom}}(A, K \otimes B). \tag{4-3}$$

The pull-back along $L_{h,K}$ induces for every ∞-category $\mathbf{D}$ an equivalence

$$L^*_{h,K} : \mathbf{Fun}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}), \tag{4-4}$$

where the additional subscript indicates the full subcategory of $\mathbf{Fun}^h(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$ of stable functors (see Definition 2.4(2)). For any symmetric monoidal ∞-category the pull-back along the symmetric monoidal refinement of $L_{h,K}$ induces an equivalence

$$L^*_{h,K} : \mathbf{Fun}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{4-5}$$

For every left-exact ∞-category $\mathbf{D}$ the pull-back along $L_{h,K}$ induces an equivalence

$$L^*_{h,K} : \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{4-6}$$

If $\mathbf{D}$ is in addition symmetric monoidal, then we have an equivalence

$$L^*_{h,K} : \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}).$$

If $\mathbf{C}$ is a pointed ∞-category with products and coproducts, then for any two objects C and C' in $\mathbf{C}$ we have a canonical morphism

$$(c \mapsto (c, 0)) \sqcup (c' \mapsto (0, c')) : C \sqcup C' \to C \times C'.$$

A pointed ∞-category $\mathbf{C}$ with products and coproducts is called *semiadditive* if this canonical map is an equivalence for every two objects C and C'; see [Lurie 2017, Definition 6.1.6.13].

If $\mathbf{C}$ is semiadditive, then its mapping spaces $\mathtt{Map}_{\mathbf{C}}(C, D)$ have canonical refinements to commutative monoids in $\mathbf{Spc}$. In particular, every object of $\mathbf{C}$ naturally becomes a commutative monoid and a commutative comonoid object in $\mathbf{C}$ [Lurie 2017, Remark 6.1.6.14].

Note that $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is pointed and admits products and coproducts by Corollary 4.3.

Proposition 4.4. *The ∞-category $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is semiadditive.*

Proof. We consider two C^*-algebras A and B. We then have a canonical homomorphism

$$c : A * B \to A \oplus B$$

induced via the universal property of the free product by the homomorphisms $A \to A \oplus B$, $a \mapsto (a, 0)$ and $B \to A \oplus B$, $b \mapsto (0, b)$.

Lemma 4.5 [Cuntz 1987, Proposition 3.1; Meyer 2000, Proposition 5.3]. *$L_{h,K}(c) : L_{h,K}(A * B) \to L_{h,K}(A \oplus B)$ is an equivalence in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$.*

A proof of Lemma 4.5 will be given below after the completion of the argument for Proposition 4.4. We consider C^*-algebras A and B. Then the canonical map from the coproduct to the product of $L_{h,K}(A)$ and $L_{h,K}(B)$ in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ has the following factorization over equivalences:

$$\begin{aligned} L_{h,K}(A) \sqcup L_{h,K}(B) &\simeq L_{h,K}(A * B) \\ &\simeq L_{h,K}(A \oplus B) \\ &\simeq L_K(L_h(A) \times L_h(B)) \\ &\simeq L_{h,K}(A) \times L_{h,K}(B), \end{aligned}$$

where the first, second, third and fourth equivalences are by Corollary 4.3(3), Lemma 4.5, Proposition 3.9, Corollary 4.3(2), respectively. This finishes the proof of Proposition 4.4 assuming Lemma 4.5. ☐

Proof of Lemma 4.5. Since we need some details of the proof of Lemma 4.5 later we recall the argument. We first observe, using that $L_{h,K} \simeq L_{h,K} \circ \mathtt{Mat}_2(-)$, that for any C^*-algebra C the left-upper-corner inclusion $C \to \mathtt{Mat}_2(C)$ is an equivalence in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$.

In the following we consider the C^*-algebras A and B as subsets of $A * B$. We define a homomorphism

$$f : A \oplus B \xrightarrow{(a,b)\mapsto(a,b)} (A * B) \oplus (A * B) \xrightarrow{\mathrm{incl}} \mathtt{Mat}_2(A * B), \quad f(a, b) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Then $f \circ c : A * B \to \mathtt{Mat}_2(A * B)$ is determined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

We consider the family of unitaries

$$U_t := \begin{pmatrix} \cos \frac{\pi t}{2} & -\sin \frac{\pi t}{2} \\ \sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{pmatrix} \tag{4-7}$$

in $\mathtt{Mat}_2(M(A*B))$ and define $h_t : A*B \to \mathtt{Mat}_2(A*B)$ by

$$h_t(a) := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad h_t(b) = U_t^* \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} U_t .$$

Then $h_0 = f \circ c$ and h_1 is an upper-corner inclusion. We have

$$\mathtt{Mat}_2(c) \circ f : A \oplus B \to \mathtt{Mat}_2(A \oplus B), \quad (a,b) \mapsto \begin{pmatrix} (a,0) & 0 \\ 0 & (0,b) \end{pmatrix}.$$

We define the homotopy $h_t : A \oplus B \to \mathtt{Mat}_2(A \oplus B)$ by

$$h_t(a,b) := \begin{pmatrix} \left(a, b\sin^2 \frac{\pi t}{2}\right) & \left(0, b \sin\frac{\pi t}{2}\cos\frac{\pi t}{2}\right) \\ \left(0, b \sin\frac{\pi t}{2}\cos\frac{\pi t}{2}\right) & \left(0, b\cos^2\frac{\pi t}{2}\right) \end{pmatrix}.$$

Then $h_0 = \mathtt{Mat}_2(c) \circ f$ and h_1 is an upper-corner inclusion. □

Remark 4.6. For two C^*-algebras A and B the mapping space $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$ in the semiadditive ∞-category $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is a commutative monoid in **Spc**. One is often interested in its group completion. In this case the observation Corollary 10.13 might be helpful. □

Lemma 4.5 can be generalized to countable coproducts and sums as follows. Recall that $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ admits small coproducts by Corollary 4.3.(3). For a small family $(A_i)_{i \in I}$ of C^*-algebras we can form the sum

$$\bigoplus_{i \in I} A_i := \operatorname*{\mathtt{colim}}_{F \subseteq I, |F| < \infty} \bigoplus_{i \in F} A_i, \tag{4-8}$$

which for infinite I should not be confused with the coproduct or product in $C^*\mathbf{Alg}^{\mathrm{nu}}$. We still have a canonical map $c : *_{i \in I} A_i \to \bigoplus_{i \in I} A_i$.

Proposition 4.7. *If I is countable, then the canonical map induces an equivalence*

$$L_{h,K}(c) : L_{h,K}(*_{i \in I} A_i) \to L_{h,K}\left(\bigoplus_{i \in I} A_i\right).$$

Proof. For finite I this follows by a finite induction from Lemma 4.5. We now assume that $I = \mathbb{N}$. We define the map

$$f : \bigoplus_{i \in I} A_i \to K \otimes *_{i \in I} A_i , \quad f((a_i)_i) := \mathtt{diag}(a_0, a_1, a_2, \dots).$$

Note that $\mathtt{lim}_{i \to \infty} \|a_i\| = 0$ so that this diagonal matrix really belongs to $K \otimes *_{i \in I} A_i$.

The composition $f \circ c : *_{i \in I} A_i \to K \otimes *_{i \in I} A_i$ is determined by $a_i \mapsto \mathtt{diag}(0, \dots, 0, a_i, 0, \dots)$ with a_i at the i-th place. We define a homotopy $h_t : *_{i \in I} A_i \to K \otimes *_{i \in I} A_i$ such that for $t \in \left[1 - \frac{1}{i+1}, 1 - \frac{1}{i+2}\right]$ it rotates in the coordinates 0 and i as in the proof of Lemma 4.5. Then h_t is continuous as a map $[0,1] \to \underline{\mathtt{Hom}}(*_{i \in I} A_i, K \otimes *_{i \in I} A_i)$. Indeed, h_t is continuous on the subalgebras $*_{i=1}^n A_i$ for all n in $\mathbb{N}$. Since their union is dense and h_t is uniformly bounded we conclude continuity. We have $h_0 = f \circ c$ and h_1 is a left-upper-corner embedding. This implies that $L_{h,K}(f) \circ L_{h,K}(c) \simeq L_{h,K}(\mathtt{id}_{*_{i \in I} A_i})$.

The composition $(\mathtt{id}_K \otimes c) \circ f : \bigoplus_{i \in I} A_i \to K \otimes \bigoplus_{i \in I} A_i$ is determined by

$$(a_i)_i \mapsto \mathtt{diag}((a_0, 0, \dots), (0, a_1, 0, \dots), (0, 0, a_2, \dots), \dots).$$

We define the homotopy $l_t : \bigoplus_{i \in I} A_i \to K \otimes \bigoplus_{i \in I} A_i$ such that for $t \in \left[1 - \frac{1}{i+1}, 1 - \frac{1}{i+2}\right]$ it rotates in the coordinates $(0, i)$ and (i, i) as in the proof of Lemma 4.5. Then l_t is continuous as a map $[0, 1] \to \underline{\mathtt{Hom}}\big(\bigoplus_{i \in I} A_i, L_K\big(\bigoplus_{i \in I} A_i\big)\big)$. Indeed, l_t is continuous on the subalgebras $\bigoplus_{i=1}^n A_i$ for all n in $\mathbb{N}$. Since their union is dense and l_t is uniformly bounded we can conclude continuity. We have $l_0 = (\mathtt{id}_K \otimes c) \circ f$ and l_1 is a left-upper-corner embedding. This implies that $L_{h,K}(c) \circ L_{h,K}(f) \simeq L_{h,K}(\mathtt{id}_{\bigoplus_{i \in I} A_i})$. ☐

Example 4.8. For any C^*-algebra A we let $\mathrm{Proj}(A)$ denote the topological space of projections in A. The functor $\mathrm{Proj}(-) : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Top}$ is corepresented by the C^*-algebra $\mathbb{C}$. Indeed, for a C^*-algebra A we have a homeomorphism

$$\underline{\mathtt{Hom}}(\mathbb{C}, A) \xrightarrow{\cong} \mathrm{Proj}(A), \quad f \mapsto f(1).$$

We define the topological space of stable projections in A by

$$\mathrm{Proj}^s(A) := \underline{\mathtt{Hom}}(\mathbb{C}, K \otimes A),$$

which becomes an H-space with respect to the block sum operations. Using the semiadditivity of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}$ its underlying space

$$\mathcal{P}\mathrm{roj}^s(A) := \ell \mathrm{Proj}^s(A) \overset{(4\text{-}3)}{\simeq} \mathtt{Map}_{L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}}(\mathbb{C}, A)$$

has a refinement to an object of $\mathbf{CMon}(\mathbf{Spc})$, i.e., a commutative monoid object in spaces. It will be called the monoid of *stable projections* in A.

We will see in Corollary 10.8 that for unital A the group completion of the commutative monoid $\mathcal{P}\mathrm{roj}^s(A)$ is equivalent to the K-theory space of A. ☐

Example 4.9. For any unital C^*-algebra A we let $U(A)$ denote the topological group of unitaries in A. The functor $U : C^*\mathbf{Alg} \to \mathbf{Groups}(\mathbf{Top})$ is corepresented by the C^*-algebra $C(S^1)$. Let $u : S^1 \to \mathbb{C}$ be the inclusion considered as an element of $U(C(S^1))$. Then we have an isomorphism

$$\underline{\mathtt{Hom}}_u(C(S^1), A) \xrightarrow{\cong} U(A), \quad f \mapsto f(u),$$

of topological groups, where the subscript u indicates that we consider the subspace of $\underline{\mathtt{Hom}}(C(S^1), A)$ of unit-preserving homomorphisms.

Using the unitalization functor $(-)^u : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}$ we define the functor

$$U^s : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Groups}(\mathbf{Top}), \quad A \mapsto \{U \in U((K \otimes A)^u) \mid U - 1_{(K \otimes A)^u} \in K \otimes A\}, \tag{4-9}$$

which associates to A the topological group of *stable unitaries*. The stable unitaries functor is corepresented by the suspension $S(\mathbb{C})$ of $\mathbb{C}$. Indeed, using the split exact sequence

$$0 \to S(\mathbb{C}) \xrightarrow{i} C(S^1) \xrightarrow{\mathtt{ev}_1} \mathbb{C} \to 0, \tag{4-10}$$

we identify $C(S^1)$ with the unitalization of $S(\mathbb{C})$. Then the unitalization functor induces an identification

$$\underline{\mathrm{Hom}}(S(\mathbb{C}), K \otimes A) \cong U^s(A). \tag{4-11}$$

By (4-3) we have an equivalence of spaces

$$\mathcal{U}^s(A) := \mathrm{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(S(\mathbb{C}), A) \simeq \ell U^s(A), \tag{4-12}$$

which equips the commutative monoid on the left with a second group structure. These two structures distribute and therefore coincide by an Eckmann–Hilton-type argument. It follows that the commutative monoid $\mathcal{U}^s(A)$ is already an object of **CGroups**(**Spc**), i.e., a commutative group object in spaces. It will be called the *group of stable unitaries*.

Unfolding definitions we see that the monoid structure on $\mathcal{U}^s(A)$ comes from the block sum of unitaries in $(K \otimes A)^u$ while the other group structures are given by the multiplication of unitaries. In Proposition 9.4 we will show that $\mathcal{U}^s(A)$ is equivalent to a one-fold delooping of the K-theory space of A. $\square$

Example 4.10. We consider two C^*-algebras A and B and homomorphisms $f, f' : A \to B$. We say that $\mathrm{im}(f) \perp \mathrm{im}(f')$ if $f(a)f'(a') = 0 = f'(a')f(a)$ for all a, a' in A. In this case we can define a homomorphism

$$f + f' : A \to B, \quad a \mapsto f(a) + f(a').$$

We then have an equivalence

$$L_{h,K}(f) + L_{h,K}(f') \simeq L_{h,K}(f + f')$$

in $\mathrm{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$, where the sum on the left is the monoid structure on the mapping space coming from the semiadditivity of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ (see Example 6.1). Indeed, this sum is represented by

$$A \xrightarrow{\mathrm{diag}(f,f')} \mathrm{Mat}_2(B) \to K \otimes B.$$

There is a rotation homotopy ϕ_t from $\mathrm{diag}(0, f')$ to $\mathrm{diag}(f', 0)$ such that $\mathrm{im}(\phi_t) \perp \mathrm{im}(\mathrm{diag}(f, 0))$ for all t. Hence $\mathrm{diag}(f, 0) + \phi_t$ is a homotopy from $\mathrm{diag}(f, f')$ to $\mathrm{diag}(f + f', 0)$. This implies the assertion. $\square$

Since K is separable the functor L_K restricts to

$$L_{\mathrm{sep},K} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$$

with an essentially small target. Together with Corollaries 3.22 and 4.3 and Proposition 4.4 this implies the following statements:

Corollary 4.11. (1) *We have a commutative square*

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \arrow[r] \arrow[d, "L_{\mathrm{sep},K}"] & C^*\mathbf{Alg}^{\mathrm{nu}}_h \arrow[d, "L_K"] \\ L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \arrow[r] & L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \end{tikzcd}$$

of ∞-categories where the vertical arrows are left-Bousfield localizations and the horizontal arrows are fully faithful.

(2) *The square in (1) has a refinement to a diagram of symmetric monoidal ∞-categories and symmetric monoidal functors for $\otimes_?$ with $? \in \{\min, \max\}$ such that $L_{\mathrm{sep},K}$ becomes a symmetric monoidal localization.*

(3) *The localization $L_{\mathrm{sep},K}$ is left-exact.*

(4) $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ *admits countable coproducts and $L_{\mathrm{sep},K}$ preserves them.*

(5) *The functor $-\otimes_? - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \times L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ is bileft-exact.*

(6) $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ *is semiadditive.*

The pull-back along $L_{\mathrm{sep},h,K} := L_{\mathrm{sep},K} \circ L_{\mathrm{sep},h}$ induces for every ∞-category $\mathbf{D}$ an equivalence

$$L^*_{\mathrm{sep},h,K} : \mathbf{Fun}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}).$$

For any symmetric monoidal ∞-category the pull-back along the symmetric monoidal refinement of $L_{\mathrm{sep},h,K}$ induces an equivalence

$$L^*_{\mathrm{sep},h,K} : \mathbf{Fun}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}).$$

For every left-exact ∞-category $\mathbf{D}$ the pull-back along $L_{\mathrm{sep},h,K}$ induces an equivalence

$$L^*_{\mathrm{sep},h,K} : \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}).$$

If $\mathbf{D}$ is, in addition, symmetric monoidal, then we have the equivalence

$$L^*_{\mathrm{sep},h,K} : \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}).$$

5. Forcing exactness

We describe left-exact Dwyer–Kan localizations

$$L_! : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$$

for ! in {splt, se, ex} designed such that the composition

$$L_{h,K,!} := L_! \circ L_{h,K} : C^*\mathbf{Alg}^{\mathrm{nu}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$$

(see (4-2) for $L_{h,K}$) sends exact (for ! = ex) or semisplit exact (for ! = se) or split exact (for ! = splt) sequences of C^*-algebras to fiber sequences. We further analyze the compatibility of $L_!$ with the symmetric monoidal structures.

Given an exact sequence $0 \to A \to B \xrightarrow{f} C \to 0$ of C^*-algebras, using the mapping cylinder construction described in Remark 3.16 we can produce a diagram of exact sequences of C^*-algebras

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "\iota_f"] & B \arrow[r, "f"] \arrow[d, "h_f"] & C \arrow[r] \arrow[l, bend right, dotted, "s"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & C(f) \arrow[r] \arrow[d, "\pi_f"] & Z(f) \arrow[r, "\tilde{f}"] & C \arrow[r] \arrow[l, bend left, dotted, "\tilde{s}"] & 0 \\
 & Q(f) \arrow[d] \arrow[u, bend left, dotted, "\hat{s}"] & & & \\
 & 0 & & &
\end{tikzcd}
\tag{5-1}
$$

We write elements of the mapping cone $C(f)$ of f as pairs (b, γ) with b in B and γ in PC (see (3-6)) such that $f(b) = \gamma(0)$ and $\gamma(1) = 0$ and consider A as a subset of B. With this notation the map ι_f is given by $\iota_f(a) := (a, 0)$. This map is the inclusion of an ideal and the C^*-algebra $Q(f)$ is defined as the quotient. We have an isomorphism

$$Q(f) \cong \{(c, \gamma) \in C \times PC \mid \gamma(0) = c, \gamma(1) = 0\} \cong C_0([0, 1)) \otimes C,$$

which implies that Q is contractible. If f admits a cpc split s (or a split), then we get induced cpc splits (or splits) $\tilde{s}(c) := (s(c), \mathtt{const}_c)$ and $\hat{s}(c, \gamma) = (s(c), \gamma)$ as indicated.

We consider a functor $F : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{C}$ to a pointed ∞-category. Versions of the following observation were basic to the approaches in [Higson 1990a; Cuntz and Skandalis 1986].

Proposition 5.1. *We assume that F is homotopy invariant and reduced, and that it sends mapping cone sequences to fiber sequences. Then the following statements are equivalent*:

(1) *F is exact (semiexact, or split exact, respectively).*

(2) *For every exact (semisplit exact, or split exact, respectively) sequence of C^*-algebras $0 \to I \xrightarrow{i} A \xrightarrow{\pi} Q \to 0$ with Q contractible the map $F(i)$ is an equivalence.*

Proof. We assume that F satisfies (1). Then F sends the exact (semisplit exact, or split exact) sequence $0 \to I \xrightarrow{i} A \xrightarrow{\pi} Q \to 0$ to a fiber sequence

$$F(I) \to F(A) \to F(Q)$$

with $F(Q) \simeq 0$. Consequently, $F(i)$ is an equivalence.

Conversely we assume that F satisfies (2). We consider a general exact (semisplit exact or split exact) sequence $0 \to A \to B \xrightarrow{f} C \to 0$ and form the diagram (5-1). By assumption F sends the lower-horizontal sequence to a fiber sequence and $F(\iota_f)$ is an equivalence. By homotopy invariance of F also $F(h_f)$ is an equivalence. We can therefore conclude that $F(A) \to F(B) \to F(C)$ is fiber sequence, too. □

Recall the Definition 3.14 of Schochet-exactness of a functor in terms of Schochet fibrant squares. By our experience, Schochet-exactness is either obvious from the construction or very difficult to verify. The following result shows that, for stable target categories, homotopy invariance and semiexactness together imply Schochet-exactness. We consider homotopy invariance and semiexactness as properties which are much closer to the usual C^*-algebraic business.

Lemma 5.2. *A homotopy invariant and semiexact functor from $C^*\mathbf{Alg}^{\mathrm{nu}}$ to a stable ∞-category is Schochet-exact.*

Proof. Let $F : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{D}$ be a homotopy invariant and semiexact functor to a stable ∞-category. Then F is reduced. Since $\mathbf{D}$ is stable, in order to show Schochet exactness by Remark 3.15 it suffices to show that F sends any Schochet-exact sequence $0 \to A \to B \to C \to 0$ to a fiber sequence.

We apply F to the diagram (5-1) and get a diagram

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, "F(\iota_f)"] & F(B) \arrow[r] \arrow[d, "F(h_f)"] & F(C) \arrow[d, equal] \\ F(C(f)) \arrow[r] \arrow[d] & F(Z(f)) \arrow[r] & F(C) \\ F(Q(f)) & & \end{tikzcd}$$

The middle-horizontal line is a fiber sequence since F is semiexact and the middle-horizontal sequence in (5-1) is semisplit exact. The functor L_h sends both horizontal sequences in (5-1) to fiber sequences since they are Schochet-exact. Since h_f is a homotopy equivalence we see that $L_h(h_f)$ and hence also $L_h(\iota_f)$ are equivalences. Since F is homotopy invariant it factorizes over L_h, and therefore $F(\iota_f)$ is also an equivalence. We can now conclude that the upper-horizontal sequence in the diagram above is a fiber sequence. □

We consider exact sequences

$$0 \to I \xrightarrow{i} A \xrightarrow{\pi} Q \to 0,$$

and define the following sets of morphisms in $C^*\mathbf{Alg}^{\mathrm{nu}}$:

$$\begin{aligned} \widehat{W}_{\mathrm{splt}} &:= \{i \mid \text{for all split exact sequences with } Q \text{ contractible}\}, \\ \widehat{W}_{\mathrm{se}} &:= \{i \mid \text{for all semisplit exact sequences with } Q \text{ contractible}\}, \\ \widehat{W}_{\mathrm{ex}} &:= \{i \mid \text{for all exact sequences with } Q \text{ contractible}\}. \end{aligned} \tag{5-2}$$

We denote the closures in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ under equivalences of their images by $L_{h,K}$ by the same symbols.

Remark 5.3. A natural idea would be to form the Dwyer–Kan localizations of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ at the sets $\widehat{W}_!$ defined above. But there is no reason that these localizations are left-exact. In order to produce left-exact localizations we must localize at the closures of the $\widehat{W}_!$ under pull-backs and the 2-out-of-3 property. □

In the following a set W of morphisms in an ∞-category $\mathbf{C}$ is always assumed to be closed under equivalences. The set W is *closed under pull-backs* if for every cartesian square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] & B \arrow[d, "f"] \\ D \arrow[r] & C \end{tikzcd}$$

in $\mathbf{C}$ with f in W also g belongs to W.

The set W has the 2-*out-of*-3 *property* if the fact that two out of f, g, $g \circ f$ belong to W implies that the third also belongs to W.

Example 5.4. If $F : \mathbf{C} \to \mathbf{D}$ is a functor between ∞-categories, then the set W of morphisms in $\mathbf{C}$ which are sent by F to equivalences has the 2-out-of-3 property. If F is a left-exact functor between left-exact ∞-categories, then W is also closed under pull-backs. □

Let $\mathbf{C}$ be a left-exact ∞-category with a set of morphisms W.

Lemma 5.5. *If W has the* 2*-out-of-*3 *property and is closed under pull-backs, then the Dwyer–Kan localization $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ is left-exact.*

Proof. The triple $(\mathbf{C}, W, \mathbf{C})$ is a category of fibrant objects in the sense of [Cisinski 2019, Definitions 7.4.12 and 7.5.7]. The assertion (1) therefore follows from Proposition 7.5.6 in the same reference. □

Let $\mathbf{C}$ be a left-exact ∞-category with a set of morphisms $\widehat{W}$, and let W be the minimal subset of morphisms containing $\widehat{W}$ which has the 2-out-of-3 property and is closed under pull-backs. Then for every left-exact ∞-category $\mathbf{D}$ the canonical inclusion

$$\mathbf{Fun}^{\mathrm{lex}, W}(\mathbf{C}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{\mathrm{lex}, \widehat{W}}(\mathbf{C}, \mathbf{D}) \tag{5-3}$$

is an equivalence.

For ! in {ex, se, splt} we define $W_!$ as the smallest set of morphisms in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ which is closed under pull-backs and has the 2-out-of-3 property, and which contains $\widehat{W}_!$ from (5-2).

Definition 5.6. We define the Dwyer–Kan localization

$$L_! : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$$

at the set $W_!$.

Note that by construction $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ is a large ∞-category. As we localize at a large set of morphisms we lose the property of being locally small at this point.

We define the functor

$$L_{h,K,!} : C^*\mathbf{Alg}^{\mathrm{nu}} \xrightarrow{L_h} C^*\mathbf{Alg}^{\mathrm{nu}}_h \xrightarrow{L_K} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \xrightarrow{L_!} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} .$$

We let $\mathbf{Fun}^{h,s,!+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$ denote the full subcategory of homotopy invariant and stable functors which are Schochet-exact and exact for $! = \mathrm{ex}$ (semiexact for $! = \mathrm{se}$ or split exact for $! = \mathrm{splt}$, respectively).

Proposition 5.7. (1) *The localization $L_!$ is left-exact.*

(2) *Pull-back along $L_{h,K,!}$ induces for every left-exact ∞-category $\mathbf{D}$ an equivalence*

$$L^*_{h,K,!} : \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{5-4}$$

(3) *$L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ is semiadditive and $L_!$ preserves coproducts.*

Proof. The assertion (1) follows from Lemma 5.5.

We next show (2). For any ∞-category $\mathbf{D}$ we have restriction functors

$$\mathbf{Fun}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \overset{L^*_!}{\simeq} \mathbf{Fun}^{W_!}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h}, \mathbf{D}) \xrightarrow{L^*_{h,K}} \mathbf{Fun}^{h,s}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}),$$

where the first is an equivalence by the universal property of the Dwyer–Kan localization $L_!$. The second functor is the restriction of the equivalence (4-4) and hence fully faithful. If $\mathbf{D}$ is left-exact, then by (1) the first functor restricts to the first equivalence in

$$\mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \overset{L^*_!}{\simeq} \mathbf{Fun}^{\mathrm{lex},W_!}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h}, \mathbf{D}) \overset{(5\text{-}3)}{\simeq} \mathbf{Fun}^{\mathrm{lex},\widehat{W}_!}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h}, \mathbf{D}). \tag{5-5}$$

Finally, by Proposition 5.1 the equivalence (4-6) restricts to an equivalence

$$\mathbf{Fun}^{\mathrm{lex},\widehat{W}_!,}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h}, \mathbf{D}) \overset{L^*_{h,K}}{\simeq} \mathbf{Fun}^{h,s,!+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}).$$

The composition of these equivalences gives (5-4).

Assertion (3) is a general fact about left-exact localizations of left-exact ∞-categories which are semiadditive. ☐

Next we consider the symmetric monoidal structures. We allow the following combinations of ! and ?:

! \ ?	min	max
splt	yes	yes
se	yes	yes
ex	no	yes

The combination (ex, min) is excluded since the minimal tensor product does not preserve exact sequences.

Proposition 5.8. (1) *The localization $L_!$ has a symmetric monoidal refinement.*

(2) *The functor $-\otimes_? - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \times L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ is bileft-exact.*

(3) *Pull-back along $L_{h,K,!}$ induces for every left-exact ∞-category an equivalence*

$$L^*_{h,K,!} : \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}) \tag{5-6}$$

for any symmetric monoidal and left-exact ∞-category $\mathbf{D}$.

Proof. We first observe that the functor

$$A \otimes_? - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$$

preserves the set $\widehat{W}_!$ defined in (5-2). Indeed if $! = \mathrm{ex}$ and $? = \max$, then we use that this functor preserves exact sequences and contractible objects. It is at this point where we must exclude the combination $! = \mathrm{ex}$ and $? = \min$.

If $!$ is in $\{\mathrm{se}, \mathrm{splt}\}$, then this functor preserves semisplit exact or split exact sequences and contractible objects for both $? = \min$ and $? = \max$.

Let $\widetilde{W}_!$ be set of morphisms in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ which are sent to equivalences by $L_!$. Then for every left-exact ∞-category $\mathbf{D}$ we have an equivalence

$$\mathbf{Fun}^{\mathrm{lex}, \widehat{W}_!}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \overset{(5\text{-}3)}{\simeq} \mathbf{Fun}^{\mathrm{lex}, W_!}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \simeq \mathbf{Fun}^{\mathrm{lex}, \widetilde{W}_!}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}). \tag{5-7}$$

By Corollary 4.3(5) and Proposition 5.7(1) the composition

$$L_! \circ (A \otimes_? -) : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \tag{5-8}$$

is left-exact. By the discussion above it inverts $\widehat{W}_!$. It then follows from (5-7) that $A \otimes_? -$ preserves the set $\widetilde{W}_!$. Since $L_!$ is also the Dwyer–Kan localization of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ at $\widetilde{W}_!$ we conclude that the localization $L_!$ has a symmetric monoidal refinement, hence (1) is true.

For (2) we note that the induced functor

$$A \otimes_? - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$$

is left-exact since it is the preimage of (5-8) under the equivalence

$$L_!^* : \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}) \xrightarrow{\simeq} \mathbf{Fun}^{\mathrm{lex}, W_!}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!})$$

(see Proposition 5.8(1)).

We finally show (3). Assertion (1) implies the first equivalence in

$$\mathbf{Fun}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \overset{L_!^*}{\simeq} \mathbf{Fun}^{W_!}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \to \mathbf{Fun}^{h,s}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}),$$

whose second arrow is a restriction of the equivalence (4-5) and hence fully faithful. We now restrict the domain to (lax) symmetric monoidal functors which are in addition left-exact and get

$$\mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \overset{L_!^*}{\simeq} \mathbf{Fun}^{W_!,\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}).$$

The second arrow indeed takes values in the indicated subcategory by Proposition 5.7(2). To see that it is essentially surjective consider a functor F in $\mathbf{Fun}^{h,s,!+Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D})$. Then by (4-5) there is an essentially unique (lax) symmetric monoidal functor $\tilde{F} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to \mathbf{D}$ such that $L^*_{h,K}\tilde{F} \simeq F$. By Proposition 5.7(2) the functor $\tilde{F}$ is left-exact and inverts $W_!$. Thus $\tilde{F}$ belongs to $\mathbf{Fun}^{W_!,\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \mathbf{D})$. $\square$

The constructions above have versions for the category of separable C^*-algebras. We let $\widehat{W}_{\mathrm{sep},!}$ denote the analogue of $\widehat{W}_!$ from (5-2) for separable algebras and $W_{\mathrm{sep},!}$ be the smallest subset of morphisms containing $\widehat{W}_{\mathrm{sep},!}$ which has the 2-out-of-3 property and is closed under pull-backs. For ! in {ex, se, splt} we define the Dwyer–Kan localization

$$L_{\mathrm{sep},!} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!} \tag{5-9}$$

at $W_{\mathrm{sep},!}$. Since $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ is locally small and essentially small, the set of equivalence classes in $W_{\mathrm{sep},!}$ is small. This implies that $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}$ is still essentially small and locally small.

We define the composition

$$L_{\mathrm{sep},h,K,!} := L_{\mathrm{sep},!} \circ L_{\mathrm{sep},h,K} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}. \tag{5-10}$$

Then we have the following statements.

Proposition 5.9. (1) *The localization $L_{\mathrm{sep},!}$ is left-exact.*

(2) *$L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}$ is semiadditive and $L_{\mathrm{sep},!}$ preserves finite coproducts.*

(3) *Pull-back along $L_{\mathrm{sep},h,K,!}$ induces for every left-exact ∞-category $\mathbf{D}$ an equivalence*

$$L^*_{\mathrm{sep},h,K,!} : \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}). \tag{5-11}$$

(4) *The localization $L_{\mathrm{sep},!}$ has a symmetric monoidal refinement.*

(5) *We have a commutative square of symmetric monoidal functors*

$$\begin{tikzcd} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \arrow[r] \arrow[d, "L_{\mathrm{sep},!}"] & L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h} \arrow[d, "L_!"] \\ L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!} \arrow[r] & L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \end{tikzcd} \tag{5-12}$$

(6) *The functor $-\otimes_? - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!} \times L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}$ is bileft-exact.*

(7) *Pull-back along $L_{\mathrm{sep},h,K,!}$ induces for every left-exact ∞-category an equivalence*

$$L^*_{\mathrm{sep},h,K,!} : \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \tag{5-13}$$

for any symmetric monoidal and left-exact ∞-category $\mathbf{D}$.

Remark 5.10. In contrast to Corollary 4.3(3) we do no not know whether $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ admits infinite coproducts.

In contrast to the upper-horizontal arrow in (5-12) the lower-horizontal arrow in this square is not known to be fully faithful; see Remark 3.21. □

Remark 5.11. If the target category $\mathbf{D}$ is stable and we consider ! in {se, ex}, then by Lemma 5.2 (or its separable version) we could remove the superscripts $+Sch$ on the right-hand sides of (5-4), (5-6), (5-11), and (5-13). □

6. Bott periodicity

We analyze the Toeplitz extension

$$0 \to K \to \mathcal{T} \to C(S^1) \to 0$$

from the homotopy-theoretic point of view. The section is essentially an ∞-categorical version of [Cuntz 1984, Section 4]. The main result is Corollary 6.11.

We start with recalling some generalities on group objects in ∞-categories. If $\mathbf{C}$ is an ∞-category admitting cartesian products, then we can consider the ∞-category of commutative algebras in $\mathbf{C}$ for the cartesian monoidal structure which will be called the ∞-category of *commutative monoids* $\mathbf{CMon}(\mathbf{C})$. Let C be a commutative monoid with multiplication map $m : C \times C \to C$. It is called a *commutative group* if the shear map

$$C \times C \xrightarrow{(c,c') \mapsto (c,m(c,c'))} C \times C$$

is an equivalence. We let $\mathbf{CGroups}(\mathbf{C})$ denote the full subcategory of $\mathbf{CMon}(\mathbf{C})$ of commutative groups. Note that we have used these notions already for $\mathbf{C} = \mathbf{Spc}$.

Dually, if $\mathbf{C}$ admits coproducts, then we can consider the ∞-categories of *cocommutative comonoids* $\mathbf{coCMon}(\mathbf{C})$ and its full subcategory *cocommutative cogroups* $\mathbf{coCGroup}(\mathbf{C})$.

Example 6.1. In a semiadditive ∞-category $\mathbf{C}$ every object is naturally a commutative monoid and a commutative comonoid. The functors forgetting the commutative monoid or comonoid structures are equivalences:

$$\mathbf{CMon}(\mathbf{C}) \xrightarrow{\simeq} \mathbf{C} \xleftarrow{\simeq} \mathbf{coCMon}(\mathbf{C}).$$

Let C be in an object of $\mathbf{C}$. Then the multiplication and comultiplication maps of the corresponding monoid or comonoid are given by

$$C \times C \xleftarrow{\simeq} C \sqcup C \xrightarrow{\text{codiag}} C, \quad C \xrightarrow{\mathtt{diag}} C \times C \xleftarrow{\simeq} C \sqcup C.$$

The conditions of being a group or a cogroup are equivalent.

For any two objects C, C' in $\mathbf{C}$ the mapping space $\mathtt{Map}_{\mathbf{C}}(C, C')$ has a natural refinement to an object of $\mathbf{CMon}(\mathbf{Spc})$. The object C' is a group if and only if $\mathtt{Map}_{\mathbf{C}}(C, C')$ is a group for all objects C. □

Example 6.2. The last assertion in Example 6.1 reduces the verification of the group property for an object in a semiadditive category to the case of monoids in spaces. In this case we have a simple criterion. An object X in $\mathbf{CMon}(\mathbf{Spc})$ is a group if an only if the monoid $\pi_0 X$ is a group. □

Lemma 6.3. *If $\mathbf{C}$ is semiadditive and left-exact, then $\Omega : \mathbf{C} \to \mathbf{C}$ (see Example 3.19) takes values in commutative groups.*

Proof. Let C' be an object of $\mathbf{C}$. We must show that $\Omega C'$ is a group. To this end we will show that $\mathtt{Map}_{\mathbf{C}}(C, \Omega C') \simeq \Omega\mathtt{Map}_{\mathbf{C}}(C, C')$ is a group in $\mathbf{Spc}$ for any object C of $\mathbf{C}$. We now use Example 6.2 in order to reduce the problem to the set of components.

Note that $\pi_0\Omega\mathtt{Map}_{\mathbf{C}}(C, C') \cong \pi_1\mathtt{Map}_{\mathbf{C}}(C, C')$ clearly has a group structure $\sharp$ as a fundamental group. This structure distributes over the commutative monoid structure $+$ on $\pi_0\mathtt{Map}_{\mathbf{C}}(C, \Omega C')$ coming from the semiadditivity in the sense that

$$(a+b)\,\sharp\,(c+d) = (a\,\sharp\,c) + (b\,\sharp\,d).$$

The Eckmann–Hilton argument implies that both structures coincide. In particular, the commutative monoid structure $+$ is a commutative group structure. $\square$

To every unital C^*-algebra A we functorially associate the topological space

$$I(A) := \{v \in A \mid v^*v = 1_A\}$$

of isometries in A. By definition, the Toeplitz algebra $\mathcal{T}$ is the isometries classifier in $C^*\mathbf{Alg}$. It contains a universal isometry v such that

$$\underline{\mathtt{Hom}}_u(\mathcal{T}, A) \xrightarrow{\cong} I(A), \quad f \mapsto f(v),$$

is a homeomorphism for every unital C^*-algebra A.

Recall from Example 4.9 that $C(S^1)$ is the unitaries classifier in $C^*\mathbf{Alg}$ with the universal unitary u. Since unitaries are in particular isometries we have a canonical unital homomorphism

$$\pi : \mathcal{T} \to C(S^1), \quad \pi(v) = u.$$

Since $C(S^1)$ is generated by u, the homomorphism π is surjective. We let K denote the kernel of π. We thus have the *Toeplitz exact sequence*

$$0 \to K \to \mathcal{T} \xrightarrow{\pi} C(S^1) \to 0. \tag{6-1}$$

It is known that $\mathcal{T}$ is separable and nuclear. The projection $e := 1_{\mathcal{T}} - vv^*$ belongs to K, and the algebra K is generated by the family of minimal pairwise orthogonal projections $(v^n e v^{*,n})_{n\in\mathbb{N}}$. This provides an identification of K with the algebra of compact operators on a separable Hilbert space and justifies the notation. Note that e is a minimal projection in K.

Using the universal property of $\mathcal{T}$ and the unit we define homomorphisms

$$q : \mathcal{T} \to \mathbb{C}, \quad v \mapsto 1, \qquad j : \mathbb{C} \to \mathcal{T}, \quad 1 \mapsto 1_{\mathcal{T}}.$$

We consider a functor $F : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{C}$ to a semiadditive ∞-category.

Proposition 6.4. *If F is homotopy invariant, stable, split exact and takes values in group objects, then $F(j)$ and $F(q)$ are mutually inverse equivalences.*

Proof. The equality $j \circ q = \mathtt{id}_{\mathbb{C}}$ implies that $F(j) \circ F(q) \simeq \mathtt{id}_{F(\mathbb{C})}$. It remains to show that $F(q) \circ F(j) \simeq \mathtt{id}_{F(\mathcal{T})}$. To this end we construct the following diagram of C^*-algebras:

$$\begin{tikzcd}
 & \mathcal{T} \arrow[d, "\kappa"] & & & & \\
0 \arrow[r] & K \otimes \mathcal{T} \arrow[r, "\iota"] \arrow[d, equal] & \bar{\mathcal{T}} \arrow[r, "p"] \arrow[d, "r"] & \mathcal{T} \arrow[r] \arrow[d, "\tau"] \arrow[ll, dashed, bend right, "j\circ q, \mathtt{id}_{\mathcal{T}}"'] \arrow[l, dotted, bend right, "s, \psi_t"'] \arrow[dl, dashed, "\alpha, \phi_t"'] & 0 & \\
0 \arrow[r] & K \otimes \mathcal{T} \arrow[r] & \mathcal{T} \otimes \mathcal{T} \arrow[r, "\pi\otimes\mathtt{id}_{\mathcal{T}}"] & C(S^1) \otimes \mathcal{T} \arrow[r] & 0 &
\end{tikzcd}$$

The lower-horizontal sequence is the tensor product of the Toeplitz sequence with $\mathcal{T}$. The algebra $\overline{\mathcal{T}}$ is defined such that the right square is a pull-back. This determines the homomorphisms r and p. The maps κ, τ and α are determined by the universal property of $\mathcal{T}$ and the relations

$$\kappa : v \mapsto e \otimes v, \quad \alpha : v \mapsto v(1-e) \otimes 1_{\mathcal{T}}, \quad \tau : v \mapsto u \otimes 1_{\mathcal{T}}.$$

Since $(\pi \otimes \mathtt{id}_{\mathcal{T}}) \circ \alpha = \tau$ we can define the map s by the universal property of the pull-back such that $r \circ s = \alpha$ and $p \circ s = \mathtt{id}_{\mathcal{T}}$. The last equality implies that the upper-horizontal sequence is split exact. By the split exactness of F we get an equivalence

$$F(\iota) \oplus F(s) : F(K \otimes \mathcal{T}) \oplus F(\mathcal{T}) \xrightarrow{\simeq} F(\overline{\mathcal{T}}).$$

In particular we can conclude that $F(\iota)$ is monomorphism.

Since $\mathtt{im}(r\iota\kappa) \perp \mathtt{im}(\alpha)$ (see Example 4.10) we can define homomorphisms

$$\phi_0 := \alpha + r\iota\kappa\,\mathtt{id}_{\mathcal{T}}, \quad \phi_1 := \alpha + r\iota\kappa(j \circ q) \tag{6-2}$$

from $\mathcal{T}$ to $\mathcal{T} \otimes \mathcal{T}$. It has been shown in [Cuntz 1984, Section 4] (see [Fritz 2010] for a nice presentation, reproduced in Remark 6.5 below) that ϕ_0 and ϕ_1 are homotopic by a homotopy $\phi_t : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$ such that $(\pi \otimes \mathtt{id}_{\mathcal{T}}) \circ \phi_t(v) = \tau$ for all t. By the universal property of the pull-back we get a homotopy $\psi_t : \mathcal{T} \to \overline{\mathcal{T}}$ from $s + \iota\kappa\,\mathtt{id}_{\mathcal{T}}$ to $s + \iota\kappa(j \circ q)$ such that $r \circ \psi_t = \phi_t$ and $p \circ \psi_t = \mathtt{id}_{\mathcal{T}}$. By the homotopy invariance of F we get

$$F(s + \iota\kappa\,\mathtt{id}_{\mathcal{T}}) \simeq F(s + \iota\kappa(j \circ q)).$$

In view of Example 4.10 and the fact that by (5-4) for $! = \mathrm{splt}$ the functor F has a left-exact, and hence additive, factorization

$$\begin{tikzcd}
C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[rr, "F"] \arrow[dr, "L_{h,K,!}"'] & & \mathbf{C} \\
 & L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \arrow[ur, dotted] &
\end{tikzcd}$$

we conclude that

$$F(s) + F(\iota\kappa\,\mathtt{id}_{\mathcal{T}}) \simeq F(s) + F(\iota\kappa(j \circ q))$$

as morphisms with target $F(\overline{\mathcal{T}})$. Since $F(\overline{\mathcal{T}})$ is a group object we can cancel $F(s)$ and obtain the equivalence

$$F(\iota\kappa\,\mathtt{id}_{\mathcal{T}}) \simeq F(\iota\kappa(j \circ q)).$$

Since, as seen above, $F(\iota)$ is a monomorphism, and $F(\kappa)$ is an equivalence by stability of F (note that κ is a left-upper-corner inclusion), we can conclude that

$$F(\mathtt{id}_{\mathcal{T}}) \simeq F(j \circ q) \simeq F(j) \circ F(q)$$

as desired. □

Remark 6.5. For completeness of the presentation, following [Fritz 2010] we sketch the construction of the homotopy between ϕ_0 and ϕ_1 from (6-2). These homomorphisms are determined via the universal property of $\mathcal{T}$ by

$$\phi_0(v) = v(1-e) \otimes 1_{\mathcal{T}} + e \otimes v, \quad \phi_1(v) = v(1-e) \otimes 1_{\mathcal{T}} + e \otimes 1_{\mathcal{T}}.$$

We will employ an explicit realization of $\mathcal{T}$ by bounded operators on $L^2(\mathbb{N})$. If $(\xi_i)_{i\in\mathbb{N}}$ denotes the standard basis, then v is the isometry given by $v\xi_i = \xi_{i+1}$ for all i in $\mathbb{N}$, and e is the projection onto the subspace generated by ξ_0.

We realize $\mathcal{T} \otimes \mathcal{T}$ correspondingly on $L^2(\mathbb{N} \times \mathbb{N})$ with basis $(\xi_{i,j})_{i,j\in\mathbb{N}\times\mathbb{N}}$. We define the selfadjoint unitaries

$$u_0 := v(1-e)v^* \otimes 1_{\mathcal{T}} + ev^* \otimes v + ve \otimes v^* + e \otimes e$$

and

$$u_1 := v(1-e)v^* \otimes 1_{\mathcal{T}} + ev^* \otimes 1_{\mathcal{T}} + ve \otimes 1_{\mathcal{T}}.$$

One then checks that

$$u_0(v \otimes 1_{\mathcal{T}})u_0^* = \phi_0(v), \quad u_1(v \otimes 1_{\mathcal{T}})u_1^* = \phi_1(v)$$

in $\mathcal{T} \otimes \mathcal{T}$. On basis vectors the unitary u_0 is given by

$$\xi_{i,j} \mapsto \begin{cases} \xi_{0,0}, & (i,j) = (0,0), \\ \xi_{1,j-1}, & i = 0,\ j \geq 1, \\ \xi_{0,j+1}, & i = 1,\ j \geq 0, \\ \xi_{i,j}, & i \geq 2. \end{cases}$$

We connect u_0 by a homotopy $u_{0,t}$ with $1_{\mathcal{T}\otimes\mathcal{T}}$ by a path in $\mathcal{T} \otimes \mathcal{T}$ which rotates (with constant speed) in each of the two-dimensional subspaces $\mathbb{C}\langle \xi_{1,j-1}, \xi_{0,j} \rangle$ for $j \geq 1$ from flip to the identity. Similarly, the action of u_1 on basis vectors is given by

$$\xi_{i,j} \mapsto \begin{cases} \xi_{1,j}, & i = 0, \\ \xi_{0,j}, & i = 1, \\ \xi_{i,j}, & i \geq 2. \end{cases}$$

We connect u_1 by a homotopy $u_{1,t}$ with $1_{\mathcal{T}\otimes\mathcal{T}}$ by a path in $\mathcal{T}\otimes\mathcal{T}$ which rotates (with constant speed) in each of the two-dimensional subspaces $\mathbb{C}\langle\xi_{0,j},\xi_{1,j}\rangle$ for $j\geq 0$ from the flip to the identity. Then $u^*_{0,t}\phi_0 u_{0,t}$ is a homotopy from ϕ_0 to the map determined by $v\mapsto v\otimes 1_{\mathcal{T}}$. Similarly, $u^*_{1,t}\phi_1 u_{1,t}$ is a homotopy from ϕ_1 to the same map. The concatenation of the first with the inverse of the second homotopy is the desired homotopy ϕ_t. One checks from the explicit formulas that $(\pi\otimes\mathtt{id}_{\mathcal{T}})\circ\phi_t(v)=\tau$ for all t.

Note that the selfadjointness of u_i is not relevant here, but it would be important for a version for real C^*-algebras. □

We now consider the split exact sequence

$$0\to\mathcal{T}_0\to\mathcal{T}\xrightarrow{q}\mathbb{C}\to 0 \tag{6-3}$$

defining $\mathcal{T}_0$ as an ideal in $\mathcal{T}$. As above we consider a functor $F: C^*\mathbf{Alg}^{\mathrm{nu}}\to\mathbf{C}$ to a semiadditive ∞-category.

Corollary 6.6. *If F is homotopy invariant, stable, split exact and takes values in group objects, then $F(\mathcal{T}_0)\simeq 0$.*

Proof. Since F is split exact it sends the split exact sequence (6-3) to a fiber sequence. Since $F(q)$ is an equivalence by Proposition 6.4 we conclude that its fiber $F(\mathcal{T}_0)$ is a zero object. □

The Toeplitz sequence (6-1) is semisplit exact. This can be seen either by an application of the Choi–Effros lifting theorem [1976] using that K is nuclear, or by an explicit construction of a cpc right-inverse s of π; see Remark 6.7.

Remark 6.7. For completeness of the presentation we provide a cpc split for the Toeplitz extension (6-1). We consider $L^2(\mathbb{Z})$ with the standard basis $(\xi_i)_{i\in\mathbb{Z}}$ and realize the Toeplitz algebra $\mathcal{T}$ on the subspace $L^2(\mathbb{N})$ as in Remark 6.5. We let w be the unitary shift operator determined by $\xi_i\mapsto\xi_{i+1}$ for all i in $\mathbb{Z}$ and $P: L^2(\mathbb{Z})\to L^2(\mathbb{N})$ be the orthogonal projection. Then we have $v=PwP$. Since $C(S^1)$ classifies unitaries (see Example 4.9) we have a unique unital homomorphism $\phi: C(S^1)\to B(L^2(\mathbb{Z}))$ determined by $\phi(u):=w$. By an explicit calculation of its action on basis vectors one checks that $[P,\phi(u^n)]$ is finite-dimensional and therefore belongs to K for all n in $\mathbb{Z}$. Since u generates $C(S^1)$ we conclude that $[P,\phi(f)]\in K$ for all f in $C(S^1)$. We define the linear map $s: C(S^1)\to B(L^2(\mathbb{N}))$ by $s(f)=P\phi(f)P$. Using the discussion above one checks that it takes vales in $\mathcal{T}$. Moreover, since $\pi(s(u))=\pi(v)=u$ we conclude that

$$\pi\circ s=\mathtt{id}_{\mathcal{T}}.$$

Since it is the compression of a homomorphism it is completely positive. □

Remark 6.8. The Toeplitz extension does not admit a split. For this reason in the constructions below we must assume that ! belongs to {se, ex} and exclude the case splt. □

We consider the diagram of vertical exact sequence

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
 & \mathcal{T}_0 \arrow[r] \arrow[d, "\pi_0"] & \mathcal{T} \arrow[d, "\pi"] & \\
 & S(\mathbb{C}) \arrow[r, "i"] \arrow[u, dotted, bend left, "s_0"] \arrow[d] & C(S^1) \arrow[u, dotted, bend left, "s"] \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd} \tag{6-4}$$

where the lower square is cartesian and the map i is as in (4-10). The cpc split s induces a cpc split s_0 as indicated.

We consider ! in {se, ex} and apply the semiexact functor $L_{h,K,!}$ to the left-vertical semisplit exact sequence in order to get a fiber sequence

$$L_{h,K,!}(S^2(\mathbb{C})) \xrightarrow{\beta_{!,\mathbb{C}}} L_{h,K,!}(\mathbb{C}) \to L_{h,K,!}(\mathcal{T}_0) \to L_{h,K,!}(S(\mathbb{C})) \tag{6-5}$$

defining $\beta_{!,\mathbb{C}}$, where we used $L_{h,K,!}(\mathbb{C}) \simeq L_{h,K,!}(K)$ by stability of $L_{h,K,!}$.

Since we want to speak about the two-fold loop functor in different left-exact ∞-categories we add subscripts indicating which category is meant in each case. Note that for all k in $\mathbb{N}$, by Example 3.19 and the left-exactness of $L_! \circ L_K$, the k-fold loop functor $\Omega^k_!$ on $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ can be represented by the k-fold suspension on the level of C^*-algebras:

$$\Omega^k_!(-) \simeq L_{h,K,!}(S^k(\mathbb{C})) \otimes - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}.$$

Recall the tensor unit constraint

$$\mathtt{id}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}} \simeq L_{h,K,!}(\mathbb{C}) \otimes - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}.$$

The following definition implicitly uses these identifications.

Definition 6.9. We define a natural transformation of endofunctors

$$\beta_! := \beta_{!,\mathbb{C}} \otimes - : \Omega^2_! \to \mathtt{id}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}.$$

We consider a functor $E : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!} \to \mathbf{C}$ to a semiadditive ∞-category and let A be an object of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$.

Corollary 6.10. *If E is left-exact and $E(-\otimes A)$ takes values in group objects, then $E(\beta_{!,A}) : E(\Omega^2_!(A)) \to E(A)$ is an equivalence.*

Proof. We consider the functor $F(-) := E(- \otimes_{\max} A)$. Then $F(\beta_{!,\mathbb{C}}) \simeq E(\beta_{!,A})$. Using Proposition 5.8(2) we observe that $L^*_{h,K,!}F$ belongs to $\mathbf{Fun}^{h,s,!}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{C})$. Since it also takes values in group objects we can apply Corollary 6.6 in order to conclude that $F(L_{h,K,!}(\mathcal{T}_0)) \simeq 0$.

The functor F sends the fiber sequence (6-5) to a fiber sequence

$$F(\Omega^2 L_{h,K,!}(\mathbb{C})) \xrightarrow{F(\beta_{!,\mathbb{C}})} F(L_{h,K,!}(\mathbb{C})) \to F(L_{h,K,!}(\mathcal{T}_0)).$$

Hence $F(\beta_{!,\mathbb{C}})$ is an equivalence. □

Recall that ! is in {se, ex}.

Corollary 6.11. *If A is a group object in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$, then $\beta_{!,A} : \Omega^2_!(A) \to A$ is an equivalence.*

Proof. We apply Corollary 6.10 to the identity functor in place of E. We further use that if A is a group object, then so is $B \otimes A$ for every B in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$. □

The statements of Proposition 6.4 and Corollaries 6.6, 6.10, and 6.11 all have separable versions which are obtained by adding subscripts sep appropriately.

7. Group objects and $\mathbf{KK}_{\mathrm{sep}}$ and $\mathbf{E}_{\mathrm{sep}}$

We consider the full subcategories of group objects in the semiadditive ∞-categories $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ for ! in {se, ex} and their separable versions. They are the targets of the two-fold loop functor and turn out the be stable ∞-categories. This two-fold loop functor is the right-adjoint of a right-Bousfield localization. It is the last step of the chain of localizations described in Section 1. In the separable case, the composition of all four localizations yields the functors

$$\mathrm{kk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}} \quad \text{and} \quad \mathrm{e}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}}$$

whose universal properties will be stated in (7-7) and (7-9).

Dually to the situation described at the beginning of Section 4 let $\mathbf{C}$ be an ∞-category with an endofunctor $R : \mathbf{C} \to \mathbf{C}$ and a natural transformation $\beta : R \to \mathtt{id}_{\mathbf{C}}$. If for every object C the morphisms

$$\beta_{R(C)}, R(\beta_C) : R(R(C)) \to R(C)$$

are equivalences, then R is the right-adjoint of a right-Bousfield localization with counit β. The functor $R : \mathbf{C} \to R(\mathbf{C})$ is also a Dwyer–Kan localization at the set of morphisms $W_R := \{\beta_C \mid C \in \mathbf{C}\}$. If $\mathbf{C}$ is left-exact, then the localization R is automatically left-exact.

If $\mathbf{C}$ is semiadditive, then we let $\mathbf{C}^{\mathrm{group}}$ denote the full subcategory of group objects in $\mathbf{C}$. A full subcategory of a semiadditive ∞-category which is closed under products is again semiadditive. A semiadditive ∞-category is called *additive* if all its objects are groups. If $\mathbf{C}$ is semiadditive, then $\mathbf{C}^{\mathrm{group}}$ is additive.

Example 7.1. A stable ∞-category is additive. □

We consider ! in {se, ex} and ? in {min, max} allowing the following combinations:

! \ ?	min	max
se	yes	yes
ex	no	yes

We consider the two-fold loop endofunctor $\Omega^2_!$ on $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ and the natural transformation $\beta_! : \Omega^2_! \to \mathtt{id}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}}$ from Definition 6.9.

Proposition 7.2. (1) *The essential image of $\Omega^2_!$ is $L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}$.*

(2) *The functor $\Omega^2_!$ is the right-adjoint of a right-Bousfield localization with counit $\beta_! : \Omega^2_! \to \mathtt{id}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}}$. The localization $\Omega^2_!$ is left-exact.*

(3) *The ∞-category $L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}$ is stable.*

(4) *For every left-exact and additive ∞-category $\mathbf{D}$ we have an equivalence*

$$(\Omega^2_! \circ L_{h,K,!})^* : \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{7-1}$$

(5) *The localization $\Omega^2_!$ admits a symmetric monoidal refinement.*

(6) *The functor*

$$- \otimes_? - : L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!} \times L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}$$

is biexact.

(7) *For every symmetric monoidal, left-exact and additive ∞-category $\mathbf{D}$ we have an equivalence*

$$(\Omega^2_! \circ L_{h,K,!})^* : \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{7-2}$$

Proof. We start with (1). By Lemma 6.3 the functor $\Omega^2_!$ takes values in group objects. If A belongs to $L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}$, then $\beta_{!,A} : \Omega^2_!(A) \to A$ is an equivalence by Corollary 6.11. Hence the essential image of $\Omega^2_!$ is precisely $L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}$.

For (2) we first note that $\beta_{!,\Omega^2_!(A)}$ is an equivalence again by Corollary 6.11. We furthermore employ the symmetry of the tensor product and $\beta_{!,A} \simeq \beta_{!,\mathbb{C}} \otimes A$ in order to see that $\Omega^2_!(\beta_{!,A}) \simeq \beta_{!,\Omega^2_!(A)}$ is an equivalence, too.

In order to show (3) we show that the loop functor

$$\Omega_! : L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}$$

is an equivalence. Indeed, by Corollary 6.11 the restriction of the natural transformation $\beta_! : \Omega^2_! \to \mathtt{id}$ from Definition 6.9 to group objects is an equivalence which exhibits $\Omega_!$ as its own inverse.

Assertion (4) follows from

$$\mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu\,group}}_{h,!}, \mathbf{D}) \overset{\Omega^{2,*}_!}{\simeq} \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \overset{L^*_{h,K,!}}{\simeq} \mathbf{Fun}^{h,s,!+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{7-3}$$

The functor $\Omega_!^{2,*}$ preserves left-exact functors by (2). It is fully faithful and essentially surjective since any left-exact functor from $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ to an additive category $\mathbf{D}$ automatically inverts by Corollary 6.10 the set $W_{\Omega_!^2} := \{\beta_{A,!} \mid A \in L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}\}$ of generators of the Dwyer–Kan localization $\Omega_!^2$. The second equivalence in (7-3) is (5-4).

For (5) we observe that the equivalence $\mathtt{id}_A \otimes \beta_{!,B} \simeq \beta_{!,A\otimes B}$ for all A and B in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ implies that the endofunctor $A \otimes -$ of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$ preserves the set $W_{\Omega_!^2}$. Consequently, $\Omega_!^2$ admits a symmetric monoidal refinement. Furthermore, for A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}$, the endofunctor $A \otimes -$ descends to a left-exact endofunctor on $L_K C^*\mathbf{Alg}^{\mathrm{nu}\,\mathrm{group}}_{h,!}$. This implies (6).

Finally, (7) follows from

$$\mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}\,\mathrm{group}}_{h,!}, \mathbf{D}) \overset{\Omega_!^{2,*}}{\simeq} \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,!}, \mathbf{D}) \overset{L^*_{h,K,!}}{\simeq} \mathbf{Fun}^{h,s,!+Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}),$$

where the first follows from the left-exactness of $\Omega_!^2$ shown in (5), and the second is (5-6). $\square$

Proposition 7.2 has a separable version which we state for later reference.

Proposition 7.3. (1) *The essential image of $\Omega^2_{\mathrm{sep},!}$ is $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}$.*

(2) *The functor $\Omega^2_{\mathrm{sep},!}$ is the right-adjoint of a right-Bousfield localization with counit $\beta_{\mathrm{sep},!} : \Omega^2_{\mathrm{sep},!} \to \mathtt{id}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}}$. The localization $\Omega^2_{\mathrm{sep},!}$ is left-exact.*

(3) *The ∞-category $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}$ is stable.*

(4) *For every left-exact and additive ∞-category $\mathbf{D}$ we have an equivalence*

$$(\Omega^2_{\mathrm{sep},!} \circ L_{\mathrm{sep},h,K,!})^* : \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}). \tag{7-4}$$

(5) *The localization $\Omega^2_{\mathrm{sep},!}$ admits a symmetric monoidal refinement.*

(6) *The functor*

$$- \otimes_? - : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}} \times L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}$$

is biexact.

(7) *For every symmetric monoidal, left-exact and additive ∞-category $\mathbf{D}$ we have an equivalence*

$$(\Omega^2_{\mathrm{sep},!} \circ L_{\mathrm{sep},h,K,!})^* : \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!+Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}). \tag{7-5}$$

We note that $L_K C^*\mathbf{Alg}^{\mathrm{nu}\,\mathrm{group}}_{h,!}$ is a large ∞-category while $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}$ is essentially small and locally small.

Remark 7.4. If $\mathbf{D}$ is stable, then by Lemma 5.2, in the right-hand sides of (7-1), (7-2), (7-4) and (7-5) we can omit the superscript $+Sch$. $\square$

Proposition 7.5. *The functors*

$$\Omega_!^2 \circ L_{h,K,!} : C^*\mathbf{Alg}^{\mathrm{nu}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}\,\mathrm{group}}_{h,!}, \quad \Omega^2_{\mathrm{sep},!} \circ L_{\mathrm{sep},h,K,!} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}$$

are Dwyer–Kan localizations.

Proof. We write out the details in the separable case. The nonseparable case is analogous. The functor $L_{\mathrm{sep},h,K,!} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}$ is constructed as an iterated Dwyer–Kan localization at sets of morphisms in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$. By definition, the last step $\Omega^2_{\mathrm{sep},!} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}$ is a Dwyer–Kan localization at the set of morphisms $\beta_{\mathrm{sep},!,A} : L_{\mathrm{sep},h,K,!}(S^2(A)) \to L_{\mathrm{sep},h,K,!}(A)$ for all A in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$. These morphisms only exist in the localization $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}$. But we can replace them by a collection of images of morphisms in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$. For A in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ we have the commutative diagram

$$\begin{tikzcd}
L_{\mathrm{sep},h,K,!}(S^2(A)) \arrow[r, "\beta_{\mathrm{sep},!,A}"] \arrow[d, "L_{\mathrm{sep},h,K,!}(\lambda_A)"] & L_{\mathrm{sep},h,K,!}(A) \arrow[d, "L_{\mathrm{sep},h,K,!}(\kappa_A)"', "\simeq"] \\
L_{\mathrm{sep},h,K,!}(C(\pi_A)) & L_{\mathrm{sep},h,K,!}(A \otimes K) \arrow[l, "\simeq", "L_{\mathrm{sep},h,K,!}(\iota_{\pi_A})"']
\end{tikzcd}$$

in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}$, where κ_A is the left-upper-corner inclusion (4-1), ι_{π_A} is the canonical inclusion that is associated to the semisplit exact sequence

$$0 \to A \otimes K \to A \otimes \mathcal{T}_0 \xrightarrow{\pi_A} S(A) \to 0$$

(the tensor product of the left column in (6-4) with A) as in (5-1), and $\lambda_A : S^2(A) \to C(\pi_A)$ is the canonical inclusion. Hence $\Omega^2_{\mathrm{sep},!}$ is also the Dwyer–Kan localization at the collection of morphisms $(L_{\mathrm{sep},h,K,!}(\lambda_A))_{A \in C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}}$. We can conclude that the composition $\Omega^2_{\mathrm{sep},!} \circ L_{\mathrm{sep},h,K,!}$ is a Dwyer–Kan localization. □

Definition 7.6. We define the *KK-theory functor for separable C^*-algebras*

$$\mathrm{kk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}} \tag{7-6}$$

to be the functor

$$\Omega^2_{\mathrm{sep},\mathrm{se}} \circ L_{\mathrm{sep},h,K,\mathrm{se}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,\mathrm{se}}{}^{\mathrm{group}} .$$

So $\mathrm{KK}_{\mathrm{sep}}$ is a locally small and essentially small stable ∞-category. The functor $\mathrm{kk}_{\mathrm{sep}}$ has the universal property that

$$\mathrm{kk}^*_{\mathrm{sep}} : \mathbf{Fun}^{\mathrm{lex}}(\mathrm{KK}_{\mathrm{sep}}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,se+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \tag{7-7}$$

for any left-exact and additive ∞-category $\mathbf{D}$. Since we know by Proposition 7.3(3) that $\mathrm{KK}_{\mathrm{sep}}$ is stable the restriction of this universal property to stable ∞-categories $\mathbf{D}$ (where by Remark 7.4 we can omit the superscript $+Sch$) already characterizes the functor $\mathrm{kk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}}$ up to equivalence.

Remark 7.7. By [Bunke et al. 2021, Theorem 1.5] the functor denoted by the same symbol in [Bunke et al. 2021, Definition 1.2] (for the trivial group G) has the same universal property and therefore is canonically equivalent to the functor defined above. We can conclude by [Bunke et al. 2021, Theorem 1.3] that the functor

$$\mathrm{ho} \circ \mathrm{kk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{hoKK}_{\mathrm{sep}}$$

with values in the triangulated category $\mathrm{hoKK_{sep}}$ is canonically equivalent to the classical functor considered in [Meyer and Nest 2006] and the C^*-literature elsewhere. As explained in the introduction, we will give an independent proof for this fact; see Corollary 1.3. □

Remark 7.8. Since $\mathrm{KK_{sep}}$ is stable it admits all finite colimits. After some hard work, in Corollary 12.3 below we will see that $\mathrm{KK_{sep}}$ admits countable colimits and is thus idempotent complete. We do not have a direct proof of this fact just from the constructions. □

Definition 7.9. We define the *E-theory functor for separable C^*-algebras*

$$\mathrm{e_{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{E_{sep}} \tag{7-8}$$

to be the functor

$$\Omega^2_{\mathrm{sep,ex}} \circ L_{\mathrm{sep},h,K,\mathrm{ex}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,\mathrm{ex}}{}^{\mathrm{group}} .$$

So $\mathrm{E_{sep}}$ is a locally small and essentially small stable ∞-category. The functor in (7-8) is the initial homotopy invariant, stable and exact functor to a left-exact and additive ∞-category, i.e., for any left-exact and additive ∞-category $\mathbf{D}$ we have the equivalence

$$\mathrm{e}^*_{\mathrm{sep}} : \mathbf{Fun}^{\mathrm{lex}}(\mathrm{E_{sep}}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,ex+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}). \tag{7-9}$$

Remark 7.10. The justification for calling the functor defined in Definition 7.9 the E-theory functor is that it has an analogous universal property as the additive 1-category-valued E-theory functors considered in [Higson 1990a; Connes and Higson 1990]. In fact, in Theorem 13.16 we show that after going to the homotopy category the functor $\mathrm{e_{sep}}$ becomes equivalent to the classical E-theory functor for separable algebras.

Asymptotic morphisms will be discussed in Section 14 below. □

Remark 7.11. Since $\mathrm{KK_{sep}}$ and $\mathrm{E_{sep}}$ are stable, at a first glance it looks more natural to formulate the universal properties for stable targets $\mathbf{D}$. But we will take advantage of the more general version for left-exact additive ∞-categories in Section 10 below. □

8. *s*-finitary functors

We extend the *KK*- and *E*-theory functors from separable to all C^*-algebras and characterize these extensions by their universal properties.

To any essentially small and locally small stable ∞-category $\mathbf{C}$ we can associate its `Ind`-completion

$$y : \mathbf{C} \to \mathtt{Ind}(\mathbf{C}).$$

As a model, using that $\mathbf{C}$ has mapping spectra, one can take the Yoneda embedding

$$y : \mathbf{C} \to \mathbf{Fun}^{\mathrm{lex}}(\mathbf{C}, \mathbf{Sp}). \tag{8-1}$$

The large stable ∞-category $\mathtt{Ind}(\mathbf{C})$ is presentable, and the fully faithful and exact functor y has the universal property that for any cocomplete stable ∞-category $\mathbf{D}$ the pull-back along y is an equivalence

$$y^* : \mathbf{Fun}^{\mathtt{colim}}(\mathtt{Ind}(\mathbf{C}), \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{\mathrm{lex}}(\mathbf{C}, \mathbf{D}), \tag{8-2}$$

where the superscript $\mathtt{colim}$ indicates small colimit preserving functors. If $\mathbf{C}$ has a biexact symmetric monoidal structure, then $\mathtt{Ind}(\mathbf{C})$ has a natural symmetric monoidal structure and y has a symmetric monoidal refinement such that for any cocomplete bicocontinuous symmetric monoidal ∞-category $\mathbf{D}$ the pull-back along y^* induces an equivalence

$$y^* : \mathbf{Fun}^{\mathtt{colim}}_{\otimes/\mathrm{lax}}(\mathtt{Ind}(\mathbf{C}), \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(\mathbf{C}, \mathbf{D}). \tag{8-3}$$

The inverse of the restriction is given by left-Kan extension. In the model (8-1) the symmetric monoidal structure on the functor category is the Day convolution structure on the functor category.

Let ! be in {se, ex} and ? be in {min, max}. As before we allow the following combinations:

! \ ?	min	max
se	yes	yes
ex	no	yes

For the moment we use the abbreviations

$$\mathrm{KK}_{\mathrm{sep},!} := L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}{}^{\mathrm{group}}, \quad \mathrm{kk}_{\mathrm{sep},!} := \Omega^2_{\sharp,!} \circ L_{\mathrm{sep},h,K,!} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep},!} \tag{8-4}$$

instead of $\mathrm{KK}_{\mathrm{sep}}$ or $\mathrm{E}_{\mathrm{sep}}$ order to discuss KK- and E-theory in a parallel manner.

Definition 8.1. We define $\mathrm{KK}_! := \mathtt{Ind}(\mathrm{KK}_{\mathrm{sep},!})$ and the functor

$$\mathrm{kk}_! : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}_!$$

as the left-Kan extension

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[r, "\mathrm{kk}_{\mathrm{sep},!}"] \arrow[dr, "\mathrm{incl}"'] & \mathrm{KK}_{\mathrm{sep},!} \arrow[r, "y"] & \mathrm{KK}_! \\ & C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[ur, dotted, "\mathrm{kk}_!"'] & \end{tikzcd}$$

of $y \circ \mathrm{kk}_{\mathrm{sep},!}$ along incl.

Since the inclusion functor incl is fully faithful, the triangle commutes up to a natural equivalence.

The following properties of the functor $\mathrm{kk}_! : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}_!$ are immediate from the definition.

Corollary 8.2. (1) *We have an equivalence* $\mathrm{kk}_! \circ \mathrm{incl} \simeq y \circ \mathrm{kk}_{\mathrm{sep},!} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_!$.

(2) $\mathrm{KK}_!$ *is a large presentable stable ∞-category compactly generated by the image of* y.

(3) *The functor* $\mathrm{kk}_! : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}_!$ *is s-finitary.*

(4) $\mathrm{KK}_!$ *admits a bicontinuous symmetric monoidal structure* $\otimes_?$ *and* y^* *has a symmetric monoidal refinement such that*

$$y^* : \mathbf{Fun}^{\mathtt{colim}}_{\otimes/\mathrm{lax}}(\mathrm{KK}_!, \mathbf{C}) \to \mathbf{Fun}^{\otimes,\mathrm{lex}}_{\otimes/\mathrm{lax}}(\mathrm{KK}_{\mathrm{sep},!}, \mathbf{D})$$

is an equivalence for any cocomplete bicontinuous symmetric monoidal ∞*-category with* $\mathbf{D}$.

(5) *The tensor product with the image*

$$b : \mathrm{kk}_!(S^2(\mathbb{C})) \to \mathrm{kk}_!(\mathbb{C}) \tag{8-5}$$

of the equivalence $\beta_{!,\mathbb{C}}$ *in* $\mathrm{KK}_!$ *induces an equivalence* $\Omega^2 \xrightarrow{\simeq} \mathtt{id}_{\mathrm{KK}_!}$ *of endofunctors of* $\mathrm{KK}_!$

Remark 8.3. The main point of Corollary 8.2(5) is that the loop functor on $\mathrm{KK}_!$ is two-periodic, and that this periodicity is implemented by the product with an (necessarily invertible) element b in $\pi_{-2}\mathrm{KK}_!(\mathbb{C}, \mathbb{C}) \simeq \pi_0\mathrm{KK}_!(S^2(\mathbb{C}), \mathbb{C})$, where $\mathrm{KK}_!(\mathbb{C}, \mathbb{C})$ is the commutative endomorphism ring spectrum of the tensor unit of $\mathrm{KK}_!$. This will be used for the calculation of the ring spectrum in Remark 9.19. □

The following results prepare the verification of the universal property of the functor $\mathrm{kk}_!$. We consider a functor $F_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathbf{C}$ and assume that it admits a left-Kan extension

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[rr, "F_{\mathrm{sep}}"] \arrow[dr, "\mathrm{incl}"'] & & \mathbf{C} \\ & C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[ur, dotted, "F"'] & \end{tikzcd}$$

By Remark 2.5, the functor F is s-finitary. Recall the notions introduced in Definition 2.4.

Proposition 8.4. *F inherits, from F_{sep}, the properties*

(1) *homotopy invariance*,

(2) *stability*,

(3) !*-exactness for* ! *in* $\{\mathrm{splt}, \mathrm{se}, \mathrm{ex}\}$, *provided in* $\mathbf{C}$ *filtered colimits preserve fiber sequences.*

Proof. This is shown in [Bunke et al. 2021, Lemma 3.2]; see also Remark 8.8. The case of $! = \mathrm{splt}$ (not discussed in the reference) is analogous to the case $! = \mathrm{se}$. □

Theorem 8.5. (1) *The functor* $\mathrm{kk}_!$ *is homotopy invariant, stable, and* !*-exact.*

(2) *The restriction along* $\mathrm{kk}_!$ *induces for every cocomplete stable* ∞*-category* $\mathbf{D}$ *an equivalence*

$$\mathrm{kk}_!^* : \mathbf{Fun}^{\mathtt{colim}}(\mathrm{KK}_!, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!,sfin}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}).$$

(3) $\mathrm{kk}_!$ *has a natural symmetric monoidal refinement such that restriction along* $\mathrm{kk}_!$ *induces for every cocomplete symmetric monoidal stable* ∞*-category* $\mathbf{D}$ *with bicontinuous symmetric monoidal structure an equivalence*

$$\mathrm{kk}_!^* : \mathbf{Fun}^{\mathtt{colim}}_{\otimes/\mathrm{lax}}(\mathrm{KK}_!, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,!,sfin}_{(\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}).$$

Proof. In order to see (1) note that $\mathrm{kk}_{\mathrm{sep},!}$ is homotopy invariant, stable, and !-exact by Proposition 7.2(4). Since y is exact, the composition $y \circ \mathrm{kk}_{\mathrm{sep},!}$ has these properties, too. The assertion now follows from Proposition 8.4.

Assertion (2) follows from the commutativity of

$$\begin{tikzcd} \mathbf{Fun}^{\mathtt{colim}}(\mathrm{KK}_!, \mathbf{D}) \arrow[r, "y^*", "\simeq,(8\text{-}2)"'] \arrow[d, "\mathrm{kk}_!^*"] & \mathbf{Fun}^{\mathrm{lex}}(\mathrm{KK}_{\mathrm{sep},!}, \mathbf{D}) \arrow[d, "\mathrm{kk}^*_{\mathrm{sep},!}", "\simeq,(7\text{-}1)"'] \\ \mathbf{Fun}^{h,s,!,sfin}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}) \arrow[r, "\mathrm{incl}^*", "\simeq"'] & \mathbf{Fun}^{h,s,!}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \end{tikzcd}$$

where by Proposition 8.4 the inverse of the lower-horizontal functor is the left-Kan extension functor along incl.

The functor $\mathrm{kk}_!$ is defined as a left-Kan extension of a symmetric monoidal functor $y \circ \mathrm{kk}_{\mathrm{sep},!}$ along another symmetric monoidal functor incl. It therefore (see [Bunke et al. 2021, Lemma 3.6]) has a lax symmetric monoidal refinement. As shown in [Bunke et al. 2021, Proposition 3.8] this structure is actually symmetric monoidal. Assertion (3) now follows from the commutativity of

$$\begin{tikzcd} \mathbf{Fun}_{\otimes/\mathrm{lax}}(\mathrm{KK}_!, \mathbf{D}) \arrow[r, "y^*", "\simeq,(8\text{-}3)"'] \arrow[d, "\mathrm{kk}_!^*"] & \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(\mathrm{KK}_{\mathrm{sep},!}, \mathbf{D}) \arrow[d, "\mathrm{kk}^*_{\mathrm{sep},!}", "\simeq,(7\text{-}2)"'] \\ \mathbf{Fun}^{h,s,!,sfin}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}) \arrow[r, "\mathrm{incl}^*", "\simeq"'] & \mathbf{Fun}^{h,s,!}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \end{tikzcd}$$

The inverse of the lower-horizontal morphism is the left-Kan extension functor. It preserves symmetric monoidal functors by same argument as in [Bunke et al. 2021, Proposition 3.8]. ☐

Definition 8.6. (1) We define the *KK*-theory for C^*-algebras by

$$\mathrm{KK} := \mathrm{KK}_{\mathrm{se}}, \quad \mathrm{kk} := \mathrm{kk}_{\mathrm{se}} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}.$$

(2) We define the E-theory for C^*-algebras by

$$\mathrm{E} := \mathrm{KK}_{\mathrm{ex}}, \quad \mathrm{e} := \mathrm{kk}_{\mathrm{ex}} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}.$$

Remark 8.7. The universal properties of *KK*- and E-theory are given by Theorem 8.5.

Thus $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ is the universal functor to a cocomplete stable ∞-category which is homotopy invariant, stable, semiexact and s-finitary. The category KK has presentably symmetric monoidal structures $\otimes_?$ for ? in {min, max}, and the functor kk has corresponding symmetric monoidal refinements which have an analogous universal property for cocomplete stable test categories with bicocontinuous symmetric monoidal structures.

The functor $\mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}$ is the universal functor to a cocomplete stable ∞-category which is homotopy invariant, stable, exact and s-finitary. The category E has a presentably symmetric monoidal structure $\otimes_{\max}$, and the functor e has a corresponding symmetric monoidal refinement which has an analogous universal property for cocomplete stable test categories with bicocontinuous symmetric monoidal structures.

Since exactness is a stronger condition than semiexactness we have a canonical comparison functor fitting into a triangle

$$\begin{tikzcd} & C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[dl, "\mathrm{kk}"'] \arrow[dr, "\mathrm{e}"] & \\ \mathrm{KK} \arrow[rr] & & \mathrm{E} \end{tikzcd}$$

which commutes up to a natural transformation. Under certain conditions it induces an equivalence on mapping spaces; see Corollary 9.16 for a detailed statement.

If A, B are separable C^*-algebras, then we have equivalences

$$\mathrm{KK}_{\mathrm{sep}}(A, B) \simeq \mathrm{KK}(A, B), \quad \mathrm{E}_{\mathrm{sep}}(A, B) \simeq \mathrm{E}(A, B), \tag{8-6}$$

by Corollary 8.2(1). □

Remark 8.8. We apologize for introducing an incompleteness of the presentation by deferring the proof of Proposition 8.4 to the reference [Bunke et al. 2021, Lemma 3.2]. But let us point out that the argument for [Bunke et al. 2021, Lemma 3.2] only employs elementary facts about C^*-algebras and their tensor products and not any deeper parts from *KK*-theory. It therefore should be directly accessible for readers having reached this point of the present paper. The same applies to the argument that $\mathrm{kk}_!$ is actually symmetric monoidal (in contrast to being lax symmetric monoidal) which is deferred to [Bunke et al. 2021, Proposition 3.8]. This argument is also by elementary C^*-algebra theory, but the case of the maximal tensor product is more involved since it uses [Bunke et al. 2021, Lemma 7.18] which does not seem to be so standard. □

9. *K*-theory and the stable group of unitaries

Using *E*-theory we can give a simple construction of a highly structured version of the topological *K*-functor for C^*-algebras. The main goal of this section is to relate this *K*-theory functor with the stable unitary group functor from (4-12).

In order to construct the *K*-theory functor we shall use the following general facts.

Remark 9.1. If $\mathbf{C}$ is a stable symmetric monoidal ∞-category with tensor unit $\mathbb{1}$, then the functor $\mathtt{map}_{\mathbf{C}}(\mathbb{1}, -) : \mathbf{C} \to \mathbf{Sp}$ is lax symmetric monoidal. Since $\mathbb{1}$ is naturally a commutative algebra object in $\mathbf{C}$ we get a commutative ring spectrum $R := \mathtt{map}_{\mathbf{C}}(\mathbb{1}, \mathbb{1})$. The ∞-category $\mathbf{C}$ has then a natural R-linear structure. In particular, its mapping spectra $\mathtt{map}_{\mathbf{C}}(C, D)$ naturally refine to objects of $\mathbf{Mod}(R)$ such that the composition is R-bilinear. □

We will apply this to the symmetric monoidal ∞-category E. Its tensor unit is given by $\mathbb{1}_{\mathrm{E}} := \mathrm{e}(\mathbb{C})$.

Definition 9.2. We define the commutative ring spectrum $\mathrm{KU} := \mathrm{E}(\mathbb{C}, \mathbb{C})$ in $\mathbf{CAlg}(\mathbf{Sp})$.

We refer to Remark 9.19 for a justification of the notation. The stable ∞-category E becomes a KU-linear stable ∞-category. In particular, its mapping spectra $\mathrm{E}(A, B)$ naturally belong to $\mathbf{Mod}(\mathrm{KU})$.

For the moment we consider the maximal tensor product on $C^*\mathbf{Alg}^{\mathrm{nu}}$. In order to incorporate the minimal tensor product, see Corollary 9.14.

Definition 9.3. The lax symmetric monoidal *topological K-theory functor* for C^*-algebras is defined by

$$K := \mathrm{E}(\mathbb{C}, -) : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Mod}(\mathrm{KU}).$$

By construction, K is homotopy invariant, stable and exact. Since $\mathbb{C}$ is separable, the object $\mathrm{e}(\mathbb{C})$ in E is compact. Hence, s-finitaryness of $\mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}$ implies that the K-theory functor is also s-finitary.

Recall the stable unitary group functor $\mathcal{U}^s$ from (4-12).

Proposition 9.4. *We have a canonical equivalence of functors*

$$\mathcal{U}^s \simeq \Omega^{\infty-1} K : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{CGroups}(\mathbf{Spc}). \tag{9-1}$$

Proof. Using Definition 9.3, stability of the ∞-category E and Example 3.19 we get an equivalence

$$\mathtt{Map}_{\mathrm{E}}(S(\mathbb{C}), -) \simeq \Omega^{\infty-1}\mathrm{E}(\mathbb{C}, -) \simeq \Omega^{\infty-1} K(-). \tag{9-2}$$

We furthermore have a transformation of $\mathbf{CGroups}(\mathbf{Spc})$-valued functors

$$\mathcal{U}^s_{\mathrm{sep}}(-) \overset{(4\text{-}12)}{\simeq} \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}}(S(\mathbb{C}), -) \xrightarrow{\Omega^2_{\mathrm{sep,ex}} \circ L_{\mathrm{sep,ex}}} \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(S(\mathbb{C}), -) \overset{(8\text{-}6)}{\simeq} \mathtt{Map}_{\mathrm{E}}(S(\mathbb{C}), -)_{|C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}}, \tag{9-3}$$

where $\mathcal{U}^s_{\mathrm{sep}}$ is the restriction of $\mathcal{U}^s$ to separable algebras. We now employ the following facts.

Lemma 9.5. *The composition (9-3) is an equivalence.*

Lemma 9.6. *The functor $\mathcal{U}^s$ preserves small filtered colimits and is in particular s-finitary.*

Combining both results we get the desired equivalence (9-1) by left-Kan extending the equivalence (9-3) along $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ and composing with (9-2). ☐

Corollary 9.7. *The K-theory functor $K : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Sp}$ preserves small filtered colimits.*

Proof. We combine Lemma 9.6 with Proposition 9.4 and two-periodicity. ☐

Remark 9.8. Using that classical E-theory for separable C^*-algebras preserves countable sums [Guentner et al. 2000, Proposition 7.1] one can show using Theorem 13.16 that $\mathrm{e}_{\mathrm{sep}}$ preserves countable sums. This implies by [Bunke and Duenzinger 2024, Proposition 3.17] that $\mathrm{e}_{\mathrm{sep}}$ preserves all countable filtered colimits. Since $\mathrm{e}(\mathbb{C})$ is a compact object of E this would give an alternative argument for the fact that K preserves filtered small colimits. ☐

Remark 9.9. In the proof of Lemma 9.5 we will employ the following general fact about mapping spaces in a Dwyer–Kan localization $\ell : \mathbf{C} \to \mathbf{C}[W^{-1}]$ of ∞-categories. We call an object C of $\mathbf{C}$ *colocal for W* if the functor $\mathtt{Map}_{\mathbf{C}}(C, -)$ sends the elements of W to equivalences. The following assertion is an easy consequence of the Yoneda lemma. If C is colocal for W, then $\ell : \mathtt{Map}_{\mathbf{C}}(C, -) \to \mathtt{Map}_{\mathbf{C}[W^{-1}]}(\ell(C), \ell(-))$ is an equivalence of functors from $\mathbf{C}$ to $\mathbf{Spc}$. ☐

Proof of Lemma 9.5. We must show that the composition

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}}(S(\mathbb{C}), -) \xrightarrow{L_{\mathrm{sep,ex}}} \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,\mathrm{ex}}}(S(\mathbb{C}), -) \xrightarrow{\Omega^2_{\mathrm{sep,ex}}} \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(S(\mathbb{C}), -)$$

is an equivalence. Since $S(\mathbb{C})$ represents a group in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,\mathrm{ex}}$ and $\Omega^2_{\mathrm{sep,ex}}$ is by Proposition 7.2 a right-Bousfield localization at the groups the second morphism is an equivalence. By Remark 9.9, in order to show that the Dwyer–Kan localization $L_{\mathrm{sep,ex}}$ induces an equivalence of mapping spaces it suffices to show that $L_{\mathrm{sep},h,K}(S(\mathbb{C}))$ is colocal for $W_{\mathrm{sep,ex}}$. Since $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}}(S(\mathbb{C}), -)$ is left-exact, by (5-3) it suffices to show that $L_{\mathrm{sep},h,K}(S(\mathbb{C}))$ is colocal for $\widehat{W}_{\mathrm{sep,ex}}$ from (5-2). By Proposition 5.1 it suffices to show that $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}}(S(\mathbb{C}), -)$ sends exact sequences to fiber sequences. In view of (4-12) this follows from the following lemma since ℓ sends Serre fiber sequences to fiber sequences.

Lemma 9.10. *If $0 \to A \to B \to C \to 0$ is an exact sequence in $C^*\mathbf{Alg}^{\mathrm{nu}}$, then $U^s(B) \to U^s(C)$ is a Serre fibration with fiber $U^s(A)$.*

Proof. This lemma is surely well-known in C^*-algebra theory. For completeness of the presentation we add a proof.

It is clear from the definition (4-9) that $U^s(A)$ is the fiber of the map $U^s(B) \to U^s(C)$. In order to show that this map is a Serre fibration we will solve the lifting problem

$$\begin{tikzcd} X \arrow[r] \arrow[d, "x\mapsto(0,x)"'] & U^s(B) \arrow[d] \\ {[0,1]\times X} \arrow[r] \arrow[ur, dotted] & U^s(C) \end{tikzcd}$$

for all compact X. By (2-2) and (4-11) this lifting problem is equivalent to

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & U^s(C(X)\otimes B) \arrow[d] \\ {[0,1]} \arrow[r] \arrow[ur, dotted] & U^s(C(X)\otimes C) \end{tikzcd}$$

It thus suffices to solve the path lifting problems

$$\begin{tikzcd} \{0\} \arrow[r, "u"] \arrow[d] & U^s(B) \arrow[d] \\ {[0,1]} \arrow[r, "\gamma"] \arrow[ur, dotted, "\tilde{\gamma}"] & U^s(C) \end{tikzcd} \tag{9-4}$$

for all surjections $B \to C$.

We call a path $\sigma : [0, 1] \to U^s(C)$ short if $\sigma(0) = 1$ and $\|\sigma(t) - 1\| < 1$ for all t. For the moment we assume that we can lift short paths to paths that start at 1 in $U^s(B)$.

Let $\gamma : [0, 1] \to U^s(C)$ be a general path. Then we can find a natural number n such that the segment $\gamma\left(\frac{i}{n}\right)^{-1}\gamma_{|[i/n,(i+1)/n]}$ is short for all $i = 0, \ldots, n-1$ (we implicitly reparametrize). We can now lift γ

inductively. We are given the lift u of $\gamma_{|[0,0]}$. Assume that have found a lift $\tilde{\gamma}$ of $\gamma_{|[0,i/n]}$. We choose a lift $\tilde{\sigma}$ of the short path $\gamma^{-1}(\frac{i}{n})\gamma_{|[i/n,(i+1)/n]}$ and define an extension of $\tilde{\gamma}$ on $[\frac{i}{n},\frac{i+1}{n}]$ by $\tilde{\gamma}(\frac{i}{n})\tilde{\sigma}$.

It remains to solve the lifting problem for short paths. We observe that the exponential map of $(K\otimes A)^u$ restricts to $\exp : i(K\otimes A)^{\mathrm{sa}} \to U^s(A)$ with the partial inverse $\log : \{U \in U^s(A) \mid \|U-1\| < 1\} \to i(K\otimes A)^{\mathrm{sa}}$. Since the tensor product preserves surjections, the map $C_0((0,1])\otimes K\otimes B \to C((0,1])\otimes K\otimes C$ and its restriction to antiselfadjoint elements are surjective. We view $\log\sigma$ as an element in $i(C_0((0,1])\otimes K\otimes C)^{\mathrm{sa}}$ and can thus choose a lift $\widehat{\log\sigma}$ in $i(C_0((0,1])\otimes K\otimes B)^{\mathrm{sa}}$. Then $\tilde{\sigma} := \exp(\widehat{\log\sigma}) : [0,1] \to U^s(B)$ is the desired lift of σ. We have thus shown Lemma 9.10. $\square$

This finishes the proof of Lemma 9.5. $\square$

Remark 9.11. Using that $S(\mathbb{C})$ is a semiprojective C^*-algebra, we could deduce the path lifting in (9-4) from [Blackadar 2016, Theorem 5.1]. $\square$

Proof of Lemma 9.6. In view of Remark 2.5 it suffices to show that the functor $\mathcal{U}^s : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{CGroups}(\mathbf{Spc})$ preserves small filtered colimits. Since the forgetful functor $\mathbf{CGroups}(\mathbf{Spc}) \to \mathbf{Spc}$ preserves small filtered colimits and is conservative it suffices to show that the underlying $\mathbf{Spc}$-valued functor of $\mathcal{U}^s$ preserves small filtered colimits.

Let I be a small filtered poset, $(B_i)_{i\in I}$ be an I-indexed family in $C^*\mathbf{Alg}^{\mathrm{nu}}$, and set $B := \mathtt{colim}_{i\in I} B_i$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$. Then we must show that the canonical maps

$$\pi_n(\mathop{\mathtt{colim}}_{i\in I} \ell U^s(B_i)) \to \pi_n(\ell U^s(B))$$

are isomorphisms at all choices of base points and for all n in $\mathbb{N}$. Since taking homotopy groups/sets on $\mathbf{Spc}$ commute with filtered colimits it suffices to show that the canonical maps $\mathtt{colim}_{i\in I}\pi_n(U^s(B_i)) \to \pi_n(U^s(B))$ are isomorphisms. This will be an immediate consequence of Lemma 9.12 below applied to the inclusions $S^n \to D^{n+1}$ or $\varnothing \to S^n$.

For i, j in I with $i \le j$ let $\phi_{j,i} : (K\otimes B_i)^u \to (K\otimes B_j)^u$ and $\phi_i : (K\otimes B_i)^u \to (K\otimes B)^u$ denote the connecting map and the canonical homomorphism.

Let X be any compact metrizable space and Y be a closed subspace. We fix i_0 in I and assume that we are given a square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & U^s(B_{i_0}) \arrow[d, "\phi_{i_0}"] \\ X \arrow[r, "g"] & U^s(B) \end{tikzcd}$$

Lemma 9.12. *There exists i in I with $i \ge i_0$ and $h : X \to U^s(B_i)$ such that $h_{|Y} = \phi_{i,i_0}\circ f$ and $\phi_i \circ h$ is homotopic to g rel Y.*

Proof. For any C^*-algebra A we set $G^s(A) := \{a \in GL_1((K\otimes A)^u) \mid 1-a \in K\otimes A\}$. Then $U^s(A) \subseteq G^s(A)$ and the polar decomposition provides a retraction $W : G^s(A) \to U^s(A)$. There exists a c in $(0,1)$ such that $\max\{\|a^*a-1\|, \|aa^*-1\|\} \le c$ implies $\|W(a)-a\| \le \frac{1}{10}$. We interpret a map $X \to G^s(A)$ as a point in $G^s(C(X)\otimes A)$. We will write ϕ_{ij} and ϕ_i instead of $\mathtt{id}_{C(X)}\otimes\phi_{ij}$ and $\mathtt{id}_{C(X)}\otimes\phi_i$. We use the general fact that a filtered colimit in $C^*\mathbf{Alg}^{\mathrm{nu}}$ is formed by taking the completion of the pre-C^*-algebra given by the filtered colimit of underlying sets equipped with the induced algebraic structures; see (2-1). We use that the (maximal) tensor product preserves filtered colimits, and that we can calculate norms in a filtered colimit as limits of norms. The last statement says, e.g., that for h in $C(X)\otimes K\otimes B_i$ we have $\|\phi_i(h)\| = \mathtt{lim}_{j\in I, i\leq j}\|\phi_{j,i}(h)\|$.

We use Dugundji's extension theorem in order to find an extension f_0 of f in $(C(X)\otimes K\otimes B_{i_0})^u$ such that $f_0-1\in C(X)\otimes K\otimes B_{i_0}$. Then $g_0:=g-\phi^u_{i_0}(f_0)\in C_0(X\setminus Y)\otimes K\otimes B$. We can now find i_1 in I such that there exists r in $C_0(X\setminus Y)\otimes K\otimes B_{i_1}$ with $\|\phi_{i_1}(r)-g_0\|\leq\frac{c}{100}$. We set $f_1:=\phi^u_{i_1,i_0}(f_0)+r$. Then $\|\phi_{i_1}(f_1)-g\|\leq\frac{c}{100}$, and hence $\max\{\|\phi_{i_1}(f_1f_1^*)-1\|, \|\phi_{i_1}(f_1^*f_1)-1\|\}\leq\frac{c}{10}$. We can then find i in I with $i\geq i_1$ such that $\max\{\|\phi_{i,i_1}(f_1)\phi_{i,i_1}(f_1^*)-1\|, \|\phi_{i,i_1}(f_1^*)\phi_{i,i_1}(f_1)-1\|\}\leq\frac{c}{3}$. We define $h:=W(\phi_{i,i_1}(f_1))$ in $U^s(C(X)\otimes B_i)$. Then $h_{|Y}=\phi_{i,i_0}(f)$ and $\|\phi_i(h)-g\|\leq\frac{1}{2}$. We get a homotopy $(W((1-s)\phi_{i_0}(h)+sg))_{s\in[0,1]}$ from $\phi_i(h)$ to g rel Y in $U^s(B)$. This finishes the proof of Lemma 9.12. □

Remark 9.13. Using that $S(\mathbb{C})$ is semiprojective, we could deduce Lemma 9.12 directly from the proof of [Bunke and Duenzinger 2024, Proposition 3.8], in particular from the existence of the lift in equation (3.6) of the same reference. □

We have now finished the proof of Lemma 9.6. □

The reason for using E-theory in order to construct the K-theory functor for C^*-algebras was that then exactness of the latter is true by construction. We have a canonical transformation

$$\Omega^{\infty-1}\mathrm{KK}(\mathbb{C},-)\to\Omega^{\infty-1}\mathrm{E}(\mathbb{C},-)\simeq\Omega^{\infty-1}K(-).$$

The arguments above work equally well for *KK*-theory (just replace ex by se) and show that the canonical transformation $\mathcal{U}^s(-)\to\Omega^{\infty-1}\mathrm{KK}(\mathbb{C},-)$ is an equivalence. Using Bott periodicity we can then conclude an equivalence of functors

$$\mathrm{KK}(\mathbb{C},-)\xrightarrow{\simeq}K(-):C^*\mathbf{Alg}^{\mathrm{nu}}\to\mathbf{Sp}. \tag{9-5}$$

As a consequence, the stable ∞-category KK also naturally acquires a KU-linear structure. We conclude:

Corollary 9.14. *The K-theory functor for C^*-algebras $K:C^*\mathbf{Alg}^{\mathrm{nu}}\to\mathbf{Mod}(\mathrm{KU})$ has a lax symmetric monoidal refinement for the minimal tensor product on $C^*\mathbf{Alg}^{\mathrm{nu}}$.*

The following is true for both tensor products on KK. We have a limit-preserving symmetric monoidal functor

$$\mathcal{K}:=\mathrm{KK}(\mathbb{C},-):\mathrm{KK}\to\mathbf{Mod}(\mathrm{KU})$$

such that $K\simeq\mathcal{K}\circ\mathrm{kk}$. Since $\mathrm{kk}(\mathbb{C})$ is compact in KK this functor also preserves colimits. It is the right-adjoint of a symmetric monoidal right-Bousfield localization

$$\mathcal{B}:\mathbf{Mod}(\mathrm{KU})\leftrightarrows\mathrm{KK}:\mathcal{K}. \tag{9-6}$$

Definition 9.15. The essential image of the left-adjoint $\mathcal{B}$ in (9-6) is called the UCT *class*.

Equivalently, the UCT-class is the localizing subcategory of KK generated by the tensor unit $\mathrm{kk}(\mathbb{C})$.

Corollary 9.16. *If B is in the* UCT*-class, then we have the following assertions*:

(1) *The natural transformation* $-\otimes_{\max} B \to -\otimes_{\min} B$ *of endofunctors on* KK *is an equivalence.*

(2) *The transformation* $\mathrm{KK}(B,-) \to \mathrm{E}(B,-)$ *of functors* $C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Mod}(\mathrm{KU})$ *is an equivalence.*

(3) (*UCT*): *We have an equivalence* $\mathrm{KK}(B,-) \simeq \mathtt{map}_{\mathbf{Mod}(\mathrm{KU})}(\mathcal{K}(B), \mathcal{K}(-))$ *of functors from* KK *to* $\mathbf{Mod}(\mathrm{KU})$.

(4) (*Künneth formula*): *We have an equivalence* $\mathcal{K}(-) \otimes_{\mathrm{KU}} \mathcal{K}(B) \simeq \mathcal{K}(-\otimes B)$ *of functors from* KK *to* $\mathbf{Mod}(\mathrm{KU})$.

Proof. In all cases the equivalence is induced by an obvious natural transformation. One argues that the full subcategory of objects B in KK for which is transformation is an equivalence is localizing, and that it contains $\mathrm{kk}(\mathbb{C})$. □

Remark 9.17. The kk-functor from Definition 8.6(1) is not compatible with filtered colimits on the level of C^*-algebras. It does not even preserve countable sums. The reason is that the functor $y : \mathrm{KK}_{\mathrm{sep}} \to \mathrm{KK}$ does not preserve countable sums. One could improve on this point by observing that $\mathrm{kk}_{\mathrm{sep}}$ preserves countable sums Corollary 12.3(2), and then working with the $\aleph_1$-Ind-completion instead of the Ind-completion in Definition 8.1. We refer to [Bunke and Duenzinger 2024, Section 3.4] where the details of such a construction have been worked out in the case of E-theory; see also [Bunke et al. 2021, Remark 3.4]

In the context of the present paper, in order to discuss the relation of the UCT-class with the classical definition, it is better to consider the separable version $\mathrm{UCT}_{\mathrm{sep}}$ defined as a smallest countably cocomplete stable ∞-category of $\mathrm{KK}_{\mathrm{sep}}$ (which is known to be countably cocomplete by Corollary 12.3(1)) containing the tensor unit; see also [Bunke et al. 2023, Section 5.5]. It is a famous question whether for every nuclear separable algebra A we have $\mathrm{kk}_{\mathrm{sep}}(A) \in \mathrm{UCT}_{\mathrm{sep}}$.

The analogues of the statements of Corollary 9.16 in the separable case hold true. □

Remark 9.18. In Corollary 9.16(1) we can replace the condition that B is in the UCT-class by the condition that B is represented by a separable and nuclear C^*-algebra. Indeed, by definition of nuclearity, the transformation $-\otimes_{\max} B \to \otimes_{\min} B$ of endofunctors of $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ is an isomorphism.

Classically it is known that for separable algebras the map from *KK*- to *E*-theory is an isomorphism if the first argument is a nuclear algebra [Higson 1990a, Theorem 3.5]. By stability and Theorem 13.16 we can conclude that in Corollary 9.16(2) we can replace the UCT-condition on B by the condition that B is represented by a separable and nuclear C^*-algebra. □

Remark 9.19. In this remark we justify Definition 9.2. By a similar argument as in the proof of Lemma 9.6 we get a weak equivalence

$$\mathop{\mathtt{colim}}_{n\in\mathbb{N}} \ell U(n) \cong \mathcal{U}^s(\mathbb{C})$$

in $\mathbf{Groups}(\mathbf{Spc})$. The left-hand side is $\Omega^{\infty-1}$ of one of the classical versions of the KU-spectrum.

We continue the justification of Definition 9.2 by calculating the ring $\pi_*\mathrm{KU}$. Since the homotopy groups of $\mathcal{U}^s(\mathbb{C}) \simeq \Omega^{\infty-1}K(\mathbb{C})$ are two-periodic, so are the homotopy groups on the left-hand side. We thus deduce the classical Bott periodicity theorem. For an explicit calculation we use, from classical topology, the additional information

$$\pi_i(\operatorname*{colim}_{n\in\mathbb{N}} U(n)) \cong \begin{cases} *, & i=0, \\ \mathbb{Z}, & i=1, \\ 0, & i=2, \end{cases}$$

in order to conclude that

$$\pi_i\Omega^\infty K(\mathbb{C}) \cong \begin{cases} \mathbb{Z}, & i \in 2\mathbb{N}, \\ 0, & i \in 2\mathbb{N}+1. \end{cases} \tag{9-7}$$

We know further that $\pi_*\mathrm{KU}$ is a ring and that the Bott periodicity is implemented by the multiplication with the invertible element b in $\pi_{-2}\mathrm{KU} \simeq \pi_0\mathrm{KK}(S^2(\mathbb{C}), \mathbb{C})$ from (8-5). Consequently, b^{-1} must be a generator of $\pi_2\mathrm{KU}$ and we get a ring isomorphism

$$\mathbb{Z}[b, b^{-1}] \xrightarrow{\cong} \pi_*\mathrm{KU}. \qquad \square$$

10. *K*-theory and the group completion of the space of projections

For a unital C^*-algebra A the abelian group $K_0(A)$ is classically defined as the Grothendieck group of the monoid of unitary equivalence classes of projections in $K \otimes A$, where the unitaries belong to the multiplier algebra $U(K \otimes A)$. Thereby the monoid operation is induced by the block sum. One then observes that the relation of unitary equivalence between projections is equivalent to homotopy. Using the notation from Example 4.8 we thus get an isomorphism

$$\pi_0(\mathcal{P}\mathrm{roj}^s(A))^{\mathrm{group}} \cong K_0(A). \tag{10-1}$$

We will show a space-level refinement of this isomorphism. Unfolding definitions we obtain a natural map

$$\mathcal{P}\mathrm{roj}^s(A) \to \Omega^\infty K(A)$$

of commutative monoids in spaces. Then Corollary 10.8 asserts that this map presents its target as a group completion. Note that the unitality assumption on A is crucial for this statement; see Example 10.1. The modification for general algebras is formulated as Theorem 10.7. Though it looks like an obvious K-theoretic statement its detailed verification is surprisingly long.

Example 10.1. Let X be a locally compact Hausdorff space which is connected and not compact. Then we have $\mathrm{Proj}^s(C_0(X)) = \{0\}$. Indeed, a projection p in $\mathrm{Proj}^s(C(X))$ can be interpreted as a function $p : X \to \mathrm{Proj}^s(\mathbb{C})$. The function $x \mapsto \|p(x)\|$ is continuous and takes values in $\{0, 1\}$. Since it vanishes at infinity, the assumptions on X imply that it vanishes identically.

We know from Remark 9.19 that $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$, and this contradicts (10-1) whose left-hand side would be the zero group. $\square$

The following two statements enable us to study the space of projections within the homotopy theory developed in the present notes. They are the analogues of Lemmas 9.10 and 9.6. Recall from Example 4.8 that $\mathrm{Proj}(A) := \underline{\mathtt{Hom}}(\mathbb{C}, A)$ denotes the topological space of projections in A and that $\mathcal{P}\mathrm{roj}(A) := \ell\mathrm{Proj}(A)$ is the associated space.

Proposition 10.2. *For every surjective map $B \to C$ of C^*-algebras the map $\mathrm{Proj}(B) \to \mathrm{Proj}(C)$ of topological spaces is a Serre fibration.*

Proposition 10.3. *The functor $\mathcal{P}\mathrm{roj} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Spc}$ preserves small filtered colimits.*

We defer the technical proofs of these statements to the end of the section.

Proposition 10.4. *The functor $\mathcal{P}\mathrm{roj}^s : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{CMon}(\mathbf{Spc})$ is homotopy invariant, stable, Schochet-exact, and s-finitary. It sends cartesian squares*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

in $C^*\mathbf{Alg}^{\mathrm{nu}}$ with the property that f is a surjection to cartesian squares.

Proof. The functor $\mathcal{P}\mathrm{roj}^s(-) \overset{(4\text{-}3)}{\simeq} \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(\mathbb{C}, -)$ is homotopy invariant and stable by definition. It furthermore sends Schochet fibrant cartesian squares to cartesian squares since $L_{h,K}$ does so.

Since the functor $K \otimes -$ preserves filtered colimits, it follows from Proposition 10.3 that $\mathcal{P}\mathrm{roj}^s(-) \simeq \mathcal{P}\mathrm{roj}(K \otimes -)$ preserves filtered colimits and is in particular s-finitary.

Since

$$\begin{tikzcd} K \otimes A \arrow[r] \arrow[d] & K \otimes B \arrow[d, "K \otimes f"] \\ K \otimes C \arrow[r] & K \otimes D \end{tikzcd}$$

is again cartesian and $K \otimes f$ is still surjective, the functor $\mathrm{Proj} := \underline{\mathtt{Hom}}(\mathbb{C}, -)$ sends this square to a cartesian square in **Top** which is in addition Serre fibrant by Proposition 10.2. We now apply ℓ and get the desired cartesian square

$$\begin{tikzcd} \mathcal{P}\mathrm{roj}^s(A) \arrow[r] \arrow[d] & \mathcal{P}\mathrm{roj}^s(B) \arrow[d, "\mathcal{P}\mathrm{roj}^s(f)"] \\ \mathcal{P}\mathrm{roj}^s(C) \arrow[r] & \mathcal{P}\mathrm{roj}^s(D) \end{tikzcd}$$

□

The group completion functor $(-)^{\mathrm{group}}$ is defined as the left-adjoint of a Bousfield localization

$$(-)^{\mathrm{group}} : \mathbf{CMon}(\mathbf{Spc}) \leftrightarrows \mathbf{CGroups}(\mathbf{Spc}) : \mathrm{incl}. \tag{10-2}$$

Recall that to any C^*-algebra A we can functorially associate the split unitalization sequence

$$0 \to A \to A^u \to \mathbb{C} \to 0. \tag{10-3}$$

Definition 10.5. We define the functor

$$\widetilde{\mathcal{P}\mathrm{roj}}^s : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{CGroups}(\mathbf{Spc}), \quad A \mapsto \mathrm{Fib}(\mathcal{P}\mathrm{roj}^s(A^u)^{\mathrm{group}} \to \mathcal{P}\mathrm{roj}^s(\mathbb{C})^{\mathrm{group}}). \tag{10-4}$$

Remark 10.6. Observe that in the definition of $\widetilde{\mathcal{P}\mathrm{roj}}^s$ we take the group completion first and then the fiber. We cannot reverse the order since the group completion does not preserve fiber sequences in general. □

We define the natural transformation

$$e_h : \mathcal{P}\mathrm{roj}^s(-) \simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,K}}(\mathbb{C}, -) \xrightarrow{\Omega^2_{\mathrm{ex}} \circ L_{\mathrm{ex}}} \mathtt{Map}_{\mathrm{E}}(\mathbb{C}, -) \simeq \Omega^\infty K(-) \tag{10-5}$$

of functors from $C^*\mathbf{Alg}^{\mathrm{nu}}$ to $\mathbf{CMon}(\mathbf{Spc})$. Since $\Omega^\infty K$ takes values in groups, by the universal property of the group completion we get the dotted arrow in the commutative diagram of functors $C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{CMon}(\mathbf{Spc})$

$$\begin{tikzcd} \mathcal{P}\mathrm{roj}^s \arrow[rr, "\mathrm{e}_h"] \arrow[dr] & & \Omega^\infty K \\ & \mathcal{P}\mathrm{roj}^{s,\mathrm{group}} \arrow[ur, dotted, "\hat{\mathrm{e}}_h"] & \end{tikzcd}$$

where the down-right arrow is the counit of the adjunction (10-2).

We next construct a natural transformation

$$\tilde{\mathrm{e}}_h : \widetilde{\mathcal{P}\mathrm{roj}}^s \to \Omega^\infty K. \tag{10-6}$$

Applying the exact functor $\Omega^\infty K$ to the split exact unitalization sequence (10-3) we get the split fiber sequence

$$\Omega^\infty K(A) \to \Omega^\infty K(A^u) \to \Omega^\infty K(\mathbb{C})$$

in $\mathbf{CGroups}(\mathbf{Spc})$. We now form the diagram of vertical fiber sequences of functors from $C^*\mathbf{Alg}^{\mathrm{nu}}$ to $\mathbf{CGroups}(\mathbf{Spc})$

$$\begin{tikzcd} \widetilde{\mathcal{P}\mathrm{roj}}^s(-) \arrow[r, dotted, "\tilde{\mathrm{e}}_h"] \arrow[d] & \Omega^\infty K(-) \arrow[d] \\ \mathcal{P}\mathrm{roj}((-)^u)^{\mathrm{group}} \arrow[r, "\hat{\mathrm{e}}_h"] \arrow[d] & \Omega^\infty K((-)^u) \arrow[d] \\ \mathcal{P}\mathrm{roj}(\mathbb{C})^{\mathrm{group}} \arrow[r, "\hat{\mathrm{e}}_h"] & \Omega^\infty K(\mathbb{C}) \end{tikzcd} \tag{10-7}$$

defining $\tilde{\mathrm{e}}_h$ as the natural extension of the lower square to a map of fibers.

The following is the main theorem of the present section.

Theorem 10.7. *The natural transformation $\tilde{\mathrm{e}}_h : \widetilde{\mathcal{P}\mathrm{roj}}^s \to \Omega^\infty K$ is an equivalence.*

Before we start with the proof we consider the specialization to unital algebras.

Corollary 10.8. *For every unital C^*-algebra A the map*

$$\hat{e}_{h,A} : \mathcal{P}\mathrm{roj}^s(A) \to \Omega^\infty K(A) \tag{10-8}$$

presents its target as a group completion.

Proof. Since the composition

$$\mathcal{P}\mathrm{roj}^s(A)^{\mathrm{group}} \to \mathcal{P}\mathrm{roj}^s(A^u)^{\mathrm{group}} \to \mathcal{P}\mathrm{roj}^s(\mathbb{C})^{\mathrm{group}}$$

vanishes, by the universal property of the fiber in (10-4) we get a canonical morphism

$$i_A : \mathcal{P}\mathrm{roj}^s(A)^{\mathrm{group}} \to \widetilde{\mathcal{P}\mathrm{roj}}^s(A). \tag{10-9}$$

If A is unital, then the identity of A canonically extends to a homomorphism $A^u \to A$. The composition $\widetilde{\mathcal{P}\mathrm{roj}}^s(A) \to \mathcal{P}\mathrm{roj}^s(A^u)^{\mathrm{group}} \to \mathcal{P}\mathrm{roj}^s(A)^{\mathrm{group}}$ provides an inverse of i_A. The map in (10-8) is then equivalent to

$$\mathcal{P}\mathrm{roj}^s(A) \to \mathcal{P}\mathrm{roj}^s(A)^{\mathrm{group}} \overset{i_A}{\simeq} \widetilde{\mathcal{P}\mathrm{roj}}^s(A) \overset{\tilde{e}_{h,A}}{\simeq} \Omega^\infty K(A),$$

where the last equivalence is given by Theorem 10.7. This shows Corollary 10.8. ☐

All of the above has a version for separable algebras. The following is the separable version of Theorem 10.7.

Proposition 10.9. *The natural transformation $\tilde{e}_{\mathrm{sep},h} : \widetilde{\mathcal{P}\mathrm{roj}}^s_{\mathrm{sep}} \to \Omega^\infty K_{\mathrm{sep}}$ is an equivalence.*

Proof of Theorem 10.7 assuming Proposition 10.9. We claim that $\widetilde{\mathcal{P}\mathrm{roj}}^s$ is s-finitary. Since $\Omega^\infty K$ is also s-finitary we then obtain the equivalence in Theorem 10.7 as a left-Kan extension of the equivalence in Proposition 10.9 along the inclusion $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$.

In order to see the claim we note that $(-)^{\mathrm{group}}$ in (10-2) is left-adjoint and preserves all colimits. By Proposition 10.4 the functor $\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}$ is also s-finitary. Finally we use that the fiber of a filtered colimit of maps in $\mathbf{CGroups}(\mathbf{Spc})$ is the filtered colimit of the fibers. ☐

The following result prepares the proof of Proposition 10.9.

Proposition 10.10. *The functor $\widetilde{\mathcal{P}\mathrm{roj}}^s : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{CGroups}(\mathbf{Spc})$ is homotopy invariant, stable and exact.*

Proof. Homotopy invariance and stability are obvious from the definition. Exactness is much deeper. It cannot be concluded simply from the exactness properties of $\mathcal{P}\mathrm{roj}^s$ stated in Proposition 10.4 since the group completion functor does not preserve fiber sequences in general. The basic insight in our special case is that for unital algebras A the group completion of $\mathcal{P}\mathrm{roj}^s(A)$ can be expressed by a specific filtered colimit.

We first recall some generalities on group completions following [Nikolaus 2017; Randal-Williams 2013]. We consider a commutative monoid X in $\mathbf{CMon}(\mathbf{Spc})$. For an element s in X we can form

$$X_s := \mathtt{colim}(X \xrightarrow{+s} X \xrightarrow{+s} X \xrightarrow{+s} \cdots)$$

in X-modules. For any finite ordered set $\{s_1, \dots, s_n\}$ of elements in X we define inductively X-modules

$$X_{\{s_1,\dots,s_n\}} := (X_{\{s_1,\dots,s_{n-1}\}})_{s_n}.$$

We choose a well ordering on $\pi_0(X)$ and a representative s in X for any component. We then define the space

$$X_\infty := \operatorname*{\mathtt{colim}}_{S\subseteq \pi_0(X)} X_S. \tag{10-10}$$

Proposition 10.11 [Nikolaus 2017, Proposition 6]. *If the fundamental group of every component of X_∞ is abelian, then $X \to X_\infty$ is equivalent to the underlying map of $X \to X^{\mathrm{group}}$.*

An element t in $\pi_0(X)$ is called *cofinal* if for every s in $\pi_0(X)$ there exists s' in $\pi_0(X)$ and n in $\mathbb{N}$ such that $s+s'=nt$. One easily checks that if t is cofinal, then the canonical map $X_t \to X_\infty$ is an equivalence. In particular if the fundamental groups of all components of X_t are abelian, then $X \to X_t$ is equivalent to the underlying map of the group completion $X \to X^{\mathrm{group}}$.

The following results enable us to apply Proposition 10.11 to our problem. For C^*-algebras A and B we have the commutative monoid $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$ and can form the space $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)_\infty$ as in (10-10).

Lemma 10.12. *If $\pi_0\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$ contains a cofinal element, then the fundamental groups of the components of $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)_\infty$ are abelian.*

Proof. In the following we use that $\ell\underline{\mathtt{Hom}}(A, K\otimes B) \simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$. Let $[t]$ be a cofinal element in $\pi_0\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$ represented by a map t in $\underline{\mathtt{Hom}}(A, K\otimes B)$. Then

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)_\infty \simeq \mathtt{colim}(\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B) \xrightarrow{[t]+-} \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B) \xrightarrow{[t]+-} \cdots).$$

We consider a component x in $\pi_0\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)_\infty$. Then there exists k in $\mathbb{N}$ and f in the topological space $\underline{\mathtt{Hom}}(A, K\otimes B)$ contributing to $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$ in the k-th stage of the $\mathbb{N}$-indexed diagram above which represents x.

We now consider elements $[\gamma]$ and $[\sigma]$ in $\pi_1(\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)_\infty, x)$. We must show that $[\gamma]\circ[\sigma] = [\sigma]\circ[\gamma]$. After going further in the diagram we can assume that $[\gamma]$ and $[\sigma]$ are represented by loops γ and σ at f in $\underline{\mathtt{Hom}}(A, K\otimes B)$, respectively.

By cofinality of $[t]$ there is a map f' and an integer n such that $[f]+[f']=n[t]$ in $\pi_0\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$. The $+$ sign in the following denotes choices of block sums. We know that $\gamma+nt$ and $\sigma+nt$ viewed as points in $\underline{\mathtt{Hom}}(A, K\otimes B)$ contributing to $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A, B)$ in the $(k+n)$-th stage of the diagram also represent $[\gamma]$ and $[\sigma]$. We now have homotopies $\gamma+nt \sim \gamma+f+f'$ and $\sigma+nt\sim\sigma+f+f'$. It thus suffices to show that $(\gamma+f)\sharp(\sigma+f)\sim(\sigma+f)\sharp(\gamma+f)$, where $\sharp$ denotes concatenation.

If we conjugate $\sigma+f$ with a two-dimensional rotation of blocks, we get a homotopy between $(\gamma+f)\sharp(\sigma+f)$ and $(\gamma+f)\sharp(f+\sigma)$. Now $(\gamma+f)\sharp(f+\sigma)$ is homotopic to $\gamma+\sigma$. Using the commutativity of $+$ up to homotopy we get a homotopy $\gamma+\sigma\sim\sigma+\gamma$. By reversing the first part we finally get a homotopy from $\gamma+\sigma$ to $(\sigma+f)\sharp(\gamma+f)$. □

Corollary 10.13. *If* $\pi_0\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A,B)$ *contains a cofinal element, then the map of spaces*

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A,B) \to \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A,B)_\infty$$

is equivalent to the underlying map of the group completion

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A,B) \to \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A,B)^{\mathrm{group}}.$$

Let now A be a unital C^*-algebra with unit 1_A. In the following we show that we can apply Corollary 10.13 to $\mathcal{P}\mathrm{roj}^s(A) \simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(\mathbb{C},A)$ by exhibiting a cofinal component. Let e be a minimal projection in K. Then we define $t_A := [e \otimes 1_A]$ in $\pi_0\mathcal{P}\mathrm{roj}^s(A)$. The following lemma is well-known.

Lemma 10.14. *The element* t_A *is cofinal.*

Proof. We consider the separable Hilbert space $H := L^2(\mathbb{N})$ and let $K := K(H)$. For n in $\mathbb{N}$ we let e_n in $K(H)$ denote the projection onto the n-th basis vector of H. We further consider the projection $P_n = \sum_{i=0}^n e_n$.

We consider a component $[p]$ in $\pi_0\mathcal{P}\mathrm{roj}^s(A)$ with p in $\mathrm{Proj}(K \otimes A)$. Then there exists n in $\mathbb{N}$ such that $\|p-(P_n\otimes 1_A)p(P_n\otimes 1_A)\| < \frac{1}{2}$. Using function calculus we get a homotopy between p and a projection p' with $p' = (P_n \otimes 1_A)p'(P_n \otimes 1_A)$. We then have $[p]=[p']$ in $\pi_0\mathcal{P}\mathrm{roj}^s(A)$ and with $q' := (P_n \otimes 1_A) - p'$ we get $[p'] + [q'] = [P_n \otimes 1_A] = nt_A$. □

To any unital C^*-algebra A we can functorially (for unital morphisms) associate the $\mathbb{N}$-indexed diagram

$$\hat{\mathcal{F}}(A) : \mathcal{P}\mathrm{roj}^s(A) \xrightarrow{-+t_A} \mathcal{P}\mathrm{roj}^s(A) \xrightarrow{-+t_A} \mathcal{P}\mathrm{roj}^s(A) \xrightarrow{-+t_A} \mathcal{P}\mathrm{roj}^s(A) \xrightarrow{-+t_A} \cdots$$

of spaces. We define the functor

$$\mathcal{F} := \underset{\mathbb{N}}{\mathtt{colim}}\, \hat{\mathcal{F}} : C^*\mathbf{Alg} \to \mathbf{Spc}. \tag{10-11}$$

We conclude that the natural transformation $\mathcal{P}\mathrm{roj}^s(-) \to \mathcal{F}(-)$ of $\mathbf{Spc}$-valued functors is equivalent to the transformation $\mathcal{P}\mathrm{roj}^s \to \mathcal{P}\mathrm{roj}^{s,\mathrm{group}}$ of $\mathbf{CMon}(\mathbf{Spc})$-valued functors after forgetting the commutative monoid structure.

We can now finally show the asserted exactness of the functor $\widetilde{\mathcal{P}\mathrm{roj}}^s$. We must show that this functor sends an exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of C^*-algebras to a fiber sequence in $\mathbf{CGroups}(\mathbf{Spc})$. We first form the square

$$\begin{tikzcd} A^u \arrow[r, "\pi"] \arrow[d, "i^u"] & \mathbb{C} \arrow[d, "j"] \\ B^u \arrow[r, "p^u"] & C^u \end{tikzcd} \tag{10-12}$$

in $C^*\mathbf{Alg}_{/\mathbb{C}}$, where π is induced by the canonical projection of the unitalization sequence (10-3), the homomorphism $j : \mathbb{C} \to C^u$ is induced by the identity of C^u, and we do not write the structure maps to $\mathbb{C}$

given by the projections of the unitalization sequences of A, B and C, respectively, and the identity of $\mathbb{C}$. This square is cartesian and p^u is surjective. Applying the functor $\mathcal{P}\mathrm{roj}^s$ we get a diagram in $\mathbf{Spc}_{/\mathcal{P}\mathrm{roj}^s(\mathbb{C})}$, and by Proposition 10.4 a cartesian square

$$\begin{tikzcd}
\mathcal{P}\mathrm{roj}^s(A^u) \arrow[r, "\mathcal{P}\mathrm{roj}^j(\pi)"] \arrow[d, "\mathcal{P}\mathrm{roj}^s(i^u)"] & \mathcal{P}\mathrm{roj}^s(\mathbb{C}) \arrow[d, "\mathcal{P}\mathrm{roj}^s(j)"] \\
\mathcal{P}\mathrm{roj}^s(B^u) \arrow[r, "\mathcal{P}\mathrm{roj}^s(p^u)"] & \mathcal{P}\mathrm{roj}^s(C^u)
\end{tikzcd} \tag{10-13}$$

We now apply the functor $\mathcal{F}$ from (10-11) to the square (10-12). This amounts to forming a filtered colimit of a diagram of squares of the form (10-13). Since a filtered colimit of cartesian squares in **Spc** is again a cartesian square and since the forgetful functor $\mathbf{CGroups}(\mathbf{Spc}) \to \mathbf{Spc}$ detects limits we can conclude that

$$\begin{tikzcd}
\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(A^u) \arrow[r, "\mathcal{P}\mathrm{roj}^j(\pi)"] \arrow[d, "\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(i^u)"] & \mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(\mathbb{C}) \arrow[d, "\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(j)"] \\
\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(B^u) \arrow[r, "\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(p^u)"] & \mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(C^u)
\end{tikzcd}$$

is a cartesian square in $\mathbf{CGroups}(\mathbf{Spc})$. Together with its unwritten structure maps it is also a diagram in $\mathbf{CGroups}(\mathbf{Spc})_{/\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(\mathbb{C})}$. We finally take the fiber of the structure maps to $\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}(\mathbb{C})$ and get the desired cartesian square (or fiber sequence)

$$\begin{tikzcd}
\widetilde{\mathcal{P}\mathrm{roj}}^s(A) \arrow[r] \arrow[d, "\widetilde{\mathcal{P}\mathrm{roj}}^s(i)"] & 0 \arrow[d] \\
\widetilde{\mathcal{P}\mathrm{roj}}^s(B) \arrow[r, "\widetilde{\mathcal{P}\mathrm{roj}}^s(p)"] & \widetilde{\mathcal{P}\mathrm{roj}}^s(C)
\end{tikzcd}$$

This finishes the proof of Proposition 10.10. □

Proof of Proposition 10.9. In order to define an inverse transformation $\Omega^\infty K \to \widetilde{\mathcal{P}\mathrm{roj}}^s$ we plan to apply the Yoneda lemma for $\mathrm{E}_{\mathrm{sep}}$. We therefore need a factorization of $\widetilde{\mathcal{P}\mathrm{roj}}^s$ through a functor $\overline{\mathcal{P}\mathrm{roj}}^s$ defined on $\mathrm{E}_{\mathrm{sep}}$. Since $\mathbf{CGroups}(\mathbf{Spc})$ is left-exact and additive, the universal property (7-9) of $\mathrm{e}_{\mathrm{sep}}$ and Proposition 10.10 together provide the dotted arrow in

$$\begin{tikzcd}
C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[rr, "\widetilde{\mathcal{P}\mathrm{roj}}^s_{\mathrm{sep}}"] \arrow[dr, "\mathrm{e}_{\mathrm{sep}}"'] & & \mathbf{CGroups}(\mathbf{Spc}) \\
& \mathrm{E}_{\mathrm{sep}} \arrow[ur, dotted, "\overline{\mathcal{P}\mathrm{roj}}"'] &
\end{tikzcd} \tag{10-14}$$

Furthermore, using Definition 9.3, the pull-back along $\mathrm{e}_{\mathrm{sep}}$ induces an equivalence

$$\mathrm{e}^*_{\mathrm{sep}} : \mathrm{Nat}(\overline{\mathcal{P}\mathrm{roj}}(-), \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C}, -)) \xrightarrow{\simeq} \mathrm{Nat}(\widetilde{\mathcal{P}\mathrm{roj}}^s_{\mathrm{sep}}(-), \Omega^\infty K_{\mathrm{sep}}(-)). \tag{10-15}$$

We define a point a_* in $\overline{\mathcal{P}\mathtt{roj}}(\mathbb{C})$ as the image of $\mathtt{id}_{\mathbb{C}}$ under

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(\mathbb{C},\mathbb{C}) \simeq \mathcal{P}\mathtt{roj}^s_{\mathrm{sep}}(\mathbb{C}) \to \mathcal{P}\mathtt{roj}^s_{\mathrm{sep}}(\mathbb{C})^{\mathrm{group}} \xrightarrow{i_{\mathbb{C}},(10\text{-}9)} \widetilde{\mathcal{P}\mathtt{roj}}^s_{\mathrm{sep}}(\mathbb{C}) \overset{(10\text{-}14)}{\simeq} \overline{\mathcal{P}\mathtt{roj}}(\mathbb{C}). \quad (10\text{-}16)$$

Via the Yoneda lemma this point determines a natural transformation

$$\tilde{a} : \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C}, -) \to \overline{\mathcal{P}\mathtt{roj}}(-)$$

of functors from $\mathrm{E}_{\mathrm{sep}}$ to $\mathbf{CGroups}(\mathbf{Spc})$ characterized by $\tilde{a}_{\mathbb{C}}(\mathrm{e}_{\mathrm{sep}}(\mathtt{id}_{\mathbb{C}})) \simeq a_*$. Its pull-back along $\mathrm{e}_{\mathrm{sep}}$ is a natural transformation

$$a := \mathrm{e}^*_{\mathrm{sep}}\tilde{a} : \Omega^\infty K_{\mathrm{sep}} \to \widetilde{\mathcal{P}\mathtt{roj}}_{\mathrm{sep}}$$

of functors from $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ to $\mathbf{CGroups}(\mathbf{Spc})$. We have already a natural transformation

$$b := \tilde{\mathrm{e}}_{\mathrm{sep},h} : \widetilde{\mathcal{P}\mathtt{roj}}_{\mathrm{sep}} \to \Omega^\infty K_{\mathrm{sep}}$$

defined by means of the diagram (10-7). In view of (10-15) there is an essentially unique natural transformation

$$\tilde{b} : \overline{\mathcal{P}\mathtt{roj}} \to \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C}, -)$$

such that $\mathrm{e}^*_{\mathrm{sep}}\tilde{b} \simeq b$.

The following proposition implies Proposition 10.9 asserting that b is an equivalence.

Proposition 10.15. *The natural transformations a and b are mutually inverse to each other.*

Proof. The assertion follows from the following two lemmas.

Lemma 10.16. *We have $b \circ a \simeq \mathtt{id}_{\Omega^\infty K_{\mathrm{sep}}}$.*

Proof. It suffices to show

$$\tilde{b} \circ \tilde{a} : \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C}, -) \to \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C}, -)$$

is equivalent to the identity. Via the Yoneda lemma this map is determined by the point $\tilde{b}_{\mathbb{C}}(a_*)$ in $\mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C},\mathbb{C})$. We therefore must show that $\tilde{b}_{\mathbb{C}}(a_*) \simeq \mathrm{e}_{\mathrm{sep}}(\mathtt{id}_{\mathbb{C}})$. In order to verify this equivalence we consider the diagram

$$\begin{tikzcd}
\mathcal{P}\mathtt{roj}^s_{\mathrm{sep}}(\mathbb{C}) \arrow[r] \arrow[rrd, dotted, "\mathcal{P}\mathtt{roj}^s(j)", "\mathtt{id}_{\mathbb{C}} \mapsto \tilde{a}_*"'] & \mathcal{P}\mathtt{roj}^s_{\mathrm{sep}}(\mathbb{C})^{\mathrm{group}} \arrow[r, "i_{\mathbb{C}}"] & \overline{\mathcal{P}\mathtt{roj}}^s_{\mathrm{sep}}(\mathbb{C}) \arrow[r, "\tilde{b}_{\mathbb{C}}"] \arrow[d] & \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C},\mathbb{C}) \arrow[d, "j_*"] \\
& & \mathcal{P}\mathtt{roj}^s_{\mathrm{sep}}(\mathbb{C}^u)^{\mathrm{group}} \arrow[r, "\hat{\mathrm{e}}_{\mathrm{sep},h,\mathbb{C}}"] \arrow[d] & \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C},\mathbb{C}^u) \arrow[d] \\
& & \mathcal{P}\mathtt{roj}_{\mathrm{sep}}(\mathbb{C})^{\mathrm{group}} \arrow[r, "\hat{\mathrm{e}}_{\mathrm{sep},h,\mathbb{C}}"] & \mathtt{Map}_{\mathrm{E}_{\mathrm{sep}}}(\mathbb{C},\mathbb{C})
\end{tikzcd}$$

obtained by merging (10-7) with (10-16), where $j : \mathbb{C} \to \mathbb{C}^u$ is the inclusion.

The two upper-left-horizontal arrows send $\mathtt{id}_{\mathbb{C}}$ to a_*. We let $\tilde{a}_*$ denote its image in $\mathcal{P}\mathrm{roj}^s_{\mathrm{sep}}(\mathbb{C}^u)^{\mathrm{group}}$. Since $\hat{\mathrm{e}}_{\mathrm{sep},h,\mathbb{C}}(\tilde{a}_*) \simeq j_*(\mathrm{e}_{\mathrm{sep}}(\mathtt{id}_{\mathbb{C}}))$ and j_* is a monomorphism we can conclude that $\tilde{b}_{\mathbb{C}}(a_*) \simeq \mathrm{e}_{\mathrm{sep}}(\mathtt{id}_{\mathbb{C}})$. □

Lemma 10.17. *We have* $a \circ b \simeq \mathtt{id}_{\widetilde{\mathcal{P}\mathrm{roj}}^s_{\mathrm{sep}}}$.

Proof. It suffices to show for every A in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ that

$$\pi_0\widetilde{\mathcal{P}\mathrm{roj}}_{\mathrm{sep}}(A) \xrightarrow{b_A} \pi_0\Omega^\infty K_{\mathrm{sep}}(A) \xrightarrow{a_A} \pi_0\widetilde{\mathcal{P}\mathrm{roj}}_{\mathrm{sep}}(A)$$

is an isomorphism. In fact, in order to deduce the isomorphism for π_i with $i > 0$ we apply this result for A replaced by $S^i(A)$ and use the left-exactness of the functors and the fact that they take values in groups.

For the calculation in π_0 introduce the simplified notation (borrowed from [Blackadar 1998, 5.3])

$$K_0 := \pi_0\Omega^\infty K_{\mathrm{sep}}, \quad \tilde{K}_0 := \pi_0\widetilde{\mathcal{P}\mathrm{roj}}^s_{\mathrm{sep}}$$

and

$$V_{00} := \pi_0\mathcal{P}\mathrm{roj}^s_{\mathrm{sep}}, \quad \tilde{K}_{00} := \pi_0\mathcal{P}\mathrm{roj}^{s,\mathrm{group}}_{\mathrm{sep}}.$$

We form the commutative diagram

$$\begin{tikzcd}
 & \tilde{K}_0(A) \arrow[r, "b_A"] \arrow[d] & K_0(A) \arrow[r, "a_A"] \arrow[d] & \tilde{K}_0(A) \arrow[d] \\
V_{00}(A^u) \arrow[r] \arrow[d, "(3)"] & \tilde{K}_{00}(A^u) \arrow[r, "\hat{e}_{h,\mathrm{sep}}"] \arrow[d] & K_0(A^u) \arrow[r, "a_{A^u}"] & \tilde{K}_0(A^u) \arrow[d, "(2)"] \\
V_{00}((A^u\otimes K)^u) \arrow[r, "(1)"] & \tilde{K}_{00}((A^u\otimes K)^u) \arrow[rr, equal] & & \tilde{K}_{00}((A^u\otimes K)^u)
\end{tikzcd}$$

The vertical maps are inclusions of summands and all cells except the lower right commute obviously. We will show that this cell also commutes. We can then conclude that

$$a_A \circ b_A = \mathtt{id}_{\tilde{K}_0(A)}.$$

The transformation $V_{00} \to \tilde{K}_{00}$ is the algebraic group completion. In view of its universal property it suffices to show that the composition of the two lower cells commutes.

We consider a point $[p]$ in $V_{00}(A^u)$ given by a map $p : \mathbb{C} \to A^u \otimes K$. We first calculate its image k_{rd} under the right-down composition. The horizontal map sends it to the point in $\tilde{K}_0(A^u)$ given by $a_{A^u}(\mathrm{e}_{\mathrm{sep}}(p)) \simeq \tilde{K}_0(p)(a_*)$. The element k_{rd} is the image of $\tilde{K}_0(p)(a_*)$ in $\tilde{K}_{00}((A^u \otimes K)^u)$ under the map (2). As seen in the proof of Lemma 10.16 the map $\tilde{K}_0(\mathbb{C}) \to \tilde{K}_{00}(\mathbb{C}^u)$ sends a_* to the image of $i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^u$ in $V_{00}(\mathbb{C}^u)$ under the group completion map $V_{00}(\mathbb{C}^u) \to \tilde{K}_{00}(\mathbb{C}^u)$. The image k_{rd} of $\tilde{K}_0(p)(a_*)$ in $\tilde{K}_{00}((A^u \otimes K)^u)$ is then the image of $p^u \circ i_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^u \to (A^u \otimes K)^u$ in $V_{00}((A^u \otimes K)^u)$ under the map (1).

We now calculate the image k_{dr} of p under the down-right composition. The image of p under (3) is $i_{A^u\otimes K}\circ p$ in $V_{00}((A^u\otimes K)^u)$. We now use that the following square commutes:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "i_{\mathbb{C}}"] \arrow[d, "p"] & \mathbb{C}^u \arrow[d, "p^u"] \\ A^u\otimes K \arrow[r, "i_{A^u\otimes K}"] & (A^u\otimes K)^u \end{tikzcd}$$

Consequently we have $i_{A^u\otimes K}\circ p\simeq p^u\circ i_{\mathbb{C}}$ in $V_{00}((A^u\otimes K)^u)$. This implies that $k_{rd}\simeq k_{dr}$. □

This finishes the proof of Proposition 10.15. □

We finally have completed the proof of Proposition 10.9. □

We finish this section with the proofs of the two technical results Propositions 10.2 and 10.3.

Proof of Proposition 10.2. The statement of Proposition 10.2 is surely a known fact in C^*-algebra theory. But for the sake of completeness we will provide an argument.

We start with two facts taken from [Blackadar 1998, Section 4.3]. We consider a C^*-algebra B and a projection p in B. Note that projections are always assumed to be selfadjoint. Assume that b is an invertible element in the multiplier algebra $M(B)$ and $b=ur$ is its polar decomposition.

Lemma 10.18. *If bpb^{-1} is a projection, then $bpb^{-1}=upu^*$.*

Proof. By assumption $q:=bpb^{-1}$ is a projection, so it is in particular selfadjoint. Writing $q=urpr^{-1}u^*$ we see that $u^*qu=rpr^{-1}$ is selfadjoint, too. This implies $rpr^{-1}=r^{-1}pr$. We multiply with r on the left and right and get the equality $r^2p=pr^2$. We now use that r is positive and therefore $r=\sqrt{r^2}$. We conclude that also $rp=pr$ holds, and this implies $q=upu^*$. □

In the following we let $B(b,r)$ denote the open r-ball at b in B.

Lemma 10.19. *There exists a constant c in $\left(0,\frac{1}{2}\right)$ and a map*

$$w_p:\mathrm{Proj}(B)\cap B(p,c)\to U(B^u)\cap B(1,1)$$

such that

$$w_p(q)pw_p(q)^{-1}=q$$

for all q in $\mathrm{Proj}(B)\cap B(p,c)$.

Proof. We define $v_p(q):=\frac{1}{2}((2q-1)(2p-1)+1)$ and note that $v_p(q)\in B^u$. If $\|p-q\|<\frac{1}{2}$, then $\|v_p(q)-1\|<1$ and $v_p(q)$ is invertible. Furthermore we have $v_p(p)=1$ and $v_p(q)pv_p^{-1}(q)=q$. We now form the polar decomposition $v_p(q)=w_p(q)r_p(q)$ in B^u. By Lemma 10.18 we then have $w_p(q)pw_p(q)^{-1}=q$. By continuity of w_p and $w_p(p)=1$ we can find a constant c in $(0,1)$ such that $w_p(B(p,c))\subseteq B(1,1)$. □

We consider a surjection $B \to C$ of C^*-algebras. We must show that the induced map $\mathrm{Proj}(B) \to \mathrm{Proj}(C)$ is a Serre fibration. To this end we will solve the lifting problem

$$\begin{tikzcd} X \arrow[r] \arrow[d, "x\mapsto(0,x)"'] & \mathrm{Proj}(B) \arrow[d] \\ {[0,1]\times X} \arrow[r] \arrow[ur, dotted] & \mathrm{Proj}(C) \end{tikzcd}$$

for all compact spaces X. This problem is equivalent to the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathrm{Proj}(C(X)\otimes B) \arrow[d] \\ {[0,1]} \arrow[r] \arrow[ur, dotted] & \mathrm{Proj}(C(X)\otimes C) \end{tikzcd}$$

We must therefore solve the path lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathrm{Proj}(B) \arrow[d] \\ {[0,1]} \arrow[r, "\gamma"] \arrow[ur, dotted, "\tilde{\gamma}"] & \mathrm{Proj}(C) \end{tikzcd}$$

for all surjections $B \to C$.

A path $\sigma : [0, 1] \to \mathrm{Proj}(C)$ is called short if $\|\sigma(t) - \sigma(0)\| < c$ and $\|w_{\sigma(0)}(\sigma(t)) - 1\| < 1$ for all t in $[0, 1]$ (with c and $w_{-}(-)$ from Lemma 10.19) . For the moment we assume that we have a solution for the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r, "\tilde{\sigma}(0)"] \arrow[d] & \mathrm{Proj}(B) \arrow[d] \\ {[0,1]} \arrow[r, "\sigma"] \arrow[ur, dotted, "\tilde{\sigma}"] & \mathrm{Proj}(C) \end{tikzcd} \tag{10-17}$$

for all short paths σ.

By continuity and compactness of the interval there exists n in $\mathbb{N}$ such that $\gamma_{|[i/n,(i+1)/n]}$ is short for all $i = 0, \ldots, n-1$. We then solve the lifting problem inductively by solving the lifting problem for the short paths $\gamma_{|[i/n,(i+1)/n]}$ with initial $\tilde{\gamma}\left(\frac{i}{n}\right)$.

We finally solve the lifting problem (10-17) for short paths σ. We set $u(t) := w_{\sigma(0)}(\sigma(t))$. Then $u(0) = 1$, $\sigma(t) = u(t)\sigma(0)u(t)^*$, and $\|u(t) - 1\| < 1$. We get a path $\log u : [0, 1] \to iC^{\mathrm{sa}}$ with $\log u(0) = 0$. We interpret $\log u$ as an element $i(C_0((0, 1]) \otimes C)^{\mathrm{sa}}$. Since $C_0((0, 1]) \otimes B \to C_0((0, 1]) \otimes C$ is surjective we can find a preimage b in $i(C_0((0, 1]) \otimes B)^{\mathrm{sa}}$. We then set $v := \exp(b) : [0, 1] \to U(B^u)$. Then $\tilde{\sigma}(t) := v(t)\tilde{\sigma}(0)v(t)^*$ is the desired lift of σ with initial $\tilde{\sigma}(0)$ in (10-17). □

Remark 10.20. Using that $\mathbb{C}$ is a semiprojective C^*-algebra, we could deduce the path lifting in (10-17) from [Blackadar 2016, Theorem 5.1]. □

Proof of Proposition 10.3. Similarly as in the proof of Lemma 9.6 we reduce the assertion to a consideration of homotopy groups and eventually to the following lemma.

We consider a small filtered family $(B_i)_{i\in I}$ of C^*-algebras indexed by a poset and set $B := \mathtt{colim}_{i\in I} B_i$. For i, j in I, $i \le j$ we let $\phi_{j,i} : B_i \to B_j$ be the structure map and $\phi_i : B_i \to B$ be the canonical homomorphism.

Let X be a compact metrizable space and Y be a closed subspace. Let i_0 be in I and assume that we are given a square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & \mathrm{Proj}(B_{i_0}) \arrow[d, "\phi_{i_0}"] \\ X \arrow[r, "g"] & \mathrm{Proj}(B) \end{tikzcd}$$

Lemma 10.21. *There exists i in I with $i \ge i_0$ and $h : X \to \mathrm{Proj}(B_i)$ such that $h_{|Y} = \phi_{i,i_0} \circ f$ and $\phi_i \circ h$ is homotopic to g rel Y.*

Proof. Let A be any C^*-algebra. For a selfadjoint element q in A we let $\sigma(q)$ denote the spectrum of q. We fix a number c in $\big(0, \frac{1}{2}\big)$ and consider the subspace

$$P(A) := \big\{q \in A^{\mathrm{sa}} \mid d\big(\sigma(q), \tfrac{1}{2}\big) > c\big\}$$

of selfadjoints in A with a spectral gap at $\frac{1}{2}$. We observe that it contains the space of projections $\mathrm{Proj}(A)$. We fix a function $\chi \in C(\mathbb{R})$ with $\chi_{|(-\infty,1/2-c]} \equiv 0$ and $\chi_{|[1/2+c,\infty)} \equiv 1$. The map $q \mapsto \chi(q)$ defined using the function calculus is a retraction $W : P(A) \to \mathrm{Proj}(A)$. By continuity we can choose a constant c_1 in $(0, \infty)$ such that $\|q^2 - q\| \le c_1$ implies $\|W(q) - q\| \le c$.

We interpret f as a function $Y \to B^{\mathrm{sa}}_{i_0}$. Using Dugundji's extension theorem we find an extension $h_0 : X \to B^{\mathrm{sa}}_{i_0}$.

We then have $g - \phi_{i_0}(h_0) \in (C_0(X \setminus Y) \otimes B)^{\mathrm{sa}}$. We can thus find i_1 in I with $i_1 \ge i_0$ such that there exists r in $(C_0(X \setminus Y) \otimes B_{i_1})^{\mathrm{sa}}$ with $\|g - \phi_{i_0}(h_0) - \phi_{i_1}(r)\| \le c_1/20$. We set $h_1 := \phi_{i_1,i_0}(h_0) + r$ in $(C(X) \otimes B_{i_1})^{\mathrm{sa}}$. Since g is a projection, we have $\|\phi_{i_1}(h_1)^2 - \phi_{i_1}(h_1)\| \le c_1/2$. We now find i in I with $i \ge i_1$ such that such that $\|\tilde{h}^2 - \tilde{h}\| \le c_1$, where $\tilde{h} := \phi_{i,i_1}(h_1)$.

We finally define $h := W(\tilde{h})$ in $\mathrm{Proj}^s(C(X) \otimes B_i)$. By construction we have $h_{|Y} = \phi_{i,i_0}(f)$ and $\|\phi_i(h) - g\| \le c$. Then $(W((1-s)\phi_i(h) + sg))_{s\in[0,1]}$ is a homotopy from $\phi_i(h)$ to g rel Y. This finishes the proof of Lemma 10.21. □

Remark 10.22. Using that $\mathbb{C}$ is semiprojective we could deduce Lemma 10.21 directly from the proof of [Bunke and Duenzinger 2024, Proposition 3.8], in particular from the existence of the lift in equation (3.6) in the same reference. □

Hence we have completed the proof of Proposition 10.3. □

11. The q-construction

The *q-construction* introduced by Cuntz [1987] is an effective tool to capture Kasparov modules in terms of homomorphisms; see Remark 11.12. Using the q-construction one can express the classical KK-theory groups in terms of homotopy classes of maps. The crucial formula states that for two separable C^*-algebras A and B we have

$$\pi_0\underline{\mathtt{Hom}}(qA, K \otimes B) \cong \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep},0}(A, B). \tag{11-1}$$

In [Cuntz 1987, Definition 1.5] this isomorphism is actually the definition of the right-hand side. From this formula the composition

$$\mathrm{KK}^{\mathrm{class}}_{\mathrm{sep},0}(A, B) \otimes \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep},0}(B, C) \to \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep},0}(A, C)$$

is not obvious. However, one can show that the left-hand side in (11-1) is naturally isomorphic to $\pi_0\underline{\mathtt{Hom}}(K \otimes qA, K \otimes qB)$ and this makes the composition obvious.

The final goal of the present section and Section 12 together is to give a selfcontained proof that, for all separable C^*-algebras A and B,

$$\pi_0\underline{\mathtt{Hom}}(qA, K \otimes B) \cong \pi_0\mathrm{KK}_{\mathrm{sep}}(A, B). \tag{11-2}$$

Remark 11.1. The comparison of (11-1) and (11-2) provides a proof of the KK-theory version of Theorem 13.16 below which does not depend on the knowledge of the universal property of $\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$. $\square$

We will start this section with recalling the q-construction. We then continue to study those of its homotopical properties that are easily accessible without going deeper into C^*-algebra theory. We shall see that inverting the images in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ of the canonical morphisms $\iota_A : qA \to A$ for all separable C^*-algebras produces a Dwyer–Kan localization of

$$L_q : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$$

which is equivalent to composition of the localizations L_{splt} from Definition 5.6 (enforcing split exactness) and the right-Bousfield localization at the subcategory of group objects; see Propositions 11.8 and 11.9.

All of the above has a separable version. At the end of the present section we go deeper into C^*-algebra theory. In Theorem 11.13 we reproduce the proof of [Cuntz 1987, Theorem 1.6]. As a consequence, for separable C^*-algebras A and B we can simplify the formula for the mapping spaces in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ to

$$\ell\underline{\mathtt{Hom}}(qA, K \otimes B) \simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(A, B), \tag{11-3}$$

which is already very close to (11-2). The final step towards this formula, discussed in Section 12, is to show that the canonical functor $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q} \to \mathrm{KK}_{\mathrm{sep}}$ is an equivalence.

We now start with the description of Cuntz' q-construction. To every C^*-algebra A we can functorially associate a diagram

$$\begin{tikzcd} 0 \arrow[r] & qA \arrow[r, "i"] & A*A \arrow[r, "d"] & A \arrow[r] \arrow[d, equal] \arrow[dl, dotted, bend right, "\iota_i"'] & 0 \\ 0 \arrow[r] & qA \arrow[r, "i"] \arrow[u, "\bar\sigma"] \arrow[dr, "\iota_A"] & A*A \arrow[r, "d"] \arrow[u, "\sigma"] \arrow[d, "p_0"] & A \arrow[r] & 0 \\ & & A & & \end{tikzcd} \tag{11-4}$$

where the horizontal sequences are exact. Recall that the free product $A*A$ together with the two canonical maps $\iota_i : A \to A*A$, $i=0,1$ represents the coproduct in $C^*\mathbf{Alg}^{\mathrm{nu}}$. The map d (often called the fold map) is determined via the universal property of the free product by $d\circ\iota_i = \mathtt{id}_A$ for $i=0,1$. The C^*-algebra qA is defined as the kernel of d. The two maps ι_i determine splits of the exact sequence. The maps $p_i : A*A \to A$ are determined by the conditions $p_i\circ\iota_i = \mathtt{id}_A$ and $p_i\circ\iota_{1-i} = 0$. We can then define the map $\iota_A := p_0\circ i : qA \to A$. The flip of the two factors of the free product defines an automorphism $\sigma : A*A \to A*A$. Since $d\circ\sigma = d$ it restricts to an involutive automorphism $\bar\sigma : qA \to qA$. In principle we should add an index A also to the notation for the maps $d, i, \sigma, \dots$ as they are all components of natural transformations but we refrain from doing so in order to shorten the notation.

Since qA is an ideal in $A*A$ we have a canonical map $m : A*A \to M(qA)$, where $M(qA)$ denotes the multiplier algebra of qA. We define $m_i := m\circ\iota_i$.

Remark 11.2. Let B be any C^*-algebra. In order to give a map $f : qA \to B$ we could give a map $\hat f : A*A \to M(B)$ and set $f := \hat f\circ i$. We must ensure that this composition takes values in the ideal B of $M(B)$. To this end we consider the *components* $\hat f_i := \hat f\circ\iota_i$ of $\hat f$. We must require that $\hat f_1(a) - \hat f_0(a) \in B$ for all a in A. Under this condition f is a well-defined homomorphism with values in B. If $\hat f_0 = \hat f_1$, then it follows that $f = 0$. We will call $\hat f$ the *associated homomorphism*.

This construction can be reversed. Assume that $f : qA \to B$ is a homomorphism. We define $B' := f(qA)$ and the map $\hat f : A*A \to M(qA) \xrightarrow{M(f)} M(B')$, where we must restrict the codomain of f to B' in order to apply the multiplier algebra functor M which is only functorial for nondegenerate morphisms. The components of $\hat f$ are then given by $\hat f_i := \hat f\circ\iota_i : A \to M(B')$. The datum

$$A \rightrightarrows^{\hat f_0}_{\hat f_1} M(B') \triangleright B' \to B$$

is called a prequasihomomorphism in [Cuntz 1987]. □

The functor $q : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ is continuous with respect to the topological enrichment of $C^*\mathbf{Alg}^{\mathrm{nu}}$. It therefore preserves homotopy equivalences and descends to a functor $q : C^*\mathbf{Alg}^{\mathrm{nu}}_h \to C^*\mathbf{Alg}^{\mathrm{nu}}_h$. Since q does not preserve K-stability we are led to define the functor

$$q^s : C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h, \quad A \mapsto K\otimes qA. \tag{11-5}$$

In the following we use the notation

$$(-)^s := L_K(-) : C^*\mathbf{Alg}_h^{\mathrm{nu}} \to L_K C^*\mathbf{Alg}_h^{\mathrm{nu}} \tag{11-6}$$

for the stabilization functor from Corollary 4.3(1). We define the natural transformation $\iota^s : q^s \to (-)^s$ such that its component at A in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is given by

$$\iota^s_A : q^s A = K \otimes qA \xrightarrow{\mathtt{id}_K \otimes \iota_A} K \otimes A = A^s. \tag{11-7}$$

Recall from Proposition 4.4 that $L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is semiadditive. By Example 6.1 every object in this category is naturally a commutative monoid object. For A in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ the object $q^s A = K \otimes qA$ is thus a commutative monoid object of $L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$. We define $\bar{\sigma}^s := \mathtt{id}_K \otimes \bar{\sigma} : q^s A \to q^s A$, where $\bar{\sigma}$ is as in (11-4).

The following is a version of [Cuntz 1987, Proposition 1.4].

Lemma 11.3. *For A in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$ the object $q^s A$ is a commutative group in $L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$ whose inversion map is given by $\bar{\sigma}^s$.*

Proof. We must show that $\mathtt{id}_{q^s A} + \bar{\sigma}^s \simeq 0$. We have the following chain of equivalences:

$$\begin{aligned}
\mathtt{Map}_{L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}}(q^s A, q^s A) &\simeq \mathtt{Map}_{C^*\mathbf{Alg}_h^{\mathrm{nu}}}(K \otimes qA,\, K \otimes qA) \\
&\simeq \mathtt{Map}_{C^*\mathbf{Alg}_h^{\mathrm{nu}}}(qA,\, K \otimes qA) \\
&\simeq \ell\,\underline{\mathtt{Hom}}(qA,\, K \otimes qA),
\end{aligned}$$

where the last line is by Corollary 3.8. The first reflects the definition of $q^s A$ and that L_K is a left-Bousfield localization; see Corollary 4.3.(1). The second equivalence is induced by left-upper-corner inclusion $qA \to K \otimes qA$ which induces an equivalence since $K \otimes qA$ is a local object in this localization. Under this equivalence the sum is determined by the block sum of morphisms $qA \to K \otimes qA$. Hence $\mathtt{id}_{q^s A} + \bar{\sigma}^s$ is induced by the composition

$$\mathtt{diag}(\mathtt{id}_{qA}, \bar{\sigma}) : qA \to \mathtt{Mat}_2(qA) \to K \otimes qA.$$

It suffices to show that $\mathtt{diag}(\mathtt{id}_{qA}, \bar{\sigma}) : qA \to \mathtt{Mat}_2(qA)$ is homotopic to zero.

The composition $m \circ i : qA \to M(qA)$ is the inclusion; hence it has the associated homomorphism $\widehat{\mathtt{id}_{qA}} = m : A * A \to M(qA)$ (using the notation introduced in Remark 11.2) and the components $\widehat{\mathtt{id}_{qA}}_{\,i} = m_i : A \to M(qA)$. Furthermore, the associated homomorphism $\widehat{\bar{\sigma}} = i \circ \sigma$ of $\bar{\sigma}$ has the components $\widehat{\bar{\sigma}}_{\,i} = m_{1-i}$. We identify $M(\mathtt{Mat}_2(qA)) \cong \mathtt{Mat}_2(M(qA))$ in the natural way. Then $\mathtt{diag}(\mathtt{id}_{qA}, \bar{\sigma})$ has the associated homomorphism

$$\widehat{\mathtt{diag}(\mathtt{id}_{qA}, \bar{\sigma})} = \mathtt{diag}(m, m \circ \sigma) : A * A \to M(\mathtt{Mat}_2(qA))$$

and the components $\widehat{\mathtt{diag}(\mathtt{id}_{qA}, \bar{\sigma})}_i = \mathtt{diag}(m_i, m_{1-i}) : A \to \mathtt{Mat}_2(M(qA))$.

For t in $[0, 1]$ we consider the scalar unitary

$$U_t := \begin{pmatrix} \cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2} \\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{pmatrix}$$

in $\mathtt{Mat}_2(M(qA))$. Since $m_0(a)-m_1(a)\in qA$ for all a in A we have

$$U_t\mathtt{diag}(m_1(a),m_0(a))U_t^*-\mathtt{diag}(m_0(a),m_1(a))\in\mathtt{Mat}_2(qA)$$

for all t in [0, 1]. Hence for every t in [0, 1], as explained in Remark 11.2, the pair

$$\mathtt{diag}(m_0,m_1),\ U_t\mathtt{diag}(m_1,m_0)U_t^*:A\to\mathtt{Mat}_2(M(qA))$$

gives the components of a map $h_t:qA\to\mathtt{Mat}_2(qA)$. We have $h_0=\mathtt{diag}(\mathtt{id}_{qA},\bar\sigma)$ and h_1 is the map with equal components $(\hat h_1)_i=\mathtt{diag}(m_0,m_1)$ for $i=0,1$. Hence $h_1=0$. □

Let $E:L_KC^*\mathbf{Alg}^{\mathrm{nu}}_h\to\mathbf{C}$ be a left-exact functor to a semiadditive ∞-category. We say that E is split exact if it sends the images under $L_{h,K}$ of split exact sequences of C^*-algebras to fiber sequences.

The following lemma is a version of [Cuntz 1987, Proposition 3.1.b]. Let A be a K-stable C^*-algebra. We will use the same symbol for the corresponding object of $L_KC^*\mathbf{Alg}^{\mathrm{nu}}_h$, i.e., we will write A instead of $L_{h,K}(A)$ in order to simplify the notation.

Lemma 11.4. *If E is split exact and $E(A)$ is a group, then the map $\iota^s_A:q^sA\to A^s$ from (11-7) induces an equivalence $E(\iota^s_A):E(q^sA)\xrightarrow{\simeq}E(A^s)$.*

Proof. The middle-horizontal exact sequence in (11-4) is split and induces the fiber sequence

$$E(q^sA)\xrightarrow{E(i^s)}E((A*A)^s)\xrightarrow{E(d^s)}E(A^s). \tag{11-8}$$

Since A is K-stable we have $A\simeq A^s$, and hence $E(A)\simeq E(A^s)$. In view of our assumptions $E(A^s)$ is a group. Since q^sA is a group by Lemma 11.3 and E preserves products (since it is split exact) we see that also $E(q^sA)$ is a group. Finally, by Lemma 4.5 we have an equivalence

$$E((A*A)^s)\xrightarrow{E(p^s_0)\times E(p^s_1)}E(A^s)\times E(A^s)$$

whose inverse is $E(\iota^s_0)\circ\mathtt{pr}_0+E(\iota^s_1)\circ\mathtt{pr}_1$. This implies that $E((A*A)^s)$ is a group as well. Using that the objects in the sequence (11-8) are commutative groups we see that

$$E(q^sA)\times E(A^s)\xrightarrow{E(i^s_A)+E(\iota_1)}E((A*A)^s)$$

is an equivalence. Hence

$$\begin{pmatrix}E(\iota^s_A)&0\\E(p^s_1\circ i^s)&\mathtt{id}_{E(A^s)}\end{pmatrix}:E(q^sA)\times E(A^s)\xrightarrow{E(i^s)+E(\iota^s_1)}E((A*A)^s)\xrightarrow{E(p^s_0)\times E(p^s_1)}E(A^s)\times E(A^s)$$

is an equivalence. Again using that the factors are groups this implies that $E(\iota^s_A)$ is an equivalence. □

We consider the set of morphisms

$$\widehat W_q:=\{\iota^s_A:q^sA\to A^s\mid A\in C^*\mathbf{Alg}^{\mathrm{nu}}_h\} \tag{11-9}$$

in $L_KC^*\mathbf{Alg}^{\mathrm{nu}}_h$.

Definition 11.5. We define the Dwyer–Kan localization

$$L_q : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q} \tag{11-10}$$

at the set $\widehat{W}_q$.

We consider the composition

$$L_{h,K,q} : C^*\mathbf{Alg}^{\mathrm{nu}} \xrightarrow{L_h} C^*\mathbf{Alg}^{\mathrm{nu}}_h \xrightarrow{L_K} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \xrightarrow{L_q} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}. \tag{11-11}$$

Proposition 11.6. *The functor $L_{h,K,q} : C^*\mathbf{Alg}^{\mathrm{nu}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$ is a Dwyer–Kan localization.*

Proof. This follows from the fact that $L_{h,K,q}$ is a composition of Dwyer–Kan localizations which are all determined by images of collections of morphisms in $C^*\mathbf{Alg}^{\mathrm{nu}}$, namely homotopy equivalences, left-upper-corner inclusions κ_A from (4-1), and the morphisms $\mathtt{id}_K \otimes \iota_A : K \otimes qA \to K \otimes A$ for all A in $C^*\mathbf{Alg}^{\mathrm{nu}}$. □

In Proposition 11.8(2) we will show that the mapping spaces in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$ can be easily understood in terms of a calculus of fractions. In a sense this result is of intermediate nature since in Corollary 11.14 and Proposition 11.15 we will state a much better result at the cost of using more of C^*-algebra theory.

For every A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ we consider the diagram $W(A) : \mathbb{N}^{\mathrm{op}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ given by

$$\cdots \xrightarrow{\iota^s_{q^3A}} (q^3A)^s \xrightarrow{\iota^s_{q^2A}} (q^2A)^s \xrightarrow{\iota^s_{qA}} (qA)^s \xrightarrow{\iota^s_A} A^s. \tag{11-12}$$

We furthermore let W_q be the subcategory of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ generated by $\widehat{W}_q$. The diagram $W(A)$ is a *putative right-calculus of fractions at* A for W_q in the sense of [Cisinski 2019, Definition 7.2.2]. In fact, $A \simeq A^s$ is the final object of the diagram and all morphisms belong to W_q.

We fix A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ and consider the functor

$$H_A : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to \mathbf{CGroups}(\mathbf{Spc}), \quad B \mapsto \operatorname*{\mathtt{colim}}_{n\in\mathbb{N}} \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}((q^nA)^s, B), \tag{11-13}$$

i.e., we insert the diagram (11-12) into the first argument of the mapping space and take the colimit. The following result is a version of [Cuntz 1987, Proposition 2.1].

Proposition 11.7. *The functor H_A is split exact.*

Proof. We consider a split exact sequence

$$0 \to I \xrightarrow{j} B \xrightarrow{\pi} Q \to 0 \tag{11-14}$$

of C^*-algebras with split $s : Q \to B$. We must show that

$$H_A(I^s) \oplus H_A(Q^s) \xrightarrow{H_A(j^s)+H_A(s^s)} H_A(B^s)$$

is an equivalence, where the superscript s stands for K-stabilization as in (11-6).

We first observe that it suffices to show that this map induces an isomorphism of groups of connected components. For i in $\mathbb{N}$ we write $H_{A,i}(-) := \pi_i H_A(-)$ for the corresponding abelian-group-valued functor. For every i in $\mathbb{N}$ we have a canonical isomorphism $H_{A,i}(-) \cong H_{A,0}(S^i(-))$, where $S^i(-) := C_0(\mathbb{R}^i) \otimes -$

is the i-fold suspension functor. Using the fact that the functor $S^i(-)$ preserves exact sequences we see that it suffices to show that

$$H_{A,0}(I^s) \oplus H_{A,0}(Q^s) \xrightarrow{H_{A,0}(j^s)\oplus H_{A,0}(s^s)} H_{A,0}(B^s) \tag{11-15}$$

is an isomorphism for all split exact sequences (11-14).

We now use that L_K is a left-Bousfield localization and Corollary 3.8, or directly (4-3), in order see that

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}((q^n A)^s, (-)^s) \simeq \mathtt{Map}_{C^*\mathbf{Alg}^{\mathrm{nu}}_h}(q^n A, (-)^s) \simeq \ell\underline{\mathtt{Hom}}(q^n A, (-)^s) \tag{11-16}$$

as a functors from $C^*\mathbf{Alg}^{\mathrm{nu}}$ to $\mathbf{Spc}$. Combining this with (11-13) we get an equivalence

$$H_A(B) \simeq \mathtt{colim}(\ell\underline{\mathtt{Hom}}(A, B^s) \xrightarrow{\iota_A^*} \ell\underline{\mathtt{Hom}}(qA, B^s) \xrightarrow{\iota_{qA}^*} \ell\underline{\mathtt{Hom}}(q^2 A, B^s) \xrightarrow{\iota_{q^2A}^*} \cdots).$$

We define a map

$$l_n : \underline{\mathtt{Hom}}(q^n A, B^s) \to \underline{\mathtt{Hom}}(q^{n+1} A, I^s)$$

as follows. Let $f : q^n A \to B^s$ be an element of $\underline{\mathtt{Hom}}(q^n A, B^s)$. Then we get an associated map $(f, s^s\pi^s f) : q^n A * q^n A \to B^s$ with the components as indicated; see Remark 11.2. We observe that

$$\pi^s \circ (f, s^s\pi^s f) \circ i_{q^n A} = 0,$$

where $i_{q^n A} : q^{n+1} A \to q^n A * q^n A$ is the canonical inclusion. Therefore we can define

$$l_n(f) := (f, s^s\pi^s f) \circ i_{q^n A} : q^{n+1} A \to I^s.$$

For $g : q^n A \to I^s$ we have $l_n(j^s \circ g) = g \circ \iota_{q^n A}$. Finally, for $h : q^n A \to Q^s$ we have $l_n(s^s \circ h) = (s^s \circ h, s^s \circ h) \circ i_{q^n A} = 0$.

We next show that $H_{A,0}(j^s)$ is injective. Let $g : q^n A \to I^s$ represent an element $[g]$ in $H_{A,0}(I^s)$ such that $H_{A,0}(j^s)([g]) = 0$. Then there exists m in $\mathbb{N}$ with $n \le m$ such that $[j^s \circ g \circ \iota_{q^n A} \circ \cdots \circ \iota_{q^{m-1} A}] = 0$ in $\pi_0\underline{\mathtt{Hom}}(q^m A, B^s)$. But then

$$0 = [l_m(j^s \circ g \circ \iota_{q^n A} \circ \cdots \circ \iota_{q^{m-1} A})] = [g \circ \iota_{q^n A} \circ \cdots \circ \iota_{q^{m-1} A} \circ \iota_{q^m A}]$$

in $\pi_0\underline{\mathtt{Hom}}(q^{m+1} A, I^s)$. This implies that $[g] = 0$.

We now show that the family of maps $(l_n)_n$ determines a well-defined map $l : H_{A,0}(B^s) \to H_{A,0}(I^s)$. Let $[f]$ in $H_{A,0}(B^s)$ be represented by a map $f : q^n A \to B^s$. Then

$$H_{A,0}(j^s)([l_n(f)]) = [f] - H_{A,0}(s^s \circ \pi^s)([f]).$$

The right-hand side does not depend on the choice of the representative. By the injectivity statement above we conclude that $[l_n(f)]$ is well-defined.

We thus have

$$H_{A,0}(j^s) \circ l + H_{A,0}(s^s) \circ H_{A,0}(\pi^s) = \mathtt{id}_{H_{A,0}}(B),$$

$$l \circ H_{A,0}(j^s) = \mathtt{id}_{H_{A,0}(I^s)}, \quad l \circ H_{A,0}(s^s) = 0, \quad H_{A,0}(p^s) \circ H_{A,0}(s^s) = \mathtt{id}_{H_{A,0}(Q^s)}.$$

These equalities imply that (11-15) is an isomorphism. □

The putative right-calculus of fractions is called a *right-calculus of fractions* in the sense of [Cisinski 2019, Definition 7.2.6] if the functor $H_A(-)$ from (11-13) sends the morphisms from W_q to equivalences.

Proposition 11.8. (1) *$W(A)$ is a right-calculus of fractions.*

(2) *We have*

$$H_A(B) \simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}}(A, B). \tag{11-17}$$

(3) *The ∞-category $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$ is additive.*

(4) *The localization L_q is left-exact.*

(5) *We have an essentially unique commutative diagram*

$$\begin{tikzcd} & C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[dl, "L_{h,K,\mathrm{splt}}"'] \arrow[d, "L_{h,K}"] \arrow[dr, "L_{h,K,q}"] & \\ L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}} \arrow[rr, dotted, "L"] & L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \arrow[l, "L_{\mathrm{splt}}"'] \arrow[r, "L_q"] & L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q} \end{tikzcd} \tag{11-18}$$

where L is a left-exact functor.

Proof. The assertion (1) follows from Proposition 11.7 and Lemma 11.4.

The assertion (2) follows from (1) and the general formula [Cisinski 2019, Definition 7.2.8] for the mapping spaces in a localization in the presence of a right-calculus of fractions.

For (3) note that by (11-17) the mapping spaces in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$ are commutative groups.

Assertion (4) is a consequence of the formula (combine (11-13) and (11-17))

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}}(A, B) \simeq \operatorname*{\mathtt{colim}}_{\mathbb{N}} \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}((q^n A)^s, B) \tag{11-19}$$

for the mapping space as a colimit over the filtered poset $\mathbb{N}$ and the fact that filtered colimits in **Spc** commute with finite limits.

We finally show (5). The two upper triangles in (11-18) reflect the definitions of the maps. By (2) the functor $H_A(-)$ represents the mapping space in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$. Since it is split exact by Proposition 11.7 the localization L_q sends (the images under $L_{h,K}$ of) split exact sequences to fiber sequences. The composition $L_{h,K,q}$ is thus homotopy invariant, stable, Schochet-exact and split exact. By the universal property of $L_{h,K,\mathrm{splt}}$ stated in Proposition 5.7(2) we get the map L and the two-morphism filling of the outer triangle in (11-18). We finally use the universal property (4-4) of $L_{h,K}$ in order to define the two-morphism filling the lower triangle. □

Proposition 11.9. *The functor L from (11-18) is equivalent to the right-adjoint of a right-Bousfield localization*

$$\mathrm{incl} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}{}^{\mathrm{group}} \leftrightarrows L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}} : R.$$

Proof. We call a functor on $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ split exact if it sends the elements of W_{splt} to equivalences. We first show that $L_{\mathrm{splt}} \circ q^s$ is split exact and left-exact and therefore by Proposition 5.7(1) descends to a left-exact functor

$$R : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}.$$

The functor $L_{\mathrm{splt}} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}$ is split exact by definition, and it is left-exact by Proposition 5.7(1). The functor $L_{\mathrm{splt}} \circ q^s$ sends A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ to the fiber of $L_{\mathrm{splt}}((A * A)^s) \xrightarrow{d^s_A} L_{\mathrm{splt}}(A^s)$, where $d_A : A * A \to A$ is the fold map. Since $L_{\mathrm{splt}}((A * A)^s) \simeq L_{\mathrm{splt}}(A^s) \times L_{\mathrm{splt}}(A^s)$ by semiadditivity we can identify $L_{\mathrm{splt}} \circ q^s$ with the functor which sends A to the fiber of some natural map $L_{\mathrm{splt}}(A^s) \times L_{\mathrm{splt}}(A^s) \to L_{\mathrm{splt}}(A^s)$ between split exact and left-exact functors. We conclude that $L_{\mathrm{splt}} \circ q^s$ itself is split exact and left-exact.

As a consequence of Lemma 11.3 and the fact that L_{splt} preserves products the functor R takes values in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}{}^{\mathrm{group}}$. We have a natural transformation

$$\kappa := L_{\mathrm{splt}}(\iota^s) : R \to \mathtt{id}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}. \tag{11-20}$$

If A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ has the property that $L_{\mathrm{splt}}(A)$ is a group, then $\kappa_{L_{\mathrm{splt}}(A)} \simeq L_{\mathrm{splt}}(\iota^s_A)$ is an equivalence by Lemma 11.4 since L_{splt} is split exact. Since L_{splt}, being a Dwyer–Kan localization, is essentially surjective we can conclude that the essential image of R is $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}{}^{\mathrm{group}}$.

In particular we see that $\kappa_{R(A)}$ is an equivalence for every A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}$. Since $R \circ L_{\mathrm{splt}} \simeq L_{\mathrm{splt}} \circ q^s$ is split exact and takes values in groups we can conclude again by Lemma 11.4 that $R(\kappa_A)$ is also an equivalence for every A.

As explained at the beginning of Section 7 this implies that R is the right-adjoint of a right-Bousfield localization

$$\mathrm{incl} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}{}^{\mathrm{group}} \leftrightarrows L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}} : R \tag{11-21}$$

with counit κ.

The functor L inverts the morphisms κ_A since $L(\kappa_A) \simeq L_q(\iota^s_A)$. This gives the factorization L' in the diagram

$$\begin{tikzcd}
& L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \arrow[dl, "L_{\mathrm{splt}}"'] \arrow[dr, "L_q"] & \\
L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}} \arrow[rr, "L"] \arrow[dr, "R"'] & & L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q} \arrow[dl, dotted, bend left, "L'^{,-1}"] \\
& L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}{}^{\mathrm{group}} \arrow[ur, dotted, "L'"] &
\end{tikzcd}$$

Since $R \circ L_{\mathrm{splt}}$ sends the morphisms ι^s_A to the equivalences $R(\kappa_A)$ for all A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$ we get an inverse to L' from the universal property of L_q. □

Recall the definition (11-11) of $L_{h,K,q} : C^*\mathbf{Alg}^{\mathrm{nu}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$.

Proposition 11.10. *For any left-exact and additive ∞-category $\mathbf{D}$ we have an equivalence*

$$L^*_{h,K,q} : \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,splt+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{11-22}$$

Proof. This follows from the chain of equivalences

$$\begin{aligned}\mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}, \mathbf{D}) &\simeq \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}{}^{\mathrm{group}}, \mathbf{D})\\ &\simeq \mathbf{Fun}^{\mathrm{lex}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}, \mathbf{D})\\ &\simeq \mathbf{Fun}^{h,s,splt+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}),\end{aligned}$$

where the first, second and third equivalences are given by $L^{\prime,*}$, R^* and $L^*_{h,K,\mathrm{splt}}$ (defined in (5-4)), respectively. In order to see that R^* is an equivalence note that the components $\kappa_A : R(A) \to A$ of the natural transformation (11-20) generate the Dwyer–Kan localization R. As a consequence of Lemma 11.4 any left-exact functor $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}} \to \mathbf{D}$ to an additive ∞-category $\mathbf{D}$ sends these components to equivalences. □

Proposition 11.11. *For $?$ in $\{\min, \max\}$ the localization L_q has a symmetric monoidal refinement and the tensor product $\otimes_?$ on $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$ is bileft-exact.*

Proof. Since the localization L_{splt} has a symmetric monoidal refinement with a bileft-exact tensor product by Proposition 7.2 it suffices to show that the functor L has one. As seen in the proof of Proposition 11.9 we have a functor which sends A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}$ to the diagram

$$\begin{tikzcd} R(A) \arrow[r] \arrow[dr, "\kappa_A"] & A \times A \arrow[r] \arrow[d, "\mathtt{pr}_0"] & A \\ & A & \end{tikzcd}$$

where the upper sequence is a fiber sequence. Since $\otimes$ is biexact on $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}$, for B in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,\mathrm{splt}}$ we get a similar diagram

$$\begin{tikzcd} R(A)\otimes B \arrow[r] \arrow[dr, "\kappa_A\otimes B"] & (A \times A)\otimes B \arrow[r] \arrow[d, "\mathtt{pr}_0\otimes B"] & A\otimes B \\ & A\otimes B & \end{tikzcd}$$

We can conclude that $\kappa_{A\otimes B} \simeq \kappa_A \otimes B$. In particular, $-\otimes B$ preserves the generators of the Dwyer–Kan localization L which therefore has a symmetric monoidal refinement. Furthermore, $-\otimes B$ descends to a left-exact functor on $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}$. □

For every symmetric monoidal additive ∞-category we thus get an equivalence

$$L^*_{h,K,q} : \mathbf{Fun}^{\mathrm{lex}}_{\otimes/\mathrm{lax}}(L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{h,q}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,splt+Sch}_{\otimes/\mathrm{lax}}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{D}). \tag{11-23}$$

Remark 11.12. We provide the bridge to Kasparov modules. We refer to [Kasparov 1988; Blackadar 1998] for a detailed theory. Let $f : qA \to B\otimes K$ be a homomorphism with components $\hat{f}_i : A \to M(B\otimes K)$. The corresponding (A, B)-bimodule (H, ϕ, F) is the $\mathbb{Z}/2\mathbb{Z}$-graded B-Hilbert C^*-module $L^2(B)\oplus L^2(B)$ with the odd endomorphism

$$F := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\phi := \begin{pmatrix} \hat{f}_0 & 0 \\ 0 & \hat{f}_1 \end{pmatrix} : A \to \mathtt{Mat}_2(B(L^2(B)), \tag{11-24}$$

where in order to interpret (11-24) we identify $B(L^2(B)) \cong M(B\otimes K)$ in the canonical way. For separable C^*-algebras A, B we interpret $\underline{\mathtt{Hom}}(qA, K\otimes B)$ as the topological space of (A, B)-Kasparov modules in the spirit of Cuntz. Its underlying space

$$\ell\,\underline{\mathtt{Hom}}(qA, K\otimes B) \simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(qA, B) \simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(q^s A, B) \tag{11-25}$$

has a natural refinement to a commutative monoid in spaces. This monoid structure reflects the direct sum of Kasparov modules. Since groups and cogroups in a semiadditive ∞-category coincide, Lemma 11.3 implies that $\ell\,\underline{\mathtt{Hom}}(qA, K\otimes B)$ is actually a commutative group. We have a natural map of commutative groups

$$\ell\,\underline{\mathtt{Hom}}(qA, K\otimes B) \to \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(A, B), \tag{11-26}$$

given by the composition of (11-25) with the canonical map from the right-hand side of this equivalence to the second stage of the colimit in (11-19). By Corollary 11.14 below we see that this map is actually an equivalence presenting the commutative mapping groups in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ in terms of spaces of Kasparov modules. □

For completeness of the presentation we now discuss [Cuntz 1987, Theorem 1.6]. All of the above has a version for separable algebras which we will indicate by an additional subscript sep. Let A be a separable C^*-algebra.

Theorem 11.13 [Cuntz 1987, Theorem 1.6]. *There exists a homomorphism $\phi : qA \to \mathtt{Mat}_2(q^2A)$ such that $\mathtt{Mat}_2(\iota_{qA})\circ\phi : qA \to \mathtt{Mat}_2(qA)$ and $\phi\circ\iota_{qA} : q^2A \to \mathtt{Mat}_2(q^2A)$ are homotopic to the left-upper-corner inclusions.*

Before we sketch the proof we derive the consequences of Theorem 11.13.

Corollary 11.14. *For separable C^*-algebras A and B the morphism (11-26) is an equivalence.*

Proof. This is an immediate consequence of (11-19), (11-16) and Theorem 11.13 which implies that the colimit in (11-19) stabilizes from $n = 1$ on. □

Proposition 11.15. *The functor $q^s_{\mathrm{sep}} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to q^s_{\mathrm{sep}} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ is the right-adjoint of a Bousfield localization*

$$\mathrm{incl} : q^s_{\mathrm{sep}} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \leftrightarrows L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} : q^s_{\mathrm{sep}}$$

and

$$q^s_{\mathrm{sep}} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to q^s_{\mathrm{sep}} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$$

represents its target as the Dwyer–Kan localization at the set $\widehat{W}_{\mathrm{sep},q}$ from the separable version of (11-9).

Proof. Let $\mathrm{incl} : q^s_{\mathrm{sep}} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ denote the inclusion of the full subcategory of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ on the image of q^s_{sep}. We have a natural transformation

$$\iota^s : \mathrm{incl} \circ q^s_{\mathrm{sep}} \to \mathtt{id}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}. \tag{11-27}$$

For A in $q^s_{\mathrm{sep}} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ and B be in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ the binatural transformation

$$\mathtt{Map}_{q^s_{\mathrm{sep}} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(A, q^s_{\mathrm{sep}} B) \xrightarrow{\mathrm{incl}} \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(\mathrm{incl}\, A, \mathrm{incl}(q^s_{\mathrm{sep}} B)) \xrightarrow{\iota^s_{B,*}} \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(\mathrm{incl}\, A, B)$$

is an equivalence. To this end we set $A = q^s_{\mathrm{sep}} A'$ for some A' in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ and factorize the map as a composition of equivalences

$$\begin{aligned}
\mathtt{Map}_{q^s_{\mathrm{sep}} L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(A, q^s_{\mathrm{sep}} B) &\simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(q^s_{\mathrm{sep}} A', q^s_{\mathrm{sep}} B) \\
&\simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(A', q^s_{\mathrm{sep}} B) \\
&\simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(A', B) \\
&\simeq \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(\mathrm{incl}\, A, B),
\end{aligned}$$

where the first and third equivalences are incl and $\iota^s_{B,*}$, respectively, and the second and fourth equivalences are by Corollary 11.14. We conclude that (11-27) is the counit of a right-Bousfield localization. Since the right-Bousfield localization is a Dwyer–Kan localization at the set of the components of its counit we conclude the second assertion by a comparison with (11-9). □

Recall the construction (4-8) of a sum of a family of C^*-algebras.

Corollary 11.16. *The category $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ admits countable coproducts which are represented by the free product and also by the sum in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$.*

Proof. Since $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ is a right-Bousfield localization of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$ it inherits all colimits from the latter category and the inclusion functor, being a left-adjoint, preserves them. The assertion now follows from Corollary 4.11(4). □

Theorem 11.13 is crucial for understanding the nature of the localization L_q which in turn implies the important categorical property of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ of being countably cocomplete and of course the simple formula (11-3) for the mapping spaces. Because of its relevance, for completeness of the presentation we decided to repeat the proof of Theorem 11.13 from [Cuntz 1987].

Proof of Theorem 11.13. We will use as a fact:

Lemma 11.17. *If $I \to B$ is the inclusion of an ideal, then $I * I \to B * B$ is injective.*

We abbreviate $QA := A * A$. By Lemma 11.17 the map $qA * qA \to QA * QA$ is injective. We construct the diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathtt{Mat}_2(qQA) \arrow[r] & \mathtt{Mat}_2(Q^2A) \arrow[r] & \mathtt{Mat}_2(QA) \arrow[r] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[ur] \arrow[dd, bend right, equal] & D \arrow[r, "\pi"] \arrow[ur] & S \arrow[r] \arrow[ur] & 0 & \\
 & 0 \arrow[r] & \mathtt{Mat}_2(q^2A) \arrow[r] \arrow[uu, bend right] & \mathtt{Mat}_2(QqA) \arrow[r] \arrow[uu, bend right] & \mathtt{Mat}_2(qA) \arrow[r] \arrow[uu, bend right] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[ur] & R \arrow[r] \arrow[ur] \arrow[uu, bend left] & \mathtt{Mat}_2(qA) \arrow[r] \arrow[ur, equal] \arrow[uu, bend left] & 0 &
\end{tikzcd}$$

All vertical maps are injective. The C^*-algebra R is defined as the subalgebra of $\mathtt{Mat}_2(QqA)$ generated by matrices of the form

$$\begin{pmatrix} \eta_0(qA) & \eta_0(qA)\eta_1(qA) \\ \eta_1(qA)\eta_0(qA) & \eta_1(qA) \end{pmatrix},$$

where we use the notation η_0 and η_1 for the canonical inclusions ι_0 and ι_1 of qA into QqA. We observe that the projection $R \to \mathtt{Mat}_2(qA)$ is surjective and defines the ideal J as its kernel. We let D be the subalgebra of $\mathtt{Mat}_2(Q^2A)$ generated by the image of R and the elements

$$\begin{pmatrix} \eta_0(\iota_0(a)) & 0 \\ 0 & \eta_1(\iota_0(a)) \end{pmatrix}, \quad a \in A.$$

One checks by an explicit calculation that R is an ideal in D, and hence J is also an ideal in D. Then S is the subalgebra of $\mathtt{Mat}_2(QA)$ generated by $\mathtt{Mat}_2(qA)$ and the diagonal elements

$$\begin{pmatrix} \iota_0(a) & 0 \\ 0 & \iota_0(a) \end{pmatrix}, \quad a \in A.$$

We let U_t be the rotation matrix from (4-7). We note that conjugation by U_t on $\mathtt{Mat}_2(QA)$ preserves the subalgebra S. The derivative of this action is a bounded derivation $\bar{\delta}$ of S.

By Pedersen's derivation lifting theorem [1976; 1979, Theorem 8.6.15] there exists a derivation δ of D such that $\pi \circ \delta = \bar{\delta} \circ \pi$. It is at this point where separability of A is important. There are counterexamples to the derivation lifting theorem for nonseparable algebras.

We define the family $\sigma_t := e^{t\delta}$ of automorphisms of D and set $\sigma := \sigma_1$.

We define $\phi : qA \to \mathtt{Mat}_2(q^2A)$ as the homomorphism with the components $\hat{\phi}_i : A \to \mathtt{Mat}_2(QqA) \to M(\mathtt{Mat}_2(q^2A))$ given by

$$\hat{\phi}_0 := \begin{pmatrix} cc\eta_0 \circ \iota_0 & 0 \\ 0 & \eta_1 \circ \iota_0 \end{pmatrix}, \quad \hat{\phi}_1 := \sigma \begin{pmatrix} cc\eta_0 \circ \iota_1 & 0 \\ 0 & \eta_1 \circ \iota_0 \end{pmatrix}. \tag{11-28}$$

In order to see that the application of σ is well-defined we rewrite

$$\begin{pmatrix} \eta_0 \circ \iota_1 & 0 \\ 0 & \eta_1 \circ \iota_0 \end{pmatrix} = \begin{pmatrix} \eta_0 \circ \iota_0 & 0 \\ 0 & \eta_1 \circ \iota_0 \end{pmatrix} - \begin{pmatrix} \eta_0 \circ (\iota_1 - \iota_0) & 0 \\ 0 & 0 \end{pmatrix},$$

which obviously takes values in D. Using in addition a similar rewriting of $\hat{\phi}_0$ one then checks that $\hat{\phi}_0 - \hat{\phi}_1$ takes values in $\mathtt{Mat}_2(q^2A)$.

The homotopy γ_t from the left-upper-corner inclusion $q^2A \to \mathtt{Mat}_2(q^2A)$ to $\phi \circ \iota_{qA}$ has the components $\hat{\gamma}_i : qA \to \mathtt{Mat}_2(Q^2A)$ given by the map (again given by a pair of components)

$$\hat{\gamma}_0 := \left(\begin{pmatrix} cc\eta_0 \circ \iota_0 & 0 \\ 0 & \eta_1 \circ \iota_0 \end{pmatrix}, \sigma_t \begin{pmatrix} \eta_0 \circ \iota_1 & 0 \\ 0 & \eta_1 \circ \iota_1 \end{pmatrix} \right)$$

and the map

$$\hat{\gamma}_1 := \left(\begin{pmatrix} \eta_1 \circ \iota_0 & 0 \\ 0 & \eta_1 \circ \iota_1 \end{pmatrix}, U_t \begin{pmatrix} \eta_1 \circ \iota_1 & 0 \\ 0 & \eta_1 \circ \iota_0 \end{pmatrix} U_t^* \right).$$

Similarly, a homotopy λ_t from the left-upper-corner inclusion $qA \to \mathtt{Mat}_2(qA)$ to $\iota_{qA} \circ \phi$ has the components $\mathtt{Mat}_2(p_0) \circ \tilde{\lambda}_t : qA \to \mathtt{Mat}_2(qA)$, where $p_0 : QqA \to qA$ and $\tilde{\lambda}_t : qA \to R \to \mathtt{Mat}_2(QqA)$ is given by

$$\left(\begin{pmatrix} \eta_0 \circ \iota_0 & 0 \\ 0 & \eta_1 \circ \iota_1 \end{pmatrix}, \sigma_t \begin{pmatrix} \eta_0 \circ \iota_1 & 0 \\ 0 & \eta_1 \circ \iota_0 \end{pmatrix} \right).$$

We leave the justifications for these formulas to the interested reader or refer to the proof of [Cuntz 1987, Theorem 1.6]. □

12. The automatic semiexactness theorem

Since the symmetric monoidal functor $\mathrm{kk}_{\mathrm{sep}}$ is homotopy invariant, stable and split exact, it belongs to the right-hand side of the separable version of the equivalence (11-23) describing the universal property of $L_{\mathrm{sep},h,K,q}$ for $\mathbf{D} := \mathrm{KK}_{\mathrm{sep}}$. Its preimage under this equivalence is the left-exact and symmetric monoidal functor h depicted by the lower-horizontal arrow in the commutative triangle

$$\begin{tikzcd} & C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[dl, "L_{\mathrm{sep},h,K,q}"'] \arrow[dr, "\mathrm{kk}_{\mathrm{sep}}"] & \\ L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q} \arrow[rr, dotted, "h"] & & \mathrm{KK}_{\mathrm{sep}} \end{tikzcd} \tag{12-1}$$

The functor h will be called the *comparison functor*.

In Theorem 12.1 we claim that this comparison functor is an equivalence. We will give two immediate proofs which at least implicitly assume the formulas (11-1), (11-2) and Theorem 13.16. They therefore involve more than just the simple homotopy-theoretic considerations from the present notes. In order to provide a selfcontained proof we will formulate two equivalent statements: Theorems 12.4 and 12.5. Note that Theorem 12.4 is just an assertion about functors defined on the category of separable C^*-algebras and does not require any K-theoretic element at all. On the other hand, the argument for Theorem 12.5 is quite accessible to the methods developed here so that we write out the details of the argument for this version. This finally verifies (11-2) in a noncircular manner.

The fact that the comparison functor is an equivalence has the important consequence that $\mathrm{KK}_{\mathrm{sep}}$ admits countable colimits and is idempotent complete; see Corollary 12.3. We do not have a direct proof of this fact just from the construction of $\mathrm{KK}_{\mathrm{sep}}$.

We start with formulating the main result of the present section.

Theorem 12.1. *The comparison functor* $h : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q} \to \mathrm{KK}_{\mathrm{sep}}$ *is an equivalence.*

As said above, we will give two proofs which should convince the reader that the assertion is true. On the other hand both involve deep facts from the classical KK-theory which are not easily provable on the basis of the approach taken in the present paper.

1. Proof via universal properties. We will show that $\mathrm{kk}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep}}$ has the same universal property as $L_{\mathrm{sep},h,K,q} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ stated in the separable version of (11-22). It appears as the upper-horizontal equivalence in the diagram below, where $\mathbf{D}$ is any left-exact and additive ∞-category:

$$\begin{tikzcd}
\mathbf{Fun}^{\mathrm{lex}}(\mathrm{KK}_{\mathrm{sep}}, \mathbf{D}) \arrow[r, dashed, "\simeq", "\mathrm{kk}^*_{\mathrm{sep}}"'] \arrow[d] & \mathbf{Fun}^{h,s,splt+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \arrow[d] \\
\mathbf{Fun}^{\coprod}(\mathrm{KK}_{\mathrm{sep}}, \mathbf{D}) \arrow[r, "\simeq", "\mathrm{kk}^*_{\mathrm{sep}}"'] \arrow[d] & \mathbf{Fun}^{h,s,splt}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \arrow[d, "!"] \\
\mathbf{Fun}(\mathrm{KK}_{\mathrm{sep}}, \mathbf{D}) \arrow[r, "\simeq", "\mathrm{kk}^*_{\mathrm{sep}}"'] & \mathbf{Fun}^{\widetilde{W}_{\mathrm{sep,se}}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D})
\end{tikzcd}, \tag{12-2}$$

where the superscript $\coprod$ indicates finite coproduct preserving functors. The lower square has been discussed in the proof of [Bunke et al. 2021, Theorem 2.23]; see (2.31) in the same reference. The lower-horizontal equivalence reflects the fact (see Proposition 7.5) that $\mathrm{kk}_{\mathrm{sep}}$ is the Dwyer–Kan localization at the set $\widetilde{W}_{\mathrm{sep,se}}$ of morphisms in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ inverted by $\mathrm{kk}_{\mathrm{sep}}$. The crucial point is the existence of the arrow marked by !. To see that it exists in [Bunke et al. 2021] we used the comparison of $\mathrm{hoKK}_{\mathrm{sep}}$ with the classical theory and the fact that the latter has a universal property involving the condition of split exactness [Bunke et al. 2021, Corollary 2.4]. The middle-horizontal equivalence has been discussed in the proof of [Bunke et al. 2021, Theorem 2.23]. For our present purpose we need the dashed equivalence which is obtained by an analogous argument explained in the subsequent paragraph.

All vertical arrows in the diagram above are fully faithful functors. Since $\mathrm{kk}_{\mathrm{sep}}$ is homotopy invariant, stable, Schochet-exact and semiexact, it is Schochet-exact and split exact. Therefore the dashed arrow exists. We must show that it is essentially surjective. Thus consider a functor F in $\mathbf{Fun}^{h,s,splt+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D})$. It gives rise to a functor $\hat{F}$ in $\mathbf{Fun}(\mathrm{KK}_{\mathrm{sep}}, \mathbf{D})$ such that $\mathrm{kk}^*_{\mathrm{sep}}\hat{F} \simeq F$. It remains to show that $\hat{F}$ is left-exact. It is clearly reduced. Every cartesian square in $\mathrm{KK}_{\mathrm{sep}}$ can be represented as the image under $\mathrm{kk}_{\mathrm{sep}}$ of a

Schochet fibrant cartesian square, and by assumption F sends this square to a cartesian square in $\mathbf{D}$. This implies left-exactness of $\hat{F}$. See the proof of Proposition 3.17(3) for an analogous argument. □

2. *Proof based on* (11-2). For any two separable C^*-algebras A and B the comparison map induces the second map in

$$\pi_0 \underline{\mathtt{Hom}}(qA, K \otimes B) \simeq \pi_0 \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(A, B) \xrightarrow{h} \mathrm{KK}_{\mathrm{sep},0}(A, B),$$

where the equivalence is by Corollary 11.14. In view of (11-2) it is a bijection. Using the left-exactness of the comparison functor we can upgrade this to obtain an isomorphism between the higher homotopy groups of mapping spaces by inserting the suspension $S^i(B)$ in the place of B for i in $\mathbb{N}$. Since the comparison functor is clearly essentially surjective it is an equivalence. □

Remark 12.2. We note that the two proofs are not independent. In order to obtain the marked arrow in (12-2) we used [Bunke et al. 2021, Corollary 2.4] which is based on the universal property of $\mathrm{hoKK}_{\mathrm{sep}}$ as the initial functor to an additive category which is homotopy invariant, stable and split exact. The verification of this universal property [Higson 1987, Theorem 4.5] also uses the formula (11-2). □

Corollary 12.3. (1) *The category* $\mathrm{KK}_{\mathrm{sep}}$ *admits all countable colimits.*

(2) *For a countable family of separable* C^**-algebras* $(B_i)_{i\in I}$ *we have an equivalence*

$$\coprod_{i\in I} \mathrm{kk}_{\mathrm{sep}}(B_i) \simeq \mathrm{kk}_{\mathrm{sep}}\Big(\bigoplus_{i\in I} B_i\Big).$$

(3) $\mathrm{KK}_{\mathrm{sep}}$ *is idempotent complete and the inclusion* $\mathrm{KK}_{\mathrm{sep}} \to \mathrm{KK}$ *identifies* $\mathrm{KK}_{\mathrm{sep}}$ *with the full subcategory of compact objects of* KK.

Proof. Since $\mathrm{KK}_{\mathrm{sep}}$ is stable by Proposition 7.2(3) it admits all finite colimits. For (1), it thus suffices to show that $\mathrm{KK}_{\mathrm{sep}}$ admits countable coproducts. But this immediately follows from Theorem 12.1 and Corollary 11.16.

The same results imply (2).

Assertion (3) is a general fact about $\mathtt{Ind}$-completions of stable ∞-categories admitting countable colimits and thus an immediate consequence of Definition 8.1 and (1). □

By comparing the universal properties of $\mathrm{kk}_{\mathrm{sep}}$ and $L_{\mathrm{sep},h,K,q}$ stated in the separable version of (11-22) and (7-7) we see that Theorem 12.1 is equivalent to the *automatic semiexactness theorem*.

Theorem 12.4. *For every left-exact and additive* ∞*-category* $\mathbf{D}$ *the canonical inclusion is an equivalence*

$$\mathbf{Fun}^{h,s,se+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,splt+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}).$$

A priori semiexactness is a much stronger condition than split exactness.

Recall from (5-1) that for every exact sequence of C^*-algebras

$$0 \to A \to B \xrightarrow{f} C \to 0 \tag{12-3}$$

we have defined a map $\iota_f : A \to C(f)$ from A to the mapping cone $C(f)$ of f. The following theorem was shown in [Cuntz and Skandalis 1986].

Theorem 12.5. *For every semisplit exact sequence (12-3) of separable C^*-algebras the morphism $L_{\mathrm{sep},h,K,q}(\iota_f)$ in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ is an equivalence.*

In view of Proposition 5.1, Theorem 12.5 is equivalent to Theorem 12.4 and hence to Theorem 12.1. Since the automatic semiexactness theorem is absolutely crucial in order to see that our construction of $\mathrm{KK}_{\mathrm{sep}}$ coincides with the classical constructions, and the proof of Theorem 12.5 in [Cuntz and Skandalis 1986] implicitly already uses this comparison, we must give an independent argument in order to avoid a circularity.

The remainder of the present section is devoted to the proof of Theorem 12.5. We will closely follow the outline given in the appendix of [Cuntz and Skandalis 1986]. We reduce the argument to a single calculation, namely Proposition 12.12, in $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, \mathbb{C})$ verifying that the composition of two explicit candidates for the Bott element and its inverse is really the identity. We use this argument also as a chance to present a calculus which allows us to manipulate morphisms in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ by constructions with semisplit exact sequences.

In the following discussion it is important to remember in which categories the morphisms live. We will therefore be more precise with the notation. We will abbreviate $L := L_{\mathrm{sep},h,K,q}$. In contrast to the conventions in the rest of the text, e.g., a C^*-algebra considered as an object of $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ will be denoted by $L(A)$ instead of simply by A. By abusing the notation, for a morphism $f : qA \to K \otimes B$ we will also use the notation $L(f)$ for the induced element in $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(A, B)$ under the map (11-26).

Following [Skandalis 1985, Section 1] and the appendix of [Cuntz and Skandalis 1986] we start with a construction which associates to every semisplit exact sequence

$$\mathcal{S} : 0 \to I \to A \xrightarrow{q} Q \to 0 \tag{12-4}$$

of separable C^*-algebras a morphism

$$f_{\mathcal{S}} : L(Q) \to L(S(I)) \tag{12-5}$$

in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$. This construction is necessarily of analytic nature since it must take the existence of the cpc split and separability into account. We provide the details since we merge the approaches of [Skandalis 1985, Section 1] and [Cuntz and Skandalis 1986]. In particular we want to work out in detail that the morphism $f_{\mathcal{S}}$ is independent of the choices.

Construction 12.6. We fix a cpc split $s : Q \to A$. By Kasparov's version of Stinespring's theorem there exists a countably generated A^u-Hilbert C^*-module E_0 and a homomorphism $\phi : Q \to B(A^u \oplus E_0)$ such that $s(x) = P\phi(x)P$ for all x in Q, where P in $B(A^u \oplus E_0)$ is the projection onto A^u and we consider A as a subset of $B(A^u \oplus E_0)$ in the canonical way. The point of taking A^u instead of A is that P becomes a compact operator on $A^u \oplus E_0$.

After making E_0 smaller if necessary we can assume that E_0 is generated as an A^u-Hilbert C^*-module by the elements of the form $(1-P)\phi(x)Pa$ for all x in Q and a in A^u. We refer to Remark 12.7 for a sketch of a direct construction of E_0 and ϕ which explains the essence of the proof of Stinespring's theorem mentioned above.

The pair (E_0, ϕ) is uniquely determined up to canonical isomorphism. Let (E_0', ϕ') be another choice. Then we define a map $E_0 \to E_0'$ sending the generator $(1-P)\phi(x)Pa$ to the generator $(1-P')\phi'(x)P'a$. In order to see that this map is well-defined we note that

$$\begin{aligned} \Big\langle \sum_i (1-P)\phi(x_i)Pa_i, \sum_j (1-P)\phi(x_j)Pa_j \Big\rangle &= \sum_{i,j} a_i^* P\phi(x_i)^*(1-P)\phi(x_j)Pa_j \\ &= \sum_{i,j} a_i^*(s(x_i^* x_j) - s(x_i)^* s(x_j))a_j \end{aligned} \tag{12-6}$$

does only depend on the split s, but not on E_0 and ϕ. In addition we observe that the right-hand side takes values in the ideal I. Hence the A^u-Hilbert C^*-module E_0 becomes an I-Hilbert C^*-module $E_{0|I}$ when we restrict the right A^u-action to I.

Remark 12.7. Here is a direct construction of E_0 and ϕ starting from the datum of the split s. One can consider the right A^u-module $Q \otimes A^u$ with the A^u-valued (actually I-valued) scalar product

$$\langle x \otimes a, x' \otimes a' \rangle := a^*(s(x^* x') - s(x)^* s(x'))a'.$$

Using that s is completely positive one checks that this is nonnegative. We then let E_0 be the completion of $Q \otimes A^u$ with respect to the induced seminorm. Note that this involves factoring out vectors with zero norm. We write suggestively $(1-P)\phi(x)Pa$ for the image of $x \otimes a$ in E_0. For y in Q we then define $\phi(y)$ in $B(A^u \oplus E_0)$ by

$$\phi(y) \begin{pmatrix} a \\ (1-P)\phi(x)Pb \end{pmatrix} = \begin{pmatrix} s(y)a + s(yx)b - s(y)s(x)b \\ (1-P)\phi(y)Pa + (1-P)\phi(yx)Pb + (1-P)\phi(y)Ps(x)b \end{pmatrix}.$$

One checks that this is a $*$-homomorphism. □

We let B denote the unital subalgebra of $B(A^u \oplus E_0)$ generated by $\phi(Q)$ and P. We further let J denote the ideal in B generated by $[\phi(Q), P]$. We finally let E_1 be the sub-A^u-Hilbert C^*-module of $A^u \oplus E_0$ generated by $J(A^u \oplus E_0)$. We then have a canonical homomorphism $B \to M(J) \to B(E_1)$.

Note that $[\phi(x), P] = (1-P)\phi(x)P - P\phi(x)(1-P)$. Combining this formula with (12-6) we see that for j in J and all e and e' in $A^u \oplus E_0$ we have $\langle je, e' \rangle \in I$. Hence the scalar product of E_1 takes values in I, and E_1 becomes an I-Hilbert C^*-module $E_{1|I}$ after restricting the right-module structure to I. Furthermore, since $J \subseteq K(A^u \oplus E_0)$ (because P was compact), we can conclude that $J \subseteq K(E_{1|I})$. In detail, consider an element j in J. It can be approximated by finite sums $\sum_i \theta_{\xi_i, \eta_i}$ of one-dimensional operators on $A^u \oplus E_0$. We can find members u and u' of an approximate identity of J such that uju' approximates j. But then j is also approximated by the finite sums $\sum_i \theta_{u\xi_i, u'^{,*}\eta_i}$ of one-dimensional operators on $E_{1|I}$.

Below, we identify the suspension $S(A)$ of a C^*-algebra with $C_0(S^1\setminus\{1\}, A)$ and $C_b(S^1\setminus\{1\}, M(A))$ with a subalgebra of the multiplier algebra $M(S(A))$. We consider the family $F : S^1 \mapsto B(E_{1|I})$ given by $F(u) := P + u(1-P)$. Since ϕ takes values in B we can consider ϕ as a homomorphism $Q \to B(E_{1|I})$. We then define a homomorphism

$$f : qQ \to S(K(E_{1|I})) \tag{12-7}$$

whose associated homomorphism (see Remark 11.2) has the components

$$\hat{f}_0, \hat{f}_1 : Q \to C_b(S^1\setminus\{1\}, B(E_{1|I})) \subseteq M(S(K(E_{1|I})))$$

given by

$$\hat{f}_0(x)(u) := \phi(x), \quad \hat{f}_1(x)(u) := F(u)\phi(q)F(u)^*.$$

Then $\hat{f}_0(x)(u) - \hat{f}_1(x)(u)$ belongs to $K(E_{1|I})$ for every u and $\hat{f}_0(x)(1) - \hat{f}_1(x)(1) = 0$. Thus $\hat{f}_0(x) - \hat{f}_1(x)$ belongs to $S(K(E_{1|I}))$ and f is well-defined. This homomorphism does not yet take values in the desired target $S(K \otimes I)$. We will employ Kasparov's stabilization theorem in order to produce a homomorphism $K(E_{1|I}) \to K \otimes I$ which is unique up to homotopy.

We let $H_I := \bigoplus_{\mathbb{N}} I$ denote the standard I-Hilbert C^*-module. We then have a canonical isomorphism $K(H_I) \cong K \otimes I$. Using that $E_{1|I}$ is countably generated (it is here where we use separability) and Kasparov's stabilization theorem [1980, Theorem 2] we can choose an isomorphism $E_{1|I} \oplus H_I \cong H_I$ which is unique up to homotopy since the unitary group of $B(H_I)$ is connected, even contractible. Using this isomorphism we get an embedding $E_{1|I} \to H_I$ of I-Hilbert C^*-modules which is also well-defined up to homotopy. It induces a homomorphism $K(E_{1|I}) \to K(H_I) \cong K \otimes I$, and hence $S(K(E_{1|I})) \to S(K \otimes I)$. Postcomposing (12-7) with this map we get a map

$$f'_{\mathcal{S}} : qQ \to S(K \otimes I), \tag{12-8}$$

which represents the desired map (12-5). Up to homotopy it only depends on choice of the cpc split. We finally see that $f_{\mathcal{S}}$ is independent of the choice of the cpc split since any two splits can be joined by a path. □

We now interpret the pre- or post-composition of $f_{\mathcal{S}}$ with a homomorphism in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ and its tensor product with an auxiliary C^*-algebra in terms of operations with semisplit exact sequences [Skandalis 1985, Lemma 1.5]. We consider a map of semisplit exact sequences of separable C^*-algebras

$$\begin{tikzcd}
\tilde{\mathcal{S}} : 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & \tilde{A} \arrow[r, "\tilde{q}"] \arrow[d] & \tilde{Q} \arrow[r] \arrow[d, "e"] & 0 \\
\mathcal{S} : 0 \arrow[r] & I \arrow[r] & A \arrow[r, "q"] & Q \arrow[r] & 0
\end{tikzcd}$$

where the right square is a pull-back.

Lemma 12.8. *We have an equivalence $f_{\tilde{\mathcal{S}}} \simeq f_{\mathcal{S}} \circ L(e)$.*

Proof. The split $s : Q \to A$ canonically induces a split $\tilde{s} : \tilde{Q} \to \tilde{A}$. Working with this split, by an inspection of the constructions we see that the resulting I-Hilbert C^*-module $\tilde{E}_{1|I}$ is canonically isomorphic to $E_{1|I}$. With this identification we get an equality $f \circ q(e) = \tilde{f} : q\tilde{Q} \to S(K(E_{1|I}))$ of maps in (12-7). This implies the desired equivalence. $\square$

We now consider a diagram of semisplit exact sequences

$$\begin{tikzcd} \mathcal{S} : 0 \arrow[r] & I \arrow[r] \arrow[d, "h"] & A \arrow[r] \arrow[d, "\tilde{h}"] & Q \arrow[r] \arrow[d, equal] & 0 \\ \tilde{\mathcal{S}} : 0 \arrow[r] & \tilde{I} \arrow[r] & \tilde{A} \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

Lemma 12.9. *We have an equivalence* $f_{\tilde{\mathcal{S}}} \simeq L(S(h)) \circ f_{\mathcal{S}}$.

Proof. The split $s : Q \to A$ induces a split $\tilde{s} := \tilde{h} \circ s : Q \to \tilde{A}$. Working with this split we get a canonical isomorphism $\tilde{E}_{i|\tilde{I}} := E_{i|I} \otimes_I \tilde{I}$. Then the resulting map $qQ \to S(K(\tilde{E}_{1|\tilde{I}}))$ in (12-7) is $qQ \to S(K(E_{1|I})) \xrightarrow{\mathtt{id}\otimes\tilde{I}} S(K(\tilde{E}_{1|\tilde{I}}))$. This implies the desired equivalence. $\square$

In the following $\otimes$ can be the minimal or the maximal tensor product. Recall that $\mathcal{S}$ denotes a semisplit exact sequence (12-4). If B is any C^*-algebra, then

$$\mathcal{S} \otimes B : 0 \to I \otimes B \to I \otimes A \to Q \otimes B \to 0$$

is semisplit exact again.

Lemma 12.10. *We have an equivalence* $f_{\mathcal{S}\otimes B} \simeq f_{\mathcal{S}} \otimes L(B)$.

Proof. The split s induces a split $s \otimes B$. With this choice the composition

$$qQ \otimes B \xrightarrow{\text{can}} q(Q \otimes B) \xrightarrow{!} S(K \otimes I \otimes B) \cong S(K \otimes I) \otimes B$$

with the marked map constructed from $\mathcal{S} \otimes B$ is equal to the map $f \otimes B$ constructed from s, and the morphism can induces an equivalence in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ since L is symmetric monoidal. This implies the assertion. $\square$

The semisplit exact sequence

$$\mathcal{R} : 0 \to S(\mathbb{C}) \to C(\mathbb{C}) \to \mathbb{C} \to 0 \tag{12-9}$$

gives rise to a map $f_{\mathcal{R}} : L(\mathbb{C}) \to L(S^2(\mathbb{C}))$ in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$. The crucial fact is that it admits a left-inverse, a posteriori even an inverse. This left-inverse $L(\beta) : L(S^2(\mathbb{C})) \to L(\mathbb{C})$, called the *Bott element*, will be given by an explicit homomorphism $\beta : qS^2(\mathbb{C}) \to \mathtt{Mat}_2(K)$ in Construction 12.17.

Remark 12.11. In [Cuntz and Skandalis 1986] the map $f_{\mathcal{R}}$ is called the Bott element. We prefer to call β the Bott element because of its role in Proposition 7.2. It is actually the crucial point that the Bott element β constructed in Definition 6.9 in the semiexact situation has a lift to the split exact world considered in the present section. This fact is the heart of the automatic semiexactness theorem. $\square$

Proposition 12.12. *We have* $L(\beta)\circ f_{\mathcal{R}} \simeq \pm \mathtt{id}_{L(\mathbb{C})}$.

Remark 12.13. Proposition 12.12 is shown in [Cuntz and Skandalis 1986] by calculating a Kasparov product. This argument is therefore not part of the theory developed in the present note.

Of course, the obvious approach would be to calculate the composition of the two representatives explicitly. This would result in a map $q^2\mathbb{C} \to K \otimes \mathbb{C}$ which would have to be compared with the composition of a left-upper-corner inclusion $\mathbb{C} \to K$ with $\iota_{\mathbb{C}} \circ \iota_{q\mathbb{C}} : q^2\mathbb{C} \to \mathbb{C}$. To go this path seems to be quite tricky.

Further below we will therefore provide argument for Proposition 12.12 which avoids going through Kasparov products or making the homomorphism ϕ in Theorem 11.13 explicit. □

For the moment we assume Proposition 12.12. For simplicity we will adjust the sign of β such that $L(\beta)\circ f_{\mathcal{R}} \simeq \mathtt{id}_{L(\mathbb{C})}$.

Proof of Theorem 12.5. We reproduce the argument from the appendix of [Cuntz and Skandalis 1986]. We consider a semisplit exact sequence

$$\mathcal{S} : 0 \to I \to A \xrightarrow{q} Q \to 0$$

and the map $\iota_q : I \to C(q)$ as in (5-1). We want to show that $L(\iota_q)$ is an equivalence.

We have a semisplit exact sequence

$$\mathcal{T} : 0 \to S(I) \to C(A) \to C(q) \to 0,$$

where the second map sends an element of $C(A)$ given by a path σ in A with $\sigma(1) = 0$ to the pair $(\sigma(0), q\circ\sigma)$ in $C(q)$; see Remark 3.16. The kernel of the map consists of paths σ in I with $\sigma(0)=0=\sigma(1)$ and is hence isomorphic to $S(I)$. We let $f_{\mathcal{T}} : L(C(q)) \to L(S^2(I))$ be the associated morphism. Then we define

$$u := (L(\beta)\otimes L(I))\circ f_{\mathcal{T}} : L(C(q)) \to L(I). \tag{12-10}$$

We now calculate the composition $u \circ L(\iota_q) : L(I) \to L(I)$. Using (12-9) we have a map of exact sequences

$$\begin{tikzcd}
\mathcal{R}\otimes I : 0 \arrow[r] & S(I) \arrow[r] \arrow[d, equal] & C(I) \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, "\iota_q"] & 0 \\
\mathcal{T} : 0 \arrow[r] & S(I) \arrow[r] & C(A) \arrow[r] & C(q) \arrow[r] & 0
\end{tikzcd}$$

where the right square is a pull-back. Then by Lemmas 12.8 and 12.10 we have

$$f_{\mathcal{T}} \circ L(\iota_q) \simeq f_{\mathcal{R}\otimes I} \simeq f_{\mathcal{R}} \otimes L(I) : L(I) \to L(S^2(I)). \tag{12-11}$$

Using Proposition 12.12 we conclude that

$$\begin{aligned} u \circ L(\iota_q) &\simeq (L(\beta) \otimes L(I)) \circ f_{\mathcal{T}} \circ L(\iota_q) \\ &\simeq (L(\beta) \otimes L(I)) \circ (f_{\mathcal{R}} \otimes L(I)) \\ &\simeq (L(\beta) \circ f_{\mathcal{R}}) \otimes L(I) \\ &\simeq \mathtt{id}_{L(I)}, \end{aligned}$$

where the first, second and fourth equivalences are by (12-10), (12-11) and Proposition 12.12, respectively.

Hence ι_q is a split monomorphism with left-inverse u.

We have the semisplit exact sequence

$$\mathcal{U} : 0 \to C(\iota_q) \to C(C(q)) \xrightarrow{\phi} C(Q) \to 0.$$

Using Remark 3.16 recall that $C(C(q))$ consists of pairs (σ, γ) with σ a path in A, $\gamma = (\gamma(-, t))_t$ a path of paths in Q such that $\gamma(s, 1) = 0$ for all s, $\gamma(1, t) = 0$ for all t, $q(\sigma(s)) = \gamma(s, 0)$ for all s and $\sigma(1) = 0$. The map ϕ sends (σ, γ) to $\gamma(0, -)$. Its kernel consists of pairs (σ, γ) where σ is a path in A with $\sigma(0) \in I$ and $\sigma(1) = 0$, and γ is path of paths in Q with $\gamma(0, t) = 0$, $\gamma(1, t) = 0$, $q(\sigma(s)) = \gamma(s, 0)$, and $\gamma(s, 1) = 0$ for all s, t. This is precisely the description of a point in $C(\iota_q)$.

Applying the above to the semisplit exact sequence $\mathcal{U}$ we conclude that $L(\iota_\phi) : L(C(\iota_q)) \to L(C(\phi))$ is a split monomorphism. Since L is Schochet-exact we see that $L(C(\phi)) \simeq 0$ since it is the fiber of a map between objects which are equivalent to zero (cones are contractible). This implies that $L(C(\iota_q)) \simeq 0$.

Again using that L is Schochet-exact we can conclude that $L(S(\iota_q)) : L(S(I)) \to L(S(C(q)))$ is an equivalence. Then also $L(S^2(\iota_q))$ is an equivalence. But $L(\iota_q)$ is a retract of $L(S^2(\iota_q))$ by

$$\begin{tikzcd} L(I) \arrow[r, "f_{\mathcal{R}} \otimes L(I)"] \arrow[d, "L(\iota_q)"] \arrow[rr, bend left, "\mathtt{id}_{L(I)}"] & L(S^2(I)) \arrow[r, "\beta \otimes L(I)"] \arrow[d, "L(S^2(\iota_q))"] & L(I) \arrow[d, "L(\iota_q)"] \\ C(q) \arrow[r, "f_{\mathcal{R}} \otimes L(C(q))"] \arrow[rr, bend right, "\mathtt{id}_{L(C(q))}"'] & L(S^2(C(q))) \arrow[r, "\beta \otimes L(C(q))"] & L(C(q)) \end{tikzcd}$$

and hence an equivalence, too. ☐

Assume that

$$\mathcal{S} : 0 \to I \to A \xrightarrow{q} Q \to 0$$

is semisplit exact. By semiexactness of $\mathrm{kk}_{\mathrm{sep}}$ we get a boundary map $\partial_{\mathcal{S}}^{\mathrm{kk}} : \mathrm{kk}_{\mathrm{sep}}(S(Q)) \to \mathrm{kk}_{\mathrm{sep}}(I)$. By the automatic semiexactness theorem we know that there is a boundary map $\partial_{\mathcal{S}} : L(S(Q)) \to L(I)$ such that $h(\partial_{\mathcal{S}}) \simeq \partial_{\mathcal{S}}^{\mathrm{kk}}$, where h is the left-exact comparison functor in (12-1). The following proposition clarifies its relation with $f_{\mathcal{S}} : L(Q) \to L(S(I))$ stated in the last sentence of [Cuntz and Skandalis 1986].

Recall the exact sequence (12-9).

Proposition 12.14. *We have an equivalence*

$$f_{\mathcal{S}} \simeq S(\partial_{\mathcal{S}}) \circ f_{\mathcal{R}} \otimes L(Q) : L(Q) \to L(S(I)).$$

Proof. We consider the following diagram

$$\begin{tikzcd}
\mathcal{S}: 0 \arrow[r] & I \arrow[r] \arrow[d, "\iota_q"] & A \arrow[r, "q"] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{U}: 0 \arrow[r] & C(q) \arrow[r] & Z(q) \arrow[r, "\tilde{q}"] & Q \arrow[r] & 0 \\
\mathcal{R} \otimes Q: 0 \arrow[r] & S(Q) \arrow[r] \arrow[u, "\partial_q"] & C(Q) \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

It implies by Lemma 12.9 that

$$L(S(\iota_q)) \circ f_{\mathcal{S}} \simeq f_{\mathcal{U}} \simeq L(S(\partial_q)) \circ (f_{\mathcal{R}} \otimes L(Q)).$$

We now use that $L(\iota_q)$ is an equivalence and that $L(\iota_q) \circ \partial_{\mathcal{S}} \simeq L(\partial_q)$ in order to deduce the desired equivalence. □

We now prepare the proof of Proposition 12.12. We will freely use results from Sections 9 and 10. We start with showing a partial case of Theorem 12.1.

Lemma 12.15. *For any separable C^*-algebra B the comparison functor h in (12-1) induces an equivalence*

$$\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, B) \xrightarrow{h} \Omega^\infty \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, B).$$

Proof. Replacing $\mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to E$ by $L_{\mathrm{sep},h,K,q} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$ and correspondingly $\Omega^\infty K(-)$ by $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, -)$ we can construct a natural transformation

$$\tilde{q}_{\mathrm{sep},h} : \widetilde{\mathcal{P}\mathrm{roj}}^{s}_{\mathrm{sep}}(-) \to \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, -)$$

in complete analogy to the construction of $\tilde{\mathrm{e}}_h$ in (10-6). We start from

$$q_h := L_{\mathrm{sep},q} : \mathcal{P}\mathrm{roj}^{s}(-) \to \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, -)$$

in place of e_h in (10-5), use that $L_{\mathrm{sep},h,K,q}$ is split exact and that the unitalization sequence (10-3) is split exact, and that $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, -)$ takes values in groups. The proof of Proposition 10.9 goes through word by word and shows that $\tilde{q}_{\mathrm{sep},h}$ is an equivalence. Here instead of (7-9) we use, of course, the universal property of $L_{\mathrm{sep},h,K,q}$ given by the separable version of (11-22) in order to construct the factorization

$\overline{\mathcal{P}\mathrm{roj}} : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q} \to \mathbf{CGroups}(\mathbf{Spc})$ in a way analogous to (10-14). The commutativity of

$$\begin{tikzcd}
 & \widetilde{\mathcal{P}\mathrm{roj}}^s_{\mathrm{sep}}(B) \arrow[dl, "\tilde{q}_{\mathrm{sep},h}"', "\simeq"] \arrow[dr, "\tilde{e}_{\mathrm{sep}}", "\simeq"'] & \\
\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, B) \arrow[rr, "h"] & & \Omega^\infty \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, B)
\end{tikzcd}$$

implies that h is also an equivalence. $\square$

Example 12.16. By the calculation of the spectrum $\mathrm{KU} \simeq \mathrm{KK}_{\mathrm{sep}}(\mathbb{C}, \mathbb{C})$ in Remark 9.19 (and implicitly using Corollary 9.16(2) in order to go from E- to KK-theory) we know that, for any i in $\mathbb{N}$,

$$\pi_0 \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, S^i(\mathbb{C})) \simeq \mathrm{KK}_{\mathrm{sep},0}(\mathbb{C}, S^i(\mathbb{C})) \cong \pi_i \mathrm{KU} \simeq \begin{cases} \mathbb{Z}, & i \in 2\mathbb{N}, \\ 0, & \text{else}, \end{cases}$$

where the first and third equivalences are by Lemma 12.15 and (9-7), respectively. The proof of Lemma 12.15 together with the fact that π_0 sends the homotopy-theoretic group completion to the algebraic group completion implies that

$$\pi_0 \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(\mathbb{C}, \mathbb{C}) \to \pi_0 \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, \mathbb{C})$$

is the algebraic group completion. This will let us detect elements in the group $\pi_0 \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, \mathbb{C})$ represented by maps $\mathbb{C} \to K$ or $q\mathbb{C} \to K$ in a simple manner.

We identify maps $p : \mathbb{C} \to K$ with the projections p in K given by the image of 1. A map $p : \mathbb{C} \to K$ is determined up to homotopy by the dimension $\dim(p)$ of the range of p. We therefore have an isomorphism of monoids

$$\pi_0 \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}}(\mathbb{C}, \mathbb{C}) \to \mathbb{N}, \quad p \mapsto \dim(p).$$

We now consider a map $(p_0, p_1) : q\mathbb{C} \to K$ given in terms of an associated map with components $p_i : \mathbb{C} \to M(K) = B$ for $i = 0, 1$ such that $p_0 - p_1 \in K$. Then $p_1 p_0 : \mathtt{im}(p_0) \to \mathtt{im}(p_1)$ is a Fredholm operator and we can define the relative index

$$I(p_0, p_1) := \mathtt{index}(p_1 p_0 : \mathtt{im}(p_0) \to \mathtt{im}(p_1)).$$

If p_i is compact for $i = 0, 1$, then $I(p_0, p_1) = \dim(p_0) - \dim(p_1)$.

We now show that the relative index is homotopy invariant for homotopies of pairs $(p_{0,t}, p_{1,t})$ such that $p_{0,t} - p_{1,t}$ is norm continuous. In particular there is no continuity condition on the families $p_{i,t}$ separately. We follow [Avron et al. 1994] and define the norm continuous families of selfadjoint operators $A_t := p_{0,t} - p_{1,t}$ and $B_t := 1 - A_t$. Then $A_t^2 + B_t^2 = 0$ and $A_t B_t + B_t A_t = 0$ (see [Avron et al. 1994, Theorem 2.1]). Since A_t is selfadjoint and compact the spectrum of A_t away from 0 is discrete and consists of eigenvalues of finite multiplicity. The relations above imply that for $\lambda \neq 1$ the operator B_t induces an isomorphism between $\mathtt{ker}(A_t - \lambda)$ and $\mathtt{ker}(A_t + \lambda)$. By [Avron et al. 1994, Proposition 3.1] we have the first equality in

$$\begin{aligned}\mathtt{index}(p_{0,t}, p_{1,t}) &= \dim(\mathtt{ker}(A_t - 1)) - \dim(\mathtt{ker}(A_t + 1)) \\ &= \dim(E_{A_t}((1-\epsilon, 1+\epsilon))) - \dim(E_{A_t}((-1-\epsilon, -1+\epsilon)))\end{aligned}$$

for any ϵ in $(0, 1)$, where E_{A_t} is the family of spectral projections for A_t. In order to see the second equality note that the contributions of the eigenspaces to the eigenvalues different from ± 1 cancel out by the consideration above. The right-hand side is continuous in t and hence constant. To this end, we consider a point t_0 in $[0, 1]$. Then we choose ϵ such that $1 \pm \epsilon$ do not belong to the spectrum of A_{t_0}. By the norm continuity of $t \mapsto A_t$ there exists δ in $(0, \infty)$ such that for all t in $[t_0 - \delta, t_0 + \delta] \cap [0, 1]$ the points $1 \pm \epsilon$ do not belong to the spectrum of A_t. But then the right-hand side is constant on $[t_0 - \delta, t_0 + \delta] \cap [0, 1]$.

Using the homotopy invariance and the additivity of the relative index for block sums we can conclude that the map $(p_0, p_1) \mapsto I(p_0, p_1)$ induces a group homomorphism $\widetilde{\dim} : \pi_0\underline{\mathtt{Hom}}(q\mathbb{C}, K) \to \mathbb{Z}$ such that the bold part of

$$\begin{tikzcd}
\pi_0\underline{\mathtt{Hom}}(\mathbb{C}, K) \arrow[r, "\dim", "\cong"'] \arrow[d, "\iota_{\mathbb{C}}^*"] \arrow[dd, bend right=60, "\pi_0 L"'] & \mathbb{N} \arrow[d, "i"] \\
\pi_0\underline{\mathtt{Hom}}(q\mathbb{C}, K) \arrow[r, "\widetilde{\dim}"] \arrow[d, "!"', "(11\text{-}26)"] & \mathbb{Z} \\
\pi_0\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C}, K) \arrow[ur, dashed, "\cong"'] &
\end{tikzcd}$$

commutes. The dashed arrow is obtained from the universal property of the arrow denoted by $\pi_0 L$ as a group completion, since the right-down map $i \circ \dim$ is a homomorphism to a group. It remains to show that the lower triangle commutes.

We claim that the arrow marked by ! is an isomorphism. Assuming the claim we know that $\iota_{\mathbb{C}}^*$ also represents a group completion. We can then argue that the two ways to go from $\pi_0\underline{\mathtt{Hom}}(q\mathbb{C}, K)$ to $\mathbb{Z}$ must agree since group completions are initial in homomorphisms to groups.

To see the claim we can appeal to Corollary 11.14. But as this implicitly uses Theorem 11.13 one could alternatively show directly that $\widetilde{\dim}$ is an isomorphism and then conclude that ! is an isomorphism. $\square$

Construction 12.17. We describe the Bott element $L(\beta)$ in the mapping space $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(S^2(\mathbb{C}), \mathbb{C})$. Instead of reproducing the construction from [Cuntz and Skandalis 1986] we describe a version which is more amenable to explicit calculations.

We consider the closed smooth manifold $\mathbb{CP}^1 \cong S^2$. It will be equipped with a constant scalar curvature Riemannian metric and the orientation determined by the complex structure. We consider $\mathbb{CP}^1$ as a Riemannian spin manifold and let $\not{D}$ be the spin Dirac operator. It acts as a first-order elliptic differential operator on the sections of the $\mathbb{Z}/2\mathbb{Z}$-graded spinor bundle $S \cong S^+ \oplus S^-$ which is odd with respect to the grading and therefore represented by a matrix

$$\begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix}. \tag{12-12}$$

The Schrödinger–Lichnerowicz formula states that $\not{D}^2 = \Delta + \frac{s}{4}$, where Δ is the canonical Laplacian on the spinor bundle associated to the connection and s is the scalar curvature. Since the Laplacian is nonnegative and s is positive we see that $\not{D}^2$ is positive and hence invertible as an unbounded operator on $H := L^2(\mathbb{CP}^1, S)$ with domain $C^\infty(\mathbb{CP}^1, S)$. Using function calculus we obtain the odd- and zero-order pseudodifferential unitary operator

$$U := \not{D}\,|\not{D}|^{-1}$$

in $B(H)$. The principal symbol of U is the unitary part of the polar decomposition of the principal symbol of $\not{D}$. If f in $C(\mathbb{CP}^1)$ acts as multiplication operator on H, then $[f, U]$ is compact. Indeed, let f be smooth for the moment and consider it as a zero-order pseudodifferential operator. Then the principal symbols of f and U commute and the commutator is a pseudodifferential operator of order -1 and hence compact. Since $C^\infty(\mathbb{CP}^1)$ is dense in $C(\mathbb{CP}^1)$ in the norm and the compact operators are closed in norm we see that $[f, U]$ is compact for all f in $C(\mathbb{CP}^1)$.

The grading of S gives a decomposition $H = H^+ \oplus H^-$ and we represent U as a matrix

$$\begin{pmatrix} 0 & U^- \\ U^+ & 0 \end{pmatrix}.$$

We have two homomorphisms $\hat{\phi}_\pm : C(\mathbb{CP}^1) \to B(H^\pm)$ such that for f in $C(\mathbb{CP}^1)$ the operator $\hat{\phi}_\pm(f)$ is the multiplication operator by f on $H^\pm$.

We define two homomorphisms $\hat{\phi}_i : C(\mathbb{CP}^1) \to B(H^+)$ for $i = 0, 1$ by

$$\hat{\phi}_0 := \hat{\phi}_+, \quad \hat{\phi}_1 := U^- \hat{\phi}_- U^+ . \tag{12-13}$$

Then we have $\hat{\phi}_0(f) - \hat{\phi}_1(f) \in K(H^+)$ for all f in $C(\mathbb{CP}^1)$. The homomorphisms $\hat{\phi}_i$ for $i = 0, 1$ are therefore the components of the associated homomorphism of a homomorphism

$$\hat{\beta} : qC(\mathbb{CP}^1) \to K(H^+).$$

It represents a point $L(\hat{\beta})$ in $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(C(\mathbb{CP}^1), \mathbb{C})$.

We fix a base point $*$ in $\mathbb{CP}^1$. Using an orientation-preserving diffeomorphism $\mathbb{R}^2 \cong \mathbb{CP}^1 \setminus \{*\}$ we identify $S^2(\mathbb{C})$ with the subalgebra $C_0(\mathbb{CP}^1 \setminus \{*\})$ of $C(\mathbb{CP}^1)$ of functions vanishing at $*$. We let $\iota : S^2(\mathbb{C}) \to C(\mathbb{CP}^1)$ denote the inclusion. We define the Bott element as the composition

$$\beta : qS^2(\mathbb{C}) \xrightarrow{q(\iota)} qC(\mathbb{CP}^1) \xrightarrow{\hat{\beta}} K(H^+).$$

Then $L(\beta) \simeq L(\hat{\beta}) \circ L(\iota)$ is a point in $\pi_0\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(S^2(\mathbb{C}), \mathbb{C})$ which is our candidate for the Bott element. $\square$

Proof of Proposition 12.12. We have the commutative diagram

$$\begin{tikzcd}
\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(S^2(\mathbb{C}),\mathbb{C}) \times \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C},S^2(\mathbb{C})) \arrow[r, "\circ"] \arrow[d] & \mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C},\mathbb{C}) \arrow[d, "\simeq"] \\
\Omega^\infty \mathrm{KK}_{\mathrm{sep}}(S^2(\mathbb{C}),\mathbb{C}) \times \Omega^\infty \mathrm{KK}_{\mathrm{sep}}(\mathbb{C},S^2(\mathbb{C})) \arrow[r, "\circ"] & \Omega^\infty \mathrm{KK}_{\mathrm{sep}}(\mathbb{C},\mathbb{C})
\end{tikzcd} \tag{12-14}$$

where the vertical morphisms are induced by h and the horizontal morphisms are given by composition. The right-vertical morphism is an equivalence by Lemma 12.15.

We have $\mathrm{KK}_{\mathrm{sep},0}(S^2(\mathbb{C}),\mathbb{C}) \cong \pi_{-2}\mathrm{KU} \cong \mathbb{Z}$. We furthermore know that under the identification $\mathrm{KK}_{\mathrm{sep},0}(S^i(\mathbb{C}),S^j(\mathbb{C})) \cong \pi_{j-i}\mathrm{KU}$ the composition

$$\mathrm{KK}_{\mathrm{sep},0}(S^j(\mathbb{C}),S^k(\mathbb{C})) \times \mathrm{KK}_{\mathrm{sep},0}(S^i(\mathbb{C}),S^j(\mathbb{C})) \to \mathrm{KK}_{\mathrm{sep},0}(S^i(\mathbb{C}),S^k(\mathbb{C}))$$

is identified with the multiplication in the ring $\pi_*\mathrm{KU} \cong \mathbb{Z}[b,b^{-1}]$.

So in order to show that $L(\beta)$ is a left-inverse of $f_{\mathcal{R}}$ up to sign it suffices to show that the images $h(L(\beta))$ and $h(f_{\mathcal{R}})$ of these elements in $\pi_{-2}\mathrm{KU}$ and $\pi_2\mathrm{KU}$ are generators. This is the content of Lemmas 12.18 and 12.19.

Lemma 12.18. *The class $h(L(\beta))$ in $\mathrm{KK}_{\mathrm{sep},0}(S^2(\mathbb{C}),\mathbb{C}) \cong \mathbb{Z}$ is a generator.*

Proof. It suffices to provide an element $L(p)$ in $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C},S^2(\mathbb{C}))$ such that $L(\beta)\circ L(p)$ represents a generator of $\pi_0\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$.

We keep the conventions from Construction 12.17. We have a tautological line bundle $L \to \mathbb{CP}^1$ which is naturally a subbundle of the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}^2$. We let $L^\perp$ be the orthogonal complement so that $L \oplus L^\perp \cong \mathbb{CP}^1 \times \mathbb{C}^2$. We consider the projection P in $\mathtt{Mat}_2(C(\mathbb{CP}^1))$ such that the value P_x is the orthogonal projection onto the fiber of $L^\perp_x$ of $L^\perp$ for all x in $\mathbb{CP}^1$. We interpret P as a homomorphism $P : \mathbb{C} \to \mathtt{Mat}_2(C(\mathbb{CP}^1))$ such that $1 \mapsto P$. We get $L(P)$ in $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}}(\mathbb{C},C(\mathbb{CP}^1))$. We can now calculate the composition $L(\hat\beta)\circ L(P)$ which is represented by

$$q\mathbb{C} \xrightarrow{qP} q\mathtt{Mat}_2(C(\mathbb{CP}^1)) \xrightarrow{\mathtt{Mat}_2(\hat\beta)} \mathtt{Mat}_2(K(H^+)), \tag{12-15}$$

where $\mathtt{Mat}_2(\hat\beta)$ is the map whose associated homomorphism has the components $\mathtt{Mat}_2(\hat\phi_i)$ with $\hat\phi_i$ as in (12-13). We set $\hat H := H^+ \otimes \mathbb{C}^2$ and identify $\mathtt{Mat}_2(K(H^+)) \cong K(\hat H)$. The components of the map (12-15) are then given by the projections $Q_i := \mathtt{Mat}_2(\hat\phi_i)(P)$ in $B(\hat H)$. Note that the difference $Q_0 - Q_1$ is compact. By Example 12.16 the class of the composition in (12-15) is detected by the relative index $I(Q_0,Q_1)$ on $\mathbb{Z}$, i.e., the index of the Fredholm operator $Q_1Q_0 : Q_0\hat H \to Q_1\hat H$. We note that $Q_0 = P_+$ and $Q_1 = \mathtt{Mat}_2(U^-)P_-\mathtt{Mat}_2(U^+)$, where we consider $P_\pm$ as a multiplication operator P on $H^\pm \otimes \mathbb{C}^2 \cong L^2(\mathbb{CP}^1, S^\pm \otimes \mathbb{C}^2)$. Multiplying with the unitary $\mathtt{Mat}_2(U^-)$ from the left we can thus identify the Fredholm operator Q_1Q_0 with

$$P_-\mathtt{Mat}_2(U^+)P_+ : L^2(\mathbb{CP}^1, S^+ \otimes \mathbb{C}^2) \to L^2(\mathbb{CP}^1, S^- \otimes \mathbb{C}^2).$$

This is a zero-order pseudodifferential operator whose symbol is the symbol of the twisted Dirac operator $\not{D}_L^+$ (see (12-12)) made unitary. In particular we have $I(Q_0, Q_1) = \mathtt{index}(\not{D}_L^+)$. By the Atiyah–Singer index theorem we have

$$\mathtt{index}(\not{D}_L^+) = \int_{\mathbb{CP}^1} \hat{A}(S^2)\mathbf{ch}(L^\perp) = -\int_{\mathbb{CP}^1} c_1(L^\perp) = 1.$$

The base point of $\mathbb{CP}^1$ gives a decomposition $L_* \oplus L_*^\perp \cong \mathbb{CP}^1 \otimes \mathbb{C}^2$. We let P_* in $\mathtt{Mat}_2(C(\mathbb{CP}^1))$ be the corresponding constant projection onto $L_*^\perp$. The same calculations as above show that the composition $L(\hat{\beta}) \circ L(P_*)$ represented by

$$q\mathbb{C} \xrightarrow{qP_*} q\mathtt{Mat}_2(C(S^2)) \xrightarrow{\mathtt{Mat}_2(\hat{\beta})} \mathtt{Mat}_2(K(H^+))$$

represents the zero element.

The projections P and P_* in $\mathtt{Mat}_2(C(S^2)) \cong M(\mathtt{Mat}_2(S^2(\mathbb{C})))$ can be considered as components of the associated homomorphism of a homomorphism $p : q\mathbb{C} \to \mathtt{Mat}_2(S^2(\mathbb{C}))$ since $P - P_* \in \mathtt{Mat}_2(S^2(\mathbb{C}))$. The composition

$$q\mathbb{C} \xrightarrow{p} \mathtt{Mat}_2(S^2(\mathbb{C})) \xrightarrow{\mathtt{Mat}_2(\iota)} \mathtt{Mat}_2(C(S^2))$$

represents the difference, i.e., $L(\iota) \circ L(p) \simeq L(P) - L(P_*)$. This implies that

$$L(\beta) \circ L(p) \simeq L(\hat{\beta}) \circ L(\iota) \circ L(p) \simeq L(\hat{\beta}) \circ (L(P) - L(P_*)) \simeq L(\hat{\beta}) \circ L(P) \simeq L(\mathtt{id}_{\mathbb{C}})$$

is a generator. □

Lemma 12.19. *The class $h(f_{\mathcal{R}})$ in $\mathrm{KK}_{\mathrm{sep},0}(\mathbb{C}, S^2(\mathbb{C})) \cong \mathbb{Z}$ is a generator.*

Proof. We first make Construction 12.6 explicit in order to describe an explicit representative of the map $f_{\mathcal{R}}$. Let t be the coordinate on $[0, 1]$. We identify the cone over $\mathbb{C}$ as $C(\mathbb{C}) \cong C_0((0, 1])$. We define the cpc map $s : \mathbb{C} \to C(\mathbb{C})$ such that $s(1) = t$, where t is the coordinate function of the interval acting by multiplication on $C_0((0, 1])$. We consider $E_0 := S(\mathbb{C}) \cong C_0((0, 1))$ as a $C(\mathbb{C})^u$-Hilbert C^*-module. We use the identification $C(\mathbb{C})^u \cong C([0, 1])$ and define $\phi : \mathbb{C} \to B(C([0, 1]) \oplus E_0)$ such that

$$\phi(1) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Note that the right-hand side is a projection. We set

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $(1 - P)\phi(1)Pa = a\sqrt{t(1-t)}$ for a in $C([0, 1])$. These elements span a dense subset of E_0. The unital algebra B spanned by P and $\phi(1)$ is given by

$$\begin{pmatrix} C([0, 1]) & S(\mathbb{C}) \\ S(\mathbb{C}) & C(S^1) \end{pmatrix} \subseteq B(C([0, 1]) \oplus S(\mathbb{C})).$$

The ideal J generated by $[P,\phi(1)]$ is $\mathtt{Mat}_2(S(\mathbb{C}))$, and $E_1 = S(\mathbb{C})\oplus S(\mathbb{C})$. We get a map $q\mathbb{C}\to S(\mathtt{Mat}_2(S(\mathbb{C})))$ whose associated map has the components

$$\hat{f}_0(1)(u) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}, \quad \hat{f}_0(1)(u) = \begin{pmatrix} t & u^{-1}\sqrt{t(1-t)} \\ u\sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

This map represents $f_{\mathcal{R}}$.

In order to check that $h(f_{\mathcal{R}})$ is a generator, using Swan's theorem we will translate the problem into a calculation in usual topological K-theory of compact spaces defined in terms of vector bundles. We interpret t and u as longitude and latitude coordinates on S^2 such that $t=0$ is the south pole and $t=1$ is the north pole. By Swan's theorem we have an isomorphism between $\mathrm{KK}_{\mathrm{sep},0}(\mathbb{C}, C(S^2))\cong K_0(C(S^2))$ and the K-theory $K^0(S^2)$ of the sphere defined in terms of vector bundles as usual. Under this isomorphism and with $S^2(\mathbb{C})\cong C_0(S^2\setminus\{n\})$ we identify $\mathrm{KK}_{\mathrm{sep},0}(\mathbb{C}, S^2(\mathbb{C}))$ with the reduced K-theory $\tilde{K}^0(S^2)$ relative to the north pole $\{n\}$.

Since $\hat{f}_0(1)$ does not depend on u it is obviously a projection P_0 in $\mathtt{Mat}_2(C(S^2))$. We now observe that $\hat{f}_1(1)$ does not depend on u if $t=0$ or $t=1$. We can therefore also interpret $\hat{f}_1(1)$ as a projection P_1 in $\mathtt{Mat}_2(C(S^2))$. Identifying projections with vector bundles (actually subbundles of the trivial bundle $S^2\times\mathbb{C}^2$) we get a class $[P_0]-[P_1]$ in $K^0(S^2)$. Since P_0 and P_1 coincide at the north pole this difference is actually a reduced class in $\tilde{K}^0(S^2)\cong\mathbb{Z}$. Our task is to show that it is a generator.

Since P_0 does not depend on the u-coordinate, it comes from a projection in $C([0,1])$. Since $[0,1]$ is contractible we can conclude that P_0 is homotopic to a constant projection and the corresponding vector bundle can be trivialized. The matrix function

$$(t,u)\mapsto U(t,u) := \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$$

defines an isomorphism of vector bundles $P_0\to P_1$ (considered as subbundles of the two-dimensional trivial bundle) on the subspace $\{t\neq 0\}$. Since

$$P_{0,n} = \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

this isomorphism extends across the north pole. Away from the north pole this isomorphism sends the section

$$(u,t)\mapsto \begin{pmatrix} \sqrt{t(1-t)} \\ (1-t) \end{pmatrix}$$

of P_0 to the section

$$(u,t)\mapsto \begin{pmatrix} \sqrt{t(1-t)} \\ u(1-t) \end{pmatrix}.$$

Note that if t becomes small this is essentially multiplication by u. So P_1 is obtained from the trivial bundle by cutting at the equator $S^1\subseteq S^2$ and regluing with a map $S^1\to U(1)$ of degree one. This implies that $[P_0]-[P_1]$ generates $\tilde{K}^0(S^2)$. □

This finishes the proof of Proposition 12.12. □

13. Half-exact functors

In classical KK- and E-theory universal properties are formulated in terms of half-exact functors to ordinary additive categories. We will recall this language and state the universal properties of the homotopy category versions of the functors constructed in the previous sections in these terms. This will be used to show that they are equivalent to the classical KK- and E-theory functors. This comparison is the main objective of this section. For simplicity we will restrict to separable algebras.

Recall that an additive 1-category is an ordinary category which is pointed, admits finite coproducts and products such for any two objects D, D' the canonical morphism $D \sqcup D' \to D \times D'$ is an isomorphism, and which has the property that the commutative monoids $\mathtt{Hom}_{\mathbf{D}}(D, D')$ are abelian groups for all objects D, D'. An additive category is automatically enriched in abelian groups. A functor $\mathbf{D} \to \mathbf{D}'$ between additive categories is additive if it preserves coproducts and products. It is then compatible with the enrichments in abelian groups. We let $\mathbf{Fun}^{\mathrm{add}}(\mathbf{D}, \mathbf{D}')$ be the category of additive functors.

Example 13.1. If $\mathbf{C}$ is an additive ∞-category, its homotopy category $\mathrm{ho}\mathbf{C}$ is an additive 1-category. □

We next introduce the notion of a half-exact additive category. A half-exact structure looks like a glimpse of a triangulated structure. We will use this notion mainly in order to match the formulation of the universal properties of KK- and E-theory in the classical literature, in particular in [Higson 1990a].

Let $A \to B \to C$ be a sequence of maps in an additive 1-category $\mathbf{D}$ and $\mathcal{S}$ be a set of objects of $\mathbf{D}$.

Definition 13.2. We say that this sequence is *half-exact with respect to* $\mathcal{S}$ if the induced sequences

$$\mathtt{Hom}_{\mathbf{D}}(D, A) \to \mathtt{Hom}_{\mathbf{D}}(D, B) \to \mathtt{Hom}_{\mathbf{D}}(D, C)$$

and

$$\mathtt{Hom}_{\mathbf{D}}(C, D) \to \mathtt{Hom}_{\mathbf{D}}(B, D) \to \mathtt{Hom}_{\mathbf{D}}(A, D)$$

of abelian groups are exact for every D in $\mathcal{S}$.

Definition 13.3. A *marking* on an additive 1-category is a subset of objects $\mathcal{S}$ which is closed under isomorphisms. A *half-exact additive category* is an additive 1-category with a marking and a collection of distinguished sequences which are required to be half-exact with respect to the given marking.

Let $\mathbf{D}$ and $\mathbf{D}'$ be half-exact additive categories with markings $\mathcal{S}$ and $\mathcal{S}'$, respectively.

Definition 13.4. A *half-exact functor* $\phi : \mathbf{D} \to \mathbf{D}'$ is an additive functor such that $\phi(\mathcal{S}) \subseteq \mathcal{S}'$ and ϕ sends the distinguished half-exact sequences in $\mathbf{D}$ to distinguished half-exact sequences in $\mathbf{D}'$.

We let $\mathbf{Fun}^{\mathrm{add},(1/2)\mathrm{ex}}(\mathbf{D}, \mathbf{D}')$ be the category of half-exact additive functors between half-exact additive categories.

Example 13.5. Every additive category has the *canonical half-exact structure* with $\mathcal{S} = \mathrm{Ob}(\mathbf{D})$ and the collection of all sequences which are half-exact with respect to this marking. In this case we denote the half-exact additive category by $\mathbf{D}_{\mathrm{can}}$. □

Example 13.6. Recall that a sequence $A \xrightarrow{i} B \xrightarrow{p} C$ in an additive 1-category $\mathbf{D}$ is *split exact* if p admits a left-inverse $s : C \to B$ such that $(i, s) : A \oplus C \to B$ is an isomorphism. The split exact sequences are half-exact for the maximal marking. We write $\mathbf{D}_{\mathrm{splt}}$ for $\mathbf{D}$ equipped with the maximal marking and the collection of split exact sequences. An additive functor $\mathbf{D} \to \mathbf{D}'$ automatically preserves split exact sequences. It therefore belongs to $\mathbf{Fun}^{\mathrm{add},(1/2)\mathrm{ex}}(\mathbf{D}_{\mathrm{splt}}, \mathbf{D}'_{\mathrm{splt}})$. □

Remark 13.7. Let $\mathbf{D}$ be a half-exact additive category with marking $\mathcal{S}$. Then a sequence

$$D_0 \to D_1 \to D_2 \to D_3 \to D_4$$

in $\mathbf{D}$ is called half-exact if each segment $D_{i-1} \to D_i \to D_{i+1}$ is half-exact. If $D_0 \to D_1$ and $D_3 \to D_4$ are isomorphisms and D_2 belongs to $\mathcal{S}$, then we can conclude that $D_2 \cong 0$ by showing that $\mathtt{id}_{D_2} = 0$.

If $D_1 \cong 0$ and $D_4 \cong 0$ and D_2 and D_3 belong to $\mathcal{S}$, then we can conclude that $D_2 \to D_3$ is an isomorphism by constructing left- and right-inverses. □

Example 13.8. In order to have a nontrivial half-exact structure at hand consider the category of abelian groups and let $\mathcal{S}$ be the set of uniquely divisible abelian groups. We distinguish all sequences which are half-exact with respect to $\mathcal{S}$. Then, e.g., the sequence $0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to 0$ is half-exact. Note that in this case we cannot conclude that 5 is an isomorphism since $\mathbb{Z}$ does not belong to the marking. □

Let $\mathbf{D}$ be a half-exact additive category. A functor $F : \mathbf{C} \to \mathbf{D}$ from a left-exact ∞-category will be called *half-exact* if the marking of $\mathbf{D}$ contains $F(\mathrm{Ob}(\mathbf{C}))$ and F sends fiber sequences to distinguished half-exact sequences.

Definition 13.9. A functor $F : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathbf{D}$ is called *half-exact* (*half-semiexact*) if the marking of $\mathbf{D}$ contains $F(\mathrm{Ob}(C^*\mathbf{Alg}^{\mathrm{nu}}))$, and if F sends exact (semisplit exact) sequences to distinguished half-exact sequences. A functor $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathbf{D}$ is *split exact* if it sends split exact sequences of C^*-algebras to split exact sequences.

We indicate such functors by superscripts $\frac{1}{2}ex$, $\frac{1}{2}se$, or *splt*.

Example 13.10. If $\mathbf{C}$ is a stable ∞-category, then $\mathrm{ho}\mathbf{C}$ is a triangulated 1-category. The sequences $A \to B \to C$ for any distinguished triangle $A \to B \to C \to \Sigma A$ are half-exact with respect to the maximal marking. We call the half-exact structure consisting of the maximal marking and these sequences the *triangulated half-exact structure*. The corresponding half-exact additive category will be denoted by $\mathrm{ho}\mathbf{C}_\Delta$.

If we consider $\mathbf{C}$ as a left-exact ∞-category, then according to our conventions the functor $\mathrm{ho} : \mathbf{C} \to \mathrm{ho}\mathbf{C}_\Delta$ is half-exact. □

We consider ! in $\{\mathrm{ex}, \mathrm{se}, q\}$. For $! \in \{\mathrm{ex}, \mathrm{se}\}$ the functor $\mathrm{kk}_{\mathrm{sep},!} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep},!}$ is as in (8-4). For $! = q$ we set

$$\mathrm{kk}_{\mathrm{sep},q} := L_{\mathrm{sep},h,K,q} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}_{\mathrm{sep},q} := L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q}$$

using the separable versions of (11-11). We note that the targets of these functors are a left-exact additive ∞-category for $! = q$ (separable version of Proposition 11.8(3)) or even stable ∞-categories

(Proposition 7.3(3)) for $! \in \{\mathrm{ex}, \mathrm{se}\}$. In particular, their homotopy categories are additive 1-categories. We consider the functor

$$\mathrm{hokk}_{\mathrm{sep},!} := \mathrm{ho} \circ \mathrm{kk}_{\mathrm{sep},!} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{hoKK}_{\mathrm{sep},!}.$$

We will equip the additive 1-category $\mathrm{hoKK}_{\mathrm{sep},!}$ with the triangulated half-exact structure in case $! \in \{\mathrm{ex}, \mathrm{se}\}$, and the split half-exact structure if $! = q$. In all cases the marking is the maximal one.

Corollary 13.11. *The functor* $\mathrm{hokk}_{\mathrm{sep},!}$ *is homotopy invariant and stable. It is split exact in the case* $! = q$, *half-semiexact in the case* $! = \mathrm{se}$, *and half-exact in the case* $! = \mathrm{ex}$.

Proof. Homotopy invariance and stability are clear by construction.

In the case $! = \mathrm{ex}$ ($! = \mathrm{se}$) we use that $\mathrm{kk}_{\mathrm{sep},!}$ sends exact (semisplit exact) sequences of C^*-algebras to fiber sequences, and the half-exactness of ho.

In the case $! = q$ we use that $\mathrm{kk}_{\mathrm{sep},q}$ and ho preserve split exact sequences. □

In order to make uniform statements in all three cases we will call a functor $F : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathbf{D}$ to a half-exact additive category !-exact if it is split exact in the case $! = q$, half-semiexact in the case $! = \mathrm{se}$, or exact in the case $! = \mathrm{ex}$. We call the functor *suspension stable* if for every morphism $f : A \to B$ in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ the fact that $F(f)$ is an isomorphism is equivalent to the fact that $F(S(f))$ is an isomorphism.

Example 13.12. For ! in $\{\mathrm{se}, \mathrm{ex}\}$ the functors $\mathrm{hokk}_{\mathrm{sep},!} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{hoKK}_{\mathrm{sep},!}$ are suspension stable. This fact is due to the triangulated structure on $\mathrm{hoKK}_{\mathrm{sep},!}$ which is a consequence if the stability of $\mathrm{KK}_{\mathrm{sep},!}$. We further use Example 3.19 in order to identify looping in $\mathrm{KK}_{\mathrm{sep},!}$ with suspension on the level of algebras. □

We let $\widetilde{W}_{\mathrm{sep},!}$ denote the set of morphisms in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ which are sent to equivalences by $\mathrm{kk}_{\mathrm{sep},!}$.

We consider a half-exact additive category $\mathbf{D}$ and a functor $F : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathbf{D}$.

Proposition 13.13. *If F is homotopy invariant, stable, (suspension stable in the cases* $! \in \{\mathrm{ex}, \mathrm{se})\}$), *and* !*-exact, then F sends* $\widetilde{W}_{\mathrm{sep},!}$ *to isomorphisms.*

Proof. We first consider the cases $! \in \{\mathrm{se}, \mathrm{ex}\}$. In order to show that F sends $\widetilde{W}_{\mathrm{sep},!}$ to equivalences it suffices to show that it admits a sequence of factorizations

$$\begin{tikzcd}
C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[d, "L_{h,K}"] \arrow[dr, "F"] \arrow[ddd, bend right=60, "\mathrm{kk}_{\mathrm{sep},!}"'] & \\
L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \arrow[d, "L_{\mathrm{sep},!}"] \arrow[r, dotted, "\bar{F}"] & \mathbf{D} \\
L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!} \arrow[d, "\Omega_{\mathrm{sep},!}"] \arrow[ur, dotted, "\tilde{F}"] & \\
L_K C^*\mathbf{Alg}^{\mathrm{nu}\ \ \mathrm{group}}_{\mathrm{sep},h,!} \arrow[uur, dotted] &
\end{tikzcd}$$

Since F is homotopy invariant and stable, it has a factorization $\bar{F}$ as indicated. The !-exactness of F implies that the functor $\bar{F}$ is actually half-exact. We now claim that $\bar{F}$ sends the morphisms in the separable version $\widehat{W}_{\mathrm{sep},!}$ of (5-2) to equivalences. In the case $! = \mathrm{ex}$ this is precisely [Blackadar 1998, Proposition 21.4.1]. In the case $! = \mathrm{se}$ the proof of [Blackadar 1998, Proposition 21.4.1] goes though word by word since all exact sequences used in that proof are then semisplit exact.

We now claim that $\bar{F}$ also inverts the closures $W_{\mathrm{sep},!}$ of $\widehat{W}_{\mathrm{sep},!}$ under 2-out-of-3 and pull-back. It suffices to show that the collection of morphisms inverted by $\bar{F}$ is preserved by pull-backs. We will use [Blackadar 1998, Theorem 21.4.4] saying that F admits long half-exact sequences

$$\cdots \to F(S(I)) \to F(S(A)) \xrightarrow{F(S(f))} F(S(B)) \to F(I) \to F(A) \xrightarrow{F(f)} F(B) \tag{13-1}$$

associated to exact (or semisplit exact, respectively) sequences $0 \to I \to A \to B \to 0$. For the semiexact case again note that all exact sequences appearing in the proof are semisplit exact. Alternatively in both cases, this also follows from the half-exactness of $\bar{F}$ by applying it to the image under L_K of the Puppe sequence (3-17) associated to the map $A \to B$. Here we use that $F(I) \xrightarrow{\cong} F(C(f))$, which follows from the fact that F inverts $\widehat{W}_{\mathrm{sep},!}$, and the analogue of Proposition 5.1 for half-exact functors.

We consider the case $! = \mathrm{se}$. The case of $! = \mathrm{ex}$ is simpler and obtained from the $! = \mathrm{se}$ case by removing all mentions of cpc-splits. We consider a diagram

$$\begin{tikzcd} & L_{\mathrm{sep},h,K}(A) \arrow[d, "\bar{f}"] \\ L_{\mathrm{sep},h,K}(B') \arrow[r, "\bar{g}"] & L_{\mathrm{sep},h,K}(B) \end{tikzcd}$$

in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$. We can assume (see, e.g., the proof of Proposition 3.17) that up to equivalence the diagram is the image under $L_{\mathrm{sep},h,K}$ of the bold part of a cartesian diagram

$$\begin{tikzcd} A \arrow[r, dotted, "g'"] \arrow[d, dotted, "f'"] & A \arrow[d, "f"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ where f admits a cpc split. The map f' again admits a cpc split. We can extend the vertical maps to exact sequences

$$0 \to I \to A \xrightarrow{f} B \to 0, \quad 0 \to I' \to A' \xrightarrow{f} B' \to 0$$

in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ such that the induced map $I \to I'$ is an isomorphism. Since $F(f)$ is an isomorphism, $F(S(f))$ and $F(S^2(f))$ are isomorphisms by suspension stability. By the long half-exact sequence (13-1) for f we conclude that $F(I) \cong 0$ and $F(S(I)) \cong 0$; see Remark 13.7. Using the version of this long half-exact sequence for f' and $F(S^2(I')) \cong 0$ and $F(S(I')) \cong 0$ we conclude (again by Remark 13.7) that $F(S(f'))$ is an isomorphism. Finally, again using suspension stability we see that $F(f') = \bar{F}(\bar{f}')$ is an isomorphism.

We thus get a factorization $\tilde{F}$. In $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,!}$ we have the morphisms

$$\beta_{\mathrm{sep},!,A} : L_{\mathrm{sep},h,K,!}(S^2(A)) \to L_{\mathrm{sep},h,K,!}(A).$$

Since F takes values in groups, using similar arguments as for Corollary 6.10 (replacing left-exactness by the existence of the long half-exact sequences) we know that $\tilde{F}(\beta_{\mathrm{sep},!,A})$ is an isomorphism. In detail, we consider the functor $F_A(-) := F(- \otimes_{\max} A)$. By Corollary 6.6 we can conclude that $F_A(\mathcal{T}_0) \cong 0$. Then the boundary map $F_A(S^2(\mathbb{C})) \to F_A(\mathbb{C})$ of the long half-exact sequence for $0 \to K \to \mathcal{T}_0 \to S(\mathbb{C}) \to 0$ is an isomorphism. But this map is precisely $\tilde{F}(\beta_{\mathrm{sep},!,A})$.

Consequently we get the last factorization as indicated. This finishes the proof in the cases $! = \mathrm{se}$ and $! = \mathrm{ex}$.

In the case of $! = q$ we construct a sequence of factorizations

$$\begin{tikzcd}
C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[d, "L_{h,K}"] \arrow[rrd, "F"] \arrow[dd, bend right=60, "\mathrm{kk}_{\mathrm{sep},q}"'] & & \\
L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h} \arrow[d, "L_{\mathrm{sep},!}"] \arrow[rr, dotted, "\bar{F}"] & & \mathbf{D} \\
L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h,q} \arrow[rru, dotted] & &
\end{tikzcd}$$

Since F is split exact and takes values in groups, by Lemma 11.4 the functor $\bar{F}$ sends the morphisms $\iota^s_A : q^s A \to A^s$ to isomorphisms for all A in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep},h}$. This yields the last factorization in this case. □

In the following we remove the assumption of suspension stability in Proposition 13.13. Consider a half-exact additive category $\mathbf{D}$.

Proposition 13.14. *A homotopy invariant, stable, and half-exact (or half-semiexact) functor* $F : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathbf{D}$ *inverts* $\widetilde{W}_{\mathrm{sep},\mathrm{ex}}$ *(or* $\widetilde{W}_{\mathrm{sep},\mathrm{se}}$, *respectively).*

Proof. We start with the case $! = \mathrm{ex}$. We consider the classical E-theory functor $\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}} \to \mathrm{E}^{\mathrm{class}}_{\mathrm{sep}}$ constructed in [Higson 1990a], where we equip the additive 1-category $\mathrm{E}^{\mathrm{class}}_{\mathrm{sep}}$ with the canonical half-exact structure from Example 13.5. The functor $\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}$ is homotopy invariant, stable, half-exact, and suspension stable. In view of Proposition 13.13 the functor $\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}$ inverts $\widetilde{W}_{\mathrm{sep},\mathrm{ex}}$.

Let now $F : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathbf{D}$ be a homotopy invariant, stable and half-exact functor. By the universal property of $\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}$ stated in [Higson 1990a, Theorem 3.6] we get a factorization

$$\begin{tikzcd}
C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[rr, "F"] \arrow[dr, "\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}"'] & & \mathbf{D} \\
& \mathrm{E}^{\mathrm{class}}_{\mathrm{sep}} \arrow[ur, "\hat{F}"'] &
\end{tikzcd}$$

This implies that F also inverts $\widetilde{W}_{\mathrm{sep},\mathrm{ex}}$.

We now consider the case $! = \mathrm{se}$. It would be natural to argue as in the exact case using a corresponding universal property of $\mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}}$ involving half-semiexactness. But since we do not know a reference for this we will argue differently invoking the automatic semicontinuity theorem. Being a half-semiexact functor, F is in particular split exact. By Proposition 13.13 it inverts $\tilde{W}_{\mathrm{sep},q}$. As a consequence of the automatic semicontinuity result Theorem 12.1 we have $\widetilde{W}_{\mathrm{sep},q} = \tilde{W}_{\mathrm{sep,se}}$. $\square$

We can now state the universal property of $\mathrm{hokk}_{\mathrm{sep},!}$.

Proposition 13.15. *Pull-back along* $\mathrm{hokk}_{\mathrm{sep},!}$ *induces an equivalence*

$$\mathbf{Fun}^{\mathrm{add},(1/2)\mathrm{ex}}(\mathrm{hoKK}_{\mathrm{sep},!},\mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,(1/2)!}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}},\mathbf{D}), \quad ! \in \{\mathrm{ex}, \mathrm{se}\},$$

for any additive half-exact category $\mathbf{D}$ *or*

$$\mathbf{Fun}^{\mathrm{add}}(\mathrm{hoKK}_{\mathrm{sep},q},\mathbf{D}) \xrightarrow{\simeq} \mathbf{Fun}^{h,s,splt}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}},\mathbf{D}), \quad ! = q,$$

for any additive 1*-category.*

Proof. It follows from Corollary 13.11 that the pull-back along $\mathrm{hokk}_{\mathrm{sep},!}$ takes values in the indicated category of functors.

We first discuss the case $! \in \{\mathrm{ex}, \mathrm{se}\}$. We consider the diagram

$$\begin{tikzcd}
 & \mathbf{Fun}^{\mathrm{add},(1/2)\mathrm{ex}}(\mathrm{hoKK}_{\mathrm{sep},!},\mathbf{D}) \arrow[r, "(2)"] \arrow[d] & \mathbf{Fun}^{h,s,(1/2)!}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}},\mathbf{D}) \arrow[d] \\
\mathbf{Fun}(\mathrm{KK}_{\mathrm{sep},!},\mathbf{D}) \arrow[r, "\mathrm{ho}^*", "\simeq"'] \arrow[rr, bend right, "\mathrm{kk}^*_{\mathrm{sep},!}", "\cong"'] & \mathbf{Fun}(\mathrm{hoKK}_{\mathrm{sep},!},\mathbf{D}) \arrow[r, "(1)", "\simeq"'] & \mathbf{Fun}^{\tilde{W}_{\mathrm{sep},!}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}},\mathbf{D})
\end{tikzcd}$$

The lower functor is an equivalence by the universal property of the Dwyer–Kan localization $\mathrm{kk}_{\mathrm{sep},!}$; see Proposition 7.5. The lower-left equivalence is the universal property of ho. As a consequence we see that the functor marked by (1) is an equivalence.

The vertical functors are fully faithful. Therefore the functor marked by (2) is fully faithful, too. It remains to show that it is essentially surjective. If F is any functor in $\mathbf{Fun}^{h,s,(1/2)!}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}},\mathbf{D})$, then by Proposition 13.14 there exists a functor $\hat{F}$ in $\mathbf{Fun}(\mathrm{KK}_{\mathrm{sep},!},\mathbf{D})$ such that $\hat{F} \circ \mathrm{kk}_{\mathrm{sep},!} \simeq F$. We furthermore have a functor $\bar{F}$ in $\mathbf{Fun}(\mathrm{hoKK}_{\mathrm{sep},!},\mathbf{D})$ such that $\mathrm{ho}^*\bar{F} \simeq \hat{F}$. It remains to show that $\bar{F}$ is additive and half-exact.

Since F sends finite sums to products we can conclude that $\bar{F}$ is additive. Since all triangles in $\mathrm{hoKK}_{\mathrm{sep},!}$ come from exact (semisplit exact) sequences in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ and F is half-exact (or half-semiexact) the functor $\bar{F}$ sends the half-exact sequences in $\mathrm{hoKK}_{\mathrm{sep},!}$ (with the triangulated half-exact structure) to half-exact sequences in $\mathbf{D}$. Hence $\bar{F}$ is also half-exact.

In the case $! = q$ we argue similarly with

$$\begin{tikzcd} & \mathbf{Fun}^{\mathrm{add}}(\mathrm{hoKK}_{\mathrm{sep},q}, \mathbf{D}) \arrow[r, "(2)"] \arrow[d] & \mathbf{Fun}^{h,s,splt}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \arrow[d] \\ \mathbf{Fun}(\mathrm{KK}_{\mathrm{sep},q}, \mathbf{D}) \arrow[r, "\mathrm{ho}^*", "\simeq"'] \arrow[rr, bend right, "\mathrm{kk}^*_{\mathrm{sep},q}", "\cong"'] & \mathbf{Fun}(\mathrm{hoKK}_{\mathrm{sep},q}, \mathbf{D}) \arrow[r, "q", "\simeq"'] & \mathbf{Fun}^{\tilde{W}_{\mathrm{sep},q}}(C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}, \mathbf{D}) \end{tikzcd}$$

□

Let $\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}} : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \to \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}}$ denote the classical additive-category-valued functor described by the universal property [Higson 1990a, Theorem 3.4]. We equip $\mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}}$ with the split half-exact structure. Then $\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$ is split exact. Since $\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$ is also homotopy invariant and stable, by Proposition 13.13 we get a dotted factorization

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[r, "\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}"] \arrow[d, "\mathrm{kk}_{\mathrm{sep},q}"] & \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}} \\ \mathrm{KK}_{\mathrm{sep},q} \arrow[r, "\mathrm{ho}"] \arrow[ur, dotted] & \mathrm{hoKK}_{\mathrm{sep},q} \arrow[u, dashed, "\psi"] \end{tikzcd}$$

The dashed factorization is induced by the universal property of ho since $\mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}}$ is an ordinary category. Since $\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$ is split exact, the dotted arrow is half-exact, and the dashed arrow is additive.

Since $\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}$ inverts $\widetilde{W}_{\mathrm{sep,ex}}$ (as seen in the proof of Proposition 13.14) we also have a factorization

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}} \arrow[r, "\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}"] \arrow[d, "\mathrm{e}_{\mathrm{sep}}"] & \mathrm{E}^{\mathrm{class}}_{\mathrm{sep}} \\ \mathrm{E}_{\mathrm{sep}} \arrow[r] \arrow[ur, dotted] & \mathrm{hoE}_{\mathrm{sep}} \arrow[u, dashed, "\psi"] \end{tikzcd}$$

Since $\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}$ is half-exact we can conclude that the dashed arrow is additive and half-exact.

Theorem 13.16. *The comparison functors* $\psi : \mathrm{hoKK}_{\mathrm{sep},q} \to \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}}$ *and* $\psi : \mathrm{hoE}_{\mathrm{sep}} \to \mathrm{E}^{\mathrm{class}}_{\mathrm{sep}}$ *are equivalences.*

Proof. By Propositions 7.5 or 11.6, respectively, we know that $\mathrm{e}_{\mathrm{sep}}$ and $\mathrm{kk}_{\mathrm{sep},q}$ are Dwyer–Kan localizations. The composition of a Dwyer–Kan localization with the canonical functor to the homotopy category is again a localization, in this case in the sense of ordinary categories. We conclude that $\mathrm{hoe}_{\mathrm{sep}}$ and $\mathrm{hokk}_{\mathrm{sep},q}$ are localizations. Note that such a localization is determined uniquely up to equivalence under $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$, and that two choices of such equivalences under $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ are isomorphic by a unique isomorphism.

Since the universal properties of $\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$ [Higson 1990a, Theorem 3.4] and $\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}$ [Higson 1990a, Theorem 3.6] are formulated in terms of equalities of functors (instead of isomorphisms), it will be useful to choose $\mathrm{hokk}_{\mathrm{sep},q}$ and $\mathrm{hoe}_{\mathrm{sep}}$ such that these functors are bijective on objects.

We write down the details of the argument for KK-theory. Since $\mathrm{hokk}_{\mathrm{sep},q}$ is homotopy invariant, stable and split exact, the universal property [Higson 1990a, Theorem 3.4] provides an additive factorization

$$\begin{tikzcd} C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[rr, "\mathrm{hokk}_{\mathrm{sep},q}"] \arrow[dr, "\mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}"'] & & \mathrm{hoKK}_{\mathrm{sep},q} \\ & \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}} \arrow[ur, dotted, "\phi"'] & \end{tikzcd}$$

which strictly commutes.

The pull-back along $\mathrm{hokk}_{\mathrm{sep},q}$ of the composition $\mathrm{hoKK}_{\mathrm{sep},q} \xrightarrow{\psi} \mathrm{KK}^{\mathrm{class}}_{\mathrm{sep}} \xrightarrow{\phi} \mathrm{hoKK}_{\mathrm{sep},q}$ is equivalent to $\mathrm{hokk}_{\mathrm{sep},q}$. Therefore by Proposition 13.15 this composition $\phi \circ \psi$ itself is an equivalence.

We now show that the composition $\psi \circ \phi$ is also an equivalence invoking the uniqueness statement of [Higson 1990a, Theorem 3.4]. This requires an equality $\psi \circ \phi \circ \mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}} = \mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$. By the construction of ϕ using [Higson 1990a, Theorem 3.4] we have an equality $\phi \circ \mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}} = \mathrm{kk}_{\mathrm{sep},q}$. But the construction of ψ only ensures a natural isomorphism $f : \psi \circ \mathrm{hokk}_{\mathrm{sep},q} \xrightarrow{\cong} \mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$ which is not necessarily an equality. We now use the freedom to replace ψ by an isomorphic functor and our special choice of $\mathrm{hokk}_{\mathrm{sep},q}$ to be bijective on objects.

For every A in $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$ we have an isomorphism $f_A : \psi(\mathrm{hokk}_{\mathrm{sep},q}(A)) \xrightarrow{\cong} \mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}(A)$. We define $\psi' : \mathrm{hoKK}_{\mathrm{sep},q} \to \mathrm{KK}^{\mathrm{class}}$ on objects such that $\psi'(\mathrm{hokk}_{\mathrm{sep},q}(A)) := \mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}(A)$. For a morphism $h : A \to B$ in $\mathrm{hoKK}_{\mathrm{sep},q}$ we then define $\psi'(h) := f_B \phi(h) f_A^{-1}$. The family $(f_A)_A$ also implements an isomorphism $\psi \cong \psi'$. Now $\psi' \circ \phi \circ \mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}} = \mathrm{kk}^{\mathrm{class}}_{\mathrm{sep}}$ which implies that $\psi' \circ \phi = \mathtt{id}$.

In particular, ϕ has a right- and a left-inverse equivalence and is hence itself an equivalence. But then also ψ is an equivalence.

The case of E-theory is completely analogous. We use the universal property [Higson 1990a, Theorem 3.6] of $\mathrm{e}^{\mathrm{class}}_{\mathrm{sep}}$. □

Note that the proof of Theorem 13.16 does not use the case of Proposition 13.14 for $! = \mathrm{se}$ and is therefore independent of the automatic semicontinuity theorem.

14. Asymptotic morphisms in E-theory

The first construction of an additive 1-category representing E-theory was given in [Higson 1990a] by enforcing universal properties. This construction was the blueprint for the ∞-categorical version considered in the present note. Shortly after in [Connes and Higson 1990] the E-theory groups were represented as equivalence classes of asymptotic morphisms; see also [Guentner et al. 2000]. Recall that we construct KK-theory for separable algebras by a sequence of Dwyer–Kan localizations applied to $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{sep}}$. In view of [Connes and Higson 1990; Guentner et al. 2000] a natural idea would be to apply a similar construction to the category of C^*-algebras and asymptotic morphisms. The first obstacle one encounters in this approach is that the composition of asymptotic morphisms is only well-defined after going over to homotopy classes. By now[3] we think that the correct way to relate E-theory with asymptotic morphisms is the one worked out recently in [Bunke and Duenzinger 2024, Section 3.5]. It is based on the shape theory of [Blackadar 1985; Dădărlat 1994] and goes beyond the scope of the present paper. We will therefore just show that asymptotic morphisms also give rise to morphisms in our version E-theory in a way which is compatible with the composition.

We consider the endofunctors

$$T, F : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}$$

defined by

$$T(A) := C_b([0, \infty), A), \quad F(A) := C_b([0, \infty), A)/C_0([0, \infty), A).$$

We have a natural transformation $\alpha : T \to F$ such that $\alpha_A : T(A) \to F(A)$ is the projection onto the quotient. We have the natural transformations

$$\beta : \mathtt{id}_{C^*\mathbf{Alg}^{\mathrm{nu}}} \to T, \quad \mathtt{ev}_0 : T \to \mathtt{id}_{C^*\mathbf{Alg}^{\mathrm{nu}}}$$

such that $\beta_A : A \to T(A)$ sends a in A to the constant function with value a, and $\mathtt{ev}_{0,A} : T(A) \to A$ evaluates the function $t \mapsto f(t)$ in $T(A)$ at $t = 0$. We finally define the natural transformation

$$\gamma := \alpha \circ \beta : \mathtt{id}_{C^*\mathbf{Alg}^{\mathrm{nu}}} \to F(A).$$

Note that the sequence

$$0 \to C_0([0, \infty), A) \to T(A) \xrightarrow{\alpha_A} F(A) \to 0$$

is exact and that $C_0([0, \infty), A)$ is contractible. Since $\mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}$ is reduced, homotopy invariant and exact we see that $\mathrm{e}(\alpha_A)$ is an equivalence for every A in $C^*\mathbf{Alg}^{\mathrm{nu}}$. We define a natural transformation $\delta : \mathrm{e} \circ F \to \mathrm{e}$ by

$$\delta_A := \mathrm{e}(\mathtt{ev}_{0,A}) \circ \mathrm{e}(\alpha_A)^{-1} : \mathrm{e}(F(A)) \to \mathrm{e}(A).$$

Following [Guentner et al. 2000, Section 2] we adopt the following definition.

Definition 14.1. For n in $\mathbb{N}$ an asymptotic morphism $f : A \rightsquigarrow_n B$ is a morphism $f : A \to F^n(B)$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

Remark 14.2. Note that asymptotic morphisms for $n = 0$ are the usual morphisms, and the case of $n = 1$ corresponds to the notion of an asymptotic morphism in [Connes and Higson 1990]. As in [Guentner et al. 2000, Section 2] we include the case of bigger n in order to have a simple definition of a composition of asymptotic morphisms which also works for nonseparable algebras. □

[3] 2024.

If $f : A \rightsquigarrow_n B$ is an asymptotic morphism, then we define

$$\mathrm{e}_n(f) := \delta_B \circ \cdots \circ \delta_{F^{n-1}(B)} \circ \mathrm{e}(f).$$

If $n = 0$, then this formula is interpreted as $\mathrm{e}_0(f) := \mathrm{e}(f)$.

Let $f' : A \rightsquigarrow_{n+1} B$ be given by $\gamma_{F^n(B)} \circ f$. Then we say that f' and f are *related*.

Lemma 14.3. *If f' is related to f, then $\mathrm{e}_n(f) \simeq \mathrm{e}_{n+1}(f')$.*

Proof. This follows from $\mathrm{e}(\gamma_{F^n(B)}) \simeq \mathrm{e}(\alpha_{F^n(B)}) \circ \mathrm{e}(\beta_{F^n(B)})$ and $\mathrm{e}(\mathtt{ev}_0) \circ \mathrm{e}(\beta_{F^n(B)}) \simeq \mathtt{id}_{\mathrm{e}(F^n(B))}$. □

The argument implies that $\mathrm{e}_1(\gamma_A) \simeq \mathtt{id}_A$ for every C^*-algebra A.

We define the *composition* of two asymptotic morphisms $f : A \rightsquigarrow_n B$ and $g : B \rightsquigarrow_m C$ as $g \sharp f : A \rightsquigarrow_{n+m} C$ given by $F^n(g) \circ f$.

Lemma 14.4. *We have $\mathrm{e}_{n+m}(g \sharp f) \simeq \mathrm{e}_m(g) \circ \mathrm{e}_n(f)$.*

Proof. We consider the diagram

$$\begin{tikzcd}
 & & \mathrm{e}(F^{n+m}(C)) \arrow[dr, "\delta_{F^m(C)}\cdots\delta_{F^{n+m-1}(C)}"] & & \\
 & \mathrm{e}(F^n(B)) \arrow[ur, "\mathrm{e}(F^n(g))"] \arrow[dr, "\delta_B\cdots\delta_{F^{n-1}(B)}"] & & \mathrm{e}(F^m(C)) \arrow[dr, "\delta_C\cdots\delta_{F^{m-1}(C)}"] & \\
\mathrm{e}(A) \arrow[ur, "\mathrm{e}(f)"] \arrow[rr, "\mathrm{e}_n(f)"] & & \mathrm{e}(B) \arrow[ur, "e(g)"] \arrow[rr, "\mathrm{e}_m(g)"] & & \mathrm{e}(C)
\end{tikzcd}$$

The square commutes since δ is a natural transformation. The lower triangles reflect the definitions of the lower-horizontal arrows. □

We say $f_0, f_1 : A \rightsquigarrow_n B$ are *homotopic* if there exists $f : A \rightsquigarrow_n C([0,1], B)$ such that $F^n(\mathtt{ev}_i) \circ f = f_i$.

Lemma 14.5. *If f_0 and f_1 are homotopic, then $\mathrm{e}_n(f_0) \simeq \mathrm{e}_n(f_1)$.*

Proof. We have $\mathrm{e}_n(f_i) \simeq \mathrm{e}_0(\mathtt{ev}_i) \sharp \mathrm{e}_n(f)$. The assertion now follows since $e_0 = e$ and e is homotopy invariant. □

In the remainder of this section we relate the E-theory constructed in the present note with the version from [Guentner et al. 2000], called the classical E-theory $\mathrm{E}^{\mathrm{class}}$. In [Guentner et al. 2000, Definition 2.13] (even in the equivariant case) a *homotopy category* $\mathfrak{A}$ *of asymptotic morphisms* is introduced. Its objects are C^*-algebras, and its morphisms are equivalence classes of asymptotic morphisms, where the equivalence relation is generated by the relations of being related and homotopy introduced above. The results above show that the functor $\mathrm{ho} \circ \mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{hoE}$ factorizes over $\mathfrak{A}$.

Corollary 14.6. *We have a commutative triangle*

$$\begin{tikzcd}
 & C^*\mathbf{Alg}^{\mathrm{nu}} \arrow[dl] \arrow[dr, "\mathrm{ho}\circ\mathrm{e}"] & \\
\mathfrak{A} \arrow[rr, "c"] & & \mathrm{hoE}
\end{tikzcd}$$

Proof. The down-left arrow sends a morphism $f : A \to B$ to the equivalence class represented by $f \rightsquigarrow_0 B$, and the lower-horizontal *comparison arrow* c sends the C^*-algebra A to $\mathrm{e}(A)$ and the class of an asymptotic morphism $f \rightsquigarrow_n B$ to $\mathrm{e}_n(f)$. □

In [Guentner et al. 2000, Definition 2.13] the *classical E-theory category* $\mathrm{E}^{\mathrm{class}}$ is defined as the category whose objects are C^*-algebras, and whose morphisms are given by

$$\mathtt{Hom}_{\mathrm{E}^{\mathrm{class}}}(A, B) := \mathtt{Hom}_{\mathfrak{A}}(K \otimes S(A), K \otimes S(B)). \tag{14-1}$$

It should not be confused with the separable version $E^{\mathrm{class}}_{\mathrm{sep}}$ from [Higson 1990a]. There is a canonical functor

$$i : \mathfrak{A} \to \mathrm{E}^{\mathrm{class}},$$

which is the identity on objects and sends the class of an asymptotic morphism $f : A \rightsquigarrow_n B$ to the class of $i(f) : K \otimes S(A) \rightsquigarrow_n K \otimes S(B)$ given by the composition

$$K \otimes S(A) \xrightarrow{f} K \otimes S(F^n(B)) \to F^n(K \otimes S(B)).$$

(Note that the second map is not an isomorphism.)

Corollary 14.7. *We have a commutative triangle*

$$\begin{tikzcd} & \mathfrak{A} \arrow[dl, "i"'] \arrow[dr, "c"] & \\ \mathrm{E}^{\mathrm{class}} \arrow[rr, "\hat{c}"] & & \mathrm{hoE} \end{tikzcd} \tag{14-2}$$

Proof. The lower-horizontal map sends a C^*-algebra A to $\mathrm{e}(A)$ and the class of a morphism $f : K \otimes S(A) \to F^n(K \otimes S(B))$ to the image under

$$\mathtt{Hom}_{\mathfrak{A}}(K \otimes S(A), K \otimes S(B)) \xrightarrow{c} \mathrm{E}_0(K \otimes S(A), K \otimes S(B)) \cong \mathrm{E}_0(A, B),$$

where we use stability of the functor e and stability of the ∞-category E for the second isomorphism. □

Remark 14.8. The functor $\hat{c}$ in (14-2) is not an equivalence. In fact the classical E-theory functor preserves countable sums by [Guentner et al. 2000, Proposition 7.1]. In contrast, the functor e does not preserve countable sums, since $y : \mathrm{E}_{\mathrm{sep}} \to \mathrm{E}$ does not preserve countable sums.

But note that it is shown in [Bunke and Duenzinger 2024] that the restriction of $\hat{c}$ to the full subcategory of separable algebras induces an equivalence $\hat{c}_{\mathrm{sep}} : \mathrm{E}^{\mathrm{class}}_{\mathrm{sep}} \to \mathrm{hoE}_{\mathrm{sep}}$. In particular the formula (14-1) gives an explicit description of the morphism groups in $\mathrm{hoE}_{\mathrm{sep}}$ in terms of homotopy classes of asymptotic morphisms. □

Remark 14.9. Let

$$\mathcal{S} : 0 \to A \to B \to C \to 0 \tag{14-3}$$

be an exact sequence of separable C^*-algebras. The E-theory analogue of Construction 12.6 is [Guentner et al. 2000, Proposition 5.5], where a morphism $\sigma_{\mathcal{S}}$ in $\mathtt{Hom}_{\mathfrak{A}}(S(C), A)$ was constructed. It follows from

[Guentner et al. 2000, Proposition 5.15] (this is an analogue of Proposition 12.14) that the image of σ_S in $\mathrm{E}_0(S(C), A)$ is the boundary map ∂_S of the fiber sequence in E associated to the exact sequence (14-3). This shows that the comparison functor $\hat{c}$ is compatible with the long exact sequences associated to exact sequences of separable C^*-algebras. □

Acknowledgements

The author was supported by the SFB 1085 (Higher Invariants) funded by the Deutsche Forschungsgemeinschaft (DFG). The author profited from discussions with M. Land und U. Pennig which started this project. Special thanks go to B. Dünzinger for carefully reading the draft.

References

[Avron et al. 1994] J. Avron, R. Seiler, and B. Simon, "The index of a pair of projections", *J. Funct. Anal.* **120**:1 (1994), 220–237. MR Zbl

[Blackadar 1985] B. Blackadar, "Shape theory for C^*-algebras", *Math. Scand.* **56**:2 (1985), 249–275. MR Zbl

[Blackadar 1998] B. Blackadar, *K-theory for operator algebras*, 2nd ed., Mathematical Sciences Research Institute Publications **5**, Cambridge University Press, 1998. MR Zbl

[Blackadar 2016] B. Blackadar, "The homotopy lifting theorem for semiprojective C^*-algebras", *Math. Scand.* **118**:2 (2016), 291–302. MR Zbl

[Brown and Ozawa 2008] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics **88**, American Mathematical Society, Providence, RI, 2008. MR Zbl

[Bunke 2020] U. Bunke, "Non-unital C^*-categories, (co)limits, crossed products and exactness", preprint, 2020. arXiv 2008.06257

[Bunke and Duenzinger 2024] U. Bunke and B. Duenzinger, "E-theory is compactly assembled", preprint, 2024. arXiv 2402.18228

[Bunke et al. 2003] U. Bunke, M. Joachim, and S. Stolz, "Classifying spaces and spectra representing the K-theory of a graded C^*-algebra", pp. 80–102 in *High-dimensional manifold topology*, edited by F. T. Farrell and W. Lück, World Sci. Publ., River Edge, NJ, 2003. MR Zbl

[Bunke et al. 2021] U. Bunke, A. Engel, and M. Land, "A stable ∞-category for equivariant KK-theory", preprint, 2021. Zbl arXiv 2102.13372

[Bunke et al. 2023] U. Bunke, M. Land, and U. Pennig, "A survey on operator K-theory via homotopical algebra", preprint, 2023. arXiv 2311.17191

[Choi and Effros 1976] M. D. Choi and E. G. Effros, "The completely positive lifting problem for C^*-algebras", *Ann. of Math.* (2) **104**:3 (1976), 585–609. MR Zbl

[Cisinski 2019] D.-C. Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics **180**, Cambridge University Press, 2019. MR Zbl

[Connes and Higson 1990] A. Connes and N. Higson, "Déformations, morphismes asymptotiques et K-théorie bivariante", *C. R. Acad. Sci. Paris Sér. I Math.* **311**:2 (1990), 101–106. MR Zbl

[Cortiñas and Thom 2007] G. Cortiñas and A. Thom, "Bivariant algebraic K-theory", *J. Reine Angew. Math.* **610** (2007), 71–123. MR Zbl

[Cuntz 1984] J. Cuntz, "K-theory and C^*-algebras", pp. 55–79 in *Algebraic K-theory, number theory, geometry and analysis* (Bielefeld, Germany, 1982), edited by A. Bak, Lecture Notes in Math. **1046**, Springer, 1984. MR Zbl

[Cuntz 1987] J. Cuntz, "A new look at KK-theory", *K-Theory* **1**:1 (1987), 31–51. MR Zbl

[Cuntz 1998] J. Cuntz, "A general construction of bivariant K-theories on the category of C^*-algebras", pp. 31–43 in *Operator algebras and operator theory* (Shanghai, 1997), edited by L. Ge et al., Contemp. Math. **228**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Cuntz and Skandalis 1986] J. Cuntz and G. Skandalis, "Mapping cones and exact sequences in *KK*-theory", *J. Operator Theory* **15**:1 (1986), 163–180. MR Zbl

[Dadarlat and Pennig 2015] M. Dadarlat and U. Pennig, "Unit spectra of K-theory from strongly self-absorbing C^*-algebras", *Algebr. Geom. Topol.* **15**:1 (2015), 137–168. MR Zbl

[Dadarlat et al. 2018] M. Dadarlat, R. Willett, and J. Wu, "Localization C^*-algebras and K-theoretic duality", *Ann. K-Theory* **3**:4 (2018), 615–630. MR Zbl

[Dell'Ambrogio et al. 2011] I. Dell'Ambrogio, H. Emerson, T. Kandelaki, and R. Meyer, "A functorial equivariant K-theory spectrum and an equivariant Lefschetz formula", preprint, 2011. arXiv 1104.3441

[Dixmier 1977] J. Dixmier, *C^*-algebras*, North-Holland Mathematical Library **15**, North-Holland, Amsterdam, 1977. MR Zbl

[Dădărlat 1994] M. Dădărlat, "Shape theory and asymptotic morphisms for C^*-algebras", *Duke Math. J.* **73**:3 (1994), 687–711. MR Zbl

[Ellis 2014] E. Ellis, "Equivariant algebraic kk-theory and adjointness theorems", *J. Algebra* **398** (2014), 200–226. MR Zbl

[Fritz 2010] T. Fritz, "Cuntz' proof of Bott periodicity for C^*-algebras", Lecture notes, 2010, http://tobiasfritz.science/2010/bott_periodicity.pdf.

[Garkusha 2014] G. Garkusha, "Algebraic Kasparov K-theory, I", *Doc. Math.* **19** (2014), 1207–1269. MR Zbl

[Garkusha 2016] G. Garkusha, "Algebraic Kasparov K-theory, II", *Ann. K-Theory* **1**:3 (2016), 275–316. MR Zbl

[Guentner et al. 2000] E. Guentner, N. Higson, and J. Trout, *Equivariant E-theory for C^*-algebras*, Mem. Amer. Math. Soc. **703**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Higson 1987] N. Higson, "A characterization of *KK*-theory", *Pacific J. Math.* **126**:2 (1987), 253–276. MR Zbl

[Higson 1988] N. Higson, "Algebraic K-theory of stable C^*-algebras", *Adv. in Math.* **67**:1 (1988), 1–140. MR Zbl

[Higson 1990a] N. Higson, "Categories of fractions and excision in *KK*-theory", *J. Pure Appl. Algebra* **65**:2 (1990), 119–138. MR Zbl

[Higson 1990b] N. Higson, "A primer on *KK*-theory", pp. 239–283 in *Operator theory: operator algebras and applications*, 1 (Durham, NH, 1988), edited by W. B. Arveson and R. G. Douglas, Proc. Sympos. Pure Math. **51**, Amer. Math. Soc., Providence, RI, 1990. MR Zbl

[Hinich 2016] V. Hinich, "Dwyer–Kan localization revisited", *Homology Homotopy Appl.* **18**:1 (2016), 27–48. MR Zbl

[Joachim 2003] M. Joachim, "K-homology of C^*-categories and symmetric spectra representing K-homology", *Math. Ann.* **327**:4 (2003), 641–670. MR Zbl

[Joachim and Stolz 2009] M. Joachim and S. Stolz, "An enrichment of *KK*-theory over the category of symmetric spectra", *Münster J. Math.* **2** (2009), 143–182. MR Zbl

[Kasparov 1980] G. G. Kasparov, "Hilbert C^*-modules: theorems of Stinespring and Voiculescu", *J. Operator Theory* **4**:1 (1980), 133–150. MR Zbl

[Kasparov 1988] G. G. Kasparov, "Equivariant *KK*-theory and the Novikov conjecture", *Invent. Math.* **91**:1 (1988), 147–201. MR Zbl

[Land and Nikolaus 2018] M. Land and T. Nikolaus, "On the relation between K- and L-theory of C^*-algebras", *Math. Ann.* **371**:1-2 (2018), 517–563. MR Zbl

[Lurie 2009] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies **170**, Princeton University Press, 2009. MR Zbl

[Lurie 2017] J. Lurie, "Higher algebra", preprint, Institute for Advanced Study, 2017, https://people.math.harvard.edu/~lurie/papers/HA.pdf.

[Mahanta 2015] S. Mahanta, "Noncommutative stable homotopy and stable infinity categories", *J. Topol. Anal.* **7**:1 (2015), 135–165. MR Zbl

[Meyer 2000] R. Meyer, "Equivariant Kasparov theory and generalized homomorphisms", *K-Theory* **21**:3 (2000), 201–228. MR Zbl

[Meyer and Nest 2006] R. Meyer and R. Nest, "The Baum–Connes conjecture via localisation of categories", *Topology* **45**:2 (2006), 209–259. MR Zbl

[Mitchener 2002] P. D. Mitchener, "KK-theory of C^*-categories and the analytic assembly map", preprint, 2002. arXiv math/0202037

[Nikolaus 2017] T. Nikolaus, "The group completion theorem via localizations of ring spectra", Expository notes, 2017, https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Group_completion.pdf.

[Østvær 2010] P. A. Østvær, *Homotopy theory for C^*-algebras*, Birkhäuser, Basel, 2010.

[Pedersen 1976] G. K. Pedersen, "Lifting derivations from quotients of separable C^*-algebras", *Proc. Nat. Acad. Sci. U.S.A.* **73**:5 (1976), 1414–1415. MR Zbl

[Pedersen 1979] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs **14**, Academic Press, London, 1979. MR Zbl

[Pisier 2020] G. Pisier, *Tensor products of C^*-algebras and operator spaces*, London Mathematical Society Student Texts **96**, Cambridge University Press, 2020. Zbl

[Randal-Williams 2013] O. Randal-Williams, "'Group-completion', local coefficient systems and perfection", *Q. J. Math.* **64**:3 (2013), 795–803. MR Zbl

[Rosenberg and Schochet 1987] J. Rosenberg and C. Schochet, "The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor", *Duke Math. J.* **55**:2 (1987), 431–474. MR Zbl

[Schochet 1984] C. Schochet, "Topological methods for C^*-algebras, III: Axiomatic homology", *Pacific J. Math.* **114**:2 (1984), 399–445. MR Zbl

[Skandalis 1985] G. Skandalis, "Exact sequences for the Kasparov groups of graded algebras", *Canad. J. Math.* **37**:2 (1985), 193–216. MR Zbl

[Skandalis 1988] G. Skandalis, "Une notion de nucléarité en K-théorie (d'après J. Cuntz)", *K-Theory* **1**:6 (1988), 549–573. MR Zbl

[Thomsen 1999] K. Thomsen, "Asymptotic homomorphisms and equivariant KK-theory", *J. Funct. Anal.* **163**:2 (1999), 324–343. MR Zbl

[Uuye 2013] O. Uuye, "Homotopical algebra for C*-algebras", *J. Noncommut. Geom.* **7**:4 (2013), 981–1006. MR Zbl

[Williams 2007] D. P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs **134**, American Mathematical Society, Providence, RI, 2007. MR Zbl

Received 4 May 2023. Revised 20 Mar 2024.

ULRICH BUNKE:

ulrich.bunke@mathematik.uni-regensburg.de
Fakultät für Mathematik, Universität Regensburg, Regensburg, Germany

msp

ORBITA
MATHEMATICAE

msp.org/om

JOURNAL OF **UMALCA**

Unión Matemática
de América Latina y el Caribe

See inside back cover or msp.org/om for submission instructions.

Orbita Mathematicae (ISSN 2993-6152 electronic, 2993-6144 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

MT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
https://msp.org/

ORBITA MATHEMATICAE

vol. 1 no. 2 2024

Umalca
Unión Matemática de América Latina y el Caribe