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# Semisimplification of contragredient Lie algebras

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We describe the structure and different features of Lie algebras in the Verlinde category, obtained as semisimplification of contragredient Lie algebras in characteristic $p$ with respect to the adjoint action of a Chevalley generator. In particular, we construct a root system for these algebras that arises as a parabolic restriction of the known root system for the classical Lie algebra. This gives a lattice grading with simple homogeneous components and a triangular decomposition for the semisimplified Lie algebra. We also obtain a nondegenerate invariant form that behaves well with the lattice grading. As an application, we exhibit concrete new examples of Lie algebras in the Verlinde category.



## 1. Introduction

In recent times, symmetric tensor categories have attracted significant attention. The work of Deligne [1990; 2002] has been a foundational stone for this theory, and it is the major influence in the modern development of the subject. By Deligne's results, every pre-Tannakian category (that is, a symmetric tensor category with objects of finite length) of moderate growth over an algebraically closed field $\Bbbk$ of characteristic zero admits a fiber functor to $\mathsf{sVec}_{\Bbbk}$, the category of finite-dimensional super vector spaces. In other words, any such category is equivalent to the category of representations of a (pro)algebraic supergroup.

Over a field $\Bbbk$ of characteristic $p > 0$ the situation is completely different. For example, the Verlinde category $\mathsf{Ver}_p$, introduced in [Gelfand and Kazhdan 1992; Georgiev and Mathieu 1992] as the semisimplification of $\mathsf{Rep}(\mathbb{Z}_p)$, does not admit a fiber functor to $\mathsf{sVec}_{\Bbbk}$. Ostrik [2020] initiated the quest of providing analogs to Deligne's theorem in positive characteristic. Ostrik's main result states that, in characteristic $p$, any pre-Tannakian category which is fusion (thus, of moderate growth) admits a fiber functor to $\mathsf{Ver}_p$. This result opens the door to different directions of research concerning pre-Tannakian categories of moderate growth, such as the notion of incompressible tensor categories [Benson et al. 2023; Coulembier et al. 2023a], necessary and sufficient conditions for such a category to fiber over $\mathsf{Ver}_p$ [Coulembier et al.

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2023b], and the study of affine group schemes in $\mathsf{Ver}_p$ [Venkatesh 2022; Coulembier 2023]. This work focuses on the latter direction, as we study Lie algebras in $\mathsf{Ver}_p$.

Finite-dimensional Lie superalgebras over fields of characteristic zero were classified by Kac [1977] in the seventies. A distinguished class in the classification is that of contragredient Lie algebras. These are Lie algebras defined from a matrix by generators and relations, slightly generalizing the construction of Kac–Moody algebras but still preserving many desired properties, such as a triangular decomposition [Kac 1990]. This family includes analogs of classical Lie algebras as well as special, orthogonal, and symplectic Lie superalgebras, and some exceptions in low rank. Once again the situation is a bit more complicated for characteristic $p > 0$. In this context, the classification of finite-dimensional contragredient Lie superalgebras was achieved in [Bouarroudj et al. 2009] and contains several exceptions when $p = 3, 5$, partially constructed *by hand*.

A possible explanation for the existence of some of the exceptions comes from the *super magic square* [Cunha and Elduque 2007], which generalizes the Freudenthal magic square for Lie algebras. More recently a different approach was given by Kannan [2022], who constructed these exceptional Lie superalgebras using the modern theory of symmetric tensor categories in positive characteristic, see also [Daza-García et al. 2024] for an explanation of the connection between these two constructions. More precisely, Kannan showed that all the exotic examples can be obtained via the semisimplification process. The starting point is the realization of $\mathsf{Ver}_p$ as the semisimplification of the representation category for the commutative Hopf algebra $\Bbbk[t]/(t^p)$, where $t$ is primitive. Thus, there is a semisimplification functor $\mathsf{Rep}(\Bbbk[t]/(t^p)) \to \mathsf{Ver}_p$ which is symmetric. In this way, one can obtain Lie algebras in $\mathsf{Ver}_p$ by applying that semisimplification functor to Lie algebras in $\mathsf{Rep}(\Bbbk[t]/(t^p))$. Now, a Lie algebra in the latter category is just a pair $(\mathfrak{g}, \partial)$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\partial$ is a derivation of $\mathfrak{g}$ such that $\partial^p = 0$. In characteristic $p = 3$, we have $\mathsf{Ver}_3 = \mathsf{sVec}_\Bbbk$ so the process always gives a Lie superalgebra. For $p = 5$ (and higher), we have a factorization $\mathsf{Ver}_5 = \mathsf{sVec}_\Bbbk \boxtimes \mathsf{Ver}_5^+$ as symmetric categories, thus the semisimplification procedure still gives Lie superalgebras after projecting to $\mathsf{sVec}_\Bbbk$. Applying this construction to particular pairs $(\mathfrak{g}, \partial)$, where $\mathfrak{g}$ is a classical Lie algebra and $\partial$ is a certain inner derivation, Kannan recovered all exceptional examples, which only exist in characteristics 3 and 5.

However, the semisimplification process for general $p$ is, at least in principle, a possible source of new examples of Lie algebras in $\mathsf{Ver}_p$, not necessarily supported in $\mathsf{sVec}_\Bbbk$, which are much desired. That idea is the starting point of this work. We study the structure of Lie algebras in $\mathsf{Ver}_p$ obtained as the semisimplification of a pair $(\mathfrak{g}, \partial)$, where $\mathfrak{g}$ is a contragredient Lie algebra and $\partial$ is an inner derivation associated to a homogeneous element with respect to the lattice grading. This homogeneity assumption is mild enough to still produce new examples of Lie algebras in $\mathsf{Ver}_p$, yet simultaneously manageable to assure nice features on these algebras. Most importantly, we obtain a grading by a suitable free abelian group which resembles that of Lie superalgebras. Indeed, contragredient Lie superalgebras admit a grading which leads to (generalized) root systems in the sense of [Heckenberger and Yamane 2008; Heckenberger and Schneider 2020]; see, e.g., [Andruskiewitsch and Angiono 2017]. In our case, we induce a grading on the semisimplified Lie algebra in $\mathsf{Ver}_p$ coming from that on the original Lie algebra $\mathfrak{g}$: because of the choice of a homogeneous element, the *root system* on the Lie algebra in $\mathsf{Ver}_p$ is the parabolic restriction (in the sense of [Cuntz and Lentner 2017]) of the original one. One may wonder which other Lie algebras in $\mathsf{Ver}_p$ have root systems. This is the content of a forthcoming paper.

The organization of the paper is the following. In Section 2 we recall the construction of contragredient Lie superalgebras since they appear along the work in two ways: the particular case of Lie algebras is a source of examples to being semisimplified, and Lie superalgebras, mainly in characteristic 3, are the images of the semisimplification functor. In particular, we recall the notion of root system, some properties, and the parabolic restriction from [Cuntz and Lentner 2017]. Later, in Section 3, we recall the notions of symmetric tensor categories, Lie algebras on these categories, the semisimplification functor, and the Verlinde category $\mathsf{Ver}_p$. The main results are part of Section 4. Here we consider a contragredient Lie algebra $\mathfrak{g}(A)$ attached to a matrix $A$ and a homogeneous element $x$. Up to an isomorphism of $\mathfrak{g}(A)$, we may assume that $x = e_i$, a generator of the *positive* part of $\mathfrak{g}(A)$. We then study this case in detail, first by understanding the structure of $\mathfrak{g}(A)$ as a module over $\Bbbk[t]/(t^p)$, and later by studying its image under the semisimplification functor. If $A$ is of rank $\theta$, it is known that $\mathfrak{g}(A)$ is $\mathbb{Z}^\theta$-graded. We show that the semisimplification of $\mathfrak{g}(A)$ is $\mathbb{Z}^{\theta-1}$-graded, where the grading comes from the projection which annihilates the $i$-th entry and gives consequently a grading coming from the parabolic restriction of the root system of $\mathfrak{g}(A)$. We finish the paper with some explicit examples in low rank.

## 2. Contragredient Lie superalgebras

**2A.** ***Conventions.*** We denote by $\mathbb{N}$ the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given $\theta \in \mathbb{N}$, let $\mathbb{I}_\theta = \{1, \dots, \theta\}$. If $\theta$ is implicit we just write $\mathbb{I} = \mathbb{I}_\theta$; in this case, we make no distinction between $\mathbb{Z}^{\mathbb{I}}$ and $\mathbb{Z}^\theta$. The canonical basis of $\mathbb{Z}^{\mathbb{I}}$ is $(\alpha_i)_{i\in\mathbb{I}}$; we use the expression $1^{a_1} \dots \theta^{a_\theta}$ to denote $a_1\alpha_1 + \cdots + a_\theta\alpha_\theta \in \mathbb{Z}^{\mathbb{I}}$. For each $\beta = 1^{a_1} \dots \theta^{a_\theta} \in \mathbb{N}_0^\theta$, the height of $\beta$ is $\mathsf{ht}(\beta) := a_1 + \cdots + a_\theta \in \mathbb{N}_0$.

We work over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. We denote by $\mathsf{Vec}_\Bbbk$ (respectively $\mathsf{sVec}_\Bbbk$) the symmetric fusion category of finite dimensional $\Bbbk$-vector spaces, (respectively super vector spaces).

**2B.** ***Root systems and Weyl groupoids.*** We recall here the notions of root systems and Weyl groupoids from [Heckenberger and Schneider 2020, Chapter 9], a generalization of the classical definitions of root systems and Weyl groups for Lie algebras.

Fix a natural number $\theta$ and a nonempty set $\mathcal{X}$. A *semi-Cartan graph* of *rank* $\theta$ and a set of *points* $\mathcal{X}$ is a quadruple $\mathcal{G} := \mathcal{G}(\mathbb{I}, \mathcal{X}, (A^X)_{X\in\mathcal{X}}, (r_i)_{i\in\mathbb{I}})$, where

- for each $i \in \mathbb{I}$, $r_i : \mathcal{X} \to \mathcal{X}$ is a function such that $r_i^2 = \mathrm{id}_\mathcal{X}$;
- $A^X = (a_{ij}^X)_{i,j\in\mathbb{I}} \in \mathbb{Z}^{\theta\times\theta}$ is a generalized Cartan matrix [Kac 1990] for all $X \in \mathcal{X}$, and the following identities hold:

$$a_{ij}^X = a_{ij}^{r_i(X)}, \quad \text{for all } X \in \mathcal{X},\ i, j \in \mathbb{I}. \tag{2.1}$$

The exchange graph of $\mathcal{G}$ is a graph with $\mathcal{X}$ as set of vertices, and an arrow labeled with $i \in \mathbb{I}$ between $X$ and $r_i(X)$ if $X \neq r_i(X)$, for each $i \in \mathbb{I}$ and $X \in \mathcal{X}$. An example of an exchange graph and semi-Cartan graph for $i = 2$ and $\mathcal{X} = \{X_1, X_2, X_3\}$ is

$$\underset{X_1}{\circ} \overset{1}{\text{———}} \underset{X_2}{\circ} \overset{2}{\text{———}} \underset{X_3}{\circ}, \quad A^{X_1} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \quad A^{X_2} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}, \quad A^{X_3} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}.$$

In this example, the reflections $r_i$ are given by

$$r_1(1) = 2, \quad r_1(2) = 1, \quad r_1(3) = 3, \quad r_2(1) = 1, \quad r_2(2) = 3, \quad r_2(3) = 2.$$

Given a monoid $\mathcal{M}$, we may consider the small category $\mathcal{D}(\mathcal{X}, \mathcal{M})$ whose set of objects is $\mathcal{X}$ and the set of morphisms between any two objects is $\mathcal{M}$. We use the notation

$$\mathsf{Hom}(X, Y) = \{(Y, f, X) \mid f \in \mathcal{M}\} \quad \text{for each pair } X, Y \in \mathcal{X}.$$

The composition is then written as

$$(Z, f, Y) \circ (Y, g, X) = (Z, fg, X) \quad \text{for any } X, Y, Z \in \mathcal{X},\ f, g \in \mathcal{M}.$$

The Weyl groupoid $\mathcal{W} := \mathcal{W}(\mathbb{I}, \mathcal{X}, (A^x)_{x \in \mathcal{X}}, (r_i)_{i \in \mathbb{I}})$ of the semi-Cartan graph $\mathcal{G}$ is defined as the full subcategory of $\mathcal{D}(\mathcal{X}, \mathrm{GL}(\mathbb{Z}^\theta))$ generated by

$$\sigma_i^X := (r_i(X), s_i^X, X), \quad i \in \mathbb{I},\ X \in \mathcal{X},$$

where $s_i^X \in \mathrm{GL}(\mathbb{Z}^\theta)$ is given by $s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i$. Notice that $\sigma_i^{r_i(X)} \sigma_i^X = (X, \mathrm{id}_X, X)$ for all $i \in \mathbb{I}$ and $X \in \mathcal{X}$, so $\mathcal{W}$ is indeed a groupoid.

For a semi-Cartan graph $\mathcal{G}$ as above, we define the set of *real roots* of $\mathcal{G}$ at $X \in \mathcal{X}$ as

$$\Delta^{X,\mathrm{re}} := \big\{ w(\alpha_i) \in \mathbb{Z}^\theta \mid i \in \mathbb{I},\ Y \in \mathcal{X},\ (X, w, Y) \in \mathsf{Hom}_{\mathcal{W}}(X, Y) \big\}.$$

We say that $\mathcal{G}$ is *finite* if $\Delta^{X,\mathrm{re}}$ is finite for all $X \in \mathcal{X}$ (equivalently, for some $X \in \mathcal{X}$).

The set of positive and negative real roots are, respectively,

$$\Delta_+^{X,\mathrm{re}} := \Delta^{X,\mathrm{re}} \cap \mathbb{N}_0^\theta, \quad \Delta_-^{X,\mathrm{re}} := \Delta^{X,\mathrm{re}} \cap (-\mathbb{N}_0^\theta).$$

Given $X \in \mathcal{X}$, $i \neq j \in \mathbb{I}$, we set $m_{ij}^X := |\Delta^{X,\mathrm{re}} \cap (\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j)| \in \mathbb{N} \cup \{\infty\}$. We say that a semi-Cartan graph $\mathcal{G}$ is a *Cartan graph* if in addition the following hold:

- For all $X \in \mathcal{X}$, $\Delta^{X,\mathrm{re}} = \Delta_+^{X,\mathrm{re}} \cup \Delta_-^{X,\mathrm{re}}$.
- For all $X \in \mathcal{X}$, $i \neq j \in \mathbb{I}$ such that $m_{ij}^X < \infty$, we have that $(r_i r_j)^{m_{ij}^X}(X) = (X)$.

A *root system* over $\mathcal{G}$ is a family $\mathcal{R} = (\Delta^X)_{X \in \mathcal{X}}$ of subsets $\Delta^X \subset \mathbb{Z}^\theta$ such that

$$0 \notin \Delta^X, \quad \alpha_i \in \Delta^X, \quad \Delta^X \subset \mathbb{N}_0^\theta \cup (-\mathbb{N}_0^\theta), \quad s_i^X(\Delta^X) = \Delta^{r_i(X)},$$

for all $i \in \mathbb{I}$ and all $X \in \mathcal{X}$. Positive and negative roots are defined as usual, and $\mathcal{R}$ is *finite* if every $\Delta^X$ is so. Also, $\mathcal{R}$ is *reduced* if $\mathbb{Z}\alpha \cap \Delta^X = \{\pm\alpha\}$ for all $\alpha \in \Delta^X$, $X \in \mathcal{X}$.

According to the definition, a Cartan graph $\mathcal{G}$ might support different root systems over it. But this is not the case when $\mathcal{G}$ is finite.

**Theorem 2.2** [Heckenberger and Schneider 2020, 10.4.7]. *If $\mathcal{G}$ is a finite Cartan graph, then $\mathcal{R} = (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}}$ is the only reduced root system over $\mathcal{G}$.*

Finite root systems are in correspondence with crystallographic arrangements: a subset of hyperplanes in a finite-dimensional $\mathbb{R}$-vector space satisfying certain properties. We refer to [Cuntz and Heckenberger 2015; Cuntz and Lentner 2017] for the precise definition and the correspondence.

The next result about root systems will be useful throughout the article.

**Theorem 2.3** [Cuntz and Heckenberger 2012, Theorem 2.4]. *Let $\gamma_1, \dots, \gamma_k \in \Delta^X_+$ be linearly independent roots. Then there exist $Y \in \mathcal{X}$, $w \in \mathsf{Hom}(X, Y)$, and $\sigma \in \mathbb{S}_{\mathbb{I}}$ such that the support of $w(\gamma_i) \in \Delta^Y_+$ is contained in $\{\sigma(1), \dots, \sigma(i)\}$ for each $1 \le i \le k$.* □

In other words, this result allows us to reduce computations on a set of $k$ linearly independent roots to computations on a root system of rank $k$, obtained as a subsystem of another object in the Weyl class. Moreover, notice that

$$w(\gamma_1) = \alpha_{\sigma(1)}, \quad \text{since } w(\gamma_1) \in \mathbb{Z}\alpha_{\sigma(1)} \cap \Delta^Y_+ = \{\alpha_{\sigma(1)}\}.$$

There are some standard constructions that produce new finite root systems from old ones, as described in [Cuntz and Lentner 2017]. More relevant for us is the restriction construction; at the level of arrangements, this process fixes a root and then projects all the hyperplanes in the arrangement onto its orthogonal component. We are interested in the description of the associated root system. First, we fix some notation:

- Let $i \in \mathbb{I}_\theta$. We denote by $\pi_i : \mathbb{Z}^\theta \to \mathbb{Z}^{\theta-1}$ the projection given by

$$\pi_i(a_1, \dots, a_\theta) = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_\theta), \quad a_i \in \mathbb{Z}.$$

- For each $\beta = (b_1, \dots, b_{\theta-1}) \in \mathbb{Z}^{\theta-1}$, let $\beta' = (1/\gcd(b_i \,|\, i \in \mathbb{I}_{\theta-1}))\beta$.

**Lemma 2.4** [Cuntz and Lentner 2017, 3.3]. *Let $\mathcal{R} = (\mathcal{C}, (\Delta^a)_{a\in A})$ be a finite root system of rank $\theta$. Fix $a \in A$ and $i \in \mathbb{I}_\theta$. Then $\bar{\Delta} := \{\pi_i(\alpha)' \mid \alpha \in \Delta^a\}$ is the set of roots corresponding to the restriction of the hyperplane arrangement of $\mathcal{R}$ to $\alpha_i^\perp$.* □

The classification of finite root systems was achieved in [Cuntz and Heckenberger 2015]. Roughly speaking, there exist families of arbitrary rank corresponding to Lie superalgebras and Lie algebras of types $A$, $B$, $C$, $D$, infinite examples in rank two corresponding to triangulations of $n$-gons, and several exceptions in ranks $3 \le \theta \le 8$. According to [Cuntz and Lentner 2017, Theorem 3.7], most of these root systems come from a classical one by restriction.

**2C. *Contragredient Lie superalgebras.*** Here we recall the construction of contragredient Lie superalgebras over $\Bbbk$ from [Andruskiewitsch and Angiono $\geq$ 2025; Bouarroudj et al. 2009; Kac 1990] and introduce notation. We fix the following *contragredient data*:

(D1) a matrix $A = (a_{ij}) \in \Bbbk^{\mathbb{I}\times\mathbb{I}}$;

(D2) a *parity vector* $\boldsymbol{p} = (p_i) \in \mathbb{G}_2^{\mathbb{I}}$ where $\mathbb{G}_2 = \{\pm 1\}$ is the (multiplicative) group with 2 elements;[1]

(D3) a $\Bbbk$-vector space $\mathfrak{h}$ of dimension $2\theta - \operatorname{rank} A$;

(D4) linearly independent subsets $(\xi_i)_{i\in\mathbb{I}} \subset \mathfrak{h}^*$ and $(h_i)_{i\in\mathbb{I}} \subset \mathfrak{h}$ *realizing* the matrix $A$, that is, $\xi_j(h_i) = a_{ij}$ for all $i, j \in \mathbb{I}$.

The set $(h_i)_{i\in\mathbb{I}}$ in (D4) is completed to a basis $(h_i)_{1\le i\le 2\theta-\operatorname{rank} A}$ of $\mathfrak{h}$. The Lie superalgebra $\widetilde{\mathfrak{g}} := \widetilde{\mathfrak{g}}(A, \boldsymbol{p})$ is presented by generators $e_i$, $f_i$, $i \in \mathbb{I}$, and $\mathfrak{h}$, with parity given by

$$|e_i| = |f_i| = |i|, \quad i \in \mathbb{I}, \qquad |h| = 0 \quad \text{for all } h \in \mathfrak{h},$$

subject to the relations, for all $i, j \in \mathbb{I}$, $h, h' \in \mathfrak{h}$,

$$[h, h'] = 0, \quad [h, e_i] = \xi_i(h)e_i, \quad [h, f_i] = -\xi_i(h) f_i, \quad [e_i, f_j] = \delta_{ij} h_i. \tag{2.5}$$

There is a unique $\mathbb{Z}$-grading $\widetilde{\mathfrak{g}} = \bigoplus_{k\in\mathbb{Z}} \widetilde{\mathfrak{g}}_k$ such that $e_i \in \widetilde{\mathfrak{g}}_1$, $f_i \in \widetilde{\mathfrak{g}}_{-1}$, $\mathfrak{h} = \widetilde{\mathfrak{g}}_0$. As usual, we denote $\widetilde{\mathfrak{n}}_+ = \bigoplus_{k>0} \widetilde{\mathfrak{g}}_k$ and $\widetilde{\mathfrak{n}}_- = \bigoplus_{k<0} \widetilde{\mathfrak{g}}_k$. The family of $\mathbb{Z}$-homogeneous ideals trivially intersecting $\mathfrak{h}$ admits a unique maximal ideal $\mathfrak{r}$; clearly $\mathfrak{r}$ splits as a sum of its positive and negative parts: $\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-$.

The *contragredient Lie superalgebra* associated to the pair $(A, \boldsymbol{p})$ is the Lie superalgebra quotient $\mathfrak{g}(A, \boldsymbol{p}) := \widetilde{\mathfrak{g}}(A, \boldsymbol{p})/\mathfrak{r}$. For sake of brevity, set $\mathfrak{g} := \mathfrak{g}(A, \boldsymbol{p})$, still denote by $e_i$, $f_i$, $h_i$ the images in $\mathfrak{g}$ of the generators for $\widetilde{\mathfrak{g}}$, and identify $\mathfrak{h}$ with its image in $\mathfrak{g}$. By homogeneity, the grading of $\widetilde{\mathfrak{g}}$ induces one in $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$; thus we have $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ as usual.

**Remark 2.6.** The Lie subsuperalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ admits a complement $\mathfrak{h}_{>\theta}$ such that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{h}_{>\theta}$, where $\mathfrak{h}_{>\theta}$ is the subspace spanned by all $h_j$ with $j > \theta$. In particular, $\mathfrak{g} = \mathfrak{g}'$ if $A$ is nondegenerate.

As in [Kannan 2022], it will be easier to work with $\mathfrak{g}'$ rather than $\mathfrak{g}$, because the former is generated by the Chevalley generators. We denote by $\mathfrak{h}_{\le\theta}$ the subspace of $\mathfrak{h}$ spanned by all $h_j$ with $j \le \theta$, thus we have $\mathfrak{g}' = \mathfrak{n}_+ \oplus \mathfrak{h}_{\le\theta} \oplus \mathfrak{n}_-$.

**Remark 2.7.** By [Bouarroudj et al. 2009], if a matrix $B$ is obtained from $A$ by rescaling some rows by nonzero scalars, then $\mathfrak{g}(A, \boldsymbol{p}) \simeq \mathfrak{g}(B, \boldsymbol{p})$. Following the convention adopted in [Hoyt and Serganova 2007], we always assume that $a_{ii} \in \{0, 2\}$. Given $i \in \mathbb{I}$, denote by $\mathfrak{g}_i$ the Lie subsuperalgebra of $\mathfrak{g}$ generated by $e_i$, $f_i$ and $h_i$. Hence, for $\mathfrak{g}_i$ we have four possibilities:

$a_{ii} = 2,\ p_i = 1:$ as usual, $\mathfrak{g}_i \simeq \mathfrak{sl}(2)$;

$a_{ii} = 0,\ p_i = 1:$ now $\mathfrak{g}_i$ is isomorphic to $\mathfrak{H}_3$, the Heisenberg algebra;

$a_{ii} = 2,\ p_i = -1:$ here $\mathfrak{g}_i \simeq \mathfrak{osp}(2)$;

$a_{ii} = 0,\ p_i = -1:$ in this case $\mathfrak{g}_i$ is isomorphic to $\mathfrak{sl}(1|1)$.

Now we recall some features of $\mathfrak{g}$ extracted from [Andruskiewitsch and Angiono $\ge$ 2025]:

[1] For our formulas it is more suitable to work with $\mathbb{G}_2$ rather than $\mathbb{Z}/2$.

(i) There exists an involution $\widetilde{\omega}$ of $\widetilde{\mathfrak{g}}$ such that

$$\widetilde{\omega}(e_i) = -f_i, \quad \widetilde{\omega}(f_i) = -p_i e_i, \quad \widetilde{\omega}(h) = -h, \qquad \text{for all } i \in \mathbb{I},\ h \in \mathfrak{h}. \tag{2.8}$$

Clearly $\widetilde{\omega}(\mathfrak{r}) = \mathfrak{r}$, so $\widetilde{\omega}$ gives rise to $\omega : \mathfrak{g} \to \mathfrak{g}$, the *Chevalley involution*.

(ii) The center $\mathfrak{c}$ of $\mathfrak{g}$ coincides with that of $\mathfrak{g}'$; we have $\mathfrak{c} = \{h \in \mathfrak{h} : \xi_i(h) = 0 \text{ for all } i \in \mathbb{I}\}$.

(iii) The unique $x \in \mathfrak{n}_+$ (respectively, $x \in \mathfrak{n}_-$) such that $[x, f_i] = 0$ (respectively, $[x, e_i] = 0$) for all $i \in \mathbb{I}$ is $x = 0$.

(iv) The Lie superalgebra $\widetilde{\mathfrak{g}}$ has a $\mathbb{Z}^{\mathbb{I}}$-grading determined by

$$\deg f_i = -\alpha_i, \quad \deg h = 0, \quad \deg e_i = \alpha_i, \qquad \text{for all } i \in \mathbb{I},\ h \in \mathfrak{h}.$$

One can show that $\mathfrak{r}$ is $\mathbb{Z}^{\mathbb{I}}$-homogeneous, hence $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbb{Z}^{\mathbb{I}}, \alpha \neq 0} \mathfrak{g}_\alpha$.

**Definition 2.9.** The set of roots of $(A, \boldsymbol{p})$ is $\nabla^{(A,\boldsymbol{p})} := \{\alpha \in \mathbb{Z}^{\mathbb{I}} - 0 : \mathfrak{g}_\alpha \neq 0\}$. The set of positive, respectively negative, roots is $\nabla_\pm^{(A,\boldsymbol{p})} = \nabla^{(A,\boldsymbol{p})} \cap (\pm\mathbb{N}_0^{\mathbb{I}})$.

Since the Chevalley involution satisfies $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for any $\alpha \in \mathbb{Z}^{\mathbb{I}}$, we get

$$\nabla^{(A,\boldsymbol{p})} = \nabla_+^{(A,\boldsymbol{p})} \cup \nabla_-^{(A,\boldsymbol{p})}, \quad \nabla_-^{(A,\boldsymbol{p})} = -\nabla_+^{(A,\boldsymbol{p})}. \tag{2.10}$$

**Remark 2.11.** Given contragredient data (D1)–(D4) and $\mathbb{J} \subset \mathbb{I}$, consider

- $A_{\mathbb{J}} := (a_{ij})_{i,j \in \mathbb{J}}$, $\boldsymbol{p}_{\mathbb{J}} := (p_i)_{i \in \mathbb{J}}$;
- $\mathfrak{g}_{\mathbb{J}}$ the subalgebra of $\mathfrak{g} = \mathfrak{g}(A, \boldsymbol{p})$ generated by $\mathfrak{h}$, $e_i$, $f_i$, for $i \in \mathbb{J}$;
- $\mathfrak{h}_{\mathbb{J}}$ the subspace of $\mathfrak{h}$ spanned by $h_i$, for $i \in \mathbb{J}$;
- $\mathfrak{h}'_{\mathbb{J}}$ the maximal subspace of $\bigcap_{i \in \mathbb{J}} \ker \xi_i$ trivially intersecting $\mathfrak{h}_{\mathbb{J}}$.

By [Hoyt and Serganova 2007, Lemma 2.1], there is an isomorphism $\mathfrak{g}(A_{\mathbb{J}}, \boldsymbol{p}_{\mathbb{J}}) \oplus \mathfrak{h}'_{\mathbb{J}} \cong \mathfrak{g}_{\mathbb{J}}$ which identifies the positive and negative parts of $\mathfrak{g}(A_{\mathbb{J}}, \boldsymbol{p}_{\mathbb{J}})$ with the subalgebras of $\mathfrak{g}(A, \boldsymbol{p})$ generated by $e_i$, respectively $f_i$, for $i \in \mathbb{J}$. In particular, we get $\nabla_\pm^{(A_{\mathbb{J}},\boldsymbol{p}_{\mathbb{J}})} = \nabla_\pm^{(A,\boldsymbol{p})} \cap \mathbb{Z}^{\mathbb{J}}$.

**2D. *Root systems for contragredient Lie superalgebras.*** Next we extract from [Andruskiewitsch and Angiono ≥ 2025] the construction of the Weyl groupoid associated to a contragredient Lie superalgebra. Let $(A, \boldsymbol{p})$ be as in (D1), (D2). Following [Kac 1990; Bouarroudj et al. 2009], we assume from now on that $A$ satisfies

$$a_{ij} = 0 \quad \text{if and only if} \quad a_{ji} = 0, \qquad \text{for all } j \neq i \in \mathbb{I}. \tag{2.12}$$

From $(A, \boldsymbol{p})$, we build:

(1) A matrix $C^{(A,\boldsymbol{p})} = (c_{ij}^{(A,\boldsymbol{p})})_{i,j \in \mathbb{I}}$ by $c_{ii}^{(A,\boldsymbol{p})} := 2$ for each $i \in \mathbb{I}$ and

$$c_{ij}^{(A,\boldsymbol{p})} := -\min\{m \in \mathbb{N}_0 : (\operatorname{ad} f_i)^{m+1} f_j = 0\}, \quad i \neq j \in \mathbb{I}. \tag{2.13}$$

Since $\operatorname{ad} f_i$ is locally nilpotent, $C^{(A,\boldsymbol{p})}$ is a well-defined generalized Cartan matrix.

(2) For each $i \in \mathbb{I}$, an involution $s_i^{(A,\boldsymbol{p})} \in GL(\mathbb{Z}^{\mathbb{I}})$ given by

$$s_i^{(A,\boldsymbol{p})}(\alpha_j) := \alpha_j - c_{ij}^{(A,\boldsymbol{p})}\alpha_i, \qquad j \in \mathbb{I}. \tag{2.14}$$

(3) For each $i \in \mathbb{I}$, the $i$-th *reflection* $r_i(A, \boldsymbol{p}) := (r_iA, r_i\boldsymbol{p}) = ((\hat{a}_{jk})_{j,k\in\mathbb{I}}, (\hat{p}_j)_{j\in\mathbb{I}})$, where

- $\hat{p}_j = p_j p_i^{c_{ij}^{(A,\boldsymbol{p})}}$, for all $j \in \mathbb{I}$;
- for $j = i$, let $\hat{a}_{jk} = c_{ik}^{(A,\boldsymbol{p})}a_{ii} - a_{ik}$, for all $k \in \mathbb{I}$;
- for $j \neq i$ with $a_{ij} = 0$, let $\hat{a}_{jk} = a_{jk}$, for all $k \in \mathbb{I}$;
- for $j \neq i$ with $a_{ij} \neq 0$, put $\hat{a}_{ji} = a_{ji}(c_{ij}^{(A,\boldsymbol{p})}a_{ii} - a_{ij})$, and for all $k \neq i, j$,

$$\hat{a}_{jk} = c_{ij}^{(A,\boldsymbol{p})}c_{ik}^{(A,\boldsymbol{p})}a_{ji}a_{ii} - c_{ij}^{(A,\boldsymbol{p})}a_{ji}a_{ik} - c_{ik}^{(A,\boldsymbol{p})}a_{ji}a_{ij} + a_{ij}a_{jk}. \tag{2.15}$$

**Remark 2.16.** For each $a \in \mathbb{F}_p$, $\tilde{a} \in \mathbb{Z}$ denotes the unique integer $0 \geq \tilde{a} \geq 1 - p$ whose class in $\mathbb{F}_p$ is $a$. By [Andruskiewitsch and Angiono ≥ 2025] the integers $c_{ij}^{(A,\boldsymbol{p})}$ can be explicitly computed as follows.

(a) If $a_{ii} = 2$ and $a_{ij} \in \mathbb{F}_p$, then

$$c_{ij}^{(A,\boldsymbol{p})} = \begin{cases} \tilde{a}_{ij} & \text{if } a_{ij} \in \mathbb{F}_p,\ \ p_i = 1 \ \text{ or } \ p_i = -1,\ \tilde{a}_{ij} \text{ even;} \\ \tilde{a}_{ij} - p & \text{if } a_{ij} \in \mathbb{F}_p,\ \ p_i = -1,\ \tilde{a}_{ij} \text{ odd;} \\ 1 - \frac{3-p_i}{2}p & \text{if } a_{ij} \notin \mathbb{F}_p. \end{cases}$$

(b) If $a_{ii} = 0$, then

$$c_{ij}^{(A,\boldsymbol{p})} = \begin{cases} 0 & \text{if } a_{ij} = 0; \\ 1 - p & \text{if } a_{ij} \neq 0,\ \ p_i = 1; \\ -1 & \text{if } a_{ij} \neq 0,\ \ p_i = -1. \end{cases}$$

Another crucial result of [Andruskiewitsch and Angiono ≥ 2025] states that, in this context, we have analogs for Lusztig's isomorphisms on quantum groups.

**Theorem 2.17.** *Let $A \in \Bbbk^{\mathbb{I}\times\mathbb{I}}$ satisfying (2.12), $\boldsymbol{p} \in (\mathbb{G}_2)^{\mathbb{I}}$, and $i \in \mathbb{I}$. There is a Lie superalgebra isomorphism $T_i^{(A,\boldsymbol{p})} : \mathfrak{g}(r_iA, r_i\boldsymbol{p}) \to \mathfrak{g}(A, \boldsymbol{p})$ such that*

$$T_i^{(A,\boldsymbol{p})}(\mathfrak{g}(r_iA, r_i\boldsymbol{p})_\beta) = \mathfrak{g}(A, \boldsymbol{p})_{s_i^{(A,\boldsymbol{p})}(\beta)}, \qquad \textit{for all } \beta \in \pm\mathbb{N}_0^{\mathbb{I}}; \tag{2.18}$$

$$T_i^{(A,\boldsymbol{p})} \circ \omega = \omega \circ T_i^{(A,\boldsymbol{p})}. \tag{2.19}$$

To obtain a reduced root system, we need to disregard roots that are natural multiples of other roots. Consider

$$\Delta_+^{(A,\boldsymbol{p})} := \nabla_+^{(A,\boldsymbol{p})} - \{k\alpha : \alpha \in \nabla_+^{(A,\boldsymbol{p})},\ k \in \mathbb{N},\ k \geq 2\}. \tag{2.20}$$

Now we are ready to state another key result from [Andruskiewitsch and Angiono ≥ 2025].

**Theorem 2.21.** *$\mathcal{R}(\mathcal{C}_\theta, (\Delta^{(A,\boldsymbol{p})})_{(A,\boldsymbol{p})\in\mathcal{X}})$ is a reduced root system.* □

To fully describe $\nabla^{(A,\boldsymbol{p})}$ we need to take into account all the multiples of roots in $\Delta^{(A,\boldsymbol{p})}$.

**Definition 2.22.** We say that a root $\beta \in \varDelta^{(A,\boldsymbol{p})}$ is odd nondegenerate if there exist $i \in \mathbb{I}$ and an element $(B,\boldsymbol{q}) = r_{i_1}\cdots r_{i_k}(A,\boldsymbol{p})$ in the Weyl class of $(A,\boldsymbol{p})$ with $b_{ii} = 2$, $\boldsymbol{q}_i = -1$ such that $s_{i_1}^{(A,\boldsymbol{p})}\cdots s_{i_k} \in \operatorname{Hom}((B,\boldsymbol{q}),(A,\boldsymbol{p}))$ maps $\alpha_i$ to $\beta$. We denote by $\varDelta_{\text{o,nd}}^{(A,\boldsymbol{p})}$ the set of all such roots.

The existence of odd nondegenerate roots turns out to be the reason behind the existence of integer multiples of roots.

**Proposition 2.23** [Andruskiewitsch and Angiono 2017]. *Assume that* $\dim \mathfrak{g}(A,\boldsymbol{p}) < \infty$. *Then*

$$\nabla^{(A,\boldsymbol{p})} = \varDelta^{(A,\boldsymbol{p})} \cup (2\varDelta_{\text{o,nd}}^{(A,\boldsymbol{p})}), \qquad \textit{and} \qquad \dim \mathfrak{g}(A,\boldsymbol{p})_\beta = 1 \quad \textit{for all } \beta \in \nabla^{(A,\boldsymbol{p})}. \qquad \square$$

**Example 2.24.** We describe all rank two finite-dimensional contragredient Lie superalgebras:

(i) The classical Lie algebras of types $A_2$, $B_2$ and $G_2$, with matrices $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & -3 \\ -1 & 2 \end{smallmatrix}\right)$ (here, $p > 3$). The root systems are the classical ones.

(ii) Similarly, $A(0|1)$, $B(0|1)$ are standard root systems, with Cartan matrices $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$. The root systems have the same roots as $A_2$ and $B_2$.

(iii) Let $p = 3$, $a \in \Bbbk - \mathbb{F}_3$. The *Brown algebra* $\mathfrak{br}(2,a)$ is constructed as follows: Set

$$A = \begin{bmatrix} 2 & -1 \\ a & 2 \end{bmatrix}, \quad A' = \begin{bmatrix} 2 & -1 \\ -1-a & 2 \end{bmatrix},$$

so $C^A$ and $C^{A'}$ are of type $B_2$. We can check that $r_2(A) = A'$, $r_1(A) = A$, $r_1(A') = A'$,

$$\varDelta_+^A = \varDelta_+^{A'} = \{1, 12, 12^2, 2\},$$

so $\dim \mathfrak{g}(A) = 10$.

(iv) Again take $p = 3$: we recall now the definition of the Lie superalgebra $\mathfrak{brj}(2;3)$. Set

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \boldsymbol{p} = (1,-1), \quad A' = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}, \quad A'' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \ \boldsymbol{p}'' = (-1,-1).$$

In this case,

$$C^{(A,\boldsymbol{p})} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}, \quad C^{(A',\boldsymbol{p})} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \quad C^{(A'',\boldsymbol{p}'')} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix},$$

$r_1(A,\boldsymbol{p}) = (A',\boldsymbol{p})$, $r_2(A,\boldsymbol{p}) = (A'',\boldsymbol{p}'')$, $r_2(A',\boldsymbol{p}) = (A',\boldsymbol{p})$, $r_1(A'',\boldsymbol{p}'') = (A'',\boldsymbol{p}'')$, and

$$\begin{aligned} \varDelta_+^{(A,\boldsymbol{p})} &= \{1, 1^2 2, 1^3 2^2, 1^4 2^3, 12, 2\}, & \varDelta_{\text{o,nd}}^{(A,\boldsymbol{p})} &= \{12, 1^2 2\}, \\ \varDelta_+^{(A',\boldsymbol{p})} &= \{1, 1^2 2, 1^3 2^2, 12, 12^2, 2\}, & \varDelta_{\text{o,nd}}^{(A',\boldsymbol{p})} &= \{1, 12\}, \\ \varDelta_+^{(A'',\boldsymbol{p}'')} &= \{1, 1^4 2, 1^3 2, 1^2 2, 12, 2\}, & \varDelta_{\text{o,nd}}^{(A'',\boldsymbol{p}'')} &= \{1, 1^2 2\}. \end{aligned}$$

Thus $\operatorname{sdim} \mathfrak{g}(A,\boldsymbol{p}) = 10|8$.

(v) Finally take $p=5$. The Lie superalgebra $\mathfrak{brj}(2;5)$ admits two possible realizations as contragredient Lie superalgebra. Namely,

$$A=\begin{bmatrix}2 & -3\\ 1 & 0\end{bmatrix},\ \boldsymbol{p}=(1,-1),\qquad A'=\begin{bmatrix}2 & -4\\ 1 & 0\end{bmatrix},\ \boldsymbol{p}'=(-1,-1).$$

In this case,

$$C^{(A,\boldsymbol{p})}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix},\qquad C^{(A',\boldsymbol{p}')}=\begin{bmatrix}2 & -4\\ -1 & 2\end{bmatrix},$$

$r_2(A,\boldsymbol{p})=(A',\boldsymbol{p}')$, $r_1(A,\boldsymbol{p})=(A,\boldsymbol{p})$, $r_1(A',\boldsymbol{p})=(A',\boldsymbol{p})$, and

$$\varDelta_{+}^{(A,\boldsymbol{p})}=\{1,1^32,1^22,1^52^3,1^32^2,1^42^3,12,2\},\qquad \varDelta_{\mathrm{o,nd}}^{(A,\boldsymbol{p})}=\{1^22,12\},$$
$$\varDelta_{+}^{(A',\boldsymbol{p})}=\{1,1^42,1^32,1^52^2,1^22,1^32^2,12,2\},\qquad \varDelta_{\mathrm{o,nd}}^{(A',\boldsymbol{p})}=\{1,1^22\}.$$

Thus $\operatorname{sdim}\mathfrak{g}(A,\boldsymbol{p})=10|12$.

**Remark 2.25.** By Proposition 2.23, we can fix bases $(e_\beta)_{\beta\in\nabla^{(A,\boldsymbol{p})}}$ for $\mathfrak{n}_+$ and $(f_\beta)_{\beta\in\nabla^{(A,\boldsymbol{p})}}$ for $\mathfrak{n}_-$ such that

- $e_{\alpha_i}=e_i$ and $f_{\alpha_i}=f_i$, for all $i\in\mathbb{I}$;
- $e_\beta\in\mathfrak{g}_\beta$ and $f_\beta\in\mathfrak{g}_{-\beta}$, for all $\beta\in\varDelta_{+}^{(A,\boldsymbol{p})}$; if $\beta=k\alpha_i+\alpha_j$ for some $i\neq j$ and $k\geq 0$, we can take $e_\beta=(\operatorname{ad}e_i)^ke_j$ and $f_\beta=(\operatorname{ad}f_i)^kf_j$. For small values of $k$, we denote

$$e_{ij}=(\operatorname{ad}e_i)e_j,\quad e_{iij}=(\operatorname{ad}e_i)^2e_j,\quad e_{iiij}=(\operatorname{ad}e_i)^3e_j,$$

and use analog notation $f_{ij}$, $f_{iij}$, $f_{iiij}$.
- $e_{2\beta}=[e_\beta,e_\beta]$ and $f_{2\beta}=[f_\beta,f_\beta]$, for all $\beta\in\varDelta_{+,\mathrm{o,nd}}^{(A,\boldsymbol{p})}$.

Let $\beta=\sum_{i\in\mathbb{I}}a_i\alpha_i\in\nabla_{+}^{(A,\boldsymbol{p})}$, $\xi_\beta:=\sum_{i\in\mathbb{I}}a_i\xi_i\in\mathfrak{h}^*$. Then

$$[h,e_\beta]=\xi_\beta(h)e_\beta,\quad [h,f_\beta]=-\xi_\beta(h)f_\beta,\qquad \text{for all } h\in\mathfrak{h}. \tag{2.26}$$

We end this section with a generalization of a well-known result for Lie algebras. The proof follows as an application of Theorem 2.3.

**Lemma 2.27.** *Assume that $\varDelta_{+}^{(A,\boldsymbol{p})}$ is finite and $\alpha,\beta\in\varDelta_{+}^{(A,\boldsymbol{p})}$ are such that $\alpha\notin\mathbb{Z}\beta$.*

(a) *If $n\in\mathbb{N}$ is such that $\beta+n\alpha\in\varDelta_{+}^{(A,\boldsymbol{p})}$, then $n<2p$ and there exists $c\in\Bbbk^\times$ such that*

$$(\operatorname{ad}e_\alpha)^ne_\beta=ce_{\beta+n\alpha}.$$

(b) *If $n\in\mathbb{N}$ is such that $\beta-n\alpha\in\varDelta_{+}^{(A,\boldsymbol{p})}$, then $n<2p$ and there exists $c\in\Bbbk^\times$ such that*

$$(\operatorname{ad}e_\alpha)^nf_\beta=cf_{\beta-n\alpha}.$$

(c) *Assume moreover that $\alpha\notin\varDelta_{\mathrm{o,nd}}^{(A,\boldsymbol{p})}$. If $n$ is as in any of the items above, then $n<p$.*

*Proof.* (a), (b) Since $\alpha$ and $\beta$ are linearly independent, by Theorem 2.3 there exists a pair $(B, \boldsymbol{p}')$ with two simple roots $\beta_1, \beta_2$, and $w \in \mathrm{Hom}((A, \boldsymbol{p}), (B, \boldsymbol{p}'))$ such that

$$\omega(\alpha) = \beta_1, \quad \omega(\beta) \in \Delta_+^{(B,\boldsymbol{p}')} \cap \{\mathbb{N}_0\beta_1 + \mathbb{N}_0\beta_2\}.$$

So it is enough to verify the claims for $(A, \boldsymbol{p})$ of rank $\theta = 2$, which is easily achieved from a case-by-case analysis, see Example 2.24.

(c) We can reduce to $\theta = 2$ again and use that $\beta_1 \notin \Delta_{\mathrm{o,nd}}^{(B,\boldsymbol{p}')}$. □

## 3. Lie algebras in symmetric tensor categories

We recall some notions and basic results related to symmetric tensor categories, Lie algebras in this broad context, and the Verlinde category.

**3A. *Symmetric tensor categories.*** A *symmetric tensor category* is an abelian $\Bbbk$-linear category which admits a rigid symmetric monoidal structure such that the tensor product is bilinear on Hom-spaces and the unit object has a one-dimensional endomorphism space.

A *pre-Tannakian* category is a symmetric tensor category where all objects have finite length, which then implies that all Hom-spaces are finite-dimensional. Notice that we adopt the terminology of [Coulembier et al. 2023b].

The ind-completion $\mathcal{C}^{\mathrm{ind}}$ of a symmetric tensor category $\mathcal{C}$ is defined as the closure of $\mathcal{C}$ under filtered colimits. This completion $\mathcal{C}^{\mathrm{ind}}$ is a $\Bbbk$-linear abelian category with an exact and symmetric tensor product, and there is universal exact symmetric embedding $\mathcal{C} \hookrightarrow \mathcal{C}^{\mathrm{ind}}$, see [Kashiwara and Schapira 2006]. We shall refer objects in $\mathcal{C}^{\mathrm{ind}}$ as ind-objects of $\mathcal{C}$. When $\mathcal{C}$ is moreover fusion, that is, finite and semisimple, the ind-objects are (possibly infinite) direct sums of simple objects in $\mathcal{C}$. The genuine objects of $\mathcal{C}$ are then recovered as ind-objects of finite length. If $X$ and $Y$ are ind-objects, we imprecisely write $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ instead of $\mathrm{Hom}_{\mathcal{C}^{\mathrm{ind}}}(X, Y)$.

**3B. *The category* $\mathrm{Rep}(\boldsymbol{\alpha}_p)$.** Let $\boldsymbol{\alpha}_p$ denote the Frobenius kernel of the additive group scheme. This group scheme is represented by the commutative and cocommutative Hopf algebra $\Bbbk[t]/(t^p)$, where $t$ is primitive. By self-duality, we can safely identify $\mathrm{Rep}(\boldsymbol{\alpha}_p)$ with the symmetric tensor category of finite dimensional representations of $\Bbbk[t]/(t^p)$ over $\Bbbk$.

Isomorphism classes of indecomposables in $\mathrm{Rep}(\boldsymbol{\alpha}_p)$ are represented by nilpotent Jordan blocks $L_i$ of size $i$ with $1 \le i \le p$. Explicitly, $L_i$ has an ordered basis $\{v_1, \dots, v_i\}$ such that $v_{j+1} = tv_j$ for $1 \le j < i$ and $tv_i = 0$. Such a basis is called *cyclic*; in this situation we write $L_i = \Bbbk\{v_1, \dots, v_i\} = \langle v_1 \rangle$ and say that $v_1$ is a cyclic generator. Notice that an element in $\mathrm{Hom}_{\alpha_p}(\langle v_1 \rangle, V)$ is determined by its value on $v_1$.

The indecomposable objects are self-dual, and an isotypic decomposition of $L_i \otimes L_j$ is known from [Green 1962]. In particular $L_1$ is the monoidal unit and $L_p \otimes L_p \simeq pL_p$. For later use, we record an explicit decomposition of some tensor products.

**Example 3.1.** Assume $p > 2$. Consider two copies of $L_2$, with cyclic bases $\{v_1, v_2\}$ and $\{w_1, w_2\}$ respectively. Then $L_2 \otimes L_2 = L_1 \oplus L_3$, where

$$L_1 = \langle v_1 \otimes w_2 - v_2 \otimes w_1 \rangle, \quad L_3 = \langle v_1 \otimes w_1 \rangle.$$

**Example 3.2.** Consider two copies of $L_3$, with cyclic bases $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$.

- If $p > 3$, then $L_3 \otimes L_3 = L_1 \oplus L_3 \oplus L_5$, where

$$L_1 = \langle v_1 \otimes w_3 - v_2 \otimes w_2 + v_3 \otimes w_1 \rangle, \quad L_3 = \langle v_1 \otimes w_2 - v_2 \otimes w_1 \rangle, \quad L_5 = \langle v_1 \otimes w_1 \rangle.$$

- If $p = 3$, then $L_3 \otimes L_3 = L_3^{(1)} \oplus L_3^{(2)} \oplus L_3^{(3)}$, where

$$L_3^{(1)} = \langle v_2 \otimes w_2 \rangle, \quad L_3^{(2)} = \langle v_1 \otimes w_2 - v_2 \otimes w_1 \rangle, \quad L_3^{(3)} = \langle v_1 \otimes w_1 \rangle.$$

**Example 3.3.** Consider two copies of $L_4$, with bases $\{v_1, v_2, v_3, v_4\}$ and $\{w_1, w_2, w_3, w_4\}$ respectively.

- If $p > 5$, then $L_4 \otimes L_4 = L_1 \oplus L_3 \oplus L_5 \oplus L_7$, where

$$\begin{aligned} L_1 &= \langle v_1 \otimes w_4 - v_2 \otimes w_3 + v_3 \otimes w_2 - v_4 \otimes v_1 \rangle, & L_5 &= \langle v_1 \otimes w_2 - v_2 \otimes w_1 \rangle, \\ L_3 &= \langle 3v_1 \otimes w_3 - 4v_2 \otimes w_2 + 3v_3 \otimes w_1 \rangle, & L_7 &= \langle v_1 \otimes w_1 \rangle. \end{aligned}$$

- If $p = 5$, then $L_4 \otimes L_4 = L_1 \oplus 3L_5$, where

$$L_1 = \langle v_1 \otimes w_4 - v_2 \otimes w_3 + v_3 \otimes w_2 - v_4 \otimes v_1 \rangle.$$

For reasons that will become evident later, we are particularly interested in tensor products of the form $L_3 \otimes L_j$ for $j$ at most 4. Next, we describe explicit direct sum decompositions.

**Example 3.4.** Consider indecomposables $L_3 = \Bbbk\{v_1, v_2, v_3\}$ and $L_2 = \Bbbk\{w_1, w_2\}$.

- Assume $p > 3$. Then $L_3 \otimes L_2 = L_2 \oplus L_4$, where

$$L_2 = \langle 2v_1 \otimes w_2 - v_2 \otimes w_1 \rangle, \quad L_4 = \langle v_1 \otimes w_1 \rangle.$$

- Assume $p = 3$. Then $L_3 \otimes L_2 = L_3^{(1)} \oplus L_3^{(2)}$, where

$$L_3^{(1)} = \langle v_1 \otimes w_1 \rangle, \quad L_3^{(2)} = \langle v_2 \otimes w_1 \rangle.$$

**Example 3.5.** Consider indecomposables $L_3 = \Bbbk\{v_1, v_2, v_3\}$ and $L_4 = \Bbbk\{w_1, w_2, w_3, w_4\}$.

- Assume $p > 5$. Then $L_3 \otimes L_4 = L_2 \oplus L_4 \oplus L_6$, where

$$L_2 = \langle v_1 \otimes w_3 - 2v_2 \otimes w_2 + 3v_3 \otimes w_1 \rangle, \quad L_4 = \langle 2v_1 \otimes w_2 - 3v_2 \otimes w_1 \rangle, \quad L_6 = \langle v_1 \otimes w_1 \rangle.$$

- Assume $p = 5$. Then $L_3 \otimes L_4 = L_2 \oplus 2L_5$, where

$$L_2 = \langle v_1 \otimes w_3 - 2v_2 \otimes w_2 + 3v_3 \otimes w_1 \rangle.$$

**3C.** ***Semisimplification and the Verlinde category.*** If $\mathcal{C}$ is a symmetric tensor category, a tensor ideal $\mathcal{I}$ is a collection of subspaces

$$\mathcal{I} = \big\{\mathcal{I}(X, Y) \subseteq \mathsf{Hom}_{\mathcal{C}}(X, Y) \mid X, Y \text{ objects in } \mathcal{C}\big\},$$

which is compatible with compositions and tensor products. The *quotient category* $\bar{\mathcal{C}}$ of $\mathcal{C}$ by $\mathcal{I}$ is the category whose objects are the same as $\mathcal{C}$ and the spaces of morphisms are $\mathsf{Hom}_{\bar{\mathcal{C}}}(X, Y) := \mathsf{Hom}_{\mathcal{C}}(X, Y)/\mathcal{I}(X, Y)$. Notice that this construction is compatible with the $\Bbbk$-linear monoidal structure and the tensor product of the original category.

As in [Etingof and Ostrik 2022], see also the references therein, we may take a spherical category $\mathcal{C}$ and the tensor ideal of *negligible* objects. The quotient $\bar{\mathcal{C}}$ obtained in this case is known as the *semisimplification* of $\mathcal{C}$ since it turns out to be semisimple: The simple objects of $\bar{\mathcal{C}}$ are the indecomposable objects in $\mathcal{C}$ with nonzero categorical dimension.

The Verlinde category $\mathsf{Ver}_p$ is defined as the semisimplification of the spherical tensor category $\mathsf{Rep}(\boldsymbol{\alpha}_p)$, see [Gelfand and Kazhdan 1992; Ostrik 2020]. This is a symmetric fusion category equipped with a symmetric monoidal functor $\mathsf{Rep}(\boldsymbol{\alpha}_p) \to \mathsf{Ver}_p$ which fails to be right or left exact. For instance, since the categorical dimension of $L_p$ is $p = 0$, it is mapped to zero in $\mathsf{Ver}_p$. Up to isomorphism, the simple objects of $\mathsf{Ver}_p$ are the images $\mathtt{L}_i$ of the indecomposables $L_i$ with $1 \le i \le p-1$ of $\mathsf{Rep}(\boldsymbol{\alpha}_p)$, and the fusion rules in $\mathsf{Ver}_p$ are

$$\mathtt{L}_i \otimes \mathtt{L}_j = \bigoplus_{k=1}^{\min\{i,j,p-i,p-j\}} \mathtt{L}_{|i-j|+2k-1}. \tag{3.6}$$

Also, the fusion subcategory generated by $\mathtt{L}_1$ and $\mathtt{L}_{p-1}$ is equivalent, as a symmetric tensor category, to $\mathsf{sVec}_{\Bbbk}$, see [Ostrik 2020; Kannan 2022].

**3D.** ***Lie algebras in symmetric tensor categories.*** We recall now different flavors of *Lie algebras* in a (strict) symmetric tensor category $\mathcal{C}$ following [Etingof 2018]. First, an *operadic Lie algebra* is an object $\mathfrak{g}$ in $\mathcal{C}$ equipped with a morphism $b : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ that satisfies the antisymmetric and Jacobi identities

$$b \circ (\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}}) = 0, \quad b \circ (b \otimes \mathrm{id}_{\mathfrak{g}}) \circ \big(\mathrm{id}_{\mathfrak{g}^{\otimes 3}} + (123)_{\mathfrak{g}^{\otimes 3}} + (132)_{\mathfrak{g}^{\otimes 3}}\big) = 0, \tag{3.7}$$

where the maps $(123)_{\mathfrak{g}^{\otimes 3}}, (132)_{\mathfrak{g}^{\otimes 3}} : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}^{\otimes 3}$ are obtained via the action of the symmetric group. We denote by $\mathsf{OLie}(\mathcal{C})$ the category of operadic Lie algebras in $\mathcal{C}$. Operadic Lie algebras are just called Lie algebras in [Rumynin 2013] and other previous papers. However, especially when working in positive characteristic, one needs to pay special attention to some details, as explained next.

For a Lie algebra $\mathfrak{g}$, consider two quotients of the tensor algebra $\mathrm{T}(\mathfrak{g})$:

- ◇ The *universal enveloping algebra* $\mathrm{U}(\mathfrak{g})$ is the quotient by the two-sided ideal generated by the image of the map

$$(-b, \mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}} - c_{\mathfrak{g},\mathfrak{g}}) : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \subseteq \mathrm{T}(\mathfrak{g}). \tag{3.8}$$

- ◇ The symmetric algebra $\mathrm{S}(\mathfrak{g})$ is the quotient of $\mathrm{T}(\mathfrak{g})$ by the image of $\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}} - c_{\mathfrak{g},\mathfrak{g}}$.

As in the category of vector spaces, there is a natural map $\eta : \mathrm{S}(\mathfrak{g}) \to \operatorname{gr}\mathrm{U}(\mathfrak{g})$ of ind-algebras in $\mathcal{C}$, which can fail to be an isomorphism. For example, it is known that for $\mathcal{C} = \mathsf{Vec}_{\Bbbk}$ and $p = 2$, for $\eta$ to be an isomorphism one needs to impose the additional condition $b(x \otimes x) = 0$ for all $x$. Also, for $\mathcal{C} = \mathsf{sVec}_{\Bbbk}$ and $p = 3$, we have to assume $b(b(x \otimes x) \otimes x) = 0$ for all odd $x$.

Motivated by this fact, an operadic Lie algebra $\mathfrak{g}$ is said to be *PBW* if the canonical map $\eta : \mathrm{S}(\mathfrak{g}) \to \operatorname{gr}\mathrm{U}(\mathfrak{g})$ is an isomorphism.

It is not a trivial task to decide whether a given operadic Lie algebra in a symmetric tensor category is PBW. For $\mathcal{C} = \mathsf{Ver}_p$, Etingof [2018] introduced the *$p$-Jacobi identity* for any $p \geq 5$, a degree-$p$ relation that generalizes the conditions $b(x \otimes x) = 0$ and $b(b(x \otimes x) \otimes x) = 0$ needed for characteristics 2 and 3, respectively. By [Etingof 2018, Theorem 6.6, Corollary 6.7], an operadic Lie algebra in $\mathsf{Ver}_p$ is PBW if and only if it satisfies the $p$-Jacobi identity. As a first application, we have the following result:

**Lemma 3.9.** *Contragredient operadic Lie superalgebras satisfy the PBW theorem in any characteristic. In other words, for any contragredient Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A, \boldsymbol{p})$, the natural map $\eta : \mathrm{S}(\mathfrak{g}) \to \operatorname{gr}\mathrm{U}(\mathfrak{g})$ is an isomorphism.*

*Proof.* By [Etingof 2018, Corollary 4.12], it is enough to show that if $p = 2$ then $[x, x] = 0$ for all $x \in \mathfrak{g}$ and that when $p = 3$, we have $[x, [x, x]] = 0$ for all odd $x$. In any case, we can assume that $x$ is homogeneous of nonzero degree; moreover, via the Chevalley involution, we reduce to the case $x \in \mathfrak{n}_+$.

When $p = 2$, it is enough to show that $[f_i, [x, x]] = 0$ for all $i \in \mathbb{I}$, which is a consequence of the Jacobi identity,

$$[f_i, [x, x]] = [[f_i, x], x] + [x, [f_i, x]] = 2[[f_i, x], x] = 0.$$

For the case $p = 3$, we need to show that $[f_i, [x, [x, x]]] = 0$ for $i \in \mathbb{I}$ and any odd $x \in \mathfrak{n}_+$. When $f_i$ is odd, using the Jacobi identity and the fact that $[f_i, x]$ is even, we get

$$\begin{aligned}
[f_i, [x, [x, x]]] &= [[f_i, x], [x, x]] - [x, [f_i, [x, x]]] \\
&= [[[f_i, x], x], x] + [x, [[f_i, x], x]] - [x, [[f_i, x], x] - [x, [f_i, x]]] = 0.
\end{aligned}$$

Similarly, when $f_i$ is even, we see that

$$\begin{aligned}
[f_i, [x, [x, x]]] &= [[f_i, x], [x, x]] + [x, [f_i, [x, x]]] \\
&= [[[f_i, x], x], x] - [x, [[f_i, x], x]] + [x, [[f_i, x], x] + [x, [f_i, x]]] = 0,
\end{aligned}$$

as claimed. □

Fix an operadic Lie algebra $\mathfrak{g}$ with bracket $b$ in a strict symmetric tensor category $\mathcal{C}$. A form on $\mathfrak{g}$ is a map $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{1}$ in $\mathcal{C}$. We say that the form is

- *symmetric* if it remains unchanged when composed with the braiding $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$;
- *invariant* if $B \circ (b \otimes \mathrm{id}_{\mathfrak{g}}) \circ (\mathrm{id}_{\mathfrak{g}^{\otimes 3}} + (123)_{\mathfrak{g}^{\otimes 3}}) = 0$;
- *nondegenerate* if its image under the natural adjunction $\mathsf{Hom}_{\mathcal{C}}(\mathfrak{g} \otimes \mathfrak{g}, \mathbb{1}) \simeq \mathsf{Hom}_{\mathcal{C}}(\mathfrak{g}, \mathfrak{g}^*)$ is an isomorphism.

**Lemma 3.10.** *Let* $\mathfrak{g}$ *be an operadic Lie algebra with a form* $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{1}$. *Let* $\bar{B} : \bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}} \to \mathbb{1}$ *denote the form on* $\bar{\mathfrak{g}}$ *induced by* $B$ *under semisimplification. If* $B$ *is symmetric, respectively, invariant, and nondegenerate, then so is* $\bar{B}$.

*Proof.* The property of being symmetric (respectively, invariant) is preserved because the semisimplification functor is additive. Nondegeneracy is also hereditary since the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(\mathfrak{g} \otimes \mathfrak{g}, \mathbb{1}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(\mathfrak{g}, \mathfrak{g}^*) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}}, \mathbb{1}) & \xrightarrow{\sim} & \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^*) \end{array}$$

commutes. □

## 4. Semisimplification of Lie algebras with a derivation

In this section, we work exclusively in characteristic $p > 2$. As in [Kannan 2022], if $\mathfrak{g}$ is a finite dimensional Lie algebra over $\Bbbk$ endowed with a derivation $\partial$ of order at most $p$, then $\mathfrak{g}$ becomes a Lie algebra in $\mathrm{Rep}(\boldsymbol{\alpha}_p)$ by letting $t$ act via $\partial$. Applying the semisimplification $\mathrm{Rep}(\boldsymbol{\alpha}_p) \to \mathrm{Ver}_p$, we get an operadic Lie algebra in $\mathrm{Ver}_p$ since the functor is braided monoidal.

We are interested in understanding the result of this process when the input is a contragredient Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ with an inner derivation $\partial = \mathrm{ad}\, x$, for some $x \in \mathfrak{g}$. If moreover, $x$ is homogeneous with respect to the grading by $\Delta^A$, using Theorem 2.3, we may assume that $x = e_i$ for some $i \in \mathbb{I}$. Let us fix some terminology.

**Notation 4.1.** We denote by $\mathsf{S} : \mathrm{Rep}(\boldsymbol{\alpha}_p) \to \mathrm{Ver}_p$ the semisimplification functor. If $\mathfrak{g}$ is a finite dimensional Lie algebra and $x \in \mathfrak{g}$ is such that $\mathrm{ad}\, x$ has order at most $p$, then

- $(\mathfrak{g}, x)$ denotes the Lie algebra in $\mathrm{Rep}(\boldsymbol{\alpha}_p)$ obtained from $\mathfrak{g}$ by letting $t$ act via $\mathrm{ad}\, x$;
- $\mathsf{S}(\mathfrak{g}, x)$ is the operadic Lie algebra in $\mathrm{Ver}_p$ obtained from semisimplification of $(\mathfrak{g}, x)$.

Next we verify that for a finite dimensional contragredient Lie algebra $\mathfrak{g}(A)$, the Chevalley generators $e_i$, $f_i$ yield suitable derivations.

**Lemma 4.2.** *Let* $A$ *be such that* $\dim \mathfrak{g}(A) < \infty$. *Then* $(\mathrm{ad}\, e_i)^p = (\mathrm{ad}\, f_i)^p = 0$ *for all* $i \in \mathbb{I}_\theta$.

*Proof.* By the binomial formula, $(\mathrm{ad}\, x)^p$ is a derivation for all $x \in \mathfrak{g}$. Hence, to show that $(\mathrm{ad}\, e_i)^p$ annihilates $\mathfrak{n}_+$, it is enough to check that $(\mathrm{ad}\, e_i)^p e_j = 0$ for all $j \in \mathbb{I}$, which follows from Remark 2.16. Also, for any $x \in \mathfrak{h} = \mathfrak{g}_0$, we have $(\mathrm{ad}\, e_i)^2 x \in (\mathrm{ad}\, e_i)\Bbbk e_i = 0$. Thus $(\mathrm{ad}\, e_i)^p|_{\mathfrak{n}_+ \oplus \mathfrak{h}} = 0$. Applying the Chevalley involution we get that $(\mathrm{ad}\, f_i)^p|_{\mathfrak{n}_- \oplus \mathfrak{h}} = 0$.

To show that $(\mathrm{ad}\, e_i)^p$ annihilates $\mathfrak{n}_-$, consider a nonzero homogeneous $x \in \mathfrak{g}_{-\beta}$, where $\beta \in \Delta^A_+$. If $\beta = \alpha_i$, then $x \in \Bbbk f_i$, and since

$$(\mathrm{ad}\, e_i)^3 f_i = (\mathrm{ad}\, e_i)^2 h_i = -a_{ii}[e_i, e_i] = 0,$$

we also have $(\operatorname{ad} e_i)^3 x = 0$. For the case $\beta \neq \alpha_i$, consider $\beta' := s_i(\beta) \in \Delta_+^{r_i A}$; we know that, up to a nonzero scalar, $x = T_i^A(\widehat{f_{\beta'}})$. Then

$$(\operatorname{ad} e_i)^p x = (\operatorname{ad} T_i^A(\widehat{f_i}))^p T_i^A(\widehat{f_{\beta'}}) = T_i^A((\operatorname{ad} \widehat{f_i})^p \widehat{f_{\beta'}}) = 0.$$

Thus $(\operatorname{ad} e_i)^p|_{\mathfrak{n}_-} = 0$. Applying the Chevalley involution, we also have $(\operatorname{ad} f_i)^p|_{\mathfrak{n}_+} = 0$. □

These two derivations turn out to yield isomorphic Lie algebras in $\mathsf{Rep}(\boldsymbol{\alpha}_p)$.

**Lemma 4.3.** *Let $A \in \Bbbk^{\theta\times\theta}$ such that $\mathfrak{g} = \mathfrak{g}(A)$ is finite dimensional, and fix $i \in \mathbb{I}_\theta$. Then the Chevalley involution $\omega : (\mathfrak{g}, e_i) \to (\mathfrak{g}, -f_i)$ is an isomorphism of Lie algebras in $\mathsf{Rep}(\boldsymbol{\alpha}_p)$.*

*Proof.* Since $\omega : \mathfrak{g} \to \mathfrak{g}$ is an isomorphism of Lie algebras, it is enough to show that $\omega : (\mathfrak{g}, e_i) \to (\mathfrak{g}, -f_i)$ is a morphism in $\mathsf{Rep}(\boldsymbol{\alpha}_p)$, which follows directly from $\omega(e_i) = -f_i$. □

**Remark 4.4.** Let $\mathbb{J} \subseteq \mathbb{I}$ be the connected component of the Dynkin diagram containing a fixed index $i \in \mathbb{I}_\theta$, and denote by $\hat{\mathbb{J}}$ the complement of $\mathbb{J}$ in $\mathbb{I}$. Then, as explained in Remark 2.11, we have $\mathfrak{g} = \mathfrak{g}(A_{\mathbb{J}}) \oplus \mathfrak{g}(A_{\hat{\mathbb{J}}})$, so

$$(\mathfrak{g}, e_i) = (\mathfrak{g}(A_{\mathbb{J}}), e_i) \oplus (\mathfrak{g}(A_{\hat{\mathbb{J}}}), e_i), \qquad \mathsf{S}(\mathfrak{g}, e_i) = \mathsf{S}(\mathfrak{g}(A_{\mathbb{J}}), e_i) \oplus \mathsf{S}(\mathfrak{g}(A_{\hat{\mathbb{J}}}), e_i).$$

Furthermore, since $(\operatorname{ad} e_i)|_{\mathfrak{g}(A_{\hat{\mathbb{J}}})} = 0$, we see that $\mathsf{S}(\mathfrak{g}(A_{\hat{\mathbb{J}}}), e_i) \simeq \mathfrak{g}(A_{\hat{\mathbb{J}}})$ is an ordinary Lie algebra in $\mathsf{Vec}_\Bbbk \subset \mathsf{Ver}_p$, similarly $(\mathfrak{g}(A_{\hat{\mathbb{J}}}), e_i) \in \mathsf{Vec}_\Bbbk \subset \mathsf{Rep}(\boldsymbol{\alpha}_p)$.

As a consequence, for most of the arguments in this section, we only need to consider matrices $A$ with connected Dynkin diagram.

**4A.** ***The structure of the Lie algebra $(\mathfrak{g}'(A), e_i)$.*** In this subsection, we fix a finite-dimensional contragredient Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ and a Chevalley generator $e_i$, and we describe the structure of the Lie algebra $(\mathfrak{g}', e_i)$ in $\mathsf{Rep}(\boldsymbol{\alpha}_p)$; see Remark 2.6.

As a first step, we show that the isotypic decomposition of $(\mathfrak{g}', e_i) \in \mathsf{Rep}(\boldsymbol{\alpha}_p)$ is determined by the roots. To describe it, we need to fix some terminology.

**Notation 4.5.** Fix $i \in \mathbb{I}_\theta$ and a matrix $A \in \Bbbk^{\theta\times\theta}$. For each $j \in \mathbb{I}_p$ consider

$$\Delta_{+,j}^A = \left\{\beta \neq \alpha_i \mid \beta + k\alpha_i \in \Delta_+^A \text{ for all } 0 \leq k < j \text{ and } \beta - \alpha_i, \beta + j\alpha_i \notin \Delta_+^A\right\}, \quad a_j = |\Delta_{+,j}^A|.$$

If $\beta \in \Delta_{+,j}^A$, we say that $\{\beta + k\alpha_i : 0 \leq k < j\}$ is a maximal $\alpha_i$-string. We also say that $\beta$ generates the string, and that $j$ is the length of the string.

The roots that generate $\alpha_i$-strings of length less than $p$ will play an important role. Let

$$\Delta_{+,\min}^A = \bigcup_{j \in \mathbb{I}_{p-1}} \Delta_{+,j}^A, \quad \Delta_{-,\min}^A = -\Delta_{+,\min}^A, \quad \Delta_{\min}^A = \Delta_{+,\min}^A \cup \Delta_{-,\min}^A. \tag{4.6}$$

**Remark 4.7.** Assume $\mathfrak{g}(A)$ is finite dimensional and fix $i \in \mathbb{I}_\theta$.

- Each positive root different from $\alpha_i$ belongs to a unique maximal $\alpha_i$-string. In fact, the existence of such a string follows from Lemma 2.27 (c) (with $\alpha = \alpha_i$), and the uniqueness follows from maximality.

- Given $j \neq i$, the simple root $\alpha_j$ generates a maximal $\alpha_i$-string of length $1 - c_{ij}^A$; see (2.13).

**Proposition 4.8.** *Fix $i \in \mathbb{I}_\theta$ and a matrix $A \in \Bbbk^{\theta\times\theta}$ such that $\mathfrak{g} = \mathfrak{g}(A)$ is finite dimensional.*

(a) *For each $\beta \in \Delta_{+,j}^A$, the subspaces*

$$M_\beta := \bigoplus_{k=0}^{j-1} \Bbbk e_{\beta+k\alpha_i}, \qquad N_\beta := \bigoplus_{k=0}^{j-1} \Bbbk f_{\beta+k\alpha_i},$$

*are $\boldsymbol{\alpha}_p$-submodules of $(\mathfrak{g}', e_i)$, and both are isomorphic to $L_j$.*

(b) *If $a_{ii} = 2$, then $\Bbbk f_i \oplus \Bbbk h_i \oplus \Bbbk e_i$ is a submodule of $(\mathfrak{g}', e_i)$ isomorphic to $L_3$, any one-dimensional subspace of $\ker \xi_i \cap \mathfrak{h}_{\leq\theta}$ is a submodule isomorphic to $L_1$ and*

$$(\mathfrak{g}', e_i) \simeq L_1^{2a_1+\theta-1} \oplus L_3^{2a_3+1} \oplus \Big(\bigoplus_{j\in\mathbb{I}_p-\{1,3\}} L_j^{2a_j}\Big) \text{ in } \operatorname{Rep}(\boldsymbol{\alpha}_p).$$

(c) *If $a_{ii} = 0$ and there exists $j$ such that $a_{ji} \neq 0$, then $\Bbbk f_i \oplus \Bbbk h_i$ is a submodule of $(\mathfrak{g}', e_i)$ isomorphic to $L_2$, there is $h \in \mathfrak{h}_{\leq\theta}$ such that $\Bbbk h \oplus \Bbbk e_i$ is a submodule isomorphic to $L_2$, any one-dimensional subspace of $\ker \xi_i \cap \mathfrak{h}_{\leq\theta}$ different from $\Bbbk h_i$ is a submodule isomorphic to $L_1$ and*

$$(\mathfrak{g}', e_i) \simeq L_1^{2a_1+\theta-2} \oplus L_2^{2a_2+2} \oplus \Big(\bigoplus_{j\in\mathbb{I}_p-\{1,2\}} L_j^{2a_j}\Big) \text{ in } \operatorname{Rep}(\boldsymbol{\alpha}_p).$$

*Proof.* For (a), fix $j \in \mathbb{I}_p$ and $\beta \in \Delta_{+,j}^A$. Using that $\mathfrak{g}$ is $\mathbb{Z}^\theta$-graded, since $\beta - \alpha_i$ and $\beta + j\alpha_i$ are not roots, we have

$$(\operatorname{ad} e_i) f_\beta = 0, \qquad (\operatorname{ad} e_i) e_{\beta+(j-1)\alpha_i} = 0;$$

so $N_\beta$ and $M_\beta$ are $\boldsymbol{\alpha}_p$-submodules of $\mathfrak{g}$. Using these last computations and Lemma 2.27, it becomes evident that these submodules are isomorphic to the nilpotent Jordan block $L_j$.

To prove (b) and (c), we first observe that $\bigoplus_{\beta\neq\alpha_i} \Bbbk e_\beta$ and $\bigoplus_{\beta\neq\alpha_i} \Bbbk f_\beta$ are $\boldsymbol{\alpha}_p$-submodules of $\mathfrak{g}$. Indeed, by Remark 4.7 we have

$$\bigoplus_{\beta\neq\alpha_i} \Bbbk e_\beta = \bigoplus_{j\in\mathbb{I}_p}\Big(\bigoplus_{\beta\in\Delta_{+,j}^A} M_\beta\Big), \qquad \bigoplus_{\beta\neq\alpha_i} \Bbbk f_\beta = \bigoplus_{j\in\mathbb{I}_p}\Big(\bigoplus_{\beta\in\Delta_{+,j}^A} N_\beta\Big).$$

The subspace $\Bbbk f_i \oplus \mathfrak{h} \oplus \Bbbk e_i$ is also an $\boldsymbol{\alpha}_p$-submodules of $\mathfrak{g}$, so

$$\mathfrak{g}' = \Big(\bigoplus_{\beta\neq\alpha_i} \Bbbk e_\beta\Big) \oplus \Big(\bigoplus_{\beta\neq\alpha_i} \Bbbk f_\beta\Big) \oplus (\Bbbk f_i \oplus \mathfrak{h}_{\leq\theta} \oplus \Bbbk e_i)$$

as an $\boldsymbol{\alpha}_p$-module. It only remains to compute the isotypic components of $\Bbbk f_i \oplus \mathfrak{h}_{\leq\theta} \oplus \Bbbk e_i$.

First, we assume that $a_{ii} = 2$. In this case $\mathfrak{h}_{\leq\theta} = \Bbbk h_i \oplus (\ker \xi_i \cap \mathfrak{h}_{\leq\theta})$ as vector spaces and $\{f_i, h_i, e_i\} \simeq \mathfrak{sl}_2$, hence $\Bbbk f_i \oplus \Bbbk h_i \oplus \Bbbk e_i \simeq L_3$ as an object in $\operatorname{Rep}(\boldsymbol{\alpha}_p)$. Also, $(\ker \xi_i) \cap \mathfrak{h}_{\leq\theta}$ is a sum of $\theta - 1$ copies of $L_1$, so (b) holds.

Now consider $a_{ii} = 0$. By hypothesis, there is $h \in \mathfrak{h}_{\leq\theta}$ such that $\xi_i(h) = 1$. Thus we have a linear complement $\mathfrak{h}_{\leq\theta} = \Bbbk h \oplus (\ker \xi_i \cap \mathfrak{h}_{\leq\theta})$. Since $[h, e_i] = e_i$, the subspace spanned by $h$ and $e_i$ is an $\boldsymbol{\alpha}_p$-submodule of $\mathfrak{g}$ isomorphic to $L_2$. Also, since $[h_i, e_i] = 0$, the subspace spanned by $h_i$ and $f_i$ is an $\boldsymbol{\alpha}_p$-submodule of $\mathfrak{g}$ isomorphic to $L_2$. Finally, any complement of $\Bbbk h_i$ in $\ker \xi_i \cap \mathfrak{h}_{\leq\theta}$ is a sum of $\theta - 2$ copies of $L_1$ and (c) follows. □

We show next that the submodules $M_\beta$, $N_\beta$ can be Lie-generated by submodules associated to simple roots.

**Proposition 4.9.** *If $\beta$ is an $\alpha_i$-string generator, then $M_\beta$ is contained in the* $\mathsf{Rep}(\boldsymbol{\alpha}_p)$*-Lie subalgebra of $(\mathfrak{g}', e_i)$ generated by the submodules $M_{\alpha_j}$, where $j \neq i$.*

*Proof.* The proof is by induction in $\mathsf{ht}(\beta)$. The base case $\mathsf{ht}(\beta) = 1$, i.e., $\beta$ a simple root, is clear.

For $\mathsf{ht}(\beta) > 1$, there exist positive roots $\alpha$ and $\gamma$ such that $\beta = \alpha + \gamma$; notice that both roots have smaller height and are independent with $\alpha_i$. Now, $\alpha$ and $\gamma$ belong to some $\alpha_i$-string, say $\alpha = \widetilde{\alpha} + c\alpha_i$ and $\gamma = \widetilde{\gamma} + d\alpha_i$ where $\widetilde{\alpha}$, $\widetilde{\gamma}$ are string generators. By Lemma 2.27, $e_\beta \in \Bbbk[e_\alpha, e_\gamma] \subset [M_{\widetilde{\alpha}}, M_{\widetilde{\gamma}}]$. Next, using the Jacobi identity, we see that for each $\beta + h\alpha_i$ in the string generated by $\beta$, we also have

$$e_{\beta+h\alpha_i} \in \Bbbk(\operatorname{ad} e_i)^h e_\beta \subset \Bbbk(\operatorname{ad} e_i)^h [e_\alpha, e_\gamma] \subset [M_{\widetilde{\alpha}}, M_{\widetilde{\gamma}}],$$

since $M_{\widetilde{\alpha}}$ and $M_{\widetilde{\gamma}}$ are $\operatorname{ad} e_i$-stable. This means that $M_\beta \subset [M_{\widetilde{\alpha}}, M_{\widetilde{\gamma}}]$, and now the proposition follows by induction and the Jacobi identity. □

Motivated by Propositions 4.8 and 4.9, we describe how the Lie bracket of $(\mathfrak{g}', e_i)$ behaves when restricted to tensor products of some simple submodules. We describe the adjoint action of an element in $\ker \xi_i$ as a first step.

**Lemma 4.10.** *Let $h \in \ker \xi_i$. If $\beta$ is an $\alpha_i$-string generator, then $h$ acts on $M_\beta$ by $\xi_\beta(h)$, and on $N_\beta$ by $-\xi_\beta(h)$.*

*Proof.* We verify first the claim concerning $M_\beta$. Let $\{\beta + k\alpha_i : 0 \leq k < d\}$ denote the $\alpha_i$-string through $\beta$. By (2.26), $[h, e_\beta] = \xi_\beta(h) e_\beta$. We show that $[h, e_{\beta+k\alpha_i}] = \xi_\beta(h) e_{\beta+k\alpha_i}$ for all such $k$. For $k = 1$, we write $e_{\beta+\alpha_i} = \mu[e_\beta, e_i]$ and, since $h \in \ker \xi_i$, we get

$$[h, e_{\beta+\alpha_i}] = \mu[[h, e_\beta], e_i] + \mu[e_\beta, [h, e_i]] = \mu\xi_\beta(h)[e_\beta, e_i] = \xi_\beta(h) e_{\beta+\alpha_i}.$$

The proof now follows inductively on $k$ using that $e_{\beta+k\alpha_i} \in \Bbbk[e_i, e_{\beta+(k-1)\alpha_i}]$.

The claim regarding the action on $N_\beta$ follows via the Chevalley involution. □

If $M$, $N$ are submodules of $(\mathfrak{g}', e_i)$, $[M, N]$ denotes the image of $[-,-]|_{M\otimes N} : M \otimes N \to \mathfrak{g}'$.

In the next couple of Lemmas, we describe $[-,-] \in \mathsf{Hom}_{\boldsymbol{\alpha}_p}(M_{\alpha_j} \otimes N_{\alpha_k}, \mathfrak{g}')$ for $j, k \in \mathbb{I}_\theta$, both different from $i$. The case $j \neq k$ is straightforward:

**Lemma 4.11.** *Let $j, k \in \mathbb{I}$, both different from $i$. If $j \neq k$, then $[M_{\alpha_j}, N_{\alpha_k}] = 0$.*

*Proof.* If $n, m \geq 0$, then $[(\operatorname{ad} e_i)^n e_j, (\operatorname{ad} f_i)^m f_k] = 0$ because $(n - m)\alpha_i + \alpha_j - \alpha_k$ cannot be a root. □

On the other hand, when $j=k$ a more careful analysis is needed.

**Notation 4.12.** If $a\in\Bbbk$ and $n\in\mathbb{N}$ we denote $(n)_a=n+a$ and $(n)^!_a=(n)_a(n-1)_a\cdots(1)_a$.

**Remark 4.13.** Assume that $a_{ii}=2$. Let $j\in\mathbb{I}_\theta$ different from $i$. Then the bracket in $\mathfrak{g}(A)$ satisfies

| $[-,-]$ | $f_j$ | $f_{ij}$ | $f_{iij}$ | $f_{iiij}$ |
|---|---|---|---|---|
| $e_j$ | $h_j$ | $a_{ji}f_i$ | $0$ | $0$ |
| $e_{ij}$ | $-a_{ji}e_i$ | $a_{ij}h_j+a_{ji}h_i$ | $2a_{ji}(1)_{a_{ij}}f_i$ | $0$ |
| $e_{iij}$ | $0$ | $-2a_{ji}(1)_{a_{ij}}e_i$ | $2(1)_{a_{ij}}(a_{ij}h_j+2a_{ji}h_i)$ | $6a_{ji}(2)^!_{a_{ij}}f_i$ |
| $e_{iiij}$ | $0$ | $0$ | $-6a_{ji}(2)^!_{a_{ij}}e_i$ | $6(2)^!_{a_{ij}}(a_{ij}h_j+3a_{ji}h_i)$ |

We fix cyclic generators for the indecomposable submodules from Proposition 4.8(b).

**Notation 4.14.** Assume $\mathfrak{g}$ is finite dimensional. Let $i\in\mathbb{I}_\theta$ be such that $a_{ii}=2$. Let $j\in\mathbb{I}_\theta$ be different from $i$. Then:

- $M_{\alpha_j}=\langle e_j\rangle=\Bbbk\{e_j,(\operatorname{ad}e_i)e_j,\dots,(\operatorname{ad}e_i)^{-c_{ij}}e_j\}$.
- $N_{\alpha_j}=\langle(\operatorname{ad}f_i)^{-c_{ij}}f_j\rangle$ which has (noncyclic) basis $\{(\operatorname{ad}f_i)^{-c_{ij}}f_j,(\operatorname{ad}f_i)^{1-c_{ij}}f_j,\dots,f_j\}$.
- $S$ denotes the copy of $L_3$ generated by $e_i$, which has a cyclic basis $\{f_i,h_i,-2e_i\}$.
- Write $\tilde h_j=h_j$ if $a_{ij}=0$, and $\tilde h_j=2h_j-a_{ji}h_i$ if $a_{ij}\neq0$. Then $\tilde h_j\in\ker\xi_i$ and generates a copy of $L_1$.
- For any direct summand $X$ of $\mathfrak{g}'$, denote by $\iota_X:X\hookrightarrow\mathfrak{g}'$ the canonical inclusion. If $Y$ is another object we also denote by $\iota_X$ the composition $X\oplus Y\twoheadrightarrow X\hookrightarrow\mathfrak{g}'$.

To obtain an explicit expression for the cyclic basis of $N_{\alpha_j}$, one uses the following:

**Remark 4.15.** Assume that $a_{ii}=2$. Let $j\in\mathbb{I}$ different from $i$. Then

$$[e_i,f_{ij}]=-a_{ij}f_j,\quad [e_i,f_{iij}]=-2(1)_{a_{ij}}f_{ij},\quad [e_i,f_{iiij}]=-3(2)_{a_{ij}}f_{iij}.$$

Now we can use these bases to explicitly describe $[-,-]\in\mathsf{Hom}_{\boldsymbol{\alpha}_p}(M_{\alpha_j}\otimes N_{\alpha_j},\mathfrak{g}')$.

**Lemma 4.16.** *Assume $\mathfrak{g}$ is finite dimensional and $a_{ii}=2$. Let $j\in\mathbb{I}$ different from $i$ and assume that $a_{ij}\in\mathbb{F}_p$. Consider on $M_{\alpha_j}$ and $N_{\alpha_j}$ the cyclic bases from Notation 4.14.*

(a) *If $c_{ij}=0$, then $[-,-]:M_{\alpha_j}\otimes N_{\alpha_j}\to\Bbbk\tilde h_j$ is the canonical isomorphism $L_1\otimes L_1\to L_1$.*

*For $-1\geq c_{ij}\geq-3$ we have $[M_{\alpha_j},N_{\alpha_j}]=\Bbbk\tilde h_j\oplus S=L_1\oplus L_3$. Also*

(b) *Assume $c_{ij}=-1$ and $p>2$. If we identify $M_{\alpha_j}\otimes N_{\alpha_j}=L_1\oplus L_3$ as in Example 3.1, then $[-,-]:M_{\alpha_j}\otimes N_{\alpha_j}\to\Bbbk\tilde h_j\oplus S$ acts as $\iota_{L_1}\oplus a_{ji}\iota_{L_3}$.*

(c) *Assume $c_{ij}=-2$ and $p>3$. If we identify $M_{\alpha_j}\otimes N_{\alpha_j}=L_1\oplus L_3\oplus L_5$ as in Example 3.2, then $[-,-]:M_{\alpha_j}\otimes N_{\alpha_j}\to\Bbbk\tilde h_j\oplus S$ acts as $6\iota_{L_1}\oplus4a_{ji}\iota_{L_3}$.*

(d) *Assume $c_{ij}=-3$ and $p=5$. If we identify $M_{\alpha_j}\otimes N_{\alpha_j}=L_1\oplus3L_5$ as in Example 3.3, then $[-,-]:M_{\alpha_j}\otimes N_{\alpha_j}\to\Bbbk\tilde h_j\oplus S$ restricted to $L_1$ acts as $72\iota_{L_1}$.*

(e) *Assume* $c_{ij} = -3$ *and* $p > 5$. *If we identify* $M_{\alpha_j} \otimes N_{\alpha_j} = L_1 \oplus L_3 \oplus L_5 \oplus L_7$ *as in* Example 3.3, *then* $[-,-] : M_{\alpha_j} \otimes N_{\alpha_j} \to \Bbbk\tilde{h}_j \oplus S$ *acts as* $72\iota_{L_1} \oplus 120a_{ji}\iota_{L_3}$.

*Proof.* (a) In this case $M_{\alpha_j} = \Bbbk\{e_j\}$, $N_{\alpha_j} = \Bbbk\{f_j\}$. Since $[e_j, f_j] = h_j$, the claim follows.

The inclusion $[M_{\alpha_j}, N_{\alpha_j}] \subseteq \Bbbk\{2h_j - a_{ji}h_i\} \oplus \Bbbk\{f_i, h_i, -2e_i\} = L_1 \oplus L_3$ follows directly from Remark 4.13.

(b) The cyclic bases for $M_{\alpha_j}$ and $N_{\alpha_j}$ given by Notation 4.14 are $M_{\alpha_j} = \Bbbk\{e_j, e_{ij}\}$ and $N_{\alpha_j} = \Bbbk\{f_{ij}, f_j\}$. According to Example 3.1, we have $M_{\alpha_j} \otimes N_{\alpha_j} = L_1 \oplus L_3$ with cyclic bases $L_1 = \Bbbk\{e_j \otimes f_j - e_{ij} \otimes f_{ij}\}$ and $L_3 = \Bbbk\{e_j \otimes f_{ij}, e_{ij} \otimes f_{ij} + e_j \otimes f_j, 2e_{ij} \otimes f_j\}$. Now we use Remark 4.13 to compute the action of the Lie bracket on these bases,

$$[e_j, f_j] - [e_{ij}, f_{ij}] = h_j - a_{ij}h_j - a_{ji}h_i = 2h_j - a_{ji}h_i, \qquad [e_j, f_{ij}] = a_{ji} f_i,$$
$$[e_{ij}, f_{ij}] + [e_j, f_j] = a_{ij}h_j + a_{ji} + h_j = a_{ji}h_i, \qquad 2[e_{ij}, f_j] = -2a_{ji}e_i,$$

and the claim follows.

(c) The cyclic bases for $M_{\alpha_j}$ and $N_{\alpha_j}$ given by Notation 4.14 are $M_{\alpha_j} = \Bbbk\{e_j, e_{ij}, e_{iij}\}$ and $N_{\alpha_j} = \Bbbk\{f_{iij}, 2f_{ij}, 4f_j\}$. According to Example 3.2 we have $M_{\alpha_j} \otimes N_{\alpha_j} = L_1 \oplus L_3 \oplus L_5$ with cyclic generators

$$L_1 = \langle 4e_j \otimes f_j - 2e_{ij} \otimes f_{ij} + e_{iij} \otimes f_{iij} \rangle, \quad L_3 = \langle 2e_j \otimes f_{ij} - e_{ij} \otimes f_{iij} \rangle, \quad L_5 = \langle e_j \otimes f_{iij} \rangle.$$

Next, we use Remark 4.13 to compute the Lie bracket on these generators

$$4[e_j, f_j] - 2[e_{ij}, f_{ij}] + [e_{iij}, f_{iij}] = 4h_j + 4h_j - 2a_{ji}h_i + 4h_j - 4a_{ji}h_i = 6(2h_j - a_{ji}h_i),$$

thus the bracket acts as $6\iota_{L_1}$ on $L_1$. Similarly

$$2[e_j, f_{ij}] - [e_{ij}, f_{iij}] = (2a_{ji} + 2a_{ji}) f_i = 4a_{ji} f_i$$

so the bracket acts as $4a_{ji}$ on the fixed basis of $L_3$. Finally

$$[e_j, f_{iij}] = 0.$$

Thus $L_5$ is annihilated by the Lie bracket, and the claim follows.

(e) The indecomposables $M_{\alpha_j}$ and $N_{\alpha_j}$ have cyclic bases

$$M_{\alpha_j} = \Bbbk\{e_j, e_{ij}, e_{iij}, e_{iiij}\} \quad \text{and} \quad N_{\alpha_j} = \Bbbk\{f_{iiij}, 3f_{iij}, 12f_{ij}, 36f_j\}.$$

According to Example 3.3, we can decompose $M_{\alpha_j} \otimes N_{\alpha_j} = L_1 \oplus L_3 \oplus L_5 \oplus L_7$ with cyclic generators

$$L_1 = \big\langle 36e_j \otimes f_j - 12e_{ij} \otimes f_{ij} + 3e_{iij} \otimes f_{iij} - e_{iiij} \otimes f_{iiij} \big\rangle,$$
$$L_3 = \big\langle 36e_j \otimes f_{ij} - 12e_{ij} \otimes f_{iij} + 3e_{iij} \otimes f_{iiij} \big\rangle,$$
$$L_5 = \langle 3e_j \otimes f_{iij} - e_{ij} \otimes f_{iiij} \rangle,$$
$$L_7 = \langle e_j \otimes f_{iiij} \rangle.$$

Using Remark 4.13, we compute the Lie bracket on the generators. In $L_1$ we have

$$\begin{aligned}36[e_j, f_j] - 12[e_{ij}, f_{ij}] + 3[e_{iij}, f_{iij}] - [e_{iiij}, f_{iiij}]\\ &= 36h_j - 12(-3h_j + a_{ji}h_i) - 12(-3h_j + 2a_{ji}h_i) - 12(-3h_j + 3a_{ji}h_i)\\ &= 72(2h_j - a_{ji}h_i),\end{aligned}$$

hence $[-,-]$ acts as $72\iota$ on $L_1$ (note that this argument also works for (d)). Similarly, the action on the generator of $L_3$ is

$$36[e_j, f_{ij}] - 12[e_{ij}, f_{iij}] + 3[e_{iij}, f_{iiij}] = (36 + 48 + 36)a_{ji} f_i = 120a_{ji} f_i.$$

And the generators of $L_5$ and $L_7$ are mapped to zero. □

The assumption $a_{ij} \in \mathbb{F}_p$ in Lemma 4.16 assures that $a_{ij}$ is the class of $c_{ij}$ modulo $p$, see (2.13), and is satisfied for all matrices $A$ such that $\dim \mathfrak{g}(A) < \infty$ except the Brown algebra in Example 4.21. Thus we will avoid the case when $a_{ij}$ is not in the prime field.

Next, we describe the adjoint action of $S = \{f_i, h_i, -2e_i\} \simeq L_3$ on itself and on the submodules $\tilde{h}_j$, $M_{\alpha_j}$, $N_{\alpha_j}$ from Notation 4.14. We will need the following computations:

**Remark 4.17.** Assume that $a_{ii} = 2$. Let $j \in \mathbb{I}$ different from $i$. Then

$$[f_i, e_j] = 0, \quad [f_i, e_{ij}] = -a_{ij}e_j, \quad [f_i, e_{iij}] = -2(1)_{a_{ij}}e_{ij}, \quad [f_i, e_{iiij}] = -3(2)_{a_{ij}}e_{iij}.$$

**Lemma 4.18.** *Assume $\mathfrak{g}$ is finite dimensional and $a_{ii} = 2$. Let $j \in \mathbb{I}$ different from $i$ and assume that $a_{ij} \in \mathbb{F}_p$. Consider on $S$, $M_{\alpha_j}$, and $N_{\alpha_j}$ the cyclic bases from Notation 4.14.*

(a) *We have $[S, S] = S$. If $p > 3$ and we identify $S \otimes S = L_1 \oplus L_3 \oplus L_5$ as in Example 3.2, then $[-,-] : S \otimes S \to S$ acts as $4\iota_{L_3}$.*

(b) *We have $[S, \tilde{h}_j] = 0$.*

(c) *If $c_{ij} = 0$, then $[S, M_{\alpha_j}] = [S, N_{\alpha_j}] = 0$.*

*For $-1 \geq c_{ij} \geq -3$, we have $[S, M_{\alpha_j}] = M_{\alpha_j}$ and $[S, N_{\alpha_j}] = N_{\alpha_j}$. Also*

(d) *Assume $c_{ij} = -1$ and $p > 3$. If we identify $S \otimes M_{\alpha_j} = L_2 \oplus L_4$ as in Example 3.4, then $[-,-] : S \otimes M_{\alpha_j} \to M_{\alpha_j}$ acts as $3\iota_{L_2}$.*

(e) *Assume $c_{ij} = -2$ and $p > 3$. If we identify $S \otimes M_{\alpha_j} = L_1 \oplus L_3 \oplus L_5$ as in Example 3.2, then $[-,-] : S \otimes M_{\alpha_j} \to M_{\alpha_j}$ acts as $4\iota_{L_3}$.*

(f) *Assume $c_{ij} = -3$ and $p = 5$. If we identify $S \otimes M_{\alpha_j} = L_2 \oplus 2L_5$ as in Example 3.5, then $[-,-] : S \otimes M_{\alpha_j} \to M_{\alpha_j}$ vanishes $L_2$.*

(g) *Assume $c_{ij} = -3$ and $p > 5$. If we identify $S \otimes M_{\alpha_j} = L_2 \oplus L_4 \oplus L_6$ as in Example 3.5, then $[-,-] : S \otimes M_{\alpha_j} \to M_{\alpha_j}$ acts as $15\iota_{L_4}$.*

*In (d)–(g), the exact same argument holds for $N_{\alpha_j}$ in place of $M_{\alpha_j}$ (with the same scalars).*

*Proof.* (a) According to Example 3.2, we have $S \otimes S = L_1 \oplus L_3 \oplus L_5$ with cyclic generators

$$L_1 = \langle -2f_i \otimes e_i - h_i \otimes h_i - 2e_i \otimes f_i \rangle, \quad L_3 = \langle f_i \otimes h_i - h_i \otimes f_i \rangle, \quad L_5 = \langle f_i \otimes f_i \rangle.$$

Now, it is easy to see that $[-,-]$ annihilates the generators of $L_1$ and $L_5$, and maps the generator of $L_3$ to $4f_i$.

(b) Follows immediately because $S$ is generated by $f_j$, and $\tilde{h}_j$ is in $\ker(\xi_i)$ by (c). In this case, $M_{\alpha_j} = \Bbbk\{e_j\}$. Since $f_i$, $h_i$, and $e_i$ annihilates $e_j$, we have $[S, M_{\alpha_j}] = 0$.

(d) According to Example 3.4, we have $S \otimes M_{\alpha_j} = L_2 \oplus L_4$, where

$$L_2 = \langle 2f_i \otimes e_{ij} - h_i \otimes e_j \rangle, \quad L_4 = \langle f_i \otimes e_j \rangle.$$

Now, we use Remark 4.17 to see that $[-,-]$ annihilates the generator of $L_4$ and maps that of $L_2$ to $3e_j$.

(e) From Example 3.2, we get $S \otimes M_{\alpha_j} = L_1 \oplus L_3 \oplus L_5$, with cyclic generators

$$L_1 = \langle f_i \otimes e_{iij} - h_i \otimes e_{ij} - 2e_i \otimes e_i \rangle, \quad L_3 = \langle f_i \otimes e_{ij} - h_i \otimes e_i \rangle, \quad L_5 = \langle f_i \otimes e_j \rangle.$$

Now the proof follows as in part (a).

(g) We use Example 3.2 to get $S \otimes L_3 = L_1 \oplus L_3 \oplus L_5$ with cyclic generators

$$L_2 = \langle -2f_i \otimes e_i - h_i \otimes h_i - 2e_i \otimes f_i \rangle, \quad L_4 = \langle f_i \otimes h_i - h_i \otimes f_i \rangle, \quad L_6 = \langle f_i \otimes e_i \rangle.$$

A straightforward computation using Remark 4.17 shows that $[-,-]$ annihilates the generators of $L_2$ and $L_6$, and it maps the generator of $L_4$ to $15e_j$. A similar argument works for (f). □

**4B.** ***The structure of the Lie algebra*** $\mathsf{S}(\mathfrak{g}'(A), e_i)$**.** We work towards a root system associated to the semisimplification $\mathsf{S}(\mathfrak{g}'(A), e_i)$, where $\mathfrak{g}(A)$ is a finite dimensional contragredient Lie algebra, and $i \in \mathbb{I}_\theta$ is such that $a_{ii} = 2$. Recall the sets $\Delta^A_{+,\min}$, $\Delta^A_{\min}$ from Notation 4.5.

Let $\pi_i : \mathbb{Z}^\theta \to \mathbb{Z}^{\theta-1}$ be the projection introduced in Lemma 2.4. Consider

$$\nabla_{\pm}^{\mathsf{S}(\mathfrak{g},e_i)} := \pi_i(\Delta^A_{\pm,\min}), \quad \nabla^{\mathsf{S}(\mathfrak{g},e_i)} := \pi_i(\Delta^A_{\min}) = \nabla_{+}^{\mathsf{S}(\mathfrak{g},e_i)} \cup \nabla_{-}^{\mathsf{S}(\mathfrak{g},e_i)}.$$

We will see that $\nabla^{\mathsf{S}(\mathfrak{g},e_i)}$ may contain positive multiples of roots. This behavior resembles that of contragredient Lie superalgebras, as described in Section 2D. Hence we introduce

$$\Delta_{\pm}^{\mathsf{S}(\mathfrak{g},e_i)} := \nabla_{\pm}^{\mathsf{S}(\mathfrak{g},e_i)} - \{n\beta : n \geq 2, \beta \in \nabla_{\pm}^{\mathsf{S}(\mathfrak{g},e_i)}\}, \quad \Delta^{\mathsf{S}(\mathfrak{g},e_i)} := \Delta_{+}^{\mathsf{S}(\mathfrak{g},e_i)} \cup \Delta_{-}^{\mathsf{S}(\mathfrak{g},e_i)}.$$

**Remark 4.19.** Recall the notations $\pi_i$ and $\Delta^A_{\pm,\min}$ from the beginning of Section 4B.

(i) Via $\pi_i : \mathbb{Z}^\theta \to \mathbb{Z}^{\theta-1}$, we obtain a grading of $(\mathfrak{g}, e_i)$ by $\mathbb{Z}^{\theta-1}$. Explicitly,

$$\mathfrak{g} = \bigoplus_{\gamma \in \mathbb{Z}^{\theta-1}} \mathfrak{g}_\gamma, \quad \text{where } \mathfrak{g}_\gamma := \bigoplus_{\beta \in \pi_i^{-1}(\gamma)} \mathfrak{g}_\beta.$$

(ii) For each $\beta \in \Delta^A_{+,\min}$, we have $M_\beta \subseteq \mathfrak{g}_{\pi_i(\beta)}$ and $N_\beta \subseteq \mathfrak{g}_{-\pi_i(\beta)}$. Thus $\mathsf{S}(\mathfrak{g}', e_i)$ inherits the $\mathbb{Z}^{\theta-1}$-graduation of $\mathfrak{g}'$. Moreover,

$$\mathsf{S}(\mathfrak{g}', e_i) = \mathsf{S}(\mathfrak{g}', e_i)_0 \oplus \Big( \bigoplus_{\gamma \in \nabla^{\mathsf{S}(\mathfrak{g}, e_i)}} \mathsf{S}(\mathfrak{g}, e_i)_\gamma \Big).$$

To illustrate the situation, we explicitly compute $\nabla^{\mathsf{S}(\mathfrak{g},e_i)}$ and $\Delta^{\mathsf{S}(\mathfrak{g},e_i)}$ for all finite Cartan matrices of rank 2. We show that, in all cases, $\Delta^{\mathsf{S}(\mathfrak{g},e_i)} = \{\pm\alpha_j\}$, where $j$ is the index different from $i$; however $\nabla^{\mathsf{S}(\mathfrak{g},e_i)}$ depends on the number of laces of the Dynkin diagram.

**Example 4.20.** Let $\mathfrak{g}$ be of type $A_2$; that is, $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. If we fix $i = 1$, then $\nabla_+^{\mathsf{S}(\mathfrak{g},e_1)} = \{2\}$. Indeed, $\mathfrak{g} = M_2 \oplus S \oplus \Bbbk\{2h_2 + h_1\} \oplus N_2$, where $M_2 = \Bbbk\{e_2, e_{12}\} \simeq L_2$, $N_2 = \Bbbk\{f_{12}, f_2\} \simeq L_2$, and one can easily verify using the bases from Example 3.1 that $[-,-]|_{M_2 \otimes M_2} = 0$.

**Example 4.21.** Let $p = 3$, $A = \left(\begin{smallmatrix} 2 & a \\ -1 & 2 \end{smallmatrix}\right)$, $a \notin \mathbb{F}_3$. Thus $\mathfrak{g}$ is the Brown Lie algebra $\mathfrak{br}(2, a)$.

- For $i = 1$, $\nabla_+^{\mathsf{S}(\mathfrak{g},e_1)} = \varnothing$ since $\mathfrak{g} = M_2 \oplus S \oplus \Bbbk\{2h_2 + h_1\} \oplus N_2$, where $M_2 = \Bbbk\{e_2, e_{12}, e_{112}\}$, $N_2 = \Bbbk\{f_{112}, 2f_{12}, 4f_2\}$, and $M_2, N_2 \simeq L_3$. Here we have $[-,-]|_{M_2 \otimes M_2} = 0$.
- For $i = 2$, we obtain $\nabla_+^{\mathsf{S}(\mathfrak{g},e_2)} = \{1, 1^2\}$ since

$$(\mathfrak{g}, e_2) = M_1 \oplus M_{1^22} \oplus S \oplus \Bbbk\{2h_1 + 2h_2\} \oplus N_1 \oplus N_{1^22},$$

where $M_1 = \Bbbk\{e_1, e_{12}\}$, $M_{1^22} = \Bbbk\{e_{112}\}$, $N_1 = \Bbbk\{f_{12}, f_1\}$, $N_{1^22} = \Bbbk\{f_{112}\}$.

According to Example 3.1, we have $M_1 \otimes M_1 \simeq L_1 \oplus L_3$, where the copy of $L_1$ has basis $e_1 \otimes e_{12} - e_{12} \otimes e_1$ so $[-,-] : M_1 \otimes M_1 \to M_{1^22} \simeq L_1$ is $2\iota_{L_1}$. It is straightforward to see that $[-,-]|_{M_{1^22} \otimes M_2} = 0$ and $[-,-]|_{M_{1^22} \otimes M_{1^22}} = 0$.

**Example 4.22.** Let $\mathfrak{g}$ be of type $B_2$; that is, $A = \left(\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right)$. In this case,

$$\nabla_+^{\mathsf{S}(\mathfrak{g},e_1)} = \begin{cases} \{2\}, & p > 3, \\ \varnothing, & p = 3, \end{cases} \qquad \nabla_+^{\mathsf{S}(\mathfrak{g},e_2)} = \{1, 1^2\}.$$

The proof and the description of the brackets are similar to those of Example 4.21.

**Example 4.23.** Assume $p > 3$ and consider $\mathfrak{g}$ of type $G_2$, that is, $A = \left(\begin{smallmatrix} 2 & -3 \\ -1 & 2 \end{smallmatrix}\right)$.

- For $i = 1$, we obtain $\nabla_+^{\mathsf{S}(\mathfrak{g},e_1)} = \{2, 2^2\}$. Indeed,

$$(\mathfrak{g}, e_1) = M_2 \oplus M_{1^32^2} \oplus S \oplus \Bbbk\{h_1 + 2h_2\} \oplus N_2 \oplus N_{1^32^2},$$

where

$$M_2 = \Bbbk\{e_2, e_{12}, e_{112}, e_{1112}\}, \qquad M_{1^32^2} = \Bbbk\{[e_{112}, e_{12}]\},$$
$$N_2 = \Bbbk\{f_{1112}, 3f_{112}, 12f_{12}, 36f_2\}, \qquad N_{1^32^2} = \Bbbk\{[f_{112}, f_{12}]\}.$$

Here $M_2 \otimes M_2 \simeq L_1 \oplus L_3 \oplus L_5 \oplus L_7$ when $p > 5$, while for $p = 5$ we have $M_2 \otimes M_2 \simeq L_1 \oplus 3L_5$; see Example 3.3. In any case, inside $M_2 \otimes M_2$ there is a copy of $L_1$ spanned by

$$e_2 \otimes e_{1112} - e_{12} \otimes e_{112} + e_{112} \otimes e_{12} - e_{1112} \otimes e_2.$$

Via $M_{1^32^2} \simeq L_1$, we see that $[-,-]: M_1 \otimes M_1 \to M_{1^32^2}$ is $4\iota_{L_1}$. We also have $[-,-]|_{M_{1^32^2}\otimes M_2} = 0$ and $[-,-]|_{M_{1^32^2}\otimes M_{1^32^2}} = 0$.

- If $i = 2$, we get $\nabla_+^{\mathsf{S}(\mathfrak{g},e_2)} = \{1, 1^2, 1^3\}$. In fact, we decompose

$$(\mathfrak{g}, e_2) = M_1 \oplus M_{1^22} \oplus M_{1^32} \oplus S \oplus \Bbbk\{2h_1 + 3h_2\} \oplus N_1 \oplus N_{1^22} \oplus N_{1^32},$$

where

$$M_1 = \Bbbk\{e_1, e_{12}\}, \quad M_{1^22} = \Bbbk\{e_{112}\}, \quad M_{1^32} = \Bbbk\{e_{1112}, \lambda_1[e_{112}, e_{12}]\},$$
$$N_1 = \Bbbk\{f_{12}, f_1\}, \quad N_{1^22} = \Bbbk\{f_{112}\}, \quad N_{1^32} = \Bbbk\{[f_{112}, f_{12}], \lambda_2 f_{1112}\}.$$

Here $M_1 \otimes M_1 \simeq L_1 \oplus L_3$, $M_{1^22} \simeq L_1$, and $M_{1^32} \simeq L_2$. Using Example 3.1 we see that $[-,-]: M_1 \otimes M_1 \to M_{1^22}$ is $2\iota_{L_1}$. Also, $[-,-]: M_1 \otimes M_{1^22} \to M_{1^32}$ is $4\iota_{L_2}$ and the other brackets between the $M_\alpha$ are 0.

We give another example, this time for a matrix of rank three.

**Example 4.24.** Let $p = 3$. We describe *root vectors* for semisimplifications of the Lie algebra $\mathfrak{br}(3)$ of dimension 29. There are two matrices realizing this Lie algebra, see [Skryabin 1993]:

$$A_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.25}$$

The corresponding positive roots are

$$\Delta_+^{A_1} = \{1, 12, 123, 1^22^33^4, 12^23^2, 12^23^3, 12^23^4, 12^33^4, 123^2, 2, 23^2, 23, 3\},$$
$$\Delta_+^{A_2} = \{1, 12^2, 12, 123^2, 12^33^2, 1^22^33^2, 12^23^2, 123, 12^23, 2, 23^2, 23, 3\}.$$

As $a_{33} = b_{33} = 0$, we consider semisimplifications by $e_i$ with $i \neq 3$. For $\mathfrak{g} = \mathfrak{g}(A_1)$ we have

$$\nabla_+^{\mathsf{S}(\mathfrak{g},e_1)} = \{2, 23, 23^2, 2^23^2, 2^23^3, 2^23^4, 2^33^4, 3\}, \quad \nabla_+^{\mathsf{S}(\mathfrak{g},e_2)} = \{1, 13, 1^23^4, 13^2, 13^3, 13^4, 3^2, 3\}.$$

And for $\mathfrak{g} = \mathfrak{g}(A_2)$ we have

$$\nabla_+^{\mathsf{S}(\mathfrak{g},e_1)} = \{2, 2^2, 2^33^2, 2^23^2, 2^23, 23^2, 23, 3\}, \quad \nabla_+^{\mathsf{S}(\mathfrak{g},e_2)} = \{13, 1^23^2, 3^2, 3\}.$$

We show details on the computations only for the last case. Notice that

$$(\mathfrak{g}, e_2) = M_1 \oplus M_{123} \oplus M_{123^2} \oplus M_{1^22^33^2} \oplus M_{23^2} \oplus M_3 \oplus S \oplus \Bbbk\{2h_1 + h_2\} \oplus \Bbbk\{2h_3 - h_2\}$$
$$\oplus N_1 \oplus N_{123} \oplus N_{123^2} \oplus N_{1^22^33^2} \oplus N_{23^2} \oplus N_3,$$

where

$$M_1 = \Bbbk\{e_1, e_{12}, e_{12^2}\}, \quad M_{123} = \Bbbk\{e_{123}, e_{2123}\}, \quad M_{123^2} = \Bbbk\{e_{3123}, e_{23123}, e_{223123}\},$$
$$M_{1^22^33^2} = \Bbbk\{[e_{2123}, e_{123}]\}, \quad M_{23^2} = \Bbbk\{e_{332}\}, \quad M_3 = \Bbbk\{e_3, e_{23}\}.$$

and the $N_\alpha$ have analogous descriptions. Here $M_1, M_{123^2} \simeq L_3$, so these vanish under the semisimplification functor. Also, $M_{123} \otimes M_{123} \simeq L_1 \oplus L_3 \simeq M_3 \otimes M_3$. One can see that $[-,-]: M_{123} \otimes M_{123} \to M_{1^22^33^2}$ is $\iota_{L_2}$, $[-,-]: M_3 \otimes M_3 \to M_{23^2}$ is $\iota_{L_2}$, and the other brackets between the $M_\alpha$ are 0.

**Proposition 4.26.** *Let $A$ be such that* $\dim \mathfrak{g}(A) < \infty$ *and* $i \in \mathbb{I}_\theta$ *such that* $a_{ii} = 2$.

(1) *If the Dynkin diagram of $A$ is simply laced, then* $\nabla^{\mathsf{S}(\mathfrak{g},e_i)} = \Delta^{\mathsf{S}(\mathfrak{g},e_i)}$.

(2) *If $A$ is of type $B_\theta$, then* $\nabla^{\mathsf{S}(\mathfrak{g},e_i)} = \Delta^{\mathsf{S}(\mathfrak{g},e_i)} \cup \{2\alpha_{i+1\,\theta}\}$.

(3) *If $A$ is of type $C_\theta$, then* $\nabla^{\mathsf{S}(\mathfrak{g},e_i)} = \begin{cases} \Delta^{\mathsf{S}(\mathfrak{g},e_i)}, & i \neq \theta, \\ \Delta^{\mathsf{S}(\mathfrak{g},e_\theta)} \cup \{2\alpha_{j\,\theta-1} : j \in \mathbb{I}_{\theta-1}\}, & i = \theta. \end{cases}$

(4) *If $A$ is of type $F_4$, then* $\nabla^{\mathsf{S}(\mathfrak{g},e_i)} = \begin{cases} \Delta^{\mathsf{S}(\mathfrak{g},e_i)}, & i = 1, 2, \\ \Delta^{\mathsf{S}(\mathfrak{g},e_3)} \cup \{2\beta : \beta = 2, 12\}, & i = 3, \\ \Delta^{\mathsf{S}(\mathfrak{g},e_4)} \cup \{2\beta : \beta = 23, 123, 12^23\}, & i = 4. \end{cases}$

*Proof.* For (1), it is enough to show that the entries of $\pi_i(\beta)$ are coprime for every $\beta \in \Delta_+^A$ different from $\alpha_i$, which we check case-by-case. In types $A_\theta$, $D_\theta$ and $E_6$ the roots $\beta \neq \alpha_i$ are either $\alpha_j$ with $j \neq i$ or else contain at least two coordinates equal to 1, so $\pi_i(\beta)$ keeps at least one coordinate equal to 1.

The same happens for all roots in types $E_7$ and $E_8$, except for the largest root in type $E_7$ and twelve roots in type $E_8$. However, these exceptional roots contain one coordinate equal to 1, another equal to 2 and another equal to 3.

For (2), at least one entry of the vector $\pi_i(\beta)$ equals 1, unless $\beta = \alpha_{i\theta} + \alpha_{i+1\,\theta}$: in this case, $\pi_i(\beta) = 2\alpha_{i+1\,\theta}$, and $\alpha_{i+1\,\theta} \in \Delta_+^{\mathsf{S}(\mathfrak{g},e_i)}$.

For (3), again $\pi_i(\beta)$ has at least one entry equal to one unless $i = \theta$ and $\beta$ is of the form $\beta = \alpha_{j\theta} + \alpha_{j\,\theta-1}$, for some $j \in \mathbb{I}_{\theta-1}$. In that case $\pi_\theta(\beta) = 2\alpha_{j\,\theta-1}$, and $\alpha_{j\,\theta-1} \in \Delta_+^{\mathsf{S}(\mathfrak{g},e_\theta)}$.

Finally, we consider (4), whence the positive roots are

$$\Delta_+^A = \big\{1, 12, 2, 1^22^23, 12^23, 123, 2^23, 23, 3, 1^22^43^34, 1^22^43^24, 1^22^33^24, 1^22^23^24, 1^22^234, 12^33^24, 12^23^24, 1^22^43^34^2, 12^234, 1234, 2^23^24, 2^234, 234, 34, 4\big\}. \quad (4.27)$$

If $i = 1, 2$, then $\pi_i(\beta)$ is not a multiple of another root for all $\beta \in \Delta_+^A$.

$i = 3$: The pairs $(\beta, \beta')$ such that $\pi_3(\beta) = 2\beta'$ are $(1^22^23, 12)$, $(2^23, 2)$. Otherwise $\pi_3(\beta)$ is not a multiple of another vector with integer entries.

$i = 4$: The pairs $(\beta, \beta')$ such that $\pi_4(\beta) = 2\beta'$ are $(1^22^43^24, 12^23)$, $(1^22^23^24, 123)$, $(2^23^24, 23)$.

And it is clear that for $i = 3, 4$, the unique $n \geq 2$ such that $\pi_i(\beta) = n\beta'$, for some $\beta, \beta'$ in $\Delta_+^A$, is $n = 2$. □

Let us push Notation 4.14 to $\mathsf{Ver}_p$.

**Notation 4.28.** Let $A \in \Bbbk^{\theta\times\theta}$ be such that $\mathfrak{g} = \mathfrak{g}(A)$ is finite dimensional, and fix $i \in \mathbb{I}_\theta$ with $a_{ii} = 2$. We set

$$\bar{\mathbb{I}} := \{j \in \mathbb{I} \mid j \neq i, 1 - c_{ij}^A < p\}.$$

For $\beta' \in \nabla_+^{\mathsf{S}(\mathfrak{g},e_i)}$ and $j \in \bar{\mathbb{I}}$, consider the following subobjects of $\mathsf{S}(\mathfrak{g}, e_i)$:

- $\overline{e_{\beta'}} := \mathsf{S}(M_\beta)$, $\overline{f_{\beta'}} := \mathsf{S}(N_\beta)$, where $\beta \in \Delta_{+,k}$ is such that $\pi_i(\beta) = \beta'$; both are isomorphic to $\mathtt{L}_k$.[2] In particular we set $\bar{e}_j := \mathsf{S}(M_{\alpha_j})$, $\bar{f}_j := \mathsf{S}(N_{\alpha_j})$; both are isomorphic to $\mathtt{L}_{1-c_{ij}^A}$.

[2]Here, $L_p = 0$.

- $\overline{h_j} := \mathsf{S}(\Bbbk\widetilde{h_j})$, isomorphic to $\mathtt{L}_1$.
- $\bar{S} := \mathsf{S}(S)$, isomorphic to $\mathtt{L}_3$, or 0 if $p = 3$.

Next we summarize the data describing the Lie algebras $\mathsf{S}(\mathfrak{g}, e_i)$.

**Definition 4.29.** We say that a root $\beta \in \Delta^A_{+,\min}$ is *$i$-good* if there exist $\alpha, \gamma \in \Delta^A_{+,\min}$ and a decomposition $\beta = \widetilde{\alpha} + \widetilde{\gamma}$ with $\widetilde{\alpha}$ and $\widetilde{\gamma}$ in the $i$-strings of $\alpha$ and $\gamma$, respectively.

In other words, $\beta$ is $i$-good if there exists a decomposition $\beta = \widetilde{\alpha} + \widetilde{\gamma}$ with $\widetilde{\alpha}, \widetilde{\gamma} \in \Delta^A_+$ such that $\widetilde{\alpha}$ and $\widetilde{\gamma}$ do not belong to $i$-strings of length $p$.

The existence of roots which are not $i$-good corresponds to the fact that the Lie algebra $\mathsf{S}(\mathfrak{g}, e_i)$ is not generated by $\bar{e}_j, \overline{f_j}, \overline{h_j}$. But if all roots are $i$-good, then we will see that $\mathsf{S}(\mathfrak{g}, e_i)$ is generated by $\bar{e}_j, \overline{f_j}, \overline{h_j}$.

**Example 4.30.** Assume that $p = 3$. Let $\mathfrak{g}$ be of type $F_4$ and $i = 1$; the positive roots are given in (4.27). Then $\beta = 12^33^24$ is not 1-good since the possible decompositions are

$$12^234,\ 23, \quad 1234,\ 2^23, \quad 123,\ 2^234, \quad 12^23,\ 234, \quad 12,\ 2^23^24, \quad 2,\ 12^23^24.$$

All decompositions have a root in a 1-string of length 3 since

$$M_{2^23} = \Bbbk e_{2^23} \oplus \Bbbk e_{12^23} \oplus \Bbbk e_{1^22^23}, \qquad M_{2^234} = \Bbbk e_{2^234} \oplus \Bbbk e_{12^234} \oplus \Bbbk e_{1^22^234},$$
$$M_{2^23^24} = \Bbbk e_{2^23^24} \oplus \Bbbk e_{12^23^24} \oplus \Bbbk e_{1^22^23^24}.$$

From here we can deduce that $\overline{e_{2^33^24}}$ does not belong to the subalgebra of $\mathsf{S}(\mathfrak{g}, e_i)$ generated by $\bar{e}_j$. This example corresponds to ($\star$) in [Kannan 2022, §4.6.5].

**Theorem 4.31.** *Let $A$ be such that $\dim \mathfrak{g}(A) < \infty$, $i \in \mathbb{I}_\theta$ such that $a_{ii} = 2$.*

(i) *$\mathsf{S}(\mathfrak{g}, e_i)$ has a triangular decomposition $\mathsf{S}(\mathfrak{g}, e_i) = \mathsf{S}(\mathfrak{g}, e_i)_+ \oplus \mathsf{S}(\mathfrak{g}, e_i)_0 \oplus \mathsf{S}(\mathfrak{g}, e_i)_-$, where*

$$\mathsf{S}(\mathfrak{g}, e_i)_\pm = \bigoplus_{\beta \in \nabla^{\mathsf{S}(\mathfrak{g}, e_i)}} \mathsf{S}(\mathfrak{g}, e_i)_{\pm\beta}, \qquad \mathsf{S}(\mathfrak{g}, e_i)_0 = \bar{S} \oplus \Big(\bigoplus_{j \neq i} \overline{h_j}\Big).$$

*Moreover the nontrivial homogeneous components are simple.*

(ii) *$\mathsf{S}(\mathfrak{g}, e_i)$ is a $\mathbb{Z}^{\theta-1}$-graded Lie algebra in $\mathsf{Ver}_p$.*

(iii) *$\mathsf{S}(\mathfrak{g}, e_i)$ has an invariant nondegenerate symmetric form $\mathtt{B}$ such that*

$$\mathtt{B}|_{\mathsf{S}(\mathfrak{g}, e_i)_\alpha \otimes \mathsf{S}(\mathfrak{g}, e_i)_\beta} = 0 \quad \text{if } \alpha \neq -\beta \in \mathbb{Z}^{\theta-1}.$$

(iv) *If every $\beta \in \Delta^A_{+,\min}$ is $i$-good, then $\mathsf{S}(\mathfrak{g}, e_i)$ is generated by $\bar{e}_j, \overline{f_j}, j \in \bar{\mathbb{I}}$, and $\mathsf{S}(\mathfrak{g}, e_i)_0$.*

(v) *If every $\beta \in \Delta^A_{+,\min}$ is $i$-good, then $\Delta^{\mathsf{S}(\mathfrak{g}, e_i)}$ is the (reduced) root system of the parabolic restriction orthogonal to $\alpha_i$.*

*Proof.* Notice that (i) follows from the triangular decomposition of $\mathfrak{g}$ and Proposition 2.23. Assertions (ii) and (iii) follow from the $\mathbb{Z}^\theta$-grading on $\mathfrak{g}$ and Lemma 3.10.

Next we deal with (iv). It suffices to prove that $\overline{e_{\beta'}}$ belongs to the subalgebra generated by $\overline{e_j}$, $j \in \bar{\mathbb{I}}$, for all $\beta' \in \nabla_+$, since the same argument shows that $\overline{f_{\beta'}}$ belongs to the subalgebra generated by $\overline{f_j}$, $j \in \bar{\mathbb{I}}$, for all $\beta' \in \nabla_+$. The proof is by induction on the height of $\beta'$. If $\beta'$ has height one, then $\beta' = \alpha_j$ for some $j \in \bar{\mathbb{I}}$ and the claim now follows by hypothesis. Now we assume that $\beta'$ has height $> 1$. Let $\beta \in \Delta^A_{+,\min}$ be such that $\beta' = \pi_i(\beta)$. As $\beta$ is $i$-good, there exist $\alpha, \gamma \in \Delta^A_{+,\min}$ and a decomposition $\beta = \widetilde{\alpha} + \widetilde{\gamma}$ with $\widetilde{\alpha}$ and $\widetilde{\gamma}$ in the $i$-strings of $\alpha$ and $\gamma$, respectively. Let $\alpha' = \pi_i(\alpha)$, $\gamma' = \pi_i(\gamma)$.

**Claim.** *The map* $[-,-] : \overline{e_{\alpha'}} \otimes \overline{e_{\gamma'}} \to \overline{e_{\beta'}}$ *is not zero.*

Let $a, b, c$ be the lengths of $i$-strings of $\alpha$, $\beta$ and $\gamma$, respectively. Let $j, k \geq 0$ be such that $\widetilde{\alpha} = \alpha + j\alpha_i$ and $\widetilde{\gamma} = \gamma + k\alpha_i$: we may assume that $j \leq k$ up to exchange $\alpha$ and $\gamma$. We look for the possible 5-tuples $(a, b, c, j, k)$. By Theorem 2.3, there exists an element $w$ of the Weyl group(oid) and $i_1, i_2, i_3 \in \mathbb{I}$ such that $w(\alpha_i) = \alpha_{i_1}$, $w(\alpha) \in \mathbb{N}_0\alpha_{i_1} + \mathbb{N}_0\alpha_{i_2}$, $w(\gamma) \in \mathbb{N}_0\alpha_{i_1} + \mathbb{N}_0\alpha_{i_2} + \mathbb{N}_0\alpha_{i_3}$: hence, $w(\beta) = w(\alpha) + w(\gamma) + (j+k)\alpha_{i_1} \in \mathbb{N}_0\alpha_{i_1} + \mathbb{N}_0\alpha_{i_2} + \mathbb{N}_0\alpha_{i_3}$ and $w$ sends the $i$-strings of $\alpha$, $\beta$, $\gamma$ bijectively to the $i_1$-strings of $w(\alpha)$, $w(\beta)$, $w(\gamma)$, respectively. Thus we have to look for the possible 5-tuples just in rank 3. If the submatrix of rank 3 is not connected, then the result is straightforward. For connected submatrices of rank 3 we obtain the following:

- $(1, 1, 1, 0, 0)$ for some triples of roots in type $C_3$ $(i = 3)$, and also in type $\mathfrak{br}(3)$ $(i = 1, 2)$.
- $(1, 2, 2, 0, 0)$ or $(2, 2, 1, 0, 0)$ for some triples of roots in types $A_3$ $(i = 1)$, $B_3$ $(i = 1, 2)$, $C_3$ $(i = 1, 3)$, and both matrices of type $\mathfrak{br}(3)$ $(i = 1, 2)$.
- $(2, 1, 2, 0, 1)$ for some triples of roots in types $A_3$ $(i = 2)$, $B_3$ $(i = 1, 2)$, $C_3$ $(i = 2, 3)$, and both matrices of type $\mathfrak{br}(3)$ $(i = 1, 2)$.
- $(1, 3, 3, 0, 0)$ for $\alpha = \alpha_1$, $\gamma = \alpha_2$ and $i = 3$ in type $B_3$.
- $(2, 3, 2, 0, 0)$ for $\alpha = \alpha_2$, $\gamma = \alpha_2 + \alpha_3$ and $i = 1$ in type $C_3$.
- $(2, 2, 3, 0, 1)$ for $\alpha = \alpha_1$, $\gamma = \alpha_3$ and $i = 2$ in type $C_3$.

Now we prove the claim for each possible 5-tuple. For $(1, t, t, 0, 0)$, $t \in \mathbb{I}_3$,

$$M_\alpha = \Bbbk e_\alpha \simeq L_1, \quad M_\beta = \bigoplus_{s=0}^{t-1} \Bbbk e_{\beta + s\alpha_i} \simeq L_t, \quad M_\gamma = \bigoplus_{s=0}^{t-1} \Bbbk e_{\gamma + s\alpha_i} \simeq L_t.$$

By Proposition 4.9 there exists $\mathtt{c} \neq 0$ such that $[e_\alpha, e_\gamma] = \mathtt{c} e_\beta$. As $[e_i, e_\alpha] = 0$, we have that

$$[e_\alpha, e_{\gamma + s\alpha_i}] = [e_\alpha, (\operatorname{ad} e_i)^s e_\gamma] = (\operatorname{ad} e_i)^s [e_\alpha, e_\gamma] = \mathtt{c} e_{\beta + s\alpha_i},$$

which implies that $[-,-] : \overline{e_{\alpha'}} \otimes \overline{e_{\gamma'}} = \mathrm{L}_1 \otimes \mathrm{L}_t \to \overline{e_{\beta'}} = \mathrm{L}_t$ is $\mathtt{c}$ times the canonical map.

For $(2, 1, 2, 0, 1)$, we have that $\beta = \alpha + \gamma + \alpha_i$ and

$$M_\alpha = \Bbbk e_\alpha \oplus \Bbbk (\operatorname{ad} e_i) e_\alpha \simeq L_2, \quad M_\beta = \Bbbk e_\beta \simeq L_1, \quad M_\gamma = \Bbbk e_\gamma \oplus \Bbbk (\operatorname{ad} e_i) e_\gamma \simeq L_2.$$

By Proposition 4.9, there exists $\mathsf{c} \neq 0$ such that $[e_\alpha, (\operatorname{ad} e_i)e_\gamma] = \mathsf{c} e_\beta$. As the $i$-string of $\beta$ has length 1, $\beta \pm \alpha_i \notin \Delta_+$, so $[e_i, e_\beta] = [e_\alpha, e_\gamma] = 0$. Thus

$$[(\operatorname{ad} e_i)e_\alpha, e_\gamma] = -[e_\alpha, (\operatorname{ad} e_i)e_\gamma] = -\mathsf{c} e_\beta,$$

which implies that $[-,-] : \overline{e_{\alpha'}} \otimes \overline{e_{\gamma'}} = \mathtt{L}_2 \otimes \mathtt{L}_2 \to \overline{e_{\beta'}} = \mathtt{L}_1$ is $2\mathsf{c}$ times the canonical map; see Example 3.1.

For $(2, 3, 2, 0, 0)$, $p > 3$, we have that $\beta = \alpha + \gamma$ and

$$M_\alpha = \Bbbk e_\alpha \oplus \Bbbk(\operatorname{ad} e_i)e_\alpha \simeq L_2, \quad M_\beta = \bigoplus_{0 \le s \le 2} \Bbbk(\operatorname{ad} e_i)^s e_\beta \simeq L_3, \quad M_\gamma = \Bbbk e_\gamma \oplus \Bbbk(\operatorname{ad} e_i)e_\gamma \simeq L_2.$$

By Proposition 4.9, there exists $\mathsf{c} \neq 0$ such that $[e_\alpha, e_\gamma] = \mathsf{c} e_\beta$. The copy of $L_3$ inside $M_\alpha \otimes M_\gamma$ as in Example 3.1 is spanned by $e_\alpha \otimes e_\gamma$, $e_\alpha \otimes (\operatorname{ad} e_i)e_\gamma + (\operatorname{ad} e_i)e_\alpha \otimes e_\gamma$ and $2(\operatorname{ad} e_i)e_\alpha \otimes (\operatorname{ad} e_i)e_\gamma$. Using the Jacobi identity and that $(\operatorname{ad} e_i)^2 e_\alpha = (\operatorname{ad} e_i)^2 e_\gamma = 0$,

$$\mathsf{c}(\operatorname{ad} e_i)e_\beta = (\operatorname{ad} e_i)[e_\alpha, e_\gamma] = [e_\alpha, (\operatorname{ad} e_i)e_\gamma] + [(\operatorname{ad} e_i)e_\alpha, e_\gamma],$$
$$\mathsf{c}(\operatorname{ad} e_i)^2 e_\beta = (\operatorname{ad} e_i)^2 [e_\alpha, e_\gamma] = 2[(\operatorname{ad} e_i)e_\alpha, (\operatorname{ad} e_i)e_\gamma],$$

so $[-,-] : \overline{e_{\alpha'}} \otimes \overline{e_{\gamma'}} = \mathtt{L}_2 \otimes \mathtt{L}_2 \to \overline{e_{\beta'}} = \mathtt{L}_3$ is $\mathsf{c}$ times the canonical map.

For $(2, 2, 3, 0, 1)$, $p > 3$, we have that $\beta = \alpha + \gamma + \alpha_i$ and

$$M_\alpha = \Bbbk e_\alpha \oplus \Bbbk(\operatorname{ad} e_i)e_\alpha \simeq L_2, \quad M_\beta = \Bbbk e_\beta \oplus \Bbbk(\operatorname{ad} e_i)e_\beta \simeq L_2, \quad M_\gamma = \bigoplus_{0 \le s \le 2} \Bbbk(\operatorname{ad} e_i)^s e_\gamma \simeq L_3.$$

By Proposition 4.9, there exists $\mathsf{c} \neq 0$ such that $[e_\alpha, (\operatorname{ad} e_i)e_\gamma] = \mathsf{c} e_\beta$. The copy of $L_2$ inside $M_\alpha \otimes M_\gamma$ as in Example 3.4 is spanned by $e_\alpha \otimes (\operatorname{ad} e_i)e_\gamma - 2(\operatorname{ad} e_i)e_\alpha \otimes e_\gamma$ and $e_\alpha \otimes (\operatorname{ad} e_i)^2 e_\gamma - (\operatorname{ad} e_i)e_\alpha \otimes (\operatorname{ad} e_i)e_\gamma$. Using the Jacobi identity and that $[e_\alpha, e_\gamma] = 0$, we check that $[-,-] : \overline{e_{\alpha'}} \otimes \overline{e_{\gamma'}} = \mathtt{L}_2 \otimes \mathtt{L}_3 \to \overline{e_{\beta'}} = \mathtt{L}_2$ is $3\mathsf{c}$ times the canonical map.

Finally (v) follows from Lemma 2.4 and the following claim: if $\beta \in \mathbb{Z}^{\theta-1}$ has coprime entries and $m \ge 2$ is such that $m\beta \in \pi_i(\nabla^A)$, then $\beta \in \pi_i(\nabla^A)$. The proof of the claim follows from Proposition 4.26 and Examples 4.20–4.24. □

Next we describe explicitly some examples of Lie algebras in $\mathsf{Ver}_p$ obtained by semisimplification as in Theorem 4.31. To describe them as objects in $\mathsf{Ver}_p$, we use the notation $\mathtt{L}_i^{(n)}$: a copy of $\mathtt{L}_i$ in degree $n \in \mathbb{Z}$. Also, $\mathsf{b} : \mathsf{S}(\mathfrak{g}, e_i) \otimes \mathsf{S}(\mathfrak{g}, e_i) \to \mathsf{S}(\mathfrak{g}, e_i)$ denotes the bracket.

**Example 4.32.** As in Examples 4.20–4.22, we take $\mathfrak{g}$ of type either $A_2$ or $B_2$, and consider the semisimplification with respect to $e_1$. The Cartan matrix is $\left(\begin{smallmatrix} 2 & -a \\ -1 & 2 \end{smallmatrix}\right)$, with $a$ being either 1 for type $A_2$, or $a = 2$ for $B_2$. As an object in $\mathsf{Ver}_p$, we have

$$\mathsf{S}(\mathfrak{g}, e_1) = \mathtt{L}_{a+1}^{(1)} \oplus (\mathtt{L}_3^{(0)} \oplus \bar{\mathtt{L}}_1^{(0)}) \oplus \mathtt{L}_{a+1}^{(-1)}, \quad \mathtt{L}_{a+1}^{(1)} = \overline{M_2}, \quad \mathtt{L}_3^{(0)} = \bar{S}, \quad \mathtt{L}_1^{(0)} = \overline{h_2}, \quad \mathtt{L}_{a+1}^{(-1)} = \overline{N_2}.$$

Here, $\mathtt{L}_3^{(n)} = 0$ if $p = 3$. By Lemmas 4.10, 4.16 and 4.18,

$$\mathsf{b}|_{\mathtt{L}_1^{(0)} \otimes \mathtt{L}_{a+1}^{(\pm 1)}} = \pm(4-a)\iota_{\mathtt{L}_{a+1}^{(\pm 1)}}, \quad \mathsf{b}|_{\mathtt{L}_{a+1}^{(1)} \otimes \mathtt{L}_{a+1}^{(-1)}} = 6^{a-1}\iota_{\mathtt{L}_1^{(0)}} \oplus (-2^{2a-1}\iota_{\mathtt{L}_3^{(0)}}),$$
$$\mathsf{b}|_{\mathtt{L}_3^{(0)} \otimes \mathtt{L}_{a+1}^{(\pm 1)}} = (a+2)\iota_{\mathtt{L}_{a+1}^{(\pm 1)}}, \qquad \mathsf{b}|_{\mathtt{L}_1^{(0)} \otimes \mathtt{L}_3^{(0)}} = \mathsf{b}|_{\mathtt{L}_{a+1}^{(\pm 1)} \otimes \mathtt{L}_{a+1}^{(\pm 1)}} = 0.$$

**Example 4.33.** Let $\mathfrak{g}$ be of type $B_2$ as in Example 4.22, $i=2$. As an object in $\mathsf{Ver}_p$,

$$\mathsf{S}(\mathfrak{g},e_2)=\mathsf{L}_1^{(2)}\oplus\mathsf{L}_2^{(1)}\oplus(\mathsf{L}_3^{(0)}\oplus\bar{\mathsf{L}}_1^{(0)})\oplus\mathsf{L}_2^{(-1)}\oplus\mathsf{L}_1^{(-2)}.$$

By Lemma 4.10, 4.16, 4.18 and direct computation,

$$\begin{aligned}
&\mathsf{b}|_{\mathsf{L}_1^{(0)}\otimes\mathsf{L}_2^{(\pm1)}}=\pm2\iota_{\mathsf{L}_2^{(\pm1)}}, && \mathsf{b}|_{\mathsf{L}_3^{(0)}\otimes\mathsf{L}_2^{(\pm1)}}=2\iota_{\mathsf{L}_2^{(\pm1)}}, && \mathsf{b}|_{\mathsf{L}_2^{(1)}\otimes\mathsf{L}_2^{(-1)}}=\iota_{\mathsf{L}_1^{(0)}}\oplus(-2\iota_{\mathsf{L}_3^{(0)}}),\\
&\mathsf{b}|_{\mathsf{L}_1^{(0)}\otimes\mathsf{L}_1^{(\pm2)}}=\pm4\iota_{\mathsf{L}_1^{(\pm2)}}, && \mathsf{b}|_{\mathsf{L}_2^{(\pm1)}\otimes\mathsf{L}_2^{(\pm1)}}=2\iota_{\mathsf{L}_1^{(\pm2)}}, && \mathsf{b}|_{\mathsf{L}_1^{(2)}\otimes\mathsf{L}_1^{(-2)}}=\iota_{\mathsf{L}_1^{(0)}},
\end{aligned}$$

and the remaining brackets $\mathsf{b}|_{\mathsf{L}_i^{(m)}\otimes\mathsf{L}_j^{(\pm n)}}$ are zero.

**Example 4.34.** Assume $p>3$ and consider $\mathfrak{g}$ of type $G_2$ as in Example 4.23. For $i=1$,

$$\mathsf{S}(\mathfrak{g},e_1)=\mathsf{L}_1^{(2)}\oplus\mathsf{L}_4^{(1)}\oplus(\mathsf{L}_3^{(0)}\oplus\bar{\mathsf{L}}_1^{(0)})\oplus\mathsf{L}_4^{(-1)}\oplus\mathsf{L}_1^{(-2)}.$$

By Lemma 4.10, 4.16, 4.18 and direct computation,

$$\begin{aligned}
&\mathsf{b}|_{\mathsf{L}_1^{(0)}\otimes\mathsf{L}_4^{(\pm1)}}=\pm\iota_{\mathsf{L}_2^{(\pm1)}}, && \mathsf{b}|_{\mathsf{L}_3^{(0)}\otimes\mathsf{L}_4^{(\pm1)}}=15\iota_{\mathsf{L}_4^{(\pm1)}}, && \mathsf{b}|_{\mathsf{L}_4^{(1)}\otimes\mathsf{L}_4^{(-1)}}=72\iota_{\mathsf{L}_1^{(0)}}\oplus(-120\iota_{\mathsf{L}_3^{(0)}}),\\
&\mathsf{b}|_{\mathsf{L}_1^{(0)}\otimes\mathsf{L}_1^{(\pm2)}}=\pm2\iota_{\mathsf{L}_1^{(\pm2)}}, && \mathsf{b}|_{\mathsf{L}_4^{(\pm1)}\otimes\mathsf{L}_4^{(\pm1)}}=4\iota_{\mathsf{L}_1^{(\pm2)}}, && \mathsf{b}|_{\mathsf{L}_1^{(2)}\otimes\mathsf{L}_1^{(-2)}}=3\iota_{\mathsf{L}_1^{(0)}},\\
&\mathsf{b}|_{\mathsf{L}_1^{(2)}\otimes\mathsf{L}_4^{(-1)}}=\iota_{\mathsf{L}_4^{(1)}}, && \mathsf{b}|_{\mathsf{L}_1^{(-2)}\otimes\mathsf{L}_4^{(1)}}=\iota_{\mathsf{L}_4^{(-1)}},
\end{aligned}$$

and the remaining brackets $\mathsf{b}|_{\mathsf{L}_i^{(m)}\otimes\mathsf{L}_j^{(\pm n)}}$ are zero.

Set now $i=2$. As an object in $\mathsf{Ver}_p$,

$$\mathsf{S}(\mathfrak{g},e_2)=\mathsf{L}_2^{(3)}\oplus\mathsf{L}_1^{(2)}\oplus\mathsf{L}_2^{(1)}\oplus(\mathsf{L}_3^{(0)}\oplus\bar{\mathsf{L}}_1^{(0)})\oplus\mathsf{L}_2^{(-1)}\oplus\mathsf{L}_1^{(-2)}\oplus\mathsf{L}_2^{(-3)}.$$

By Lemma 4.10, 4.16, 4.18 and direct computation,

$$\begin{aligned}
&\mathsf{b}|_{\mathsf{L}_1^{(0)}\otimes\mathsf{L}_2^{(\pm1)}}=\pm\iota_{\mathsf{L}_2^{(\pm1)}}, && \mathsf{b}|_{\mathsf{L}_3^{(0)}\otimes\mathsf{L}_2^{(\pm1)}}=3\iota_{\mathsf{L}_2^{(\pm1)}}, && \mathsf{b}|_{\mathsf{L}_2^{(1)}\otimes\mathsf{L}_2^{(-1)}}=\iota_{\mathsf{L}_1^{(0)}}\oplus(-3\iota_{\mathsf{L}_3^{(0)}}),\\
&\mathsf{b}|_{\mathsf{L}_1^{(0)}\otimes\mathsf{L}_1^{(\pm2)}}=\pm2\iota_{\mathsf{L}_1^{(\pm2)}}, && \mathsf{b}|_{\mathsf{L}_3^{(0)}\otimes\mathsf{L}_2^{(\pm3)}}=3\iota_{\mathsf{L}_2^{(\pm3)}}, && \mathsf{b}|_{\mathsf{L}_1^{(2)}\otimes\mathsf{L}_1^{(-2)}}=3\iota_{\mathsf{L}_1^{(0)}},\\
&\mathsf{b}|_{\mathsf{L}_1^{(0)}\otimes\mathsf{L}_2^{(\pm3)}}=\pm3\iota_{\mathsf{L}_2^{(\pm3)}}, && \mathsf{b}|_{\mathsf{L}_2^{(\pm1)}\otimes\mathsf{L}_2^{(\pm1)}}=\iota_{\mathsf{L}_1^{(\pm2)}}, && \mathsf{b}|_{\mathsf{L}_2^{(3)}\otimes\mathsf{L}_2^{(-3)}}=6\iota_{\mathsf{L}_1^{(0)}}\oplus(-6\iota_{\mathsf{L}_3^{(0)}}),\\
&\mathsf{b}|_{\mathsf{L}_2^{(\pm1)}\otimes\mathsf{L}_1^{(\pm2)}}=\iota_{\mathsf{L}_2^{(\pm3)}}, && \mathsf{b}|_{\mathsf{L}_1^{(\pm2)}\otimes\mathsf{L}_2^{(\mp1)}}=\iota_{\mathsf{L}_2^{(\pm1)}},\\
&\mathsf{b}|_{\mathsf{L}_2^{(\pm3)}\otimes\mathsf{L}_2^{(\mp1)}}=3\iota_{\mathsf{L}_1^{(\pm2)}}, && \mathsf{b}|_{\mathsf{L}_2^{(\pm3)}\otimes\mathsf{L}_1^{(\mp2)}}=6\iota_{\mathsf{L}_2^{(\pm1)}},
\end{aligned}$$

and the remaining brackets $\mathsf{b}|_{\mathsf{L}_i^{(m)}\otimes\mathsf{L}_j^{(\pm n)}}$ are zero.

**Corollary 4.35.** *Let $p=3$, and take $A$ and $i$ as in Theorem 4.31(iv). Then $\mathsf{S}(\mathfrak{g},e_i)\simeq\mathfrak{g}(B,\boldsymbol{p})$, where*

$$\boldsymbol{p}=(p_j)_{j\in\bar{\mathbb{I}}},\quad p_j=\begin{cases}1, & a_{ij}=0,\\ -1, & a_{ij}\neq0;\end{cases}\qquad B=(b_{jk})_{j,k\in\bar{\mathbb{I}}},\quad b_{jk}=\begin{cases}a_{jk}, & a_{ij}=0,\\ -a_{jk}-a_{ji}a_{ik}, & a_{ij}\neq0.\end{cases}$$

*Proof.* Let $\bar{\mathfrak{g}}:=\mathsf{S}(\mathfrak{g},e_i)$. The $\mathbb{Z}^{\theta-1}$-grading of $\bar{\mathfrak{g}}$ induces a $\mathbb{Z}$-grading such that $\bar{e}_j\in\bar{\mathfrak{g}}_1$, $\bar{f}_j\in\bar{\mathfrak{g}}_{-1}$, $j\in\bar{\mathbb{I}}$, and $\bar{h}_k\in\bar{\mathfrak{g}}_0$, $k\neq i$. By Proposition 4.8 (a), if $a_{ij}=0$, then $M_{\alpha_j}\simeq L_1\simeq N_{\alpha_j}$, so $\bar{e}_j$, $\bar{f}_j$ are even, while if $a_{ij}\neq0$ and $j\in\bar{\mathbb{I}}$, then $M_{\alpha_j}\simeq L_2\simeq N_{\alpha_j}$; thus the parity of the $\bar{e}_j$ and the $\bar{f}_j$ is given by $\boldsymbol{p}$.

By Theorem 4.31, $\bar{\mathfrak{g}}$ is generated by $\overline{e_j}$, $\overline{f_j}$, $j \in \bar{\mathbb{I}}$, and $\overline{h_k}$, $k \neq i$, since $S = 0$. By Lemmas 4.11 and 4.16, $[\overline{e_j}, \overline{f_k}] = \delta_{jk}\overline{h_j}$ up to rescale the $\overline{f_j}$. Following Notation 4.14,

$$[\widetilde{h_j}, e_k] = \begin{cases} a_{jk}e_k, & a_{ij} = 0, \\ -(a_{jk} + a_{ji}a_{ik})e_k, & a_{ij} \neq 0. \end{cases}$$

Thus $[\overline{h_j}, \overline{e_k}] = b_{jk}\overline{e_k}$ for all $j, k \in \bar{\mathbb{I}}$. Analogously, $[\overline{h_j}, \overline{f_k}] = -b_{jk}\overline{f_k}$ for all $j, k \in \bar{\mathbb{I}}$. As $\mathfrak{h}$ is abelian, we also have that $[\overline{h_j}, \overline{h_k}] = 0$ for all $j, k$. Thus relations (2.5) hold in $\bar{\mathfrak{g}}$, which implies that there is a surjective map $\widetilde{\mathfrak{g}}(B, \boldsymbol{p}) \twoheadrightarrow \bar{\mathfrak{g}}$.

As $\mathfrak{g}(B, \boldsymbol{p})$ is the quotient of $\widetilde{\mathfrak{g}}(B, \boldsymbol{p})$ by the maximal ideal trivially intersecting $\mathfrak{h}$, the map above factorizes through a surjective map $\pi : \bar{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}(B, \boldsymbol{p})$. As $\bar{\mathfrak{g}}$ has a nondegenerate invariant symmetric bilinear form by Theorem 4.31, the map $\pi$ is an isomorphism. □

**Remark 4.36.** This corollary is related with [Kannan 2022, Theorem 3.6.5] when the characteristic is three. It applies to those examples in [loc. cit.] where the element of the Lie algebra is homogeneous.

**Example 4.37.** We consider semisimplifications of the Lie algebra $\mathfrak{br}(3)$; see Example 4.24.

(1) The semisimplification of $\mathfrak{g}(A_1)$ under $\operatorname{ad} e_1$ is the Lie superalgebra $\mathfrak{g}(B_1, \boldsymbol{p}_1)$, where

$$B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{p}_1 = (1, 0).$$

(2) The semisimplification of $\mathfrak{g}(A_1)$ under $\operatorname{ad} e_2$ is the Lie superalgebra $\mathfrak{g}(B_2, \boldsymbol{p}_2)$, where

$$B_2 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{p}_2 = (1, 1).$$

(3) The semisimplification of $\mathfrak{g}(A_2)$ under $\operatorname{ad} e_1$ is the Lie superalgebra $\mathfrak{g}(B_3, \boldsymbol{p}_3)$, where

$$B_3 = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{p}_3 = (1, 0).$$

(4) The semisimplification of $\mathfrak{g}(A_2)$ under $\operatorname{ad} e_2$ does not fit in the context of Corollary 4.35 since $\beta = \alpha_1 + \alpha_2 + \alpha_3$ is not 2-good.

That is, the three possible semisimplifications give the Lie superalgebra $\mathfrak{brj}(2; 3)$: we recover the three possible realizations of $\mathfrak{brj}(2; 3)$ as a contragredient Lie superalgebra. This corresponds to the construction given in [Kannan 2022, §4.2].

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# The degree of a tropical root surface of type A

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We prove that the tropical surface of the root system $A_{n-1}$ has degree $\frac{1}{2}n(n-1)(n-2)$.

## 1. Introduction

Tropical geometry was developed to answer questions in classical algebraic geometry combinatorially. Tropicalization converts a projective variety $V$ into a polyhedral complex $\operatorname{trop}(V)$ that, roughly speaking, records the behavior of $V$ at infinity. The tropical variety $\operatorname{trop}(V)$ retains a surprising amount of information about $V$, such as its dimension and degree. Many important invariants of $\operatorname{trop}(V)$ can be computed using combinatorics and discrete geometry, thus giving computations of algebro-geometric invariants of $V$. For detailed introductions to tropical geometry, see [Brugallé and Shaw 2014; Maclagan and Sturmfels 2015; Mikhalkin and Rau 2010].

Initially, tropical geometry was most interested in studying tropicalizations of algebraic varieties of importance. However, a more robust theory arises when one considers abstract tropical varieties, most of which do not arise via tropicalization. This is analogous to the situation in matroid theory, where a linear subspace $V$ of a vector space gives rise to a matroid $M_V$, but a more robust theory arises when one considers all matroids, most of which do not arise from a linear subspace. (This is not just an analogy: matroids may be understood as the tropical fans of degree 1 [Ardila and Klivans 2006; Fink 2013].)

Tropical geometry is then a rich source of well motivated combinatorial problems of significance within and beyond combinatorics. A good theory needs good examples, and combinatorics is a rich source of tropical varieties. In this spirit, Ardila, Kato, McMillon, Perez, and Schindler construct tropical surfaces associated in a natural way to the classical root systems.

The geometric protagonist of this paper is the tropical surface $S(A_{n-1})$ associated to the root system $A_{n-1}$ of the special linear Lie algebra $\mathfrak{sl}_n$. Our main result is that this surface has degree $\frac{1}{2}n(n-1)(n-2)$.

## 2. Background

Let $n$ be a positive integer and write $[n] = \{1, 2, \dots, n\}$. Let $\mathsf{e}_1, \dots, \mathsf{e}_n$ be the standard basis of $\mathbb{R}^n$, and write $\mathsf{e}_S = \sum_{s \in S} \mathsf{e}_s$ for each subset $S \subseteq [n]$.

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**2.1.** ***Root systems.*** Let us begin by defining root systems and root polytopes.

**Definition 1** [Bourbaki 1968]. A *crystallographic root system* $\Phi$ is a set of vectors in $\mathbb{R}^n$ satisfying:

- For every root $\beta \in \Phi$, the set $\Phi$ is closed under reflection across the hyperplane perpendicular to $\beta$.
- For any two roots $\alpha, \beta \in \Phi$, the quantity $2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle$ is an integer, where $\langle -, - \rangle$ is the standard inner product in $\mathbb{R}^n$.
- If $\beta, c\beta \in \Phi$ for $c \in \mathbb{R}$, then $c = 1$ or $c = -1$.

**Definition 2** [Bourbaki 1968]. An *irreducible root system* is one that cannot be partitioned into the union of two proper subsets $\Delta = \Delta_1 \cup \Delta_2$, such that $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \Delta_1$ and $\beta \in \Delta_2$.

Root systems play a fundamental role in many areas of mathematics; for example, they are key to the classification of semisimple Lie algebras [Bourbaki 1968]. The irreducible root systems have been classified into four infinite *classical families* and five *exceptional* root systems. In this paper we focus on the most classical family,

$$A_{n-1} = \{\mathsf{e}_i - \mathsf{e}_j \, : \, i, j \in [n], \ i \neq j\}.$$

This is the root system of the special linear Lie algebra $\mathfrak{sl}_n$.

**Definition 3.** The *root polytope* $P(\Phi)$ of a root system $\Phi$ is the convex hull of $\Phi$.

**2.2.** ***Tropical geometry.*** To define the root surfaces $S(A_{n-1})$ that interest us, we first introduce some basic definitions from tropical geometry. Let $N_{\mathbb{Z}} \cong \mathbb{Z}^r$ be a lattice and let $N = N_{\mathbb{Z}} \otimes \mathbb{R} \cong \mathbb{R}^r$ be the corresponding vector space. A *cone* is a set of the form

$$\text{cone}(v_1, \ldots, v_n) = \{\lambda_1 v_1 + \cdots + \lambda_n v_n \, : \, \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\}$$

for vectors $v_1, \ldots, v_n$ in $N$. The cone is *rational* if it is generated by lattice vectors. A (rational) *polyhedral fan* is a nonempty finite collection $\Sigma$ of (rational) cones in $N$ such that every face of a cone in $\Sigma$ is also in $\Sigma$, and the intersection of any two cones in $\Sigma$ is a face of both of them. A fan is *pure* of dimension $d$ if all maximal faces are $d$-dimensional. We let $\Sigma^i$ denote the set of cones of $\Sigma$ of dimension $i$. Tropical fans are those that meet the following balancing condition.

**Definition 4** [Maclagan and Sturmfels 2015]. Let $\Sigma \subseteq N$ be a rational polyhedral fan, pure of dimension $d$, with a choice of weight $w(\sigma) \in \mathbb{N}$ for each maximal cone $\sigma \in \Sigma^d$.

For each $(d-1)$-cone $\tau \in \Sigma^{d-1}$, consider the $(d-1)$-subspace $L_\tau \subseteq N$ spanned by $\tau$, the induced $(d-1)$-lattice $L_{\tau,\mathbb{Z}} = L_\tau \cap N_{\mathbb{Z}}$, and the quotient $(n-d+1)$-lattice $N(\tau) = N_{\mathbb{Z}} / L_{\tau,\mathbb{Z}}$. Each $d$-cone $\sigma \in \Sigma^d$ with $\sigma \supset \tau$ determines a ray $(\sigma + L_\tau)/L_\tau$ in $N/L_\tau$. This ray is rational with respect to the lattice $N(\tau)$; let $u_{\sigma/\tau}$ be the first lattice point on this ray. The fan $\Sigma$ is *balanced at* $\tau$ if

$$\sum_{\sigma \in \Sigma^d : \sigma \supset \tau} w(\sigma) u_{\sigma/\tau} = 0 \quad \text{in } N/L_\tau.$$

The fan $\Sigma$ is a *tropical fan* if it is balanced at all faces of dimension $d - 1$.[1]

[1]Notice that the balancing condition for $\Sigma$ at $\tau$ only depends on the fan $\text{Star}_\Sigma(\tau)$, with weights inherited from $\Sigma$.

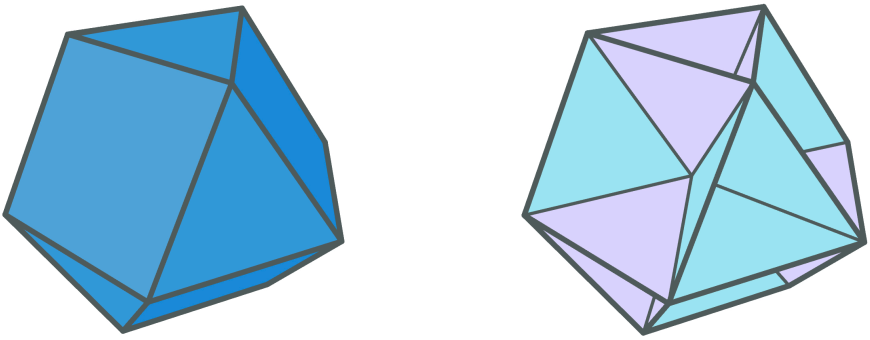

**Figure 1.** The root polytope $P(A_3)$ and the tropical root surface $S(A_3)$.

Tropical varieties are more general than tropical fans; see [Mikhalkin and Rau 2010] for a definition. *Tropical surfaces* are tropical varieties that are pure of dimension 2. In particular, 2-dimensional tropical fans are tropical surfaces.

**Definition 5.** The *tropical root surface* $S(\Phi)$ of a root system $\Phi$ is the cone over the 1-skeleton of $P(\Phi)$ with unit weights on all facets. It consists of:

- Rays: $\mathrm{cone}(r)$ for each $r \in \Phi$.
- Facets: $\mathrm{cone}(r, s)$ for each $r, s \in \Phi$ such that $rs$ is an edge of the root polytope $P(\Phi)$.
- Weights: $w(\sigma) = 1$ for every facet $\sigma$.

Since the root system $A_{n-1}$ is $(n-1)$-dimensional, we regard $S(A_{n-1})$ as a tropical root surface in the lattice $N_{\mathbb{Z}} = \mathbb{Z}^n/\mathbb{Z}$ with $N = \mathbb{R}^n/\mathbb{R}$.

Tropical root surfaces were introduced in [Perez 2019; Schindler 2017; Ardila et al. $\geq$ 2025] by Federico Ardila, Chiemi Kato, Jewell McMillon, Maria Isabel Perez, and Anna Schindler. Figure 1 shows the root polytope and the tropical surface of the root system $A_3$, with its cones truncated for visibility.

In classical algebraic geometry, the degree of an irreducible projective variety of dimension $d$ is obtained by counting its intersection points with a generic linear space of codimension $d$. In tropical geometry, degree is defined similarly. The analog of a generic linear space is the standard tropical linear space of codimension $d$, which we now define. Let us write $\mathsf{e}_1, \ldots, \mathsf{e}_n$ for the image of the unit vectors of $\mathbb{R}^n$ in $\mathbb{R}^n/\mathbb{R}$.

**Definition 6** [Ardila and Klivans 2006; Maclagan and Sturmfels 2015]. The *standard tropical linear space* $\Sigma_{n,n-d}$ is the tropical fan in $\mathbb{R}^n/\mathbb{R}$ whose facets are the $(n-1-d)$-dimensional cones

$$C_I := \{x \in \mathbb{R}^n/\mathbb{R} : \min_{1 \leq j \leq n} x_j = x_i \text{ for all } i \in I\} = \mathrm{cone}\{\mathsf{e}_i : i \notin I\}$$

for each choice of a $(d+1)$-subset $I \subseteq [n]$, where every facet has weight 1. Its support (i.e., the union of all of its cones) is

$$|\Sigma_{n,n-d}| = \{x \in \mathbb{R}^n : \text{ the smallest } d+1 \text{ entries of } x \text{ are equal to each other}\}. \tag{1}$$

The fan $\Sigma_{n,n-d}$ is the *coarse subdivision* of the *Bergman fan* of the uniform matroid $U_{n,n-d}$; it is the tropicalization of any sufficiently generic $(n-d)$-dimensional linear space in $n$-space, as shown in [Ardila and Klivans 2006]. We remark that we are using the *min* convention of tropical geometry, although the *max* convention would give the same results, since tropical root surfaces are symmetric across the origin.

**Definition 7.** Consider two tropical fans $\Sigma_1$ and $\Sigma_2$ of complementary dimensions $d_1$ and $d_2$ in a vector space $N$; that is, $d_1 + d_2 = \dim N$. We say $\Sigma_1$ and $\Sigma_2$ *intersect transversally* if $\Sigma_1 \cap \Sigma_2$ is a finite union of points, and each such point $p$ can be written uniquely as $p = \sigma_1 \cap \sigma_2$ for facets $\sigma_1, \sigma_2$ of $\Sigma_1$, $\Sigma_2$, respectively. The *weight* of each intersection point $p$ is

$$w(p) \coloneqq w(\sigma_1)w(\sigma_2)\,[N_{\mathbb{Z}} : L_{\sigma_1,\mathbb{Z}} + L_{\sigma_2,\mathbb{Z}}].$$

We call $\operatorname{index}(p) \coloneqq [N_{\mathbb{Z}} : L_{\sigma_1,\mathbb{Z}} + L_{\sigma_2,\mathbb{Z}}]$ the *index* of $p$. The *degree of the transversal intersection at $p$* is

$$\Sigma_1 \cdot \Sigma_2 \coloneqq \sum_{p \in \Sigma_1 \cap \Sigma_2} w(p).$$

If $\Sigma_1$ and $\Sigma_2$ are balanced but do not necessarily intersect transversally, then $\Sigma_1$ and $\Sigma_2 + v$ do intersect transversally for generic vectors $v \in N$, and the balancing condition implies that the degree of their transversal intersection does not depend on $v$ [Mikhalkin and Rau 2010, Propositions 4.3.3, 4.3.6]. Thus we define the *degree of the intersection* to be

$$\Sigma_1 \cdot \Sigma_2 \coloneqq (v + \Sigma_1) \cdot \Sigma_2$$

for generic $v$.

**Definition 8.** The *degree of a tropical fan* $\Sigma$ in $N = \mathbb{R}^n/\mathbb{R}$ of dimension $d$ is the degree of its intersection with the standard tropical linear space of codimension $d$:

$$\deg \Sigma \coloneqq \Sigma \cdot \Sigma_{n,n-d}.$$

In practice, to find the degree of a tropical fan $\Sigma$, one chooses a convenient generic vector $v \in \mathbb{R}^n/\mathbb{R}$ and performs the following steps.

(1) Find the intersections of $v + \Sigma$ with $\Sigma_{n,n-d}$.

(2) For each intersection point $p$ identify the cones $v + \sigma_1$ of $v + \Sigma$ and $\sigma_2$ of $\Sigma_{n,n-d}$ containing it, and find the weight of that intersection.

(3) Find the degree of $\Sigma$ by adding the weights of the intersection points above.

Step (2) can be carried out by choosing lattice bases $\{\alpha_i\}$ and $\{\beta_i\}$ for $L_{\sigma_1,\mathbb{Z}}$ and $L_{\sigma_2,\mathbb{Z}}$, so that $L_{\sigma_1,\mathbb{Z}} = \mathbb{Z}\langle \alpha_1, \dots, \alpha_{d_1}\rangle$ and $L_{\sigma_2,\mathbb{Z}} = \mathbb{Z}\langle \beta_1, \dots, \beta_{d_2}\rangle$. Then the index of the intersection $p$ can be computed as

$$\operatorname{index}(p) = |\det(\alpha_1, \dots, \alpha_{d_1}, \beta_1, \dots, \beta_{d_2})|.$$

## 3. The tropical root surface of type A and its degree

The following result first appeared in [Schindler 2017], in a slightly different form. We include a proof for completeness.

**Proposition 9.** *The tropical root surface $S(A_{n-1})$ is a tropical surface.*

*Proof.* We verify the balancing condition for an arbitrary ray $r = \operatorname{cone}(\mathsf{e}_i - \mathsf{e}_k)$. The maximal cones of $S(A_{n-1})$ containing $r$ are the cones over the edges of the root polytope $P(A_{n-1})$ containing the vertex $\mathsf{e}_i - \mathsf{e}_k$; these are known [Ardila et al. 2011; Cellini and Marietti 2015] to be

$$\mathcal{A}_{i,jk} = \operatorname{cone}(\mathsf{e}_i - \mathsf{e}_j, \mathsf{e}_i - \mathsf{e}_k), \quad \mathcal{A}_{ij,k} = \operatorname{cone}(\mathsf{e}_i - \mathsf{e}_k, \mathsf{e}_j - \mathsf{e}_k), \qquad \text{for } j \neq i, k. \tag{2}$$

The primitive vectors in these cones with respect to $r$ are $\overline{\mathsf{e}_i - \mathsf{e}_j}$ and $\overline{\mathsf{e}_j - \mathsf{e}_k}$ in $N/\mathbb{Z}(\mathsf{e}_i - \mathsf{e}_k)$, respectively. Then the balancing condition for $r$ says

$$\sum_{\sigma \in \Sigma^2 : \sigma \supset r} w(\sigma) u_{\sigma/r} = \sum_{j \neq i,k} \overline{(\mathsf{e}_i - \mathsf{e}_j)} + \sum_{j \neq i,k} \overline{(\mathsf{e}_j - \mathsf{e}_k)} = (n-2)\overline{(\mathsf{e}_i - \mathsf{e}_k)} = 0,$$

as desired. It follows that $S(A_{n-1})$ is indeed a tropical surface. □

We can now state and prove our main result.

**Theorem 10.** *The degree of the tropical root surface $S(A_{n-1})$ is $\frac{1}{2}n(n-1)(n-2)$.*

*Proof.* We follow the approach outlined at the end of Section 2.2, studying the intersection of $v + S(A_{n-1})$ with $\Sigma_{n,n-2}$, where $v$ is the *superincreasing* translation vector

$$v = (0, 1, 10, 100, 1000, \ldots).$$

It can be verified that this vector is generic by adding a small vector $\epsilon$ to it, and verifying that the intersection of $v + S(A_{n-1})$ with $\Sigma_{n,n-2}$, described below, has the same combinatorial structure as the intersection of $(v+\epsilon) + S(A_{n-1})$ with $\Sigma_{n,n-2}$.

(A) First we find the intersection points of $v + S(A_{n-1})$ and $\Sigma_{n,n-2}$.

For each 2-cone $\sigma \in S(A_{n-1})$ we need to find the points $v + s$ for $s \in \sigma$ whose three smallest entries $I$ are equal, so that $v + s \in C_I \subseteq \Sigma_{n,n-2}$ by (1). The maximal cones of $S(A_{n-1})$ are of the form $\mathcal{A}_{i,jk}$ and $\mathcal{A}_{ij,k}$ for $i, j, k$ pairwise distinct, as defined in (2). We consider these two types of cones separately.

(A1) Let us find the intersection points of $v + \mathcal{A}_{i,jk}$ and $\Sigma_{n,n-2}$ for $i, j, k$ pairwise distinct.

Let $s = a(\mathsf{e}_i - \mathsf{e}_j) + b(\mathsf{e}_i - \mathsf{e}_k) = (a+b)\mathsf{e}_i - a\mathsf{e}_j - b\mathsf{e}_k \in \mathcal{A}_{i,jk}$ for $a, b \geq 0$. To make the three smallest entries of $v + s$ equal, we need to choose one entry $i$ of $v = (0, 1, 10, 100, \ldots)$ to add $(a+b)$ to, and two entries $j$ and $k$ to subtract $a$ and $b$ from, respectively. Let $m = \min_{1 \leq i \leq n}(v+s)_i$ be the smallest entry of $v + s$, which appears at least three times. This places constraints on $a$ and $b$, as well as $i$, $j$, and $k$, as we now explain in detail. Consider the following cases.

Case 1.1. $m < 0$. To achieve this minimum we would have to subtract from at least 3 entries of $v$. Since we can only subtract from 2 entries, this case does not contribute any intersection points.

Case 1.2. $m = 0$. To achieve $m = 0$, we must leave entry $v_1 = 0$ unchanged, subtract from any two other entries $j, k > 1$ (necessarily subtracting $a = v_j = 10^{j-2}$ and $b = v_k = 10^{k-2}$), and add (necessarily $a+b$) to any of the remaining entries $i \neq 1, j, k$. The set of minimal coordinates is $I = \{1, j, k\}$. There are $\binom{n-1}{2}(n-3)$ possible intersection points in this case.

Case 1.3. $0 < m < 1$. To achieve such a value of $m$, we would have to add $a+b = m$ to $v_1$ and subtract $a = 10^{j-2} - m$ and $b = 10^{k-2} - m$ to two other entries $j, k > 1$. This would imply that $10^{j-2} + 10^{k-2} = 3m$, which is impossible because the left hand side is at least 11 and the right hand side is less than 3. Thus this case does not contribute any intersection points.

Case 1.4. $m = 1$. In this case we must add at least 1 to entry $v_1 = 0$, leave entry $v_2 = 1$ unchanged, and subtract from two other entries $j, k > 2$. At least one of those two entries, say $j$, must lead to a minimum coordinate $(v+s)_j = 1$, so $a = 10^{j-2} - 1$. This means that $(v+s)_1 = a+b > 1$ is not a minimum coordinate, so $(v+s)_k = 1$ must be the other minimum coordinate, and $b = 10^{k-2} - 1$. The set of minimal coordinates is $I = \{2, j, k\}$. There are $\binom{n-2}{2}$ possible intersection points in this case.

Case 1.5. $m > 1$. In this case we would have to add to the entries $v_1 = 0$ and $v_2 = 1$ to make them greater than or equal to $m$. Since we can only add to one entry, this case does not contribute any intersection points.

(A2) Now let us find the intersection points of $v + \mathcal{A}_{ij,k}$ and $\Sigma_{n,n-2}$ for $i, j, k$ pairwise distinct.

Let $s = a(\mathsf{e}_i - \mathsf{e}_k) + b(\mathsf{e}_j - \mathsf{e}_k) = a\mathsf{e}_i + b\mathsf{e}_j - (a+b)\mathsf{e}_k \in \mathcal{A}_{ij,k}$ for $a, b \geq 0$. To make the three smallest entries of $v+s$ equal to each other, we need to choose two entries $i$ and $j$ of $v = (0, 1, 10, 100, \ldots)$ to add $a$ and $b$ to, respectively, and one entry $k$ to subtract $(a+b)$ from. Let $m = \min_{1 \leq i \leq n}(v+s)_i$ be the smallest coordinate of $v+s$, which appears at least three times. Consider the following cases:

Case 2.1. $m < 1$. To achieve this value of $m$ we would need to subtract from two of the original entries of $v$, which is impossible. Thus, this case does not contribute any intersection points.

Case 2.2. $m = 1$. A value of $m = 1$ can only be achieved in entries $1, 2, k$ of $v+s$ for some $k > 2$. We must add $a = 1$ to $v_1$, leave $v_2$ unchanged, subtract $a+b = 10^{k-2} - 1$ from $v_k$, and hence add $b = 10^{k-2} - 2$ to some other entry $j \neq 1, 2, k$. The set of minimal coordinates is $I = \{1, 2, k\}$. This case contributes $(n-2)(n-3)$ intersection points.

Case 2.3. $1 < m < 10$. Again, such a value of $m$ can only be achieved in entries $1, 2, k$ of $v+s$ for some $k > 2$. Now, for these three new entries to equal $m$, we must add $a = m$ to $v_1$, add $b = m-1$ to $v_2$, and subtract $a+b = 10^{k-2} - m$ from $v_k$. This forces $m + (m-1) = 10^{k-2} - m$, which gives $m = \frac{1}{3}(10^{k-2}+1)$. Since $1 < m < 10$, we must have $k = 3$. Thus, the set of minimal coordinates is $I = \{1, 2, 3\}$, and this case contributes a single intersection point.

Case 2.4. $m = 10$. In order to make the three smallest entries of $v+s$ equal to 10, we have the following three options. (1) Add $a = 10$ to $v_1$, add $b = 9$ to $v_2$, leave $v_3$ untouched, and subtract 19 from any of the remaining entries $k > 3$. Here $I = \{1, 2, k\}$. (2) Add $a = 10$ to $v_1$, subtract $a + b = 10^{k-1} - 10$ from $v_k$ with $k > 3$, and add $b = 10^{k-1} - 20$ to $v_2$. Here $I = \{1, 3, k\}$. (3) Add $a = 9$ to $v_2$, subtract $a + b = 10^{k-1} - 10$ from $v_k$ with $k > 3$, and add $b = 10^{k-1} - 19$ to $v_1$. Here $I = \{2, 3, k\}$. In each of these options $k$ can be any number between 4 and $n$. This case contributes $3(n-3)$ intersection points.

Case 2.5. $m > 10$. To achieve this value of $m$, we would need to add to the three smallest entries of $v$, which is impossible. Thus, this case does not contribute any intersection points.

(B) We now find the multiplicity of each of the intersection points $p$ that we found in (A).

In each case we have identified the cones of $v + S(A_{n-1})$ and $\Sigma_{n,n-2}$ that the intersection $p = v + s$ belongs to. We now compute the multiplicity as the absolute value of the determinant of the matrix whose columns are lattice bases for the planes generated by these cones.

Suppose that $p$ is the intersection point of cone $v + \mathcal{A}_{i,jk}$ (or $v + \mathcal{A}_{ij,k}$) of $v + S(A_{n-1})$ and cone $C_I$ of $\Sigma_{n,n-2}$, where $I = \{i_{n-2}, i_{n-1}, i_n\}$ and $[n] - I = \{i_1, \ldots, i_{n-3}\}$, as described in (2) and Definition 6. Then the index of that intersection is

$$\begin{aligned}\text{index}(p) &= |\det(\mathsf{e}_{i_1}, \mathsf{e}_{i_2}, \mathsf{e}_{i_3} \ldots, \mathsf{e}_{i_{n-3}}, \mathsf{e}_i - \mathsf{e}_j, \mathsf{e}_i - \mathsf{e}_k, \mathsf{e}_{[n]})| \\ &= |\det(\mathsf{e}_{i_1}, \mathsf{e}_{i_2}, \mathsf{e}_{i_3} \ldots, \mathsf{e}_{i_{n-3}}, \mathsf{e}_{i_{n-2}} + \mathsf{e}_{i_{n-1}} + \mathsf{e}_{i_n}, \mathsf{e}_i - \mathsf{e}_j, \mathsf{e}_i - \mathsf{e}_k)|.\end{aligned}$$

The analogous result holds for the cones of the form $v + \mathcal{A}_{ij,k}$.[2]

(B1) Case 1.2. In this case we have $I = \{1, j, k\}$ so

$$\begin{aligned}\text{index}(p) &= |\det(\widehat{\mathsf{e}_1}, \mathsf{e}_2, \mathsf{e}_3, \ldots, \mathsf{e}_i, \ldots, \widehat{\mathsf{e}_j}, \ldots, \widehat{\mathsf{e}_k}, \ldots, \mathsf{e}_n, \mathsf{e}_1 + \mathsf{e}_j + \mathsf{e}_k, \mathsf{e}_i - \mathsf{e}_j, \mathsf{e}_i - \mathsf{e}_k)| \\ &= |\det(\widehat{\mathsf{e}_1}, \mathsf{e}_2, \mathsf{e}_3, \ldots, \mathsf{e}_i, \ldots, \widehat{\mathsf{e}_j}, \ldots, \widehat{\mathsf{e}_k}, \ldots, \mathsf{e}_n, \mathsf{e}_1 + \mathsf{e}_j + \mathsf{e}_k, \mathsf{e}_j, \mathsf{e}_k)| \\ &= |\det(\widehat{\mathsf{e}_1}, \mathsf{e}_2, \mathsf{e}_3, \ldots, \mathsf{e}_i, \ldots, \widehat{\mathsf{e}_j}, \ldots, \widehat{\mathsf{e}_k}, \ldots, \mathsf{e}_n, \mathsf{e}_1, \mathsf{e}_j, \mathsf{e}_k)| \\ &= 1,\end{aligned}$$

where in the first step we subtract column $\mathsf{e}_i$ from columns $\mathsf{e}_i - \mathsf{e}_j$ and $\mathsf{e}_i - \mathsf{e}_k$ and change their signs, and in the second step we subtract $\mathsf{e}_j$ and $\mathsf{e}_k$ from $\mathsf{e}_1 + \mathsf{e}_j + \mathsf{e}_k$. Thus all intersections in this case have multiplicity 1.

Case 1.4. Here $I = \{2, j, k\}$ and $i = 1$ so

$$\begin{aligned}\text{index}(p) &= |\det(\mathsf{e}_1, \widehat{\mathsf{e}_2}, \mathsf{e}_3, \ldots, \widehat{\mathsf{e}_j}, \ldots, \widehat{\mathsf{e}_k}, \ldots, \mathsf{e}_n, \mathsf{e}_2 + \mathsf{e}_j + \mathsf{e}_k, \mathsf{e}_1 - \mathsf{e}_j, \mathsf{e}_1 - \mathsf{e}_k)| \\ &= |\det(\mathsf{e}_1, \widehat{\mathsf{e}_2}, \mathsf{e}_3, \ldots, \widehat{\mathsf{e}_j}, \ldots, \widehat{\mathsf{e}_k}, \ldots, \mathsf{e}_n, \mathsf{e}_2 + \mathsf{e}_j + \mathsf{e}_k, \mathsf{e}_j, \mathsf{e}_k)| \\ &= |\det(\mathsf{e}_1, \widehat{\mathsf{e}_2}, \mathsf{e}_3, \ldots, \widehat{\mathsf{e}_j}, \ldots, \widehat{\mathsf{e}_k}, \ldots, \mathsf{e}_n, \mathsf{e}_2, \mathsf{e}_j, \mathsf{e}_k)| \\ &= 1.\end{aligned}$$

It follows that each intersection in case 1.4 has multiplicity 1.

[2] We need the additional $n$-th entry $\mathsf{e}_{[n]}$ in these formulas because the $n-1$ generating vectors live in $\mathbb{R}^n/\mathbb{R}\mathsf{e}_{[n]}$, and we wish to regard them as vectors in $\mathbb{R}^n$ in order to compute their determinant.

(B2) Case 2.2. Here $I = \{1, 2, k\}$ and $i = 1$ so

$$\begin{aligned}
\operatorname{index}(p) &= |\det(\widehat{e_1}, \widehat{e_2}, e_3, e_4, \dots, \widehat{e_k}, \dots, e_n, e_1 + e_2 + e_k, e_1 - e_k, e_j - e_k)| \\
&= |\det(\widehat{e_1}, \widehat{e_2}, e_3, e_4, \dots, \widehat{e_k}, \dots, e_n, e_1 + e_2 + e_k, e_1 - e_k, e_k)| \\
&= |\det(\widehat{e_1}, \widehat{e_2}, e_3, e_4, \dots, \widehat{e_k}, \dots, e_n, e_1 + e_2 + e_k, e_1, e_k)| \\
&= |\det(\widehat{e_1}, \widehat{e_2}, e_3, e_4, \dots, \widehat{e_k}, \dots, e_n, e_2, e_1, e_k)| \\
&= 1,
\end{aligned}$$

where we first subtract $e_j$ from $e_j - e_k$ and change the sign to $e_k$, then we add $e_k$ to $e_1 - e_k$, and then we subtract $e_1$ and $e_k$ from $e_1 + e_2 + e_k$. Again, it follows that each one of these intersections has multiplicity 1.

Case 2.3. In this case $I = \{1, 2, 3\}$ and $i = 1$, $j = 2, k = 3$, so

$$\begin{aligned}
\operatorname{index}(p) &= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \dots, e_n, e_1 + e_2 + e_3, e_1 - e_3, e_2 - e_3)| \\
&= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \dots, e_n, 3e_3, e_1 - e_3, e_2 - e_3)| \\
&= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \dots, e_n, 3e_3, e_1, e_2)| \\
&= 3,
\end{aligned}$$

where we first subtract $e_1 - e_3$ and $e_2 - e_3$ from $e_1 + e_2 + e_3$, and then we add one third of $3e_3$ to $e_1 - e_3$ and $e_2 - e_3$. Thus this intersection has multiplicity 3.

Case 2.4. Here we had three options: In option 1 we had $I = \{1, 2, 3\}$ and $i = 1$, $j = 2$, so

$$\begin{aligned}
\operatorname{index}(p) &= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \dots, e_k, \dots, e_n, e_1 + e_2 + e_3, e_1 - e_k, e_2 - e_k)| \\
&= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \dots, e_k, \dots, e_n, e_1 + e_2 + e_3, e_1, e_2)| \\
&= |\det(\widehat{e_1}, \widehat{e_2}, \widehat{e_3}, e_4, \dots, e_k, \dots, e_n, e_3, e_1, e_2)| \\
&= 1,
\end{aligned}$$

where we first add $e_k$ to $e_1 - e_k$ and $e_2 - e_k$, and then subtract $e_1$ and $e_2$ from $e_1 + e_2 + e_3$. These intersections then have multiplicity 1.

In option 2 we had $I = \{1, 3, k\}$ and $i = 1$, $j = 2$, so

$$\begin{aligned}
\operatorname{index}(p) &= |\det(\widehat{e_1}, e_2, \widehat{e_3}, e_4, \dots, \widehat{e_k}, \dots, e_n, e_1 + e_3 + e_k, e_1 - e_k, e_2 - e_k)| \\
&= |\det(\widehat{e_1}, e_2, \widehat{e_3}, e_4, \dots, \widehat{e_k}, \dots, e_n, e_1 + e_3 + e_k, e_1 - e_k, e_k)| \\
&= |\det(\widehat{e_1}, e_2, \widehat{e_3}, e_4, \dots, \widehat{e_k}, \dots, e_n, e_1 + e_3 + e_k, e_1, e_k)| \\
&= |\det(\widehat{e_1}, e_2, \widehat{e_3}, e_4, \dots, \widehat{e_k}, \dots, e_n, e_3, e_1, e_k)| \\
&= 1,
\end{aligned}$$

where we first subtract $e_2$ from $e_2 - e_k$ and change the sign of the result, then add $e_k$ to $e_1 - e_k$, and finally subtract $e_1$ and $e_k$ from $e_1 + e_3 + e_k$. These intersections then have multiplicity 1.

Option 3 is analogous to option 2, reversing the roles of 1 and 2, so these intersections have multiplicity 1 as well.

| case | number of intersections | intersection multiplicity | contribution to degree |
|---|---|---|---|
| 1.1 | 0 | — | 0 |
| 1.2 | $\binom{n-1}{2}(n-3)$ | 1 | $\frac{1}{2}(n-1)(n-2)(n-3)$ |
| 1.3 | 0 | — | 0 |
| 1.4 | $\binom{n-2}{2}$ | 1 | $\frac{1}{2}(n-2)(n-3)$ |
| 1.5 | 0 | — | 0 |
| 2.1 | 0 | — | 0 |
| 2.2 | $(n-2)(n-3)$ | 1 | $(n-2)(n-3)$ |
| 2.3 | 1 | 3 | 3 |
| 2.4 | $3(n-3)$ | 1 | $3(n-3)$ |
| 2.5 | 0 | — | 0 |

**Table 1.** Intersection points of $v+S(A_{n-1})$ and $\Sigma_{n,n-2}$ with their multiplicities.

(C) Finally, we collect in Table 1 all the intersections points and their multiplicities, as computed in (A) and (B). Putting them together, we conclude that the degree of the tropical root surface of type $A_{n-1}$ is

$$\deg S(A_{n-1}) = \binom{n-1}{2}(n-3) + \binom{n-2}{2} + (n-2)(n-3) + 3 + 3(n-3) = \tfrac{1}{2}n(n-1)(n-2),$$

as desired. □

## 4. Remarks and future work

Nayeong Kim computed the degrees of the tropical surfaces of the remaining classical root systems [Kim 2023]. It is natural to ask whether these tropical surfaces can be obtained as tropicalizations of classical varieties. This is a subtle question: the surface $S(A_{n-1})$ *is* a tropicalization of a projective variety, but we do not know whether that is the case for any root system. This will be explained in a future paper.

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# Moduli spaces of quasitrivial sheaves

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A torsion-free sheaf $E$ on a projective variety $X$ is called *quasitrivial* if $E^{\vee\vee} = \mathcal{O}_X^{\oplus r}$. While such sheaves are always $\mu$-semistable, they may not be semistable. We study the Gieseker–Maruyama moduli space $\mathcal{N}_X(r, n)$ of rank-$r$ semistable quasitrivial sheaves on $X$ with $E^{\vee\vee}/E$ being a 0-dimensional sheaf of length $n$ via the Quot scheme of points $\mathrm{Quot}(\mathcal{O}_X^{\oplus r}, n)$. We show that when $X$ is a suitable projective variety, $\mathcal{N}_X(r, n)$ is empty for $r > n$, while $\mathcal{N}_X(n, n)$ has no stable points and is isomorphic to the symmetric product $\mathrm{Sym}^n(X)$. Our main result is the construction of an irreducible component of $\mathcal{N}_X(r, n)$ of dimension $n(d+r-1)-r^2+1$, where $d = \dim(X)$ when $r < n$. Furthermore, if we restrict to $X = \mathbb{P}^3$, this is the only irreducible component when $n \le 10$.



## 1. Introduction

Let $(X, A)$ be a polarized projective variety, let $P$ be a polynomial in $\mathbb{Q}[t]$, and denote by $\mathcal{M}_X(P)$ the Gieseker–Maruyama moduli space of semistable sheaves on $X$ with Hilbert polynomial $P$. Maruyama [1977] proved that the space $\mathcal{M}_X(P)$ is a projective scheme. However, especially when $d := \dim(X) \ge 3$, the geometry of such a scheme remains largely unknown, despite the efforts of many authors in the past four decades, and questions about connectedness, irreducibility, the number of irreducible components, and so on, remain open even when we restrict to work over projective spaces.

For instance, set $X = \mathbb{P}^3$ and write $\mathcal{M}_{\mathbb{P}^3}(P) = \mathcal{M}_{\mathbb{P}^3}(r, c_1, c_2, c_3)$, where $c_1$, $c_2$, and $c_3$ are the first, second, and third Chern classes of $E$, respectively. When $r = 1$ and $c_1 = 0$ (which can always be achieved after twisting by an appropriate line bundle), one gets that $\mathcal{M}_{\mathbb{P}^3}(1, 0, c_2, c_3)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^{a,g}(\mathbb{P}^3)$ of 1-dimensional schemes of degree $a = -c_2$ and genus $g = c_3 - 2c_2$

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[Kuznetsov et al. 2018, Lemma B.5.6], which is known always to be connected [Hartshorne 1966]. Not much is known in general when $r \geq 2$, though:

(1) $\mathcal{M}_{\mathbb{P}^3}(2, c_1, c_2, c_3)$ is irreducible for $c_3 = c_2^2 - c_2 + 2$ when $c_1 = 0$, or $c_3 = c_2^2$ when $c_1 = -1$; see [Schmidt 2020, Theorem 1.1].

(2) $\mathcal{M}_{\mathbb{P}^3}(2, 0, 2, c_3)$ has 2 irreducible components when $c_3 = 2$ and it has 3 irreducible components when $c_3 = 0$ [Jardim et al. 2017, Section 6].

(3) $\mathcal{M}_{\mathbb{P}^3}(2, -1, 2, c_3)$ has 2 irreducible components when $c_3 = 2$ and it has 4 irreducible components when $c_3 = 0$ [Almeida et al. 2022, Main Theorem 3].

Moreover, the moduli spaces in items (2) and (3) are connected. For higher values of $c_2$, one can check that the number of irreducible components of $\mathcal{M}_{\mathbb{P}^3}(2, c_1, c_2, 0)$ grows with $c_2$; see [Ein 1988, Proposition 3.6]. It is not known whether $\mathcal{M}_{\mathbb{P}^3}(2, c_1, c_2, c_3)$ is always connected.

The goal of this paper is to explore a somewhat exotic case, namely

$$\mathcal{N}_X(r, n) := \mathcal{M}_X(r \cdot P_X(t) - n) \quad \text{with } r \geq 1 \text{ and } n \in \mathbb{Z},$$

whose points correspond to *quasitrivial* rank-$r$ sheaves, that is, sheaves $E$ on $X$ such that $E^{\vee\vee} = \mathcal{O}_X^{\oplus r}$; this nomenclature is borrowed from Artamkin [1991]. When $X = \mathbb{P}^3$, we have $\mathcal{N}_{\mathbb{P}^3}(r, n) = \mathcal{M}_{\mathbb{P}^3}(r, 0, 0, -n)$.

The motivation comes from its close relationship, described in the body of the paper, between $\mathcal{N}_X(r, n)$ and the Hilbert and Quot schemes of points in $X$. Moreover, even though the main focus of this paper is the moduli space of semistable quasitrivial sheaves, we also provide some results regarding $\mu$-semistable quasitrivial sheaves.

First, we study $\mu$-semistable sheaves $E$ on $(X, A)$ with $\mathrm{rk}(E) \geq 1$ and vanishing Chern classes, and show that they are always extensions of ideal sheaves of subschemes of $X$ of codimension at least 3; see Lemma 2.3 below. In addition, when $(X, A)$ is a *good* projective variety (see Definition 2.4), we prove that the moduli space of such sheaves is a GIT quotient of a Quot scheme $\mathrm{Quot}(\mathcal{O}_X^{\oplus r}, u)$, where $u$ is a polynomial of degree less than or equal to $d - 3$; see Theorem 3.6.

**Main Theorem.** *Let $(X, A)$ be a good projective variety of dimension $d \geq 3$.*

(1) *$\mathcal{N}_X(r, n)$ is empty whenever $r > n$ or $n < 0$.*

(2) *$\mathcal{N}_X(n, n)$ is isomorphic to* $\mathrm{Sym}^n(X)$.

(3) *If $r < n$, then $\mathcal{N}_X(r, n)$ has an irreducible component of dimension $n(d + r - 1) - r^2 + 1$. Moreover, if $X = \mathbb{P}^3$ and $n \leq 10$, then $\mathcal{N}_{\mathbb{P}^3}(r, n)$ is irreducible.*

The upper bound on $n$ in the last claim comes from the fact that the variety $\mathcal{C}(n)$ of triples of $n \times n$ commuting matrices is known to be irreducible precisely for $n \leq 10$; in fact, we conclude that $\mathcal{N}_{\mathbb{P}^3}(r, n)$ is irreducible whenever $\mathcal{C}(n)$ is.

The paper is organized as follows. We start by studying reflexive sheaves with vanishing Chern classes in Section 2; we prove a key technical result about the triviality of these sheaves, which is subsequently used in the following sections. In Section 3 we give a criterion that tells when a torsion-free sheaf arising as the kernel of a point in $\mathrm{Quot}(\mathcal{O}_X^{\oplus r}, u)$ is (semi)stable, and explain the relation between sheaf (semi)stability and the GIT-stability with respect to the natural action of $\mathrm{GL}_r$ on $\mathrm{Quot}(\mathcal{O}_X^{\oplus r}, u)$.

Section 4 is where we establish items (1) and (2) of the Main Theorem, and we prove many technical results that will be used in Sections 5 and 6. In Section 5 we construct an irreducible component of $\mathcal{N}_X(2, n)$ that we will use as an induction step to construct an irreducible component for $\mathcal{N}_X(r, n)$. In the last section, we restrict to $X = \mathbb{P}^3$ and prove that $\mathcal{N}_{\mathbb{P}^3}(r, n)$ is irreducible for $n < 10$.

We work over the complex numbers $\mathbb{C}$. For a torsion-free sheaf $E$, (semi)stable always means Gieseker (semi)stability, while $\mu$-(semi)stability refers to stability in the sense of Mumford–Takemoto. As usual, we denote by the lower capital letters the dimension of the respective cohomology or Ext group: $h^i(F) := \dim(H^i(F))$ and $\mathrm{ext}^i(F, G) := \dim(\mathrm{Ext}^i(F, G))$.

## 2. Semistable reflexive sheaves with vanishing Chern classes

Recall that a torsion-free sheaf $E$ on a smooth polarized projective variety $(X, A)$ of dimension $d$ is said to be *$\mu$-(semi)stable* with respect to an ample divisor $A$ if

$$\frac{c_1(F)\cdot A^{d-1}}{\mathrm{rk}(F)} < (\leq) \frac{c_1(E)\cdot A^{d-1}}{\mathrm{rk}(E)}$$

for every proper nontrivial subsheaf $F \subset E$. Furthermore, $E$ is *Gieseker (semi)stable* if every proper nontrivial subsheaf $F \subset E$ satisfies

$$\frac{P_F(t)}{\mathrm{rk}(F)} < (\leq) \frac{P_E(t)}{\mathrm{rk}(E)},$$

where

$$P_G(t) := \chi(G \otimes \mathcal{O}_X(t\cdot A)) = \sum_{k=0}^{d} \alpha_k(G)t^k$$

denotes the Hilbert polynomial of the (possibly not torsion-free) sheaf $G$ with respect to the polarization $A$.

In general, one has the chain of strict implications

$$\mu\text{-stability} \implies \text{Gieseker stability} \implies \text{Gieseker semistability} \implies \mu\text{-semistability};$$

see [Huybrechts and Lehn 2010, Lemma 1.2.13]. Moreover, $E$ is $\mu$-(semi)stable if and only if the dual sheaf $E^\vee$ is $\mu$-(semi)stable.

The main characters of the present paper are presented in the following definition, borrowed from Artamkin [1991, p. 452].

**Definition 2.1.** A torsion-free sheaf $E$ on a projective variety $X$ is said to be *quasitrivial* if $E^{\vee\vee} \simeq \mathcal{O}_X^{\oplus r}$.

Quasitrivial sheaves of rank 1 are just ideal sheaves of subschemes of codimension at least 2. Clearly, quasitrivial sheaves are properly $\mu$-semistable (i.e., $\mu$-semistable but not $\mu$-stable) when $\mathrm{rk}(E) > 1$.

Notice that if $E$ is quasitrivial and $\mathrm{codim}(Q_E) \geq 3$, where $Q_E := E^{\vee\vee}/E$, then $c_1(E) = c_2(E) = 0$. We will now investigate to what extent the converse is true; since we will frequently use the following condition, it is worth fixing it in a definition.

**Definition 2.2.** A torsion-free sheaf $F$ on a polarized projective variety $(X, A)$ is said to have *vanishing Chern classes* if $c_1(F)\cdot A^{d-1} = c_2(F)\cdot A^{d-2} = 0$.

By Grothendieck–Riemann–Roch, $F$ has vanishing Chern classes if and only if

$$\deg(P_F(t) - r\cdot P_X(t)) \leq d-3,$$

where $r = \mathrm{rk}(F)$ and $P_X(t) = \chi(\mathcal{O}_X(t\cdot A))$ is the Hilbert polynomial of the structural sheaf.

A particular case of a result due to Simpson [1992, Theorem 2] establishes that if $F$ is a $\mu$-semistable reflexive sheaf with vanishing Chern classes on $(X, A)$, then $F$ is an extension of $\mu$-stable locally free sheaves with vanishing Chern classes. This has the following interesting consequence.

**Lemma 2.3.** *Let $(X, A)$ be a smooth polarized projective variety such that every $\mu$-stable reflexive sheaf with vanishing Chern classes is a line bundle. If $E$ is a semistable reflexive sheaf with vanishing Chern classes, then its Jordan–Hölder filtration has factors in* $\mathrm{Pic}^0(X)$.

By the Corlette–Simpson correspondence [Simpson 1992, Corollary 1.3], the hypothesis of the lemma holds whenever $\pi_1(X)$ is abelian. Examples of such varieties are Fano varieties, rational surfaces, abelian varieties, K3 surfaces, products of the previous ones, and quotients of simply connected varieties by finite abelian groups (e.g., Enriques surfaces). One can also check that the conclusion of the lemma holds when $(X, A)$ is a smooth projective surface with $K_X\cdot A \leq 0$ and satisfying $\chi(\mathcal{O}_X) = 1$.

*Proof.* We argue by induction on $\mathrm{rk}(E) = r$. If $E$ is a rank-2 semistable reflexive sheaf with $c_1(E)\cdot A^{d-1} = c_2(E)\cdot A^{d-2} = 0$, then by Simpson's result, $E$ must be an extension of $\mu$-stable locally free sheaves with vanishing Chern classes; that is, we can write $E$ in the short exact sequence

$$0 \to L \to E \to L' \to 0,$$

where $L$ and $L'$ are in $\mathrm{Pic}^0(X)$. Thus we can take

$$0 \subset L \subset E$$

as the desired Jordan–Hölder filtration.

Now suppose the result is valid for a rank less than $r$ and let $E$ be a semistable reflexive sheaf with $c_1 = c_2 = 0$. Again, by Simpson's result, there exists a filtration of $E$,

$$0 = G_0 \subset G_1 \subset \cdots \subset G_k \subset E,$$

such that each quotient is a $\mu$-stable locally free sheaf with vanishing Chern classes. Let $F := G_k$ and $F' := E/G_k$, and consider the exact sequence

$$0 \to F \to E \to F' \to 0.$$

Since $F'$ is a $\mu$-stable locally free sheaf with vanishing Chern classes, the hypothesis implies that $F' \in \mathrm{Pic}^0(X)$. Now we can apply the induction hypothesis to $F$, which has rank $r-1$, and we get a Jordan–Hölder filtration of $F$,

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l \subset F,$$

whose factors are in $\operatorname{Pic}^0(X)$. Finally, we take

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l \subset F \subset E,$$

and this is a Jordan–Hölder filtration for $E$ which satisfies our requirements. □

Since varieties satisfying the hypotheses of Lemma 2.3 will play an important role in the present paper, we introduce the following definition.

**Definition 2.4.** A smooth polarized projective variety $(X, A)$ is said to be *good* if $h^1(\mathcal{O}_X) = 0$, $\operatorname{Pic}(X) = \mathbb{Z}$, and every $\mu$-stable reflexive sheaf $F$ with vanishing Chern classes is a line bundle.

Fano manifolds with Picard rank 1 are perhaps the most relevant examples of good projective varieties. We complete this section with the following application of Simpson's result.

**Lemma 2.5.** *Let $E$ be a $\mu$-semistable sheaf of rank $r$ on a good polarized projective variety $(X, A)$ of dimension $d \geq 3$. If $c_1(E) \cdot A^{d-1} = c_2(E) \cdot A^{d-2} = 0$, then $E$ is quasitrivial and* $\operatorname{codim}(Q_E) \geq 3$.

*Proof.* Since $E$ is torsion-free, we have $\dim(Q_E) \leq d-2$ by [Okonek et al. 1980, II, Corollary of Lemma 1.1.8]. It follows that we can write the Hilbert polynomial of $Q_E$ as $P_{Q_E}(t) = at^{d-2} + q(t)$ for some $a \in \mathbb{Q}_{\geq 0}$ and $q \in \mathbb{Q}[t]$ with $\deg(u) \leq d-3$. Since $c_1(E) \cdot A^{d-1} = c_2(E) \cdot A^{d-2} = 0$, it follows that the Hilbert polynomial of $E$ is of the form $P_E(t) = r \cdot P_X(t) + v(t)$, where $v \in \mathbb{Q}[t]$ with $\deg(v) \leq d-3$ and $r := \operatorname{rk}(E)$. Hence, we can use the standard exact sequence

$$0 \to E \to E^{\vee\vee} \xrightarrow{\varphi_E} Q_E \to 0 \tag{1}$$

to note that

$$P_{E^{\vee\vee}}(t) = P_E(t) + P_{Q_E}(t) = r \cdot P_X(t) + at^{d-2} + q(t) + v(t).$$

A simple calculation with Grothendieck–Riemann–Roch yields

$$a = -\frac{1}{(d-2)!} c_2(E^{\vee\vee}) \cdot A^{d-2},$$

since $c_1(E^{\vee\vee}) \cdot A^{d-1} = c_1(E) \cdot A^{d-1} = 0$. Since $E^{\vee\vee}$ is $\mu$-semistable, we can apply the Bogomolov inequality [Huybrechts and Lehn 2010, Theorem 7.1], obtaining the inequality

$$\Delta(E^{\vee\vee}) = 2r c_2(E^{\vee\vee}) \cdot A^{d-2} \geq 0,$$

and thus $a \leq 0$. It follows that $a = 0$, so $E^{\vee\vee}$ is a $\mu$-semistable reflexive sheaf of rank $r$ on $X$ with $c_1(E^{\vee\vee}) \cdot A^{d-1} = c_2(E^{\vee\vee}) \cdot A^{d-2} = 0$. According to Simpson's result, $E^{\vee\vee}$ must be a successive extension of copies of $\mathcal{O}_X$, since $\operatorname{Pic}^0(X) = \{\mathcal{O}_X\}$. But the hypothesis $h^1(\mathcal{O}_X) = 0$ implies that extensions of $\mathcal{O}_X$ by itself are trivial, so we conclude that $E^{\vee\vee} = \mathcal{O}_X^{\oplus r}$. Moreover, $P_{Q_E}(t) = q(t)$ so $\deg(P_{Q_E}) \leq d-3$ and $\operatorname{codim}(Q_E) \geq 3$. □

## 3. Moduli spaces of semistable quasitrivial sheaves

Let $(X, A)$ be a smooth polarized variety of dimension $d$, and let $V_r$ denote a complex vector space of dimension $r$.

If $E$ is a quasitrivial sheaf of rank $r$ on $X$, then the standard exact sequence (1) provides a point $(\varphi_E, Q_E)$ in the Quot scheme $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$, where $u(t) = r \cdot P_X(t) - P_E(t)$ is a polynomial with rational coefficients of degree $\leq d-2$. Applying the functor $\mathrm{Hom}(-, \mathcal{O}_X)$ to the exact sequence in (1), we obtain

$$V_r \simeq \mathrm{Hom}(E, \mathcal{O}_X) \tag{2}$$

since $\mathrm{Hom}(Q_E, \mathcal{O}_X) = 0$ (because $Q$ is a torsion sheaf) and

$$\mathrm{Ext}^1(Q_E, \mathcal{O}_X) \simeq H^{n-1}(Q_E \otimes \omega_X)^* = 0$$

since $\dim(Q_E) \leq n-2$.

Conversely, consider a point $[(\varphi, Q)]$ in the Quot scheme $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ with $u \in \mathbb{Q}[t]$, which can be represented by an epimorphism $\varphi : V_r \otimes \mathcal{O}_X \twoheadrightarrow Q$ onto a sheaf $Q$ with Hilbert polynomial $P_Q(t) = u(t)$. Set $E := \ker \varphi$; we have a short exact sequence

$$0 \to E \to V_r \otimes \mathcal{O}_X \to Q \to 0. \tag{3}$$

Clearly, $E$ is a torsion-free sheaf of rank $r$; note that

$$P_E(t) = P_{V_r \otimes \mathcal{O}_X}(t) - P_Q(t) = r \cdot P_X(t) - u(t).$$

In addition, if $(\varphi', Q')$ is another representative for the same point, then $Q'$ is isomorphic to $Q$ and $\ker \varphi' \simeq \ker \varphi$. Since the isomorphism class of the kernel sheaf $E$ is independent of the choice of representative for a point in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$, we will, from now on, simply denote a point in the Quot scheme by a pair $(\varphi, Q)$.

**Proposition 3.1.** *Given* $(\varphi, Q) \in \mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ *with* $\deg(u) \leq d-2$, *the sheaf* $E := \ker \varphi$ *is a quasitrivial sheaf of rank* $r$.

*Proof.* Applying $\mathcal{H}om(-, \mathcal{O}_X)$ to sequence (3) we have

$$0 \to Q^\vee \to (V_r \otimes \mathcal{O}_X)^\vee \to E^\vee \to \mathcal{E}xt^1(Q, \mathcal{O}_X).$$

Since $\mathrm{codim}(Q) \geq 2$, we have that $Q^\vee = \mathcal{E}xt^1(Q, \mathcal{O}_X) = 0$ by [Huybrechts and Lehn 2010, Proposition 1.1.6], so that $E^\vee \cong V_r^* \otimes \mathcal{O}_X$. Since $E^\vee$ is properly $\mu$-semistable, so is $E$. □

However, different points in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ may lead to isomorphic quasitrivial sheaves. Indeed, consider the following action of $\mathrm{GL}(V_r) \simeq \mathrm{Aut}(V_r \otimes \mathcal{O}_X)$ on $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$:

$$g \cdot (\varphi, Q) := (\varphi \circ g, Q). \tag{4}$$

Note that $g \cdot (\varphi, Q)$ is clearly in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ again. Previous considerations lead to the following theorem characterizing the set of isomorphism classes of quasitrivial sheaves. Note that scalar multiples of the identity $\mathbf{1} \in \mathrm{GL}(V_r)$ act trivially on the Quot scheme.

**Theorem 3.2.** *There is a bijection between the set of isomorphism classes of rank-$r$ quasitrivial sheaves with Hilbert polynomial $P(t)$ on $X$ and the set of orbits*

$$\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)\,/\,\mathrm{PGL}_r \quad \textit{where } u(t) := r \cdot P_X(t) - P(t).$$

*Proof.* Let $(\varphi, Q)$ and $(\varphi', Q')$ be points in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ lying in the same $\mathrm{GL}_r$-orbit; that is, there is some $g \in \mathrm{GL}_r$ such that $(\varphi, Q) = g \cdot (\varphi', Q') = (\varphi' \circ g, Q')$. Hence, there is an isomorphism $f : Q \to Q'$ such that $f \circ \varphi = \varphi' \circ g$. Let $E = \ker \varphi$ and $E' = \ker \varphi'$ be the corresponding $\mu$-semistable sheaves given by Proposition 3.1. Thus we can consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & V_r \otimes \mathcal{O}_X & \xrightarrow{\varphi} & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & E' & \longrightarrow & V_r \otimes \mathcal{O}_X & \xrightarrow{\varphi'} & Q' & \longrightarrow & 0 \end{array} \tag{5}$$

Since $g$ and $f$ are isomorphisms, by the snake lemma, it follows that $h : E \to E'$ is also an isomorphism, as desired.

Now let $E$ and $E'$ be two isomorphic quasitrivial sheaves; let $h : E \to E'$ be an isomorphism. As we pointed out in the first paragraph of this section, we obtain corresponding points $(\varphi, Q)$ and $(\varphi', Q')$ in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$, where $u(t) := r \cdot P_X(t) - P(t)$. Since $E^{\vee\vee} \simeq (E')^{\vee\vee} \simeq V_r \otimes \mathcal{O}_X$, we obtain an induced morphism $h^{\vee\vee} : V_r \otimes \mathcal{O}_X \to V_r \otimes \mathcal{O}_X$; if this is not a monomorphism, then the kernel of the induced morphism $\ker h^{\vee\vee} \to Q$ is injective, contradicting the fact that $\ker h^{\vee\vee}$ is torsion-free; it follows that $h^{\vee\vee}$ must be an isomorphism.

We then construct a commutative diagram like that in (5), with $g = h^{\vee\vee}$. In particular, we get an isomorphism $f : Q \to Q'$ and a commutative diagram

$$\begin{array}{ccc} V_r \otimes \mathcal{O}_X & \xrightarrow{\varphi} & Q \\ & {\scriptstyle \varphi' \circ g} \searrow & \downarrow{\scriptstyle f} \\ & & Q' \end{array}$$

That is, $(\varphi, Q) = (\varphi' \circ g, Q')$ in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, n)$. Therefore $(\varphi, Q) = g \cdot (\varphi', Q')$. □

**Remark 3.3.** Cazzaniga and Ricolfi [2022] studied sheaves on $\mathbb{P}^d$ framed along a fixed hyperplane $D \subset \mathbb{P}^d$, that is, pairs $(E, \phi)$ consisting of a torsion-free sheaf $E$ on $\mathbb{P}^d$ together with an isomorphism $\phi : E|_D \xrightarrow{\sim} V_r \otimes \mathcal{O}_D$, where $r = \mathrm{rk}\, E$. They showed the moduli space $F_{r,n}(\mathbb{P}^d)$ of $D$-framed sheaves on $\mathbb{P}^d$ with Chern character $v_{r,n} := (r, 0, \dots, 0, -n)$ is isomorphic to the Quot scheme of points $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{A}^d}, n)$ for $d \geq 3$.

By [Cazzaniga and Ricolfi 2022, Corollary 1.6], for every $D$-framed sheaf $(E, \phi)$ with Chern character $v_{r,n}$, we can fit $E$ in a short exact sequence

$$0 \to E \to V_r \otimes \mathcal{O}_{\mathbb{P}^d} \to Q \to 0,$$

where $Q$ has finite support contained in $\mathbb{A}^d = \mathbb{P}^d \setminus D$. It follows that $E^{\vee\vee} = V_r \otimes \mathcal{O}_{\mathbb{P}^d}$, and thus $E$ is a quasitrivial sheaf.

We believe that similar results should also hold for framed sheaves on more general pairs $(X, D)$ of varieties with divisor.

It is easy to see that there are quasitrivial sheaves that are not semistable: just consider $\mathcal{O}_X \oplus \mathcal{I}_Z$, where $\mathcal{I}_Z$ denotes the ideal sheaf of a subscheme $Z \subset X$ of codimension at least 2. Our next task is to characterize semistable quasitrivial sheaves in terms of the corresponding point in a Quot scheme.

**Lemma 3.4.** *Let $E$ be a quasitrivial sheaf. Every saturated subsheaf $F \hookrightarrow E$ with $\mu(F) = 0$, and every torsion-free quotient $E \twoheadrightarrow G$ with $\mu(G) = 0$, is also quasitrivial.*

*Proof.* We prove the claim about subsheaves; the proof of the claim regarding quotients is similar.

Let $E$ be a quasitrivial sheaf, and let $F \hookrightarrow E$ be a saturated subsheaf with $\mu(F) = 0$ so that $D := E/F$ is torsion-free and $\mu(D) = 0$. We then obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & F^{\vee\vee} & \longrightarrow & Q_F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & V_r \otimes \mathcal{O}_X & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & D^{\vee\vee} & \longrightarrow & Q_D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $F^{\vee\vee}$ is a saturated subsheaf of $V_r \otimes \mathcal{O}_X$ with $\mu(F^{\vee\vee}) = \mu(F) = 0$, Corollary 1.6.11 of [Huybrechts and Lehn 2010] implies that $F^{\vee\vee} = V_s \otimes \mathcal{O}_X$ for some $s \in \{1, \dots, r-1\}$. It then follows that $D^{\vee\vee} \simeq V_{r-s} \otimes \mathcal{O}_X$. □

As a first application of the previous lemma, we characterize the Jordan–Hölder filtration of quasitrivial sheaves.

**Proposition 3.5.** *If $E$ is a quasitrivial sheaf of rank $r$, then $E$ admits a filtration whose factors are ideal sheaves $\mathcal{I}_{Z_i}$ of subschemes $Z_i \subset X$, $i = 1, \dots, r$, of codimension at least 2. Moreover, $\sum_i P_{\mathcal{O}_{Z_i}}(t) = r \cdot P_X(t) - P_E(t)$.*

*Proof.* We proceed by induction on the rank $r$. The claim is true when $r = 1$ because, in that case, $E = \mathcal{I}_Z$. Assuming the claim also holds for rank $r \geq 2$, we show that it holds for rank $r + 1$.

Since $E$ is properly $\mu$-semistable, let $F \hookrightarrow E$ be the maximal $\mu$-destabilizing subsheaf. The quotient sheaf $G := E/F$ is a $\mu$-stable sheaf with $\mu(G) = 0$, so $G = \mathcal{I}_{Z_{r+1}}$ for some subscheme $Z_{r+1} \subset X$ of codimension at least 2. Notice that $F$ is a quasitrivial sheaf of rank $r$ so, by induction, $F$ admits a filtration whose factors are $r$ ideal sheaves of subschemes of codimension at least 2. It follows that $E$ also admits such a filtration, with factors being all the factors of $F$ plus $\mathcal{I}_{Z_{r+1}}$.

For the last formula, just note that

$$P_E(t) = \sum_{i=1}^{r} P_{\mathcal{I}_{Z_i}}(t) = r \cdot P_X(t) - \sum_{i=1}^{r} P_{\mathcal{O}_{Z_i}}(t). \qquad \square$$

Now let $(\varphi, Q)$ be an point in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ and set $E = \ker \varphi$. We can relate the (semi)stability of $E$ with the GIT-stability of $(\varphi, Q)$ with respect to the $\mathrm{GL}_r$-action on $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ given by (4). To do this, we use [Huybrechts and Lehn 2010, Lemma 4.4.5]: a closed point $(\varphi, Q)$ in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ is GIT-(semi)stable if, and only if, for every nontrivial proper linear subspace $V_s \subset V_r$ the induced subsheaf $Q' := \varphi(V_s \otimes \mathcal{O}_X) \hookrightarrow Q$ satisfies the inequality

$$\frac{P_{Q'}}{s} > (\geq) \frac{P_Q}{r}. \tag{6}$$

**Theorem 3.6.** *Let $(\varphi, Q)$ be an point in $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ and let $E = \ker \varphi$. Then $(\varphi, Q)$ is GIT-(semi)stable with respect to the $\mathrm{GL}_r$ action in (4) if, and only if, $E$ is (semi)stable.*

*Proof.* Assume that $(\varphi, Q) \in \mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ is not GIT-(semi)stable, and let $V_s \subset V_r$ be the destabilizing subspace. The kernel sheaf

$$F := \ker\{V_s \otimes \mathcal{O}_X \hookrightarrow V_r \otimes \mathcal{O}_X \xrightarrow{\varphi} Q\}$$

is a subsheaf of $E$, and setting $Q'$ as above, we get

$$\frac{P_F(t)}{\mathrm{rk}(F)} = \frac{s \cdot P_X(t) - P_{Q'}(t)}{s} = P_X(t) - \frac{P_{Q'}(t)}{s} > (\geq)\ P_X(t) - \frac{P_Q(t)}{r} = \frac{P_E(t)}{\mathrm{rk}(E)};$$

therefore, $E$ is not (semi)stable.

Conversely, assume that $E$ is not (semi)stable, and let $F \hookrightarrow E$ be a destabilizing subsheaf; since $E$ is $\mu$-semistable, we may assume that $\mu(F) = 0$, and thus $F$ is also quasitrivial by Lemma 3.4. Therefore, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & V_s \otimes \mathcal{O}_X & \longrightarrow & Q_F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & V_r \otimes \mathcal{O}_X & \xrightarrow{\varphi} & Q & \longrightarrow & 0 \end{array} \tag{7}$$

where $Q_F = \varphi(V_s \otimes \mathcal{O}_X)$. Since $P_F(t)/\mathrm{rk}(F) > (\geq)\ P_E(t)/\mathrm{rk}(E)$, we conclude that

$$\frac{P_{Q'}}{s} < (\leq) \frac{P_Q}{r} = \frac{\mathrm{rk}(F) \cdot P_Q}{\mathrm{rk}(E)},$$

showing that $(\varphi, Q)$ is not GIT-(semi)stable. $\square$

Let $V_r \otimes \mathcal{O}_{X \times \mathrm{Quot}} \xrightarrow{\boldsymbol{\varphi}} \boldsymbol{Q}$ be the universal object for the Quot scheme $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$. The kernel sheaf $\boldsymbol{E}$ provides a flat family of quasitrivial sheaves parametrized by $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$. Restricting this family to the open subset $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)^{\mathrm{ss}}$ consisting of GIT-semistable elements, we obtain a modular morphism[1]

$$\widetilde{\Psi} : \mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)^{\mathrm{ss}} \to \mathcal{M}_X(P),$$

where $\mathcal{M}_X(P)$ denotes the Gieseker–Maruyama moduli space of semistable sheaves on $X$ with fixed Hilbert polynomial $P(t) = r \cdot P_X(t) - u(t)$. By virtue of Theorems 3.2 and 3.6, $\widetilde{\Psi}$ factors through an injective morphism

$$\Psi : \mathrm{Quot}(V_r \otimes \mathcal{O}_X, u) /\!\!/ \mathrm{PGL}(V_r) \to \mathcal{M}_X(P)$$

whose image is precisely the subset of quasitrivial sheaves in $\mathcal{M}_X(P)$. What is more, when $(X, A)$ is a good polarized variety and $\deg(u) \leq n-3$, Lemma 2.5 implies that $\Psi$ is bijective. Our next goal is to study this situation more clearly.

**Lemma 3.7.** *Assume $h^1(\mathcal{O}_X) = 0$. If $E$ is a stable quasitrivial sheaf of rank $r$ such that $h^{n-2}(Q_E \otimes \omega_X) = 0$, then*

$$\mathrm{ext}^1(E, E) = \hom(E, Q_E) - r^2 + 1.$$

*Proof.* Note that $\mathrm{Ext}^1(E, \mathcal{O}_X) \simeq H^{n-1}(E \otimes \omega_X)^*$ by Serre duality. Therefore, twisting the exact sequence in (3) by $\omega_X$ and taking cohomology, the hypotheses imply that $H^{n-1}(E \otimes \omega_X) = 0$.

Applying the functor $\mathrm{Hom}(E, -)$ to sequence (3), we obtain

$$0 \to \mathrm{Hom}(E, E) \to \mathrm{Hom}(E, \mathcal{O}_X) \otimes V_r \to \mathrm{Hom}(E, Q) \to \mathrm{Ext}^1(E, E) \to 0,$$

providing the desired formula. □

Note that the left-hand side of the equality in Lemma 3.7 is the dimension of the tangent space $T_E \mathcal{M}_X(P)$ of the moduli space $\mathcal{M}_X(P)$ at $E$, while the right-hand side is the dimension of the tangent space $T_{(Q_E, \varphi)} \mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ of the Quot scheme $\mathrm{Quot}(V_r \otimes \mathcal{O}_X, u)$ at $V_r \otimes \mathcal{O}_X \xrightarrow{\varphi} Q_E$ minus the dimension of $\mathrm{PGL}(V_r)$.

## 4. Semistable sheaves with vanishing Chern classes

Let $(X, A)$ be a polarized variety of dimension at least 3. We are finally ready to focus on the main character of this paper, namely the Gieseker–Maruyama moduli space

$$\mathcal{N}_X(r, n) := \mathcal{M}_X(r \cdot P_X(t) - n) \quad \text{with } r, n \geq 1. \tag{8}$$

Note that $E$ has rank $r$ and vanishing Chern classes; when $(X, A)$ is good (in the sense of Definition 2.4), every $E \in \mathcal{N}_X(r, n)$ is quasitrivial, and $Q_E := E^{\vee\vee}/E$ is a 0-dimensional sheaf of length $n$. Let $\mathcal{N}_X(r, n)^{\mathrm{st}}$ denote the (open, possibly empty) subset consisting of stable sheaves.

[1]The fact that $\mathcal{M}_X(P)$ is a coarse moduli space for a moduli functor $\mathfrak{M}_X^P$ implies that for any for scheme of finite type $S$ there is a map from the set $\mathfrak{M}_X^P(S)$ of families of semistable sheaves with Hilbert polynomial $X$ parametrized by $S$ to $\mathrm{Hom}(S, M)$; in other words, every $S$-family yields a modular morphism $S \to \mathcal{M}_X(P)$.

The first observation is that $\mathcal{N}_X(1,n)$ coincides with the Hilbert scheme $\mathrm{Hilb}^n(X)$ of 0-dimensional subschemes of $X$ with length $n$. We will therefore focus on $r \geq 2$, and our initial task is to check whether $\mathcal{N}_X(r,n)$ is nonempty; here is a first easy observation:

**Proposition 4.1.** *If $(X, A)$ is a good polarized variety, then $\mathcal{N}_X(r,n) = \varnothing$ for $r > n$.*

*Proof.* Since $E \in \mathcal{N}_X(r,n)$ is semistable, we must have $H^0(E) = 0$; indeed, any nontrivial morphism $\mathcal{O}_X \to E$ would destabilize $E$ since $p_E(t) < p_{\mathcal{O}_X}$.

Since $E$ is quasitrivial, we can take cohomology in the short exact sequence in (3), obtaining

$$0 \to H^0(V_r \otimes \mathcal{O}_X) \to H^0(Q),$$

and therefore $r \leq n$. □

The case $r = n$ turns out to be quite special as well.

**Proposition 4.2.** *If $(X, A)$ is a good polarized variety, then $\mathcal{N}_X(n,n) = \mathrm{Sym}^n(X)$, and $\mathcal{N}_X(n,n)^{\mathrm{st}} = \varnothing$ when $n \geq 2$.*

*Proof.* Let $E$ be a sheaf in $\mathcal{N}_X(n,n)$; by Proposition 3.5, $E$ admits a $n$-step filtration whose factors are sheaves of ideals $\mathcal{I}_{Z_i}$, where each $Z_i$ is 0-dimensional, and whose lengths sum to $n$. None of the $Z_i$'s can be empty, otherwise, $E$ could not be semistable, and thus each $Z_i$ must have length equal to 1. This means that $E$ cannot be stable and that it is S-equivalent to $\bigoplus_{i=1}^n \mathcal{I}_{Z_i}$, which is the graded object associated with the Jordan–Hölder filtration of $E$. □

Our next task is to show that $\mathcal{N}_X(r,n) \neq \varnothing$ when $r < n$, but this will follow from several partial results.

**Definition 4.3.** A quasitrivial sheaf $E$ on polarized variety $(X, A)$ is called *unbalanced* if it admits an epimorphism $E \twoheadrightarrow \mathcal{I}_q$ to the ideal sheaf of a point $q \in X$, and it does not admit a morphism $\mathcal{I}_q \to E$ such that the composition $E \to \mathcal{I}_q \to E$ is the identity.

Note that if $E$ is unbalanced, then one can consider the kernel of the epimorphism $E \twoheadrightarrow \mathcal{I}_q$ and the exact sequence

$$0 \to F \to E \to \mathcal{I}_q \to 0;$$

note that $F$ is also quasitrivial; moreover, the second condition in Definition 4.3 is equivalent to the nonsplitting of this previous short exact sequence.

**Proposition 4.4.** *Let $F$ and $G$ be quasitrivial sheaves on $(X, A)$ defined by*

$$0 \to F \to V_r \otimes \mathcal{O}_X \to Q_F \to 0, \tag{9}$$

$$0 \to G \to V_s \otimes \mathcal{O}_X \to Q_G \to 0, \tag{10}$$

*where $Q_F$ and $Q_G$ are supported on 0-dimensional subschemes of $X$. If $\mathrm{Supp}(Q_F) \cap \mathrm{Supp}(Q_G) = \varnothing$, then*

- $\mathrm{ext}^1(F,G) = \mathrm{rk}(F) \cdot h^1(G)$,
- $\mathrm{ext}^2(F,G) = \mathrm{rk}(G) \cdot h^0(Q_F)$,
- $\mathrm{ext}^3(F,G) = 0$.

*Proof.* First, taking cohomologies on the sequences defining $F$ and $G$ we obtain that $h^2(F(t\cdot A)) = h^2(G(t\cdot A)) = 0$ for all $t\in\mathbb{Z}$. Now, apply $\mathrm{Hom}(Q_F,-)$ to sequence (10) and we obtain $\mathrm{ext}^i(Q_F,G)=0$ for $i=0,1,2$ and $\mathrm{ext}^3(Q_F,G)\cong \mathrm{rk}(G)\cdot h^0(Q_F)$. Apply $\mathrm{Hom}(-,G)$ to sequence (9) and we get $\mathrm{ext}^3(F,G)=0$, $\mathrm{ext}^2(F,G)=\mathrm{rk}\,G\cdot h^0(Q_G)$ and $\mathrm{ext}^1(F,G)=\mathrm{rk}(F)\cdot h^1(G)$, as desired. □

**Remark 4.5.** Let $F$ be a torsion-free sheaf on $(X,A)$ defined by sequence (9). Note that if $F$ is stable, then $H^0(F)=0$. Therefore, taking cohomologies on sequence (9), we get that $H^0(Q_F) = V_r\otimes H^0(\mathcal{O}_X)\oplus H^1(F)$; that is,

$$h^1(F) = h^0(Q_F) - \mathrm{rk}(F). \tag{11}$$

The next statement guarantees the existence of stable unbalanced sheaves of rank 2.

**Lemma 4.6.** *For each pair $(q,Z)$ consisting of a point $q\in X$ and a reduced 0-dimensional scheme $Z\subset X$ not containing $q$ with $h^0(\mathcal{O}_Z)\geq 2$, there exists a stable unbalanced sheaf $E\in\mathrm{Ext}^1(\mathcal{I}_q,\mathcal{I}_Z)$ such that for every $p\in Z$ there is an epimorphism $\varepsilon: E\to\mathcal{I}_p$ with $\ker\varepsilon\simeq\mathcal{I}_{Z'}$, where $Z'=(Z\setminus\{p\})\cup\{q\}$. In particular, $\mathcal{N}_X^{\mathrm{st}}(2,n)\neq\varnothing$ for every $n\geq 3$.*

*Proof.* Let $\widetilde{Z}=\{p_1,\dots,p_n\}\subset X$ be a reduced 0-dimensional subscheme, and set

$$Z_j := \{p_1,\dots,p_n\}\setminus\{p_j\}.$$

Consider the morphism $\varphi: V_2\otimes\mathcal{O}_X\to\mathcal{O}_{\widetilde{Z}}$ given by the choice of vectors

$$\alpha_i\in V_2 \ \text{ for } \ i=1,\dots,n \quad \text{ with } \ \alpha_i\neq\lambda\cdot\alpha_j\ (\lambda\in\mathbb{C}) \ \text{ for all } \ i\neq j,$$

and let $E:=\ker\varphi$. Fix $j\in\{1,\dots,n\}$ and choose $\beta\in V_2$ such that

$$\beta\in\langle\alpha_j\rangle^{\perp} \qquad \text{and} \qquad \beta\notin\langle\alpha_i\rangle^{\perp} \quad \text{whenever } \ i\neq j.$$

The choice of such vector $\beta$ gives a linear map of $V_1\hookrightarrow V_2$ such that the image of the composition

$$V_1\otimes\mathcal{O}_X \xrightarrow{\beta} V_2\otimes\mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_{\widetilde{Z}}$$

is precisely $\mathcal{O}_{Z_j}$, leading to the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z_j} & \longrightarrow & V_1\otimes\mathcal{O}_X & \longrightarrow & \mathcal{O}_{Z_j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & V_2\otimes\mathcal{O}_X & \xrightarrow{\varphi} & \mathcal{O}_{\widetilde{Z}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{p_j} & \longrightarrow & W_1\otimes\mathcal{O}_X & \longrightarrow & \mathcal{O}_{p_j} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

That is, $E\in\mathrm{Ext}^1(\mathcal{I}_{p_j},\mathcal{I}_{Z_j})$ for all $j=1,\dots,n$. In particular, $E$ is unbalanced; we claim that $E$ is stable.

Indeed, if $E$ is not stable, then it must be destabilized by an ideal sheaf $\mathcal{I}_Y$ such that $h^0(\mathcal{O}_Y) \leq \frac{1}{2}n$. Let $p_k \in \widetilde{Z}$ such that $p_k \notin Y$ and consider $E$ as an element of $\operatorname{Ext}^1(\mathcal{I}_{p_k}, \mathcal{I}_{Z_k})$. Hence we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_Y & = & \mathcal{I}_Y & & & & \\
 & & \downarrow & & \downarrow & \overset{0}{\searrow} & & & \\
0 & \longrightarrow & \mathcal{I}_{Z_k} & \longrightarrow & E & \longrightarrow & \mathcal{I}_{p_k} & \longrightarrow & 0
\end{array}$$

The composition $\mathcal{I}_Y \to E \to \mathcal{I}_{p_k}$ is zero by the choice of $p_k \notin Y$. Thus the monomorphism $\mathcal{I}_Y \hookrightarrow E$ must factor through $\mathcal{I}_{Z_k}$; however, there can be no nontrivial morphism $\mathcal{I}_Y \to \mathcal{I}_{Z_k}$ since $h^0(\mathcal{O}_{Z_k}) = n-1 > \frac{1}{2}n = h^0(\mathcal{O}_Y)$ when $n \geq 3$. It follows that $E$ must be stable. □

Next, we prove the nonemptiness of $\mathcal{N}_X(r,n)$ via induction on the rank.

**Proposition 4.7.** *Let $(X, A)$ be a good polarized variety. Given positive integers $r < n$, there is $E \in \mathcal{N}_X(r,n)^{\mathrm{st}}$ such that $Q_E$ is the structure sheaf of a reduced 0-dimensional subscheme of $X$. Moreover, when $r \geq 2$ every such sheaf is unbalanced.*

*Proof.* We argue by induction on $r$. For $r = 1$, just let $E = \mathcal{I}_Z$, where $Z$ is a reduced 0-dimensional subscheme; it clearly satisfies the desired properties.

When $r \geq 2$, the induction hypothesis guarantees that there is an unbalanced sheaf $F \in \mathcal{N}_X(r-1, n-1)^{\mathrm{st}}$ such that $Q_F$ is the structure sheaf of a reduced 0-dimensional subscheme $Z \subset X$ whenever $n > r$. Let $p \notin Z$, and consider an extension of the form

$$0 \to F \to E \to \mathcal{I}_p \to 0.$$

We start by showing $\operatorname{Ext}^1(\mathcal{I}_p, F) \neq 0$, so that nonsplit extensions as above do exist. Indeed, Proposition 4.4 yields $\operatorname{ext}^1(\mathcal{I}_p, F) = \operatorname{rk}(\mathcal{I}_p) \cdot h^1(F) = h^1(F)$. To see that $h^1(F) \neq 0$, consider the sequence

$$0 \to F \to V_{r-1} \otimes \mathcal{O}_X \to Q_F \to 0;$$

taking cohomologies, noting that $h^1(X, \mathcal{O}_X) = 0$, we have

$$h^1(F) = h^0(F) + h^0(Q_F) - \dim(V_{r-1}) = n - r > 0$$

since $h^0(F) = 0$ because $F$ is stable and $h^0(Q_F) = n-1$.

By the induction hypothesis and the construction in the proof of Lemma 4.6, it is easy to see that $E$ is a quasitrivial sheaf and that $Q_E \simeq Q_F \oplus \mathcal{O}_p = \mathcal{O}_{Z'}$, where $Z' := Z \cup \{p\}$. Now, we claim that for every point $q \in Z'$, we have $\hom(E, \mathcal{I}_q) = 1$. Indeed, apply the functor $\operatorname{Hom}(-, \mathcal{I}_q)$ to sequence

$$0 \to E \to V_r \otimes \mathcal{O}_X \to Q_E \to 0,$$

and we get

$$0 \to \operatorname{Hom}(Q_E, \mathcal{I}_q) \to \operatorname{Hom}(V_r \otimes \mathcal{O}_X, \mathcal{I}_q) \to \operatorname{Hom}(E, \mathcal{I}_q) \to \operatorname{Ext}^1(Q_E, \mathcal{I}_q) \to \operatorname{Ext}^1(V_r \otimes \mathcal{O}_X, \mathcal{I}_q) \to \cdots.$$

From $\mathrm{Hom}(V_r \otimes \mathcal{O}_X, \mathcal{I}_q) = V_r \otimes H^0(X, \mathcal{I}_q) = 0$ and $\mathrm{Ext}^1(V_r \otimes \mathcal{O}_X, \mathcal{I}_q) = V_r \otimes H^1(X, \mathcal{I}_q) = 0$, it follows that

$$\mathrm{Hom}(E, \mathcal{I}_q) \cong \mathrm{Ext}^1(Q_E, \mathcal{I}_q) = \mathrm{Ext}^1(\mathcal{O}_{Z'}, \mathcal{I}_q).$$

Next, apply the functor $\mathrm{Hom}(-, \mathcal{I}_q)$ to the sequence

$$0 \to \mathcal{I}_{Z'} \to \mathcal{O}_X \to \mathcal{O}_{Z'} \to 0.$$

Using the same idea as before, we conclude that $\mathrm{Ext}^1(\mathcal{O}_{Z'}, \mathcal{I}_q) \cong \mathrm{Hom}(\mathcal{I}_{Z'}, \mathcal{I}_q)$; since $q \in Z'$, we know that $\hom(\mathcal{I}_{Z'}, \mathcal{I}_q) = 1$.

Since $F$ is unbalanced, consider an epimorphism $F \twoheadrightarrow \mathcal{I}_q$ and its kernel, forming the short exact sequence

$$0 \to G \to F \to \mathcal{I}_u \to 0$$

for some $u \in Z$. The same argument that we used to show that $\hom(E, \mathcal{I}_q) = 1$ works for every quasitrivial sheaf; that is, $\hom(F, \mathcal{I}_u) = 1$ for every $u \in Z$. Since $\hom(F, \mathcal{I}_u) = \hom(E, \mathcal{I}_u) = 1$, we conclude that the composition $F \hookrightarrow E \to \mathcal{I}_u$ must be equal to the morphism $F \to \mathcal{I}_u$, and we can form the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & K & \longrightarrow & \mathcal{I}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & \mathcal{I}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_u & = & \mathcal{I}_u & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It is not hard to see that $K$ is indeed a quasitrivial sheaf. So this proves that we can write $E$ as an element of $\mathrm{Ext}^1(F, \mathcal{I}_p)$ for every $p \in \mathrm{Supp}(Q_E)$ and some quasitrivial sheaf $F$. In particular, $E$ is also unbalanced.

We have just proved that $\mathrm{Hom}(E, \mathcal{I}_q) = \mathrm{Hom}(\mathcal{I}_{Z'}, \mathcal{I}_q)$, so if $q \notin Z'$, then $\mathrm{Hom}(\mathcal{I}_{Z'}, \mathcal{I}_q)$ is zero. It follows that $\mathrm{Hom}(E, \mathcal{I}_q) = 0$ whenever $q$ is not in $\mathrm{Supp}(Q_E)$.

Finally, we show that $E$ is stable. Assume, by contradiction, $E$ is not stable; by Theorem 3.6, $E$ admits a quasitrivial subsheaf $G \hookrightarrow E$ of rank $s$ with $h^0(Q_G) < (s \cdot n)/r < n$. Choose $p \in \mathrm{Supp}(Q_E)$ such that $p \notin \mathrm{Supp}(Q_G)$ and write $E$ as an element of $\mathrm{Ext}^1(F, \mathcal{I}_p)$. By what we have just proved, the composition $G \hookrightarrow E \to \mathcal{I}_p$ is a morphism in $\mathrm{Hom}(G, I_p)$, and therefore, it must be equal to zero because $p \notin \mathrm{Supp}(Q_G)$. In this case, we can consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & \searrow^{0} & & & \\
0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & \mathcal{I}_p & \longrightarrow & 0
\end{array}$$

But $F$ is stable and $G \hookrightarrow F$ with $p(G) > p(F)$. This is a contradiction, and therefore $E$ is stable. □

**Proposition 4.8.** *Let $E \in \mathcal{N}_X(r,n)^{\mathrm{st}}$ be such that $Q_E = \mathcal{O}_Z$ for some 0-dimensional scheme $Z$ of length $n$, and let $\mathcal{I}_Z$ be the corresponding sheaf of ideals. Then*

$$\mathrm{ext}^1(E,E) = \hom(\mathcal{I}_Z, \mathcal{O}_Z) + (r-1)(n-1).$$

*Proof.* Note that $E$ satisfies the hypothesis of Lemma 3.7, so we know that

$$\mathrm{ext}^1(E,E) = \hom(E, Q_E) - r^2 + 1.$$

Now to compute $\hom(E, Q_E)$ we consider the sequence

$$0 \to E \to V_r \otimes \mathcal{O}_X \to Q_E \to 0,$$

and apply the functor $\mathrm{Hom}(-, Q_E)$ to obtain the following exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}(Q_E, Q_E) \to \mathrm{Hom}(V_r \otimes \mathcal{O}_X, Q_E) \to \mathrm{Hom}(E, Q_E) \\ \to \mathrm{Ext}^1(Q_E, Q_E) \to \mathrm{Ext}^1(V_r \otimes \mathcal{O}_X, Q_E) \to \cdots . \end{aligned} \tag{12}$$

Because $\hom(V_r \otimes \mathcal{O}_X, Q_E) = r \cdot h^0(Q_E)$ and $\mathrm{ext}^1(V_r \otimes \mathcal{O}_X, Q_E) = r \cdot h^1(Q_E) = 0$, we have $\hom(E, Q_E) = rn + \mathrm{ext}^1(Q_E, Q_E) - \hom(Q_E, Q_E)$.

Next, we consider the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to Q_E \to 0, \tag{13}$$

where we replace $\mathcal{O}_Z = Q_E$. Applying $\mathrm{Hom}(-, Q_E)$, we have the exact sequence

$$0 \to \mathrm{Hom}(Q_E, Q_E) \to \mathrm{Hom}(\mathcal{O}_X, Q_E) \to \mathrm{Hom}(\mathcal{I}_Z, Q_E) \to \mathrm{Ext}^1(Q_E, Q_E) \to \mathrm{Ext}^1(\mathcal{O}_X, Q_E) \to \cdots .$$

Using the same idea as before, we conclude that

$$\hom(Q_E, Q_E) - n + \hom(\mathcal{I}_Z, Q_E) - \mathrm{ext}^1(Q_E, Q_E) = 0,$$

which implies that

$$\mathrm{ext}^1(Q_E, Q_E) - \hom(Q_E, Q_E) = \hom(\mathcal{I}_Z, Q_E) - n.$$

Now we combine the above equalities:

$$\begin{aligned} \mathrm{ext}^1(E,E) &= \hom(E, Q_E) - r^2 + 1 \\ &= \hom(\mathcal{I}_Z, Q_E) + rn - n - r^1 + 1 \\ &= \hom(\mathcal{I}_Z, Q_E) + n(r-1) - (r^2 - 1) \\ &= \hom(\mathcal{I}_Z, Q_E) + n(r-1) - (r+1)(r-1) \\ &= \hom(\mathcal{I}_Z, \mathcal{O}_Z) + (n-r-1)(r-1). \end{aligned}$$

□

**Remark 4.9.** When $Z$ is reduced in the above proposition, $\hom(\mathcal{I}_Z, \mathcal{O}_Z) = nd$, where $d = \dim(X)$.

Continuing the long exact sequence in (12), we get that $\mathrm{Ext}^1(E, Q_E) \cong \mathrm{Ext}^2(Q_E, Q_E)$; assuming, as in the previous proposition that $Q_E = \mathcal{O}_Z$, and using the short exact sequence (13) we obtain

$$\mathrm{Ext}^2(Q_E, Q_E) \cong \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) \hookrightarrow \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z).$$

If $Z$ is unobstructed as a point in $\mathrm{Hilb}^n(X)$ (e.g., if $Z$ is reduced), then $\mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z) = 0$ and it follows that $\mathrm{Ext}^1(E, Q_E) = 0$ as well. By [Huybrechts and Lehn 2010, Proposition 2.2.8] we can conclude that $V_r \otimes \mathcal{O}_X \twoheadrightarrow Q_E$ is smooth as a point in $\mathrm{Quot}^n(V_r \otimes \mathcal{O}_X)$.

## 5. An irreducible component of $\mathcal{N}_X(2, n)$

Returning to the case $r = 2$, Proposition 4.8 indicates that we should construct an irreducible family of stable rank-2 quasitrivial sheaves on $(X, A)$ of dimension $n(d+1) - 3$, where $d = \dim(X)$.

To do this, we will need certain general results regarding relative Ext sheaves. To be precise, let $f : X \to Y$ be a morphism of schemes, and let $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ be the categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules, respectively. Let $F \in \mathfrak{M}(X)$. We define $\mathcal{E}xt^p_f(F, -)$ to be the right-derived functors of the left-exact functor $f_*\mathcal{H}om_{\mathcal{O}_X}(F, -) : \mathfrak{M}(X) \to \mathfrak{M}(Y)$.

Let $f : X \to Y$ a flat projective morphism of noetherian schemes and $F$, $G$ coherent $\mathcal{O}_X$-modules, flat over $Y$. For every $u : Y' \to Y$ of noetherian schemes, we have the base-change morphism

$$\tau^i(u) : u^*\mathcal{E}xt^i_f(F, G) \to \mathcal{E}xt^i_{p_2}(p_1^*F, p_1^*G),$$

where $p_1$ and $p_2$ are the projections in the following diagram:

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

If $y \in Y$ and $u : \mathrm{Spec}\, k(y) \to Y$ is the respective map, we denote the base-change morphism by

$$\tau^i(y) : \mathcal{E}xt^i_f(F, G) \otimes_Y k(y) \to \mathrm{Ext}^i_{X_y}(F_y, G_y).$$

The following result due to Lange will be used several times; see [Lange 1983, Theorem 1.4].

**Theorem 5.1.** *Let $y \in Y$ be a point and assume the base-change morphism $\tau^i(y) : \mathcal{E}xt^i_f(F, G) \otimes_Y k(y) \to \mathrm{Ext}^i_{X_y}(F_y, G_y)$ to be surjective. Then*

(i) *there is a neighborhood $U$ of $y$ such that $\tau^i(y')$ is an isomorphism for all $y' \in U$;*

(ii) *$\tau^{i-1}(y)$ is surjective if and only if $\mathcal{E}xt^i_f(F, G)$ is locally free in a neighborhood of $y$.*

Our next lemma is the crucial technical fact to be explored in the desired irreducible family of quasitrivial sheaves.

**Lemma 5.2.** *Let $f : X \to Y$ be a projective morphism of noetherian schemes with $Y$ reduced, and let $F$ and $G$ be coherent sheaves on $X$ flat over $Y$. If $\mathrm{Ext}^3_{X_y}(F_y, G_y) = 0$ and the dimension of $\mathrm{Ext}^i_{X_y}(F_y, G_y)$ is constant for $i = 1, 2$ for all $y \in Y$, then the base-change morphism $\tau^1(y)$ is an isomorphism for every $y \in Y$, and $\mathcal{E}xt^1_f(F, G)$ is locally free.*

*Proof.* Since $\mathrm{Ext}^3_{X_y}(F_y, G_y) = 0$, the base-change morphism is $\tau^i(y)$ is trivially surjective. The first item of Theorem 5.1 implies that the sheaf $\mathcal{E}xt^3_f(F, G)$ is zero and item (ii) shows that $\tau^2(y)$ is surjective. Applying Theorem 5.1 again for $i = 2$, we get that $\tau^2(y)$ is an isomorphism. Moreover, [Mumford 1974, Lemma 1, p. 51] implies that $\mathcal{E}xt^2_f(F, G)$ is locally free. Finally, applying Theorem 5.1 again for $i = 1$, we obtain that $\tau^1(y)$ is an isomorphism, and since $\mathrm{ext}^1_{X_y}(F_y, G_y)$ is constant, [Mumford 1974, Lemma 1, p. 51] implies that $\mathcal{E}xt^1_f(F, G)$ is locally free. □

We are finally in the position to construct the family of quasitrivial sheaves we are looking for; let $\mathcal{H}^i$ the universal sheaf for the Hilbert scheme of $i$ points on $X \times \mathrm{Hilb}^i(X)$. Consider the following diagram:

$$\begin{array}{ccc} & \mathrm{Hilb}^1(X) \times \mathrm{Hilb}^{n-1}(X) & \\ & \uparrow{\scriptstyle f} & \\ & X \times \mathrm{Hilb}^1(X) \times \mathrm{Hilb}^{n-1}(X) & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X \times \mathrm{Hilb}^1(X) & & X \times \mathrm{Hilb}^{n-1}(X) \end{array}$$

Let

$$U := \{(q, Z) \in \mathrm{Hilb}^1(X) \times \mathrm{Hilb}^{n-1}(X) \mid q \notin Z,\ Z = Z_{\mathrm{red}}\},$$

and set $X_U := X \times U$ with $\pi : X_U \to U$ being the canonical projection; note that $U$ is open in $\mathrm{Hilb}^1(X) \times \mathrm{Hilb}^{n-1}(X)$. Define $\mathcal{E}^i := \mathcal{E}xt^i_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1})$. Let $(q, Z)$ be a point in $U$ and let $\tau^i(q, Z)$ be the corresponding base-change morphism, that is,

$$\tau^i(q, Z) : \mathcal{E}xt^i_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1}) \otimes k(q, Z) \to \mathrm{Ext}^i_X(\mathcal{I}_q, \mathcal{I}_Z).$$

Proposition 4.4 and Lemma 5.2 shows us that $\mathcal{E}^1$ is locally free on $U$. Note that, by Theorem 5.1, $\tau^0(q, Z)$ is an isomorphism; that is, for every $(q, z) \in U$ we have

$$\tau^0(q, Z) : \mathcal{E}xt^0_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1}) \otimes k(q, Z) \xrightarrow{\sim} \mathrm{Hom}_X(\mathcal{I}_q, \mathcal{I}_Z).$$

However, $\mathrm{Hom}_X(\mathcal{I}_q, \mathcal{I}_Z) = 0$ for every pair $(q, Z) \in U$, which implies that

$$\mathcal{E}xt^0_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}^{n-1}) = 0$$

as well. In this case, by [Lange 1983, Corollary 4.5], there is a universal extension $\mathcal{H}$ on $X \times H$ with $H := \mathbb{P}((\mathcal{E}^1)^\vee)$ such that for every $h \in H$, the restriction $E_h := \mathcal{H}|_{X\times\{h\}}$ is a nonsplit extension of two sheaves of ideals of 0-dimensional subschemes of $X$ of the form

$$0 \to \mathcal{I}_Z \to E \to \mathcal{I}_q \to 0. \tag{14}$$

In other words, every member $E_h$ of the family $\mathcal{H}$ satisfies the exact sequence in (14), and therefore is an unbalanced sheaf; since stability is an open condition, Lemma 4.6 guarantees that there is an open subset $H' \subset H$ whose projection $H' \to U$ is surjective and such that $E_h$ is stable for every $h \in H'$. Therefore, $\mathcal{H}|_{H'}$ is a family of stable rank-2 quasitrivial sheaves parametrized by the scheme $H'$, whose dimension can be easily computed as follows:

$$\dim(H') = \dim(U) + \mathrm{ext}^1(\mathcal{I}_q, \mathcal{I}_Z) - 1 = d + d(n-1) + n - 3 = n(d+1) - 3.$$

Moreover, Proposition 4.8 yields $\mathrm{ext}^1(E, E) = n(d+1) - 3$.

**Theorem 5.3.** *For every $n \geq 3$, $\mathcal{N}_X(2, n)$ contains an irreducible component of dimension $n(d+1) - 3$.*

*Proof.* $\mathcal{N}_X(2, n)$ is a coarse moduli space, so our family $\mathcal{H}$ on $X \times H'$ gives us a modular morphism $\Psi : H' \to \mathcal{N}_X(2, n)$ whose image is precisely the subset of stable unbalanced sheaves. However, as we have seen in Lemma 4.6, the representation of an unbalanced sheaf as an extension of the ideal sheaf of a point by the ideal sheaf of a 0-dimensional scheme is not unique, meaning that the morphism $\Psi$ is not injective. Nonetheless, we argue that it is a finite map.

Indeed, note that the Proposition 4.7 shows that an unbalanced sheaf can be represented as an extension of an ideal sheaf of a point by an ideal sheaf of a reduced 0-dimensional scheme in at most $n$ different ways. In other words, if $E \in \mathrm{Im}\,\Phi \subset \mathcal{N}_X(2, n)$, then $\Phi^{-1}(E)$ consists of at most $n$ different points.

This means that the dimension of the image of $\Psi$ in $\mathcal{N}_X(2, n)$ is equal to the dimension of $H$. Since every $E \in \mathrm{Im}\,\Psi$ satisfies

$$\dim(T_E \mathcal{N}_X(2, n)) = \mathrm{ext}^1(E, E) = n(d+1) - 3 = \dim(\mathrm{Im}\,\Psi),$$

we conclude that the closure of $\mathrm{Im}\,\Psi$ within $\mathcal{N}_X(2, n)$ is an irreducible component of $\mathcal{N}_X(2, n)$. □

Since $H'$ is the projectivization of the locally free sheaf $\mathcal{E}^1$ over the integral scheme $U$, we get that $H'$ is also integral. As a consequence, we can conclude that the irreducible component of $\mathcal{N}_X(2, n)$ that we just constructed (as the closure of the image of the modular morphism $\Psi$) is generically reduced.

## 6. An irreducible component of $\mathcal{N}_X(r, n)$

In this section, we will construct an irreducible component of $\mathcal{N}_X(r, n)^{\mathrm{st}}$ whenever $r < n$. We will use Section 5 as an induction step, along with results from earlier sections.

We will argue by induction. The case $r = 2$ is already done, so suppose we have a family $\mathcal{H}_{r-1,n-1}$ that gives us an generically reduced, irreducible component $H_{r-1,n-1}$ of $\mathcal{N}_X(r-1, n-1)$ with dimension $(n-1)(d+r-2) - (r-1)^2 + 1$; that is, $\mathcal{H}_{r-1,n-1}$ is sheaf on $X \times H_{r-1,n-1}$ such that for every $h \in H_{r-1,n-1}$, $\mathcal{H}_{r-1,n-1}|_h \in \mathcal{N}_X(r-1, n-1)^{\mathrm{st}}$. Consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Hilb}^1(X) \times H_{r-1,n-1} & & \\ & & \uparrow f & & \\ & & X \times \mathrm{Hilb}^1(X) \times H_{r-1,n-1} & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ X \times \mathrm{Hilb}^1(X) & & & & X \times H_{r-1,n-1} \end{array}$$

where $H_{r-1,n-1}$ denotes the irreducible component given by the induction hypothesis on $\mathcal{N}_X(r-1, n-1)$.

Define

$$U := \{(y, F) \in \mathrm{Hilb}^1 \times H_{r-1,n-1} \mid y \notin \mathrm{Supp}\, Q_F,\ \mathrm{Supp}\, Q_F = (\mathrm{Supp}\, Q_F)_{\mathrm{red}}\}$$

as an open subset of $\mathrm{Hilb}^1 \times H_{r-1,n-1}$, and let $\mathbb{X} := X \times U$ with $\pi : \mathbb{X} \to U$ the projection onto the second factor. Note that $U$ is also a generically reduced, irreducible scheme.

Let $\mathcal{E}^i := \mathcal{E}xt^i_\pi(p_1^*\mathcal{H}^1, p_2^*\mathcal{H}_{r-1,n-1})$. By Proposition 4.4 and Lemma 5.2, $\mathcal{E}^1$ is a locally free sheaf on $\mathrm{Hilb}^1 \times H_{r-1,n-1}$ whose fibers over a point $(y, F) \in X \times \mathcal{N}_X(r-1, n-1)^{\mathrm{st}}$ is $\mathrm{Ext}^1(\mathcal{I}_y, F)$ for every $y \notin \mathrm{Supp}\, Q_F$.

Note that $\mathrm{Hom}_X(\mathcal{I}_p, F) = 0$ for every $F \in \mathcal{N}_X(r-1, n-1)^{\mathrm{st}}$. So, by Theorem 5.1, $\mathcal{E}^0 = 0$, and [Lange 1983, Corollary 4.2] implies that there is an universal extension $\mathcal{H}$ on $X \times H$ with $H = \mathbb{P}((\mathcal{E}^1)^\vee)$ such that for every $h \in H$, the restriction $\mathcal{H}|_h$ is a nonsplit extension of the form

$$0 \to F \to E \to \mathcal{I}_y \to 0. \tag{15}$$

In other words, every member $E_h$ of the family $\mathcal{H}$ satisfies the exact sequence in (15), and therefore is an unbalanced sheaf; since stability is an open condition, Proposition 4.7 guarantees that there is an open subset $H' \subset H$ whose projection $H' \to U$ is surjective and such that $E_h$ is stable for every $h \in H'$. Therefore, $\mathcal{H}|_{H'}$ is a family of stable rank-$r$ quasitrivial sheaves parametrized by the generically reduced, irreducible scheme $H'$, whose dimension can be computed as follows:

$$\begin{aligned}\dim(H') &= \dim(U) + \mathrm{ext}^1(\mathcal{I}_p, F) - 1\\ &= d + (n-1)(d + r - 1 - 1) - (r-1)^2 + 1 + (n - 1 - (r-1)) - 1\\ &= n(d + r - 1) - r^2 + 1.\end{aligned}$$

Note that Proposition 4.8 implies $\mathrm{ext}^1(E, E) = n(d + r - 1) - r^2 + 1$.

**Theorem 6.1.** *Let $r < n$ be positive integers. $\mathcal{N}_X(r, n)$ contains a generically reduced and irreducible component of dimension $n(d + r - 1) - r^2 + 1$.*

*Proof.* $\mathcal{N}_X(r, n)$ is a coarse moduli space, so our family $\mathcal{H}$ on $X \times H'$ gives us a modular morphism $\Psi : H' \to \mathcal{N}_X(r, n)$ whose image is precisely the subset of stable unbalanced sheaves. However, as we have seen in Proposition 4.7, the representation of an unbalanced sheaf as an extension of the ideal sheaf of a point by a quasitrivial sheaf supported in a 0-dimensional scheme is not unique, meaning that the morphism $\Psi$ is not injective. Nonetheless, we argue that it is a finite map.

Indeed, note that the proof of Proposition 4.7 shows that an unbalanced sheaf can be represented as an extension as in (15) in at most $n$ different ways. In other words, if $E \in \mathrm{Im}\, \Phi \subset \mathcal{N}_X(r, n)$, then $\Phi^{-1}(E)$ consists of at most $n$ different points.

This means that the dimension of the image of $\Psi$ in $\mathcal{N}_X(r, n)$ is equal to the dimension of $H$. Since every $E \in \mathrm{Im}\, \Psi$ satisfies

$$\dim(T_E \mathcal{N}_X(r, n)) = \mathrm{ext}^1(E, E) = n(d + r - 1) - r^2 + 1 = \dim(\mathrm{Im}\, \Psi),$$

we conclude the closure of $\mathrm{Im}\, \Psi$ within $\mathcal{N}_X(r, n)$ is an irreducible component of $\mathcal{N}_X(r, n)$, as desired. □

## 7. Projective spaces

In this section we prove that $\mathcal{N}_{\mathbb{P}^3}(r,n)$ is irreducible for $n \le 10$. Our starting point is the fact that the affine Quot scheme $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{A}^3}, n)$ is irreducible for $n \le 10$; see [Henni and Guimarães 2021, Corollary 6.1]. We will use the following technical lemma to see that the same holds for the projective Quot scheme $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{P}^3}, n)$.

**Lemma 7.1.** *Let $X$ be the union of two irreducible subschemes $A$ and $B$, and let $C$ be the intersection of $A$ and $B$. Assume $C$ is nonempty and open in $A$ and $B$. Then $X$ is irreducible.*

*Proof.* The closure of $A$ in $X$ is the same as the closure of $C$ in $X$. Indeed, a function vanishing on $C$ vanishes on $A$ because $C$ is open in $A$. Also, the closure of $B$ in $X$ is the same as the closure of $C$ in $X$. Since $X = A \cup B$, we conclude that $X$ is the closure of $C$ in $X$. But $C$ is an open subset of an irreducible scheme $A$, so $X$ is irreducible. □

As a simple consequence, we have:

**Lemma 7.2.** *If $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{A}^d}, n)$ is irreducible, then $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{P}^d}, n)$ is irreducible.*

*Proof.* Fix coordinates $[x_0 : x_1 : \cdots : x_d]$ for $\mathbb{P}^d$ and let $H_i = \{x_i = 0\}$ so $A_i := \mathbb{P}^d \setminus H_i \cong \mathbb{A}^d$. For $(\varphi, Q) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, n)$, if $\mathrm{Supp}(Q) \cap H_i = \varnothing$, then we can restrict $\varphi$ to $A_i$ and we obtain a point in $\mathrm{Quot}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$. Since we are assuming $\mathrm{Quot}(\mathcal{O}_{\mathbb{A}^d}^{\oplus r}, n)$ to be irreducible, we can write $\mathrm{Quot}(\mathcal{O}_{\mathbb{P}^d}^{\oplus r}, n)$ as the union of irreducible subschemes such that the two-by-two intersection is open inside those two. Therefore, by induction and the previous lemma, we get the statement. □

In particular, we conclude that $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{P}^3}, n)$ is irreducible for $n \le 10$.

As it was pointed out in Section 3, right before Lemma 3.7, there exists a surjective morphism

$$\widetilde{\Psi} : \mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{P}^3}, n)^{\mathrm{ss}} \to \mathcal{N}_{\mathbb{P}^3}(r,n),$$

where the superscript ss indicated the open subset of $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{P}^3}, n)$ consisting of GIT-semistable points in the sense of Theorem 3.6. It follows that $\mathcal{N}_{\mathbb{P}^3}(r,n)$ is irreducible whenever $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{P}^3}, n)$ is irreducible. We have therefore established the following claim.

**Corollary 7.3.** *$\mathcal{N}_{\mathbb{P}^3}(r,n)$ is irreducible for $n \le 10$.*

Note that when $n \le 10$, Corollary 7.3 tells us that the component constructed in Theorems 5.3 and 6.1 is the only one, thus providing an explicit description of $\mathcal{N}_{\mathbb{P}^3}(r,n)$ for $n \le 10$.

**Remark 7.4.** Let $\mathcal{C}(d,n)$ the variety consisting of $d$-tuples of $n \times n$ commuting matrices. Jelisiejew and Šivic [2022] classified the components of $\mathcal{C}(d,n)$ for $n \le 7$, which gives a classification of the components of $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{A}^d}, n)$ for $n \le 7$, and Henni and Guimarães [2021, Proposition 6.1] proved that the number of irreducible components of the affine Quot scheme $\mathrm{Quot}(V_r \otimes \mathcal{O}_{\mathbb{A}^d}, n)$ is always smaller than or equal to the number of irreducible components of $\mathcal{C}(d,n)$ (regardless of the value of $r$). Therefore, we can conclude that $\mathcal{N}_{\mathbb{P}^d}(r,n)$ is irreducible whenever $\mathcal{C}(d,n)$ is.

Determining whether $\mathcal{C}(3, n)$ is irreducible is an interesting open problem. Currently, $\mathcal{C}(3, n)$ is known to be irreducible for $n \leq 10$ but reducible for $n \geq 29$; see [Han 2005; Holbrook and Omladič 2001; Omladič 2004; O'Meara et al. 2011; Šivic 2008; 2012]. In addition, note that $\mathcal{N}_{\mathbb{P}^3}(1, n) = \mathrm{Hilb}^n(\mathbb{P}^3)$ has been shown to be irreducible for $n \leq 11$; see [Henni and Jardim 2018] for $n \leq 10$ and [Douvropoulos et al. 2017] for the case $n = 11$.

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# Inverse theorems for discretized sums and $L^q$ norms of convolutions in $\mathbb{R}^d$

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We prove inverse theorems for the size of sumsets and the $L^q$ norms of convolutions in the discretized setting, extending to arbitrary dimension an earlier result of ours. These results have applications to the dimensions of dynamical self-similar sets and measures, and to the higher-dimensional fractal uncertainty principle. The proofs are based on a structure theorem for the entropy of convolution powers due to M. Hochman.

## 1. Introduction and main results

In order to state and discuss our main results, we introduce some notation. We let $\mathcal{D}_k$ be the family of half-open dyadic cubes of side length $2^{-k}$ in $\mathbb{R}^d$ (the value of $d$ will be implicit from context). For $x \in \mathbb{R}^d$, the only element of $\mathcal{D}_k$ containing $x$ is denoted by $\mathcal{D}_k(x)$. More generally, if $A \subset \mathbb{R}^d$, then $\mathcal{D}_k(A)$ denotes the family of cubes in $\mathcal{D}_k$ intersecting $A$, and we write $|A|_{2^{-k}} = |\mathcal{D}_k(A)|$.

We write $[S] = \{0, 1, \dots, S-1\}$ for short; if $S$ is not an integer, then $[S] := [\lfloor S \rfloor]$. Logarithms are always to base 2.

**Definition 1.1.** Given $m \in \mathbb{N}$, a $2^{-m}$*-set* is a subset of $2^{-m}\mathbb{Z}^d \cap [0,1)^d$, and a $2^{-m}$*-measure* is a probability measure supported on $2^{-m}\mathbb{Z}^d \cap [0,1)^d$.

A $2^{-SL}$-set is called $(L, S)$*-uniform* if there is a sequence $(R_s)_{s\in[S]}$ such that for each $s$ and each $I \in \mathcal{D}_{sL}(A)$ it holds that

$$|A \cap I|_{2^{-(s+1)L}} = R_s.$$

In this case, we refer to $R_s$ as the *branching numbers* of $A$, and we say that $A$ is $(L, (R_s)_{s\in[S]})$-uniform.

Given a finitely supported measure $\mu$ and $q \in [1, +\infty)$, we define the (discrete) $L^q$ norm of $\mu$ as

$$\|\mu\|_q^q = \sum_x \mu(x)^q. \tag{1-1}$$

We denote the Grassmannians of linear and affine $k$-dimensional planes in $\mathbb{R}^d$ by $\mathbb{G}(d,k)$ and $\mathbb{A}(d,k)$, respectively. Let $\pi_W$ denote the orthogonal projection onto a subspace $W \le \mathbb{R}^d$. If $c > 0$ and $I$ is a cube, we denote by $cI$ the cube with the same center as $I$ and side length $c$ times the side length of $I$.

We can now state our main result, which we informally announced in [Shmerkin 2023, Section 3.8.3].

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**Theorem 1.2.** *For each $q \in (1, \infty)$ and $\delta > 0$ the following holds if $L \geq L(\delta) \in \mathbb{N}$ and $\varepsilon < \varepsilon(L)$, for all sufficiently large $S = S(\delta, L, \varepsilon)$:*

*Let $m = SL$. Suppose $\mu, \nu$ are $2^{-m}$-measures on $[0, 1)^d$ such that*

$$\|\mu * \nu\|_q \geq 2^{-\varepsilon m} \|\mu\|_q.$$

*Then there exist $2^{-m}$-sets $A \subset \operatorname{supp} \mu$ and $B \subset \operatorname{supp} \nu$ and a sequence $(k_s)_{s\in[S]}$ taking values in $[0, d]$ such that:*

(A1) $\|\mu|_A\|_q \geq 2^{-\delta m} \|\mu\|_q$.

(A2) *$\mu(x) \leq 2\mu(y)$ for all $x, y \in A$.*

(A3) *$A$ is $(L, (R_s)_{s\in[S]})$-uniform for some sequence $(R_s)$.*

(A4) *For each $s \in [S]$ and each $I \in \mathcal{D}_{sL}(A)$, there is $W_I \in \mathbb{G}(d, d - k_s)$ such that*

$$R_s \geq 2^{(k_s - \delta)L} |\pi_{W_I}(A \cap I)|_{2^{-(s+1)L}}.$$

(B1) $\|\nu|_B\|_1 = \nu(B) \geq 2^{-\delta m}$.

(B2) *$\nu(x) \leq 2\nu(y)$ for all $x, y \in B$.*

(B3) *$B$ is $(L, S)$-uniform.*

(B4) *For each $s \in [S]$ and each $I \in \mathcal{D}_{sL}(B)$, there is $V_I \in \mathbb{A}(d, k_s)$ such that*

$$I \cap B \subset \mathcal{D}_{(s+1)L}(V_I).$$

*Moreover, if $A \cup B \subset \left[\frac{1}{3}, \frac{2}{3}\right)^d$ then, after translating $A$ and $B$ by suitable vectors in $\left[-\frac{1}{3}, \frac{1}{3}\right)^d$, we may also assume that*

(C1) *$x \in \frac{1}{3}\mathcal{D}_{sL}(x)$ for each $x \in A \cup B$ and $s \in [S]$.*

We make some remarks on the meaning and novelty of this statement.

**Remark 1.3.** Condition (A4) says that $A \cap I$ is densely contained in the product of a $k_s$-dimensional subspace and another (arbitrary) set — we refer to this behavior as "almost saturation". On the other hand, (B4) states that $B \cap I$ is contained in a $k_s$-dimensional plane at the appropriate scale. Thus, Theorem 1.2 can be roughly summarized as saying that if no $L^q$ flattening occurs in $\mu * \nu$ then, after passing to regular subsets $A$ and $B$ that capture, respectively, a "large" proportion of the $L^q$ norm and the mass, at each scale $2^{-sL}$ the measure $\mu$ is "almost saturated" on a $k_s$-dimensional subspace, while $\nu$ is "almost contained" in a $k_s$-dimensional affine plane.

**Remark 1.4.** The one-dimensional version of this result is [Shmerkin 2019, Theorem 2.1]. In the case $d = 1$, the only possibilities are $k_s = 0$ and $k_s = 1$, corresponding to $B$ having "no branching" and $A$ having "almost full branching", respectively. In higher dimensions, the possibility that $0 < k_s < d$ makes the situation more complicated. Lack of smoothing can indeed happen due to saturation on intermediate-dimensional subspaces, for example, if $\mu = \nu$ is the uniform measure $\lambda_k$ on some subspace of dimension $k \in [1, d-1]$ or, more generally, if $\mu$ is (roughly) a product measure $\lambda_k \times \mu'$, where $\mu'$ is arbitrary, and $\nu$ is concentrated on supp $\lambda_k$. It is also possible to construct examples where the conclusion of Theorem 1.2 fails, and $\mu = \nu$ is roughly the uniform measure on a plane at each scale, with the dimension of the plane varying according to the scale; see the discussion in [Shmerkin 2019, p. 338], which extends to higher dimensions in a straightforward manner. This can be extended to show that (A4) and (B4) are necessary in Theorem 1.2 even for $\mu$ and $\nu$ of very different sizes. In this sense, Theorem 1.2 is conceptually fairly tight, but we underline that we are making no claims on how the different subspaces are related to each other — one would expect that in some rather strong sense, all the planes arising at a given scale are "essentially the same" up to translation, but we are currently unable to prove this.

Theorem 1.2 can be seen as an $L^q$ analog of Hochman's higher-dimensional inverse theorem for entropy [2017, Theorem 2.8]. Indeed, our proof is based on Hochman's machinery, but we do not apply Theorem 2.8 directly. Instead, we appeal to [Hochman 2017, Theorem 4.12] (recalled as Theorem 3.3 below) to prove the following inverse theorem for sumsets first. We denote the $r$-neighborhood of a set $X$ by $X(r)$.

**Theorem 1.5.** *For each $\delta > 0$ the following holds if $L \geq L_0(\delta) \in \mathbb{N}$, $0 < \sigma < \sigma(L)$, and $S \geq S_0(\delta, L, \sigma)$: Let $m = SL$. Let $A$ be a $2^{-m}$ set in $[0, 1)^d$ such that*

$$|A + A| \leq 2^{\sigma m}|A|.$$

*Then there exists a subset $A' \subset A$ such that the following holds:*

(i) *$A'$ is $(L, (R_s)_{s\in[S]})$-uniform for some sequence $(R_s)_{s\in[S]}$.*

(ii) $|A'| \geq 2^{-\delta m}|A|$.

(iii) *For each $s \in [S]$, there is $k_s \in \{0, 1, \ldots, d\}$ such that*

$$\log R_s \geq L(k_s - \delta),$$

*and for each $I \in \mathcal{D}_{sL}(A')$ there is a $k_s$-dimensional affine plane $W_I$ such that*

$$A' \cap I \subset W_I(2^{-(s+1)L}).$$

**Remark 1.6.** Theorem 1.5 is closely related to Freiman's theorem in additive combinatorics [Tao and Vu 2006, Theorem 5.32], which roughly states that a set with small sumset has large intersection with a generalized arithmetic progression. However, even the strongest known quantitative version of Freiman's theorem [Sanders 2013, Theorem 1.4] does not seem, by itself, to be sufficient for our purposes.

## 2. Uniformization lemmas

In this section we collect some "uniformization lemmas" that will be used in the proof of Theorem 1.5. The following basic result appears implicitly in several papers of J. Bourgain and was explicitly recorded in [Keleti and Shmerkin 2019, Lemma 3.4].

**Lemma 2.1.** *Let $L, S \in \mathbb{N}$. Let $A$ be a $2^{-m}$-set in $[0,1)^d$, where $m = SL$. Then there is an $(L,S)$-uniform subset $A' \subset A$ such that*

$$|A'| \geq (2Ld)^{-S}|A| = 2^{(-\log(2Ld)/L)m}|A|.$$

Given a set $A \subset [0,1)^d$ and a dyadic cube $I \in \mathcal{D}_k$, we define the renormalization $A^I$ as

$$A^I = \boldsymbol{H}_I(A \cap I),$$

where $\boldsymbol{H}_I$ is the orientation-preserving homothety mapping $I$ to $[0,1)^d$. If $\ell \leq k$, we also let $A^I_\ell$ be the $2^{-\ell}$-set corresponding to $A^I$:

$$A^I_\ell = \{ j2^{-\ell} : 2^{-\ell}[j + [0,1)^d) \cap A^I \neq \varnothing \}.$$

The same idea behind Lemma 2.1 yields the following more general result:

**Lemma 2.2.** *Let $(F_s)_{s\in[S]}$ be functions defined on $2^{-L}$-sets and taking $\leq V$ values. For any $2^{-SL}$-set $A_0$, there exists a $(L,S)$-uniform subset $A$ such that*:

- $|A| \geq (2LVd)^{-S}|A_0|$.
- *For each $s \in [S]$, the function $F_s$ is constant on $A^I_L$ over all $I \in \mathcal{D}_{sL}(A)$.*

*Proof.* The idea is to prune the $A_0$-tree backwards, ensuring the desired uniformity and constancy at each scale.

To begin, we pigeonhole a value $k \in [dL]$ such that

$$\sum \{ |A_0 \cap I| : I \in \mathcal{D}_{(S-1)L}, |A_0 \cap I|_L \in (2^{k-1}, 2^k] \} \geq (dL)^{-1}|B|.$$

For each $I$ with $|A_0 \cap I|_L \in (2^{k-1}, 2^k]$, let $B^I_L$ be a subset of $(A_0)^I_L$ of size $2^{k-1}$ (so, the size drops by at most a factor of 2), and let $\widetilde{B}_{S-1}$ be the union of all such $B^I_L$. Then

$$|\widetilde{B}_{S-1}| \geq (2dL)^{-1}|A_0|.$$

Next, we pigeonhole a value $v_{S-1}$ in the range of $F_{S-1}$ such that

$$\sum \{ |B \cap I| : I \in \mathcal{D}_{(S-1)L}, F_{S-1}((\widetilde{B}_{S-1})^I_L) = v_{S-1} \} \geq V^{-1}|\widetilde{B}_{S-1}|.$$

Let $B_{S-1}$ be the union of the $\widetilde{B}_{S-1} \cap I$ appearing in the left-hand side. Thus,

$$|B_{S-1}| \geq V^{-1}|\widetilde{B}_{S-1}| \geq (2dVL)^{-1}|A_0|.$$

We replace $A_0$ by $B_{S-1}$ and continue by backwards induction in $s$. Eventually we reach a set $B_0$, and define $A := B_0$. Clearly, $|A| \geq (2LVd)^{-S}|A_0|$.

By construction, if $I \in \mathcal{D}_{sL}(A)$ for $0 \leq s \leq S-1$, then $I \in \mathcal{D}_{sL}(B_s)$, and $A^I_L = (B_s)^I_L$. Thus, the constancy of $F$ and the branching number at each level is preserved throughout the process, and holds for $A$ as well. □

Given a set $A \subset [0,1)^d$ and $L \in \mathbb{N}$, let $D_L(A)$ be the smallest integer $j$ such that $A \subset W((\sqrt{d}+1)2^{-L})$ for some $W \in \mathbb{A}(d,j)$. Since $j = d$ always works, this is well defined and takes values in $[d+1]$. Applying Lemma 2.2 to $F = D_L$ and $V = d+1$, we get:

**Lemma 2.3.** *Let $L, S \in \mathbb{N}$, and let $A_0$ be a $2^{-SL}$-set in $[0,1)^d$. Then there exists a $(L,S)$-uniform subset $A \subset A_0$ such that*

$$|A| \geq (2d(d+1)L)^{-S}|A_0|,$$

*and, for each $s \in [S]$, the dimension $D_L(A_L^I)$ is constant over all $I \in \mathcal{D}_{sL}(A)$.*

The same idea yields the following lemma, asserting that one can always find a large subset for which all points are near the center of dyadic cubes, after translation.

**Lemma 2.4.** *Let $A \subset \left[\frac{1}{3},\frac{2}{3}\right)^d$ be a $2^{-SL}$-set. There are $A' \subset A$ with $|A'| \geq 3^{-2dS}|A|$ and a vector $y \in \left[-\frac{1}{3},\frac{1}{3}\right)^d$ such that $x \in \frac{1}{3}\mathcal{D}_{sL}(x)$ for all $x \in A' + y$ and $s \in [S]$.*

*Proof.* Given a cube $I$ in $[0,1)^d$, let $\{I^{(j)}\}_{j\in\{-1,0,1\}^d}$ be the partition of $I$ into $3^d$ nonoverlapping cubes of side length a third that of $I$, indexed in the natural way. For example, $I^{(0,\ldots,0)} = \frac{1}{3}I$.

Let $A_S = A$. Once $A_{s+1}$ has been defined for $s \in [S]$, we define $A_s$ as follows: For each $I \in \mathcal{D}_{sL}(A_{s+1})$, pigeonhole an index $j(I) \in \{-1,0,1\}^d$ such that

$$|A_{s+1} \cap I^{(j(I))}| \geq 3^{-d}|A_{s+1} \cap I|.$$

Then select $j(s)$ such that

$$\sum\{|A_{s+1} \cap I| : j(I) = j(s)\} \geq 3^{-d}|A_{s+1}|.$$

Set

$$A_s = \bigcup\{A_{s+1} \cap I^{(j(s))} : j(I) = j(s)\}.$$

Then $|A_s| \geq 3^{-2d}|A_{s+1}|$. The claim holds with $A' = A_1$ and $y = -\sum_{s=1}^{S-1} 3^{-(s+1)} j(s)$. □

We conclude this section by recalling another simple but useful result, stating that one can always "collapse the branching" of a uniform set at a subset of scales.

**Lemma 2.5.** *Given $L, S \in \mathbb{N}$ the following holds. Let $A_0$ be $(L,S,R_s)$-uniform. Let $\mathcal{S} \subset [S]$ be any set, and for each $s \in \mathcal{S}$ and $I \in \mathcal{D}_{sL}(A_0)$ let $X^I \subset (A_0)_L^I$ be a subset with $|X^I| = R_s'$ (independent of $I$).*

*Then there exists a $(L,S,R_s'')$-uniform set $A \subset A_0$ such that:*

(a) *If $s \in \mathcal{S}$, then $A_L^I = X^I$ for all $I \in \mathcal{D}_{sL}(A)$. In particular, $R_s'' = R_s'$.*

(b) *If $s \notin \mathcal{S}$, then $A_L^I = (A_0)_L^I$ for all $I \in \mathcal{D}_{sL}(A)$. In particular, $R_s'' = R_s$.*

(c) $|A| \geq \left(\prod_{s\in\mathcal{S}} \dfrac{R_s'}{R_s}\right) \cdot |A_0|.$

*Proof.* The case where $X^I$ is a singleton (equivalently, $R_s' = 1$) for $s \in \mathcal{S}$ is [Shmerkin 2019, Lemma 3.7]. The general case follows in the same way, pruning the tree backwards, replacing $A_L^I$ by $X^I$ at each step when $s \in \mathcal{S}$. □

## 3. Proof of Theorem 1.5

**3.1.** ***Hochman's theorem on saturation of self-convolutions.*** The proof of Theorem 1.5 is based on Hochman's work on inverse theorems for the entropy of convolutions [2017] and specifically [Hochman 2017, Theorem 4.2]. We start by reviewing the concepts required to state this result.

We write $\mathcal{P}_d$ to denote the Borel probability measures on $[0,1)^d$. Given $\mu \in \mathcal{P}_d$ and $x, k$ such that $\mu(\mathcal{D}_k(x)) > 0$, we write

$$\mu^{x,k} = \frac{1}{\mu(\mathcal{D}_k(x))} \boldsymbol{H}_{x,k}(\mu|_{\mathcal{D}_k(x)}),$$

where $\boldsymbol{H}_{x,k}$ is the (orientation-preserving) homothety mapping $\mathcal{D}_{x,k}$ to $[0,1)^d$.

Given a finite set $I \subset \mathbb{N}$ and a family of measures $\mathcal{M} \subset \mathcal{P}([0,1)^d)$, we use the notation

$$\mathbb{P}_{i\in I}(\mu^{x,i} \in \mathcal{M}) = \frac{1}{|I|}\sum_{i\in I} \mu\{x : \mu^{x,i} \in \mathcal{M}\}.$$

Likewise, if $F : \mathcal{P}([0,1)^d) \to \mathbb{R}$ is a Borel function, we write

$$\mathbb{E}_{i\in I}(F(\mu^{x,i})) = \frac{1}{|I|}\sum_{i\in I} \int F(\mu^{x,i})\, d\mu(x).$$

Given a measure $\mu \in \mathcal{P}([0,1)^d)$ and $m \in \mathbb{N}$, we denote by

$$H_m(\mu) = \frac{1}{m} H(\mu, \mathcal{D}_m)$$

the *normalized* entropy of $\mu$ on the dyadic partition $\mathcal{D}_m$.

The following simple but fundamental lemma relates global and local entropies.

**Lemma 3.1** [Hochman 2017, Section 3.2]. *Let* $\mu \in \mathcal{P}([0,1)^d)$. *Then*

$$H_{SL}(\mu) = \mathbb{E}_{i\in L[S]} H_L(\mu^{x,i}).$$

**Definition 3.2.** Fix a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$, a linear subspace $V \le \mathbb{R}^d$, and $\varepsilon > 0$, $L \in \mathbb{N}$.

We say that $\mu$ is $(V, \varepsilon)$-*concentrated* if there is $x$ such that

$$\mu(x + V^{(\varepsilon)}) \ge 1 - \varepsilon,$$

where $X^{(\varepsilon)}$ denotes the open $\varepsilon$-neighborhood of $X$.

We say that $\mu$ is $(V, L)$-*saturated* if

$$H_L(\mu) \ge H_L(\Pi_{V^\perp}\mu) + \dim(V) - \frac{C}{L},$$

where $C$ is a suitable constant depending only on the ambient dimension $d$.[1]

The following corollary of [Hochman 2017, Theorem 4.12] is our main tool in the proof of the inverse theorem:

[1] In [Hochman 2017], the definition of saturation has an additional parameter $\varepsilon > 0$; however, in order for the proof of [Hochman 2017, Theorem 4.2] to work, one has to take $\varepsilon = O(1/L)$, which matches our definition.

**Theorem 3.3.** *Given $\eta > 0$ and $L \in \mathbb{N}$, there is a number $k = k(\eta, L) > 0$ such that the following holds for large enough $S$: Let $\mu \in \mathcal{P}([0, 1)^d)$ and let $\tau = \mu^{*k}$ be its $k$-th convolution power. Then there are subspaces $(V_{sL})_{s\in[S]}$ of $\mathbb{R}^d$ such that*

$$\mathbb{P}_{L[S]}(\tau^{x,i} \text{ is } (V_i, L)\text{-saturated }) \geq 1 - \eta,$$
$$\mathbb{P}_{L[S]}(\mu^{x,i} \text{ is } (V_i, \eta)\text{-concentrated }) \geq 1 - \eta.$$

This is the same statement of [Hochman 2017, Theorem 4.12], except that we have $[S]L$ instead of $[SL + 1]$. The version above follows from the observation that if $I \subset J$ then

$$\mathbb{P}_I(\nu^{x,i} \in \mathcal{M}) \leq \frac{|J|}{|I|}\mathbb{P}_J(\nu^{x,i} \in \mathcal{M}).$$

**3.2. *Proof of Theorem 1.5.*** We use standard $O(\cdot)$ notation to hide unimportant constants. All the implicit constants below may depend on the ambient dimension $d$. Let $L$ be large enough to be fixed later. Let $k = k(L)$ be the number given by Theorem 3.3 for

$$\eta := 2^{-(d+1)L}. \tag{3-1}$$

Suppose $|A + A| \leq 2^{\sigma m}|A|$. By the Plünnecke–Ruzsa inequalities (see, e.g., [Tao and Vu 2006, Corollary 6.28]), this implies

$$|kA| \leq 2^{\sigma km}|A|. \tag{3-2}$$

Applying Lemma 2.3 and recalling that $m = SL$, let $A_0$ be an $L$-uniform subset of $A$ such that

$$|A_0| \geq (2d(d+1)L)^{-S}|A|, \tag{3-3}$$

with numbers $k_s \in [d + 1]$ such that for all $I \in \mathcal{D}_{sL}(A_0)$ there is a minimal affine subspace $W_I$ of dimension $k_s$ satisfying

$$A_0 \cap I \subset W_I((\sqrt{d} + 1)2^{-(s+1)L}). \tag{3-4}$$

Recalling that $m = LS$, we get that

$$\log|kA_0| \overset{(3\text{-}2)}{\leq} \sigma km + \log|A| \overset{(3\text{-}3)}{\leq} \left(\sigma k + \frac{O_d(1) + \log L}{L}\right)m + \log|A_0|.$$

We pick $\sigma = \sigma(L) = (kL)^{-1}\log L$ (recall that $k = k(L)$). Hence, we have

$$|kA_0| \leq 2^{O(\log L/L)m}|A_0|. \tag{3-5}$$

Let $\nu$ be the uniform probability measure on $A_0$. Set also $\tau = \nu^{*k}$. By Theorem 3.3, there exist subspaces $V_0, V_L, \ldots, V_{(S-1)L} \leq \mathbb{R}^d$ such that

$$\mathbb{P}_{i\in L[S]}(\tau^{x,i} \text{ is } (V_i, L)\text{-saturated }) > 1 - \eta, \tag{3-6}$$
$$\mathbb{P}_{i\in L[S]}(\nu^{x,i} \text{ is } (V_i, \eta)\text{-concentrated }) > 1 - \eta. \tag{3-7}$$

Let

$$\mathcal{S} = \{s \in [S] : \nu^{x,sL} \text{ is } (V_{sL}, \eta)\text{-concentrated for some } x\}.$$

It follows from (3-7) that

$$|[S]\setminus\mathcal{S}|\le \eta S. \tag{3-8}$$

Since the set $A_0$ is $(L,S)$-uniform, the measures $\nu^{x,sL}$ are all uniform (they give the same mass to all points in their support). We claim that if $\nu^{x,sL}$ is $(V_{sL},\eta)$-concentrated, then $\nu^{x,sL}$ must be supported on a $(\sqrt{d}+1)2^{-(s+1)L}$-neighborhood of a translate of $V_{sL}$. Indeed, otherwise there is a cube $J\in\mathcal{D}_L$ with $\nu^{x,sL}(J)>0$ that does not meet the $2^{-(d+1)L}$ neighborhood of the translate of $V_{sL}$ given by the definition of concentration. But then

$$\nu^{x,sL}(J)\ge 2^{-dL}\overset{(3\text{-}1)}{>}\eta,$$

as all cubes in $\mathcal{D}_L(\operatorname{supp}(\nu^{x,sL}))$ have equal mass and there are at most $2^{dL}$ of them. This yields a contradiction with the definition of concentration.

In light of the claim and the minimality of $W_I$ in (3-4), we deduce that

$$k_s\le\dim(V_{sL})\quad\text{for all } s\in\mathcal{S}. \tag{3-9}$$

It follows directly from Definition 3.2 that if $\rho$ is $(V,L)$-saturated then $H_L(\rho)\ge\dim(V)-O(1/L)$. Applying Lemma 3.1 and (3-6), and recalling that $\eta=2^{-(d+1)L}\le 1/L$, we get that

$$H_m(\tau)\ge(1-\eta)\biggl(\frac{1}{S}\sum_{s\in[S]}\dim(V_{sL})-O(1/L)\biggr)\ge -O(1/L)+\frac{1}{S}\sum_{s\in[S]}\dim(V_{sL}).$$

Since $\tau$ is supported on $kA_0$,

$$\log|kA_0|\ge H(\tau,\mathcal{D}_m)\ge -O(1/L)m+L\sum_{s\in[S]}\dim(V_{sL}).$$

Recalling (3-5), we deduce that

$$\log|A_0|\ge -O(\log L/L)m+L\sum_{s\in\mathcal{S}}\dim(V_{sL}).$$

Writing $(R_s)_{s\in[S]}$ for the branching numbers of $A_0$, we also have

$$\log|A_0|=\prod_{s\in[S]}\log R_s\le|[S]\setminus\mathcal{S}|dL+\sum_{s\in\mathcal{S}}\log R_s.$$

Since

$$|[S]\setminus\mathcal{S}|dL\overset{(3\text{-}8)}{\le}\eta dSL\overset{(3\text{-}1)}{<}(\log L/L)m,$$

we see that

$$\sum_{s\in\mathcal{S}}\log R_s\ge -O(\log L/L)m+\sum_{s\in\mathcal{S}}L\dim(V_{sL}).$$

On the other hand, we get from (3-4) and (3-9) that

$$\log R_s\le O(1)+L\dim(V_{sL}),\qquad s\in\mathcal{S}.$$

Thus, by Markov's inequality,

$$\bigl|\{s\in\mathcal{S}:\log R_s<L(\dim V_{sL}-L^{-\frac{1}{2}})\}\bigr|\le O(L^{-\frac{1}{2}}\log L)S.$$

In light of (3-8), the above also holds if we replace $\mathcal{S}$ by $[S]$ on the left-hand side. The desired set $A'$ is obtained from $A_0$ by using Lemma 2.5 to "collapse the branching" of $A_0$ to 1 at the scales $s$ for which $\log R_s < L(\dim V_{sL} - L^{-\frac{1}{2}})$. We redefine $R_s$ to be 1 and $k_s$ to be 0 for these collapsed scales.

To conclude, take $L$ large enough that $CL^{-\frac{1}{2}} \log L \le \delta$ for a large constant $C$ (possibly depending on $d$). Property (i) holds by construction. To verify (ii), note that (using (3-8) once again), we have

$$|A'| \ge 2^{O(L^{-\frac{1}{2}} \log L)m} |A_0| \ge 2^{O(L^{-\frac{1}{2}} \log L)m} |A| \ge 2^{-\delta m} |A|.$$

Finally, (iii) follows from (3-9) and that by construction either $k_s = 0$ or $\log R_s \ge L(\dim V_{sL} - L^{-\frac{1}{2}})$.

**3.3. *Proof of Theorem 1.2.*** Equipped with Theorem 1.5, the proof of Theorem 1.2 follows roughly the same steps as the proof of the inverse theorem from [Shmerkin 2019, Theorem 2.1]. We apply several lemmas from that work; some were stated there only in dimension 1 but the same proof works in higher dimensions. We keep allowing all implicit constants to depend on the ambient dimension $d$.

Let $\tau > 0$ be a small parameter to be chosen shortly in terms of $L$, and define $\varepsilon$ by

$$\varepsilon = \frac{\tau}{2 \max(q, q')},$$

where $q'$ is the dual exponent of $q$ (i.e., $q = q/(q-1)$). Assume that $\mu$, $\nu$ are as in the statement of Theorem 1.2, and

$$\|\mu * \nu\|_q \ge 2^{-\varepsilon m} \|\mu\|_q.$$

Applying [Shmerkin 2019, Lemmas 3.3 and 3.4], we get $2^{-m}$-sets $A_1$, $B_1$ such that:

(a) $\mu, \nu$ are constant up to a factor of 2 on $A_1$, $B_1$ respectively.

(b) $\|\mathbf{1}_{A_1} * \mathbf{1}_{B_1}\|_2^2 \ge 2^{-\tau m} |A_1||B_1|^2$, where $\mathbf{1}_X$ denotes the indicator function of a set $X$, and $\|\cdot\|_2$ is the $L^2$ norm of a finitely supported measure as defined in (1-1).

(c) $\|\mu|_{A_1}\|_q \ge 2^{-2\varepsilon m} \|\mu\|_q$ and $\|\nu|_{B_1}\|_1 \ge 2^{-2\varepsilon m}$.

Let $L$ be large enough in terms of $\delta$ that Theorem 1.5 applies, and let $\sigma = \sigma(L)$ be the number provided by Theorem 1.5. Let $\tau = \tau(\sigma)$ be the number provided by the asymmetric Balog–Szemerédi–Gowers theorem in the form given in [Shmerkin 2019, Theorem 3.2] (with $\sigma$ in place of $\kappa$). Even though stated there in the real line, this theorem holds in any abelian group. The inequality (b) is precisely the assumption of the asymmetric Balog–Szemerédi–Gowers theorem. Note that the parameters follow the chain of dependencies $\delta \to L \to \sigma \to \tau \to \varepsilon$ (and $\varepsilon$ depends additionally on $q$); later we will impose additional constraints of some on the parameters that respect these dependencies.

Applying the asymmetric Balog–Szemerédi–Gowers theorem, we get $2^{-m}$-sets $H_0$ and $X$ such that:

(i) $|H_0 + H_0| \le 2^{\sigma m} |H_0|$.

(ii) $|A_1 \cap (X + H_0)| \ge 2^{-\sigma m} |A_1| \ge 2^{-2\sigma m} |X||H_0|$.

(iii) $|B_1 \cap H_0| \ge 2^{-\sigma m} |B_1|$.

Let $H$ be the set obtained by applying Theorem 1.5 to $H_0$, with branching numbers $(R_s)_{s\in[S]}$, and dimension sequence $(k_s)_{s\in[S]}$. Our choice of parameters ensures that Theorem 1.5 is indeed applicable.

At this stage we apply Lemma 2.4 to $A_1\cap(X+H_0)$ and $B_1\cap H_0$; since the sets $A$, $B$ will ultimately be subsets of these sets, this ensures that (C1) holds. This refinement only decreases the size of $A_1\cap(X+H_0)$ and $B_1\cap H_0$ by a factor that can be absorbed into $2^{-\sigma m}$ (at the cost of halving the value of $\sigma$), so for notational simplicity we keep denoting these refinements by $A_1\cap(X+H_0)$ and $B_1\cap H_0$.

Now comes the most significant change with respect to [Shmerkin 2019]. Instead of splitting the scales $s$ according to a full branching/no-branching dichotomy, we need to consider all the possible dimensions $k_s$ coming from Theorem 1.5. Given $s\in[S]$ and a $2^{-L}$-set $Y$, we let

$$F_s(Y)=\big\lfloor\big(\inf\big\{\log|\pi_V Y|_{2^{-L}}:V\in\mathbb{G}(d,d-k_s)\big\}\big)\big\rfloor\in[0,Ld].$$

Unpacking the definition, this means that

$$|\pi_V Y|_{2^{-L}}\geq 2^{F_s(Y)}\quad\text{for all } V\in\mathbb{G}(d,d-k_s),$$

and there is $V\in\mathbb{G}(d,d-k_s)$ such that $|\pi_V Y|_{2^{-L}}\leq 2^{F_s(Y)+1}$.

Applying Lemma 2.2 to $A_1\cap(X+H_0)$, we get a uniform subset $A\subset A_1\cap(X+H_0)$ such that $F_s$ is constant on each $A_L^I$, $I\in\mathcal{D}_{sL}(A)$, and

$$|A|\geq 2^{-O(\log L/L)m}|A_1\cap(X+H_0)|\overset{\text{(ii)}}{\geq} 2^{-(O(\log L/L)+\sigma)m}|A_1|. \tag{3-10}$$

We make $\sigma$ smaller if needed to ensure that $\sigma\leq\log L/L$. Since $A+H\subset X+H_0+H_0$, it follows that

$$|A+H|\leq|X||H_0+H_0|\overset{\text{(i)}}{\leq}2^{\sigma m}|X||H_0|\overset{\text{(ii)}}{\leq}2^{2\sigma m}|A_1|\overset{\text{(3-10)}}{\leq}2^{O(\log L/L)m}|A|. \tag{3-11}$$

Let $M_s$ be the constant value of $F_s$ at the sets $A_L^I$, $I\in\mathcal{D}_{sL}(A)$.

**Lemma 3.4.** *There exists a constant $c\in(0,1)$ such that*

$$|A+H|\geq c^S\prod_{s=0}^{S-1}2^{(1-\delta)Lk_s}\cdot 2^{M_s}. \tag{3-12}$$

*Proof.* Fix $s\in[S]$, $I\in\mathcal{D}_{sL}(A)$, $J\in\mathcal{D}_{sL}(H)$. By Theorem 1.5, $R_s\geq 2^{(1-\delta)Lk_s}$ and there is $W_J\in\mathbb{A}(d,k_s)$ such that $H^J\subset W_J(C_d 2^{-L})$. Since the projection of $A^I$ to $W_J^\perp$ has $2^{-L}$-covering number $\geq 2^{M_s}$, it follows that

$$|(A\cap I)+(H\cap J)|_{2^{-(s+1)L}}=|A^I+H^J|_{2^{-L}}\gtrsim R_s 2^{M_s}\geq 2^{(1-\delta)Lk_s}2^{M_s}.$$

We emphasize that this bound is independent of $I$ and $J$ (for a given $s$). It follows that, given $I\in\mathcal{D}_{sL}(A)$, $J\in\mathcal{D}_{sL}(H)$, there are $\gtrsim 2^{(1-\delta)Lk_s}2^{M_s}$ pairwise disjoint sets of the form $I'+J'$, where $I'\in\mathcal{D}_{(s+1)L}(A\cap I)$, $J'\in\mathcal{D}_{(s+1)L}(H\cap J)$.

Applying this with $s=0$, we obtain a collection $\mathcal{K}_0\subset\mathcal{D}_L(A)\times\mathcal{D}_L(H)$ such that

(a) $\{I+J:(I,J)\in\mathcal{K}_0\}$ are pairwise disjoint,

(b) $|\mathcal{K}_0|\geq c\cdot 2^{(1-\delta)Lk_0}2^{M_0}$.

For each $(I, J) \in \mathcal{K}_0$, we apply the same argument to obtain a family

$$\mathcal{K}_{(I,J)} \subset \mathcal{D}_{2L}(A \cap I) \times \mathcal{D}_{2L}(H \cap J)$$

such that the analogous properties hold. Let $\mathcal{K}_1 = \bigcup_{(I,J)\in\mathcal{K}_0} \mathcal{K}_{(I,J)}$. Then $\mathcal{K}_1 \subset \mathcal{D}_{2L}(A) \times \mathcal{D}_{2L}(H)$, and

(a) $\{I + J : (I, J) \in \mathcal{K}_1\}$ are pairwise disjoint,

(b) $|\mathcal{K}_1| \geq c2^{(1-\delta)Lk_0}2^{M_0} \cdot c2^{(1-\delta)Lk_1}2^{M_1}$.

The claim follows by iterating this argument. □

Let $(R'_s)_{s\in[S]}$ be the branching numbers of $A$. By definition of $M_s$, for any $I \in \mathcal{D}_{sL}(A)$, there is an orthogonal projection $\pi$ onto a $(d-k_s)$-dimensional plane such that $|\pi(A^I)|_{2^{-L}} \leq 2^{M_s+1}$, so we have

$$\log R'_s \leq Lk_s + M_s + O(1). \tag{3-13}$$

Combining (3-11) and (3-12), and taking $L$ large enough in terms of $\delta$ that $\log L/L \leq \delta$, we get

$$\sum_{s\in[S]} Lk_s + M_s - \log R'_s + O(1) \leq O(\delta)m.$$

Recalling (3-13) and applying Markov's inequality, we deduce that

$$\log R'_s \geq (1 - \delta^{\frac{1}{2}})(Lk_s + M_s)$$

for all $s$ outside an exceptional set $\mathcal{E}$ with

$$\sum_{s\in\mathcal{E}} \log R'_s \leq \delta^{\frac{1}{2}} \log |A|.$$

Collapsing the branching for scales in $\mathcal{E}$ to 1 via Lemma 2.5, we get all the desired properties for $A$ (with $\delta^{\frac{1}{2}}$ in place of $\delta$).

To construct $B$ (from $B_1$) we proceed in a similar fashion. We uniformize $B_1 \cap H_0$ so that $F_s$ (same function as above) is constant on each scale $s$. Then, since $B \subset B_1 \cap H_0 \subset H_0$,

$$|B + H| \leq |H_0 + H_0| \leq 2^{\delta m}|H_0| \leq 2^{(\delta+\sigma)m}|H| \leq 2^{(\delta+\sigma)m} \prod_{s=0}^{S-1} 2^{Lk_s+O(1)}.$$

Let us denote again by $M_s$ the value of $F$ at scale $s$, now for the set $B$. Then Lemma 3.4, this time applied to $B$, yields that

$$|B + H| \gtrsim \prod_{s=0}^{S-1} 2^{(1-\delta)Lk_s} \cdot 2^{M_s}.$$

Thus, comparing upper and lower bounds for $|B + H|$, and taking $\sigma \leq \delta$ as we may,

$$\prod_{s=0}^{S-1} 2^{M_s} \leq O(\delta)m. \tag{3-14}$$

Note that, by definition of $M_s$, for each $I \in \mathcal{D}_{sL}(B)$ there is a $(d-k_s)$-dimensional subspace $W_I$ such that $|\pi_{W_I} B_L^I|_{2^{-L}} \le 2^{M_s+1}$. By pigeonholing, there is an affine plane $V_I$ of dimension $k_s$ such that

$$\left|B_L^I \cap V_I((\sqrt{d}+1)2^{-L})\right| \gtrsim 2^{-M_s}|B_L^I|.$$

Thus, we can apply Lemma 2.5, replacing each $B_L^I$ by a suitable subset, so that (B4) holds for this choice of $V_I$ (and uniformity is preserved). Doing so reduces the cardinality of each $B_L^I$ by a factor $\lesssim 2^{M_s}$, and so by (3-14) the total loss in cardinality is an acceptable factor $2^{O(\delta)m}$. This finishes the proof of Theorem 1.2.

## 4. Connections and applications

**4.1.** ***$L^q$ dimensions of dynamical self-similar measures, and applications.*** In [Shmerkin 2019], we applied the one-dimensional version of Theorem 1.2 to compute the $L^q$ dimensions of a class of dynamically driven self-similar measures under a very mild exponential separation assumption. In a joint work with E. Corso [Corso and Shmerkin 2024], we make crucial use of Theorem 1.2 to extend this result to arbitrary dimensions, under suitable assumptions. Just as in the one-dimensional case, this has a broad range of applications to the dimension theory of self-similar sets and measures, their projections and their slices. We refer the reader to [Corso and Shmerkin 2024] for details, and to [Shmerkin 2023, Section 3.8] for an announcement of some of these applications.

**4.2.** ***Khalil's inverse theorem.*** Recently, O. Khalil [2023, Proposition 11.10] proved a related inverse theorem for the $L^q$ norms of convolutions, motivated by a striking application to exponential mixing of geodesic flows. Roughly speaking, Khalil considers the case in which $\nu$ is a $2^{-m}$-measure such that

$$\nu\big(H(\eta r) \cap B(x,r)\big) \le C\eta^{\beta}\nu(B(x,r)) \tag{4-1}$$

for some $C, \beta > 0$, all $\eta > 0$, all hyperplanes $H$ in $[0,1)^d$, and "almost all" points $x$ and scales $r \in [2^{-m}, 1]$ (recall that $X(r)$ stands for the $r$-neighborhood of a set $X$). He shows that under this assumption, if $\mu$ is any $2^{-m}$-measure with $\|\mu\|_q \le 2^{-\varepsilon qm}$ for some $\varepsilon > 0$, then

$$\|\mu * \nu\|_q \le 2^{-\sigma m}\|\mu\|_q, \tag{4-2}$$

where $\sigma = \sigma(q, \beta, \varepsilon) > 0$. The nature of the "almost all" points and scales condition in Khalil's theorem has no direct correlate in Theorem 1.2. Nevertheless, at least at a moral level Theorem 1.2 is more general: the assumption (4-1) ensures that (in the context of Theorem 1.2), if (4-2) fails, then $k_s = d$ for "almost all" scales $s$, and this can be seen to imply that $\|\mu\|_q > 2^{-\varepsilon qm}$.

Khalil's proof also relies on the results of Hochman [2017], but he applies a different result of Hochman [2017, Theorem 2.8]. As pointed out in the introduction, this is a result that can be seen as an entropy version of Theorem 1.2.

We remark that the application of Theorem 1.2 to the $L^q$ dimensions of dynamical self-similar measures in [Corso and Shmerkin 2024] requires the full strength of Theorem 1.2, and cannot be deduced from Khalil's result. Indeed, in the application of Theorem 1.2 in [Corso and Shmerkin 2024], the only information we have about $\nu$ is that $\|\nu\|_q \le 2^{-\rho m}$ for some small parameter $\rho > 0$; in particular, nothing prevents $\nu$ from being supported on a single line, let alone a hyperplane.

**4.3.** ***Additive energy of discretized sets.*** Given a finite set $X \subset \mathbb{R}^d$, we define its *additive energy* by

$$\mathcal{E}(X, X) = \#\{(x_1, x_2, y_1, y_2) \in X^4 : x_1 + y_1 = x_2 + y_2\}.$$

This is a key concept in additive combinatorics, with applications throughout mathematics; see [Tao and Vu 2006, Section 2] for an introduction.

When $X$ is a $2^{-m}$-set, the additive energy can be expressed as the $L^2$ norm of convolutions. Indeed, if $\mu$ is the counting measure on $X$ (*not* normalized), then

$$\mathcal{E}(X, X) = \|\mu * \mu\|_2^2.$$

A trivial bound for $\mathcal{E}(X, X)$ is given by $|X|^3$ (given $x_1$, $y_1$, $x_2$, the parameter $y_2$ is uniquely determined). In many applications one would like to know that there is a gain over this trivial bound. The results of this article provide mild conditions under which an exponential gain over $|X|^3$ is achieved. While it is possible to deduce this from Theorem 1.2, the statement follows directly by combining Theorem 1.5 with the (standard) Balog–Szemerédi–Gowers theorem.

**Corollary 4.1.** *For each $\delta > 0$ the following holds if $L \ge L_0(\delta) \in \mathbb{N}$ and $0 < \sigma < \sigma(L)$, for all sufficiently large $S = S(\delta, L, \sigma)$:*

*Let $m = SL$. Let $X$ be a $2^{-m}$ set in $[0, 1)^d$ such that*

$$\mathcal{E}(X, X) \ge 2^{-\sigma m}|X|^3.$$

*Then there exists a subset $A \subset X$ such that the following holds:*

(i) *$A$ is $(L, S, (R_s))$-uniform for some sequence $(R_s)_{s\in[S]}$.*

(ii) $|A| \ge 2^{-\delta m}|X|$.

(iii) *For each $s \in [S]$, there is $k_s \in [0, d]$ such that*

$$\log R_s \ge L(k_s - \delta),$$

*and for each $I \in \mathcal{D}_{sL}(A')$ there is $W_I \in \mathbb{A}(d, k_s)$ such that*

$$A \cap I \subset W_I(2^{-(s+1)L}).$$

*Proof.* By the Balog–Szemerédi–Gowers theorem [Tao and Vu 2006, Theorem 2.29], the assumption on the additive energy implies that there is $Y \subset X$ with $|Y| \gtrsim 2^{-O(\sigma)m}|X|$ such that

$$|Y + Y| \lesssim 2^{O(\sigma)m}|Y|.$$

The claim follows by applying Theorem 1.5 to $Y$. □

Roughly speaking, the above corollary says that if $X$ has nearly maximal additive energy in the exponential sense, then at almost all scales it looks locally like an affine plane (with the dimension of the plane depending only on the scale).

We say that a $2^{-m}$-set $X$ is Ahlfors-regular between scales $2^{-m}$ and 1 if there exist constants $C \geq 1$, $t \in [0, d]$ such that, for each $x \in X$ and $r \in [2^{-m}, 1]$,

$$C^{-1} r^t |X| \leq |X \cap B(X, r)|_{2^{-m}} \leq C r^t |X|.$$

In dimension $d = 1$, S. Dyatlov and J. Zahl [2016] proved that Ahlfors-regular sets in the sense above with $t \in (0, 1)$ admit an exponential additive energy improvement, with the exponent depending quantitatively on the constants $C$, $t$. In an unpublished manuscript, B. Murphy [2019] obtained nearly sharp bounds for the exponential decay exponent in this context. L. Cladek and T. Tao [2021] developed a different quantitative approach in $d = 1$ and obtained the first results in higher dimension: they proved that Ahlfors-regular sets have a quantitative exponential additive energy gain if $t \notin \mathbb{Z}$ (this condition is necessary in general — a counterexample is a set that looks nearly like a $d$-dimensional subspace). It is not hard to recover a nonquantitative form of this result using Corollary 4.1. As a further example, we deduce additive energy flattening for sets that are "porous on $k$-planes". If $B$ is a ball, we denote its radius by $r(B)$.

**Definition 4.2.** Let $k \in \{1, \dots, d\}$ and $\rho, \eta \in (0, 1)$. We say that a set $X \subset \mathbb{R}^d$ is $(k, \rho)$*-porous between scales* $\eta$ *and* 1 if for every $W \in \mathbb{A}(d, k)$, and every ball $B$ with radius $r(B) \in [\eta, 1]$ centered in $W$, there is $y \in B \cap W$ such that

$$B(y, \rho \cdot r) \cap X \cap W = \varnothing.$$

We say that $X$ is $k$-porous between scales $\eta$ and 1 if it is $(k, \rho)$-porous between scales $\eta$ and 1 for some $\rho > 0$, and drop reference to the scales if $\eta = 0$ or is understood from context.

This definition is equivalent to $X \cap W$ being upper $\rho$-porous (in the classical sense) for all $k$-planes $W$. If $X \subset \mathbb{R}^{d-k+1}$ is 1-porous (for example, we can take $X = C^{d-k+1}$, where $C$ is the middle-thirds Cantor set), then $[0, 1]^{k-1} \times X$ is $k$-porous, but it is not $(k-1)$-porous.

**Proposition 4.3.** *Given* $k \in \{1, \dots, d\}$, $\rho \in (0, 1)$ *and* $\lambda > 0$, *there is* $\sigma = \sigma(k, \rho, \lambda) > 0$ *such that the following holds for all sufficiently large* $m \geq m_0(k, \rho, \lambda)$: *Let* $X \subset [0, 1)^d$ *be a* $2^{-m}$ *set which is* $(k, \rho)$*-porous between scales* $2^{-m}$ *and* 1, *and such that*

$$|X| \geq 2^{(k-1+\lambda)m}.$$

*Then*

$$\mathcal{E}(X, X) \leq 2^{-\sigma m} |X|^3. \tag{4-3}$$

*Proof.* To begin, we claim that, for every $j \in [m]$, each $W \in \mathbb{A}(d, k)$, and every $I \in \mathcal{D}_j(W)$,

$$\frac{1}{L} \log \left| W \cap I \cap X(2^{-j+L}) \right|_{2^{-(j+L)}} \leq k - c_d \frac{\rho^k}{\log(1/\rho)} + \frac{O(d, \rho)}{L}. \tag{4-4}$$

Let $Q_0$ be a cube in $W$ of side length $\ell_0 = d^{\frac{1}{2}} 2^{-j}$ such that $W \cap I \subset Q_0$. Let $p$ be the smallest integer such that $2^{-p} \le \frac{1}{4} d^{-\frac{1}{2}} \rho$. We consider the $2^{-p}$-adic grid $(\mathcal{Q}_i)_{i\ge 0}$ of subcubes of $Q_0$ (thus, $Q_0$ is the only element of $\mathcal{Q}_0$, and cubes in $\mathcal{Q}_i$ have side length $2^{-ip}\ell_0$).

Fix $i$ such that $2^{-pi}\ell_0\rho/2 > 2^{-(j+L)}$ and $Q' \in \mathcal{Q}_i$. Then $Q'$ contains a ball $B_Q$ with $r(B_Q) = 2^{-pi}\ell_0/2$. Applying the definition of porosity to $B_Q$, we deduce that there is a ball $B'_Q \subset B_Q \subset Q'$ with $r(B'_q) = \rho \cdot r(B_q)$ such that

$$B'_Q \cap X \cap W = \varnothing.$$

By the assumption on $i$, if $B''_Q$ is the ball concentric with $B'_Q$ and half the radius (thus, $r(B''_Q) = \rho 2^{-pi}\ell_0/4$), then

$$B''_Q \cap X(2^{-(j+L)}) \cap W = \varnothing.$$

In turn, by our choice of $p$, the ball $B''_Q$ contains a cube $Q'' \in \mathcal{Q}_{i+1}$. In short, each cube in $\mathcal{Q}_i$ contains a cube in $\mathcal{Q}_{i+1}$ which is disjoint from $X(2^{-(j+L)}) \cap W$. Iterating this, we deduce that $Q_0 \cap X$ can be covered by $(2^{pk}-1)^i$ cubes in $\mathcal{Q}_i$. The largest allowed value $i_{\max}$ of $i$ satisfies $2^{-pi_{\max}}\ell_0\rho \sim_p 2^{-(j+L)}$; recalling the definitions of $\ell_0$ and $p$, we get that $i_{\max} \ge L/p - O_{d,\rho}(1)$. In turn, each cube in $\mathcal{Q}_{i_{\max}}$ can be covered by $2^d$ standard dyadic cubes of side length $\sim_d 2^{-i_{\max}p}\ell_0$. We deduce that

$$|W \cap I \cap X(2^{-(j+L)})|_{2^{-(j+L)}} \le |Q_0 \cap X|_{2^{-(j+L)}} \lesssim_{d,\rho} [(2^{pk}-1)^{1/p}]^L.$$

From here a small calculation yields the claim.

Now let

$$\delta = \min\left\{ \frac{c_d}{4} \frac{\rho^k}{\log(1/\rho)}, \frac{\lambda}{2} \right\},$$

where $c_d$ is the constant from (4-4). Let $L_0 = L_0(\delta)$ be the constant from Corollary 4.1 and take $L \ge L_0$ large enough that

$$2d\, O(d, \rho) \le \delta L,$$

where $O(d, \rho)$ is the implicit constant from (4-4). We have arranged things so that, for each $I \in \mathcal{D}_{sL}$ and each $k$-plane $W$,

$$\left| W \cap I \cap X(2^{-(s+1)L}) \right|_{2^{-(s+1)L}} \le 2^{(k-2d\delta)L}. \tag{4-5}$$

Assume for sake of contradiction that (4-3) does not hold, with $\sigma$ given by Corollary 4.1 applied with the chosen values of $\delta$ and $L$. Let $Y$ be the set provided by the corollary, with branching numbers $(R_s)_{s\in[S]}$ and dimension sequence $(k_s)_{s\in S}$.

Note that

$$L \sum_{s\in[S]} k_s \ge \sum_{s\in S} \log R_s = \log|Y| \ge \log|X| - \delta m = (k - 1 + \lambda - \delta)m.$$

Thus, since $\delta \le \lambda/2$,

$$\sum_{s\in[S]} k_s \ge (k - 1 + \lambda/2)S.$$

In particular, there is some $k_s$ such that $k_s \geq k$. By Corollary 4.1(iii), there are $s \in [S]$, $I \in \mathcal{D}_s$ and a $k'$-plane $W'$ with $k' \geq k$ such that $Y \cap I \subset W'(2^{-(s+1)L})$. Since $|Y \cap I| \geq R_s \geq 2^{(1-\delta)Lk'}$, after translating $W'$ if needed we get

$$|W' \cap I \cap X^{2(-(s+1)L)}|_{2^{-(s+1)L}} \geq 2^{(1-\delta)Lk'}.$$

Pigeonholing, there is a $k$-plane $W \subset W'$ such that

$$|W \cap I \cap X^{(}2^{-(s+1)L})|_{2^{-(s+1)L}} \gtrsim 2^{(1-\delta)Lk}.$$

This contradicts (4-5). This contradiction finishes the proof. □

The assumption that $|X| \geq 2^{(k-1+\lambda)m}$ is required since a $(k-1)$-plane is $k$-porous and has essentially maximal additive energy. Proposition 4.3 can be improved in several ways. An inspection of the proof shows that one does not need to have porosity at all scales, but only at *most* scales, with "most" depending on the value of $\lambda$. Under further assumptions on $X$, for example, if $X$ is Ahlfors-regular between scales $2^{-m}$ and 1, one only needs porosity at a positive proportion of scales. One can deduce these facts from Corollary 4.1 using locally entropy averages in the spirit of [Shmerkin 2012] in order to establish analogs of (4-4) at a subset of scales. We defer a detailed discussion to a future work.

We remark that Theorem 1.2 can be used to show flattening of additive energy in more general settings. Given two Borel probability measures $\mu$, $\nu$ on $\mathbb{R}^d$, we define their *additive energy at scale* $r > 0$ by

$$\mathcal{E}_r(\mu, \nu) = (\mu^2 \times \nu^2)\{(x_1, x_2, y_1, y_2) : |x_1 + y_1 - x_2 - y_2| \leq r\}.$$

Again this is, up to constants, the same as the $L^2$ norm of convolutions. We leave the proof of the following standard lemma to the reader. Given a Radon measure $\mu$ on $\mathbb{R}^d$, we denote by $\mu^{(m)}$ any reasonable $2^{-m}$-discretization of $\mu$; for concreteness, let

$$\mu^{(m)}(x) = \mu\big(x + 2^{-m}[-\tfrac{1}{2}, \tfrac{1}{2})^d\big).$$

**Lemma 4.4.** *Let* $\mu$, $\nu$ *be Radon measures on* $[0, 1]^d$. *Then, for every* $m \in \mathbb{N}$,

$$\mathcal{E}_{2^{-m}}(\mu, \nu) \sim_d \|\mu^{(m)} * \nu^{(m)}\|_2^2.$$

**4.4.** ***Application to the fractal uncertainty principle.*** One particular problem in which additive energy in the fractal setting is relevant is the *fractal uncertainty principle* (FUP). Given a small scale $h \in (0, 1)$, let

$$\mathscr{F}_h f(x) = (2\pi h)^{-d/2} \int e^{i(1/h)x \cdot y} f(y)\, dy$$

be the semiclassical Fourier transform on $\mathbb{R}^d$. A FUP is a nontrivial estimate of the form

$$\|\mathbf{1}_X \mathscr{F}_h \mathbf{1}_Y\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq h^\beta \quad \text{as } h \to 0,$$

under suitable assumptions on the (typically Cantor-type) sets $X, Y \subset \mathbb{R}^d$ defined at resolution $h$. Here the "trivial" estimate is

$$\|\mathbf{1}_X \mathscr{F}_h \mathbf{1}_Y\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \min\{1, h^{d/2} |X|_h^{\frac{1}{2}} |Y|_h^{\frac{1}{2}}\}; \tag{4-6}$$

see, e.g., [Dyatlov 2019, before (2.7) and (6.3)]. Note that a FUP for $X$ and $Y$ says that $f$ cannot be supported on $Y$ in physical space, and simultaneously on $X$ in frequency space. See [Dyatlov 2019] for an introduction to the subject, and [Han and Schlag 2020; Cladek and Tao 2021; Backus et al. 2023; Cohen 2023] for recent progress on the FUP in dimension $d \geq 2$.

It was shown by S. Dyatlov and J. Zahl [2016, Theorem 4.2] (see also [Dyatlov 2019, Proposition 5.4]) that additive energy estimates imply FUP estimates near the critical threshold $|X|_h|Y|_h = h^{-d}$. While the results of [Dyatlov and Zahl 2016; Dyatlov 2019] are stated in the context of Ahlfors-regular sets in the line, the proof of [Dyatlov 2019, Proposition 5.4] yields the following general estimate:

**Proposition 4.5.** *Let $h$ be a small dyadic number. Let $X, Y \subset [0, 1]^d$ be $h$-sets, and suppose*

$$\mathcal{E}(X, X) \leq h^{\sigma} |X|_h^3$$

*for some $\sigma > 0$. Then, writing $Z_h = Z + [0, h]$,*

$$\|\mathbf{1}_{X_h} \mathscr{F}_h \mathbf{1}_{Y_h}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq h^{\beta} \quad \text{for } \beta = \frac{3}{8}\left(d - \frac{\log(|X|_h|Y|_h)}{\log h}\right) + \frac{\sigma}{8}.$$

Note that Proposition 4.5 improves upon the trivial estimate (4-6) whenever

$$h^{\sigma/3} < h^d |X|_h^{-1} |Y|_h^{-1} < h^{-\sigma}.$$

It then follows from the discussion in Section 4.3 that Proposition 4.5 gives a FUP near the critical parameter for wide classes of sets, for example, whenever $X$ is covered by Proposition 4.3.

We remark that A. Cohen [2023] proved that, in the regime $|X|_h|Y|_h \geq h^{-d}$, the FUP holds whenever $X$ is porous on lines and $Y$ is $d$-porous between scales $h$ and 1. Porosity on lines in the sense of [Cohen 2023] is slightly stronger than 1-porosity in the sense of Definition 4.2. Proposition 4.3 applies to more general pairs $X$ and $Y$, since $X$ is allowed to contain lines as long as $|X|_h > h^{-1-\lambda}$ for some $\lambda > 0$, and $Y$ is arbitrary. Of course, the price to pay is that the FUP is only established very close to the critical value.

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# The three graces in the Tits–Kantor–Koecher category

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A metaphor of Loday describes Lie, associative, and commutative associative algebras as "the three graces" of the operad theory. We study the three graces in the category of $\mathfrak{sl}_2$-modules that are sums of copies of the trivial and the adjoint representation. That category is not symmetric monoidal, and so one cannot apply the wealth of results available for algebras over operads. Motivated by a recent conjecture of the second author and Mathieu, we embark on the exploration of the extent to which that category "pretends" to be symmetric monoidal. To that end, we examine various homological properties of free associative algebras and free associative commutative algebras, and study the Lie subalgebra generated by the generators of the free associative algebra.

## 1. Introduction

For $\Bbbk$-linear algebras, at least over a field of characteristic zero, it is nowadays sufficiently standard to place them in the context of algebras over an operad in a symmetric monoidal category. This paper is concerned with one situation where this intuition fails in a very notable way, but many features that operadic algebras exhibit are nevertheless present.

Let us be a bit more specific about it. Kashuba and Mathieu [2021] proposed a conjecture which, if true, would lead to important new insight into the free Jordan algebra in several generators. The universe in which the story unfolds is the category $\mathfrak{T}$ whose objects are completely reducible $\mathfrak{sl}_2$-modules that decompose as direct sums of copies of trivial modules and adjoint modules. We can talk about Lie algebras in $\mathfrak{T}$, which are objects of $\mathfrak{T}$ which are Lie algebras whose Lie bracket is $\mathfrak{sl}_2$-equivariant, and even about free Lie algebras in $\mathfrak{T}$. Kashuba and Mathieu [2021] proved that the free Lie algebra in $\mathfrak{T}$ generated by $n$ copies of the adjoint representation can be described via a version of the celebrated Tits–Kantor–Koecher construction [1962; 1964; 1967] due to Allison and Gao [Allison and Gao 1996; Caveny and Smirnov 2014]: it is the Lie algebra obtained by that construction from the free (nonunital) Jordan algebra on $n$ generators.

In this paper, we start a systematic study of the following conjecture. (For $U$ the direct sum of several copies of the adjoint representation, this is the central conjecture of [Kashuba and Mathieu 2021].)

**Main conjecture.** *For an object $U$ of the category $\mathfrak{T}$, let us consider the homology with trivial coefficients of the free Lie algebra in $\mathfrak{T}$ generated by $U$ with its natural $\mathfrak{sl}_2$-action, and let us truncate it by quotienting out all irreducible $\mathfrak{sl}_2$-submodules different from the trivial and the adjoint representation. That truncation is concentrated in degrees* 0 *and* 1.

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In a sense, this suggests that in certain aspects the category $\mathfrak{T}$ "pretends" to be a symmetric monoidal category. Indeed, for any symmetric monoidal category $\mathcal{C}$ containing the symmetric monoidal category of vector spaces over a field $\Bbbk$ of zero characteristic as a full monoidal subcategory, the homology of any free Lie algebra in $\mathcal{C}$ is concentrated in degrees 0 and 1, a result which is essentially equivalent to the Koszul property of the Lie operad [Ginzburg and Kapranov 1994, Theorem 4.2.5]. Thus, if $\mathfrak{T}$ were the quotient of the symmetric monoidal category of $\mathfrak{sl}_2$-modules by a monoidal ideal, the main conjecture would follow automatically.

In this paper, we prove that the obvious analogue of the main conjecture does not hold for free associative commutative algebras in $\mathfrak{T}$ and holds for free associative algebras in $\mathfrak{T}$. We also discuss the main conjecture for free Lie algebras in $\mathfrak{T}$, and prove a somewhat surprising result stating that for the free associative algebra in $\mathfrak{T}$ generated by $n$ copies of the adjoint representation, the Lie subalgebra generated by its generators is the Tits–Kantor–Koecher construction of the free *special* Jordan algebra on $n$ generators.

## 2. Preliminaries and recollections

For simplicity we shall work over a field $\Bbbk$ of zero characteristic, though some of our results are available in greater generality. To simplify some formulas, we shall slightly abuse notation for free algebras: our $\mathrm{Com}(V)$ and $\mathrm{Ass}(V)$ will be the free *unital* commutative associative algebra and the free *unital* associative algebra, which, in classical terms, are, respectively, the symmetric algebra $S(V)$ and the tensor algebra $T(V)$. In all other aspects, our operadic conventions correspond to those of the monograph by Loday and Vallette [2012]. For $n \geq 0$, we shall denote by $L(n)$ the irreducible $\mathfrak{sl}_2$-module of the highest weight $n$. In particular, $L(0)$ is the trivial module whose basis element we shall often denote by $x$, and $L(2)$ is the adjoint module $\mathfrak{sl}_2$ with the usual basis $e, f, h$. We shall normalize the Killing form of $\mathfrak{sl}_2$ so that its nonzero values are $K(e, f) = K(f, e) = \frac{1}{2}$ and $K(h, h) = 1$.

**2.1.** ***The Tits–Kantor–Koecher category.*** As mentioned in the introduction, this paper studies algebras in the category $\mathfrak{T}$ whose objects are completely reducible $\mathfrak{sl}_2$-modules that decompose as direct sums of copies of trivial modules and adjoint modules, and whose morphisms are $\mathfrak{sl}_2$-module morphisms. In other words, each object of $\mathfrak{T}$ is of the form

$$L(0) \otimes A \oplus L(2) \otimes B,$$

where $A$ and $B$ are vector spaces of multiplicities of the corresponding modules. We shall call this category the *Tits–Kantor–Koecher category*.

Let $\mathcal{P}$ be an operad in Vect. Since objects of $\mathfrak{T}$ are vector spaces, we can talk about $\mathcal{P}$-algebras in $\mathfrak{T}$ in the following straightforward way.

**Definition 2.1.** A $\mathcal{P}$-algebra in $\mathfrak{T}$ is an object $U$ of $\mathfrak{T}$ equipped with a map

$$\mathcal{P}(U) \to U$$

which is a $\mathcal{P}$-algebra structure in Vect and, additionally, a morphism of $\mathfrak{sl}_2$-modules for the obvious extension of the $\mathfrak{sl}_2$-module structure on $U$ to $\mathcal{P}(U)$. All $\mathcal{P}$-algebras in $\mathfrak{T}$ form a category, where morphisms are maps that respect both the algebra structure and the $\mathfrak{sl}_2$-module structure.

Moreover, we can talk about free $\mathcal{P}$-algebras in $\mathfrak{T}$.

**Definition 2.2.** The free $\mathcal{P}$-algebra in $\mathfrak{T}$ generated by an object $U$, denoted by $\mathcal{P}^{\mathfrak{T}}(U)$, is the left adjoint of the forgetful functor from the category of $\mathcal{P}$-algebras in $\mathfrak{T}$ to the category $\mathfrak{T}$.

Existence of free algebras is justified by the following obvious result.

**Proposition 2.3.** *The free algebra $\mathcal{P}^{\mathfrak{T}}(U)$ is isomorphic to the quotient of the free algebra $\mathcal{P}(U)$ by the ideal generated by all "wrong" isotypic components for the $\mathfrak{sl}_2$-action (components of the form $L(k)$ for $k \neq 0, 2$).*

*Proof.* Let $A$ be a $\mathcal{P}$-algebra in the category $\mathfrak{T}$. Note that the universal property of the free $\mathcal{P}$-algebras implies that any linear map $f : U \to A$ extends to a unique $\mathcal{P}$-algebra map $f_* : \mathcal{P}(U) \to A$. The $\mathfrak{sl}_2$-module structure on $U$ induces an $\mathfrak{sl}_2$-module structure on $\mathcal{P}(U)$, and if the map $f$ were $\mathfrak{sl}_2$-equivariant, the map $f_*$ is also $\mathfrak{sl}_2$-equivariant. We also note that the map $f_*$ has to annihilate all the "wrong" isotypic components, since its codomain only contains the isotypic components of the highest weights 0 and 2. Thus, if we define $\mathcal{P}^{\mathfrak{T}}(U)$ as the quotient of $\mathcal{P}(U)$ by the ideal generated by all "wrong" isotypic components, and denote by $\phi : \mathcal{P}(U) \to \mathcal{P}^{\mathfrak{T}}(U)$ the canonical projection, the $\mathcal{P}$-algebra $\mathcal{P}^{\mathfrak{T}}(U)$ is an algebra in the category $\mathfrak{T}$, and each map $f_*$ factors through $\ker \phi$, so we may write $f_* = f_*^{\mathfrak{T}} \circ \phi$ for a suitable $\mathcal{P}$-algebra map $f_*^{\mathfrak{T}}$, which proves the universal property of $\mathcal{P}^{\mathfrak{T}}(U)$. $\square$

In the following sections, various examples of free $\mathcal{P}$-algebras in $\mathfrak{T}$ appear for $\mathcal{P}$ being one of the "three graces". It may be interesting to study these algebras for other operads $\mathcal{P}$.

At this point, a crucial warning is an order. Since the category $\mathfrak{T}$ is not symmetric monoidal (it is a quotient of the symmetric monoidal category of all completely reducible $\mathfrak{sl}_2$-modules supported at finitely many weights by a subcategory which is not a monoidal ideal), $\mathcal{P}$-algebras in $\mathfrak{T}$, even though they have an operad behind their definition, are not algebras over an operad! Of course, what they really are is algebras over the free algebra monad $\mathcal{P}^{\mathfrak{T}}$. However, the free $\mathcal{P}$-algebra in $\mathfrak{T}$ generated by an object $U$, while defined in a sufficiently straightforward way, is not obtained from $U$ by any tractable standard formula.

**2.2.** ***Some recollections on Jordan algebras.*** Let us briefly recall some of the results on Jordan algebras and their relationship to Lie algebras in $\mathfrak{T}$; we refer the reader to [Kashuba and Mathieu 2021] for details.

A *Jordan algebra* is a commutative not necessarily associative algebra $J$ satisfying the identity $(x^2y)x = x^2(yx)$ for all $x, y \in J$. It is known that for every Jordan algebra $J$, the associator $(x, y, z) = (xy)z - x(yz)$ is a derivation with respect to its argument $y$. A derivation obtained in this way for some $x, z \in J$ is denoted by $\partial_{x,z}$ and is called an *inner derivation* of $J$. It is known that the vector space $\mathrm{Inner}(J)$ of all inner derivations is a subalgebra of the Lie algebra of all derivations of $J$ [Zhevlakov et al. 1982].

Over a field of characteristic zero, every identity is equivalent to a multilinear identity, and in particular the Jordan identity is equivalent to the multilinear identity

$$((xy)z)t + ((yt)z)x + ((xt)z)y = (xy)(zt) + (xz)(yt) + (xt)(yz).$$

This means that Jordan algebras can be described as algebras over an operad, which we shall denote by Jord. It is well known that there is a morphism of operads Jord $\to$ Ass sending the generator of the Jordan operad to the symmetrization of the associative product; in more classical terms, for every associative algebra, the operation

$$a \circ b = \tfrac{1}{2}(ab + ba) \tag{1}$$

satisfies the Jordan identity. This morphism has a kernel; in other words, Jordan algebras arising as subalgebras of associative algebras in the way indicated above always satisfy some extra identities [Albert and Paige 1959; Glennie 1966]. The image of that morphism is denoted by SJord and called the special Jordan operad. As for any class of algebras over an operad, one can consider the free Jordan algebra generated by a vector space $V$, denoted by Jord$(V)$, and the free special Jordan algebra generated by a vector space $V$, denoted by SJord$(V)$.

**Remark 2.4.** One important property of the Jordan operad that was established and meaningfully used by Kashuba and Mathieu [2021] states that it is *cyclic*, which means that the natural $S_n$-action on its $n$-th component extends to a $S_{n+1}$-action (we refer the reader to the foundational paper [Getzler and Kapranov 1995] for further details on cyclic operads). In fact, in parallel to the long conceptual proof of that result in [Kashuba and Mathieu 2021], there exists a very short argument proving it: inspecting the Jordan identity, one notes that the $S_4$-module generated by it is preserved by the $S_5$-action arising from the standard cyclic operad structure on the free operad with one generator, and hence the Jordan operad is cyclic, being a quotient of the cyclic operad by a cyclically invariant ideal. By a similar argument, the special Jordan operad is cyclic, being a suboperad of a cyclic operad generated by a cyclically invariant subspace of the space of generators.

There is a relationship between Jordan algebras and Lie algebras in $\mathfrak{T}$ going back to the work of Tits [1962], who noted that for each Lie algebra $\mathfrak{g}$ in $\mathfrak{T}$ one can define a Jordan algebra structure on the vector space of weight 2 elements in $\mathfrak{g}$ by the formula $x, y \mapsto \frac{1}{2}[x, f(y)]$, where $f(-)$ refers to the action of $f \in \mathfrak{sl}_2$ on $\mathfrak{g}$. This construction is clearly a functor from the category $\mathfrak{T}$ to the category of Jordan algebras, which we shall call the *Tits functor*. Tits also noticed that there is a construction in the opposite direction, which however is not functorial: to a Jordan algebra $J$, one can associate the object $L(0) \otimes \text{Inner}(J) \oplus L(2) \otimes J$ of $\mathfrak{T}$, equipped with a Lie algebra structure defined as follows: for two elements of $L(0) \otimes \text{Inner}(J) \cong \text{Inner}(J)$, their Lie bracket is the Lie bracket of derivations, for an element of $L(0) \otimes \text{Inner}(J) \cong \text{Inner}(J)$ and an element of $L(2) \otimes J$, their Lie bracket comes from the action of $\text{Inner}(J)$ on $L(2) \otimes J$, and finally for $u_1 \otimes z_1, u_2 \otimes z_2 \in L(2) \otimes J$, one defines

$$[u_1 \otimes z_1, u_2 \otimes z_2] := K(u_1, u_2)\partial_{z_1,z_2} + [u_1, u_2] \otimes (z_1 z_2).$$

That construction was further investigated and generalized by Kantor [1964] and Koecher [1967], and is usually referred to as the *Tits–Kantor–Koecher construction*.

A functorial version of the Tits–Kantor–Koecher construction was found by Allison and Gao [Allison and Gao 1996; Caveny and Smirnov 2014]. It is defined as follows. Given a Jordan algebra $J$, we define the *Tits–Allison–Gao functor* by

$$\mathrm{TAG}(J) := L(0) \otimes \mathcal{B}(J) \oplus L(2) \otimes J,$$

where $\mathcal{B}(J) := \Lambda^2(J)/\Bbbk\{z \wedge z^2 : z \in J\}$. Note that $\partial_{z,z^2}(y) = (z, y, z^2) = 0$ due to the Jordan identity, so the natural map $\Lambda^2(J) \to \mathrm{Inner}(J)$ sending $x \wedge y$ to $\partial_{x,y}$ induces a surjective map $\mathcal{B}(J) \to \mathrm{Inner}(J)$, and $\mathcal{B}(J)$ can be viewed as a natural functorial replacement of the nonfunctorial $\mathrm{Inner}(J)$ in the Tits–Kantor–Koecher construction. The map $\mathcal{B}(J) \to \mathrm{Inner}(J)$ leads to an action of $\mathcal{B}(J)$ on $J$ by derivations, and one can show that if one extends it to $\Lambda^2(J)$, that action induces a Lie algebra structure on $\mathcal{B}(J)$, thus leading to a Lie algebra structure on $\mathrm{TAG}(J)$ analogous to that of the Tits–Kantor–Koecher construction. Kashuba and Mathieu [2021] showed that the Tits functor and the Tits–Allison–Gao functor form a pair of adjoint functors.

**2.3.** ***Gröbner–Shirshov bases.*** In this article, we make extensive use of Gröbner–Shirshov bases in all kinds of algebras that we consider. The purpose of such bases is to consider particularly useful systems of generators of ideals in free algebras, allowing one to have normal forms for elements in the quotient by an ideal. We note that in different types of algebras, one encounters two qualitatively different situations. For commutative associative algebras and associative algebras, free algebras admit bases (of commutative monomials and of words, respectively), for which the product of two basis elements is always another basis element, whereas for Lie algebras and superalgebras, it is not possible to introduce a basis for which the Lie bracket of two basis elements is always a basis element, so that much more intricate considerations are required. To distinguish between such situations, we shall talk about *Gröbner bases* in contexts of the first kind (and call the corresponding basis elements *monomials*) and about *Shirshov bases* in context of the second kind, though, as we see, Shirshov bases of ideals in Lie algebras and superalgebras can be related to Gröbner bases of ideals in the corresponding universal enveloping algebras. In this section, we only give very brief recollections, referring the reader to [Bokut et al. 1999; Bokut and Malcolmson 1999; Bremner and Dotsenko 2016; Ufnarovskij 1995] for details.

Let us begin with the combinatorially more transparent case of commutative associative algebras and associative algebras. A total order of monomials in the free algebra is said to be *admissible* if it is a well-order, and the product is an increasing function of its arguments: replacing one of the monomials in the product by a greater one increases the result. Given an admissible order of the free algebra, one can define a *Gröbner basis* of an ideal $I$ as a subset $G \subset I$ for which the leading monomial of every element of $I$ is divisible by a leading monomial of an element of $G$. The primary reason to look for Gröbner bases is dictated by considerations of linear algebra: a Gröbner basis for an ideal gives extensive information on the quotient modulo $I$. For a set $S$ of polynomials, a monomial is said to be *normal* with respect to $S$ if it is not divisible by any of the leading monomials of elements of $S$. It is easy to show that the normal monomials with respect to any set $S$ of generators of an ideal $I$ always form a spanning set of the quotient modulo $I$. One can prove that $S$ is a Gröbner basis if and only if the cosets of monomials that are normal with respect to $S$ form a basis of the quotient modulo $I$.

In the Lie case, we shall mostly encounter Lie superalgebras, and so we discuss Shirshov bases in that generality. We start with explaining what kind of monomial bases in free algebras we consider. Suppose that $X$ is a set equipped with a well-order. We can consider the free monoid $\langle X\rangle$ generated by $X$, consisting of all words in the alphabet $X$ with the associative product of each two elements given by concatenation. We shall moreover assume that there is a parity function $X \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ allowing us to write $X = X_0 \sqcup X_1$; we extend the parity to $\langle X\rangle$ additively, so that we can talk about even and odd words in the alphabet $X$. A nonempty word $w$ is said to be a *Lyndon–Shirshov word* if it is the strictly largest one (with respect to the graded lexicographic order of $\langle X\rangle$ induced by the order of $X$) among all its cyclic shifts. Furthermore, a nonempty word $w$ is said to be a *super-Lyndon–Shirshov word* if it is a Lyndon–Shirshov word or a square of an odd Lyndon–Shirshov word (a square of a Lyndon–Shirshov word clearly is not a Lyndon–Shirshov word itself).

To deal with normal forms, we invoke universal enveloping algebras. For a Lie superalgebra $\mathfrak{g}$ with generators $X$ and relations $R$, let us interpret elements of $R$ as linear combinations of commutators in the free associative algebra $\Bbbk\langle X\rangle$, so that these elements are defining relations of the universal enveloping algebra $U(\mathfrak{g})$. Then if $R$ is a Gröbner basis of those defining relations of $U(\mathfrak{g})$, the Lie superalgebra $\mathfrak{g}$ has a basis whose leading terms, once we expand everything inside $U(\mathfrak{g})$ into combinations of associative monomials, are super-Lyndon–Shirshov words that are normal with respect to $G$.

For algebras in $\mathfrak{T}$, we shall always choose generators in a way compatible with the $\mathfrak{sl}_2$-weights, so that for $U = L(0)\otimes A\oplus L(2)\otimes B$ we choose a basis $\{x_i : i = 1, \dots, \dim(A)\}\sqcup\{e_j, f_j, h_j : j = 1, \dots, \dim(B)\}$ (we are going to assume our algebras finitely generated; most of our arguments require only a very mild modification for the infinite number of generators).

## 3. Free commutative associative algebras

In this section, we shall describe free commutative associative algebras in the category $\mathfrak{T}$, and show that the main conjecture does not hold for them.

We shall start by presenting free commutative associative algebras in $\mathfrak{T}$ by generators and relations.

**Proposition 3.1.** *Suppose that* $U = L(0) \otimes A \oplus L(2) \otimes B$. *The algebra* $\mathrm{Com}^{\mathfrak{T}}(U)$ *is the quotient of* $S(U)$ *by the relations given by the* $\mathfrak{sl}_2$*-submodule in* $S^2(U)$ *generated by* $S^2(e \otimes B)$, *where* $e$ *is the highest weight vector of* $L(2)$. *In particular, we have an isomorphism of commutative associative algebras*

$$\mathrm{Com}^{\mathfrak{T}}(U) \cong S(A) \otimes \mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B).$$

*Proof.* For a basis $\{x_i : i = 1, \dots, \dim(A)\}\sqcup\{e_j, f_j, h_j : j = 1, \dots, \dim(B)\}$ of $U$, we see that our relations must contain all monomials $e_i e_j$ (for there are no elements of weight 4 in objects of $\mathfrak{T}$), which are precisely the basis elements of the vector space $S^2(e\otimes B)$. Since we work with commutative associative algebras, the quotient by these relations is spanned by monomials that contain at most one generator $e_i$. Consequently, the quotient by the relations given by $\mathfrak{sl}_2$-submodule generated by these elements already has no elements of weight 4 or more, and hence is an object of $\mathfrak{T}$, namely, the free commutative associative algebra.

In particular, we do not impose any relations on the generators $x_i$, so

$$\mathrm{Com}^{\mathfrak{T}}(U) \cong S(A) \otimes \mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B). \qquad \square$$

Proposition 3.1 implies that in order to describe the free commutative associative algebras in $\mathfrak{T}$, it is essentially enough to describe free commutative associative algebras generated by an object of the form $L(2) \otimes B$, that is, by several copies of the adjoint $\mathfrak{sl}_2$-module. We shall use Gröbner bases to study those algebras.

**Proposition 3.2.** *We have*

$$\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B) \cong L(0) \oplus L(2) \otimes B \oplus \big(L(0) \otimes S^2(B) \oplus L(2) \otimes \Lambda^2(B)\big) \oplus L(0) \otimes \Lambda^3(B),$$

*where each product of more than three generators from $L(2) \otimes B$ vanishes, and products of two or three generators are given, respectively, by the map*

$$S^2(L(2) \otimes B) \to L(0) \otimes S^2(B) \oplus L(2) \otimes \Lambda^2(B)$$

*sending $(u_1 \otimes b_1)(u_2 \otimes b_2)$ to $K(u_1, u_2) \otimes (b_1 b_2) + [u_1, u_2] \otimes (b_1 \wedge b_2)$ and by the map*

$$S^3(L(2) \otimes B) \to L(0) \otimes \Lambda^3(B)$$

*sending $(u_1 \otimes b_1)(u_2 \otimes b_2)(u_3 \otimes b_3)$ to $K([u_1, u_2], u_3) \otimes (b_1 \wedge b_2 \wedge b_3)$.*

*Proof.* According to Proposition 3.1, the relations of the algebra $\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B)$ are given by the $\mathfrak{sl}_2$-submodule generated by $S^2(e \otimes B)$. In terms of the generators $\{e_j, f_j, h_j : j = 1, \dots, \dim(B)\}$, this submodule has a basis of elements

$$e_i e_j, \quad f_i f_j, \quad h_i e_j + h_j e_i, \quad h_i f_j + h_j f_i, \quad f_i e_j + e_i f_j - h_i h_j$$

for all $i \leq j \in \{1, \dots, \dim(B)\}$.

Let us consider the order of generators

$$e_{\dim(B)} < \cdots < e_1 < h_{\dim(B)} < \cdots < h_1 < f_{\dim(B)} < \cdots < f_1,$$

and the inverse lexicographical order associated to it; to compare two monomials in our generators with respect to that order, we choose the smallest generator in which they differ, and declare the monomials with the *smaller* exponent to be *larger*. Let us reproduce the above relations with their leading monomials underlined:

$$\begin{aligned} \underline{e_i e_j},\ \underline{f_i f_j}, &\quad i \leq j \in \{1, \dots, \dim(B)\}, \\ h_i e_j + \underline{h_j e_i},\ \underline{h_i f_j} + h_j f_i, &\quad i < j \in \{1, \dots, \dim(B)\}, \\ 2\underline{h_i e_i},\ 2\underline{h_i f_i}, &\quad i \in \{1, \dots, \dim(B)\}, \\ f_i e_j + e_i f_j - \underline{h_i h_j}, &\quad i \leq j \in \{1, \dots, \dim(B)\}. \end{aligned}$$

This means that the quadratic monomials that are normal with respect to the defining relations of our algebra are

$$h_i e_j,\ h_j f_i, \quad i < j \in \{1, \dots, \dim(B)\} \qquad \text{and} \qquad e_i f_j, \quad i, j \in \{1, \dots, \dim(B)\}.$$

Consequently, the monomials of degree three that are normal with respect to the defining relations of our algebra are precisely all the monomials

$$f_i h_j e_k, \quad i < j < k \in \{1, \dots, \dim(B)\},$$

and there are no such normal monomials of degree four or higher. As we mentioned above, the cosets of monomials that are normal with respect to the defining relations of an algebra always span the algebra, and form a basis if and only if the defining relations form a Gröbner basis of the ideal of relations. The spanning property means that we have the following upper bounds on dimensions of nonzero homogeneous components of that algebra:

$$1, \quad \dim(B), \quad 2\binom{\dim(B)}{2} + \dim(B)^2, \quad \binom{\dim(B)}{3}.$$

It remains to note that the product on

$$L(0) \oplus L(2) \otimes B \oplus \big(L(0) \otimes S^2(B) \oplus L(2) \otimes \Lambda^2(B)\big) \oplus L(0) \otimes \Lambda^3(B)$$

defined in the statement of the proposition is commutative and associative; the only nontrivial case is the product of three generators, where the associativity is an immediate consequence of the invariance of the Killing form. Moreover, this commutative associative algebra is generated by $L(2) \otimes B$ and thus admits a surjective map from the free algebra $\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B)$, giving the lower bounds

$$1, \quad \dim(B), \quad \binom{\dim(B)+1}{2} + 3\binom{\dim(B)}{2}, \quad \binom{\dim(B)}{3}$$

on dimensions of homogeneous components of that algebra. Since we have

$$\binom{\dim(B)+1}{2} + 3\binom{\dim(B)}{2} = 2\binom{\dim(B)}{2} + \dim(B)^2,$$

the two algebras are isomorphic (and the defining relations listed above form a Gröbner basis). □

In general, computing homology of commutative algebras presented by generators and relations is not an easy task. However, we shall now see that the algebras $\mathrm{Com}^{\mathfrak{T}}(U)$ possess sufficiently good homological properties.

**Corollary 3.3.** *For every object $U$ of the category $\mathfrak{T}$, the free algebra $\mathrm{Com}^{\mathfrak{T}}(U)$ is a quadratic Koszul algebra.*

*Proof.* First of all, we recall that according to Proposition 3.1, we have

$$\mathrm{Com}^{\mathfrak{T}}(U) \cong S(A) \otimes \mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B),$$

so since the symmetric algebra is well known to be Koszul and since the tensor product of Koszul algebras is Koszul [Polishchuk and Positselski 2005, Chapter 3, Corollary 1.2], it is enough to show that all algebras $\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B)$ are Koszul.

As we saw in the proof of Proposition 3.2, for a certain ordering of monomials, every algebra $\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B)$ admits a quadratic Gröbner basis. It follows from [Polishchuk and Positselski 2005, Chapter 4, Section 8] that these algebras are Koszul. □

It is easy to see that the Koszul dual algebra of a commutative associative algebra $A$ is the universal enveloping algebra of a Lie superalgebra, and, if the algebra $A$ is Koszul, that the latter Lie superalgebra is the cohomology of $A$, and its linear dual coalgebra is the homology of $A$. Thus, to test the main conjecture in the case of free commutative associative algebras, we should compute the underlying $\mathfrak{sl}_2$-module of the corresponding Lie superalgebra, which we shall do using Shirshov bases for Lie superalgebras.

**Theorem 3.4.** *Let $U = L(2)$. We have*

$$H_1(\mathrm{Com}^{\mathfrak{T}}(U))^{\mathfrak{T}} \cong L(2), \quad H_2(\mathrm{Com}^{\mathfrak{T}}(U))^{\mathfrak{T}} = 0, \quad H_3(\mathrm{Com}^{\mathfrak{T}}(U))^{\mathfrak{T}} = 0, \quad H_4(\mathrm{Com}^{\mathfrak{T}}(U))^{\mathfrak{T}} \cong L(2).$$

*Moreover, for every object $W$ of the category $\mathfrak{T}$ that contains at least one copy of the adjoint module, we have $H_4(\mathrm{Com}^{\mathfrak{T}}(W))^{\mathfrak{T}} \neq 0$.*

*Proof.* A direct calculation shows that the relations of the quadratic dual associative algebra of $\mathrm{Com}^{\mathfrak{T}}(U)$ are

$$\begin{aligned}
\{h_p^*, f_q^*\} - \{h_q^*, f_p^*\} &= 0, \quad p < q \in \{1, \dots, \dim(B)\},\\
\{h_p^*, e_q^*\} - \{h_q^*, e_p^*\} &= 0, \quad p < q \in \{1, \dots, \dim(B)\},\\
\{e_p^*, f_q^*\} - \{e_q^*, f_p^*\} &= 0, \quad p < q \in \{1, \dots, \dim(B)\},\\
\{e_p^*, f_q^*\} + \{f_p^*, e_q^*\} + 2\{h_p^*, h_q^*\} &= 0, \quad p \le q \in \{1, \dots, \dim(B)\},
\end{aligned}$$

where the curly brackets denote the anticommutator $\{x, y\} = xy + yx$. For $U = L_2$, this means that our algebra is generated by three elements $e^*$, $h^*$, $f^*$ subject to the only relation

$$\{e^*, f^*\} + \{h^*, h^*\} = 0.$$

If we choose the order of generators for which $e^*$ is the largest one, the element $e^* f^*$ is the leading term of this relation, and, since this monomial does not form any overlaps with itself, our relation is a Gröbner basis. A direct inspection shows that the super-Lyndon–Shirshov words of length at most 4 that are not divisible by $e^* f^*$ are as follows:

- $e^*,\ h^*,\ f^*$ of length 1,
- $(e^*)^2,\ e^*h^*,\ (h^*)^2,\ h^*f^*,\ (f^*)^2$ of length 2,
- $(e^*)^2h^*,\ e^*(h^*)^2,\ e^*h^*f^*,\ (h^*)^2f^*,\ h^*(f^*)^2$ of length 3,
- $(e^*)^3h^*,\ (e^*)^2(h^*)^2,\ (e^*)^2h^*f^*,\ e^*(h^*)^3,\ e^*(h^*)^2f^*,\ e^*h^*f^*h^*,\ e^*h^*(f^*)^2,\ (h^*)^3f^*,\ (h^*)^2(f^*)^2,$ $h^*(f^*)^3$ of length 4.

The words of length 2 are immediate to list. For lengths 3 and 4, it is sufficiently easy, and requires only very basic observations; for instance, if a super-Lyndon–Shirshov word starts with $h^*$, it cannot contain $e^*$, for otherwise it cannot be the largest among its cyclic shifts. Computing the weights of these words, we find

$$H_1(\mathrm{Com}^{\mathfrak{T}}(U)) \cong L(2), \quad H_2(\mathrm{Com}^{\mathfrak{T}}(U)) \cong L(4),$$
$$H_3(\mathrm{Com}^{\mathfrak{T}}(U)) \cong L(4), \quad H_4(\mathrm{Com}^{\mathfrak{T}}(U)) \cong L(6) \oplus L(2),$$

and the first statement follows.

Suppose now that $W$ is an object of $\mathfrak{T}$ that contains at least one copy of $L(2)$. Let $A$ be a subalgebra of $\mathrm{Com}^{\mathfrak{T}}(W)$ generated by that copy; clearly, $A \cong \mathrm{Com}^{\mathfrak{T}}(U)$, and the bar complex of $\mathrm{Com}^{\mathfrak{T}}(W)$ contains the bar complex of $\mathrm{Com}^{\mathfrak{T}}(U)$ as a direct summand, so we have $H_4(\mathrm{Com}^{\mathfrak{T}}(U))^{\mathfrak{T}} \neq 0$, as required. □

**Remark 3.5.** Note that since our algebras are Koszul, we can easily determine homology from its $\mathfrak{sl}_2$-character, which in turn can be determined from the general properties of Koszul algebras. We did the corresponding calculations using [SageMath] and obtained the following $\mathrm{GL}(B)$-module isomorphisms:

$$H_1(\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B))^{\mathfrak{T}} \cong L(2) \otimes B, \quad H_2(\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B))^{\mathfrak{T}} \cong 0,$$
$$H_3(\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B))^{\mathfrak{T}} \cong 0, \quad H_4(\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B))^{\mathfrak{T}} \cong L(2) \otimes S^4(B),$$
$$H_5(\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B))^{\mathfrak{T}} \cong L(2) \otimes (S^5(B) \oplus S^{4,1}(B) \oplus S^{3,2}(B)) \oplus L(0) \otimes S^{4,1}(B).$$

Here, for a partition $\lambda \vdash n$, we denote by $S^\lambda(B)$ the corresponding Schur functor [Macdonald 1995, Chapter 1], that is, the $\mathrm{GL}(B)$-module on the multiplicity of the irreducible $S_n$-module corresponding to $\lambda$ in the tensor product $B^{\otimes n}$.

Furthermore, $H_6(\mathrm{Com}^{\mathfrak{T}}(L(2) \otimes B))^{\mathfrak{T}}$ contains the trivial $\mathfrak{sl}_2$-module with the multiplicity $S^6(B) \oplus S^{5,1}(B) \oplus S^{4,2}(B) \oplus S^{4,2}(B) \oplus S^{3,2,1}(B)$, so the trivial $\mathfrak{sl}_2$-module also appears in the higher homology, as long as our object contains at least one copy of $L(2)$.

## 4. Free associative algebras

In this section, we shall describe free associative algebras in the category $\mathfrak{T}$, and show that the main conjecture holds for them.

We shall start with the following analogue of Proposition 3.1 which lists all relations of the free algebra.

**Proposition 4.1.** *Suppose that $U = L(0) \otimes A \oplus L(2) \otimes B$. The algebra $\mathrm{Ass}^{\mathfrak{T}}(U)$ is the quotient of $\mathrm{Ass}(U)$ by the relations given by the $\mathfrak{sl}_2$-submodule in $U^{\otimes 2}$ generated by $(e \otimes B) \otimes T(L(0) \otimes A) \otimes (e \otimes B)$, where $e$ is the highest weight vector of $L(2)$.*

*Proof.* For a basis $\{x_i : i = 1, \dots, \dim(A)\} \sqcup \{e_j, f_j, h_j : j = 1, \dots, \dim(B)\}$ of the vector space $U = L(0) \otimes A \oplus L(2) \otimes B$, we see that our relations must contain all monomials $e_i x_{k_1} \cdots x_{k_s} e_j$ (for there are no elements of weight 4 in objects of $\mathfrak{T}$), which are precisely the basis elements of the vector space $(e \otimes B) \otimes T(L(0) \otimes A) \otimes (e \otimes B)$.

It is easy to describe the $\mathfrak{sl}_2$-submodule generated by each such element: it consists of the elements

$$e_i \boldsymbol{x} e_j, \quad f_i \boldsymbol{x} f_j, \quad h_i \boldsymbol{x} e_j + e_i \boldsymbol{x} h_j, \quad h_i \boldsymbol{x} f_j + f_i \boldsymbol{x} h_j, \quad f_i \boldsymbol{x} e_j + e_i \boldsymbol{x} f_j - h_i \boldsymbol{x} h_j$$

for $i, j \in \{1, \ldots, \dim(B)\}$, where we denote for brevity $\boldsymbol{x} = x_{k_1} \cdots x_{k_s}$. Let us choose some order of generators for which all generators $e_j$ are greater than all generators $h_j$, which in turn are greater than all generators $f_j$, which are greater than all generators $x_i$, and consider the graded lexicographic order of monomials. Then the leading terms of those relations are

$$e_i \boldsymbol{x} e_j, \quad f_i \boldsymbol{x} f_j, \quad e_i \boldsymbol{x} h_j, \quad h_i \boldsymbol{x} f_j, \quad e_i \boldsymbol{x} f_j$$

for $i, j \in \{1, \ldots, \dim(B)\}$. In principle, we do not know whether our relations form a Gröbner basis, but the monomials that are normal with respect to them form a spanning set in the quotient algebra, and so we can look at them to ensure that the quotient by our relations is an object of $\mathfrak{T}$. Indeed, such a normal monomial can contain at most one element $e_j$ (since no element from $L(2) \otimes B$ can follow $e_j$, even separated by some factors $x_i$), and at most one element $f_j$ (since no element from $L(2) \otimes B$ can precede $f_j$, even separated by some factors $x_i$). Consequently, the quotient by the relations given by the $\mathfrak{sl}_2$-submodule generated by these elements already has no elements of weight 4 or more, and hence is an object of $\mathfrak{T}$, namely, the free associative algebra. $\square$

We shall now use the calculation from the previous proof to give an explicit description of the algebra $\mathrm{Ass}^{\mathfrak{T}}(L(2) \otimes B)$.

**Proposition 4.2.** *We have*

$$\mathrm{Ass}^{\mathfrak{T}}(L(2) \otimes B) = \Bbbk \oplus L(2) \otimes B \oplus \bigoplus_{p \geq 2} \mathrm{Mat}_2(B^{\otimes p}),$$

*where the product is given by simultaneously performing the matrix product and the concatenation of tensors.*

*Proof.* Note that the proof of Proposition 4.1 shows that for any order of generators for which all generators $e_j$ are greater than all generators $h_j$, which in turn are greater than all generators $f_j$, the defining relations of the algebra $\mathrm{Ass}^{\mathfrak{T}}(L(2) \otimes B)$ have the following leading terms:

$$e_i e_j, \quad f_i f_j, \quad e_i h_j, \quad h_i f_j, \quad e_i f_j.$$

Thus, for $p \geq 2$, the monomials of degree $p$ that are normal forms with respect to these relations are of the following types:

$$h_{i_1} h_{i_2} \cdots h_{i_p}, \quad h_{i_1} h_{i_2} \cdots h_{i_{p-1}} e_{i_p}, \quad f_{i_1} h_{i_2} \cdots h_{i_p}, \quad f_{i_1} h_{i_2} \cdots h_{i_{p-1}} e_{i_p}.$$

(There can be no elements after $e_j$ and no elements before $f_j$.) This gives an estimate from the above on dimensions of homogeneous components of the algebra $\mathrm{Ass}^{\mathfrak{T}}(L(2) \otimes B)$: these normal monomials form a spanning set. At the same time, the vector space

$$\Bbbk \oplus L(2) \otimes B \oplus \bigoplus_{p \geq 2} \mathrm{Mat}_2(B^{\otimes p})$$

with the algebra structure described above is an algebra in the category $\mathfrak{T}$ (since we have an isomorphism of $\mathfrak{sl}_2$-modules $\mathrm{Mat}_2 \cong L(0) \oplus L(2)$) and is easily seen to be generated by $L(2) \otimes B$, so we obtain a lower bound that coincides with the upper one, and therefore our algebra is free, and our relations form a Gröbner basis. □

**Corollary 4.3.** *The algebra* $\mathrm{Ass}^{\mathfrak{T}}(L(2) \otimes B)$ *is Koszul.*

*Proof.* This follows from the fact that this algebra has a quadratic Gröbner basis, since this implies the Koszul property [Polishchuk and Positselski 2005, Chapter 4, Theorem 3.1]. □

**Remark 4.4.** In the general case of the free associative algebra generated by $U = L(0) \otimes A \oplus L(2) \otimes B$, that algebra, as we saw in the proof of Proposition 4.1, is not at all the coproduct of two free algebras, and is not Koszul (since it is not even quadratic). It is however possible to write down a description of this algebra analogous to that of Proposition 4.2. That description, which we shall not really use in this article, is

$$\mathrm{Ass}^{\mathfrak{T}}(U) = T(A) \oplus L(2) \otimes (T(A) \otimes B \otimes T(A)) \oplus \bigoplus_{p \geq 2} \mathrm{Mat}_2\big((T(A) \otimes B \otimes T(A))^{\boxtimes p}\big),$$

where $\boxtimes = \otimes_{T(A)}$, and the product is given by simultaneously performing the matrix product and the concatenation of tensors.

We shall now prove that the main conjecture holds for associative algebras in the category $\mathfrak{T}$.

**Theorem 4.5.** *For every object $U$ of $\mathfrak{T}$, the truncated homology $H_\bullet(\mathrm{Ass}^{\mathfrak{T}}(U))^{\mathfrak{T}}$ vanishes in each degree greater than one.*

*Proof.* For an augmented associative algebra $R$, the homology $H_\bullet(R)$ is the homology of the bar construction $\mathsf{B}_{\mathrm{Ass}}(R)$, which computes $\mathrm{Tor}^R_\bullet(\Bbbk, \Bbbk)$. That Tor functor can be computed using any free resolution $(F_\bullet, d) = (X_\bullet \otimes R, d)$ of the augmentation module $\Bbbk$; precisely,

$$\mathrm{Tor}^R_\bullet(\Bbbk, \Bbbk) \cong H_\bullet(F_\bullet \otimes_R \Bbbk, d \otimes_R 1) \cong H_\bullet\big((X_\bullet \otimes R) \otimes_R \Bbbk, d \otimes_R 1\big) \cong H_\bullet(X_\bullet, \bar{d}),$$

where $\bar{d}$ is the part of $d$ that "survives" after tensoring with the augmentation module.

We shall use the so-called Anick resolution [Anick 1986; Ufnarovskij 1995] that exists for an algebra with a given Gröbner basis. Let us briefly recall its key features. Suppose $R = \Bbbk\langle X \mid G\rangle$ is an augmented associative algebra with generators $X$ and relations $R$. We shall assume that $G$ is a reduced Gröbner basis for a certain admissible order of monomials in generators $X$ (this means that the leading terms of $G$ form an antichain in the poset of monomials in $X$ with the order given by divisibility). Out of that datum, one can inductively define the notion of a (right) $k$-chain and a tail of a $k$-chain as follows.

- The 0-chains are generators (elements of $X$); each of them coincides with its tail.
- For $k > 0$, a $k$-chain is a word $c$ in the alphabet $X$ which can be written as the concatenation $c't$, where $c'$ is a $(k-1)$-chain, and $t$ is some word, called the tail of $c$, such that

- if we denote by $t'$ the tail of $c'$, there exists a factorization $t' = m_1 m_2$ such that $m_2 t$ is the leading term of an element of $G$, and there are no other divisors of $t't$ that are leading terms of $G$,
- no proper beginning of $c$ is a $k$-chain.

Anick [1986] established that there exists a resolution of the (right) augmentation $R$-module $\Bbbk$ by the free right $R$-modules of the form

$$\cdots \to \Bbbk C_k \otimes R \xrightarrow{d_k} \Bbbk C_{k-1} \otimes R \to \cdots \to \Bbbk C_1 \otimes R \xrightarrow{d_1} \Bbbk C_0 \otimes R \xrightarrow{d_0} R \to 0.$$

The differentials $d_k$ of that resolution are constructed inductively together with the splittings $i_k : \operatorname{Ker} d_{k-1} \to \Bbbk C_k \otimes R$ satisfying $d_k i_k = \mathrm{Id}_{|\operatorname{Ker} d_{k-1}}$.

In our case, $R = \mathrm{Ass}^{\mathfrak{T}}(U)$ for some object $U$ of $\mathfrak{T}$. Let us first consider the particular case $U = L(2) \otimes B$, in which we established that the algebra has a quadratic Gröbner basis. This means that each $k$-chain is a word of length $k+1$, for the only way to add a tail to a $k$-chain by "linking" it with a quadratic relation is when that tail consists of just one letter. Therefore, for each $c \in C_k \otimes R$, we have $d_k(c) \in C_{k-1} \otimes \bar{R}$, where $\bar{R}$ is the augmentation ideal of $R$, and so $\bar{d}_k(c) = 0$. Thus,

$$H_k(\mathrm{Ass}^{\mathfrak{T}}(L(2) \otimes B)) \cong \Bbbk C_{k-1}.$$

Moreover, the leading terms of the Gröbner basis found in Proposition 4.2 are

$$e_i e_j, \quad f_i f_j, \quad e_i h_j, \quad h_i f_j, \quad e_i f_j,$$

so each $k$-chain is of one the following forms:

$$e_{i_1} e_{i_2} \cdots e_{i_s} f_{j_1} f_{j_2} \cdots f_{j_t}, \quad s, t \geq 0, \ s + t = k + 1,$$
$$e_{i_1} e_{i_2} \cdots e_{i_s} h_p f_{j_1} f_{j_2} \cdots f_{j_t}, \quad s, t \geq 0, \ s + t = k.$$

Every element of the first type has the $\mathfrak{sl}_2$-weight $2s - 2(k+1-s) = 4s - 2 - 2k$, and every element of the second type has the $\mathfrak{sl}_2$-weight $2s - 2(k-s) = 4s - 2k$. It is therefore clear that for the fixed sequence of indices in $\{1, \ldots, n\}^{k+1}$ these weights contain every even number between $2k+2$ and $-2k-2$ exactly once, and we have an $\mathfrak{sl}_2$-module isomorphism

$$\Bbbk C_k \cong L(2k+2)^{n^{k+1}}.$$

Thus, $H_\bullet(\mathrm{Ass}^{\mathfrak{T}}(L(2) \otimes B))^{\mathfrak{T}} = C^{\mathfrak{T}}_{\bullet-1}$ vanishes in degrees greater than 1.

Let us now discuss the general case $R = \mathrm{Ass}^{\mathfrak{T}}(L(0) \otimes A \oplus L(2) \otimes B)$. As established in Proposition 4.1, while the relations of this algebra are not quadratic, they are "quadratic relative to the base algebra" $T(A)$: they are the elements

$$e_i \boldsymbol{x} e_j, \quad f_i \boldsymbol{x} f_j, \quad h_i \boldsymbol{x} e_j + e_i \boldsymbol{x} h_j, \quad h_i \boldsymbol{x} f_j + f_i \boldsymbol{x} h_j, \quad f_i \boldsymbol{x} e_j + e_i \boldsymbol{x} f_j - h_i \boldsymbol{x} h_j,$$

where we denote for brevity $\boldsymbol{x} = x_{k_1} \cdots x_{k_s}$. Let us show that for any order of generators for which all generators $e_j$ are greater than all generators $h_j$, which in turn are greater than all generators $f_j$, which are greater than all generators $x_i$, these relations form a Gröbner basis for the graded lexicographic order of monomials. The fastest way to justify this is to say that every S-polynomial of two elements of this set will be reduced to zero in the exact same way as every S-polynomial of two elements of the set of relations for $\mathrm{Ass}^{\mathfrak{T}}(L(2)\otimes B)$: the fact that now we have some "layers" $x_{k_1}\cdots x_{k_s}$ between the variables from the adjoint representation does not change anything in computing the reductions. In fact, the same argument applies to the Anick resolution: instead of each single chain, we shall have infinitely many chains with "layers" $x_{k_1}\cdots x_{k_s}$ between the variables from the adjoint representation. Computation of the $\mathfrak{sl}_2$-weights does not change, and we find that $H_k(\mathrm{Ass}^{\mathfrak{T}}(L(0)\otimes A\oplus L(2)\otimes B))$ is a multiple of $L(2k)$, completing the proof. □

## 5. Free Lie algebras

In this section, we discuss the original case of the main conjecture: that for free Lie algebras. That conjecture remains out of reach for the time being, but we hope that recording various results and non-results should be helpful in attacking it.

**5.1.** ***Universal enveloping algebras: the Poincaré–Birkhoff–Witt non-theorem.*** If the category $\mathfrak{T}$ were symmetric monoidal, we would be able to compute Lie algebra homology using Tor over the universal enveloping algebra. Let us consider the simplest possible example which exhibits a surprising numerical coincidence that however does not present itself in more complicated situations.

**Example 5.1.** Consider the case of the Tits–Allison–Gao construction applied to $t\Bbbk[t]$, the free Jordan algebra on one generator. Kashuba and Mathieu [2021, Section 4.1] explained that in this case one of the theorems of Garland and Lepowsky [1976] on Lie algebra homology applies, showing that $H_k(\mathrm{Lie}^{\mathfrak{T}}(L(2)))\cong L(2k)$. Similarly, examining the proof of Theorem 4.5, we find $H_k(\mathrm{Ass}^{\mathfrak{T}}(L(2)))\cong L(2k)$. Thus, in this case computing the homology using the universal enveloping algebra "works", in the weakest possible sense of giving the correct answer. However, this phenomenon does not persist: already for the free Jordan algebra on two generators, the situation is much more complicated. To see that, recall that we have seen in the proof of Theorem 4.5 that we have a $\mathfrak{sl}_2$-module isomorphism

$$H_2(\mathrm{Ass}^{\mathfrak{T}}(L(2)\oplus L(2)))\cong L(4)^4$$

and that $L(4)$ appears only in homological degree two. However, a theorem of Shirshov [Zhevlakov et al. 1982] asserts that the free Jordan algebra on two generators is special, that is, coincides with the Jordan subalgebra of $\Bbbk\langle x, y\rangle$ generated by $x$, $y$ under the Jordan product defined by (1); thus, it is not hard to compute the Euler characteristic of the $\mathfrak{sl}_2$-action on the Chevalley–Eilenberg complex of the Lie algebra $\mathrm{Lie}^{\mathfrak{T}}(L(2)\oplus L(2))$ and to see that $L(4)$ appears in the homology with multiplicity at least 10.

In the remainder of this section, we shall explain in what precise sense the universal enveloping algebras in $\mathfrak{T}$ lose some information about their Lie algebras: it turns out that they do not generally contain the original Lie algebra as a subalgebra, and so in particular there is no Poincaré–Birkhoff–Witt type theorems for universal enveloping algebras in $\mathfrak{T}$. A functorial approach to Poincaré–Birkhoff–Witt type theorems was developed by Dotsenko and Tamaroff [2021], who essentially proved that such theorem holds for all

Lie algebras if and only if it holds for all free Lie algebras. Thus, we shall focus on universal enveloping algebras of free Lie algebras in $\mathfrak{T}$, which, of course, are free associative algebras in $\mathfrak{T}$ (the composition of two left adjoint functors is a left adjoint functor). We shall find that the universal enveloping algebra of $\mathrm{Lie}^{\mathfrak{T}}(L(2)\otimes B)$ has, as the Lie subalgebra generated by the generators, the algebra $\mathrm{TAG}(\mathrm{SJord}(B))$, the Tits–Allison–Gao construction of the special Jordan algebra generated by $B$. Kashuba and Mathieu [2021, Corollary 2] proved that $\mathcal{B}(\mathrm{SJord}(B))\cong\mathrm{Inner}(\mathrm{SJord}(B))$, so that the Tits–Allison–Gao functor applied to the free special Jordan algebra gives the same result as the Tits–Kantor–Koecher construction applied to that algebra. As a consequence, the below result gives a new way to think of free special Jordan algebras and their Tits–Kantor–Koecher Lie algebras; while arguably not useful for the main conjecture, it may, for instance, help to reinterpret the celebrated Glennie identity [1966] in Lie-theoretic terms, which we hope to address elsewhere.

**Theorem 5.2.** *The Lie subalgebra $\mathfrak{L}(B)$ of the algebra $\mathrm{Ass}^{\mathfrak{T}}(L(2)\otimes B)$ generated by $L(2)\otimes B$ is isomorphic to* $\mathrm{TAG}(\mathrm{SJord}(B))$, *the Tits–Allison–Gao construction of the special Jordan algebra generated by $B$.*

*Proof.* As mentioned above, we have $\mathcal{B}(\mathrm{SJord}(B))\cong\mathrm{Inner}(\mathrm{SJord}(B))$, so we may work with the Tits–Kantor–Koecher construction.

By [Kashuba and Mathieu 2021, Lemma 5], we have $\mathrm{Inner}(\mathrm{SJord}(B))\cong[\mathrm{SJord}(B),\mathrm{SJord}(B)]$ (where the commutator is taken in the free associative algebra), and thus the Tits–Kantor–Koecher construction applied to the free special Jordan algebra is

$$L(0)\otimes[\mathrm{SJord}(B),\mathrm{SJord}(B)]\oplus L(2)\otimes\mathrm{SJord}(B),$$

with the Lie bracket given by

$$[u_1+a_1\otimes j_1,u_2+a_2\otimes j_2]=\big([u_1,u_2]+K(a_1,a_2)[j_1,j_2]\big)+\big(a_2\otimes u_1(j_2)-a_1\otimes u_2(j_1)+[a_1,a_2]\otimes j_1\circ j_2\big).$$

This means that if we denote by 1 the basis element of $L(0)$ and by $e,h,f$ the basis elements of $L(2)\cong\mathfrak{sl}_2$, the Lie bracket is given by

$$\begin{aligned}
&[1\otimes u_1,1\otimes u_2]=1\otimes[u_1,u_2], && [I\otimes u,e\otimes j]=e\otimes[u,j],\\
&[I\otimes u,h\otimes j]=h\otimes[u,j], && [I\otimes u,f\otimes j]=f\otimes[u,j],\\
&[e\otimes j_1,e\otimes j_2]=0, && [e\otimes j_1,f\otimes j_2]=h\otimes(j_1\circ j_2)+1\otimes[j_1,j_2],\\
&[f\otimes j_1,f\otimes j_2]=0, && [e\otimes j_1,h\otimes j_2]=-2e\otimes(j_1\circ j_2),\\
&[h\otimes j_1,f\otimes j_2]=2f\otimes(j_1\circ j_2), && [h\otimes j_1,h\otimes j_2]=\tfrac{1}{2}\otimes[j_1,j_2].
\end{aligned}$$

On the other hand, we recall from Proposition 4.2 that the algebra

$$\mathrm{Ass}^{\mathfrak{T}}(L(2)\otimes B)$$

is explicitly given by

$$\Bbbk\oplus L(2)\otimes B\oplus\bigoplus_{p\geq 2}\mathrm{Mat}_2(B^{\otimes p}).$$

It will be convenient to write elements of the latter algebra as combinations of the elements of the form

$$E\otimes \boldsymbol{x},\ F\otimes \boldsymbol{x},\ H\otimes \boldsymbol{x},\ I\otimes \boldsymbol{x},$$

where $E$, $F$, $H$ correspond to the standard embedding of $\mathfrak{sl}_2$ into $2\times 2$-matrices, $I$ is the identity $2\times 2$-matrix, $\boldsymbol{x}$ is an word in basis elements of $B$ (for $I\otimes \boldsymbol{x}$, this word should be of length at least 2). Let us compute the Lie brackets of such elements, taking into account the multiplication table for the matrices $E$, $F$, $H$:

$$EF=\tfrac{1}{2}(I+H),\quad FE=\tfrac{1}{2}(I-H), EH=-E,\quad HE=E,\quad FH=F,\quad HF=-F,$$
$$E^2=F^2=0,\quad H^2=I.$$

We obtain the following:

$$\begin{aligned}
[E\otimes \boldsymbol{x}, E\otimes \boldsymbol{x}'] &= [F\otimes \boldsymbol{x}, F\otimes \boldsymbol{x}'] = 0,\\
[E\otimes \boldsymbol{x}, F\otimes \boldsymbol{x}'] &= EF\otimes \boldsymbol{x}\boldsymbol{x}'-FE\otimes \boldsymbol{x}'\boldsymbol{x} = H\otimes(\boldsymbol{x}\circ \boldsymbol{x}')+I\otimes \tfrac{1}{2}[\boldsymbol{x},\boldsymbol{x}'],\\
[E\otimes \boldsymbol{x}, H\otimes \boldsymbol{x}'] &= EH\otimes \boldsymbol{x}\boldsymbol{x}'-HE\otimes \boldsymbol{x}'\boldsymbol{x} = -2E\otimes(\boldsymbol{x}\circ \boldsymbol{x}'),\\
[I\otimes \boldsymbol{x}, E\otimes \boldsymbol{x}'] &= E\otimes[\boldsymbol{x},\boldsymbol{x}'],\\
[F\otimes \boldsymbol{x}, H\otimes \boldsymbol{x}'] &= FH\otimes \boldsymbol{x}\boldsymbol{x}'-HF\otimes \boldsymbol{x}'\boldsymbol{x} = 2F\otimes(\boldsymbol{x}\circ \boldsymbol{x}'),\\
[I\otimes \boldsymbol{x}, F\otimes \boldsymbol{x}'] &= F\otimes[\boldsymbol{x},\boldsymbol{x}'],\\
[H\otimes \boldsymbol{x}, H\otimes \boldsymbol{x}'] &= H^2\otimes \boldsymbol{x}\boldsymbol{x}'-H^2\otimes \boldsymbol{x}'\boldsymbol{x} = I\otimes[\boldsymbol{x},\boldsymbol{x}'],\\
[I\otimes \boldsymbol{x}, H\otimes \boldsymbol{x}'] &= H\otimes[\boldsymbol{x},\boldsymbol{x}'],\\
[I\otimes \boldsymbol{x}, I\otimes \boldsymbol{x}'] &= I\otimes[\boldsymbol{x},\boldsymbol{x}'].
\end{aligned}$$

Examining these formulas, we note that they match precisely the Lie brackets of the Tits–Kantor–Koecher construction above. This now allows us to prove by induction on degree of elements (with respect to $B$) that the identity map of the vector space $L(2)\otimes B$ the Lie algebra map extends to a well-defined Lie algebra map from

$$\mathrm{TAG}(\mathrm{SJord}(B))\cong L(0)\otimes[\mathrm{SJord}(B),\mathrm{SJord}(B)]\oplus L(2)\otimes \mathrm{SJord}(B)$$

to the Lie subalgebra of $\mathrm{Ass}^{\mathfrak{T}}(L(2)\otimes B)$ generated by $L(2)\otimes B$. This statement is trivial for generators ($n=1$), and the coincidence of the above formulas assures the step of induction. □

**5.2.** ***Some other observations.***

**5.2.1.** *Computational evidence.* In [Kashuba and Mathieu 2021], some computational evidence is offered in favor of the main conjecture in the case of free Lie algebras. One computation that goes slightly further was performed while preparing this paper.

**Proposition 5.3.** *The multilinear component of the free Jordan algebra on* 8 *generators* (*in other words*, *the component* Jord(8) *of the Jordan operad*) *has dimension* 19089. *Moreover*, *this agrees with the* *dimension of the degree* 8 *coefficient of the Schur functor of the operad* Jord *predicted by the main conjecture.*

*Proof.* The dimension of Jord(8) is computed using the `albert` software [Jacobs 1994]. The degree 8 coefficient of the Schur functor of Jord predicted by the main conjecture is computed by implementing the recursive formula of the proof of [Kashuba and Mathieu 2021, Lemma 1] in [SageMath]. □

**5.2.2.** *Braided categories of the same size.* As we have repeatedly emphasized, if the category $\mathfrak{T}$ were symmetric monoidal, the main conjecture would have been trivial. However, it is not: braided monoidal categories with two objects 1 and $c$ satisfying $c \otimes c = 1 \oplus c$ can be classified, and neither of them is symmetric. Algebras in those categories that are braided commutative can be classified [Booker and Davydov 2012], but nothing is understood about braided Lie algebras in those categories; this would be an interesting question to investigate, though we doubt that it would help with the main conjecture.

**5.2.3.** *Koszul duality.* If our category were symmetric monoidal, the main conjecture would be equivalent to the Koszulness of the appropriate operad. In such situation, proving that conjecture for Lie algebras and commutative associative algebras would be equivalent tasks. In our setting, the same is not at all clear, and we do not dare to see what one can learn from the fact that the main conjecture does not hold for free commutative associative algebras.

**5.2.4.** *The case of superalgebras.* In [Shang 2025], the natural analogue of the main conjecture of [Kashuba and Mathieu 2021] is explored. While the latter conjecture is stated on the level of Schur functors, and therefore its validity for free algebras is manifestly equivalent to its validity for superalgebras, that work raises an interesting question of studying the superized version of the category $\mathfrak{T}$, where the usual interplay between algebras and superalgebras (see, e.g., [Shestakov 1991; Vaughan-Lee 1998; Zel'manov 1988]) is destroyed due to the lack of a symmetric monoidal structure.

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