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The degree of a tropical root surface of type A

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We prove that the tropical surface of the root system A_{n-1} has degree $\frac{1}{2}n(n-1)(n-2)$.

1. Introduction

Tropical geometry was developed to answer questions in classical algebraic geometry combinatorially. Tropicalization converts a projective variety V into a polyhedral complex $\mathrm{trop}(V)$ that, roughly speaking, records the behavior of V at infinity. The tropical variety $\mathrm{trop}(V)$ retains a surprising amount of information about V, such as its dimension and degree. Many important invariants of $\mathrm{trop}(V)$ can be computed using combinatorics and discrete geometry, thus giving computations of algebro-geometric invariants of V. For detailed introductions to tropical geometry, see [Brugallé and Shaw 2014; Maclagan and Sturmfels 2015; Mikhalkin and Rau 2010].

Initially, tropical geometry was most interested in studying tropicalizations of algebraic varieties of importance. However, a more robust theory arises when one considers abstract tropical varieties, most of which do not arise via tropicalization. This is analogous to the situation in matroid theory, where a linear subspace V of a vector space gives rise to a matroid M_V, but a more robust theory arises when one considers all matroids, most of which do not arise from a linear subspace. (This is not just an analogy: matroids may be understood as the tropical fans of degree 1 [Ardila and Klivans 2006; Fink 2013].)

Tropical geometry is then a rich source of well motivated combinatorial problems of significance within and beyond combinatorics. A good theory needs good examples, and combinatorics is a rich source of tropical varieties. In this spirit, Ardila, Kato, McMillon, Perez, and Schindler construct tropical surfaces associated in a natural way to the classical root systems.

The geometric protagonist of this paper is the tropical surface $S(A_{n-1})$ associated to the root system A_{n-1} of the special linear Lie algebra $\mathfrak{sl}_n$. Our main result is that this surface has degree $\frac{1}{2}n(n-1)(n-2)$.

2. Background

Let n be a positive integer and write $[n] = \{1, 2, \ldots, n\}$. Let $\mathsf{e}_1, \ldots, \mathsf{e}_n$ be the standard basis of $\mathbb{R}^n$, and write $\mathsf{e}_S = \sum_{s \in S} \mathsf{e}_s$ for each subset $S \subseteq [n]$.

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2.1. ***Root systems.*** Let us begin by defining root systems and root polytopes.

Definition 1 [Bourbaki 1968]. A *crystallographic root system* Φ is a set of vectors in $\mathbb{R}^n$ satisfying:

- For every root $\beta \in \Phi$, the set Φ is closed under reflection across the hyperplane perpendicular to β.
- For any two roots $\alpha, \beta \in \Phi$, the quantity $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle$ is an integer, where $\langle -, - \rangle$ is the standard inner product in $\mathbb{R}^n$.
- If $\beta, c\beta \in \Phi$ for $c \in \mathbb{R}$, then $c = 1$ or $c = -1$.

Definition 2 [Bourbaki 1968]. An *irreducible root system* is one that cannot be partitioned into the union of two proper subsets $\Delta = \Delta_1 \cup \Delta_2$, such that $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \Delta_1$ and $\beta \in \Delta_2$.

Root systems play a fundamental role in many areas of mathematics; for example, they are key to the classification of semisimple Lie algebras [Bourbaki 1968]. The irreducible root systems have been classified into four infinite *classical families* and five *exceptional* root systems. In this paper we focus on the most classical family,

$$A_{n-1} = \{\mathsf{e}_i - \mathsf{e}_j \, : \, i, j \in [n], \ i \neq j\}.$$

This is the root system of the special linear Lie algebra $\mathfrak{sl}_n$.

Definition 3. The *root polytope* $P(\Phi)$ of a root system Φ is the convex hull of Φ.

2.2. ***Tropical geometry.*** To define the root surfaces $S(A_{n-1})$ that interest us, we first introduce some basic definitions from tropical geometry. Let $N_{\mathbb{Z}} \cong \mathbb{Z}^r$ be a lattice and let $N = N_{\mathbb{Z}} \otimes \mathbb{R} \cong \mathbb{R}^r$ be the corresponding vector space. A *cone* is a set of the form

$$\operatorname{cone}(v_1, \dots, v_n) = \{\lambda_1 v_1 + \cdots + \lambda_n v_n \, : \, \lambda_1, \lambda_2, \dots, \lambda_n \geq 0\}$$

for vectors $v_1, \dots, v_n$ in N. The cone is *rational* if it is generated by lattice vectors. A (rational) *polyhedral fan* is a nonempty finite collection Σ of (rational) cones in N such that every face of a cone in Σ is also in Σ, and the intersection of any two cones in Σ is a face of both of them. A fan is *pure* of dimension d if all maximal faces are d-dimensional. We let Σ^i denote the set of cones of Σ of dimension i. Tropical fans are those that meet the following balancing condition.

Definition 4 [Maclagan and Sturmfels 2015]. Let $\Sigma \subseteq N$ be a rational polyhedral fan, pure of dimension d, with a choice of weight $w(\sigma) \in \mathbb{N}$ for each maximal cone $\sigma \in \Sigma^d$.

For each $(d-1)$-cone $\tau \in \Sigma^{d-1}$, consider the $(d-1)$-subspace $L_\tau \subseteq N$ spanned by τ, the induced $(d-1)$-lattice $L_{\tau,\mathbb{Z}} = L_\tau \cap N_{\mathbb{Z}}$, and the quotient $(n-d+1)$-lattice $N(\tau) = N_{\mathbb{Z}}/L_{\tau,\mathbb{Z}}$. Each d-cone $\sigma \in \Sigma^d$ with $\sigma \supset \tau$ determines a ray $(\sigma + L_\tau)/L_\tau$ in N/L_τ. This ray is rational with respect to the lattice $N(\tau)$; let $u_{\sigma/\tau}$ be the first lattice point on this ray. The fan Σ is *balanced at* τ if

$$\sum_{\sigma \in \Sigma^d \, : \, \sigma \supset \tau} w(\sigma) u_{\sigma/\tau} = 0 \quad \text{in } N/L_\tau.$$

The fan Σ is a *tropical fan* if it is balanced at all faces of dimension $d-1$.[1]

[1] Notice that the balancing condition for Σ at τ only depends on the fan $\operatorname{Star}_\Sigma(\tau)$, with weights inherited from Σ.

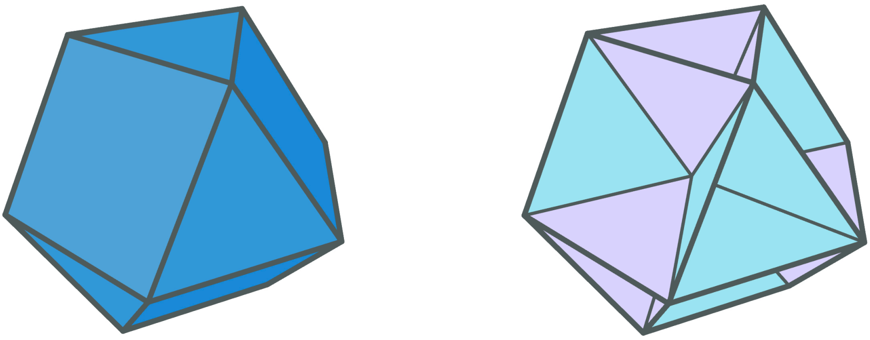

Figure 1. The root polytope $P(A_3)$ and the tropical root surface $S(A_3)$.

Tropical varieties are more general than tropical fans; see [Mikhalkin and Rau 2010] for a definition. *Tropical surfaces* are tropical varieties that are pure of dimension 2. In particular, 2-dimensional tropical fans are tropical surfaces.

Definition 5. The *tropical root surface* $S(\Phi)$ of a root system Φ is the cone over the 1-skeleton of $P(\Phi)$ with unit weights on all facets. It consists of:

- Rays: $\mathrm{cone}(r)$ for each $r \in \Phi$.
- Facets: $\mathrm{cone}(r, s)$ for each $r, s \in \Phi$ such that rs is an edge of the root polytope $P(\Phi)$.
- Weights: $w(\sigma) = 1$ for every facet σ.

Since the root system A_{n-1} is $(n-1)$-dimensional, we regard $S(A_{n-1})$ as a tropical root surface in the lattice $N_{\mathbb{Z}} = \mathbb{Z}^n/\mathbb{Z}$ with $N = \mathbb{R}^n/\mathbb{R}$.

Tropical root surfaces were introduced in [Perez 2019; Schindler 2017; Ardila et al. $\geq$ 2025] by Federico Ardila, Chiemi Kato, Jewell McMillon, Maria Isabel Perez, and Anna Schindler. Figure 1 shows the root polytope and the tropical surface of the root system A_3, with its cones truncated for visibility.

In classical algebraic geometry, the degree of an irreducible projective variety of dimension d is obtained by counting its intersection points with a generic linear space of codimension d. In tropical geometry, degree is defined similarly. The analog of a generic linear space is the standard tropical linear space of codimension d, which we now define. Let us write $\mathsf{e}_1, \ldots, \mathsf{e}_n$ for the image of the unit vectors of $\mathbb{R}^n$ in $\mathbb{R}^n/\mathbb{R}$.

Definition 6 [Ardila and Klivans 2006; Maclagan and Sturmfels 2015]. The *standard tropical linear space* $\Sigma_{n,n-d}$ is the tropical fan in $\mathbb{R}^n/\mathbb{R}$ whose facets are the $(n-1-d)$-dimensional cones

$$C_I := \{x \in \mathbb{R}^n/\mathbb{R} : \min_{1 \leq j \leq n} x_j = x_i \text{ for all } i \in I\} = \mathrm{cone}\{\mathsf{e}_i : i \notin I\}$$

for each choice of a $(d+1)$-subset $I \subseteq [n]$, where every facet has weight 1. Its support (i.e., the union of all of its cones) is

$$|\Sigma_{n,n-d}| = \{x \in \mathbb{R}^n : \text{ the smallest } d+1 \text{ entries of } x \text{ are equal to each other}\}. \quad (1)$$

The fan $\Sigma_{n,n-d}$ is the *coarse subdivision* of the *Bergman fan* of the uniform matroid $U_{n,n-d}$; it is the tropicalization of any sufficiently generic $(n-d)$-dimensional linear space in n-space, as shown in [Ardila and Klivans 2006]. We remark that we are using the *min* convention of tropical geometry, although the *max* convention would give the same results, since tropical root surfaces are symmetric across the origin.

Definition 7. Consider two tropical fans Σ_1 and Σ_2 of complementary dimensions d_1 and d_2 in a vector space N; that is, $d_1 + d_2 = \dim N$. We say Σ_1 and Σ_2 *intersect transversally* if $\Sigma_1 \cap \Sigma_2$ is a finite union of points, and each such point p can be written uniquely as $p = \sigma_1 \cap \sigma_2$ for facets σ_1, σ_2 of Σ_1, Σ_2, respectively. The *weight* of each intersection point p is

$$w(p) := w(\sigma_1)w(\sigma_2)\,[N_{\mathbb{Z}} : L_{\sigma_1,\mathbb{Z}} + L_{\sigma_2,\mathbb{Z}}].$$

We call $\operatorname{index}(p) := [N_{\mathbb{Z}} : L_{\sigma_1,\mathbb{Z}} + L_{\sigma_2,\mathbb{Z}}]$ the *index* of p. The *degree of the transversal intersection at p* is

$$\Sigma_1 \cdot \Sigma_2 := \sum_{p \in \Sigma_1 \cap \Sigma_2} w(p).$$

If Σ_1 and Σ_2 are balanced but do not necessarily intersect transversally, then Σ_1 and $\Sigma_2 + v$ do intersect transversally for generic vectors $v \in N$, and the balancing condition implies that the degree of their transversal intersection does not depend on v [Mikhalkin and Rau 2010, Propositions 4.3.3, 4.3.6]. Thus we define the *degree of the intersection* to be

$$\Sigma_1 \cdot \Sigma_2 := (v + \Sigma_1) \cdot \Sigma_2$$

for generic v.

Definition 8. The *degree of a tropical fan* Σ in $N = \mathbb{R}^n/\mathbb{R}$ of dimension d is the degree of its intersection with the standard tropical linear space of codimension d:

$$\deg \Sigma := \Sigma \cdot \Sigma_{n,n-d}.$$

In practice, to find the degree of a tropical fan Σ, one chooses a convenient generic vector $v \in \mathbb{R}^n/\mathbb{R}$ and performs the following steps.

(1) Find the intersections of $v + \Sigma$ with $\Sigma_{n,n-d}$.

(2) For each intersection point p identify the cones $v + \sigma_1$ of $v + \Sigma$ and σ_2 of $\Sigma_{n,n-d}$ containing it, and find the weight of that intersection.

(3) Find the degree of Σ by adding the weights of the intersection points above.

Step (2) can be carried out by choosing lattice bases $\{\alpha_i\}$ and $\{\beta_i\}$ for $L_{\sigma_1,\mathbb{Z}}$ and $L_{\sigma_2,\mathbb{Z}}$, so that $L_{\sigma_1,\mathbb{Z}} = \mathbb{Z}\langle \alpha_1, \dots, \alpha_{d_1}\rangle$ and $L_{\sigma_2,\mathbb{Z}} = \mathbb{Z}\langle \beta_1, \dots, \beta_{d_2}\rangle$. Then the index of the intersection p can be computed as

$$\operatorname{index}(p) = |\det(\alpha_1, \dots, \alpha_{d_1}, \beta_1, \dots, \beta_{d_2})|.$$

3. The tropical root surface of type A and its degree

The following result first appeared in [Schindler 2017], in a slightly different form. We include a proof for completeness.

Proposition 9. *The tropical root surface $S(A_{n-1})$ is a tropical surface.*

Proof. We verify the balancing condition for an arbitrary ray $r = \operatorname{cone}(\mathsf{e}_i - \mathsf{e}_k)$. The maximal cones of $S(A_{n-1})$ containing r are the cones over the edges of the root polytope $P(A_{n-1})$ containing the vertex $\mathsf{e}_i - \mathsf{e}_k$; these are known [Ardila et al. 2011; Cellini and Marietti 2015] to be

$$\mathcal{A}_{i,jk} = \operatorname{cone}(\mathsf{e}_i - \mathsf{e}_j, \mathsf{e}_i - \mathsf{e}_k), \quad \mathcal{A}_{ij,k} = \operatorname{cone}(\mathsf{e}_i - \mathsf{e}_k, \mathsf{e}_j - \mathsf{e}_k), \qquad \text{for } j \neq i, k. \tag{2}$$

The primitive vectors in these cones with respect to r are $\overline{\mathsf{e}_i - \mathsf{e}_j}$ and $\overline{\mathsf{e}_j - \mathsf{e}_k}$ in $N/\mathbb{Z}(\mathsf{e}_i - \mathsf{e}_k)$, respectively. Then the balancing condition for r says

$$\sum_{\sigma \in \Sigma^2 : \sigma \supset r} w(\sigma) u_{\sigma/r} = \sum_{j \neq i,k} \overline{(\mathsf{e}_i - \mathsf{e}_j)} + \sum_{j \neq i,k} \overline{(\mathsf{e}_j - \mathsf{e}_k)} = (n-2)\overline{(\mathsf{e}_i - \mathsf{e}_k)} = 0,$$

as desired. It follows that $S(A_{n-1})$ is indeed a tropical surface. □

We can now state and prove our main result.

Theorem 10. *The degree of the tropical root surface $S(A_{n-1})$ is $\frac{1}{2}n(n-1)(n-2)$.*

Proof. We follow the approach outlined at the end of Section 2.2, studying the intersection of $v + S(A_{n-1})$ with $\Sigma_{n,n-2}$, where v is the *superincreasing* translation vector

$$v = (0, 1, 10, 100, 1000, \ldots).$$

It can be verified that this vector is generic by adding a small vector ϵ to it, and verifying that the intersection of $v + S(A_{n-1})$ with $\Sigma_{n,n-2}$, described below, has the same combinatorial structure as the intersection of $(v + \epsilon) + S(A_{n-1})$ with $\Sigma_{n,n-2}$.

(A) First we find the intersection points of $v + S(A_{n-1})$ and $\Sigma_{n,n-2}$.

For each 2-cone $\sigma \in S(A_{n-1})$ we need to find the points $v + s$ for $s \in \sigma$ whose three smallest entries I are equal, so that $v + s \in C_I \subseteq \Sigma_{n,n-2}$ by (1). The maximal cones of $S(A_{n-1})$ are of the form $\mathcal{A}_{i,jk}$ and $\mathcal{A}_{ij,k}$ for i, j, k pairwise distinct, as defined in (2). We consider these two types of cones separately.

(A1) Let us find the intersection points of $v + \mathcal{A}_{i,jk}$ and $\Sigma_{n,n-2}$ for i, j, k pairwise distinct.

Let $s = a(\mathsf{e}_i - \mathsf{e}_j) + b(\mathsf{e}_i - \mathsf{e}_k) = (a+b)\mathsf{e}_i - a\mathsf{e}_j - b\mathsf{e}_k \in \mathcal{A}_{i,jk}$ for $a, b \geq 0$. To make the three smallest entries of $v + s$ equal, we need to choose one entry i of $v = (0, 1, 10, 100, \ldots)$ to add $(a+b)$ to, and two entries j and k to subtract a and b from, respectively. Let $m = \min_{1 \leq i \leq n}(v + s)_i$ be the smallest entry of $v + s$, which appears at least three times. This places constraints on a and b, as well as i, j, and k, as we now explain in detail. Consider the following cases.

Case 1.1. $m < 0$. To achieve this minimum we would have to subtract from at least 3 entries of v. Since we can only subtract from 2 entries, this case does not contribute any intersection points.

Case 1.2. $m = 0$. To achieve $m = 0$, we must leave entry $v_1 = 0$ unchanged, subtract from any two other entries $j, k > 1$ (necessarily subtracting $a = v_j = 10^{j-2}$ and $b = v_k = 10^{k-2}$), and add (necessarily $a+b$) to any of the remaining entries $i \neq 1, j, k$. The set of minimal coordinates is $I = \{1, j, k\}$. There are $\binom{n-1}{2}(n-3)$ possible intersection points in this case.

Case 1.3. $0 < m < 1$. To achieve such a value of m, we would have to add $a+b = m$ to v_1 and subtract $a = 10^{j-2} - m$ and $b = 10^{k-2} - m$ to two other entries $j, k > 1$. This would imply that $10^{j-2} + 10^{k-2} = 3m$, which is impossible because the left hand side is at least 11 and the right hand side is less than 3. Thus this case does not contribute any intersection points.

Case 1.4. $m = 1$. In this case we must add at least 1 to entry $v_1 = 0$, leave entry $v_2 = 1$ unchanged, and subtract from two other entries $j, k > 2$. At least one of those two entries, say j, must lead to a minimum coordinate $(v+s)_j = 1$, so $a = 10^{j-2} - 1$. This means that $(v+s)_1 = a+b > 1$ is not a minimum coordinate, so $(v+s)_k = 1$ must be the other minimum coordinate, and $b = 10^{k-2} - 1$. The set of minimal coordinates is $I = \{2, j, k\}$. There are $\binom{n-2}{2}$ possible intersection points in this case.

Case 1.5. $m > 1$. In this case we would have to add to the entries $v_1 = 0$ and $v_2 = 1$ to make them greater than or equal to m. Since we can only add to one entry, this case does not contribute any intersection points.

(A2) Now let us find the intersection points of $v + \mathcal{A}_{ij,k}$ and $\Sigma_{n,n-2}$ for i, j, k pairwise distinct.

Let $s = a(\mathsf{e}_i - \mathsf{e}_k) + b(\mathsf{e}_j - \mathsf{e}_k) = a\mathsf{e}_i + b\mathsf{e}_j - (a+b)\mathsf{e}_k \in \mathcal{A}_{ij,k}$ for $a, b \geq 0$. To make the three smallest entries of $v+s$ equal to each other, we need to choose two entries i and j of $v = (0, 1, 10, 100, \ldots)$ to add a and b to, respectively, and one entry k to subtract $(a+b)$ from. Let $m = \min_{1 \leq i \leq n}(v+s)_i$ be the smallest coordinate of $v+s$, which appears at least three times. Consider the following cases:

Case 2.1. $m < 1$. To achieve this value of m we would need to subtract from two of the original entries of v, which is impossible. Thus, this case does not contribute any intersection points.

Case 2.2. $m = 1$. A value of $m = 1$ can only be achieved in entries $1, 2, k$ of $v+s$ for some $k > 2$. We must add $a = 1$ to v_1, leave v_2 unchanged, subtract $a+b = 10^{k-2} - 1$ from v_k, and hence add $b = 10^{k-2} - 2$ to some other entry $j \neq 1, 2, k$. The set of minimal coordinates is $I = \{1, 2, k\}$. This case contributes $(n-2)(n-3)$ intersection points.

Case 2.3. $1 < m < 10$. Again, such a value of m can only be achieved in entries $1, 2, k$ of $v+s$ for some $k > 2$. Now, for these three new entries to equal m, we must add $a = m$ to v_1, add $b = m-1$ to v_2, and subtract $a+b = 10^{k-2} - m$ from v_k. This forces $m + (m-1) = 10^{k-2} - m$, which gives $m = \frac{1}{3}(10^{k-2} + 1)$. Since $1 < m < 10$, we must have $k = 3$. Thus, the set of minimal coordinates is $I = \{1, 2, 3\}$, and this case contributes a single intersection point.

Case 2.4. $m = 10$. In order to make the three smallest entries of $v+s$ equal to 10, we have the following three options. (1) Add $a = 10$ to v_1, add $b = 9$ to v_2, leave v_3 untouched, and subtract 19 from any of the remaining entries $k > 3$. Here $I = \{1, 2, k\}$. (2) Add $a = 10$ to v_1, subtract $a + b = 10^{k-1} - 10$ from v_k with $k > 3$, and add $b = 10^{k-1} - 20$ to v_2. Here $I = \{1, 3, k\}$. (3) Add $a = 9$ to v_2, subtract $a + b = 10^{k-1} - 10$ from v_k with $k > 3$, and add $b = 10^{k-1} - 19$ to v_1. Here $I = \{2, 3, k\}$. In each of these options k can be any number between 4 and n. This case contributes $3(n-3)$ intersection points.

Case 2.5. $m > 10$. To achieve this value of m, we would need to add to the three smallest entries of v, which is impossible. Thus, this case does not contribute any intersection points.

(B) We now find the multiplicity of each of the intersection points p that we found in (A).

In each case we have identified the cones of $v + S(A_{n-1})$ and $\Sigma_{n,n-2}$ that the intersection $p = v + s$ belongs to. We now compute the multiplicity as the absolute value of the determinant of the matrix whose columns are lattice bases for the planes generated by these cones.

Suppose that p is the intersection point of cone $v + \mathcal{A}_{i,jk}$ (or $v + \mathcal{A}_{ij,k}$) of $v + S(A_{n-1})$ and cone C_I of $\Sigma_{n,n-2}$, where $I = \{i_{n-2}, i_{n-1}, i_n\}$ and $[n] - I = \{i_1, \dots, i_{n-3}\}$, as described in (2) and Definition 6. Then the index of that intersection is

$$\begin{aligned} \operatorname{index}(p) &= |\det(\mathsf{e}_{i_1}, \mathsf{e}_{i_2}, \mathsf{e}_{i_3} \dots, \mathsf{e}_{i_{n-3}}, \mathsf{e}_i - \mathsf{e}_j, \mathsf{e}_i - \mathsf{e}_k, \mathsf{e}_{[n]})| \\ &= |\det(\mathsf{e}_{i_1}, \mathsf{e}_{i_2}, \mathsf{e}_{i_3} \dots, \mathsf{e}_{i_{n-3}}, \mathsf{e}_{i_{n-2}} + \mathsf{e}_{i_{n-1}} + \mathsf{e}_{i_n}, \mathsf{e}_i - \mathsf{e}_j, \mathsf{e}_i - \mathsf{e}_k)|. \end{aligned}$$

The analogous result holds for the cones of the form $v + \mathcal{A}_{ij,k}$.[2]

(B1) Case 1.2. In this case we have $I = \{1, j, k\}$ so

$$\begin{aligned} \operatorname{index}(p) &= |\det(\widehat{\mathsf{e}_1}, \mathsf{e}_2, \mathsf{e}_3, \dots, \mathsf{e}_i, \dots, \widehat{\mathsf{e}_j}, \dots, \widehat{\mathsf{e}_k}, \dots, \mathsf{e}_n, \mathsf{e}_1 + \mathsf{e}_j + \mathsf{e}_k, \mathsf{e}_i - \mathsf{e}_j, \mathsf{e}_i - \mathsf{e}_k)| \\ &= |\det(\widehat{\mathsf{e}_1}, \mathsf{e}_2, \mathsf{e}_3, \dots, \mathsf{e}_i, \dots, \widehat{\mathsf{e}_j}, \dots, \widehat{\mathsf{e}_k}, \dots, \mathsf{e}_n, \mathsf{e}_1 + \mathsf{e}_j + \mathsf{e}_k, \mathsf{e}_j, \mathsf{e}_k)| \\ &= |\det(\widehat{\mathsf{e}_1}, \mathsf{e}_2, \mathsf{e}_3, \dots, \mathsf{e}_i, \dots, \widehat{\mathsf{e}_j}, \dots, \widehat{\mathsf{e}_k}, \dots, \mathsf{e}_n, \mathsf{e}_1, \mathsf{e}_j, \mathsf{e}_k)| \\ &= 1, \end{aligned}$$

where in the first step we subtract column e_i from columns $\mathsf{e}_i - \mathsf{e}_j$ and $\mathsf{e}_i - \mathsf{e}_k$ and change their signs, and in the second step we subtract e_j and e_k from $\mathsf{e}_1 + \mathsf{e}_j + \mathsf{e}_k$. Thus all intersections in this case have multiplicity 1.

Case 1.4. Here $I = \{2, j, k\}$ and $i = 1$ so

$$\begin{aligned} \operatorname{index}(p) &= |\det(\mathsf{e}_1, \widehat{\mathsf{e}_2}, \mathsf{e}_3, \dots, \widehat{\mathsf{e}_j}, \dots, \widehat{\mathsf{e}_k}, \dots, \mathsf{e}_n, \mathsf{e}_2 + \mathsf{e}_j + \mathsf{e}_k, \mathsf{e}_1 - \mathsf{e}_j, \mathsf{e}_1 - \mathsf{e}_k)| \\ &= |\det(\mathsf{e}_1, \widehat{\mathsf{e}_2}, \mathsf{e}_3, \dots, \widehat{\mathsf{e}_j}, \dots, \widehat{\mathsf{e}_k}, \dots, \mathsf{e}_n, \mathsf{e}_2 + \mathsf{e}_j + \mathsf{e}_k, \mathsf{e}_j, \mathsf{e}_k)| \\ &= |\det(\mathsf{e}_1, \widehat{\mathsf{e}_2}, \mathsf{e}_3, \dots, \widehat{\mathsf{e}_j}, \dots, \widehat{\mathsf{e}_k}, \dots, \mathsf{e}_n, \mathsf{e}_2, \mathsf{e}_j, \mathsf{e}_k)| \\ &= 1. \end{aligned}$$

It follows that each intersection in case 1.4 has multiplicity 1.

[2] We need the additional n-th entry $\mathsf{e}_{[n]}$ in these formulas because the $n-1$ generating vectors live in $\mathbb{R}^n / \mathbb{R}\mathsf{e}_{[n]}$, and we wish to regard them as vectors in $\mathbb{R}^n$ in order to compute their determinant.

(B2) Case 2.2. Here $I = \{1, 2, k\}$ and $i = 1$ so

$$\begin{aligned}\text{index}(p) &= |\det(\widehat{e_1}, \widehat{e_2}, e_3, e_4, \ldots, \widehat{e_k}, \ldots, e_n, e_1 + e_2 + e_k, e_1 - e_k, e_j - e_k)| \\ &= |\det(\widehat{e_1}, \widehat{e_2}, e_3, e_4, \ldots, \widehat{e_k}, \ldots, e_n, e_1 + e_2 + e_k, e_1 - e_k, e_k)| \\ &= |\det(\widehat{e_1}, \widehat{e_2}, e_3, e_4, \ldots, \widehat{e_k}, \ldots, e_n, e_1 + e_2 + e_k, e_1, e_k)| \\ &= |\det(\widehat{e_1}, \widehat{e_2}, e_3, e_4, \ldots, \widehat{e_k}, \ldots, e_n, e_2, e_1, e_k)| \\ &= 1,\end{aligned}$$

where we first subtract e_j from $e_j - e_k$ and change the sign to e_k, then we add e_k to $e_1 - e_k$, and then we subtract e_1 and e_k from $e_1 + e_2 + e_k$. Again, it follows that each one of these intersections has multiplicity 1.

Case 2.3. In this case $I = \{1, 2, 3\}$ and $i = 1$, $j = 2, k = 3$, so

$$\begin{aligned}\text{index}(p) &= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \ldots, e_n, e_1 + e_2 + e_3, e_1 - e_3, e_2 - e_3)| \\ &= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \ldots, e_n, 3e_3, e_1 - e_3, e_2 - e_3)| \\ &= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \ldots, e_n, 3e_3, e_1, e_2)| \\ &= 3,\end{aligned}$$

where we first subtract $e_1 - e_3$ and $e_2 - e_3$ from $e_1 + e_2 + e_3$, and then we add one third of $3e_3$ to $e_1 - e_3$ and $e_2 - e_3$. Thus this intersection has multiplicity 3.

Case 2.4. Here we had three options: In option 1 we had $I = \{1, 2, 3\}$ and $i = 1$, $j = 2$, so

$$\begin{aligned}\text{index}(p) &= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \ldots, e_k, \ldots, e_n, e_1 + e_2 + e_3, e_1 - e_k, e_2 - e_k)| \\ &= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \ldots, e_k, \ldots, e_n, e_1 + e_2 + e_3, e_1, e_2)| \\ &= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \ldots, e_k, \ldots, e_n, e_3, e_1, e_2)| \\ &= 1,\end{aligned}$$

where we first add e_k to $e_1 - e_k$ and $e_2 - e_k$, and then subtract e_1 and e_2 from $e_1 + e_2 + e_3$. These intersections then have multiplicity 1.

In option 2 we had $I = \{1, 3, k\}$ and $i = 1$, $j = 2$, so

$$\begin{aligned}\text{index}(p) &= |\det(\widehat{e_1}, e_2, \widehat{e_3}, e_4, \ldots, \widehat{e_k}, \ldots, e_n, e_1 + e_3 + e_k, e_1 - e_k, e_2 - e_k)| \\ &= |\det(\widehat{e_1}, e_2, \widehat{e_3}, e_4, \ldots, \widehat{e_k}, \ldots, e_n, e_1 + e_3 + e_k, e_1 - e_k, e_k)| \\ &= |\det(\widehat{e_1}, e_2, \widehat{e_3}, e_4, \ldots, \widehat{e_k}, \ldots, e_n, e_1 + e_3 + e_k, e_1, e_k)| \\ &= |\det(\widehat{e_1}, e_2, \widehat{e_3}, e_4, \ldots, \widehat{e_k}, \ldots, e_n, e_3, e_1, e_k)| \\ &= 1,\end{aligned}$$

where we first subtract e_2 from $e_2 - e_k$ and change the sign of the result, then add e_k to $e_1 - e_k$, and finally subtract e_1 and e_k from $e_1 + e_3 + e_k$. These intersections then have multiplicity 1.

Option 3 is analogous to option 2, reversing the roles of 1 and 2, so these intersections have multiplicity 1 as well.

case	number of intersections	intersection multiplicity	contribution to degree
1.1	0	—	0
1.2	$\binom{n-1}{2}(n-3)$	1	$\frac{1}{2}(n-1)(n-2)(n-3)$
1.3	0	—	0
1.4	$\binom{n-2}{2}$	1	$\frac{1}{2}(n-2)(n-3)$
1.5	0	—	0
2.1	0	—	0
2.2	$(n-2)(n-3)$	1	$(n-2)(n-3)$
2.3	1	3	3
2.4	$3(n-3)$	1	$3(n-3)$
2.5	0	—	0

Table 1. Intersection points of $v+S(A_{n-1})$ and $\Sigma_{n,n-2}$ with their multiplicities.

(C) Finally, we collect in Table 1 all the intersections points and their multiplicities, as computed in (A) and (B). Putting them together, we conclude that the degree of the tropical root surface of type A_{n-1} is

$$\deg S(A_{n-1}) = \binom{n-1}{2}(n-3) + \binom{n-2}{2} + (n-2)(n-3) + 3 + 3(n-3) = \tfrac{1}{2}n(n-1)(n-2),$$

as desired. □

4. Remarks and future work

Nayeong Kim computed the degrees of the tropical surfaces of the remaining classical root systems [Kim 2023]. It is natural to ask whether these tropical surfaces can be obtained as tropicalizations of classical varieties. This is a subtle question: the surface $S(A_{n-1})$ *is* a tropicalization of a projective variety, but we do not know whether that is the case for any root system. This will be explained in a future paper.

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References

[Ardila and Klivans 2006] F. Ardila and C. J. Klivans, "The Bergman complex of a matroid and phylogenetic trees", *J. Combin. Theory Ser. B* **96**:1 (2006), 38–49. MR Zbl

[Ardila et al. 2011] F. Ardila, M. Beck, S. Hoşten, J. Pfeifle, and K. Seashore, "Root polytopes and growth series of root lattices", *SIAM J. Discrete Math.* **25**:1 (2011), 360–378. MR Zbl

[Ardila et al. $\geq$ 2025] F. Ardila, C. Kato, J. McMillon, M. I. Perez, and A. Schindler, "Spectra of tropical Laplacians of classical root polytopes", in preparation.

[Babaee and Huh 2017] F. Babaee and J. Huh, "A tropical approach to a generalized Hodge conjecture for positive currents", *Duke Math. J.* **166**:14 (2017), 2749–2813. MR Zbl

[Bourbaki 1968] N. Bourbaki, *Groupes et algèbres de Lie*: *Chapitres* 4 *à* 6, Hermann, Paris, 1968. MR Zbl

[Brugallé and Shaw 2014] E. Brugallé and K. Shaw, "A bit of tropical geometry", *Amer. Math. Monthly* **121**:7 (2014), 563–589. MR Zbl

[Cellini and Marietti 2015] P. Cellini and M. Marietti, "Root polytopes and Borel subalgebras", *Int. Math. Res. Not.* **2015**:12 (2015), 4392–4420. MR Zbl

[Cordero 2022] M. Cordero Aguilar, *The tropical degree of a tropical root surface*, master's thesis, San Francisco State University, 2022.

[Fink 2013] A. Fink, "Tropical cycles and Chow polytopes", *Beitr. Algebra Geom.* **54**:1 (2013), 13–40. MR Zbl

[Kim 2023] N. Kim, *Degrees of tropical root surfaces of classical root systems*, master's thesis, San Francisco State University, 2023.

[Maclagan and Sturmfels 2015] D. Maclagan and B. Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics **161**, American Mathematical Society, Providence, RI, 2015. MR Zbl

[Mikhalkin and Rau 2010] G. Mikhalkin and J. Rau, "Tropical geometry", notes, 2010, https://www.math.uni-tuebingen.de/user/jora/downloads/main.pdf.

[Perez 2019] M. I. Perez, *Spectra of tropical Laplacians of classical root polytopes*, master's thesis, San Francisco State University, 2019, http://hdl.handle.net/10211.3/213976.

[Schindler 2017] A. M. Schindler, *Algebraic and combinatorial aspects of two symmetric polytopes*, master's thesis, San Francisco State University, 2017, http://hdl.handle.net/10211.3/197035.

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