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Unimodal measurable pseudo-Anosov maps

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We exhibit a continuously varying family F_λ of homeomorphisms of the sphere S^2, for which each F_λ is a measurable pseudo-Anosov map. Measurable pseudo-Anosov maps are generalizations of Thurston's pseudo-Anosov maps, and also of generalized pseudo-Anosov maps (*Geom. Topol.* **8** (2004), 1127–1188). They have a transverse pair of invariant full measure turbulations, consisting of streamlines which are dense injectively immersed lines: these turbulations are equipped with measures which are expanded and contracted uniformly by the homeomorphism. The turbulations need not have a good product structure anywhere, but have some local structure imposed by the existence of tartans: bundles of unstable and stable streamline segments which intersect regularly, and on whose intersections the product of the measures on the turbulations agrees with the ambient measure.

Each map F_λ is semiconjugate to the inverse limit of the core tent map with slope λ: it is topologically transitive, ergodic with respect to a background Oxtoby–Ulam measure, has dense periodic points, and has topological entropy $h(F_\lambda) = \log\lambda$ (so that no two F_λ are topologically conjugate). For a full measure, dense G_δ set of parameters, F_λ is a measurable pseudo-Anosov map but not a generalized pseudo-Anosov map, and its turbulations are nowhere locally regular.

1. Introduction

Since their introduction by Thurston in the 1970s, pseudo-Anosov maps have played a central role in low-dimensional geometry and topology, as well as in low-dimensional dynamics. They first appeared in Thurston's classification theorem for isotopy classes of surface homeomorphisms, and are also fundamental for the statement and the proof of Thurston's hyperbolization theorem for fibered 3-manifolds. In both of these discussions, there is a finiteness hypothesis: the surfaces involved are of finite topological type (compact surfaces from which finitely many points have been removed, or marked).

In surface dynamics, the way that pseudo-Anosov maps most frequently appear is as follows: given a homeomorphism f of a surface S, find a periodic orbit O and apply the classification theorem to f on $S \setminus O$; if this isotopy class is pseudo-Anosov then one can conclude, amongst other things, that f has infinitely many other periodic orbits, of infinitely many distinct periods, and has positive topological entropy. If we adopt the point of view that the periodic orbit O consists of marked points instead of having been removed (in which case only isotopies relative to O are allowed), we can consider the collection $\mathrm{pA}(S)$ of all pseudo-Anosov homeomorphisms on a given surface S, relative to all possible

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finite invariant sets and up to topological changes of coordinates. This set $\mathrm{pA}(S)$ is countable, since there is at most one pseudo-Anosov map in any marked isotopy class up to change of coordinates.

At this point some questions arise naturally:

- What kind of set is $\mathrm{pA}(S)$?
- How does it sit inside the set $\mathrm{Homeo}(S)$ of all homeomorphisms of S?
- Can we complete it somehow, and what sort of maps would be necessary to do this?

To the best of our knowledge the nature of $\mathrm{pA}(S)$ as a subset of $\mathrm{Homeo}(S)$ remains a puzzle, and none of these questions has a satisfactory answer.

Our approach to this problem is to restrict attention to a subset of $\mathrm{pA}(S^2)$ which naturally lies within an arc in $\mathrm{Homeo}(S^2)$ — a parametrized family $\{F_\lambda\}$ of sphere homeomorphisms — and to understand what sort of homeomorphisms the F_λ are: this is expected to illuminate the general case. We show that the F_λ have enough structure to make them worthy of being called *measurable pseudo-Anosov maps*. These are analogs of pseudo-Anosov maps in which the transverse pair of invariant measured foliations is replaced with a transverse pair of invariant measured *turbulations*: full measure subsets of the surface decomposed into a disjoint union of *streamlines*, which are injectively immersed lines equipped with measures. As in the pseudo-Anosov case, there is a dilatation $\lambda > 1$ such that one of the turbulations is uniformly expanded by a factor λ and the other is uniformly contracted by a factor $1/\lambda$. In contrast to the pseudo-Anosov case, they need not have a good product structure anywhere. Instead, some structure is imposed by the existence of *tartans*: bundles of arcs of stable and unstable streamlines which intersect regularly, and in the context of which the measures on the streamlines are holonomy invariant and have a product agreeing with the ambient measure on the sphere.

By construction the family contains countably many pseudo-Anosov maps. It also contains uncountably many *generalized pseudo-Anosov maps*, as introduced in [19]: these are defined in the same way as pseudo-Anosov maps, except that their invariant foliations are permitted to have a countably infinite number of singularities, provided that they only accumulate on a finite set (Definitions 2.8).

Our main focus here is on defining measurable pseudo-Anosov maps, and showing that they are abundant in the sense that they form a continuously varying family. In [13], we show that the definition has dynamical weight by proving that any measurable pseudo-Anosov map is Devaney chaotic (it has dense periodic points and is topologically transitive); and, under an additional hypothesis, is ergodic. (For the particular family discussed in this paper, these dynamical properties follow more directly from the construction.)

The definition of measurable pseudo-Anosov maps is involved, but has three fundamental properties: it is analogous to Thurston's definition of pseudo-Anosov maps, and generalizes it; it is strong enough to have dynamical weight as explained above; and it is broad enough to contain the maps of the family F_λ (which was originally constructed with different goals in mind). We were also guided by the higher-dimensional holomorphic analogs of [7; 27; 32] — in particular, the term 'turbulation' is borrowed from comments in [27].

The following statement summarizes the main results of the paper (the fourth item depends in part on results of [15]).

Theorem. *There exists a continuously varying family of sphere homeomorphisms $F_\lambda : S^2 \to S^2$ for $\lambda \in (\sqrt{2}, 2]$ such that:*

- *F_λ is a measurable pseudo-Anosov map with dilatation λ.*
- *For a countable discrete set of parameters, F_λ is pseudo-Anosov.*
- *For an uncountable dense set of parameters, F_λ is generalized pseudo-Anosov.*
- *For a dense G_δ, full measure set of parameters, F_λ is not generalized pseudo-Anosov, its invariant turbulations are nowhere locally regular, and the complement of its unstable turbulation contains a dense G_δ set.*
- *Each F_λ is topologically transitive, ergodic with respect to a background Oxtoby–Ulam measure, has dense periodic points, and has topological entropy $h(F_\lambda) = \log \lambda$ (so that no two F_λ are topologically conjugate).*

In the remainder of the introduction we explain these statements in a bit more detail and put them in a broader context.

Connections with surface dynamics. A result of Bonatti and Jeandenans (see the appendix of [8]) relates pseudo-Anosov maps to differentiable dynamics: there is a natural quotient of a Smale (axiom A and strong transversality) surface diffeomorphism without impasses (bigons bounded by a stable segment and an unstable segment) which is pseudo-Anosov. The semiconjugacies between the Smale diffeomorphisms and their pseudo-Anosov quotients are mild in the sense that their fibers carry no entropy. This result has recently been generalized by Mello [28], who shows that, when impasses are allowed, the quotient is a generalized pseudo-Anosov map.

These results indicate that pseudo-Anosov and generalized pseudo-Anosov maps can be viewed as linear models, with constant expansion and contraction, for Smale surface diffeomorphisms. This places pseudo-Anosov and generalized pseudo-Anosov maps in a broader context in the study of surface dynamics. However, the results only apply to a collection of hyperbolic maps which is countable up to changes of coordinates, so that this semiconjugacy from the differentiable world to the pseudo-Anosov world cannot apply to all maps in a parametrized family, such as the Hénon family. The existence of a parametrized family of measurable pseudo-Anosov maps makes them candidates to fill this gap: we conjecture that the 0-entropy quotient [21] of any surface diffeomorphism with a single homoclinic class is a measurable pseudo-Anosov map. Note also that the *tartans* mentioned above, which are the key part of the definition of measurable pseudo-Anosov maps, are reminiscent of *Pesin blocks* from the theory of nonuniformly hyperbolic surface diffeomorphisms.

***A path in* Homeo(S^2).** The family $\{F_\lambda\}$ was originally constructed in [12]. Each homeomorphism F_λ is obtained as a quotient of the natural extension of the core tent map f_λ of slope λ: the quotient maps are mild (all but at most one fiber contains at most 3 points) and are described explicitly. The family intersects $\mathrm{pA}(S^2)$ at a particular collection of pseudo-Anosov maps which are isotopic to Smale's horseshoe map relative to single periodic orbits: those which are *quasi-one-dimensional*, in the sense that their invariant train tracks are intervals, and the train track maps are core tent maps. These were identified in [23]: they are parametrized by an invariant called *height*, which is a rational in $\left(0, \frac{1}{2}\right)$.

In [19] it was shown that to *every* horseshoe periodic orbit satisfying an irreducibility condition there is associated a generalized pseudo-Anosov map of the sphere whose dynamics mimics that of a core tent map. The family $\{F_\lambda\}$ is the closure of this countable collection of *unimodal generalized pseudo-Anosov maps* — thus our present results greatly extend the main application of [20], which was to obtain a limit of a *single* sequence of unimodal generalized pseudo-Anosov maps. One of the main subsidiary results of this paper (Theorem 6.20) is that there are uncountably many other parameters λ for which F_λ is a generalized pseudo-Anosov map.

Connections with one-dimensional dynamics and inverse limits. To explain the theorem statement above in more detail, we first provide some preliminary definitions and notation. These preliminaries are expanded on in Sections 3.1 (tent maps), 3.2 (inverse limits), and 3.3 (invariant measures).

The *core tent map* $f_\lambda : I_\lambda \to I_\lambda$ with slope $\lambda \in \left(\sqrt{2}, 2\right]$ is the restriction of the full tent map $T_\lambda : [0, 1] \to [0, 1]$ defined by $T_\lambda(x) = \min(\lambda x, \lambda(1 - x))$ to the interval $I_\lambda = \left[T_\lambda^2\left(\frac{1}{2}\right), T_\lambda\left(\frac{1}{2}\right)\right]$. It has *turning* or *critical point* $c = \frac{1}{2}$. For the vast majority of the paper we will only be concerned with single tent maps, so, except where absolutely necessary, the dependence of objects on the parameter λ will be suppressed. The natural extension of the tent map, acting on its inverse limit, is denoted by $\hat{f} : \hat{I} \to \hat{I}$. We write $\pi_0 : \hat{I} \to I$ for the projection of the inverse limit onto its base using the 0-th coordinate, and $[x] := \pi_0^{-1}(x)$ for the π_0-fiber of $\hat{I}$ above a point $x \in I$.

As discussed above, it is shown in [12] that there is a map $g : \hat{I} \to S^2$ which semiconjugates $\hat{f} : \hat{I} \to \hat{I}$ to a sphere homeomorphism $F : S^2 \to S^2$ (and, momentarily restoring the dependence on the parameter, the sphere homeomorphisms F_λ vary continuously with λ). The tent map f has a unique ergodic invariant measure μ of maximal entropy, which is absolutely continuous with respect to Lebesgue measure. This measure gives rise to an ergodic $\hat{f}$-invariant measure on the inverse limit, and in turn to a fully supported ergodic F-invariant one (nonatomic and positive on nonempty open sets), also denoted by μ, on S^2. When we speak of full measure on S^2, it is always with respect to this measure (which depends, of course, on the parameter λ).

The behavior of the sphere homeomorphism F depends on two pieces of information about the tent map f: its *height*, and the nature of the orbit of its turning point:

- The height can be defined dynamically either as the rotation number of the outside map, a circle endomorphism associated with f (Section 4.2); or as the prime ends rotation number of the basin of infinity when $\hat{I}$ is embedded in the sphere as an attractor of a sphere homeomorphism which extends $\hat{f}$ [12]. It can be determined algorithmically from the kneading sequence of f when it is rational, and approximated arbitrarily closely when it is irrational [23]. It is a decreasing function of the parameter λ, with an interval of parameters corresponding to each rational height, and a single parameter corresponding to each irrational.

- As far as the orbit of the turning point is concerned, the main distinction is between the postcritically finite (where the orbit of c is periodic or preperiodic) and the postcritically infinite cases.

For a typical parameter λ, the tent map f is postcritically infinite with rational height. In this case, the sphere homeomorphism is measurable pseudo-Anosov, but is neither pseudo-Anosov nor generalized pseudo-Anosov. It is important to understand that, even though the turbulations are of full measure and transverse to each other, their local structure can be very complicated: for example, when the critical orbit is dense, the streamlines which locally give a product structure are never locally of full measure, so that there are no local foliated charts anywhere on the sphere. Informally, the orbit of the critical fiber $[c]$ carries 'kinks' which become dense and preclude the existence of local product structures anywhere on the sphere.

The construction of the measurable pseudo-Anosov map proceeds as follows. The unstable streamlines correspond to the full measure collection of path components in the inverse limit which are dense injectively immersed lines (see [11]). The measure is locally the pull back of Lebesgue measure on the interval under the projection π_0. The stable turbulation is built using the fibers $\pi_0^{-1}(x)$ and measures α_x supported on these fibers, which were constructed in [11] using the density of the tent map absolutely continuous invariant measure of maximal entropy. Each fiber is a Cantor set and, with countably many exceptions, projects to an arc in the sphere carrying the push forward of α_x. The ends of all but countably many of these arcs are connected in pairs by the semiconjugacy g. The countable collection of nonconforming fibers/arcs are omitted from the stable turbulation.

While we obtain a nice global structure on a set of full measure, the inverse limits in this case are known to be exquisitely complicated topological objects. For example, when the parameter λ is such that critical orbit is dense (these parameters are contained in the rational height, postcritically infinite case), theorems of Bruin and of Raines imply that the inverse limit $\hat{I}_\lambda$ is nowhere locally the product of a Cantor set and an interval [17; 30]. Even more striking is that, for a perhaps smaller, but still dense G_δ set of parameters, Barge, Brucks and Diamond [5] (see also [1]) show that the inverse limit has a strong self-similarity: every open subset of $\hat{I}_\lambda$ contains a homeomorphic copy of $\hat{I}_t$ for every $t \in (\sqrt{2}, 2]$.

In the other cases of the height and critical orbit data, the construction follows the same general outline, but there is more regularity and the sphere homeomorphism F_λ is generalized pseudo-Anosov:

- When the height is irrational (which implies that the tent map is postcritically infinite); or when λ is an endpoint of a rational-height interval (which implies that the tent map is postcritically finite), there is a single bad point (accumulation of singularities) ∞, and a single orbit of 1-pronged singularities which is homoclinic to it: all other points are regular points of the invariant foliations.

- When λ is in the interior of a rational-height interval and the tent map is postcritically finite, then, with the exception of a single parameter λ in each rational-height interval, there is a fixed bad point ∞ and a further periodic orbit Q of bad points having the same period as the postcritical periodic orbit. Heteroclinic orbits of connected 1-pronged and 3-pronged singularities emerge from ∞ and converge to Q.

For the exceptional parameter λ in each rational-height interval, the sphere homeomorphism F_λ is pseudo-Anosov.

Open questions. Our results relate to a specific family of measurable pseudo-Anosov maps, but they raise a number of questions of a more general nature. In [19] it was shown that the generalized pseudo-Anosov maps considered here are quasiconformal with respect to a complex structure on the sphere, and that the invariant foliations are trajectories of an integrable quadratic differential. Does any of this structure exist for the more general measurable pseudo-Anosov maps and, if it does, is the quasiconformal distortion of F_λ equal to λ? A related question is whether or not measurable pseudo-Anosov maps are smoothable: topologically conjugate to, say, a C^k-diffeomorphism, for some $k \geq 1$? Another natural question concerns their prevalence amongst surface homeomorphisms. In our setting the measurable pseudo-Anosov maps occur in a continuous one-parameter family containing a countable family of pseudo-Anosov maps. How common is this: for example, given two isotopic pseudo-Anosov maps (necessarily relative to finite invariant sets), is there a continuous generic one-parameter family connecting them which consists solely of measurable pseudo-Anosov maps? More generally, is the collection of measurable pseudo-Anosov maps in some sense the completion of the collection of generalized pseudo-Anosov maps?

Structure of the paper. In Section 2 we define measurable pseudo-Anosov maps and prove some basic properties. Because the invariant turbulations of a measurable pseudo-Anosov map need not have a local product structure anywhere, their associated measures are defined on streamlines rather than on transverse arcs. For consistency we take the same approach with the measures on (generalized) pseudo-Anosov foliations, and these alternative definitions are also given in Section 2. Section 3 is a review of some necessary background material on tent maps, their invariant measures, and their inverse limits. Section 4 contains the main theoretical content of the paper: we discuss the structure of π_0-fibers of inverse limits of tent maps, identify their *extreme* elements, and model the action of the natural extension on these extreme elements by means of a circle map called the *outside* map. In Sections 5 and 6 we apply this first to the postcritically infinite rational-height case and then to the irrational-height case, proving that the associated sphere homeomorphisms are measurable pseudo-Anosov maps and generalized pseudo-Anosov maps, respectively. The paper ends with a brief summary of how the postcritically finite rational case could be treated using the same approach: we omit the details because the fact that the sphere homeomorphisms associated to these tent maps are generalized pseudo-Anosov maps was proved, using quite different techniques, in [12].

2. Generalized and measurable pseudo-Anosov maps

We will show that the sphere homeomorphisms F_λ are either examples of Thurston's pseudo-Anosov maps [33], or one of two successive generalizations of them: the *generalized pseudo-Anosov maps* of [19], which are permitted to have infinitely many pronged singularities, provided that these only accumulate at finitely many points; and *measurable pseudo-Anosov maps*, which have invariant measured *turbulations*, rather than foliations. A measured turbulation (Definitions 2.11) is a partition of a full measure subset of the sphere into *streamlines*: immersed lines equipped with measures, which play the role of the leaves of a foliation.

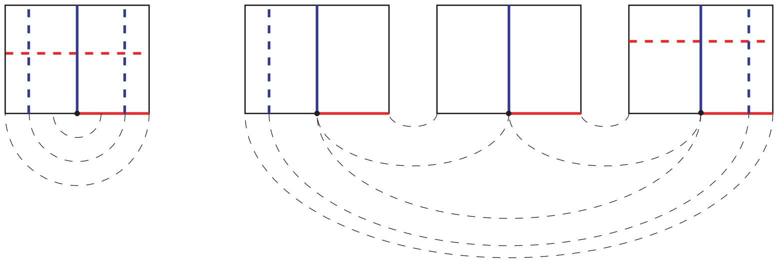

Figure 1. A 1-pronged model (left) and a 3-pronged model (right), each with two unstable (red) and two stable (blue) leaf segments. The dashed leaf segments are arcs, while the solid ones, which pass through the singular point of the model, are 1-ods and 3-ods. The curved dashed lines denote identifications.

Because the invariant turbulations of a measurable pseudo-Anosov map may not have a local product structure in any open set, it is necessary to define the associated measures on the streamlines, rather than on arcs transverse to them. For this reason, and for the sake of consistency, we take the same approach with the measures for pseudo-Anosov and generalized pseudo-Anosov foliations.

We start by defining local models for regular and singular points of pseudo-Anosov and generalized pseudo-Anosov maps. These models come equipped with measures not only on the domain, but also on each stable and unstable leaf segment. Note that the invariant measured foliations will be defined globally, with the local models serving only to ensure that they have the desired local structure: thus no transition conditions are required where the local models overlap.

Because measurable pseudo-Anosov maps cannot be defined using local models for their invariant turbulations, properties such as holonomy invariance of the measures and their compatibility with the ambient measure are expressed instead in terms of *tartans* (Definitions 2.15): unions of stable and unstable arcs which intersect regularly, each stable arc intersecting each unstable arc exactly once.

Remark 2.1. When we say that a subset of a topological space is measurable, it will always mean with respect to the Borel σ-algebra.

2.1. *Pseudo-Anosov and generalized pseudo-Anosov maps.*

Definition 2.2 (regular model). Let I and J be intervals in $\mathbb{R}$. We denote by $R_{I,J}$ the rectangle $I \times J$ in $\mathbb{R}^2$, equipped with two-dimensional Lebesgue measure M, and refer to it as a *regular model*. Each horizontal $I \times \{\beta\}$ in $R_{I,J}$ is called an *unstable leaf segment*, and is equipped with one-dimensional Lebesgue measure m_u; similarly, each vertical $\{\alpha\} \times J$, with one-dimensional Lebesgue measure m_s, is called a *stable leaf segment*.

Definition 2.3 (p-pronged model; see Figure 1). Given $a, b > 0$ and $p \in \mathbb{Z}_+ \setminus \{2\}$, we denote by $Q_{p,a,b}$ the space obtained from p copies $R_0, \dots, R_{p-1}$ of $(-a, a) \times [0, b)$ by identifying $(\alpha, 0) \in R_i$ with $(-\alpha, 0) \in R_{i+1 \bmod p}$ for each $\alpha \in [0, a)$ and $i \in \{0, \dots, p-1\}$, and refer to it as a *p-pronged model*. We equip $Q_{p,a,b}$ with the measure M induced by Lebesgue measure on each R_i. We say that the point obtained by identification of $(0, 0) \in R_i$ for all i is the *singular point* of the model.

An *unstable leaf segment* in $Q_{p,a,b}$ is either a segment $(-a, a) \times \{\beta\}$ in some R_i for $\beta \in (0, b)$; or the union of the segments $[0, a) \times \{0\}$ in each R_i after identifications. A *stable leaf segment* in $Q_{p,a,b}$ is either the union (after identifications) of $\{\alpha\} \times [0, b)$ in R_i and $\{-\alpha\} \times [0, b)$ in $R_{i+1 \bmod p}$ for some $\alpha \in (0, a)$; or the union of $\{0\} \times [0, b)$ in R_i over all i. Unstable and stable leaf segments are equipped with the measures m_u and m_s induced by disintegration of M.

Definition 2.4 (OU). We say that a measure μ on a topological space X is *OU* (for Oxtoby–Ulam) if it is Borel, nonatomic, and positive on open subsets of X. If X is a compact manifold, then it is further required that $\mu(\partial X) = 0$.

For the remainder of this section, we let Σ be a surface, and μ be an OU probability measure on Σ.

Note that the following definition of a measured foliation on Σ makes little sense in isolation (since there is no requirement for measures on distinct leaves to be compatible), and will only take on a proper meaning when constraints on the local structure are introduced in Definitions 2.7. Note also that it is not the standard definition: as explained above, we put measures on the leaves rather than on arcs transverse to them, for consistency with the definition of measurable pseudo-Anosov turbulations.

Definition 2.5 (measured foliation, image foliation). A *measured foliation* $(\mathcal{F}, \nu)$ on Σ is a partition $\mathcal{F}$ of Σ into subsets called *leaves*, together with an OU measure ν_ℓ on each nonsingleton leaf ℓ. If $F : \Sigma \to \Sigma$ is a homeomorphism, we write $F(\mathcal{F}, \nu)$ for the measured foliation $(\mathcal{F}', \nu')$ whose leaves are $\{F(\ell) : \ell \in \mathcal{F}\}$, with measures $\nu'_{F(\ell)} = F_*(\nu_\ell)$ on nonsingleton leaves.

Note that, while μ is a probability measure, there is no requirement for the leaf measures ν_ℓ to be finite, and they will generally not be.

Definitions 2.6 (regular and p-pronged charts). Let $(\mathcal{F}^s, \nu^s)$ and $(\mathcal{F}^u, \nu^u)$ be measured foliations on Σ, and let $x \in \Sigma$. We say that the foliations have a *regular chart* at x if there is a neighborhood N of x, a regular model $R = R_{I,J}$, and a measure-preserving homeomorphism $\Phi : N \to R$, such that for each $\ell \in \mathcal{F}^s$ (respectively $\ell \in \mathcal{F}^u$), and each path component L of $\ell \cap N$, $\Phi|_L$ is a measure-preserving homeomorphism onto a stable (respectively unstable) leaf segment of R. (To be clear, that Φ is measure-preserving means that $\Phi_*(\mu) = M$; and that $\Phi|_L$ is measure-preserving means that $(\Phi|_L)_*(\nu_\ell^{s/u}) = m_{s/u}$.)

We say that the foliations have a *p-pronged chart* at x if there is a neighborhood N of x, a p-pronged model $Q = Q_{p,a,b}$, and a measure-preserving homeomorphism $\Phi : N \to Q$, with $\Phi(x)$ the singular point of Q, such that for each $\ell \in \mathcal{F}^s$ (respectively $\ell \in \mathcal{F}^u$), and each path component L of $\ell \cap N$, $\Phi|_L$ is a measure-preserving homeomorphism onto a stable (respectively unstable) leaf segment of Q.

Definitions 2.7 (pseudo-Anosov and generalized pseudo-Anosov foliations). Let $(\mathcal{F}^s, \nu^s)$ and $(\mathcal{F}^u, \nu^u)$ be measured foliations on Σ. We say that they are *pseudo-Anosov foliations* if they have regular charts at all but finitely many points of Σ, and at each other point they have a pronged chart. We say that they are *generalized pseudo-Anosov foliations* if there is a countable $Y \subset \Sigma$ and a finite $Z \subset \Sigma \setminus Y$ such that the foliations have regular charts at every point of $\Sigma \setminus (Y \cup Z)$; pronged charts at every point of Y; and each point of Z is a singleton leaf of both foliations.

Definitions 2.8 (pseudo-Anosov and generalized pseudo-Anosov maps). Let $F : \Sigma \to \Sigma$ be a homeomorphism. We say that F is (*generalized*) *pseudo-Anosov* if there are (generalized) pseudo-Anosov foliations $(\mathcal{F}^s, \nu^s)$ and $(\mathcal{F}^u, \nu^u)$ on Σ and a number $\lambda > 1$ such that $F(\mathcal{F}^s, \nu^s) = (\mathcal{F}^s, \lambda\nu^s)$ and $F(\mathcal{F}^u, \nu^u) = (\mathcal{F}^u, \lambda^{-1}\nu^u)$. (Note that the λ and the λ^{-1} are the other way round in the standard definitions, in which the measures are on arcs transverse to the foliations rather than on the leaves themselves.)

Remark 2.9. Let ℓ be a leaf of one of the invariant foliations of a pseudo-Anosov or generalized pseudo-Anosov map. Since ℓ is either a singleton, or is a union of countably many leaf segments of regular or p-pronged charts, it follows from Definitions 2.6 that $\mu(\ell) = 0$.

2.2. *Measured turbulations and tartans.*

Definition 2.10 (immersed line). An *immersed line* in Σ is a continuous injective image of $\mathbb{R}$.

Definitions 2.11 (measured turbulations, streamlines, dense streamlines, stream measure, transversality). A *measured turbulation* $(\mathcal{T}, \nu)$ on Σ is a partition of a full μ-measure Borel subset of Σ into immersed lines called *streamlines*, together with an OU measure ν_ℓ on each streamline ℓ, which assigns finite measure to any closed arc in ℓ. We refer to the measures ν_ℓ as *stream measures* to distinguish them from the ambient measure μ on Σ. We say that the measured turbulation *has dense streamlines* if every streamline is dense in Σ.

Two such turbulations are *transverse* if they are topologically transverse on a full μ-measure subset of Σ (to be precise, there is a full measure subset of Σ, every point x of which is contained in streamlines ℓ and ℓ' of the two turbulations, which intersect transversely at x).

Definitions 2.12 (stream arc, measure of stream arc). Let $(\mathcal{T}, \nu)$ be a measured turbulation on Σ. Given distinct points x and y of a streamline ℓ of $\mathcal{T}$, we write $[x, y]_\ell$, or just $[x, y]$ if the streamline is irrelevant or clear from the context, for the (unoriented) closed arc in ℓ with endpoints x and y; and $[x, y)_\ell$ and $(x, y)_\ell$ for the arcs obtained by omitting one or both endpoints of $[x, y]_\ell$. We refer to these as *stream arcs* of the turbulation. The *measure* of a stream arc is its stream measure.

We will impose a regularity condition on turbulations which requires that stream arcs of small measure are small. Note that the following definition is independent of the choice of metric on Σ: since Σ is compact, it is equivalent to the topological condition that for every neighborhood N of the diagonal $\{(x, x) : x \in \Sigma\}$ in $\Sigma \times \Sigma$, there is some $\delta > 0$ such that if $[x, y]$ is a stream arc with $\nu([x, y]) < \delta$, then $(x, y) \in N$. However the metric formulation is usually easier to apply directly.

Definition 2.13 (tame turbulation). Let d be any metric on Σ compatible with its topology. A measured turbulation $(\mathcal{T}, \nu)$ on Σ is *tame* if for every $\epsilon > 0$ there is some $\delta > 0$ such that if $[x, y]$ is a stream arc with $\nu([x, y]) < \delta$, then $d(x, y) < \epsilon$.

Definition 2.14 (image turbulation). If $F : \Sigma \to \Sigma$ is a μ-preserving homeomorphism, we write $F(\mathcal{T}, \nu)$ for the measured turbulation whose streamlines are $\{F(\ell) : \ell \in \mathcal{T}\}$, with measures $\nu_{F(\ell)} = F_*(\nu_\ell)$.

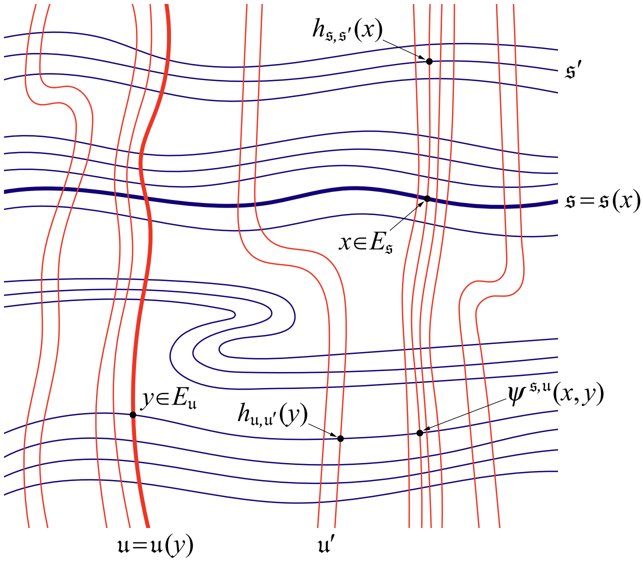

Figure 2. Illustration of Notation 2.16.

Let $(\mathcal{T}^s, \nu^s)$ and $(\mathcal{T}^u, \nu^u)$ be a transverse pair of measured turbulations on Σ. With a view to what is to come, we refer to them as *stable* and *unstable* turbulations, and similarly apply the adjectives stable and unstable to their streamlines, stream measures, stream arcs, etc. — although dynamics will not enter the picture until Section 2.3.

Definitions 2.15 (tartan R, fibers, $R^{\pitchfork}$, positive measure tartan). A *tartan* $R = (R^s, R^u)$ consists of Borel subsets R^s and R^u of Σ, which are disjoint unions of stable and unstable stream arcs, respectively, having the following properties:

(a) Every arc of R^s intersects every arc of R^u exactly once, either transversely or at an endpoint of one or both arcs.

(b) There is a consistent orientation of the arcs of R^s and R^u: that is, the arcs can be oriented so that every arc of R^s (respectively R^u) crosses the arcs of R^u (respectively R^s) in the same order.

(c) The measures of the arcs of R^s and R^u are bounded above.

(d) There is an open topological disk $U \subseteq \Sigma$ which contains $R^s \cup R^u$.

We refer to the arcs of R^s and R^u as the stable and unstable *fibers* of R, to distinguish them from other stream arcs. We write $R^{\pitchfork}$ for the set of *tartan intersection points*: $R^{\pitchfork} = R^s \cap R^u$. We say that R is a *positive measure tartan* if $\mu(R^{\pitchfork}) > 0$.

While R^s and R^u are defined as subsets of Σ, they decompose uniquely as disjoint unions of stable and unstable stream arcs, so where necessary or helpful can be viewed as the collection of their fibers.

Notation 2.16 ($\mathfrak{s}(x)$, $\mathfrak{u}(x)$, $\mathfrak{s} \pitchfork \mathfrak{u}$, $E^{\mathfrak{s}}$, $E^{\mathfrak{u}}$, $\psi^{\mathfrak{s},\mathfrak{u}}$, $\nu_{\mathfrak{s}}$, $\nu_{\mathfrak{u}}$, $\nu_{\mathfrak{s},\mathfrak{u}}$, $h_{\mathfrak{s},\mathfrak{s}'}$, and $h_{\mathfrak{u},\mathfrak{u}'}$). Let R be a tartan; x, $y \in R^{\pitchfork}$; $\mathfrak{s}$ and $\mathfrak{s}'$ be stable fibers of R; and $\mathfrak{u}$ and $\mathfrak{u}'$ be unstable fibers of R. We write (see Figure 2)

- ν_x^s and ν_x^u for the stream measures on the stable and unstable streamlines through x;
- $\mathfrak{s}(x)$ and $\mathfrak{u}(x)$ for the stable and unstable fibers of R containing x;
- $\mathfrak{s} \pitchfork \mathfrak{u} = \mathfrak{u} \pitchfork \mathfrak{s} \in R^{\pitchfork}$ for the unique intersection point of $\mathfrak{s}$ and $\mathfrak{u}$;
- $E^{\mathfrak{s}} = \mathfrak{s} \cap R^u = \mathfrak{s} \cap R^{\pitchfork}$, and $E^{\mathfrak{u}} = \mathfrak{u} \cap R^s = \mathfrak{u} \cap R^{\pitchfork}$;
- $\psi^{\mathfrak{s},\mathfrak{u}} : E^{\mathfrak{s}} \times E^{\mathfrak{u}} \to R^{\pitchfork}$ for the map $(x, y) \mapsto \mathfrak{u}(x) \pitchfork \mathfrak{s}(y)$;
- $\nu_{\mathfrak{s}}$ and $\nu_{\mathfrak{u}}$ for the (restriction of the) stream measures on $E^{\mathfrak{s}}$ and $E^{\mathfrak{u}}$;
- $\nu_{\mathfrak{s},\mathfrak{u}}$ for the product measure $\nu_{\mathfrak{s}} \times \nu_{\mathfrak{u}}$ on $E^{\mathfrak{s}} \times E^{\mathfrak{u}}$;
- $h_{\mathfrak{s},\mathfrak{s}'} : E^{\mathfrak{s}} \to E^{\mathfrak{s}'}$ (respectively $h_{\mathfrak{u},\mathfrak{u}'} : E^{\mathfrak{u}} \to E^{\mathfrak{u}'}$) for the holonomy map $x \mapsto \mathfrak{u}(x) \pitchfork \mathfrak{s}'$ (respectively $y \mapsto \mathfrak{s}(x) \pitchfork \mathfrak{u}'$).

The following is the key condition on tartans, which guarantees that the product measure on the set of tartan intersection points induced by the measures on the fibers agrees with the ambient measure.

Definition 2.17 (compatible tartan). We say that the tartan R is *compatible* (with the ambient measure μ) if, for all stable and unstable fibers $\mathfrak{s}$ and $\mathfrak{u}$, the bijection $\psi^{\mathfrak{s},\mathfrak{u}} : E^{\mathfrak{s}} \times E^{\mathfrak{u}} \to R^{\pitchfork}$ is bimeasurable, and $\psi_*^{\mathfrak{s},\mathfrak{u}} \nu_{\mathfrak{s},\mathfrak{u}} = \mu|_{R^{\pitchfork}}$.

Important consequences of compatibility, proved in [13], include:

(a) The stream measures are the disintegration of the ambient measure onto fibers: if $A \subset R^{\pitchfork}$ is Borel,
$$\mu(A) = \int_{E^{\mathfrak{s}}} \nu_{\mathfrak{u}(x)}(A \cap \mathfrak{u}(x))\, d\nu_{\mathfrak{s}}(x) = \int_{E^{\mathfrak{u}}} \nu_{\mathfrak{s}(y)}(A \cap \mathfrak{s}(y))\, d\nu_{\mathfrak{u}}(y).$$

(b) The stream measures are holonomy invariant in tartans. Let $\mathfrak{s}$ and $\mathfrak{s}'$ be stable fibers of R, and $A \subset E^{\mathfrak{s}}$ be $\nu_{\mathfrak{s}}$-measurable. Then $A' = h_{\mathfrak{s},\mathfrak{s}'}(A)$ is $\nu_{\mathfrak{s}'}$-measurable, and $\nu_{\mathfrak{s}'}(A') = \nu_{\mathfrak{s}}(A)$. The analogous statement holds for holonomies $h_{\mathfrak{u},\mathfrak{u}'}$.

The final condition on the turbulations is that the compatible tartans see a full measure subset of Σ.

Definition 2.18 (full turbulations). We say that the transverse pair $(\mathcal{T}^s, \nu^s)$ and $(\mathcal{T}^u, \nu^u)$ is *full* if

(a) there is a countable collection R_i of (positive measure) compatible tartans with $\mu\left(\bigcup_i R_i^{\pitchfork}\right) = 1$; and

(b) for every nonempty open subset U of Σ, there is a positive measure compatible tartan $R = (R^s, R^u)$ with $R^s \cup R^u \subset U$.

Part (b) of the definition is an additional regularity condition, which prevents the fibers of tartans from behaving too wildly between intersection points. In [13], a variety of conditions are presented, each of which, together with part (a) of the definition, implies part (b). Here we only state the condition which we will use later.

Definition 2.19 (regular tartan). We say that a tartan R is *regular* if for all $x \in R^{\pitchfork}$ and all neighborhoods U of x, there is some $\delta > 0$ such that if $y \in \mathfrak{s}(x) \cap R^{\pitchfork}$ and $z \in \mathfrak{u}(x) \cap R^{\pitchfork}$ with $\nu_{\mathfrak{s}(x)}([x, y]_s) < \delta$ and $\nu_{\mathfrak{u}(x)}([x, z]_u) < \delta$, then the stream arcs $[x, y]_s$, $[y, \mathfrak{s}(z) \pitchfork \mathfrak{u}(y)]_u$, $[\mathfrak{s}(z) \pitchfork \mathfrak{u}(y), z]_s$, and $[z, x]_u$ are all contained in U.

Lemma 2.20. *Suppose that part* (*a*) *of Definition 2.18 is satisfied*, *and in addition each of the tartans R_i is regular. Then part* (*b*) *of the definition is also satisfied.*

Remark 2.21. If a tartan R is regular, then so is $F(R)$.

2.3. *Measurable pseudo-Anosov maps.*

Definitions 2.22 (measurable pseudo-Anosov turbulations, measurable pseudo-Anosov map, dilatation). A pair $(\mathcal{T}^s, \nu^s)$, $(\mathcal{T}^u, \nu^u)$ of measured turbulations on Σ are said to be *measurable pseudo-Anosov turbulations* if they are transverse, tame, full, and have dense streamlines.

A μ-preserving homeomorphism $F : \Sigma \to \Sigma$ is *measurable pseudo-Anosov* if there is a pair $(\mathcal{T}^s, \nu^s)$, $(\mathcal{T}^u, \nu^u)$ of measurable pseudo-Anosov turbulations and a number $\lambda > 1$, called the *dilatation* of F, such that $F(\mathcal{T}^s, \nu^s) = (\mathcal{T}^s, \lambda \nu^s)$ and $F(\mathcal{T}^u, \nu^u) = (\mathcal{T}^u, \lambda^{-1} \nu^u)$.

In the hypotheses of the following lemma, we are not assuming that F is a measurable pseudo-Anosov map: rather, the lemma will be used to prove that the F_λ are measurable pseudo-Anosov maps, by propagating a single positive measure compatible tartan over a full measure subset of Σ using the ergodicity of F_λ.

Lemma 2.23. *Let* $(\mathcal{T}^s, \nu^s)$, $(\mathcal{T}^u, \nu^u)$ *be a transverse pair of measured turbulations on* Σ, *and suppose that* $F : \Sigma \to \Sigma$ *is μ-preserving and satisfies* $F(\mathcal{T}^s, \nu^s) = (\mathcal{T}^s, \lambda \nu^s)$ *and* $F(\mathcal{T}^u, \nu^u) = (\mathcal{T}^u, \lambda^{-1} \nu^u)$ *for some* $\lambda > 1$.

If $R = (R^s, R^u)$ *is a compatible tartan*, *then so is* $F(R)$ (*the tartan whose fibers are the F-images of the fibers of* R).

Proof. It is immediate from Definitions 2.15 that $F(R)$ is a tartan, and that $F(R)^\pitchfork = F(R^\pitchfork)$.

For compatibility, let $\mathfrak{s}$ and $\mathfrak{u}$ be any stable and unstable fibers of R, and write $\mathfrak{s}'$ and $\mathfrak{u}'$ for the image fibers of $F(R)$. Then there is a commutative diagram of bijections

$$\begin{tikzcd} E^{\mathfrak{s}} \times E^{\mathfrak{u}} \arrow[r, "\psi^{\mathfrak{s},\mathfrak{u}}"] \arrow[d, "F \times F"] & R^\pitchfork \arrow[d, "F"] \\ E^{\mathfrak{s}'} \times E^{\mathfrak{u}'} \arrow[r, "\psi^{\mathfrak{s}',\mathfrak{u}'}"] & F(R)^\pitchfork \end{tikzcd}$$

The bimeasurability of $\psi^{\mathfrak{s}',\mathfrak{u}'}$ follows from that of $\psi^{\mathfrak{s},\mathfrak{u}}$ since F and $F \times F$ are homeomorphisms. Since $F_*(\nu_{\mathfrak{s}}) = \lambda \nu_{\mathfrak{s}'}$ and $F_*(\nu_{\mathfrak{u}}) = \lambda^{-1} \nu_{\mathfrak{u}'}$, we have $(F \times F)_* \nu_{\mathfrak{s},\mathfrak{u}} = \nu_{\mathfrak{s}',\mathfrak{u}'}$, and hence $F_*(\psi_*^{\mathfrak{s},\mathfrak{u}} \nu_{\mathfrak{s},\mathfrak{u}}) = \psi_*^{\mathfrak{s}',\mathfrak{u}'} \nu_{\mathfrak{s}',\mathfrak{u}'}$. However $\psi_*^{\mathfrak{s},\mathfrak{u}} \nu_{\mathfrak{s},\mathfrak{u}} = \mu|_{R^\pitchfork}$ by compatibility of R, so that $\psi_*^{\mathfrak{s}',\mathfrak{u}'} \nu_{\mathfrak{s}',\mathfrak{u}'} = \mu|_{F(R)^\pitchfork}$ by F-invariance of μ. $\square$

It is immediate from the definitions that any pseudo-Anosov map is a generalized pseudo-Anosov map. We now show that any generalized pseudo-Anosov map $F : \Sigma \to \Sigma$ with dense nonsingular leaves is a measurable pseudo-Anosov map.

Lemma 2.24. *Let $F : \Sigma \to \Sigma$ be generalized pseudo-Anosov*, *and suppose that the nonsingular leaves of its invariant foliations are dense in Σ. Then F is also measurable pseudo-Anosov*, *with the streamlines of its* (*un*)*stable invariant turbulation given by the nonsingular leaves ℓ of the* (*un*)*stable foliation $\mathcal{F}^{s/u}$*, *carrying the measures $\nu_\ell^{s/u}$.*

Proof. The stable and unstable streamlines are obtained from the stable and unstable foliations by omitting the finitely many singleton leaves at points of Z, and the countably many leaves emanating from points of Y. These omitted leaves have zero measure by Remark 2.9, so that $(\mathcal{T}^s, \nu^s)$ and $(\mathcal{T}^u, \nu^u)$ cover full measure in Σ and hence are turbulations. Clearly F preserves μ and satisfies $F(\mathcal{T}^s, \nu^s) = (\mathcal{T}^s, \lambda\nu^s)$ and $F(\mathcal{T}^u, \nu^u) = (\mathcal{T}^u, \lambda^{-1}\nu^u)$. It remains therefore to show that the turbulations are measurable pseudo-Anosov turbulations (Definitions 2.22). The streamlines are dense by assumption, so it remains to show that the turbulations are transverse; that they are tame; and that they are full.

By Definitions 2.6, there are regular chart domains D_x about each point $x \in \Sigma \setminus (Y \cup Z)$ together with ambient and stream measure preserving homeomorphisms $\Phi_x : D_x \to M_x$ to regular models. In particular, the stable and unstable turbulations are transverse. For fullness, the stable and unstable leaf segments of D_x which are not contained in singular leaves form a tartan $R_x = (R_x^s, R_x^u)$, whose compatibility with μ is immediate using Φ_x. Moreover we have $\mu(R_x^{\pitchfork}) = \mu(D_x) > 0$ by Remark 2.9. Since $\Sigma \setminus (Y \cup Z)$ is separable, it is covered by countably many chart domains $D_i = D_{x_i}$: the corresponding tartans R_i provide a countable collection of compatible tartans with $\mu\big(\bigcup_i R_i^{\pitchfork}\big) = 1$. The second condition in the definition of fullness is immediate, since every nonempty open subset U of Σ contains a regular point x, and the chart domain D_x can be restricted to a subset of U.

It remains to show that the turbulations are tame. Let d be a metric on Σ compatible with its topology, and suppose for a contradiction that there is some $\epsilon > 0$ and a sequence $([x_i, y_i])$ of stream arcs — of either turbulation — with measures converging to zero, such that $d(x_i, y_i) \geq \epsilon$ for all i. By taking a subsequence, we can assume that $x_i \to x^* \in \Sigma$ and $y_i \to y^* \in \Sigma$.

Consider first the case $x^* \notin Z$, so that there is a chart domain D about x^* and an ambient and stream-measure preserving homeomorphism $\Phi : D \to R$ or $\Phi : D \to Q$ to a regular or pronged model. Let $\eta > 0$ be such that the leaf through x^* has segments of measure η on each side of x^* contained in D (or on each prong emanating from x^* if $x^* \in Y$). For i sufficiently large, we have that $x_i \in D$, and the streamline through x_i has measure at least $\eta/2$ on each side of x_i contained in D. Since the measure of $[x_i, y_i]$ converges to zero, we have $[x_i, y_i] \subset D$ for sufficiently large i. Then $\Phi([x_i, y_i])$ is a horizontal or vertical segment with Lebesgue measure converging to zero, so that $\Phi(y_i) \to \Phi(x^*)$, and hence $y_i \to x^*$, contradicting $d(x_i, y_i) \geq \epsilon$.

Therefore we must have $x^* \in Z$ and, analogously, $y^* \in Z$. We obtain the required contradiction by showing that there is a positive lower bound on the measure of any stream arc which connects sufficiently small neighborhoods of two distinct points of Z.

Pick $\xi > 0$ such that distinct elements z_1, z_2 of Z have $d(z_1, z_2) > 5\xi$. Let $U = \bigcup_{z \in Z} B(z, \xi)$, a union of $|Z|$ disjoint open disks. Cover $\Sigma \setminus U$ with a finite collection (D_i) of (regular or pronged) chart domains of diameter less than ξ. Any stream arc with endpoints in different components of U must intersect some D_i which is contained entirely in $\Sigma \setminus U$, and therefore must contain an entire leaf segment (or perhaps two prongs of an n-od leaf segment in the pronged case). However, in each chart domain there is a minimum measure of such leaf segments (coming from the models of Definitions 2.2 and 2.3). The required lower bound on the stream measure of any stream arc which connects two different components of U follows. $\square$

3. Background

In this section we review the material which forms the background of this work, and state some key results from old and recent papers. Section 3.1 concerns tent maps and their study using symbolic dynamics. In Section 3.2 we discuss inverse limits of tent maps. Finally, in Section 3.3, we summarize key properties of the absolutely continuous invariant measures of tent maps, and state some results from [11] which will play a key role in this paper.

3.1. *Tent maps.*

Definition 3.1 (tent map; see Figure 3). Let $\lambda \in \left(\sqrt{2}, 2\right)$. The (*core*) *tent map* $f_\lambda : [a_\lambda, b_\lambda] \to [a_\lambda, b_\lambda]$ *of slope* λ is the restriction of the map $T_\lambda : [0, 1] \to [0, 1]$ defined by $T_\lambda(x) = \min(\lambda x, \lambda(1 - x))$ to the interval $I_\lambda := [a_\lambda, b_\lambda] := \left[T_\lambda^2\left(\frac{1}{2}\right), T_\lambda\left(\frac{1}{2}\right)\right]$. We write $c = \frac{1}{2}$, the turning point of f_λ.

Throughout the paper we work with a single tent map of fixed slope and drop the subscripts λ, writing simply $f : I \to I$, where $I = [a, b]$.

Notation 3.2 (PC). The postcritical set is denoted by $\mathrm{PC} := \{f^r(c) : r \geq 1\}$.

Notice that $f(c) = b$ and $f(b) = a$, so that $a, b \in \mathrm{PC}$.

Notation 3.3 ($\hat{x}$; see Figure 3). Let $\hat{a} \in [c, b]$ satisfy $f(\hat{a}) = f(a)$. For each $x \in [a, \hat{a}]$ with $x \neq c$, we write $\hat{x}$ for the unique element of I with $\hat{x} \neq x$ and $f(\hat{x}) = f(x)$.

The fundamental idea of the symbolic approach to the dynamics of tent maps is to code orbits of f with sequences of 0s and 1s, according to whether successive points along the orbit lie to the left or to the right of the turning point. This leaves open the question of which symbol to use when the orbit passes through the turning point. We take a hybrid approach, leaving the choice open when the turning point is not periodic, and making a choice based on the parity of the orbit of the turning point when it is periodic.

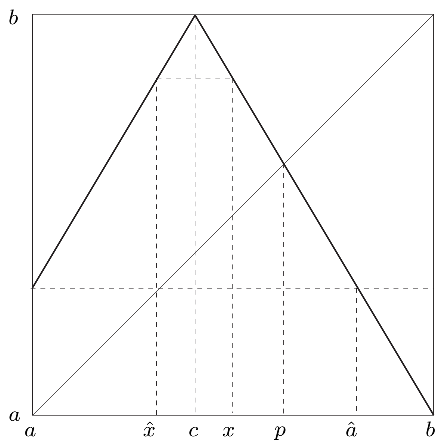

Figure 3. A tent map $f : I \to I$.

This approach has the advantage of giving clean necessary and sufficient admissibility conditions for a sequence to be an itinerary [10].

Definition 3.4 (itinerary $j(x)$, kneading sequence $\kappa(f)$). If c is a periodic point of f, of period n, write $\varepsilon(f) = 0$ (respectively $\varepsilon(f) = 1$) if an even (respectively odd) number of the points $\{f^r(c) : 1 \le r < n\}$ lie in $(c, b]$. The *itinerary* $j(x) \in \{0, 1\}^{\mathbb{N}}$ of a point $x \in I$ is then defined by

$$j(x)_r = \begin{cases} 0 & \text{if } f^r(x) \in [a, c), \\ 1 & \text{if } f^r(x) \in (c, b], \\ \varepsilon(f) & \text{if } f^r(x) = c \end{cases} \qquad \text{for each } r \in \mathbb{N}$$

(we adopt the convention that $0 \in \mathbb{N}$).

If c is not a periodic point of f, we say that a sequence $s \in \{0, 1\}^{\mathbb{N}}$ is *an itinerary of* $x \in I$ if $f^r(x) \in [a, c]$ whenever $s_r = 0$, and $f^r(x) \in [c, b]$ whenever $s_r = 1$. Therefore each $x \in I$ has exactly two itineraries if $c \in \operatorname{orb}(x, f) := \{f^r(x) : r \in \mathbb{N}\}$, and a unique itinerary otherwise.

With an abuse of notation, we write $j(x) = s$ to mean that s is an itinerary of x. With this abusive notation we have that $j(x) = s \implies j(f(x)) = \sigma(s)$, where $\sigma : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ is the shift map.

Since f is uniformly expanding on each of its branches, any sequence $s \in \{0, 1\}^{\mathbb{N}}$ is the itinerary of at most one point $x \in I$.

The *kneading sequence* $\kappa(f) \in \{0, 1\}^{\mathbb{N}}$ of f is defined to be the itinerary of b (which is well defined, since if c is not a periodic point then $c \notin \operatorname{orb}(b, f)$).

Definition 3.5 (unimodal order). The *unimodal order* (also known as the *parity-lexicographical order*) is a total order $\preceq$ on $\{0, 1\}^{\mathbb{N}}$, defined as follows: if s and t are distinct elements of $\{0, 1\}^{\mathbb{N}}$, let $r \in \mathbb{N}$ be least such that $s_r \ne t_r$. Then $s \preceq t$ if and only if $\sum_{i=0}^{r} s_i$ is even.

Remark 3.6. The unimodal order is defined precisely to reflect the usual order of points on the interval: if s and t are itineraries of x and y, then $x < y \implies s \prec t$; and, conversely, if $s \prec t$ then either $x < y$, or s and t are the two itineraries of $x = y$ in the case where c is not periodic.

Remark 3.7. As in Definition 3.1, we only consider tent maps f whose slope λ satisfies $\sqrt{2} < \lambda < 2$, that is, whose topological entropy $h(f) = \log \lambda$ satisfies $\frac{\log 2}{2} < h(f) < \log 2$. The condition $\lambda > \sqrt{2}$ is equivalent to each of the following:

- f is not renormalizable.
- $\kappa(f) \succ 10(11)^\infty$.
- $\kappa(f) = 10(11)^k 0 \ldots$ for some $k \ge 0$.

It is also equivalent (see Theorem 4.7) to the statement that $q(f) < \frac{1}{2}$, where $q(f)$ is the *height* of f as given by Definition 4.6.

The condition $\lambda < 2$ is to avoid cluttering statements with exceptions for the special case $\lambda = 2$ (in which the natural extension of f is semiconjugate to a generalized pseudo-Anosov map called the *tight horseshoe map*; see, for example, [19]).

The following lemma is dependent on the assumption that $\lambda > \sqrt{2}$.

Lemma 3.8. *$f(a) < p < \hat{a}$, where p is the fixed point of f (see Figure 3).*

Proof. By Remark 3.7 we have $j(a) = 0(11)^k0\dots$ for some $k \ge 0$, so that $j(\hat{a}) = 1(11)^k0\dots$ and $j(f(a)) = (11)^k0\dots$. Comparing these itineraries in the unimodal order with $j(p) = 1^\infty$ and using Remark 3.6 gives the result. ☐

3.2. *Inverse limits.* Inverse limits of tent maps have been intensively studied, both for their intrinsic interest as complicated topological spaces, and in dynamical systems as models for attractors [2; 4; 5; 6; 9; 14; 18; 30].

Definitions 3.9 (inverse limit $\hat{I}$, projections π_r, natural extension $\hat{f}$). The *inverse limit* $\hat{I} = \hat{I}_\lambda$ of the tent map $f : I \to I$ is the space

$$\hat{I} = \{\boldsymbol{x} \in I^{\mathbb{N}} : f(x_{r+1}) = x_r \text{ for all } r \in \mathbb{N}\} \subset I^{\mathbb{N}},$$

endowed with the metric $d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{r=0}^{\infty} |x_r - y_r|/2^r$, which induces its natural topology as a subset of the product space $I^{\mathbb{N}}$.

We denote elements of $\hat{I}$ with angle brackets, $\boldsymbol{x} = \langle x_0, x_1, x_2, \dots \rangle$.

For each $r \in \mathbb{N}$, we denote by $\pi_r : \hat{I} \to I$ the projection onto the r-th coordinate, $\pi_r(\boldsymbol{x}) = x_r$.

The *natural extension* of f is the homeomorphism $\hat{f} : \hat{I} \to \hat{I}$ defined by

$$\hat{f}(\langle x_0, x_1, x_2, \dots \rangle) = \langle f(x_0), x_0, x_1, x_2, \dots \rangle.$$

Clearly each π_r is a semiconjugacy from $\hat{f} : \hat{I} \to \hat{I}$ to $f : I \to I$. It is straightforward to see that any semiconjugacy from a homeomorphism $F : X \to X$ to f factors through each π_r.

Since $\pi_r = \pi_0 \circ \hat{f}^{-r}$, we have that, for all $\boldsymbol{x} \in \hat{I}$,

$$\boldsymbol{x} = \big\langle \pi_0(\boldsymbol{x}), \pi_0(\hat{f}^{-1}(\boldsymbol{x})), \pi_0(\hat{f}^{-2}(\boldsymbol{x})), \dots \big\rangle. \tag{1}$$

The next result is an important consequence of the condition that $\lambda \in \big(\sqrt{2}, 2\big)$.

Lemma 3.10. *For every $\boldsymbol{x} \in \hat{I}$, there are infinitely many $r \in \mathbb{N}$ with $x_r \in [f(a), b)$: that is, infinitely many $r \in \mathbb{N}$ such that x_r has two preimages.*

Proof. Let $p > c$ be the fixed point of f. If $x_r \le p$ then x_{r+1}, being a preimage of x_r, satisfies either $x_{r+1} < c$ or $x_{r+1} \ge p$ (see Figure 3). Since $\lambda < 2$, there is an upper bound on the number of consecutive entries of $\boldsymbol{x}$ which are smaller than c: hence $x_r \ge p$ for infinitely many $r \in \mathbb{N}$.

Since $p > f(a)$ by Lemma 3.8, there are infinitely many r with $x_r \in (f(a), b]$. However if $x_r = b$ for some $r \ge 2$ then $x_{r-2} = f(a)$, and the result follows. ☐

Definitions 3.11 (π_0-fibers $[x]$, cylinder sets $[x, y_1, \dots, y_r]$). For each $x \in I$ we define the *π_0-fiber* $[x]$ *of $\hat{I}$ above x* by $[x] = \pi_0^{-1}(x)$.

More generally, if $f(y_1) = x$ and $f(y_i) = y_{i-1}$ for $2 \le i \le r$, then we define the *cylinder set* $[x, y_1, \dots, y_r] \subset [x]$ by

$$[x, y_1, \dots, y_r] = \{\boldsymbol{x} \in [x] : x_i = y_i \text{ for } 1 \le i \le r\}.$$

Note that the diameter of $[x, y_1, \dots, y_r]$ is bounded above by $1/2^r$, so that the cylinder subsets of $[x]$ form a basis for its topology.

Definition 3.12 (arc). An (*open*, *half-open*, *or closed*) *arc* in $\hat{I}$ is a subset of $\hat{I}$ which is a continuous injective image of an (open, half-open, or closed) interval.

An open arc in $\hat{I}$ is therefore synonymous with an immersed line in $\hat{I}$ (Definition 2.10).

We will need the following well-known facts about the topology of $\hat{I}$ (the former is Theorem 2.2 of [30], the latter is folklore).

Lemma 3.13. (a) *If $\boldsymbol{x} \in \hat{I}$ is not in the ω-limit set $\omega([c], \hat{f})$ of the critical fiber, then it has a neighborhood homeomorphic to the product of an open interval and a Cantor set.*

(b) *Every path component of $\hat{I}$ is either a point or an arc.*

It is worth noting how badly Lemma 3.13(a) can fail for points in the ω-limit set of the critical fiber. Barge, Brucks, and Diamond [5] show that there is a dense G_δ set of parameters λ for which every open subset of $\hat{I}_\lambda$ contains a homeomorphic copy of $\hat{I}_{\lambda'}$ for every $\lambda' \in \left(\sqrt{2}, 2\right)$.

Nevertheless, the fibers of tent maps inverse limits are regular (in the case $\lambda > \sqrt{2}$).

Lemma 3.14. *$[x]$ is a Cantor set for all $x \in I$.*

Proof. $[x]$ is certainly compact and totally disconnected. To see that it is perfect, let $\boldsymbol{x} \in [x]$ and $R > 0$. By Lemma 3.10 there is some $r > R$ for which x_r has two preimages x_{r+1} and $\hat{x}_{r+1}$. Let $\boldsymbol{x}' \in [x]$ have $x'_i = x_i$ for $i \le r$ and $x'_{r+1} = \hat{x}_{r+1}$. Then $0 < d(\boldsymbol{x}, \boldsymbol{x}') < 1/2^R$. $\square$

We can introduce symbolic dynamics on $\hat{I}$ just as we did for forwards orbits of tent maps.

Definition 3.15 (itineraries on $\hat{I}$). Let $\boldsymbol{x} \in \hat{I}$. An element s of $\{0, 1\}^{\mathbb{N}}$ is said to be an *itinerary* of $\boldsymbol{x}$ if $s_r = 0 \implies x_{r+1} \le c$, and $s_r = 1 \implies x_{r+1} \ge c$. In an abuse of notation, we write $J(\boldsymbol{x}) = s$ to mean that s is an itinerary of $\boldsymbol{x}$.

Remark 3.16. (a) An element $\boldsymbol{x}$ of $[x]$ is determined by its itinerary $J(\boldsymbol{x}) = s$, since the x_r are determined inductively by $x_0 = x$; and x_{r+1} is the unique preimage of x_r which lies in $[a, c]$ if $s_r = 0$, or in $[c, b]$ if $s_r = 1$.

(b) If $x \notin \mathrm{PC}$, then every element of $[x]$ has a unique itinerary, and if c is not a periodic point of f then every element of $\hat{I}$ has at most two itineraries. More generally, for any $\boldsymbol{x} \in \hat{I}$, the set of itineraries of $\boldsymbol{x}$ is closed in $\{0, 1\}^{\mathbb{N}}$.

By Remark 3.16(b), if $x \notin \mathrm{PC}$ then the fiber $[x]$ can be totally ordered by the unimodal order on the itineraries of its elements. In particular, this makes it possible to define consecutive elements of the fiber.

Definition 3.17 (consecutive elements of $[x]$). Let $x \notin \mathrm{PC}$, so that every $\boldsymbol{x} \in [x]$ has a unique itinerary $J(\boldsymbol{x})$. We say that distinct points $\boldsymbol{x}, \boldsymbol{x}' \in [x]$ are *consecutive* if there is no point of $[x]$ whose itinerary lies strictly between $J(\boldsymbol{x})$ and $J(\boldsymbol{x}')$ in the unimodal order.

More generally, we can consider all of the itineraries which are realized in a fiber.

Definition 3.18 ($\mathcal{K}_x$, $L_x : \mathcal{K}_x \to [x]$). For each $x \in I$, write

$$\mathcal{K}_x = \{s \in \{0,1\}^{\mathbb{N}} : s \text{ is an itinerary of some } \boldsymbol{x} \in [x]\}.$$

By Remark 3.16(a), there is a function $L_x : \mathcal{K}_x \to [x]$ defined by $L_x(J(\boldsymbol{x})) = \boldsymbol{x}$ for all $\boldsymbol{x} \in [x]$.

Lemma 3.19. *For each $x \in I$, $\mathcal{K}_x$ is compact and $L_x : \mathcal{K}_x \to [x]$ is continuous.*

More generally, if $\mathcal{V} = \{(x,s) \in I \times \{0,1\}^{\mathbb{N}} : s \in \mathcal{K}_x\}$, then the function $L : \mathcal{V} \to \hat{I}$ defined by $L(x,s) = L_x(s)$ is continuous.

Proof. Compactness of $\mathcal{K}_x$ is straightforward: if $s^{(i)} \to s$ are itineraries of $\boldsymbol{x}^{(i)} \in [x]$ with $\boldsymbol{x}^{(i)} \to \boldsymbol{x}$, then s is an itinerary of $\boldsymbol{x}$.

For continuity of L_x, note that if $\boldsymbol{x}, \boldsymbol{y} \in [x]$ and $J(\boldsymbol{x})_r = J(\boldsymbol{y})_r$ for all $r \le R$, then $x_r = y_r$ for all $r \le R$.

For the final statement, note that if $\boldsymbol{x}, \boldsymbol{y} \in \hat{I}$ with $|x_0 - y_0| < \epsilon$ and $J(\boldsymbol{x})_r = J(\boldsymbol{y})_r$ for all $r \le R$, then $|x_r - y_r| < \epsilon/\lambda^r$ for all $r \le R$. $\square$

Remark 3.20. It follows that if $x \notin \mathrm{PC}$, so that $J|_{[x]} : [x] \to \mathcal{K}_x$ is well defined, then this map is continuous, being the inverse of the continuous bijection L_x.

3.3. *Invariant measures.* Recall (for example, [3; 16; 22; 24; 25; 26; 31]) that each tent map $f = f_\lambda$ has a unique invariant Borel probability measure $\mu = \mu_\lambda$ which is absolutely continuous with respect to Lebesgue measure m, and $d\mu = \varphi\, dm$, for some $\varphi \in L^1(m)$ defined on $[a,b] \setminus \mathrm{PC}$ which is bounded away from zero, of bounded variation, and satisfies, for all $x \notin \mathrm{PC}$ and $r \in \mathbb{N}$,

$$\varphi(x) = \sum_{f^r(y)=x} \frac{\varphi(y)}{\lambda^r}. \tag{2}$$

Moreover, μ is ergodic.

This measure μ induces an ergodic $\hat{f}$-invariant OU probability measure $\hat{\mu}$ on $\hat{I}$ characterized by $(\pi_r)_*(\hat{\mu}) = \mu$ for all $r \in \mathbb{N}$ [29].

The following result is from [11].

Lemma 3.21. *For each $x \notin \mathrm{PC}$, there is a Borel measure α_x on $\hat{I}$, which is supported on $[x]$ and is given on cylinder subsets of $[x]$ by*

$$\alpha_x([x, y_1, \dots, y_r]) = \frac{\varphi(y_r)}{\lambda^r}.$$

In particular, $\alpha_x([x]) = \varphi(x)$.

Remarks 3.22. (a) α_x is OU: since the cylinder subsets form a basis of $[x]$, α_x is positive on nonempty open subsets of $[x]$; and since any point $\boldsymbol{x}$ is contained in $[x_0, \dots, x_r]$ for all r, α_x is nonatomic.

(b) Since $\hat{f}([x, y_1, \dots, y_r]) = [f(x), x, y_1, \dots, y_r]$, we have $\alpha_{f(x)}(\hat{f}(A)) = \lambda^{-1}\alpha_x(A)$ for every Borel subset A of $[x]$.

(c) If c is not preperiodic, then φ can be extended inductively over PC by

$$\varphi(f^i(c)) = \sum_{f(y)=f^i(c)} \frac{\varphi(y)}{\lambda},$$

so that it satisfies (2) for all $x \in I$; and therefore the measures α_x can be defined for all $x \in I$ in such a way that they satisfy (a) and (b).

Remarks 3.22(b) says that $\hat{f}$ contracts π_0-fibers uniformly by a factor λ, with respect to the measures α_x.

The images of these fibers under the semiconjugacy $g : \hat{I} \to \Sigma = S^2$ — which are almost all arcs — will form the stable foliation or turbulation of the generalized or measurable pseudo-Anosov map semiconjugate to $\hat{f}$, with the measure on leaves/streamlines coming from the α_x. The unstable foliation/turbulation will come from the path components of $\hat{I}$, together with Lebesgue measure m on their π_0-images, which is expanded by $\hat{f}$ by a factor λ.

The remaining definitions and results in this section are from [11]: Theorem 3.26 is Lemma 8.3, Theorem 3.27 is Theorem 7.1, Theorem 3.28 is Theorem 8.1, Theorem 3.30 is Theorem 1.1(a), and Lemma 3.31 is Lemma 3.5. The reader may find it helpful to interpret these in light of the genesis of the invariant turbulations just described. The 0-boxes of Definition 3.25 will provide the unstable fibers of a tartan, whose stable fibers come from the π_0-fibers of $\hat{I}$; Theorem 3.26 will provide a tartan R with $\mu(R^{\pitchfork}) > 0$, which will enable us to show fullness of the turbulations; Theorem 3.27 translates to unstable holonomy invariance of these tartans (stable holonomy invariance is straightforward); Theorem 3.28 will be used to show that they are compatible with μ; and Theorem 3.30, together with Lemma 3.31, ensures that an unstable turbulation can be built from well-behaved path components.

Definition 3.23 (0-flat arc). An arc Γ in $\hat{I}$ is 0-*flat* (*over the subinterval J of I*) if π_0 is injective on Γ (and maps it onto J).

Remark 3.24. Alternatively, the arc Γ is 0-flat if and only if $c \notin \operatorname{Int}(\pi_r(\Gamma))$ for all $r \geq 1$. In particular, the points of a 0-flat arc all share a common itinerary.

Definition 3.25 (0-box). Let J be a subinterval of I. A 0-*box B over J* is a Borel disjoint union of 0-flat arcs over J.

Theorem 3.26. *There exists a* 0*-box B with $\hat{\mu}(B) > 0$.*

Theorem 3.27 (holonomy invariance). *Let B be a* 0*-box over a subinterval J. Then*

$$\alpha_x(B) = \alpha_y(B) \quad \textit{for all } x, y \in J \setminus \mathrm{PC}.$$

Theorem 3.28. *For every Borel subset E of $\hat{I}$, we have*

$$\hat{\mu}(E) = \int_I \alpha_x(E)\, dm(x).$$

Definition 3.29 (globally leaf regular). A point $\boldsymbol{x} \in \hat{I}$ is said to be *globally leaf regular* if its path component Γ is an open arc, and each component of $\Gamma \setminus \{\boldsymbol{x}\}$ is dense in $\hat{I}$.

Theorem 3.30 (typical path components). *$\hat{\mu}$-almost every point of $\hat{I}$ is globally leaf regular.*

Globally leaf regular path components (or, more generally, open arcs) can be decomposed into 0-flat arcs.

Lemma 3.31 (0-flat decomposition of globally leaf regular component). *Let $\Gamma \subset \hat{I}$ be an open arc. Then there is a countable (finite, infinite, or bi-infinite) sequence (Γ_i) of 0-flat arcs in $\hat{I}$, unique up to reindexing by an order-preserving or order-reversing bijection, such that:*

(a) $\Gamma = \bigcup_i \Gamma_i$.

(b) *Each Γ_i is disjoint from all other Γ_j, except that it intersects Γ_{i-1} and Γ_{i+1}, if they exist, at its endpoints.*

(c) *$\boldsymbol{x} \in \Gamma$ is an endpoint of some Γ_i if and only if $x_r = c$ for some r.*

Remark 3.32. It follows from Lemma 3.31 and Theorem 3.28 that if Γ is the path component of a globally leaf regular point, then $\hat{\mu}(\Gamma) = 0$. This is because each Γ_i of Lemma 3.31 intersects each π_0-fiber in at most one point, so that $\alpha_x(\Gamma_i) = 0$ for all x.

4. Extreme elements of π_0-fibers and the outside map

If $x \notin \mathrm{PC}$ then every element of $[x]$ has a well-defined itinerary, so that $[x]$ can be totally ordered by the unimodal order and, being compact, has minimum and maximum elements. When $x \in \mathrm{PC}$ there is no such order on the fiber, but there are nevertheless minimum and maximum elements of $\mathcal{K}_x$, the set of itineraries realized on $[x]$ (recall Definition 3.18), which correspond to unique 'minimum' and 'maximum' elements of the fiber itself. When $x = a$ or $x = b$, it turns out that these elements coincide.

These extreme elements of fibers are important because the identifications induced by the map $g : \hat{I} \to \Sigma$ which semiconjugates $\hat{f}$ to the sphere homeomorphism $F : \Sigma \to \Sigma$ take place along their orbits (the identifications will be described in Lemma 5.4). In this section we study the action of $\hat{f}$ on extreme elements, which we model with a circle map called the *outside map* associated to f. The construction [12] of the sphere homeomorphism $F : \Sigma \to \Sigma$ depends crucially on the dynamics of this circle map, and in particular on its rotation number, which is called the *height* $q(f)$ of f.

4.1. *Extreme elements.*

Recall (Definition 3.18) that

$$\mathcal{K}_x = \{s \in \{0, 1\}^{\mathbb{N}} : s \text{ is an itinerary of some } \boldsymbol{x} \in [x]\}$$

for each $x \in I$, and that $L_x : \mathcal{K}_x \to [x]$ associates to each $s \in \mathcal{K}_x$ the unique $\boldsymbol{x} \in [x]$ with that itinerary. Since $\mathcal{K}_x$ is compact (Lemma 3.19) and every nonempty subset of $\{0, 1\}^{\mathbb{N}}$ has an infimum and supremum, $\mathcal{K}_x$ has a maximum element $\mathcal{U}_x$ and a minimum element $\mathcal{L}_x$ with respect to the unimodal order.

Definition 4.1 (upper, lower, extreme elements, $\boldsymbol{e}(x_u)$, $\boldsymbol{e}(x_\ell)$). For each $x \in I$, we write $\boldsymbol{e}(x_u) = L_x(\mathcal{U}_x)$ and $\boldsymbol{e}(x_\ell) = L_x(\mathcal{L}_x)$ for the *upper* and *lower* elements of $[x]$, having as itineraries $\mathcal{U}_x$ and $\mathcal{L}_x$, respectively, and refer to them both as *extreme* elements. (The apparently idiosyncratic notation will become more natural in Section 4.2, where a map $\boldsymbol{e}$ onto the set of extreme elements will be defined.)

The following lemma describes the action of $\hat{f}$ on extreme elements.

Lemma 4.2. *Let $x \in I$.*

(a) *If $x < f(a)$, so that x has a unique f-preimage $x^1 > c$, then*

$$\boldsymbol{e}(x_\ell) = \hat{f}(\boldsymbol{e}(x^1_u)) \quad \text{and} \quad \boldsymbol{e}(x_u) = \hat{f}(\boldsymbol{e}(x^1_\ell)).$$

(b) *If $f(a) \le x < b$, so that x has preimages $x^0 < c$ and $x^1 > c$, then*

$$\boldsymbol{e}(x_\ell) = \hat{f}(\boldsymbol{e}(x^0_\ell)) \quad \text{and} \quad \boldsymbol{e}(x_u) = \hat{f}(\boldsymbol{e}(x^1_\ell)).$$

(c) *If $x = b$, so that x has unique preimage c, then*

$$\boldsymbol{e}(b_\ell) = \boldsymbol{e}(b_u) = \hat{f}(\boldsymbol{e}(c_\ell)).$$

Proof. (a) We have $[x] = \hat{f}([x^1])$; and if $\boldsymbol{x} \in [x^1]$, then the itineraries of $\hat{f}(\boldsymbol{x}) \in [x]$ are exactly the itineraries of $\boldsymbol{x}$ with a symbol 1 prepended, so that $\hat{f}|_{[x^1]}$ reverses the order on itineraries.

(b) We have $[x] = \hat{f}([x^0]) \sqcup \hat{f}([x^1])$; if $\boldsymbol{x} \in [x^0]$ (respectively $\boldsymbol{x} \in [x^1]$) then the itineraries of $\hat{f}(\boldsymbol{x}) \in [x]$ are exactly those of $\boldsymbol{x}$ with a symbol 0 (respectively 1) prepended. Therefore $\hat{f}|_{[x^0]}$ preserves the order on itineraries and $\hat{f}|_{[x^1]}$ reverses it; and all elements of $\hat{f}([x^1])$ have larger itineraries than all elements of $\hat{f}([x^0])$.

(c) We have $[b] = \hat{f}([c])$; and if $\boldsymbol{x} \in [c]$, then for each itinerary of $\boldsymbol{x}$ there are two itineraries of $\hat{f}(\boldsymbol{x}) \in [b]$, one with the symbol 0 prepended and one with the symbol 1 prepended. □

Recall from Definition 3.17 that, provided $x \notin \mathrm{PC}$, we can define the notion of consecutive elements of $[x]$. These will play an important role in the theory, since typically two elements of a fiber are identified by the semiconjugacy $g : \hat{I} \to \Sigma$ if and only if they are consecutive. The proof of Lemma 4.2 also elucidates the action of $\hat{f}$ on pairs of consecutive points, as described by the next lemma.

Lemma 4.3. *Let $x \in I \setminus \mathrm{PC}$.*

(a) *Let $r > 0$ and suppose that $f^r(x) \notin \mathrm{PC}$. If $\boldsymbol{x}$ and $\boldsymbol{x}'$ are consecutive in $[x]$, then $\hat{f}^r(\boldsymbol{x})$ and $\hat{f}^r(\boldsymbol{x}')$ are consecutive in $[f^r(x)]$.*

(b) *If $\boldsymbol{x}$ and $\boldsymbol{x}'$ are consecutive in $[x]$ and $x_1 = x'_1$, then $\hat{f}^{-1}(\boldsymbol{x})$ and $\hat{f}^{-1}(\boldsymbol{x}')$ are consecutive in $[x_1]$.*

(c) *If $\boldsymbol{x}$ and $\boldsymbol{x}'$ are consecutive in $[x]$ and $x_1 \ne x'_1$, then $\hat{f}^{-1}(\boldsymbol{x}) = \boldsymbol{e}(x_{1,u})$ and $\hat{f}^{-1}(\boldsymbol{x}') = \boldsymbol{e}(x'_{1,u})$.*

(d) *If $x \in [a, \hat{a}] \setminus \{c\}$, then $\hat{f}(\boldsymbol{e}(x_u))$ and $\hat{f}(\boldsymbol{e}(\hat{x}_u))$ are consecutive in $[f(x)]$.*

The extreme elements of a fiber $[x]$ coincide if and only if x is an endpoint of I:

Lemma 4.4. $\boldsymbol{e}(a_\ell) = \boldsymbol{e}(a_u)$, *and* $\boldsymbol{e}(b_\ell) = \boldsymbol{e}(b_u)$. *On the other hand, if $x \in I \setminus \{a, b\}$ then $\boldsymbol{e}(x_\ell) \ne \boldsymbol{e}(x_u)$.*

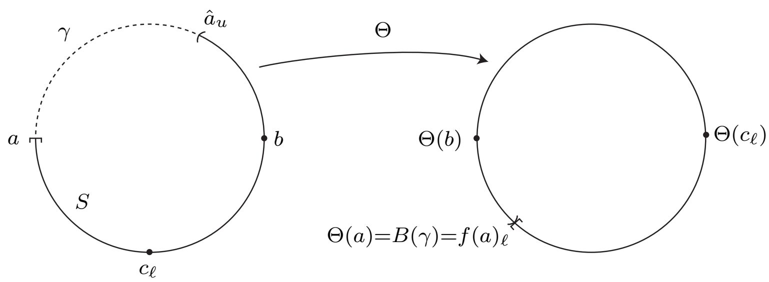

Figure 4. The circle S and the map $\Theta : [a, \hat{a}_u) \to S$.

Proof. $\boldsymbol{e}(b_\ell) = \boldsymbol{e}(b_u)$ by Lemma 4.2(c), and it follows from Lemma 4.2(a) that

$$\boldsymbol{e}(a_\ell) = \boldsymbol{e}(a_u) = \hat{f}(\boldsymbol{e}(b_u) = \boldsymbol{e}(b_\ell)).$$

Suppose then that $x \in (a, b)$. By Lemma 3.10, there is some $y \in [f(a), b)$ and $r \in \mathbb{N}$ such that $f^r(y) = x$ and $f^i(y) \in (a, f(a))$ for $1 \le i < r$. Then $\boldsymbol{e}(y_\ell) \neq \boldsymbol{e}(y_u)$ by Lemma 4.2(b); and $\{\boldsymbol{e}(x_\ell), \boldsymbol{e}(x_u)\} = \left\{\hat{f}^r(\boldsymbol{e}(y_\ell)), \hat{f}^r(\boldsymbol{e}(y_u))\right\}$ by Lemma 4.2(a), so that $\boldsymbol{e}(x_\ell) \neq \boldsymbol{e}(x_u)$ as required. $\square$

4.2. *The outside map.* We model the action of $\hat{f}$ on the set of extreme elements, as described in Lemma 4.2, with a circle map (see Figure 4).

Let S be the circle obtained by gluing together two copies of I at their endpoints. We denote the points of S by x_ℓ and x_u, for $x \in I$, depending on whether they belong to the 'lower' or 'upper' copy of I. Therefore $a_\ell = a_u$ and $b_\ell = b_u$, and we also denote these points of S with the symbols a and b, respectively. We write $\tau : S \to I$ for the projection $\tau(x_\ell) = \tau(x_u) = x$. When we use interval notation in S, we regard the interval as an arc in S which goes in the positive sense from its first endpoint to its second endpoint, in the model of Figure 4. Thus, for example, $[a, b]$ contains x_ℓ for all x, while $[b, a]$ contains x_u for all x.

Definition 4.1 provides an injective function $\boldsymbol{e} : S \to \hat{I}$ with image the set of extreme elements of $\hat{I}$. This map is clearly discontinuous, since $\hat{I}$ does not contain a circle: its discontinuity set is discussed in Lemma 4.5(d) and Remark 4.10.

Similarly, we define a symbolic version $\mathcal{E} : S \setminus \{a, b\} \to \{0, 1\}^{\mathbb{N}}$ by $\mathcal{E}(x_\ell) = \mathcal{L}_x$ and $\mathcal{E}(x_u) = \mathcal{U}_x$, so that $\mathcal{E}(y)$ is an itinerary of $\boldsymbol{e}(y)$ for each $y \in S \setminus \{a, b\}$ (recall that $\mathcal{L}_x$ and $\mathcal{U}_x$ are the minimum and maximum itineraries in the fiber $[x]$).

If $\Theta : [a, \hat{a}_u) \to S$ is defined (see Figure 4) by

$$\begin{aligned} \Theta(x_\ell) &= f(x)_\ell \quad \text{if } x \in [a, c], \\ \Theta(x_\ell) &= f(x)_u \quad \text{if } x \in [c, b], \\ \Theta(x_u) &= f(x)_\ell \quad \text{if } x \in (\hat{a}, b], \end{aligned} \tag{3}$$

then it follows from Lemma 4.2 — and of course this is the point of the definition — that

$$\hat{f}(\boldsymbol{e}(y)) = \boldsymbol{e}(\Theta(y)) \quad \text{for all } y \in [a, \hat{a}_u) \tag{4}$$

(if $y \in [\hat{a}_u, a)$ then $\hat{f}(\boldsymbol{e}(y))$ is not an extreme point by Lemma 4.2).

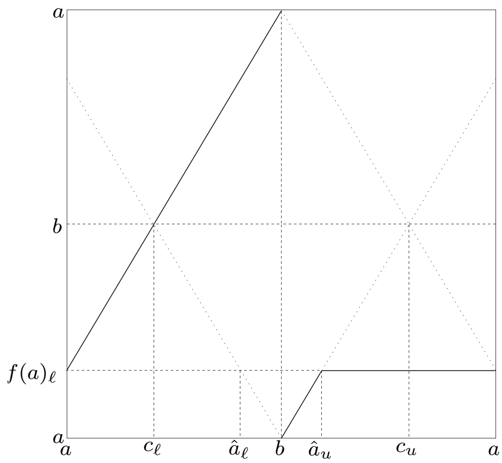

Figure 5. The outside map $B : S \to S$. Θ has the same graph, except that it is undefined on the interval $[\hat{a}_u, a)$, on which B takes the constant value $f(a)_\ell$.

Θ is a continuous bijection, whose inverse has a one-sided discontinuity at $f(a)_\ell$ — it is discontinuous as we approach $f(a)_\ell$ in the positive sense — and is continuous elsewhere.

Θ can be extended to a continuous monotone circle map $B : S \to S$ of degree 1 by setting $B(y) = f(a)_\ell$ for all $y \in \gamma := [\hat{a}_u, a]$. B is called the *outside map* associated to the tent map f (Figure 5). We write $\mathring{\gamma} := (\hat{a}_u, a)$ for the interior of γ.

The next lemma gathers some straightforward facts about Θ.

Lemma 4.5. (a) *If $\Theta^r(y)$ exists for some $r > 0$, then $\tau(\Theta^r(y)) = f^r(\tau(y))$.*

(b) *$\Theta^{-r}(y)$ exists for all $y \in S$ and all $r \in \mathbb{N}$, and $\boldsymbol{e}(\Theta^{-r}(y)) = \hat{f}^{-r}(\boldsymbol{e}(y))$.*

(c) *An explicit formula for $\boldsymbol{e}(y)$ is given by*

$$\boldsymbol{e}(y) = \big\langle \tau(y), \tau(\Theta^{-1}(y)), \tau(\Theta^{-2}(y)), \ldots \big\rangle.$$

(d) *$\boldsymbol{e}$ is discontinuous at $y \in S$ if and only if $y \in \operatorname{orb}(f(a)_\ell, \Theta)$, and these discontinuities are one-sided* (*as we approach y in the positive sense*).

(e) *$\mathcal{E} : S \setminus \{a, b\} \to \{0, 1\}^{\mathbb{N}}$ is locally constant at $y \in S \setminus \{a, b\}$ if and only if $y \notin \overline{\operatorname{orb}(f(a)_\ell, \Theta)}$.*

Proof. (a) and (b) are immediate from (3) and (4), respectively.

(c) We have $\boldsymbol{e}(y) = \big\langle \pi_0(\boldsymbol{e}(y)), \pi_0\big(\hat{f}^{-1}(\boldsymbol{e}(y))\big), \pi_0\big(\hat{f}^{-2}(\boldsymbol{e}(y))\big), \ldots \big\rangle$ by (1), and (b) gives $\hat{f}^{-r}(\boldsymbol{e}(y)) = \boldsymbol{e}(\Theta^{-r}(y)) \in \big[\tau(\Theta^{-r}(y))\big]$, so that $\pi_0\big(\hat{f}^{-r}(\boldsymbol{e}(y))\big) = \tau(\Theta^{-r}(y))$.

(d) This follows from (c) because Θ^{-1} has a one-sided discontinuity at $f(a)_\ell$ and is continuous elsewhere.

(e) Since c_u is not in the domain of Θ, it follows from (c) that $\boldsymbol{e}(y)_r = c$ for some $r \geq 1$ if and only if $y \in \operatorname{orb}(\Theta(c_\ell), \Theta) = \operatorname{orb}(b, \Theta) = \{a, b\} \cup \operatorname{orb}(f(a)_\ell, \Theta)$. $\square$

From parts (d) and (e) of this lemma, we see that the Θ-orbit of $f(a)_\ell$ plays an important role in understanding the structure of the extreme elements of $\hat{I}$. In particular, there is a strong distinction between the cases where this orbit is finite and those where it is infinite. Note carefully that the orbit can be finite not only when it is periodic or preperiodic, but also, more commonly, when $\Theta^r(f(a)_\ell) \in [\hat{a}_u, a)$ is outside the domain of Θ for some r (see Theorem 4.8(c)).

4.3. *Dynamics of the outside map.* The outside map was defined in [19], and a closely related map was introduced in [16]. In this section we state those of its dynamical properties which we will use and make some related definitions. Theorem 4.7, parts (a) and (b) of Theorem 4.8, and part (a) of Theorem 4.9 can be found in [23]; while the remaining parts of Theorems 4.8 and 4.9 are proved in [12].

Definition 4.6 (height). The *height* $q_\lambda = q(f_\lambda)$ of f is the Poincaré rotation number of the outside map $B : S \to S$.

Theorem 4.7. *q_λ is decreasing in λ, and $q_\lambda \in \left(0, \frac{1}{2}\right)$ for all λ. (In fact, for general tent maps, $q_\lambda = 0$ if and only if $\lambda = 2$, and $q_\lambda = \frac{1}{2}$ if and only if $\lambda \in \left(1, \sqrt{2}\,\right]$.)*

Theorem 4.8 (the rational-height case). *Let $m/n \in \left(0, \frac{1}{2}\right)$ be rational.*

(a) *There is a nontrivial closed interval $\Lambda_{m/n} = [\lambda^-_{m/n}, \lambda^+_{m/n}] \subset \left(\sqrt{2}, 2\right)$ of parameters such that $q_\lambda = m/n$ if and only if $\lambda \in \Lambda_{m/n}$.*

(b) *The critical orbit of f_λ is periodic (with period n) if $\lambda = \lambda^-_{m/n}$, and preperiodic (to period n) if $\lambda = \lambda^+_{m/n}$.*

(c) *For each $\lambda \in \Lambda_{m/n}$, the smallest positive integer r with $\Theta^r(a) \in \gamma = [\hat{a}_u, a]$ is $r = n$. We have $\Theta^n(a) = a$ if $\lambda = \lambda^-_{m/n}$; $\Theta^n(a) = \hat{a}_u$ if $\lambda = \lambda^+_{m/n}$; and $\Theta^n(a) \in (\hat{a}_u, a)$ if $\lambda \in \mathrm{Int}(\Lambda_{m/n})$. Moreover, there is a unique $\lambda = \lambda^*_{m/n} \in \Lambda_{m/n}$ for which $\Theta^n(a) = c_u$.*

(d) *For each $\lambda \in \Lambda_{m/n}$, the set $S \setminus \bigcup_{r\geq 0} B^{-r}(\mathring{\gamma})$ of points whose B-orbits never enter $\mathring{\gamma} = (\hat{a}_u, a)$ is a period-n orbit of B, which attracts the Θ^{-1}-orbit of every point of S.*

Note that, by (4), we have $\Theta^n(a) = f^n(a)_u$ whenever $\lambda \in \Lambda_{m/n}$.

Theorem 4.9 (the irrational-height case). *Let $q \in \left(0, \frac{1}{2}\right)$ be irrational.*

(a) *There is a parameter $\lambda_q \in \left(\sqrt{2}, 2\right)$ such that $q_\lambda = q$ if and only if $\lambda = \lambda_q$.*

(b) *If $\lambda = \lambda_q$ then f is postcritically infinite. Moreover $\mathrm{orb}(f(a)_\ell, \Theta)$ is disjoint from $\gamma = [\hat{a}_u, a]$, and the closure of this orbit is a Cantor set $\mathcal{C}_S = S \setminus \bigcup_{r\geq 0} \Theta^{-r}(\mathring{\gamma})$, the set of points whose Θ-orbit never enters $\mathring{\gamma}$. In particular $\mathcal{C}_S$ contains a and $\hat{a}_u$, since $\Theta(a) = f(a)_\ell$ has an orbit which never enters γ; and Θ is not defined at $\hat{a}_u$.*

The values of $\lambda^\pm_{m/n}$, $\lambda^*_{m/n}$, and λ_q can be specified in terms of the kneading sequences of the corresponding tent maps (see, for example, [12]), but this will not be necessary here.

Remark 4.10. Using Lemma 4.5(d) and (e), it follows from Theorem 4.8(c) that if $q_\lambda = m/n$ then $\boldsymbol{e} : S \to \hat{I}$ has exactly n discontinuities, and $\mathcal{E} : S \setminus \{a, b\} \to \{0, 1\}^{\mathbb{N}}$ is constant on each component of the complement of these discontinuities. (If $\lambda = \lambda^-_{m/n}$ then the Θ-orbit of $f(a)_\ell$ is periodic, while if $\lambda > \lambda^-_{m/n}$ then it is a finite-orbit segment.)

We can use Theorems 4.8 and 4.9 to classify tent maps: first, according to whether their height is rational or irrational; and second, in the rational case, according to whether the parameter λ is an endpoint of the rational-height interval, the special interior value for which $\Theta^n(a) = c_u$, or a general interior value.

Definitions 4.11 (type of tent map: irrational, rational, endpoint, NBT, general). We say that f_λ is of *irrational* or *rational* type based on whether q_λ is irrational or rational. In the rational case with $q_\lambda = m/n$, we say that f_λ is of *endpoint* type if $\lambda = \lambda^{\pm}_{m/n}$; of *NBT* type if $\lambda = \lambda^{*}_{m/n}$; and of *general* type otherwise.

5. The rational postcritically infinite case

Throughout this section we assume that f is of rational type with height $q = m/n$, and that the critical orbit is not periodic or preperiodic. We will show that the corresponding sphere homeomorphism $F : \Sigma \to \Sigma$ is a measurable pseudo-Anosov map. (In the case where f is postcritically finite, F is a generalized pseudo-Anosov map, as proved in [19]: see Section 6.4.)

The path that we take is as follows:

- The map $g : \hat{I} \to \Sigma$ which semiconjugates $\hat{f}$ to F sends most of the π_0-fibers $[x]$ in $\hat{I}$ — those for which x is not in the grand orbit of c — to arcs in Σ. In Section 5.1 we describe g (recalling results from [12]) and prove this result (Theorem 5.6).
- The streamlines of the stable turbulation are unions of these arcs, joined at their endpoints. The stable measure comes from the measures α_x of Lemma 3.21 supported on the π_0-fibers. In Section 5.2 we define $(\mathcal{T}^s, \nu^s)$ (Definition 5.13) and show that F contracts ν^s by a factor λ (Lemma 5.14) and that the turbulation is tame (Lemma 5.16).
- The streamlines of the unstable turbulation are the g-images of the globally leaf regular path components of $\hat{I}$ (Definition 3.29): recall from Theorem 3.30 that the union of these has full measure — we omit countably many of them, on which the identifications induced by g are nontrivial. The unstable measure comes from Lebesgue measure on the components of the 0-flat decompositions (Lemma 3.31) of these path components. In Section 5.3 we define $(\mathcal{T}^u, \nu^u)$ (Definition 5.17) and show that F expands ν^u by a factor λ (Lemma 5.18) and that the turbulation is tame (Lemma 5.19).
- In Section 5.4 we complete the proof that F is a measurable pseudo-Anosov map, by proving that the two measured turbulations are transverse and full.

5.1. ***The semiconjugacy $g : \hat{I} \to \Sigma$.*** We now describe the identifications on $\hat{I}$ realized by the semiconjugacy $g : \hat{I} \to \Sigma$, and show that, under these identifications, the π_0-fibers of points $y \in I$ which are not on the grand orbit

$$\mathrm{GO}(c) = \{x \in I : f^m(x) = f^n(c) \text{ for some } m, n \geq 0\}$$

of c become arcs in Σ.

Notation 5.1 (Π, $\boldsymbol{P}$). We write $\Pi = S \setminus \bigcup_{r\geq 0} B^{-r}(\mathring{\gamma})$, the set of points whose B-orbits never enter $\mathring{\gamma} = (\hat{a}_u, a)$, and $\boldsymbol{P} = \boldsymbol{e}(\Pi) \subset \hat{I}$.

Lemma 5.2. *Π is a period-n orbit of B which is disjoint from γ, and $\boldsymbol{P}$ is a period-n orbit of $\hat{f}$ contained in the set of extreme elements of $\hat{I}$.*

Proof. Theorem 4.8(d) states that Π is a period-n orbit of B. By definition it is disjoint from $\mathring{\gamma}$, so we need only show that the endpoints a and $\hat{a}_u$ of γ are not in Π. Since f is not postcritically finite we have $\lambda \in \operatorname{Int}(\Lambda_{m/n})$ by Theorem 4.8(b), and hence $B^n(a) = \Theta^n(a) \in \mathring{\gamma}$ by Theorem 4.8(c), so that $a \notin \Pi$. Since $B(a) = B(\hat{a}_u) = f(a)_\ell$, we have $\hat{a}_u \notin \Pi$ also.

It follows from (4) that $\boldsymbol{P} = \boldsymbol{e}(\Pi)$ is a period-n orbit of $\hat{f}$, which is clearly contained in the image of $\boldsymbol{e}$, the set of extreme elements of $\hat{I}$. □

Notation 5.3 (z, $\mathcal{Z}$). We write $z = f^n(a)$, so that $\Theta^n(a) = z_u$, and $\mathcal{Z} = \{z, \hat{z}\}$.

The following key lemma is a translation into the language of this paper of Remark 5.17(e) of [12].

Lemma 5.4. *The semiconjugacy $g : \hat{I} \to \Sigma$ is the quotient map of the equivalence relation on $\hat{I}$ with the nontrivial equivalence classes*

(EI) $\left\{\hat{f}^r(\boldsymbol{e}(x_u)),\ \hat{f}^r(\boldsymbol{e}(\hat{x}_u))\right\}$, *for each $x \in (a, c) \setminus \mathcal{Z}$ and each $r \in \mathbb{Z}$;*

(EII) $\left\{\hat{f}^r(\boldsymbol{e}(\hat{z}_u)),\ \hat{f}^{r+n}(\boldsymbol{e}(a)),\ \hat{f}^{r+n}(\boldsymbol{e}(\hat{a}_u))\right\}$, *for each $r \in \mathbb{Z}$; and*

(EIII) *the period-n orbit $\boldsymbol{P}$.*

We first use this result to understand the identifications induced by g on individual fibers of $\hat{I}$. Note that the condition $y \in I \setminus \mathrm{GO}(c)$ implies that $y \notin \mathrm{PC}$, so that the fiber $[y]$ is totally ordered, and in particular consecutive elements are well defined.

Lemma 5.5 (identifications within fibers). *Let $y \in I \setminus \mathrm{GO}(c)$.*

(a) *Nonextreme elements of $[y]$ are only identified with other (nonextreme) elements of $[y]$.*

(b) *Distinct nonextreme elements $\boldsymbol{x}$ and $\boldsymbol{x}'$ of $[y]$ are identified if and only if they are consecutive.*

(c) *The extreme elements $\boldsymbol{e}(y_u)$ and $\boldsymbol{e}(y_\ell)$ of $[y]$ are not identified with each other: $g(\boldsymbol{e}(y_u)) \neq g(\boldsymbol{e}(y_\ell))$.*

Proof. (a) Since $y \notin \mathrm{GO}(c)$, equivalence classes of type (EII) are disjoint from $[y]$. Moreover, if $r \geq 0$ then for any $x \in (a, c) \setminus \mathcal{Z}$ we have $\hat{f}^{-r}(\boldsymbol{e}(x_u)) = \boldsymbol{e}(\Theta^{-r}(x_u))$ and $\hat{f}^{-r}(\boldsymbol{e}(\hat{x}_u)) = \boldsymbol{e}(\Theta^{-r}(\hat{x}_u))$ by Lemma 4.5(b), so these are extreme elements, as are the points of $\boldsymbol{P}$. Therefore the nontrivial equivalence classes which contain nonextreme elements of $[y]$ are precisely $\left\{\hat{f}^r(\boldsymbol{e}(x_u)),\ \hat{f}^r(\boldsymbol{e}(\hat{x}_u))\right\}$ for those $x \in (a, c) \setminus \mathcal{Z}$ and $r > 0$ which satisfy $f^r(x) = y$; and, since $f^r(x) = f^r(\hat{x})$, such equivalence classes consist of two points of $[y]$.

(b) Suppose that $\boldsymbol{x}$ and $\boldsymbol{x}'$ are consecutive. Let $r > 0$ be least such that $x_r \neq x'_r$. By Lemma 4.3(b) and (c), we have $\hat{f}^{-r}(\boldsymbol{x}) = \boldsymbol{e}(x_{r,u})$ and $\hat{f}^{-r}(\boldsymbol{x}') = \boldsymbol{e}(x'_{r,u})$. Since $x'_r = \hat{x}_r$ (both have image x_{r-1}), and these are not equal to a, c, z, or $\hat{z}$ (since $y \notin \mathrm{GO}(c)$), it follows from (EI) that $g(\boldsymbol{x}) = g(\boldsymbol{x}')$ as required.

For the converse, if $g(\boldsymbol{x}) = g(\boldsymbol{x}')$ then by the proof of (a) there are $x \in (a, c) \setminus \mathcal{Z}$ and $r > 0$ such that $\{\boldsymbol{x}, \boldsymbol{x}'\} = \left\{\hat{f}^r(\boldsymbol{e}(x_u)),\ \hat{f}^r(\boldsymbol{e}(\hat{x}_u))\right\}$. These are consecutive by Lemma 4.3(d) and (a).

(c) The two extreme elements cannot both belong to the periodic orbit $\boldsymbol{P}$, since if $y < \hat{a}$ then $y_u \in \mathring{\gamma}$ (so that $y_u \notin \Pi$, and hence $\boldsymbol{e}(y_u) \notin \boldsymbol{P}$); while if $y > \hat{a}$ then $f(y) < f(a) < \hat{a}$ by Lemma 3.8, and hence $B(y_\ell) = f(y)_u \in \mathring{\gamma}$ (so that $y_\ell \notin \Pi$, and hence $\boldsymbol{e}(y_\ell) \notin \boldsymbol{P}$).

It follows from Lemma 5.4 and $y \notin \mathrm{GO}(c)$ that we could only have $g(\boldsymbol{e}(y_u)) = g(\boldsymbol{e}(y_\ell))$ if there were some $x \in (a, c)$ and $r \in \mathbb{Z}$ with $\{\boldsymbol{e}(y_u), \boldsymbol{e}(y_\ell)\}$ equal to $\left\{\hat{f}^r(\boldsymbol{e}(x_u)), \hat{f}^r(\boldsymbol{e}(\hat{x}_u))\right\}$. However if $r > 0$ then $\hat{f}^r(\boldsymbol{e}(x_u))$ and $\hat{f}^r(\boldsymbol{e}(\hat{x}_u))$ are nonextreme elements of $[f^r(x)]$. If $r = -s \le 0$, on the other hand, then we would have $\{y_u, y_\ell\} = \{\Theta^{-s}(x_u), \Theta^{-s}(\hat{x}_u)\}$ as in the proof of (a), so that s would be both least with $B^s(y_u) \in \mathring{\gamma}$, and least with $B^s(y_\ell) \in \mathring{\gamma}$. This is impossible, since, as above, if $y < \hat{a}$ then $y_u \in \mathring{\gamma}$ but $y_\ell \notin \mathring{\gamma}$; while if $y > \hat{a}$ then $B(y_\ell) \in \mathring{\gamma}$ but $B(y_u) = f(y)_\ell \notin \mathring{\gamma}$. □

Theorem 5.6. *Let $y \in I \setminus \mathrm{GO}(c)$. Then $g([y])$ is an arc in Σ.*

Proof. Let $H : \{0, 1\}^{\mathbb{N}} \to [0, 1]$ be the order-preserving homeomorphism onto the middle thirds Cantor set defined by

$$H(s) = \sum_{r=0}^{\infty} \frac{2\varepsilon_r}{3^r}, \qquad \text{where } \varepsilon_r = \left(\sum_{i=0}^{r} s_i\right) \bmod 2.$$

Then $H \circ J : [y] \to [0, 1]$ is an order-preserving embedding of the Cantor set $[y]$ into the interval. Therefore, by Lemma 5.5(b) and (c), $g([y])$ is homeomorphic to the space obtained by collapsing the complementary intervals of a Cantor subset of the interval: that is, to an arc. □

Notation 5.7 ($\mathcal{A}_y$). For each $y \in I \setminus \mathrm{GO}(c)$, we write $\mathcal{A}_y$ for the arc $g([y]) \subset \Sigma$.

Lemma 5.8. *For $y \in I \setminus \mathrm{GO}(c)$, let $\mathcal{B}_y$ be the subset of $\mathcal{A}_y$ consisting of points with a unique g-preimage. Then $g^{-1} : \mathcal{B}_y \to [y]$ is continuous.*

Proof. Suppose for a contradiction that there is a convergent sequence $x_n \to x$ in $\mathcal{B}_y$ with $g^{-1}(x_n) \not\to g^{-1}(x)$. Since $[y]$ is compact, we can assume by taking a subsequence that $g^{-1}(x_n) \to \boldsymbol{x} \ne g^{-1}(x)$. Since g is continuous, it follows that $x_n \to g(\boldsymbol{x})$, so that $x = g(\boldsymbol{x})$. The distinct elements $\boldsymbol{x}$ and $g^{-1}(x)$ of $[y]$ are therefore both mapped to x by g, contradicting the assumption that $x \in \mathcal{B}_y$. □

5.2. *The stable turbulation.* The streamlines of the stable turbulation will be formed from unions of the arcs $\mathcal{A}_y$, which are joined at their endpoints by the identifications of Lemma 5.4. In order to ensure that the endpoints are identified in pairs, we need to exclude the identifications of types (EII) and (EIII) in Lemma 5.4. Type (EII) identifications are excluded in any case, since $y \notin \mathrm{GO}(c)$; to deal with type (EIII), we will also require that y is not in the grand orbit of $\pi_0(\boldsymbol{P})$.

Notation 5.9 (P, V, μ). Write $P = \pi_0(\boldsymbol{P})$ and let $V = I \setminus (\mathrm{GO}(c) \cup \mathrm{GO}(P))$, a totally f-invariant subset of I with countable complement.

Denote by μ the OU ergodic F-invariant probability measure $g_*(\hat{\mu})$ on Σ, where $\hat{\mu}$ is the measure on $\hat{I}$ defined in Section 3.3. No confusion will arise from the same symbol having been used for the absolutely continuous f-invariant measure on I.

If $y \in V$ then, by definition of $\boldsymbol{P}$, there is some $r \geq 0$ with $B^r(y_u) \in (\hat{a}_u, a)$: say $B^r(y_u) = x_u$. Then $x \notin \mathcal{Z}$, since $y \notin \mathrm{GO}(c)$, and hence, by Lemma 5.4, $\boldsymbol{e}(y_u) = \boldsymbol{e}(B^{-r}(x_u))$ is identified with $\boldsymbol{e}(B^{-r}(\hat{x}_u))$, and only with this point.

That is, for every $y \in V$ there is a unique $x \in V \setminus \{y\}$ such that $g(\boldsymbol{e}(y_u))$ is an endpoint of $\mathcal{A}_x$; and, by an identical argument, there is a unique $x' \in V \setminus \{y\}$ such that $g(\boldsymbol{e}(y_\ell))$ is an endpoint of $\mathcal{A}_{x'}$.

It follows that for every $y \in V$, there is a bi-infinite sequence $(y_i)_{i\in\mathbb{Z}}$ in V with $y_0 = y$ and $y_i \neq y_{i+1}$ for each i, unique up to reversal, such that each $\mathcal{A}_{y_i}$ shares one endpoint with $\mathcal{A}_{y_{i-1}}$ and the other with $\mathcal{A}_{y_{i+1}}$.

Notation 5.10 (ℓ_y). For each $y \in V$ we write $\ell_y = \bigcup_{i\in\mathbb{Z}} \mathcal{A}_{y_i}$, which is either an immersed line, or, if the sequence (y_i) repeats, a simple closed curve. We shall see in Lemma 5.15 that the latter is impossible.

The ℓ_y will be the streamlines of the stable turbulation, and we now define the stream measures.

Definition 5.11 ($\nu^s_{\ell_y}$). We define a Borel measure $\nu^s_{\ell_y}$ on ℓ_y by

$$\nu^s_{\ell_y} = \sum_{i\in K} g_*(\alpha_{y_i}),$$

where the α_{y_i} are the measures of Lemma 3.21, supported on $[y_i]$; and where $K = \mathbb{Z}$ if the y_i are all distinct, and otherwise $K = \{0, \dots, r-1\}$ if $r > 0$ is least with $y_r = y_0$. Notice from Lemma 3.21 that $\nu^s_{\ell_y}(\mathcal{A}_{y_i}) = \varphi(y_i)$.

The next lemma ensures that the streamlines and stream measures meet the conditions of Definitions 2.11.

Lemma 5.12. (a) $\mu\left(\bigcup_{y\in V} \ell_y\right) = 1$.

(b) *The measures $\nu^s_{\ell_y}$ are OU and assign finite measure to closed arcs.*

Proof. Write $X = \bigcup_{y\in V} \ell_y$, and note that $X = g(\pi_0^{-1}(V))$. Since V has countable complement, Theorem 3.28 gives

$$\mu(X^s) = \hat{\mu}(\pi_0^{-1}(V)) = \int_V \alpha_x([x])\, dm(x) = \int_I \alpha_x([x])\, dm(x) = \hat{\mu}(\hat{I}) = \mu(\Sigma).$$

$\nu^s_{\ell_y}$ is OU since the same is true of each α_{y_i} (Remarks 3.22(a)). Every closed arc in ℓ_y is contained in a finite union of the $\mathcal{A}_{y_i}$, and so has finite measure. $\square$

Until we show in Lemma 5.15 that all of the ℓ_y are immersed lines, we temporarily relax the definition of measured turbulation to allow streamlines which are simple closed curves.

Definition 5.13 ($\mathcal{T}^s, \nu^s$). We define the stable turbulation $(\mathcal{T}^s, \nu^s)$ on Σ to be the measured turbulation whose streamlines are the distinct immersed lines and simple closed curves ℓ_y, for $y \in V$, with measure $\nu^s_{\ell_y}$ on the streamline ℓ_y.

Lemma 5.14.
$$F(\mathcal{T}^s, \nu^s) = (\mathcal{T}^s, \lambda\nu^s).$$

Proof. F sends streamlines to streamlines since $\hat{f}$ sends π_0-fibers into π_0-fibers and respects the identifications of Lemma 5.4.

If A is a Borel subset of a streamline ℓ_y then $\nu^s_{\ell_y}(A) = \sum_{i\in K} \alpha_{y_i}(g^{-1}(A))$, where $K = \mathbb{Z}$ or $K = \{0, \dots, r-1\}$ as in Definition 5.11. On the other hand

$$\begin{aligned}\nu^s_{\ell_{f(y)}}(F(A)) &= \sum_{i\in K} \alpha_{f(y_i)}\big(g^{-1}(F(A))\big)\\ &= \sum_{i\in K} \alpha_{f(y_i)}\big(\hat{f}(g^{-1}(A))\big)\\ &= \lambda^{-1}\sum_{i\in K} \alpha_{y_i}(g^{-1}(A)) = \lambda^{-1}\nu^s_{\ell_y}(A),\end{aligned}$$

where the first equality comes from $F(\mathcal{A}_{y_i}) \subset \mathcal{A}_{f(y_i)}$; the second is because g is almost everywhere invertible; and the third comes from Remarks 3.22(b).

Thus $F_*(\nu^s_{\ell_y}) = \lambda \nu^s_{\ell_{f(y)}}$ as required. □

Lemma 5.15. *Every streamline of $(\mathcal{T}^s, \nu^s)$ is dense in Σ. In particular, no streamlines are simple closed curves, so that $(\mathcal{T}^s, \nu^s)$ is a measured turbulation.*

Proof. Let $U \subset \Sigma$ be nonempty and open, and ℓ_y be a streamline: we shall show that they intersect.

$g^{-1}(U)$ is open in $\hat{I}$, so $\pi_0(g^{-1}(U))$ contains an interval $J \subset I$ (one way to see this is from Theorem 3.30 and Lemma 3.31: $g^{-1}(U)$ contains an open arc in $\hat{I}$, which has a 0-flat decomposition). Tent maps with slope $\lambda > \sqrt{2}$ are locally eventually onto, so that there is some $n > 0$ with $f^n(J) = I$. In particular, there is some $x \in J$ with $f^n(x) = f^n(y)$: thus $\hat{f}^n([x]) \subset [f^n(y)]$ and $\hat{f}^n([y]) \subset [f^n(y)]$, so that both $[x]$ and $[y]$ are contained in $\hat{f}^{-n}\big([f^n(y)]\big)$. Applying g, both $\mathcal{A}_x$ and $\mathcal{A}_y$ are contained in $F^{-n}(\mathcal{A}_{f^n(y)})$. Since $F^{-n}(\mathcal{A}_{f^n(y)})$ is a stream arc containing $\mathcal{A}_y$, the streamline which contains it is ℓ_y. Thus $\mathcal{A}_x \subset \ell_y$: but since $x \in J \subset \pi_0(g^{-1}(U))$ we have $\mathcal{A}_x \cap U \neq \varnothing$. □

Therefore $(\mathcal{T}^s, \nu^s)$ is a measured turbulation with dense streamlines, which satisfies $F(\mathcal{T}^s, \nu^s) = (\mathcal{T}^s, \lambda\nu^s)$. It remains to show that it is tame (Definition 2.13).

Lemma 5.16. *$(\mathcal{T}^s, \nu^s)$ is tame.*

Proof. Let ρ be a metric on Σ inducing its topology, and let $\epsilon > 0$. Let $\xi > 0$ be small enough that $d(\boldsymbol{x}, \boldsymbol{y}) < \xi \implies \rho(g(\boldsymbol{x}), g(\boldsymbol{y})) < \epsilon/2$, where d is the standard metric of Definitions 3.9.

Let $k > 0$ be a lower bound for the density φ of Section 3.3 on $V \subset I \setminus \mathrm{PC}$. Let $r > 0$ be an integer large enough that $1/2^r < \xi$, and set $\delta = k/\lambda^r$. We shall show that any stream arc $[u, v]$ with measure less than δ satisfies $\rho(u, v) < \epsilon$, establishing tameness of the turbulation.

Recall from Definitions 3.11 that, given our fixed integer $r > 0$, each fiber $[x]$ of $\hat{I}$ is partitioned into a finite number of level-r cylinder sets

$$[x, y_1, \dots, y_r] = \{\boldsymbol{x} \in [x] : x_i = y_i \text{ for } 1 \le i \le r\},$$

where $f(y_1) = x$ and $f(y_i) = y_{i-1}$ for $2 \le i \le r$. If $x \in V$, so that $x \notin \mathrm{PC}$ and itineraries are well defined on $[x]$, then each of these cylinder sets corresponds to a choice of the first r symbols of the itinerary of $\boldsymbol{x}$: in particular, given two distinct level-r cylinder subsets C_1 and C_2 of $[x]$, either $J(\boldsymbol{x}) \prec J(\boldsymbol{x}')$ for every $\boldsymbol{x} \in C_1$ and $\boldsymbol{x}' \in C_2$, or $J(\boldsymbol{x}') \prec J(\boldsymbol{x})$ for every $\boldsymbol{x} \in C_1$ and $\boldsymbol{x}' \in C_2$. It follows that the image

$g([x, y_1, \dots, y_r])$ of a cylinder set is a stream arc, which we denote by $\mathcal{A}_{x,y_1,\dots,y_r}$ and call a *level-r cylinder arc*. The stream arc $\mathcal{A}_x$ is a union of these level-r cylinder arcs, which intersect only at their endpoints.

By Lemma 3.21 the arc $\mathcal{A}_{x,y_1,\dots,y_r}$ has measure $\varphi(y_r)/\lambda^r \geq k/\lambda^r = \delta$, so that any stream arc of measure less than δ is contained in the union of at most two level-r cylinder arcs (possibly in different fibers of $\hat{I}$). Now the metric diameter of a level-r cylinder set is bounded above by $\sum_{i=r+1}^{\infty} |b-a|/2^i < 1/2^r < \xi$, so that the metric diameter of a level-r cylinder arc is bounded above by $\epsilon/2$. Therefore the endpoints $[u, v]$ of a stream arc contained in the union of at most two level-r cylinder arcs satisfy $\rho(u, v) < \epsilon$ as required. □

5.3. *The unstable turbulation.* The streamlines of the unstable turbulation will be g-images of the globally leaf regular path components (Definition 3.29) of $\hat{I}$. We will omit the countably many path components which are involved in identifications — that is, which contain points $\boldsymbol{x}$ for which $g^{-1}\big(\{g(\boldsymbol{x})\}\big) \neq \{\boldsymbol{x}\}$.

Consider the identifications of type (EI) from Lemma 5.4. By Lemma 4.5(d), $\boldsymbol{e} : S \to \hat{I}$ is discontinuous at $y \in S$ if and only if $y \in \operatorname{orb}(f(a)_\ell, \Theta)$. Since there is exactly one point of this orbit in $(\hat{a}_u, a)$, namely $\Theta^n(a) = z_u$, the set of points $\{\boldsymbol{e}(x_u) : x \in (a, \hat{a})\}$ is contained in at most two path components of $\hat{I}$. The union Γ_1 of the bi-infinite $\hat{f}$-orbits of these path components is therefore an $\hat{f}$-invariant set which contains all of the points of $\hat{I}$ involved in identifications of type (EI).

Likewise, the union Γ_2 of the bi-infinite $\hat{f}$-orbits of the path components of $\boldsymbol{e}(\hat{z}_u)$, $\boldsymbol{e}(a)$, and $\boldsymbol{e}(\hat{a}_u)$ is an $\hat{f}$-invariant set which contains all points involved in identifications of type (EII); and the union Γ_3 of the path components of the points of $\boldsymbol{P}$ is $\hat{f}$-invariant and contains all points involved in identifications of type (EIII).

Let

$$K^u = \{\boldsymbol{x} \in \hat{I} : \boldsymbol{x} \text{ is globally leaf regular}\} \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3). \tag{5}$$

Then K^u is an $\hat{f}$-invariant union of globally leaf regular path components of $\hat{I}$ with $g^{-1}\big(\{g(\boldsymbol{x})\}\big) = \{\boldsymbol{x}\}$ for all $\boldsymbol{x} \in K^u$.

Since $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ contains only countably many of the path components of $\hat{I}$, each of which has $\hat{\mu}$-measure zero by Remark 3.32, it follows from Theorem 3.30 that $\hat{\mu}(K^u) = \hat{\mu}(\hat{I})$.

Let Γ be a path component of K^u. Since Γ is an immersed line and g is injective on K^u, its image $\ell = g(\Gamma)$ is also an immersed line. Moreover, ℓ is dense in Σ, since Γ is dense in $\hat{I}$ and g is continuous and surjective. These dense lines ℓ will be the streamlines of the unstable turbulation.

We define an OU measure ν^u_ℓ on ℓ using the 0-flat decomposition $\Gamma = \bigcup_{i \in \mathbb{Z}} \Gamma_i$ of Lemma 3.31: if A is a Borel subset of ℓ, then

$$\nu^u_\ell(A) = \sum_{i \in \mathbb{Z}} m\big(\pi_0(g^{-1}(A) \cap \Gamma_i)\big), \tag{6}$$

where m is Lebesgue measure on I.

Every closed stream arc in ℓ is contained in a finite union of the arcs $g(\Gamma_i)$ coming from the 0-flat decomposition of Γ, and so has finite measure by (6).

Definition 5.17 $(\mathcal{T}^u, \nu^u)$. We define the unstable turbulation $(\mathcal{T}^u, \nu^u)$ to be the measured turbulation on Σ whose streamlines are the g-images of the path components of K^u, with measure ν^u_ℓ on the streamline ℓ.

Lemma 5.18.
$$F(\mathcal{T}^u, \nu^u) = (\mathcal{T}^u, \lambda^{-1}\nu^u).$$

Proof. Let $\ell = g(\Gamma)$ be a streamline of $\mathcal{T}^u$. Then $F(\ell) = g(\hat{f}(\Gamma))$ is also a streamline.

Let A be a Borel subset of ℓ. Write $\Gamma = \bigcup_{i\in\mathbb{Z}} \Gamma_i$ for the 0-flat decomposition of Lemma 3.31, and for each $i \in \mathbb{Z}$ define measurable subsets of A by

$$A_i^L = \{x \in A : g^{-1}(x) \in \Gamma_i \text{ and } \pi_0(g^{-1}(x)) < c\},$$
$$A_i^R = \{x \in A : g^{-1}(x) \in \Gamma_i \text{ and } \pi_0(g^{-1}(x)) > c\}.$$

It is enough to show that $\nu^u_{F(\ell)}(F(A_i^Q)) = \lambda\nu^u_\ell(A_i^Q)$ for each i and each $Q \in \{L, R\}$.

Now $\nu^u_\ell(A_i^Q) = m\big(\pi_0(g^{-1}(A_i^Q))\big)$. Since $g^{-1}(A_i^Q)$ is contained in a 0-flat arc disjoint from $[c]$, its image $\hat{f}(g^{-1}(A_i^Q)) = g^{-1}(F(A_i^Q))$ is contained in a 0-flat arc. Therefore, if $\bigcup_{i\in\mathbb{Z}} \delta_i$ is the 0-flat decomposition of $g^{-1}(F(\ell))$, there is some j for which $g^{-1}(F(A_i^Q)) \subset \delta_j$. It follows that

$$\begin{aligned}
\nu^u_{F(\ell)}(F(A_i^Q)) &= m\Big(\pi_0\big(g^{-1}(F(A_i^Q))\big)\Big)\\
&= m\Big(\pi_0\big(\hat{f}(g^{-1}(A_i^Q))\big)\Big)\\
&= m\Big(f\big(\pi_0(g^{-1}(A_i^Q))\big)\Big)\\
&= \lambda m\big(\pi_0(g^{-1}(A_i^Q))\big)\\
&= \lambda\nu^u_\ell(A_i^Q)
\end{aligned}$$

as required. Here the fourth equality follows from the fact that $\pi_0(g^{-1}(A_i^Q))$ is contained in $[a, c]$ or in $[c, b]$, according to whether $Q = L$ or $Q = R$, on which f expands uniformly by a factor λ. ☐

Therefore $(\mathcal{T}^u, \nu^u)$ is a measured turbulation with dense streamlines, which satisfies $F(\mathcal{T}^u, \nu^u) = (\mathcal{T}^u, \lambda^{-1}\nu^u)$. It remains to show that it is tame (Definition 2.13).

Lemma 5.19. *$(\mathcal{T}^u, \nu^u)$ is tame.*

Proof. If Γ is a 0-flat arc in $\hat{I}$ over the interval J, then $m(\pi_0(\Gamma)) = |J|$, while the metric diameter of Γ is $\frac{2\lambda}{2\lambda-1}|J| < 2|J|$ (see, for example, Section 4.1 of [11]).

Let ρ be a metric on Σ inducing its topology and let $\epsilon > 0$. Pick $\xi > 0$ small enough that $d(\boldsymbol{x}, \boldsymbol{y}) < \xi$ implies $\rho(g(\boldsymbol{x}), g(\boldsymbol{y})) < \epsilon$.

Let $\delta = \xi/2$, and consider a stream arc $[x, y] \subset \ell$ with $\nu^u_\ell([x, y]) < \delta$. Then $g^{-1}([x, y])$ is a union of finitely many 0-flat arcs which intersect only at their endpoints, arising from the 0-flat decomposition of $g^{-1}(\ell)$. By (6) and the remarks above, the metric diameter of $g^{-1}([x, y])$ is less than $2\delta = \xi$, so that $\rho(x, y) < \epsilon$ as required. ☐

5.4. ***F is a measurable pseudo-Anosov map.*** We have defined tame measured turbulations $(\mathcal{T}^s, \nu^s)$ and $(\mathcal{T}^u, \nu^u)$ on Σ, having dense streamlines, with the properties that $F(\mathcal{T}^s, \nu^s) = (\mathcal{T}^s, \lambda\nu^s)$ and

$F(\mathcal{T}^u, \nu^u) = (\mathcal{T}^u, \lambda^{-1}\nu^u)$. In this section we show that these turbulations satisfy the remaining conditions of Definitions 2.22, thereby establishing that $F : \Sigma \to \Sigma$ is measurable pseudo-Anosov. That is,

- the two turbulations are transverse (Lemma 5.20); and
- the turbulations are full: there is a countable collection of compatible tartans whose intersections cover a full measure subset of Σ, and every nonempty open subset of Σ contains a positive measure compatible tartan (Lemma 5.21).

Lemma 5.20. *Every intersection of $\mathcal{T}^s$ and $\mathcal{T}^u$ is topologically transverse.*

Proof. Let $g(\boldsymbol{x})$ be a point of intersection of a streamline $\ell = g(\Gamma)$ of $\mathcal{T}^u$, and a streamline ℓ' of $\mathcal{T}^s$. The streamline ℓ' is locally contained in the arc $\mathcal{A}_{x_0}$ for some $x_0 \in V$ ($g(\boldsymbol{x})$ cannot be an endpoint of $\mathcal{A}_{x_0}$, since points of ℓ have a unique g-preimage). The arc $\mathcal{A}_{x_0}$ separates any sufficiently small disk neighborhood of $g(\boldsymbol{x})$ into points $g(\boldsymbol{y})$ with $y_0 < x_0$, and points $g(\boldsymbol{y})$ with $y_0 > x_0$. It is therefore enough to show that $\pi_0|_\Gamma$ is locally injective at $\boldsymbol{x}$. This is a consequence of Lemma 3.31: since $x_0 \notin \mathrm{GO}(c)$, we have $x_r \neq c$ for all r, and hence $\boldsymbol{x}$ is in the interior of one of the components Γ_i of the 0-flat decomposition of Γ by Remark 3.24. ☐

Lemma 5.21. *The pair $(\mathcal{T}^s, \nu^s)$, $(\mathcal{T}^u, \nu^u)$ is full.*

Proof. We will prove the existence of a positive measure regular compatible tartan R. It follows from Lemma 2.23 and Remark 2.21 that $F^i(R)$ is a regular compatible tartan for each $i \geq 0$. Since F is ergodic this countable collection of tartans has $\mu\left(\bigcup_{i\geq 0} F^i(R)^{\pitchfork}\right) = 1$, and fullness follows from Lemma 2.20.

By Theorem 3.26 there is a 0-box B with $\hat{\mu}(B) > 0$: recall that B is a disjoint union of 0-flat arcs over some interval $J \subseteq I$. Without loss of generality we can assume that the arcs of B are all contained in the full measure subset K^u of (5), so that their images under g are contained in streamlines of $\mathcal{T}^u$, and g is injective on B.

Recall (Notation 5.9) that

$$V = I \setminus (\mathrm{GO}(c) \cup \mathrm{GO}(P)),$$

and write $J' = J \cap V$, which has countable complement in J. Let $R^u = g(B)$, a disjoint union of unstable stream arcs, each with finite stream measure $m(J)$; and let $R^s = \bigcup_{y\in J'} \mathcal{A}_y$ (recall Notation 5.7), a disjoint union of stable stream arcs with bounded stream measures $\nu^s_{\ell_y}(\mathcal{A}_y) = \varphi(y)$.

By definition, π_0 is a bijection from each arc of B onto J, so that each of these arcs intersects each fiber $[y]$ (with $y \in J$) exactly once. Since $g^{-1}(x)$ is a point for each $x \in R^u$, it follows that every arc of R^s intersects every arc of R^u exactly once, and these intersections are either transverse or at endpoints by Lemma 5.20. Each unstable fiber $\mathfrak{u}$ can be oriented in the direction of increasing $\pi_0 \circ g^{-1} : \mathfrak{u} \to J$, and with this orientation crosses the stable fibers $\mathcal{A}_y$ in order of increasing y. Likewise, each stable fiber can be oriented in the direction of increasing itinerary, and with this orientation crosses the unstable fibers in order of increasing itinerary (see Remark 3.24).

The open topological disk $\Sigma \setminus \{g(\boldsymbol{e}(c^u))\}$ contains $R^s \cup R^u$, so $R = (R^s, R^u)$ is a tartan (Definitions 2.15). It has positive measure by Theorem 3.28:

$$\mu(R^{\pitchfork}) = \hat{\mu}(B \cap \pi_0^{-1}(J')) = \int_{J'} \alpha_x(B)\, dm(x) = \int_J \alpha_x(B)\, dm(x) = \hat{\mu}(B) > 0.$$

It thus remains to show that R is compatible and regular.

Given $y, z \in J'$, there is a holonomy map $h_{y,z} : [y] \cap B \to [z] \cap B$, which takes the point of $[y] \cap \Gamma$ to the point of $[z] \cap \Gamma$ for each 0-flat arc Γ of B. That is, since each 0-flat arc has constant itinerary (Remark 3.24), $h_{y,z}(\boldsymbol{x}) = L(z, J(\boldsymbol{x}))$, where $L : \mathcal{V} \to \hat{I}$ is the continuous function of Lemma 3.19. In particular, $h_{y,z}$ is a homeomorphism (with inverse $h_{z,y}$). Moreover, if $X \subseteq [y] \cap B$ is Borel, then

$$\alpha_y(X) = \alpha_z(h_{y,z}(X)), \tag{7}$$

by applying Theorem 3.27 to the 0-box consisting of those 0-flat arcs of B which pass through X.

Now pick arbitrary stable and unstable fibers $\mathfrak{s} = \mathcal{A}_y = g([y])$ and $\mathfrak{u} = g(\Gamma)$ of R, where $y \in J'$ and Γ is one of the 0-flat arcs of B. Define

$$\psi : ([y] \cap B) \times (\Gamma \cap \pi_0^{-1}(J')) \to B \cap \pi_0^{-1}(J)$$

by $\psi(\boldsymbol{x}, \boldsymbol{x}') = h_{y,\pi_0(\boldsymbol{x}')}(\boldsymbol{x}) = L(\pi_0(\boldsymbol{x}'), J(\boldsymbol{x}))$, which is continuous by Lemma 3.19 and Remark 3.20, and so a homeomorphism with inverse $\psi^{-1}(\boldsymbol{x}) = \big(L(y, J(\boldsymbol{x})), L(\pi_0(\boldsymbol{x}), s)\big)$, where s is the constant itinerary of Γ. The definition is made in order that

$$\begin{tikzcd} ([y] \cap B) \times (\Gamma \cap \pi_0^{-1}(J')) \arrow[r, "\psi"] \arrow[d, "G \times G"] & B \cap \pi_0^{-1}(J) \arrow[d, "G"] \\ E^{\mathfrak{s}} \times E^{\mathfrak{u}} \arrow[r, "\psi^{\mathfrak{s},\mathfrak{u}}"] & R^{\pitchfork} \end{tikzcd}$$

commutes, where we have written $G = g|_B$, a homeomorphism by Lemma 5.8. Therefore $\psi^{\mathfrak{s},\mathfrak{u}}$ is a homeomorphism, and in particular bimeasurable. Since $G_*\hat{\mu} = \mu$ and $(G \times G)_*(\alpha_y \times M) = \nu_{\mathfrak{s},\mathfrak{u}}$ by Definition 5.11 and (6) (where we have written $M = (\pi_0|_\Gamma^{-1})_* m$), it suffices for compatibility to show that $\psi_*(\alpha_y \times M) = \hat{\mu}$.

Let E be a Borel subset of $B \cap \pi_0^{-1}(J')$. Then

$$\begin{aligned} (\alpha_y \times M)(\psi^{-1}(E)) &= \int_{\Gamma \cap \pi_0^{-1}(J)} \alpha_y\big(\{\boldsymbol{x} \in [y] \cap B : \psi(\boldsymbol{x}, z) \in E\}\big)\, dM(z) \\ &= \int_{J'} \alpha_y\big(\{\boldsymbol{x} \in [y] \cap B : h_{y,z}(\boldsymbol{x}) \in E\}\big)\, dm(z) \\ &= \int_{J'} \alpha_y\big(h_{z,y}([z] \cap E)\big)\, dm(z) \\ &= \int_{J'} \alpha_z(E)\, dm(z) = \hat{\mu}(E) \end{aligned}$$

as required, where we have used, in turn, Fubini's theorem, the definition of ψ, $h_{z,y} = h_{y,z}^{-1}$, (7), and Theorem 3.28.

For regularity, let $x = g(\boldsymbol{x}) \in R^{\pitchfork}$ and U be a neighborhood of x. Write $V = g^{-1}(U)$, a neighborhood of $\boldsymbol{x}$. By Lemma 3.19, $W = L^{-1}(V)$ is a neighborhood of $(x_0, J(\boldsymbol{x}))$ in $\{(x, s) \in I \times \{0, 1\}^{\mathbb{N}} : s \in \mathcal{K}_x\}$: that is, there is some $\epsilon > 0$ and $r \in \mathbb{N}$ such that if $\boldsymbol{x}' \in \hat{I}$ has $|x_0' - x_0| < \epsilon$, and $J(\boldsymbol{x})_i = J(\boldsymbol{x}')_i$ for $0 \le i \le r$, then $\boldsymbol{x}' \in V$.

Since J is continuous on $[x_0]$ by Remark 3.20, there is some $\eta > 0$ such that if $\boldsymbol{y} \in [x_0]$ and $\alpha_{x_0}\big(\{\boldsymbol{z} \in [x_0] : J(\boldsymbol{z}) \text{ is between } J(\boldsymbol{x}) \text{ and } J(\boldsymbol{y})\}\big) < \eta$, then $J(\boldsymbol{y})_i = J(\boldsymbol{x})_i$ for $0 \le i \le r$.

Taking $\delta = \min(\epsilon, \eta)$, it follows that if $y \in \mathfrak{s}(x) \cap R^{\pitchfork}$ and $z \in \mathfrak{u}(x) \cap R^{\pitchfork}$ with $\nu_{\mathfrak{s}(x)}([x, y]_s) < \delta$ and $\nu_{\mathfrak{u}(x)}([x, z]_u) < \delta$, then the stream arcs $[x, y]_s$, $[y, \mathfrak{s}(z) \pitchfork \mathfrak{u}(y)]_u$, $[\mathfrak{s}(z) \pitchfork \mathfrak{u}(y), z]_s$, and $[z, x]_u$ are all contained in U. This establishes regularity of R. □

This completes the proof that $F : \Sigma \to \Sigma$ is a measurable pseudo-Anosov map whenever f has rational height and is not postcritically finite.

6. The irrational case

In this section we assume that f is of irrational type. We will show that the corresponding sphere homeomorphism $F : \Sigma \to \Sigma$ is a generalized pseudo-Anosov map, having a single bi-infinite orbit of 1-pronged singularities which is homoclinic to a point $\infty = g(\mathcal{C})$ of Σ, where $\mathcal{C}$ is a Cantor set in $\hat{I}$.

6.1. *The semiconjugacy $g : \hat{I} \to \Sigma$.* Here we describe the identifications on $\hat{I}$ induced by the semiconjugacy $g : \hat{I} \to \Sigma$.

Recall from Theorem 4.9 that $\operatorname{orb}(f(a)_\ell, \Theta)$ is disjoint from γ, and that

$$\mathcal{C}_S = \overline{\operatorname{orb}(f(a)_\ell, \Theta)} = S \setminus \bigcup_{r \ge 0} \Theta^{-r}(\mathring{\gamma})$$

is a Cantor set which contains a and $\hat{a}_u$. The following lemma is a translation of Remark 5.17(a) of [12] into the language of this paper.

Lemma 6.1. *The semiconjugacy $g : \hat{I} \to \Sigma$ is the quotient map of the equivalence relation on $\hat{I}$ with the nontrivial equivalence classes*

(EI) $\left\{\hat{f}^r(\boldsymbol{e}(x_u)), \hat{f}^r(\boldsymbol{e}(\hat{x}_u))\right\}$ *for each $x \in (a, c)$ and each $r \in \mathbb{Z}$; and*

(EII) *the subset $\mathcal{C} = \overline{\boldsymbol{e}(\mathcal{C}_S)}$ of $\hat{I}$, which is a Cantor set.*

We write $\infty = g(\mathcal{C}) \in \Sigma$.

6.2. *The equivalence class $\mathcal{C}$.* We start by investigating the infinite equivalence class $\mathcal{C} = \overline{\boldsymbol{e}(\mathcal{C}_S)}$ of (EII). The key observation is that, although a is clearly an endpoint of the Θ-invariant Cantor set $\mathcal{C}_S$, its image $\Theta(a) = f(a)_\ell$ is not (Lemma 6.3): this is because Θ collapses the gap $\mathring{\gamma}$ adjacent to a. Since $\boldsymbol{e}$ is discontinuous along the orbit of $f(a)_\ell$ by Lemma 4.5(d), the image $\boldsymbol{e}(\mathcal{C}_S)$ is not compact: taking its closure can be seen (Theorem 6.4) as adding a gap adjacent to each $\Theta^r(f(a)_\ell)$, whose other endpoint corresponds to the nonextreme element $\hat{f}^{r+1}(\boldsymbol{e}(\hat{a}_u))$ of $\hat{I}$.

Notation 6.2 ($\xrightarrow{+}$, $\xrightarrow{-}$). If (y_i) is a sequence in S, we will write $y_i \xrightarrow{+} y$ (respectively $y_i \xrightarrow{-} y$) if $y_i \to y$ in the positive (respectively negative) direction: that is, if for every $z \in S$ distinct from y we have $y_i \in [z, y]$ (respectively $y_i \in [y, z]$) for sufficiently large i.

Lemma 6.3. *For every $r \geq 0$, there are sequences (y_i) and (y_i') in $\mathcal{C}_S$ with $y_i \xrightarrow{+} \Theta^r(f(a)_\ell)$ and $y_i' \xrightarrow{-} \Theta^r(f(a)_\ell)$.*

Proof. Since $a \in \mathcal{C}_S = \overline{\operatorname{orb}(f(a)_\ell, \Theta)}$ and $\operatorname{orb}(f(a)_\ell, \Theta) \cap \gamma = \varnothing$ by Theorem 4.9, there is a sequence (y_i) in $\operatorname{orb}(f(a)_\ell, \Theta) \subset \mathcal{C}_S$ with $y_i \xrightarrow{-} a$. Applying Θ^{r+1}, and using the fact that Θ is defined and continuous in a neighborhood of each point of $\operatorname{orb}(f(a)_\ell, \Theta)$ gives the required sequences converging negatively to points of this orbit. By an identical argument, there is a sequence (y_i') in $\mathcal{C}_S$ with $y_i' \xrightarrow{+} \hat{a}_u$, which gives the required sequences converging positively to points of $\operatorname{orb}(f(a)_\ell, \Theta)$. □

Theorem 6.4. *The Cantor set $\mathcal{C}$ is $\hat{f}$-invariant, and is given by*

$$\mathcal{C} = \boldsymbol{e}(\mathcal{C}_S) \cup \left\{ \hat{f}^r(\boldsymbol{e}(\hat{a}_u)) : r \geq 1 \right\}. \tag{8}$$

Moreover $\operatorname{orb}(\boldsymbol{e}(f(a)_\ell), \hat{f})$ *is dense in $\mathcal{C}$.*

Proof. $\boldsymbol{e}(\mathcal{C}_S) \subset \mathcal{C}$, so for (8) we need to show that $\mathcal{C} \setminus \boldsymbol{e}(\mathcal{C}_s) = \left\{ \hat{f}^r(\boldsymbol{e}(\hat{a}_u)) : r \geq 1 \right\}$.

Suppose that $\boldsymbol{x} \in \mathcal{C}$, so that there is a sequence (y_i) in $\mathcal{C}_S$ with $\boldsymbol{e}(y_i) \to \boldsymbol{x}$. Taking a subsequence if necessary, assume $y_i \to y \in \mathcal{C}_S$. By Lemma 4.5(d), $\boldsymbol{e}$ is discontinuous at y if and only if $y \in \operatorname{orb}(f(a)_\ell, \Theta)$; and even at points of this orbit, $\boldsymbol{e}$ is continuous as we approach y in the negative direction. Therefore, if either $y \notin \operatorname{orb}(f(a)_\ell, \Theta)$, or there is a subsequence $y_{r_i} \xrightarrow{-} y$, then $\boldsymbol{x} = \boldsymbol{e}(y) \in \boldsymbol{e}(\mathcal{C}_S)$. On the other hand, by Lemma 6.3, there is for each $r \geq 1$ a sequence (y_i) in $\mathcal{C}_S$ with $y_i \xrightarrow{+} \Theta^{r-1}(f(a)_\ell)$. Hence $\mathcal{C} \setminus \boldsymbol{e}(\mathcal{C}_s)$ is exactly the set of limits of $(\boldsymbol{e}(y_i))$ for such sequences (y_i).

Suppose then that $y_i \xrightarrow{+} \Theta^{r-1}(f(a)_\ell)$. Now $\boldsymbol{e}(y_i) = \langle \tau(y_i), \tau(\Theta^{-1}(y_i)), \dots \rangle$ by Lemma 4.5(c). Therefore

$$\begin{aligned}
\boldsymbol{e}(y_i) &\to \left\langle \tau\left(\Theta^{r-1}(f(a)_\ell)\right), \tau\left(\Theta^{r-2}(f(a)_\ell)\right), \dots, \tau(f(a)_\ell), \tau(\hat{a}_u), \tau(\Theta^{-1}(\hat{a}_u)), \dots \right\rangle \\
&= \left\langle f^r(a), f^{r-1}(a), \dots, f(a), \tau(\hat{a}_u), \tau(\Theta^{-1}(\hat{a}_u)), \dots \right\rangle \\
&= \hat{f}^r\left(\left\langle \tau(\hat{a}_u), \tau(\Theta^{-1}(\hat{a}_u)), \dots \right\rangle\right) \\
&= \hat{f}^r(\boldsymbol{e}(\hat{a}_u)),
\end{aligned}$$

where the first equality uses Lemma 4.5(a), the third uses Lemma 4.5(c), and the appearance of $\hat{a}_u$ rather than a in the first line is because y_i converges in the positive direction. This establishes (8).

Since $\hat{f}^r\left(\boldsymbol{e}(f(a)_\ell)\right) = \boldsymbol{e}\left(\Theta^r(f(a)_\ell)\right) \in \boldsymbol{e}(\mathcal{C}_S)$ by (4), we have $\operatorname{orb}\left(\boldsymbol{e}(f(a)_\ell), \hat{f}\right) \subset \mathcal{C}$. We finish by showing that this orbit is dense in $\mathcal{C}$ so that, in particular, $\mathcal{C}$ is $\hat{f}$-invariant.

It is enough to show denseness in $\boldsymbol{e}(\mathcal{C}_S)$. Since $\mathcal{C}_S = \overline{\operatorname{orb}(f(a)_\ell, \Theta)}$, every $y \in \mathcal{C}_S$ either lies on $\operatorname{orb}(f(a)_\ell, \Theta)$, or is the limit of a subsequence $\Theta^{r_i}(f(a)_\ell) \to y$ of this orbit. In the former case we have $\boldsymbol{e}(y) = \hat{f}^r\left(\boldsymbol{e}(f(a)_\ell)\right)$ for some r by (4); and in the latter case, $\hat{f}^{r_i}\left(\boldsymbol{e}(f(a)_\ell)\right) = \boldsymbol{e}\left(\Theta^{r_i}(f(a)_\ell)\right) \to \boldsymbol{e}(y)$ by (4) and Lemma 4.5(d). □

We finish the section with some key facts about the dynamics of $\hat{f}$ as it pertains to $\mathcal{C}$, and the identifications on the extreme points of $\hat{I}$.

Lemma 6.5. (a) *The ω-limit set of the critical fiber $[c]$ is equal to $\mathcal{C}$.*

(b) *The α-limit set of the set of extreme elements of $\hat{I}$ has $\alpha(\boldsymbol{e}(S), \hat{f}) \subset \mathcal{C}$.*

(c) *The $\hat{f}$-orbit of $\boldsymbol{e}(c_u)$ is homoclinic to $\mathcal{C}$, and the F-orbit of $g(\boldsymbol{e}(c_u))$ is homoclinic to ∞.*

(d) *There is no $x \in (a, b)$ for which both $\boldsymbol{e}(x_u)$ and $\boldsymbol{e}(x_\ell)$ lie in $\mathcal{C}$.*

(e) *If $x \in (a, c)$ then $g(\boldsymbol{e}(x_\ell)) \neq g(\boldsymbol{e}(\hat{x}_\ell))$.*

Proof. (a) Note first that $\omega([c], \hat{f}) = \omega(\boldsymbol{e}(c_\ell), \hat{f})$, as $\operatorname{diam}\big(\hat{f}^r([c])\big) = \operatorname{diam}([c])/2^r$. Now the first points of the Θ-orbit of c_ℓ are $c_\ell \mapsto b \mapsto a \mapsto f(a)_\ell$, which are contained in $[a, \hat{a}_u)$, so that $\hat{f}^3(\boldsymbol{e}(c_\ell)) = \boldsymbol{e}(f(a)_\ell)$ by (4). The result follows since the $\hat{f}$-orbit of $\boldsymbol{e}(f(a)_\ell)$ is a dense subset of the Cantor set $\mathcal{C}$ by Theorem 6.4.

(b) Suppose that $\boldsymbol{x} \in \alpha(\boldsymbol{e}(S), \hat{f})$, so that there is some $y \in S$ and a sequence $r_i \to \infty$ with $\hat{f}^{-r_i}(\boldsymbol{e}(y)) = \boldsymbol{e}(\Theta^{-r_i}(y)) \to \boldsymbol{x}$. Assume, taking a subsequence if necessary, that $\Theta^{-r_i}(y) \to y^* \in S$. Then $y^* \in \mathcal{C}_S$, since otherwise we would have $\Theta^r(y^*) \in \mathring{\gamma}$ for some $r \geq 0$, so that $\Theta^r(\Theta^{-r_i}(y)) \in \mathring{\gamma}$ for sufficiently large i, a contradiction as $\mathring{\gamma}$ is disjoint from the range of Θ^{-1}.

If $y^* \notin \operatorname{orb}(f(a)_\ell, \Theta)$ then $\boldsymbol{x} = \boldsymbol{e}(y^*)$ by Lemma 4.5(d); while if $y^* = \Theta^r(f(a)_\ell)$, then taking a subsequence we can assume that either $\Theta^{-r_i}(y) \xrightarrow{+} y^*$, so that $\boldsymbol{x} = \hat{f}^{r+1}(\boldsymbol{e}(\hat{a}_u))$; or $\Theta^{-r_i}(y) \xrightarrow{-} y^*$, so that $\boldsymbol{x} = \boldsymbol{e}(y^*)$. In each case we have $\boldsymbol{x} \in \mathcal{C}$ by (8).

(c) This is immediate from parts (a) and (b).

(d) If $x < \hat{a}$ then $x_u \in \mathring{\gamma}$, so that $\boldsymbol{e}(x_u) \notin \mathcal{C}$; while if $x \geq \hat{a}$ then $f(x) \leq f(\hat{a}) = f(a) < \hat{a}$ — since $j(f(a)) = (11)^k 0\ldots$ and $j(\hat{a}) = 1(11)^k 0\ldots$ for some $k \geq 0$ by Remark 3.7 — so that $\Theta(x_l) = f(x)_u \in \mathring{\gamma}$, and $\boldsymbol{e}(x_\ell) \notin \mathcal{C}$.

(e) The argument is similar to that of (d). We have $\Theta(x_\ell) = f(x)_\ell$ and $\Theta(\hat{x}_\ell) = f(x)_u$ by (3), and

- if $f(x) < \hat{a}$ then $f(x)_u \in \mathring{\gamma}$ while $f(x)_\ell \notin \mathring{\gamma}$: therefore $\hat{x}_\ell \notin \mathcal{C}$, and $\{x_\ell, \hat{x}_\ell\} \neq \{\hat{f}^{-r}(z_u), \hat{f}^{-r}(\hat{z}_u)\}$ for any $z \in (a, c)$;
- if $f(x) = \hat{a}$ then $\hat{x}_\ell \in \mathcal{C}$, but $\Theta^2(x_\ell) = f(a)_u \in \mathring{\gamma}$, so $x_\ell \notin \mathcal{C}$; and
- if $f(x) > \hat{a}$ then $f^2(x) < \hat{a}$, and $\Theta^2(x_\ell) = f^2(x)_u \in \mathring{\gamma}$ while $\Theta^2(\hat{x}_\ell) = f^2(x)_\ell \notin \mathring{\gamma}$, so that $g(x_\ell) \neq g(\hat{x}_\ell)$ as in the first case. □

6.3. ***The invariant foliations.*** The generalized pseudo-Anosov F-invariant foliations are constructed in a very similar way to the measurable pseudo-Anosov turbulations in the rational case: leaves of the stable foliation are locally g-images of the fibers $[x]$, with measure coming from $g_*(\alpha_x)$; while leaves of the unstable foliation are g-images of connected components of $\hat{I} \setminus \mathcal{C}$, with measure induced by Lebesgue measure on 0-flat arcs.

These foliations have a single bad singularity at ∞, where uncountably many stable and unstable leaves meet: in the language of Definitions 2.7, we have $Z = \{\infty\}$. They also have 1-pronged singularities along the bi-infinite orbit of $g(\boldsymbol{e}(c_u))$: the leaves emanating from these singularities are finite, terminating at ∞. All other points are regular.

In Section 6.3.1 we construct the stable measured foliation $(\mathcal{F}^s, \nu^s)$, and show that $F(\mathcal{F}^s, \nu^s) = (\mathcal{F}^s, \lambda\nu^s)$; we then deal with the unstable foliation in Section 6.3.2. Finally, in Section 6.3.3, we exhibit the charts required by Definitions 2.7 to establish that these are generalized pseudo-Anosov foliations, completing the proof that $F : \Sigma \to \Sigma$ is a generalized pseudo-Anosov map.

Remark 6.6. It will be helpful to summarize the essential features of the identifications described in Lemma 6.1.

(a) For $r \leq 0$, (EI) identifies the two extreme points $\boldsymbol{e}(\Theta^r(x_u))$ and $\boldsymbol{e}(\Theta^r(\hat{x}_u))$.

(b) For $r > 0$, (EI) identifies two nonextreme points of the same fiber $[f^r(x)]$.

(c) (EII) collapses to ∞ an uncountable collection of extreme points of distinct (Lemma 6.5(d)) fibers together with nonextreme points $\hat{f}^r(\boldsymbol{e}(\hat{a}_u))$ of the postcritical fibers $[f^r(a)]$ for $r \geq 1$.

6.3.1. *The stable foliation.* In the rational case, the images in Σ of fibers above the orbit of c are in general not arcs, because of the 3-element identifications in Lemma 5.4. By contrast, in the irrational case, where there are no such identifications, *every* π_0-fiber yields an arc $\mathcal{A}_x = g([x])$. Fibers above the postcritical set still need to be treated by a separate argument, since the itinerary, and hence the notion of consecutive points, is not well defined on such fibers, but this argument is a straightforward induction. The reader should note carefully in Theorem 6.7 that, although $\boldsymbol{e}\big(\Theta^r(f(a)_\ell)\big)$ is by definition an extreme element of $[f^{r+1}(a)]$ for $r \geq 0$, the point $\infty = g\big(\boldsymbol{e}\big(\Theta^r(f(a)_\ell)\big)\big)$ is not an endpoint of the arc $g\big([f^{r+1}(a)]\big)$. This situation arises because there is no natural order on $[f^{r+1}(a)]$, due to the lack of well-defined itineraries.

Theorem 6.7. *Let $x \in I$. Then $\mathcal{A}_x := g([x])$ is an arc in Σ. Moreover*:

(a) *The endpoints of $\mathcal{A}_x$ are $g(\boldsymbol{e}(x_\ell))$ and $g(\boldsymbol{e}(x_u))$, unless $x \in \operatorname{orb}(b, f)$, in which case, writing $x = f^r(c)$ for $r \geq 1$, one endpoint of $\mathcal{A}_x$ is $g\big(\hat{f}^r(\boldsymbol{e}(c_u))\big)$ and the other is either $g(\boldsymbol{e}(x_\ell))$ or $g(\boldsymbol{e}(x_u))$.*

(b) *$\infty = g(\mathcal{C})$ is an interior point of $\mathcal{A}_x$ if and only if $x \in \operatorname{orb}(f(a), f)$.*

Proof. Suppose first that $x \notin \operatorname{orb}(b, f)$. If $\boldsymbol{x}$ and $\boldsymbol{x}'$ are distinct elements of $[x]$, then $g(\boldsymbol{x}) = g(\boldsymbol{x}')$ if and only if $\boldsymbol{x}$ and $\boldsymbol{x}'$ are consecutive. The proof of this is essentially identical to that of Lemma 5.5(b), the only additional point to note being that if $g(\boldsymbol{x}) = g(\boldsymbol{x}')$ then, by Remark 6.6(c), it is not possible that $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{C}$.

That $\mathcal{A}_x$ is an arc for $x \notin \operatorname{orb}(b, f)$, with endpoints $g(\boldsymbol{e}(x_u))$ and $g(\boldsymbol{e}(x_\ell))$, follows exactly as in the proof of Theorem 5.6. ∞ is not an interior point of $\mathcal{A}_x$ by Remark 6.6(c).

Since $\mathcal{A}_b = F(\mathcal{A}_c)$ is a homeomorphic image of $\mathcal{A}_c$, it is also an arc, with endpoints $F\big(g(\boldsymbol{e}(c_\ell))\big) = g\big(\hat{f}(\boldsymbol{e}(c_\ell))\big) = g(\boldsymbol{e}(b)) = \infty$ and $F\big(g(\boldsymbol{e}(c_u))\big) = g\big(\hat{f}(\boldsymbol{e}(c_u))\big)$. (That $g(\boldsymbol{e}(b)) = \infty$ follows because $\Theta^2(b) = f(a)_\ell$, so $\operatorname{orb}(b, \Theta)$ is disjoint from $\mathring{\gamma}$: that is, $b \in \mathcal{C}_S$, so $\boldsymbol{e}(b) \in \mathcal{C}$.) Likewise $\mathcal{A}_a = F(\mathcal{A}_b)$ is an arc with endpoints ∞ and $g\big(\hat{f}^2(\boldsymbol{e}(c_u))\big)$.

The result for $x \in \operatorname{orb}(f(a), f) = \{f^r(a) : r \geq 1\}$ follows by induction on r. For the base case $r = 1$, we have that $\mathcal{A}_{f(a)} = F(\mathcal{A}_a) \cup F(\mathcal{A}_{\hat{a}}) = F(\mathcal{A}_a \cup \mathcal{A}_{\hat{a}})$ is the union of the F-images of the two arcs $\mathcal{A}_a$ and

$\mathcal{A}_{\hat{a}}$ (the latter is an arc as $\hat{a} \notin \mathrm{orb}(b, f)$, since $f(\hat{a}) = f(a)$ and f is injective on $\mathrm{orb}(b, f)$). Now $\mathcal{A}_a$ and $\mathcal{A}_{\hat{a}}$ intersect at their endpoints $g(\boldsymbol{e}(a)) = g(\boldsymbol{e}(\hat{a}_u)) = \infty$. They have no other points of intersection by Remark 6.6 (noting that $[a]$ has only one extreme point $\boldsymbol{e}(a)$). Therefore $\mathcal{A}_{f(a)}$ is an arc with ∞ in its interior, with endpoints $g\big(\hat{f}^3(\boldsymbol{e}(c_u))\big)$ and $g\big(\hat{f}(\boldsymbol{e}(\hat{a}_\ell))\big) = g\big(\boldsymbol{e}(f(a)_u)\big)$ as required.

Suppose then that $r > 1$.

If $f^r(a) < f(a)$ then, by the inductive hypothesis, $\mathcal{A}_{f^r(a)} = F(\mathcal{A}_{f^{r-1}(a)})$ is an arc with ∞ in its interior. Since $f^r(a) < f(a)$, both extremes of $[f^{r-1}(a)]$ map to extremes of $[f^r(a)]$, so that the endpoints of $\mathcal{A}_{f^r(a)}$ are $F\big(g\big(\boldsymbol{e}(f^{r-1}(a)_{u/\ell})\big)\big) = g\big(\boldsymbol{e}(f^r(a)_{\ell/u})\big)$ and $F\big(g\big(\hat{f}^{r-1}(\boldsymbol{e}(c_u))\big)\big) = g\big(\hat{f}^r(\boldsymbol{e}(c_u))\big)$ as required.

On the other hand, if $f(a) < f^r(a) < b$ then $\mathcal{A}_{f^r(a)}$ is the union of $F(\mathcal{A}_{f^{r-1}(a)})$ and $F(\mathcal{A}_{\widehat{f^{r-1}(a)}})$. The former is an arc with ∞ in its interior by the inductive hypothesis, and the latter is also an arc, since $\widehat{f^{r-1}(a)} \notin \mathrm{orb}(b, f)$ as f is injective on $\mathrm{orb}(b, f)$. Now $\mathcal{A}_{f^{r-1}(a)}$ and $\mathcal{A}_{\widehat{f^{r-1}(a)}}$ intersect at their endpoints $g\big(\boldsymbol{e}(f^{r-1}(a)_u)\big) = g\big(\boldsymbol{e}(\widehat{f^{r-1}(a)}_u)\big)$ by (EI) of Lemma 6.1. This is their only intersection: their other endpoints are not identified by Lemma 6.5(e); the identifications (EI) only identify extremes of fibers and points of the same fiber; and $\infty \notin \mathcal{A}_{\widehat{f^{r-1}(a)}}$ (if we had $\widehat{f^{r-1}(a)}_\ell \in \mathcal{C}$ as well as $f^{r-1}(a)_\ell \in \mathcal{C}$, then both extremes of $[f^r(a)]$ would be in $\mathcal{C}$, contradicting Lemma 6.5(d)). Therefore $\mathcal{A}_{f^r(a)}$ is an arc with ∞ in its interior, and with endpoints $F\big(g\big(\boldsymbol{e}(\widehat{f^{r-1}(a)}_\ell)\big)\big) = g\big(\boldsymbol{e}(f^r(a)_{u/\ell})\big)$ and $F\big(g\big(\hat{f}^{r-1}(\boldsymbol{e}(c_u))\big)\big) = g\big(\hat{f}^r(\boldsymbol{e}(c_u))\big)$ as required. $\square$

Definition 6.8 ($\mathcal{I}_x$, $\mathcal{J}_x$). Let $x \in I$.

If $x = f^r(a)$ for some $r \geq 1$, we define $\mathcal{J}_x$ to be the component of $\mathcal{A}_x \setminus \{\infty\}$ which contains $g\big(\hat{f}^r(\boldsymbol{e}(c_u))\big)$, and $\mathcal{I}_x$ to be the other component.

Otherwise, we define $\mathcal{I}_x = \mathcal{A}_x \setminus \{\infty\}$.

Each $\mathcal{I}_x$ and $\mathcal{J}_x$ is therefore homeomorphic to a half-open or closed interval, according to whether $\infty \in \mathcal{A}_x$ or $\infty \notin \mathcal{A}_x$.

There are three types of extreme elements of $\hat{I}$:

- points of $\mathcal{C}$, which are identified with all other points of $\mathcal{C}$;
- points of the form $\hat{f}^r(\boldsymbol{e}(c_u))$ for $r \leq 0$, which are not identified with any other points; and
- points of the form $\hat{f}^r(\boldsymbol{e}(x_u))$ for $r \leq 0$ and $x \in (a, \hat{a}) \setminus \{c\}$, which are identified with $\hat{f}^r(\boldsymbol{e}(\hat{x}_u))$ and no other points.

Therefore each endpoint of each $\mathcal{I}_x$ is either ∞, or a point of the form $\hat{f}^r(\boldsymbol{e}(c_u))$, or is an endpoint of exactly one other $\mathcal{I}_{x'}$.

We define an equivalence relation $\sim$ on I by declaring that $x \sim y$ if and only if there is a finite sequence $x = x_1, \dots, x_k = y$ such that $\mathcal{I}_{x_i}$ and $\mathcal{I}_{x_{i+1}}$ share an endpoint for $1 \leq i < k$. We denote by $\langle x \rangle$ the equivalence class containing x.

Definition 6.9 ($(\mathcal{F}^s, \nu^s)$). The stable measured foliation $(\mathcal{F}^s, \nu^s)$ on Σ has leaves ℓ of the following forms:

- $\ell = \bigcup_{y \in \langle x \rangle} \mathcal{I}_y$ for $x \in I$, with measure $\nu^s_\ell = \sum_{y \in \langle x \rangle} g_*(\alpha_y)$, where α_y is the measure of Remarks 3.22(c) on $[y]$ (restricted to $\mathcal{I}_y$);
- $\ell = \mathcal{J}_{f^r(a)}$ for $r \geq 1$, with measure $\nu^s_\ell = g_*(\alpha_{f^r(a)})$; and
- a singleton leaf $\ell = \infty$.

Lemma 6.10.
$$F(\mathcal{F}^s, \nu^s) = (\mathcal{F}^s, \lambda \nu^s).$$

Proof. We have $F(\mathcal{I}_x) \subset \mathcal{I}_{f(x)}$ for all $x \in I$, and $F(\mathcal{J}_{f^r(a)}) = \mathcal{J}_{f^{r+1}(a)}$ for all $r \geq 1$, so that $F(\ell)$ is a leaf whenever ℓ is. The proof that F contracts the stable measure by a factor λ is similar to that of Lemma 5.14: if A is a Borel subset of $\ell = \bigcup_{y \in \langle x \rangle} \mathcal{I}_y$, then, since $\langle f(x) \rangle = f(\langle x \rangle)$, we have

$$\begin{aligned}
\nu^s_{F(\ell)}(F(A)) &= \sum_{y \in \langle f(x) \rangle} \alpha_y\big(g^{-1}(F(A))\big) \\
&= \sum_{y \in f(\langle x \rangle)} \alpha_y\big(g^{-1}(F(A))\big) \\
&= \sum_{y \in \langle x \rangle} \alpha_{f(y)}\big(g^{-1}(F(A))\big) \\
&= \sum_{y \in \langle x \rangle} \alpha_{f(y)}\big(\hat{f}(g^{-1}(A))\big) \\
&= \lambda^{-1} \sum_{y \in \langle x \rangle} \alpha_y(g^{-1}(A)) \\
&= \lambda^{-1} \nu^s_\ell(A),
\end{aligned}$$

where the fifth equality is by Remarks 3.22(b). An analogous but more straightforward argument applies to the leaves $\mathcal{J}_{f^r(a)}$. □

6.3.2. *The unstable foliation.* Each path component of $\hat{I}$ is either a point or an arc by Lemma 3.13(b). Any path component of $\hat{I} \setminus \mathcal{C}$ is therefore a path connected subset of a point or an arc, and so is itself a point or an arc. Since $\mathcal{C} = \omega([c], \hat{f})$ by Lemma 6.5(a), it follows from Lemma 3.13(a) that in fact each path component of $\hat{I} \setminus \mathcal{C}$ is an open arc.

$\boldsymbol{e}$ is continuous on γ by Lemma 4.5(d) and Theorem 4.9, so that $\{\boldsymbol{e}(x_u) : x \in [a, \hat{a}]\}$ is an arc in $\hat{I}$ which intersects $\mathcal{C}$ exactly at its endpoints $\boldsymbol{e}(a)$ and $\boldsymbol{e}(\hat{a}_u)$. It follows that $S_0 := \{\boldsymbol{e}(x_u) : x \in (a, \hat{a})\}$ is a path component of $\hat{I} \setminus \mathcal{C}$.

Since $g^{-1}\big(\{g(\boldsymbol{e}(x_u))\}\big) = \{\boldsymbol{e}(x_u), \boldsymbol{e}(\hat{x}_u)\}$ for $x \in (a, \hat{a})$, the image $g(S_0)$ of S_0 in Σ is the half-open interval $\mathcal{S}_0 := \big\{g(\boldsymbol{e}(x_u)) : x \in (a, c]\big\}$ with endpoints $g(\boldsymbol{e}(c_u))$ and ∞.

Definition 6.11 (the spikes $\mathcal{S}_r$). For each $r \in \mathbb{Z}$, we write $S_r = \hat{f}^r(S_0)$, a path component of $\hat{I} \setminus \mathcal{C}$; and $\mathcal{S}_r = F^r(\mathcal{S}_0) = g(S_r)$, a half-open interval in Σ with one endpoint on the orbit of $g(\boldsymbol{e}(c_u))$ and the other at ∞. We refer to the $\mathcal{S}_r$ as *spikes*.

Every $\boldsymbol{x} \in \hat{I} \setminus \big(\mathcal{C} \cup \bigcup_{r \in \mathbb{Z}} S_r\big)$ has $g^{-1}\big(\{g(\boldsymbol{x})\}\big) = \{\boldsymbol{x}\}$. Therefore, each path component Γ of $\hat{I} \setminus \mathcal{C}$ other than the S_r has image $g(\Gamma)$ an immersed line; and these images, together with ∞ and the spikes $\mathcal{S}_r$, partition Σ.

Definition 6.12 ($(\mathcal{F}^u, \nu^u)$). The unstable measured foliation $(\mathcal{F}^u, \nu^u)$ on Σ has leaves $\ell = \{\infty\}$ and $\ell = g(\Gamma)$ for each path component Γ of $\hat{I} \setminus \mathcal{C}$.

For $\ell = g(\Gamma)$ and A a Borel subset of ℓ,

$$\nu^u_\ell(A) = \begin{cases} \frac{1}{2}\sum_i m\big(\pi_0(g^{-1}(A) \cap \Gamma_i)\big) & \text{if } \ell \text{ is a spike,} \\ \sum_i m\big(\pi_0(g^{-1}(A) \cap \Gamma_i)\big) & \text{otherwise,} \end{cases} \tag{9}$$

where $\Gamma = \bigcup_i \Gamma_i$ is the 0-flat decomposition of Lemma 3.31.

Lemma 6.13. $$F(\mathcal{F}^u, \nu^u) = (\mathcal{F}^u, \lambda^{-1}\nu^u).$$

Proof. Since $\hat{f}$ preserves $\mathcal{C}$ and permutes the path components of $\hat{I} \setminus \mathcal{C}$, mapping S_r to S_{r+1}, the leaves of $F(\mathcal{F}^u, \nu^u)$ coincide with the leaves of $(\mathcal{F}^u, \nu^u)$.

The proof that $F_*(\nu^u_\ell) = \lambda^{-1}\nu^u_{F(\ell)}$ for each ℓ is formally identical to that of Lemma 5.18, using that $F(\ell)$ is a spike if and only if ℓ is. □

6.3.3. *Charts for the foliations.* We have defined a pair $(\mathcal{F}^s, \nu^s)$, $(\mathcal{F}^u, \nu^u)$ of F-invariant measured foliations whose leaves are, respectively, contracted and expanded by a factor λ under the action of F. It remains to show that these are generalized pseudo-Anosov foliations, by constructing the charts of Definitions 2.6. We do this first around points $g(\boldsymbol{x})$ for which x_0 is not in the closure of the postcritical set $\mathrm{PC} = \mathrm{orb}(b, f)$, and then use the action of F to transfer these charts to general points of $\Sigma \setminus \{\infty\}$.

Notation 6.14 (Y). We write $Y = I \setminus \overline{\mathrm{PC}}$.

Lemma 6.15. *Let K be an interval contained in Y. Then $\pi_0^{-1}(K)$ is a disjoint union of 0-flat arcs $\gamma : K \to \hat{I}$ over K.*

Proof. Since $K \subset Y$ we have in particular that b, a, and $f(a)$ are not in K, so that either $K \subset (a, f(a))$, or $K \subset (f(a), b)$. In the former case $f^{-1}(K) \subset Y$ is a single interval mapped homeomorphically by f onto K; and in the latter case $f^{-1}(K) \subset Y$ is a union of two disjoint intervals each mapped homeomorphically onto K.

Given $\boldsymbol{x} \in \pi_0^{-1}(K)$, we can define a sequence (K_r) of intervals in Y inductively by taking $K_0 = K$, and K_r to be the component of $f^{-1}(K_{r-1})$ which contains x_r. By the previous paragraph, f maps each K_r homeomorphically onto K_{r-1}.

The function $\gamma : K \to \hat{I}$ given by $\gamma(y) = \langle y, y_1, y_2, \dots \rangle$, where $y_r \in K_r$ for each r, therefore defines a 0-flat arc over K, passing through $\boldsymbol{x}$, on which every point has the same itinerary as $\boldsymbol{x}$. It follows that $\pi_0^{-1}(K)$ is a disjoint union of such 0-flat arcs over K, that is, a 0-box over K. □

If $\boldsymbol{x} \in \hat{I}$ satisfies $x_r \in \overline{\mathrm{PC}}$ for all r, then $\boldsymbol{x} \in \omega([c], \hat{f})$, so that $\boldsymbol{x} \in \mathcal{C}$ by Lemma 6.5(a). For any other elements of $\hat{I}$, since $\overline{\mathrm{PC}}$ is forward f-invariant, there is a unique transition point in the thread between elements of $\overline{\mathrm{PC}}$ and elements of Y.

Definition 6.16 ($r(\boldsymbol{x})$). If $\boldsymbol{x} \notin \mathcal{C}$, let $r(\boldsymbol{x}) \geq 0$ be least such that $x_{r(\boldsymbol{x})} \notin \overline{\mathrm{PC}}$. Then $x_r \notin \overline{\mathrm{PC}}$ for all $r \geq r(\boldsymbol{x})$.

Suppose that $r(\boldsymbol{x}) = 0$. Let K be the component of Y containing x_0. Define $\psi : \pi_0^{-1}(K) \to \mathbb{R}$ by

$$\psi(\boldsymbol{y}) = \alpha_{y_0}\big(\{\boldsymbol{z} \in [y_0] : J(\boldsymbol{z}) \preceq J(\boldsymbol{y})\}\big). \tag{10}$$

Then ψ is constant on each 0-flat arc of $\pi_0^{-1}(K)$ by Theorem 3.27. Moreover, for each $x \in K$ and each $\boldsymbol{y}, \boldsymbol{y}' \in [x]$, we have $\psi(\boldsymbol{y}) = \psi(\boldsymbol{y}')$ if and only if $\boldsymbol{y}$ and $\boldsymbol{y}'$ are consecutive in $[x]$, since α_x is OU by Remarks 3.22(a).

Suppose in addition that $\boldsymbol{x}$ is not an extreme element of its π_0-fiber. Let $\psi_* = \max_{\boldsymbol{y}\in\pi_0^{-1}(K)} \psi(\boldsymbol{y})$, and $U_{\boldsymbol{x}} = \psi^{-1}(0, \psi_*)$, a neighborhood of $\boldsymbol{x}$. Let $R_{\boldsymbol{x}} = K \times (0, \psi_*)$, and define $\Psi_{\boldsymbol{x}} : U_{\boldsymbol{x}} \to R_{\boldsymbol{x}}$ by $\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = (y_0, \psi(\boldsymbol{y}))$. Then $\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = \Psi_{\boldsymbol{x}}(\boldsymbol{y}')$ if and only if $g(\boldsymbol{y}) = g(\boldsymbol{y}')$. Writing $N_{\boldsymbol{x}} = g(U_{\boldsymbol{x}})$, a neighborhood of $g(\boldsymbol{x})$, $\Psi_{\boldsymbol{x}}$ therefore induces a homeomorphism $\Phi_{\boldsymbol{x}} : N_{\boldsymbol{x}} \to R_{\boldsymbol{x}}$ with $\Psi_{\boldsymbol{x}} = \Phi_{\boldsymbol{x}} \circ g$.

Lemma 6.17. *Suppose that $r(\boldsymbol{x}) = 0$ and $\boldsymbol{x}$ is not an extreme element of its π_0-fiber. Then $\Phi_{\boldsymbol{x}} : N_{\boldsymbol{x}} \to R_{\boldsymbol{x}}$ is a regular chart for the invariant foliations of F at $g(\boldsymbol{x})$.*

Proof. According to Definitions 2.6 we need to show that $\Phi_{\boldsymbol{x}}$ is measure-preserving; and that for each unstable (respectively stable) leaf ℓ, and each path component L of $\ell \cap N_{\boldsymbol{x}}$, $\Phi_{\boldsymbol{x}}|_L$ is a measure-preserving homeomorphism onto $K \times \{y\}$ for some $y \in (0, \phi_*)$ (respectively $\{x\} \times (0, \phi_*)$ for some $x \in K$).

(a) Let A be a Borel subset of $N_{\boldsymbol{x}}$. Then

$$\begin{aligned}
\mu(A) = \hat{\mu}(g^{-1}(A)) &= \int_K \alpha_x(g^{-1}(A))\, dm(x) \\
&= \int_K \alpha_x(g^{-1}(A) \cap [x])\, dm(x) \\
&= \int_K m\big(\psi(g^{-1}(A) \cap [x])\big)\, dm(x) \\
&= \int_K m\big(\big\{y \in (0, \psi_*) : (x, y) \in \Psi_{\boldsymbol{x}}(g^{-1}(A))\big\}\big)\, dm(x) \\
&= \int_K m\big(\{y \in (0, \psi_*) : (x, y) \in \Phi_{\boldsymbol{x}}(A)\}\big)\, dm(x) \\
&= M(\Phi_{\boldsymbol{x}}(A)),
\end{aligned}$$

where M is two-dimensional Lebesgue measure. Here the equality in the first line is given by Theorem 3.28.

(b) Let $\ell = g(\Gamma)$ be a nontrivial leaf of $\mathcal{F}^u$, where Γ is a path component of $\hat{I} \setminus \mathcal{C}$. Then $\Gamma \cap U_{\boldsymbol{x}}$ is a countable union of 0-flat arcs over K, so that each path component L of $\ell \cap N_{\boldsymbol{x}}$ is of the form $L = g(\gamma(K))$, where $\gamma : K \to \hat{I}$ is one of the 0-flat arcs of Lemma 6.15. Then $\Phi_{\boldsymbol{x}}\big(g(\gamma(x))\big) = (x, C)$ for some constant C, so that $\Phi_{\boldsymbol{x}}|_L$ is a homeomorphism onto the unstable leaf segment $K \times \{C\}$ of $R_{\boldsymbol{x}}$.

If $A = g(\gamma(B))$ is a Borel subset of L, then $g^{-1}(A) = \gamma(B)$ if ℓ is not a spike; while $g^{-1}(A) = \gamma(B) \cup \gamma'(B)$ if ℓ is a spike, where $\gamma' : K \to \hat{I}$ is the 0-flat arc consecutive to γ. In either case, $\nu_\ell^u(A) = m(B)$ by (9), so that $\nu_\ell^u(A) = m(\Phi_{\boldsymbol{x}}|_L(A))$ as required.

(c) Let ℓ be a nontrivial leaf of $\mathcal{F}^s$.

If $\ell = \mathcal{J}_{f^r(a)}$ for some $r \geq 1$, then $\ell \subset g\big([f^r(a)]\big) \subset g(\pi_0^{-1}(\text{PC}))$, so $\ell \cap N_{\boldsymbol{x}} = \varnothing$. We can therefore assume that $\ell = \bigcup_{z\in\langle y\rangle} \mathcal{I}_z$ for some $y \in I$, a countable union of sets of the form $\mathcal{I}_z$. Since any such set which intersects $N_{\boldsymbol{x}}$ must have $\mathcal{I}_z = \mathcal{A}_z$ — as $N_{\boldsymbol{x}}$ is disjoint from $g(\pi_0^{-1}(\mathrm{PC}))$ — each component L of $\ell \cap N_{\boldsymbol{x}}$ is of the form $\mathcal{A}_x \cap N_{\boldsymbol{x}}$ for some $x \in K$: that is, it is the g-image of a fiber $[x]$ less its extremes, so that $\Phi_{\boldsymbol{x}}|_L$ is a homeomorphism onto the stable leaf segment $\{x\} \times (0, \phi_*)$ of $R_{\boldsymbol{x}}$.

For each $y \in (0, \phi_*)$ we have $\nu_\ell^s\big(\Phi_{\boldsymbol{x}}|_L^{-1}(\{x\} \times (0, y))\big) = y$ by (10) and Definition 6.9, so that $\Phi_{\boldsymbol{x}}|_L$ is measure-preserving as required. □

Suppose, on the other hand, that $r(\boldsymbol{x}) = 0$ but that $\boldsymbol{x}$ is an extreme element of its fiber, so that $\boldsymbol{x} = \hat{f}^{-r}(\boldsymbol{e}(x_u))$ for some $r \geq 0$ and $x \in (a, \hat{a})$.

Consider first the case $x \neq c$. Then $g(\boldsymbol{x}) = g(\boldsymbol{x}')$, where $\boldsymbol{x}' = \hat{f}^{-r}(\boldsymbol{e}(\hat{x}_u))$. Write $z = \pi_0\big(\hat{f}^{-r}(\boldsymbol{e}(c_u))\big)$, and choose $\epsilon > 0$ so that both $K = (x_0 - \epsilon, x_0 + \epsilon)$ and $K' = (x_0' - \epsilon, x_0' + \epsilon)$ are contained in Y and don't contain z (so, in particular, are disjoint). Suppose that $\boldsymbol{x}$ and $\boldsymbol{x}'$ are lower elements of their π_0-fibers; only minor modifications are needed to treat the case where they are upper elements.

Define maps $\psi : \pi_0^{-1}(K) \to \mathbb{R}$ and $\psi' : \pi_0^{-1}(K') \to \mathbb{R}$ by (10); let $U = \psi^{-1}([0, \psi_*))$ and $U' = \psi'^{-1}([0, \psi'_*))$, where ψ_* and ψ'_* are the maximum values of ψ and ψ'; and define $U_{\boldsymbol{x}} = U \cup U'$, and $\Psi_{\boldsymbol{x}} : U_{\boldsymbol{x}} \to R_{\boldsymbol{x}} = K \times (-\psi'_*, \psi_*)$ by

$$\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = \begin{cases} (y_0, \psi(\boldsymbol{y})) & \text{if } \boldsymbol{y} \in U, \\ (2z - y_0, -\psi'(\boldsymbol{y})) & \text{if } \boldsymbol{y} \in U'. \end{cases}$$

Then $\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = \Psi_{\boldsymbol{x}}(\boldsymbol{y}')$ if and only if $g(\boldsymbol{y}) = g(\boldsymbol{y}')$, and $\Psi_{\boldsymbol{x}}$ induces a homeomorphism $\Phi_{\boldsymbol{x}} : N_{\boldsymbol{x}} \to R_{\boldsymbol{x}}$, where $N_{\boldsymbol{x}} = g(U_{\boldsymbol{x}})$, a neighborhood of $g(\boldsymbol{x})$. The proof of the following is essentially identical to that of Lemma 6.17.

Lemma 6.18. *Suppose that $r(\boldsymbol{x}) = 0$, and that $\boldsymbol{x} = \hat{f}^{-r}(\boldsymbol{e}(x_u))$ for some $r \geq 0$ and $x \in (a, \hat{a})$ with $x \neq c$. Then $\Phi_{\boldsymbol{x}} : N_{\boldsymbol{x}} \to R_{\boldsymbol{x}}$ is a regular chart for the invariant foliations of F at $g(\boldsymbol{x})$.*

To complete the analysis of charts around $\boldsymbol{x}$ where $r(\boldsymbol{x}) = 0$, it remains to consider the case where $\boldsymbol{x} = \hat{f}^{-r}(\boldsymbol{e}(c_u))$ for some $r \geq 0$: in this case we will see that there is a 1-pronged singularity at $\boldsymbol{x}$. Choose $\epsilon > 0$ so that $K := (x_0 - \epsilon, x_0 + \epsilon) \subset Y$, and suppose that $\boldsymbol{x}$ is a lower element of its π_0-fiber; only minor modifications are needed if it is an upper element. Define $\psi : \pi_0^{-1}(K) \to \mathbb{R}$ by (10), let $U_{\boldsymbol{x}} = \psi^{-1}([0, \psi_*))$, where ψ_* is the maximum value of ψ, set $R_{\boldsymbol{x}} = (-\epsilon, \epsilon) \times [0, \psi_*)$, and define $\Psi_{\boldsymbol{x}} : U_{\boldsymbol{x}} \to R_{\boldsymbol{x}}$ by

$$\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = (y_0 - x_0, \psi(\boldsymbol{y})).$$

Then, for $\boldsymbol{y}, \boldsymbol{y}' \in U_{\boldsymbol{x}}$, we have $g(\boldsymbol{y}) = g(\boldsymbol{y}')$ if and only if either $\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = \Psi_{\boldsymbol{x}}(\boldsymbol{y}')$, or $\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = (u, 0)$ and $\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = (-u, 0)$ for some $u \in (-\epsilon, \epsilon)$. Therefore, setting $Q_{\boldsymbol{x}}$ to be the 1-pronged model $Q_{\boldsymbol{x}} = R_{\boldsymbol{x}}/(u, 0) \sim (-u, 0)$, and writing $N_{\boldsymbol{x}} = g(U_{\boldsymbol{x}})$, $\Psi_{\boldsymbol{x}}$ induces a homeomorphism $\Phi_{\boldsymbol{x}} : N_{\boldsymbol{x}} \to Q_{\boldsymbol{x}}$, which sends $g(\boldsymbol{x})$ to the singular point of $Q_{\boldsymbol{x}}$.

Lemma 6.19. *Suppose that $r(\boldsymbol{x}) = 0$, and that $\boldsymbol{x} = \hat{f}^{-r}(\boldsymbol{e}(c_u))$ for some $r \geq 0$. Then $\Phi_{\boldsymbol{x}} : N_{\boldsymbol{x}} \to Q_{\boldsymbol{x}}$ is a 1-pronged chart for the invariant foliations of F at $g(\boldsymbol{x})$.*

Proof. (a) If A is a Borel subset of $N_{\boldsymbol{x}}$ then $\mu(A) = \hat{\mu}(g^{-1}(A)) = \hat{\mu}(g^{-1}(A) \cap \psi^{-1}(0, \psi_*))$. That this is equal to $M(\Phi_{\boldsymbol{x}}(A))$ follows exactly as in (a) from the proof of Lemma 6.17.

(b) As in (b) from the proof of Lemma 6.17, if ℓ is a nontrivial leaf of $\mathcal{F}^u$ then each path component L of $\ell \cap N_{\boldsymbol{x}}$ is of the form $L = g(\gamma(K))$, where $\gamma : K \to \hat{I}$ is one of the 0-flat arcs of Lemma 6.15, so that there is some constant $C \in [0, \psi_*)$ such that $\Phi_{\boldsymbol{x}}\big(g(\gamma(x))\big)$ is the equivalence class of (x, C) in $Q_{\boldsymbol{x}}$. If $C > 0$ then this equivalence class is trivial, and the proof that $\Phi_{\boldsymbol{x}}|_L$ is a measure preserving homeomorphism onto the corresponding stable leaf segment of $Q_{\boldsymbol{x}}$ is identical to that of (b) from the proof of Lemma 6.17.

If $C = 0$ then the leaf ℓ is a spike. If A is a Borel subset of L then $A = g(\gamma(B)) = g(\gamma(B'))$, where $B \subset (x_0 - \epsilon, x_0]$ and $B' \subset [x_0, x_0 + \epsilon)$; and hence $\nu_l^u(A) = m(\Phi_{\boldsymbol{x}}|_L(A)) = m(B)$ as required.

(c) As in (c) from the proof of Lemma 6.17, if ℓ is a nontrivial leaf of $\mathcal{F}^s$, then $\ell = \bigcup_{z \in \langle y \rangle} \mathcal{I}_z$ for some $y \in I$, a countable union of sets of the form $\mathcal{I}_z$. Since any such set which intersects $N_{\boldsymbol{x}}$ must have $\mathcal{I}_z = \mathcal{A}_z$, each component L of $\ell \cap N_{\boldsymbol{x}}$ is either $\mathcal{A}_{x_0} \cap N_{\boldsymbol{x}}$, or $(\mathcal{A}_{x_0 - \delta} \cup \mathcal{A}_{x_0 + \delta}) \cap N_{\boldsymbol{x}}$ for some $\delta \in (0, \epsilon)$. Therefore $\Phi_{\boldsymbol{x}}|_L$ is a homeomorphism onto a stable leaf segment of $Q_{\boldsymbol{x}}$, which is measure-preserving as in the proof of Lemma 6.17. □

Theorem 6.20. *Let $F : \Sigma \to \Sigma$ be the sphere homeomorphism obtained from a tent map f of irrational height. Then F is a generalized pseudo-Anosov map, whose invariant foliations $(\mathcal{F}^u, \nu^u)$ and $(\mathcal{F}^s, \nu^s)$ have 1-pronged singularities along the bi-infinite orbit of $g(\boldsymbol{e}(c_u))$, and are regular at every other point of $\Sigma \setminus \{\infty\}$.*

Proof. We have defined stable and unstable foliations satisfying $F(\mathcal{F}^s, \nu^s) = (\mathcal{F}^s, \lambda\nu^s)$ and $F(\mathcal{F}^u, \nu^u) = (\mathcal{F}^u, \lambda^{-1}\nu^u)$. It remains to show that these foliations have 1-pronged charts along the orbit of $g(\boldsymbol{e}(c_u))$, and regular charts at every other point of $\Sigma \setminus \{\infty\}$.

For the former, let $x = F^s\big(g(\boldsymbol{e}(c_u))\big)$ for $s \in \mathbb{Z}$, so that $x = g(\boldsymbol{x})$ for a unique $\boldsymbol{x} \in \hat{I}$, namely $\boldsymbol{x} = \hat{f}^s(\boldsymbol{e}(c_u))$, and write $r = r(\boldsymbol{x})$. Then if $\boldsymbol{y} = \hat{f}^{-r}(\boldsymbol{x})$ we have $r(\boldsymbol{y}) = 0$, so that by Lemma 6.19 there is a 1-pronged chart $\Phi_{\boldsymbol{y}} : N_{\boldsymbol{y}} \to Q_{\boldsymbol{y}}$ for the invariant foliations at $g(\boldsymbol{y}) = F^{-r}(x)$.

Recall that $Q_{\boldsymbol{y}} = (-\epsilon, \epsilon) \times [0, \phi_*)/(u, 0) \sim (-u, 0)$ for some ϵ and ϕ_*, write

$$Q = (-\lambda^r\epsilon, \lambda^r\epsilon) \times [0, \lambda^{-r}\phi_*)/(u, 0) \sim (-u, 0),$$

a 1-pronged model, and define $\xi : Q_{\boldsymbol{y}} \to Q$ by $[(x, y)] \mapsto [(\lambda^r x, \lambda^{-r} y)]$. Let $N = F^r(N_{\boldsymbol{y}})$, and $\Phi = \xi \circ \Phi_{\boldsymbol{y}} \circ F^{-r} : N \to Q$. Then

- $\Phi(x) = \xi \circ \Phi_{\boldsymbol{y}}(g(\boldsymbol{y})) = \xi\big([(0, 0)]\big) = [(0, 0)]$ is the singular point of Q.
- Since F, $\Phi_{\boldsymbol{y}}$, and ξ are all measure-preserving homeomorphisms (with respect to the measure μ on Σ and two-dimensional Lebesgue measure M on $Q_{\boldsymbol{y}}$ and Q), Φ is a measure-preserving homeomorphism.
- If $\ell \in \mathcal{F}^s$ and L is a path component of $\ell \cap N$, then $F^{-r}(L)$ is a path component of $F^{-r}(\ell) \cap N_{\boldsymbol{y}}$, so that $\Phi_{\boldsymbol{y}}|_{F^{-r}(L)}$ is a measure-preserving homeomorphism onto a stable leaf segment of $Q_{\boldsymbol{y}}$. Since F^{-r} scales the measure on L by a factor λ^r, and ξ scales the measure on stable leaf segments by a factor λ^{-r}, $\Phi|_L$ is a measure-preserving homeomorphism onto a stable leaf segment of Q. Analogously, for each

$\ell \in \mathcal{F}^u$ and each path component L of $\ell \cap N$, $\Phi|_L$ is a measure-preserving homeomorphism onto an unstable leaf segment of Q.

Therefore $\Phi : N \to Q$ is a 1-pronged chart for the invariant foliations at x, as required.

The argument for other points $x \in \Sigma \setminus \{\infty\}$ is analogous. Let $\boldsymbol{x} \in g^{-1}(x)$, $r = r(\boldsymbol{x})$, and $\boldsymbol{y} = \hat{f}^{-r}(\boldsymbol{x})$, which satisfies $r(\boldsymbol{y}) = 0$. By Lemmas 6.17 and 6.18, there is a regular chart $\Phi_{\boldsymbol{y}} : N_{\boldsymbol{y}} \to R_{\boldsymbol{y}}$ for the invariant foliations at $y = F^{-r}(x)$. Setting $N = F^r(N_{\boldsymbol{y}})$ and $R = \xi(R_{\boldsymbol{y}})$, where $\xi(x, y) = (\lambda^r x, \lambda^{-r} y)$, it follows that $\Phi = \xi \circ \Phi_{\boldsymbol{y}} \circ F^{-r} : N \to R$ is a regular chart for the invariant foliations at x. □

6.4. ***The rational postcritically finite case.*** In the case where f is postcritically finite, and therefore has rational height $q(f) = m/n$, analogous arguments can be used to show that the sphere homeomorphism F is generalized pseudo-Anosov. We omit the details of these because the result in this case was proved in [12] (using quite different methods), and only sketch some of the main features.

Let Q be the periodic orbit, of period k, say, contained in PC, and $R = \mathrm{PC} \setminus Q$ be the preperiodic part of the postcritical orbit. Therefore Q and R are finite subsets of I, with R being empty in the case where b is itself a periodic point. The natural extension $\hat{f}$ has a corresponding period-k orbit $\boldsymbol{Q}$ lying in the fibers above Q.

If $x \notin \mathrm{PC}$, then $\mathcal{A}_x := g([x])$ is an arc in Σ, by an argument similar to that of Section 5.1. On the other hand, $\mathcal{A}_x$ is a triod for $x \in R$; while for $x \in Q$, because $F^k(\mathcal{A}_x) \subset \mathcal{A}_x$, it is a tree with infinitely many valence-1 and valence-3 vertices — corresponding to 1- and 3-pronged singularities of the invariant foliations — except in the NBT case mentioned below.

Recall from Theorem 4.8 and Definitions 4.11 that f can be of three types:

- *Endpoint type*: when $\Theta^n(a) = a$ or $\Theta^n(a) = \hat{a}_u$.
- *NBT type*: when $\Theta^n(a) = c_u$.
- *General type*: in all other cases.

The endpoint case is most similar to the irrational case. Suppose that $\Theta^n(a) = a$, so that, by Lemma 4.5(c), $\boldsymbol{e}(a) = \langle a, b = \tau(\Theta^{-1}(a)), \dots, \tau(\Theta^{-(n-1)}(a)), a, \dots \rangle$ is a period-n point of $\hat{f}$, whose orbit lies in the fibers above Q. Then $\mathcal{C} = \left\{\hat{f}^r(\boldsymbol{e}(a)) : 0 \le r < n\right\} \cup \left\{\hat{f}^r(\boldsymbol{e}(\hat{a}_u)) : r \in \mathbb{Z}\right\}$ is a countable set which plays the role of the Cantor set $\mathcal{C}$ in the irrational case. Specifically, in close analogy with Lemma 6.1, Remark 5.17(c) of [12] states that $\mathcal{C}$ is a nontrivial fiber of $g : \hat{I} \to \Sigma$, and that the only other nontrivial fibers are $\left\{\hat{f}^r(\boldsymbol{e}(x_u)), \hat{f}^r(\boldsymbol{e}(\hat{x}_u))\right\}$ for each $x \in (a, c)$ and $r \in \mathbb{Z}$. Writing $\infty = h(\mathcal{C}) \in \Sigma$, it can be shown that F is a generalized pseudo-Anosov map with 1-pronged singularities along the bi-infinite orbit of $g(\boldsymbol{e}(c_u))$ and regular points elsewhere in $\Sigma \setminus \{\infty\}$. The orbit of $g(\boldsymbol{e}(c_u))$ is homoclinic to ∞. $F^r\left(g(\boldsymbol{e}(c_u))\right)$ lies in $\mathcal{A}_x$ for some $x \in Q$ when $r \ge 0$, and is the g-image of an extreme point when $r \le 0$. When $\Theta^n(a) = \hat{a}_u$, the situation is the same with the roles of $\boldsymbol{e}(a)$ and $\boldsymbol{e}(\hat{a}_u)$ exchanged (in this case the postcritical orbit is strictly preperiodic, with $|R| = 2$).

In the general case, the identifications are given by (EI), (EII), and (EIII) of Lemma 5.4. The period-n orbit $\boldsymbol{P}$ of (EIII) is collapsed to a fixed point of F; while the identifications (EII) and the singletons $\hat{f}^r(\boldsymbol{e}(c_u))$ give rise to bi-infinite heteroclinic orbits of connected 3-pronged and 1-pronged singularities: these orbits converge to the fixed point $g(\boldsymbol{P})$ under iteration of F^{-1}, and to the periodic orbit $g(\boldsymbol{Q})$ under iteration of F. There are therefore regular or pronged charts for the invariant foliations at every point of Σ except for the $k+1$ points of $g(\boldsymbol{P}) \cup g(\boldsymbol{Q})$.

In the NBT case, the element $\hat{f}^r(\boldsymbol{e}(\hat{z}_u))$ of (EII) coincides with $\boldsymbol{e}(c_u)$, so that the sequences of 1-pronged and 3-pronged singularities cancel out. In this case $F : \Sigma \to \Sigma$ is pseudo-Anosov, with an n-pronged singularity at $g(\boldsymbol{P})$, and 1-pronged singularities at the points of $g(\boldsymbol{Q})$ (which is a period $n+2$ orbit, since $f^n(a) = c$ in this case).

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Homotopy structures realizing algebraic kk-theory

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Algebraic kk-theory, introduced by Cortiñas and Thom (2007), is a bivariant K-theory defined on the category **Alg** of algebras over a commutative unital ring ℓ. It consists of a triangulated category kk endowed with a functor from **Alg** to kk that is the universal excisive, homotopy invariant and matrix-stable homology theory. Moreover, one can recover Weibel's homotopy K-theory KH from kk since we have $kk(\ell, A) = \mathrm{KH}(A)$ for any algebra A. We prove that **Alg** with the split surjections as fibrations and the kk-equivalences as weak equivalences is a stable category of fibrant objects, whose homotopy category is kk. As a consequence of this, we prove that the Dwyer–Kan localization kk_∞ of the ∞-category of algebras at the set of kk-equivalences is a stable infinity category whose homotopy category is kk.

1. Introduction

Kasparov's KK-theory, introduced in [16], is the major tool in noncommutative topology. To every pair (A, B) of separable C^*-algebras it associates an abelian group $KK(A, B)$ that generalizes both of topological K-theory and topological K-homology. These groups were deeply studied by Cuntz, who gave in [7] an alternative description of them and provided a new perspective on the theory. The groups $KK(A, B)$ are the hom-sets of an additive category KK whose objects are separable C^*-algebras. Higson proved in [15] that KK is the target of the universal homotopy invariant, C^*-stable and split-exact functor from the category C^***Alg** of separable C^*-algebras into an additive category. Meyer and Nest proved in [20] that the category KK is actually triangulated in a natural way. Cuntz also analyzed KK-theory from an algebraic standpoint and defined in [8] a bivariant K-theory for all locally convex algebras.

Motivated by the works of Cuntz and Higson, algebraic kk-theory was introduced by Cortiñas and Thom in [6] as a completely algebraic counterpart of Kasparov's KK-theory. To every pair (A, B) of algebras over a commutative ring ℓ it associates an abelian group $kk(A, B)$ that generalizes Weibel's homotopy K-theory KH, defined in [27]. The groups $kk(A, B)$ are the hom-sets of a triangulated category kk whose objects are ℓ-algebras. This category kk is the target of the universal polynomial homotopy invariant, M_∞-stable and excisive functor from the category **Alg** of ℓ-algebras into a triangulated category.

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Triangulated categories often appear (but not always; see [21]) as the homotopy categories of stable ∞-categories. A natural question is whether there exist stable ∞-categories whose homotopy categories are the bivariant K-theory categories mentioned above. For Kasparov's KK-theory this is answered affirmatively by Land and Nikolaus in [17]. Indeed, they construct a stable ∞-category KK_∞ whose homotopy category is KK, upon performing the Dwyer–Kan localization of the ∞-category $C^*\mathbf{Alg}$ at the set of KK-equivalences. The same methods are used in [3] to establish the G-equivariant case for a countable group G. A different approach is taken in [2], where a stable ∞-category realizing KK-theory is constructed independently of the classical KK-groups. In this paper we address this question for algebraic kk-theory; the G-equivariant case [9] will be addressed in a separate work. Our main result is the following:

Theorem 1.1 (Theorem 5.3). *Let* $\mathbf{Alg}$ *be the category of* ℓ*-algebras and let* W_{kk} *be the set of* kk*-equivalences in* $\mathbf{Alg}$*. Let* $kk_\infty := \mathbf{Alg}[W_{kk}^{-1}]$ *be the Dwyer–Kan localization. Then* kk_∞ *is a stable* ∞*-category whose homotopy category is triangle equivalent to* kk*.*

To prove this result we follow the path taken in [17]. A key tool for controlling finite limits in kk_∞ is the fact that $\mathbf{Alg}$ admits the structure of a category of fibrant objects where the weak equivalences are the kk-equivalences. This fact is proved in Proposition 3.11 and can be considered as an analogue of [26, Theorem 2.29] in the algebraic context. To show that the loop functor in kk_∞ is an equivalence, we use that the homotopy category of this category of fibrant objects is the category kk defined by Cortiñas and Thom in [6]; the latter is established in Theorem 4.24. Along the way, we prove the following theorem, which is of independent interest.

Theorem 1.2 (Theorem 4.1). *Let* W_H *be the family of polynomial homotopy equivalences in* $\mathbf{Alg}$*. Let* $W_S = \{A \to M_\infty A\}_{A\in\mathbf{Alg}}$ *be the family of upper-left corner inclusions. Let* W_E *be the family of classifying maps of those extensions whose middle term is either contractible or an infinite-sum ring. Then the functor* $j : \mathbf{Alg} \to kk$ *is initial in the category of those functors* $F : \mathbf{Alg} \to \mathcal{C}$ *such that:*

- $\mathcal{C}$ *is a category with finite products and* F *preserves finite products.*
- F *sends morphisms in* $W_H \cup W_S \cup W_E$ *to isomorphisms.*

The stability of kk_∞ implies that kk-theory is naturally enriched over spectra. A point-set level construction of a spectrum representing kk-theory was presented in [12]. This was later used in [13] to prove that kk is contravariantly equivalent to a full subcategory of certain motivic stable homotopy category. Our methods are completely different from those used in [12] and [13].

In [2], a stable ∞-category representing Kasparov's KK-theory is constructed by enforcing the universal properties one by one through a sequence of localizations. It is suggested in the same paper that this method, with some modifications, should also work for algebraic kk-theory. The first localization performed in [2] is the Dwyer–Kan localization of $C^*\mathbf{Alg}$ at the homotopy equivalences; write $C^*\mathbf{Alg}_h$ for the resulting ∞-category. It is proved in [2, Proposition 3.5] that $C^*\mathbf{Alg}_h$ is equivalent to the coherent nerve of $C^*\mathbf{Alg}$, considered as a Kan-complex enriched category. In the algebraic context, we get the following result.

Proposition 1.3. *Let* $\mathbf{Alg}_h$ *be the Dwyer–Kan localization of* $\mathbf{Alg}$ *at the polynomial homotopy equivalences. Then* $\mathbf{Alg}_h$ *is equivalent to the coherent nerve of* $\mathbf{Alg}$, *considered as a Kan-complex enriched category with the hom-spaces* $\underline{\mathrm{HOM}}(A, B)$ *defined in (2.4).*

Back on the topological side, it is proved in [2, Proposition 3.17] that $C^*\mathbf{Alg}_h$ is left-exact using that $C^*\mathbf{Alg}$ admits the structure of a category of fibrant objects with homotopy equivalences as weak equivalences [26, Theorem 2.19]. At this point, an important difference arises between the topological and algebraic contexts. We show in Proposition 3.10 that there is no reasonable family of fibrations making $\mathbf{Alg}$ into a category of fibrant objects with polynomial homotopy equivalences as weak equivalences. As a consequence, we do not know whether the ∞-category $\mathbf{Alg}_h$ has pullbacks. This indicates that polynomial homotopies behave worse than continuous ones and suggests that more changes are needed to translate the methods of [2] to the algebraic context.

2. Preliminaries

We recall definitions and results that will be used later on. Throughout this paper, ℓ is a commutative ring with unit and $\mathbf{Alg}$ is the category of associative not necessarily unital ℓ-algebras.

2.1. *Homotopies.*

2.1.1. *Homotopy of continuous maps.* Let $f, g : X \to Y$ be continuous maps of topological spaces. We say that f is *homotopic* to g, denoted by $f \sim g$, if there exists a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Note that H can be viewed as a continuous map $H : X \to C([0, 1], Y)$ such that $\mathrm{ev}_0 \circ H = f$ and $\mathrm{ev}_1 \circ H = g$. Here, $C([0, 1], Y)$ denotes the space of continuous maps $[0, 1] \to Y$. It is well known that $\sim$ is an equivalence relation. Indeed, to show transitivity we can proceed as follows. Let H be a homotopy from f to g and H' be a homotopy from g to h. By the universal property of the pullback, we have a unique map H'' making the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!\, H''"] \arrow[drr, bend left, "H'"] \arrow[ddr, bend right, "H"'] & & \\
& C([0,1],Y) \times_Y C([0,1],Y) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner"] & C([0,1],Y) \arrow[d, "\mathrm{ev}_0"] \\
& C([0,1],Y) \arrow[r, "\mathrm{ev}_1"'] & Y
\end{tikzcd} \tag{2.1}$$

Since $[0, 1] \vee [0, 1] \cong \left[0, \frac{1}{2}\right] \vee \left[\frac{1}{2}, 1\right] = [0, 1]$, we have $C([0, 1], Y) \times_Y C([0, 1], Y) \cong C([0, 1], Y)$ and we can consider H'' as a continuous map $X \to C([0, 1], Y)$. This H'' is a homotopy from f to h.

2.1.2. *Homotopy of algebra homomorphisms.* Let $f, g : A \to B$ be algebra homomorphisms. We say that f is *elementary homotopic* to g, denoted by $f \sim_e g$, if there exists an algebra homomorphism

$H : A \to B[t]$ such that $\mathrm{ev}_0 \circ H = f$ and $\mathrm{ev}_1 \circ H = g$, where $\mathrm{ev}_i : B[t] \to B$ is the evaluation at i for $i = 0, 1$. Note that $\sim_e$ is reflexive and symmetric, but not transitive. Indeed, since $B[t] \times_B B[t] \not\cong B[t]$, we cannot obtain a new homotopy by glueing two homotopies. We say that f is *homotopic* to g, denoted by $f \sim g$, if there exist $h_1, \dots, h_n \in \hom_{\mathbf{Alg}}(A, B)$ such that

$$f \sim_e h_1 \sim_e \cdots \sim_e h_n \sim_e g.$$

Put $B_n[t] = \big\{(p_0, \dots, p_n) \in B[t] \times \cdots \times B[t] \mid \mathrm{ev}_1(p_i) = \mathrm{ev}_0(p_{i+1}) \text{ for } 0 \le i < n\big\}$. Define $\mathrm{ev}_i : B_n[t] \to B$ by $\mathrm{ev}_0(p_0, \dots, p_n) = \mathrm{ev}_0(p_0)$ and $\mathrm{ev}_1(p_0, \dots, p_n) = \mathrm{ev}_1(p_n)$. We have a directed system

$$B_\bullet[t] : B[t] \xrightarrow{\beta_0} B_1[t] \xrightarrow{\beta_1} \cdots \to B_n[t] \xrightarrow{\beta_n} B_{n+1}[t] \xrightarrow{\beta_{n+1}} \cdots,$$

where $\beta_n(p_0, \dots, p_n) = (p_0, \dots, p_n, \mathrm{ev}_1(p_n))$. Note that ev_i induces a morphism $\mathrm{ev}_i : B_\bullet[t] \to B$ in the category of ind-algebras. With this notation, f is homotopic to g if there exists a morphism of ind-algebras $H : A \to B_\bullet[t]$ such that $\mathrm{ev}_0 \circ H = f$ and $\mathrm{ev}_1 \circ H = g$. Homotopy is a transitive relation. Indeed, let $f, h, g \in \hom_{\mathbf{Alg}}(A, B)$ such that $f \sim h$ and $h \sim g$. Then there exist algebra homomorphisms $H : A \to B_n[t]$ and $H' : A \to B_m[t]$ such that $\mathrm{ev}_0 \circ H = f$, $\mathrm{ev}_1 \circ H = h$, $\mathrm{ev}_0 \circ H' = h$ and $\mathrm{ev}_1 \circ H' = g$. By the universal property of the pullback, there is a unique morphism $H'' : A \to B_n[t] \times_B B_m[t] = B_{n+m}[t]$ such that $\mathrm{ev}_0 \circ H'' = f$ and $\mathrm{ev}_1 \circ H'' = g$:

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists!\, H''"] \arrow[drr, bend left, "H'"] \arrow[ddr, bend right, "H"'] & & \\
& B_n[t] \times_B B_m[t] \arrow[r, "pr"] \arrow[d, "pr"'] \arrow[dr, phantom, "\lrcorner"] & B_m[t] \arrow[d, "\mathrm{ev}_0"] \\
& B_n[t] \arrow[r, "\mathrm{ev}_1"'] & B
\end{tikzcd}$$

Homotopy is compatible with composition of algebra homomorphisms. We have a category $[\mathbf{Alg}]$ whose objects are the ℓ-algebras and whose hom-sets are defined by

$$\hom_{[\mathbf{Alg}]}(A, B) = [A, B] := \hom_{\mathbf{Alg}}(A, B)/\sim .$$

2.2. ***Simplicial enrichment of algebras.*** Let us recall the simplicial enrichment in **Alg** introduced in [6]. Let B be an algebra. For $n \ge 0$, the algebra B^{Δ^n} of *B-valued polynomial functions on the standard n-simplex* is defined as $B^{\Delta^n} := B[t_0, \dots, t_n]/\langle t_0 + \cdots + t_n - 1\rangle$. For a simplicial set X, the algebra of *B-valued polynomial functions on X* is defined as $B^X := \hom_{\mathbf{sSet}}(X, B^\Delta)$ where B^Δ is the simplicial algebra $[n] \mapsto B^{\Delta^n}$.

The category **Alg** can be enriched over simplicial sets, as we proceed to recall. For a pair of algebras (A, B), let $\underline{\hom}(A, B)$ be the simplicial set defined by

$$[n] \mapsto \hom_{\mathbf{Alg}}(A, B^{\Delta^n}).$$

We have a simplicial composition

$$\circ : \underline{\mathrm{hom}}(B, C) \times \underline{\mathrm{hom}}(A, B) \to \underline{\mathrm{hom}}(A, C)$$

defined as follows. Let $f \in \underline{\mathrm{hom}}(A, B)_n$ and $g \in \underline{\mathrm{hom}}(B, C)_n$ be represented by $a : A \to B^{\Delta^n}$ and $b : B \to C^{\Delta^n}$ respectively. Then $g \circ f$ is represented by the composite

$$A \xrightarrow{a} B^{\Delta^n} \xrightarrow{b^{\Delta^n}} (C^{\Delta^n})^{\Delta^n} \xrightarrow{\mu} C^{\Delta^n \times \Delta^n} \xrightarrow{\mathrm{diag}^*} C^{\Delta^n}, \tag{2.2}$$

where μ is the morphism defined in [23, Section 3.1].

Remark 2.3. Upon identifying $C^{\Delta^n} \cong C \otimes \ell^{\Delta^n}$, the composite $\mathrm{diag}^* \circ \mu$ in (2.2) equals the morphism $\mathrm{id}_C \otimes m_{\Delta^n} : C \otimes \ell^{\Delta^n} \otimes \ell^{\Delta^n} \to C \otimes \ell^{\Delta^n}$ where m_{Δ^n} is the multiplication in the commutative algebra ℓ^{Δ^n}.

Upon applying Ex^∞ to the enrichment described above we get an enrichment of **Alg** over Kan complexes; see [6, Theorem 3.2.3]. Explicitly, we have

$$\underline{\mathrm{HOM}}(A, B) := \mathrm{Ex}^\infty \underline{\mathrm{hom}}(A, B) = \operatorname*{colim}_r \mathrm{hom}_{\mathbf{Alg}}(A, B_r^{\Delta^\bullet}). \tag{2.4}$$

Here we write $B_r^{\Delta^n}$ for $B^{\mathrm{sd}^r \Delta^n}$, where $\mathrm{sd}^r \Delta^n$ is the r-fold subdivision of Δ^n.

Remark 2.5. For $B \in \mathbf{Alg}$, $X \in \mathbf{sSet}$ and $r \geq 0$, let B_r^X denote $B^{\mathrm{sd}^r X}$, where $\mathrm{sd}^r \Delta^n$ is the r-fold subdivision of Δ^n. As explained in [6, Section 3.2], we have an ind-algebra $B_\bullet^X$ with transition morphisms induced by the last vertex map.

Remark 2.6. Two algebra homomorphisms $f, g : A \to B$ are homotopic if and only if there exist $r \in \mathbb{N}$ and $H : A \to B_r^{\Delta^1}$ such that $d_0 \circ H = f$ and $d_1 \circ H = g$.

2.3. ***Categories of fibrant objects.*** Following [1], a *category of fibrant objects* is a category $\mathcal{C}$ with terminal object $*$ and distinguished subcategories $\mathcal{F}$ and W such that:

- (F_1) The isomorphisms of $\mathcal{C}$ are in $\mathcal{F}$.
- (F_2) The pullback in $\mathcal{C}$ of a morphism in $\mathcal{F}$ exists and is in $\mathcal{F}$.
- (F_3) For any object B of $\mathcal{C}$, the morphism $B \to *$ is in $\mathcal{F}$.
- (W_1) The isomorphisms of $\mathcal{C}$ are in W.
- (W_2) If two of f, g and gf are in W, then so is the third.
- (FW_1) The pullback in $\mathcal{C}$ of a morphism in $W \cap \mathcal{F}$ is in $W \cap \mathcal{F}$.
- (FW_2) For any object B of $\mathcal{C}$, the diagonal map $B \to B \times B$ admits a factorization

$$B \xrightarrow[s]{\sim} B^I \overset{a}{\underset{d}{\twoheadrightarrow}} B \times B,$$

where $s \in W$ and $d \in \mathcal{F}$. The triple (B^I, s, d) is called a *path-object* for B.

The morphisms in $\mathcal{F}$ are called *fibrations* and are denoted by $\twoheadrightarrow$. The morphisms in W are called *weak equivalences* and are denoted by $\xrightarrow{\sim}$. The morphisms in $W \cap \mathcal{F}$ are called *trivial fibrations* and are denoted by $\overset{\sim}{\twoheadrightarrow}$.

The structure of a category of fibrant objects $\mathcal{C}$ with weak equivalences W is a tool to deal with the localization $\mathcal{C}[W^{-1}]$. The latter is called the *homotopy category* of $\mathcal{C}$ and is denoted by $\mathrm{Ho}(\mathcal{C})$.

Lemma 2.7. *Let $(\mathcal{C}, \mathcal{F}, W)$ be a category of fibrant objects. Then $\mathcal{C}[W^{-1}]$ has finite products and the localization functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ commutes with finite products.*

Proof. It suffices to show that $f \times g \in W$ whenever $f, g \in W$. Let $f : A \to A'$ and $g : B \to B'$ morphisms in W and consider the following pullbacks of algebras:

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d, "f\times\mathrm{id}"'] & A \arrow[d, "\sim"', "f"] \\ A' \times B \arrow[r, two heads] & A' \end{tikzcd} \qquad \begin{tikzcd} A' \times B \arrow[r] \arrow[d, "\mathrm{id}\times g"'] & B \arrow[d, "\sim"', "g"] \\ A' \times B' \arrow[r, two heads] & B' \end{tikzcd}$$

By [1, I.4 Lemma 2], we have that $f \times \mathrm{id}$ and $g \times \mathrm{id}$ are weak equivalences and thus $f \times g = (\mathrm{id} \times g) \circ (f \times \mathrm{id})$ is a weak equivalence as well. □

2.3.1. *Examples.* For any model category $\mathcal{M}$, let $\mathcal{M}_f$ be the full subcategory consisting of the fibrant objects. Then $\mathcal{M}_f$ is naturally a category of fibrant objects, with weak equivalences and fibrations restricted from $\mathcal{M}$. For example, let **Top** be the category of compactly generated weakly Hausdorff topological spaces together with continuous maps. Then **Top** is a category of fibrant objects with W the weak homotopy equivalences and $\mathcal{F}$ the Serre fibrations.

Two examples in the context of C^*-algebras are provided in [26]. Let $C^*\mathbf{Alg}$ be the category of separable C^*-algebras with $*$-homomorphisms. The category $C^*\mathbf{Alg}$ is naturally enriched over **Top** upon endowing $\hom_{C^*\mathbf{Alg}}(A, B)$ with the compact-open topology; see [26, Remark 2.1]. If B is a C^*-algebra and X is a compact Hausdorff space, we write B^X for the C^*-algebra $\hom_{\mathbf{Top}}(X, B) \cong C(X) \otimes B$. Let $\mathrm{ev}_x : B^X \to B$ be the evaluation at $x \in X$. Two $*$-homomorphisms $f_0, f_1 : A \to B$ are *homotopic* if there exists a $*$-homomorphism $H : A \to B^I$, with $I = [0, 1]$, such that $f_0 = \mathrm{ev}_0 \circ H$ and $f_1 = \mathrm{ev}_1 \circ H$. In this case, we write $f_0 \sim f_1$. Denote the set of homotopy classes of $*$-homomorphisms $A \to B$ by

$$[A, B] = \hom_{C^*\mathbf{Alg}}(A, B)/\!\sim .$$

We have a category $[C^*\mathbf{Alg}]$ whose objects are the separable C^*-algebras and whose hom-sets are the sets $[A, B]$. A $*$-homomorphism f is a *homotopy equivalence* if $[f]$ is invertible in $[C^*\mathbf{Alg}]$. By [26, Proposition 2.13], f is a homotopy equivalence if and only if the induced map $f_* : \hom_{C^*\mathbf{Alg}}(D, A) \to \hom_{C^*\mathbf{Alg}}(D, B)$ is a weak equivalence in **Top** for all $D \in C^*\mathbf{Alg}$. A $*$-homomorphism $p : E \to B$ is called a *Schochet fibration* [25] if the induced map $p_* : \hom_{C^*\mathbf{Alg}}(D, E) \to \hom_{C^*\mathbf{Alg}}(D, B)$ is a Serre fibration in **Top** for all $D \in C^*\mathbf{Alg}$. By [26, Theorem 2.19], we have a category of fibrant objects $(C^*\mathbf{Alg}, \mathcal{F}_{\mathrm{Sch}}, W_h)$ where the weak equivalences are the homotopy equivalences and the fibrations are the Schochet fibrations. The corresponding homotopy category is $[C^*\mathbf{Alg}]$. This example is not of the form $\mathcal{M}_f$ for any model category $\mathcal{M}$; see [26, Appendix].

Let KK be the category whose objects are the separable C^*-algebras and whose hom-sets are the Kasparov groups $KK(A, B)$. A $*$-homomorphism $f : A \to B$ is a *KK-equivalence* if the induced morphism $f_* : KK(D, A) \to KK(D, B)$ is an isomorphism for all $D \in C^*\mathbf{Alg}$. By [26, Theorem 2.29], we have a category of fibrant objects $(C^*\mathbf{Alg}, \mathcal{F}_{\mathrm{Sch}}, W_{KK})$ where the weak equivalences are the KK-equivalences and the fibrations are the Schochet fibrations.

2.4. *Bivariant algebraic K-theory.* Let us recall the details of the algebraic kk-theory introduced in [6]. An alternative construction of kk-theory was given in [11]. An *extension* is a short exact sequence of ℓ-algebras

$$\mathscr{E} : A \to B \to C \tag{2.8}$$

that splits as a sequence of ℓ-modules. An *excisive homology theory* on $\mathbf{Alg}$ consists of a functor $H : \mathbf{Alg} \to \mathscr{T}$ into a triangulated category $\mathscr{T}$ together with a morphism $\partial_{\mathscr{E}} : \Omega H(C) \to H(A)$ for every extension $\mathscr{E}$ such that:

(1) The triangle $\Omega H(C) \xrightarrow{\partial_{\mathscr{E}}} H(A) \to H(B) \to H(C)$ is distinguished for every extension $\mathscr{E}$.

(2) The morphisms $\partial_{\mathscr{E}}$ are natural with respect to morphisms of extensions — a *morphism of extensions* is just a morphism of short exact sequences in $\mathbf{Alg}$.

Here we write Ω for the desuspension functor in $\mathscr{T}$. We say that H is *homotopy invariant* if it sends polynomially homotopic morphisms to the same morphism. We say that H is *M_∞-stable* if it sends the upper-left corner inclusion $A \to M_\infty A$, $a \mapsto a \cdot e_{1,1}$, to an isomorphism for every algebra A. Here, we write $M_\infty A$ for the algebra of finitely supported matrices with coefficients in A indexed over $(\mathbb{Z}_{\geq 1})^2$.

Theorem 2.9 [6, Theorem 6.6.2]. *There exists a triangulated category kk endowed with a functor $j : \mathbf{Alg} \to kk$ that is the universal homotopy invariant and M_∞-stable excisive homology theory. That is, any homotopy invariant and M_∞-stable excisive homology theory factors uniquely through j.*

The triangulated category kk and the functor j are uniquely determined up to equivalence of triangulated categories. The category kk can be constructed so that its objects are the ℓ-algebras and the functor j is the identity on objects. We often write $kk(A, B)$ instead of $kk(j(A), j(B))$.

For $A, B \in \mathbf{Alg}$ and $n \in \mathbb{Z}$, put $kk_n(A, B) := kk(A, \Omega^n B)$. These groups fit into a long exact sequence

$$\cdots \to kk_{n+1}(D, C) \to kk_n(D, A) \to kk_n(D, B) \to kk_n(D, C) \to \cdots$$

upon applying $kk(D, -)$ to the extension (2.8).

Theorem 2.10 [6, Theorem 8.2.1]. *Let A be an algebra and let $\mathrm{KH}_*(A)$ be Weibel's homotopy K-theory groups of A [27]. Then there exists a natural isomorphism*

$$kk_*(\ell, A) \cong \mathrm{KH}_*(A).$$

For $A \in \mathbf{Alg}$, put $PA := tA[t]$ and $\Omega A := (t^2 - t)A[t]$. These are called the *path algebra* and the *loop algebra* of A respectively and fit into the *loop extension* of A; see [6, Section 4.5]. The desuspension Ω in the triangulated category kk is induced by these loop algebras; see [6, Lemma 6.3.9]. Let $f : A \to B$ be an algebra homomorphism. Let P_f be defined by the following morphism of extensions where the square on the right is a pullback:

$$\begin{tikzcd} \Omega B \arrow[r, "\iota_f"] \arrow[d, equal] & P_f \arrow[r, "\pi_f"] \arrow[d] & A \arrow[d, "f"] \\ \Omega B \arrow[r, "\mathrm{incl}"] & PB \arrow[r, "\mathrm{ev}_1"] & B \end{tikzcd} \tag{2.11}$$

By [6, Definition 6.5.1], a triangle in kk is *distinguished* if and only if it is isomorphic to one of the form

$$\Omega B \xrightarrow{j(\iota_f)} P_f \xrightarrow{j(\pi_f)} A \xrightarrow{j(f)} B$$

for some algebra homomorphism $f : A \to B$.

3. Category of fibrant objects structures on Alg

Let W_H be the class of polynomial homotopy equivalences in $\mathbf{Alg}$, and let W_{kk} be the class of kk-equivalences, that is, those algebra homomorphisms that become isomorphisms in kk. In this section we study the existence of category of fibrant objects structures on $\mathbf{Alg}$ having each of these classes as the class of weak equivalences. Consider the class Fib of surjective morphisms in $\mathbf{Alg}$ that have a linear section. We prove in Proposition 3.11 that ($\mathbf{Alg}$, Fib, W_{kk}) is a category of fibrant objects. This result can be considered as an algebraic analogue of [26, Theorem 2.29]. Moreover, in Proposition 3.10 we prove that there is no reasonable class of fibrations $\mathcal{F}$ making ($\mathbf{Alg}$, $\mathcal{F}$, W_H) into a category of fibrant objects. This shows that the algebraic counterpart of [26, Theorem 2.19] does not hold. This happens because polynomial homotopies behave worse than continuous ones, as it is shown in the following lemma.

Lemma 3.1. *Let ℓ be a domain such that $\mathrm{KH}_0(\ell) \neq 0$; this happens, for example, if ℓ is a field or a principal ideal domain. Put $\Omega_0 = (t^2 - t)\ell[t]$ and $\Omega_1 = \{(p, q) \in \ell[t] \times \ell[t] : \mathrm{ev}_1(p) = \mathrm{ev}_0(q),\ \mathrm{ev}_0(p) = \mathrm{ev}_1(q) = 0\}$. Then the morphism $\beta : \Omega_0 \to \Omega_1$ defined by $\beta(p) = (p, 0)$ is not a polynomial homotopy equivalence.*

Proof. Suppose that β is a polynomial homotopy equivalence. Then $[\beta]$ is an isomorphism in $[\mathbf{Alg}]$ and it has an inverse $[\alpha]$ where α is an algebra homomorphism $\alpha : \Omega_1 \to \Omega_0$. In particular, $[\alpha] \circ [\beta] = [\mathrm{id}_{\Omega_0}]$. We have

$$\alpha(t^2 - t, 0)\,\alpha(0, t^2 - t) = \alpha((t^2 - t, 0)(0, t^2 - t)) = \alpha(0, 0) = 0 \in \Omega_0 \subset \ell[t].$$

As $\ell[t]$ is a domain, we should have $\alpha(t^2 - t, 0) = 0$ or $\alpha(0, t^2 - t) = 0$. Suppose that $\alpha(t^2 - t, 0) = 0$. Then we have

$$\begin{aligned} \alpha\Big((t^2 - t)\Big(\sum_0^n a_i t^i\Big), 0\Big) &= \alpha\Big((a_0(t^2 - t), 0) + \Big((t^2 - t)\Big(\sum_1^n a_i t^i\Big), 0\Big)\Big) \\ &= a_0 \alpha(t^2 - t, 0) + \alpha(t^2 - t, 0)\,\alpha\Big(\sum_1^n a_i t^i, \sum_1^n a_i (1 - t)^i\Big) = 0, \end{aligned}$$

showing that $\alpha \circ \beta$ is the zero morphism. Then $[\mathrm{id}_{\Omega_0}] = [0]$ and $\Omega_0 \cong 0$ in [**Alg**]. Since $j : \mathbf{Alg} \to kk$ factors through [**Alg**], then $\Omega_0 \cong 0$ in kk. But this cannot happen since we have

$$0 = kk(\Omega_0, \Omega_0) = kk(\Omega\ell, \Omega\ell) \cong kk(\ell, \ell) \cong KH_0(\ell) \neq 0.$$

If $\alpha(0, t^2 - t) = 0$, we can show that $\alpha \circ \tilde{\beta}$ is the zero morphism, where $\tilde{\beta} : \Omega_0 \to \Omega_1$ is given by $\tilde{\beta}(p) = (0, p)$. But β and $\tilde{\beta}$ are homotopic morphisms; indeed, an elementary homotopy $H : \Omega_0 \to \Omega_1[s]$ is given by

$$H(p) = \big(p(t(1-s)),\, p(1 - s(1-t))\big)$$

Then $[\mathrm{id}_{\Omega_0}] = [\alpha] \circ [\beta] = [\alpha] \circ [\tilde{\beta}] = [\alpha \circ \tilde{\beta}] = 0$ and we get to the same contradiction as before. □

Remark 3.2. It is easily verified using excision that β is a kk-equivalence; see [6, Section 6.3].

Remark 3.3. The analogue of β in the topological context is identified with the morphism $\beta : \mathbb{C}(0, 1) \to \mathbb{C}(0, 1)$ given by

$$\beta(f)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Here, $\mathbb{C}(0, 1)$ denotes the C^*-algebra of continuous functions $[0, 1] \to \mathbb{C}$ that vanish at $t = 0, 1$. This β is easily seen to be a homotopy equivalence.

Recall the definition of the Kan complex $\underline{\mathrm{HOM}}(A, B)$ from (2.4). The following propositions will be useful later on.

Proposition 3.4 (cf. [26, Proposition 2.13]). *Let $f : A \to B$ be an algebra homomorphism. Then f is a homotopy equivalence if and only if for every algebra D, the induced morphism*

$$f_* : \underline{\mathrm{HOM}}(D, A) \to \underline{\mathrm{HOM}}(D, B) \tag{3.5}$$

is a weak equivalence of simplicial sets.

Proof. By [23, Theorem 3.10] for every $n \geq 0$ we have a natural bijection

$$\pi_n \underline{\mathrm{HOM}}(D, A) \cong [D, A_\bullet^{S^n}] = \operatorname*{colim}_r [D, A_r^{S^n}],$$

where $A_\bullet^{S^n}$ is the ind-algebra of polynomials on the n-dimensional cube with coefficients in A vanishing at the boundary of the cube; see [23, Example 2.11].

Suppose that (3.5) is a weak equivalence of simplicial sets for every algebra D. Upon applying π_0 we get bijections

$$f_* : [D, A] \to [D, B]$$

for every algebra D. By Yoneda, the latter implies that f is an isomorphism in [**Alg**], i.e., a polynomial homotopy equivalence.

Now suppose that f is a polynomial homotopy equivalence and fix an algebra D. To prove that (3.5) is a weak equivalence of simplicial sets, we must show that it induces a bijection upon applying π_n for all $n \geq 0$. By [23, Lemma 2.10], we have algebra isomorphisms $A_r^{S^n} \cong A \otimes \mathbb{Z}_r^{S^n}$ for all $n, r \geq 0$. Since tensoring with a ring preserves polynomial homotopy equivalences, we have a morphism of diagrams:

$$\begin{tikzcd}
{[D, A_0^{S^n}]} \arrow[r] \arrow[d, "f_*"', "\cong"] & {[D, A_1^{S^n}]} \arrow[r] \arrow[d, "f_*"', "\cong"] & {[D, A_2^{S^n}]} \arrow[r] \arrow[d, "f_*"', "\cong"] & \cdots \\
{[D, B_0^{S^n}]} \arrow[r] & {[D, B_1^{S^n}]} \arrow[r] & {[D, B_2^{S^n}]} \arrow[r] & \cdots
\end{tikzcd}$$

Upon taking colimit over r we get an isomorphism

$$f_* : \pi_n \underline{\mathrm{HOM}}(D, A) \to \pi_n \underline{\mathrm{HOM}}(D, B). \qquad \square$$

Proposition 3.6 (cf. [26, Corollary 2.6]). *For any algebra D, the functor $\underline{\mathrm{HOM}}(D, -) : \mathbf{Alg} \to \mathbf{sSet}$ preserves pullbacks.*

Proof. Since Ex^∞ preserves pullbacks, it suffices to show that $\underline{\mathrm{hom}}(D, -)$ preserves pullbacks. But the latter follows from the facts that $\underline{\mathrm{hom}}(D, A)_n = \hom_{\mathbf{Alg}}(D, A^{\Delta^n})$ and that $(-)^{\Delta^n} : \mathbf{Alg} \to \mathbf{Alg}$ preserves pullbacks since it is naturally isomorphic to tensoring with the polynomial algebra $\ell[s_1, \dots, s_n]$. $\square$

In [26, Theorem 2.19] it is proved that $(C^*\mathbf{Alg}, \mathcal{F}_{\mathrm{Sch}}, W_h)$ is a category of fibrant objects. One might expect a similar result to hold in the algebraic context — more precisely, that there exists a class of fibrations $\mathcal{F}$ making $(\mathbf{Alg}, \mathcal{F}, W_H)$ into a category of fibrant objects. Natural candidates for path-objects (FW_2) are the diagrams

$$B \xrightarrow[\sim]{\mathrm{const}} B_n[t] \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} B \times B, \tag{3.7}$$

where $B_n[t]$, ev_0 and ev_1 are defined in Section 2.1.2. Thus, we want the family $\mathcal{F}$ to contain the morphisms $\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1) : B_n[t] \to B \times B$. Mimicking the definition of a Schochet fibration, we could take $\mathcal{F}$ to be the class of algebra homomorphisms that induce a Kan fibration upon applying $\underline{\mathrm{HOM}}(D, -)$ for every algebra D. It is easily verified that this family $\mathcal{F}$ is closed by pullbacks but it does not contain the morphisms ev, as we proceed to explain. Define

$$P_0 = \{p \in \ell[t] : p(0) = 0\} \subset \ell_0[t], \quad P_1 = \{(p, q) \in \ell[t] \times \ell[t] : p(0) = 0,\, p(1) = q(0)\} \subset \ell_1[t],$$
$$\Omega_0 = \ker(\mathrm{ev}_1 : P_0 \to \ell), \quad \Omega_1 = \ker(\mathrm{ev}_1 : P_1 \to \ell).$$

These fit into a commutative diagram of algebras:

$$\begin{tikzcd}
& \Omega_0 \arrow[rr] \arrow[dl, "\beta"'] \arrow[dd] & & P_0 \arrow[dl, "\sim"'] \arrow[dd, two heads, "\mathrm{ev}_1"] \\
\Omega_1 \arrow[rr] \arrow[dd] & & P_1 \arrow[dd, two heads, "\mathrm{ev}_1", near start] & \\
& 0 \arrow[rr] \arrow[dl, equal] & & \ell \arrow[dl, equal] \\
0 \arrow[rr] & & \ell &
\end{tikzcd} \tag{3.8}$$

where the front and back faces are pullbacks, β is the morphism of Lemma 3.1, and $P_0 \xrightarrow{\sim} P_1$ is a homotopy equivalence since both P_0 and P_1 are contractible algebras. Suppose that the morphisms ev belong to $\mathcal{F}$. Since $\mathcal{F}$ is closed by pullbacks, the right vertical morphisms in (3.8) lie also in $\mathcal{F}$. By Propositions 3.4 and 3.6 we get the diagram of Kan complexes:

$$\begin{tikzcd}
 & \underline{\mathrm{HOM}}(D, \Omega_0) \arrow[rr] \arrow[dl, "\beta_*"'] \arrow[dd] & & \underline{\mathrm{HOM}}(D, P_0) \arrow[dl, "\sim"'] \arrow[dd, two heads] \\
\underline{\mathrm{HOM}}(D, \Omega_1) \arrow[rr, crossing over] \arrow[dd] & & \underline{\mathrm{HOM}}(D, P_1) & \\
 & \star \arrow[rr] \arrow[dl, equal] & & \underline{\mathrm{HOM}}(D, \ell) \arrow[dl, equal] \\
\star \arrow[rr] & & \underline{\mathrm{HOM}}(D, \ell) \arrow[from=uu, crossing over, two heads] &
\end{tikzcd} \tag{3.9}$$

where the front and back faces are homotopy pullbacks. It follows that β_* is a weak equivalence of simplicial sets. Then $\beta : \Omega_0 \to \Omega_1$ is a polynomial homotopy equivalence by Proposition 3.4, contradicting Lemma 3.1. This shows that $\mathcal{F}$ does not contain the morphisms ev. In fact, almost the same argument shows that there is no family of fibrations $\mathcal{F}$ that contains the morphisms ev and makes $(\mathbf{Alg}, \mathcal{F}, W_H)$ into a category of fibrant objects.

Proposition 3.10. *Let ℓ be a domain with $\mathrm{KH}_0(\ell) \neq 0$. Let W_H be the set of polynomial homotopy equivalences in* **Alg**. *Let $\mathcal{F}$ be a set of morphisms in* **Alg** *that is closed by pullbacks and that contains the evaluation map* $\mathrm{ev} : \ell[t] \to \ell \times \ell$ *defined by* $\mathrm{ev}(p) = (p(0), p(1))$. *Then* $(\mathbf{Alg}, \mathcal{F}, W_H)$ *is* ***not*** *a category of fibrant objects.*

Proof. Consider the commutative diagram of algebras (3.8). Since $\mathrm{ev} \in \mathcal{F}$ and $\mathcal{F}$ is closed by pullbacks, then the right vertical morphisms in (3.8) lie in $\mathcal{F}$ as well. If $(\mathbf{Alg}, \mathcal{F}, W_H)$ was a category of fibrant objects, then β would be a homotopy equivalence by the coglueing lemma [14, Lemma 9.10]. But β is not a homotopy equivalence by Lemma 3.1. It follows that $(\mathbf{Alg}, \mathcal{F}, W_H)$ is not a category of fibrant objects. □

Let Fib be the set of those surjective morphisms in **Alg** that have a linear section. By Proposition 3.10, $(\mathbf{Alg}, \mathrm{Fib}, W_H)$ is not a category of fibrant objects — all axioms are satisfied except for (FW_1). However, we do get a category of fibrant objects if we replace polynomial homotopy equivalences by kk-equivalences, as shown below.

Proposition 3.11. *Let W_{kk} be the set of kk-equivalences and let* Fib *be the set of those surjective morphisms in* **Alg** *that have a linear section. Then* $(\mathbf{Alg}, \mathrm{Fib}, W_{kk})$ *is a category of fibrant objects.*

Proof. Most axioms are straightforward to verify. Let us show that fibrations and trivial fibrations are preserved by pullbacks. Consider a *Milnor square* of algebras, i.e., a pullback square as follows where f

is surjective and has a linear section:

$$\begin{tikzcd} A \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

Since this square is a pullback of ℓ-modules, it follows from its universal property that $\bar{f}$ also belongs to Fib:

$$\begin{tikzcd} C \arrow[dr, dashed] \arrow[drr, bend left, "s\circ g"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\ & A \arrow[r, "\bar{g}"] \arrow[d, two heads, "\bar{f}"'] & B \arrow[d, two heads, "f"] \\ & C \arrow[r, "g"'] & D \arrow[u, bend right, "s"'] \end{tikzcd}$$

Suppose now that $f \in W_{kk}$. For each algebra E, we have a long exact Mayer–Vietoris sequence as follows [10, Lemma B.5], where $f^* : kk_*(D, E) \to kk_*(B, E)$ is an isomorphism:

$$\begin{tikzcd} \vdots \arrow[d] & & & \\ kk_*(D, E) \arrow[r, "\binom{f^*}{g^*}"] & kk_*(B, E) \oplus kk_*(C, E) \arrow[r, "(\bar{g}^* \ -\bar{f}^*)"] & kk_*(A, E) \arrow[r, "\partial"] & kk_{*-1}(D, E) \arrow[d] \\ (*) & (**) & & \vdots \end{tikzcd}$$

Let $x \in \ker \bar{f}^*$. By the exactness at $(**)$, there exists $z \in kk_*(D, E)$ such that $(f^*(z), g^*(z)) = (0, x)$. Since f^* is injective, then $z = 0$ and $x = 0$. This shows that $\bar{f}^*$ is injective.

It follows from the exactness at $(*)$ and the injectivity of f^* that ∂ is the zero morphism. Hence, $(\bar{g}^* \ -\bar{f}^*)$ is surjective.

Let $w \in kk_*(A, E)$. There exists a pair (u, v) such that

$$w = \bar{g}^*(u) - \bar{f}^*(v) = \bar{g}^*(f^*(t)) - \bar{f}^*(v) = \bar{f}^*(g^*(t)) - \bar{f}^*(v) = \bar{f}^*(g^*(t) - v).$$

This shows that $\bar{f}^*$ is surjective. Then $\bar{f}^*$ is an isomorphism and $\bar{f} \in W_{kk}$. □

Remark 3.12. Let $\mathcal{C}$ be a pointed category of fibrant objects. By [1, Theorem 3] there is a functor $\Omega : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ such that, for any object B and any path object B^I, ΩB is canonically identified with the fiber of $B^I \to B \times B$. Furthermore, ΩB has a natural group structure. The *stable homotopy category* of $\mathcal{C}$ is, by definition, the category $\mathrm{Ho}(\mathcal{C})[\Omega^{-1}]$ obtained from $\mathrm{Ho}(\mathcal{C})$ upon inverting this endofunctor Ω; see [26, Definition 1.23]. We say that $\mathcal{C}$ is *stable* if $\Omega : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ is an isomorphism.

Endow **Alg** with the category of fibrant objects structure of Proposition 3.11. A path object for an algebra B is the diagram (3.7) for any n. The loop endofunctor mentioned above is induced by $\Omega : \mathbf{Alg} \to \mathbf{Alg}$, $\Omega B = (t^2 - t)B[t]$. We will see in Theorem 4.24 that $\mathrm{Ho}(\mathbf{Alg}) \cong kk$. This implies that this category of fibrant objects is stable since $\Omega : kk \to kk$ is an isomorphism [6, Lemma 6.3.8].

4. Algebraic kk-theory as a localization of categories

In this section we prove that the functor $j : \mathbf{Alg} \to kk$ is the localization of $\mathbf{Alg}$ at the set of kk-equivalences in the sense of ordinary categories. Consider the following sets of morphisms in $\mathbf{Alg}$:

- W_H, the set of homotopy equivalences;
- $W_S := \{\iota_A : A \to M_\infty A : A \in \mathbf{Alg}\}$, the set of upper-left corner inclusions;
- W_E, the set of classifying maps of extensions

$$A \to B \to C$$

  where B is either a contractible ring or an infinite sum ring.

Write $\mathrm{Iso}(\mathcal{C})$ for the set of isomorphisms in a category $\mathcal{C}$. The main technical result of this section is the following.

Theorem 4.1 (cf. [3, Proposition 2.1; 17, Theorem 2.3). *The functor $j : \mathbf{Alg} \to kk$ is initial in the category of those functors $F : \mathbf{Alg} \to \mathcal{C}$ such that*:

- *$\mathcal{C}$ is a category with finite products and F preserves finite products.*
- $F(W_H \cup W_S \cup W_E) \subseteq \mathrm{Iso}(\mathcal{C})$.

The idea of the proof is simple. Suppose that we have a functor $F : \mathbf{Alg} \to \mathcal{C}$ satisfying the conditions stated in Theorem 4.1. We want to show that F factors uniquely through some $\tilde{F} : kk \to \mathcal{C}$. It is easy to define $\tilde{F}$ on objects: since j is the identity on objects we should have $\tilde{F}(A) = F(A)$ for all A. To define $\tilde{F}$ on morphisms, recall that any morphism $\phi : A \to B$ in kk is represented by some algebra homomorphism $f : J^v A \to M_k M_\infty B_r^{S^v}$ [6, Section 6.1]. Moreover, as we explain below, in $\mathcal{C}$ we have identifications $F(\Sigma^v J^v A) \cong F(A)$ and $F(\Sigma^v M_k M_\infty B_r^{S^v}) \cong F(B)$. We would then define $\tilde{F}(\phi)$ as the composite

$$F(A) \cong F(\Sigma^v J^v A) \xrightarrow{F(\Sigma^v f)} F(\Sigma^v M_k M_\infty B_r^{S^v}) \cong F(B).$$

The rest of this section is devoted to showing that this definition indeed makes sense and gives a well defined functor $\tilde{F} : kk \to \mathcal{C}$.

4.1. *Excision and F-equivalences.*

Definition 4.2. Let $F : \mathbf{Alg} \to \mathcal{C}$ be a functor. We say that an algebra homomorphism is an *F-equivalence* if it becomes an isomorphism upon applying $F(C \otimes -)$, for every algebra C.

Example 4.3. Let $F : \mathbf{Alg} \to \mathcal{C}$ be a functor such that $F(W_E) \subseteq \mathrm{Iso}(\mathcal{C})$ and let A be an algebra. Let $\rho_A : JA \to A^{S^1}$ be the classifying map of the loop extension of A [6, Section 4.5]. Then ρ_A is an F-equivalence. Indeed, for any algebra C, we have a morphism of extensions:

$$\begin{tikzcd} C \otimes JA \arrow[r] \arrow[d, "C\otimes\rho_A"'] & C \otimes TA \arrow[r] \arrow[d] & C \otimes A \arrow[d, equal] \\ C \otimes A^{S^1} \arrow[r] & C \otimes PA \arrow[r] & C \otimes A \end{tikzcd}$$

that induces a commutative square

$$\begin{tikzcd} J(C\otimes A) \arrow[r, "\mathrm{clas}"] \arrow[d, equal] & C\otimes JA \arrow[d, "C\otimes\rho_A"] \\ J(C\otimes A) \arrow[r, "\mathrm{clas}"] & C\otimes A^{S^1} \end{tikzcd}$$

where the horizontal morphisms belong to W_E and thus become isomorphisms upon applying F.

Lemma 4.4. *Let $F : \mathbf{Alg} \to \mathcal{C}$ be a functor such that $F(W_E) \subseteq \mathrm{Iso}(\mathcal{C})$. If $f : A \to B$ is an F-equivalence, then $J(f) : JA \to JB$ is an F-equivalence too.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} F(C\otimes JA) \arrow[r, "F(C\otimes\rho_{JA})"] \arrow[d, "F(C\otimes J(f))"'] & F(C\otimes A^{S^1}) \cong F((\ell^{S^1}\otimes C)\otimes A) \arrow[d, "F((\ell^{S^1}\otimes C)\otimes f)"] \\ F(C\otimes JB) \arrow[r, "F(C\otimes\rho_{JB})"] & F(C\otimes B^{S^1}) \cong F((\ell^{S^1}\otimes C)\otimes B) \end{tikzcd}$$

The right vertical morphism is an isomorphism since f is an F-equivalence and both horizontal morphisms are isomorphisms by Example 4.3. Then the left vertical morphism is an isomorphism too. $\square$

Lemma 4.5. *Let $F : \mathbf{Alg} \to \mathcal{C}$ be a functor such that $F(W_E) \subseteq \mathrm{Iso}(\mathcal{C})$. If f is an algebra homomorphism that fits into a morphism of extensions*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d] & A'' \arrow[d, "f''"] \\ B \arrow[r] & B' \arrow[r] & B'' \end{tikzcd} \tag{4.6}$$

where f'' is an F-equivalence and A' and B' are either contractible rings or infinite sum rings, then f is an F-equivalence. In particular, the classifying maps of extensions where the middle term is either a contractible ring or an infinite sum ring are F-equivalences.

Proof. For any algebra C, the morphism (4.6) induces a commutative square:

$$\begin{tikzcd} F(J(C\otimes A'')) \arrow[r, "F(\mathrm{clas})"] \arrow[d, "F(J(C\otimes f''))"'] & F(C\otimes A) \arrow[d, "F(C\otimes f)"] \\ F(J(C\otimes B'')) \arrow[r, "F(\mathrm{clas})"] & F(C\otimes B) \end{tikzcd}$$

The left vertical morphism is an isomorphism by Lemma 4.4, since f'' — and thus $C\otimes f''$ — is an F-equivalence. The horizontal morphisms are isomorphisms since they result from applying F to morphisms in W_E. Then the right vertical morphism is an isomorphism too. $\square$

Lemma 4.7. *Let $F : \mathbf{Alg} \to \mathcal{C}$ be a functor such that $F(W_E) \subseteq \mathrm{Iso}(\mathcal{C})$. Then the following algebra homomorphisms are F-equivalences for any algebra A:*

(1) $x_A : J\Sigma A \to M_\infty A$, *the classifying map of the cone extension of* A; *see* [6, Section 4.7].

(2) $c_A : J\Sigma A \to \Sigma JA$, *the classifying map of the extension*

$$\Sigma JA \to \Sigma TA \to \Sigma A.$$

(3) $\gamma_A^n : A_r^{S^n} \to A_{r+1}^{S^n}$, *the morphism induced by the last vertex map.*

(4) $\mu_A^{m,n} : (A_r^{S^m})_s^{S^n} \to A_{r+s}^{S^{m+n}}$, *the morphism defined in* [23, Section 3.1].

Proof. To see that c_A and x_A are F-equivalences, note that they fit into the following morphisms of extensions and use Lemma 4.5:

$$\begin{tikzcd} J\Sigma A \arrow[r] \arrow[d, "c_A"'] & T\Sigma A \arrow[r] \arrow[d] & \Sigma A \arrow[d, equal] \\ \Sigma JA \arrow[r] & \Sigma TA \arrow[r] & \Sigma A \end{tikzcd} \qquad \begin{tikzcd} J\Sigma A \arrow[r] \arrow[d, "x_A"'] & T\Sigma A \arrow[r] \arrow[d] & \Sigma A \arrow[d, equal] \\ M_\infty A \arrow[r] & \Gamma A \arrow[r] & \Sigma A \end{tikzcd}$$

To prove that γ_A^n is a weak equivalence we will proceed by induction on n. For $n = 0$ the result is obvious since $A_r^{S^0} \cong A$ for any r and γ_A^0 is identified with the identity of A. For the inductive step, recall from [24, Section 2.28] that we have a morphism of extensions

$$\begin{tikzcd} A_r^{S^{n+1}} \arrow[r] \arrow[d, "\gamma_A^{n+1}"'] & P(n, A)_r \arrow[r] \arrow[d] & A_r^{S^n} \arrow[d, "\gamma_A^n"] \\ A_{r+1}^{S^{n+1}} \arrow[r] & P(n, A)_{r+1} \arrow[r] & A_{r+1}^{S^n} \end{tikzcd}$$

where the middle terms are contractible. If γ_A^n is an F-equivalence, then γ_A^{n+1} is an F-equivalence as well by Lemma 4.5.

Let us prove that $\mu_A^{m,n}$ is an F-equivalence. First note that it suffices to consider the case $r = s = 0$ since we have a commutative square

$$\begin{tikzcd} (A_r^{S^m})_s^{S^n} \arrow[r, "\mu_A^{m,n}"] & A_{r+s}^{S^{m+n}} \\ (A_0^{S^m})_0^{S^n} \arrow[r, "\mu_A^{m,n}"] \arrow[u] & A_0^{S^{m+n}} \arrow[u] \end{tikzcd}$$

where the vertical morphisms are F-equivalences by (3). We will proceed by induction on n. For $n = 0$ there is nothing to prove, since $(A^{S^m})^{S^0} \cong A^{S^m}$ and $\mu_A^{m,0}$ identifies with the identity of A^{S^m}. For the inductive step, recall from [24, Example 2.29] that we have a morphism of extensions

$$\begin{tikzcd} (A^{S^m})^{S^{n+1}} \arrow[r] \arrow[d, "\mu_A^{m,n+1}"'] & P(n, A^{S^m}) \arrow[r] \arrow[d] & (A^{S^m})^{S^n} \arrow[d, "\mu_A^{m,n}"] \\ A^{S^{m+n+1}} \arrow[r] & P(m+n, A) \arrow[r] & A^{S^{m+n}} \end{tikzcd}$$

where the middle terms are contractible. If $\mu_A^{m,n}$ is an F-equivalence, then $\mu_A^{m,n+1}$ is an F-equivalence as well by Lemma 4.5. $\square$

4.2. *Group objects in the localization.*

Lemma 4.8. *Let $\mathcal{C}$ be a category with finite products and let $F : \mathbf{Alg} \to \mathcal{C}$ be a functor that preserves finite products and such that $F(W_H \cup W_S) \subseteq \mathrm{Iso}(\mathcal{C})$. Then*:

1. *For every algebra A, $F(A)$ is a commutative monoid object in $\mathcal{C}$.*
2. *For every algebra homomorphism $f : A \to B$, $F(f) : F(A) \to F(B)$ is a morphism of monoid objects in $\mathcal{C}$.*

Proof. Let A be an algebra and let $\overline{m}_A : A \times A \to M_2(A)$ be defined by

$$(a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Let $m_A : F(A) \times F(A) \to F(A)$ be the following composite in $\mathcal{C}$:

$$F(A) \times F(A) \cong F(A \times A) \xrightarrow{F(\overline{m}_A)} F(M_2 A) \xrightarrow[\cong]{F(\iota_A)^{-1}} F(A)$$

Note that $F(0)$ is a terminal object in $\mathcal{C}$ since F preserves products. Let $u_A : F(0) \to F(A)$ be induced by the zero morphism $0 \to A$. We claim that m_A and u_A are, respectively, a multiplication and a unit that make $F(A)$ into a monoid object. Let us prove that m_A is associative, i.e., that the following diagram in $\mathcal{C}$ commutes:

$$\begin{tikzcd}
F(A) \times F(A) \times F(A) \arrow[r, "\mathrm{id} \times m_A"] \arrow[d, "m_A \times \mathrm{id}"'] & F(A) \times F(A) \arrow[d, "m_A"] \\
F(A) \times F(A) \arrow[r, "m_A"] & F(A)
\end{tikzcd}$$

Unravelling the definition of m_A, one shows that the commutativity of the square above is equivalent to that of the outer square in the following diagram:

$$\begin{tikzcd}
F(A \times A \times A) \arrow[r, "F(\mathrm{id} \times \overline{m}_A)"] \arrow[d, "F(\overline{m}_A \times \mathrm{id})"'] & F(A \times M_2 A) \arrow[r, "F(\mathrm{id} \times \iota_A)^{-1}"] \arrow[d, "F(\iota_A \times \mathrm{id})"] & F(A \times A) \arrow[d, "F(\overline{m}_A)"] \\
F(M_2 A \times A) \arrow[r, "F(\mathrm{id} \times \iota_A)"] \arrow[d, "F(\iota_A \times \mathrm{id})^{-1}"'] & F(M_2 A \times M_2 A) \arrow[r, phantom, "(\star)"] \arrow[dr, "F(\overline{m}_{M_2 A})"] & F(M_2 A) \arrow[d, "F(\iota_{M_2 A})"] \\
F(A \times A) \arrow[r, "F(\overline{m}_A)"'] & F(M_2 A) \arrow[r, "F(\iota_{M_2 A})"'] & F(M_4 A)
\end{tikzcd} \tag{4.9}$$

Suppose for a moment that both trapezoids in (4.9) commute. Then, to prove that the outer square in (4.9) commutes it suffices to show that $\overline{m}_{M_2 A} \circ (\iota_A \times \mathrm{id}) \circ (\mathrm{id} \times \overline{m}_A)$ and $\overline{m}_{M_2 A} \circ (\mathrm{id} \times \iota_A) \circ (\overline{m}_A \times \mathrm{id})$ become equal upon applying F. The latter is easily verified, as we proceed to explain. We have

$$[\overline{m}_{M_2A} \circ (\iota_A \times \mathrm{id}) \circ (\mathrm{id} \times \overline{m}_A)](a_1, a_2, a_3) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \end{pmatrix},$$

$$[\overline{m}_{M_2A} \circ (\mathrm{id} \times \iota_A) \circ (\overline{m}_A \times \mathrm{id})](a_1, a_2, a_3) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and both matrices are conjugate by a permutation matrix. Thus, they induce the same morphism in $\mathcal{C}$ by [5, Proposition 2.2.6]. To show that both trapezoids in (4.9) commute one can proceed in a similar fashion. For example, the trapezoid marked with ($\star$) commutes since we have

$$[\iota_{M_2A} \circ \overline{m}_A](a_1, a_2) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [\overline{m}_{M_2A} \circ (\iota_A \times \mathrm{id}) \circ (\mathrm{id} \times \iota_A)](a_1, a_2) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and both matrices are conjugate by a permutation matrix. This proves the associativity of m_A. The commutativity of m_A and the fact that u_A is a unit are easily verified. This finishes the proof of (1). For (2), note that m_A and u_A are natural in A with respect to algebra homomorphisms since both $\overline{m}_A$ and ι_A are. □

Lemma 4.10. *Let $\mathcal{C}$ be a category with finite products and let $F : \mathbf{Alg} \to \mathcal{C}$ be a functor that preserves finite products and such that $F(W_H \cup W_S \cup W_E) \subseteq \mathrm{Iso}(\mathcal{C})$. Then:*

(1) *$F(A)$ is a commutative group object in $\mathcal{C}$ for every $A \in \mathbf{Alg}$ and moreover this group structure coincides with the monoid structure from Lemma 4.8.*

(2) *$F(f) : F(A) \to F(B)$ is a morphism of group objects in $\mathcal{C}$ for every morphism $f : A \to B$ in $\mathbf{Alg}$.*

(3) *The function $F^{\mathrm{ind}}(C \otimes -) : [A, B^{S^1}_\bullet] \to \hom_{\mathcal{C}}(F(C \otimes A), F(C \otimes B^{S^1}_\bullet))$ is a group homomorphism for any $A, B, C \in \mathbf{Alg}$.*

Proof. Note that (2) will follow from Lemma 4.8(2) once we prove (1).

To prove (1), recall from Lemma 4.7(3) that we have an isomorphism $F(\gamma_A) : F(A^{S^1}_r) \to F(A^{S^1}_{r+1})$ for every $r \geq 0$. This implies that $F(A^{S^1})$ and $F(A^{S^1}_\bullet)$ are isomorphic objects of $\mathcal{C}^{\mathrm{ind}}$. It is well known that $A^{S^1}_\bullet$ is a group object in $[\mathbf{Alg}]^{\mathrm{ind}}$; see [6, Theorem 3.3.2] and [23, Theorem 3.10]. Upon applying F^{ind}, we get that $F(A^{S^1}_\bullet) \cong F(A^{S^1})$ is a group object in $\mathcal{C}^{\mathrm{ind}}$. Hence $F(A^{S^1})$ is a group object in $\mathcal{C}$. We claim that the group operation in $F(A^{S^1})$ coincides with the commutative monoid operation from Lemma 4.8. Indeed, the group operation in $F(A^{S^1})$ can be described in terms of concatenation [24, Example 2.20] as the following composite:

$$F(A^{S^1}) \times F(A^{S^1}) \cong F(A^{S^1} \times A^{S^1}) \xrightarrow{F(\mathrm{concat})} F(A^{S^1}_1) \xrightarrow[\cong]{F(\gamma_A)^{-1}} F(A^{S^1}).$$

It is then clear that this composite is a morphism of monoid objects since both $F(\text{concat})$ and $F(\gamma_A)$ are by Lemma 4.8(2). It follows that both operations coincide by the Hilton–Eckmann argument. This shows that the monoid structure of $F(A^{S^1})$ from Lemma 4.8 is indeed a commutative group structure.

Now consider the following chain of isomorphisms in $\mathcal{C}$:

$$F(A) \xrightarrow[\cong]{F(\iota_A)} F(M_\infty A) \xleftarrow[\cong]{F(x_A)} F(J\Sigma A) \xrightarrow[\cong]{F(\rho_{\Sigma A})} F((\Sigma A)^{S^1}).$$

By Lemma 4.8(2), Example 4.3 and Lemma 4.7(1), each of these is a monoid isomorphism for the monoid structure from Lemma 4.8. We have shown that the monoid structure of $F((\Sigma A)^{S^1})$ is indeed a commutative group structure. It follows that the monoid structure of $F(A)$ is a commutative group structure as well.

The assertion (3) follows from the above and [24, Proposition 2.46(ii)]. $\square$

4.3. *An explicit construction of the kk-theory category.* Following [22], we proceed to recall one way to construct the algebraic kk-theory category defined by Cortiñas and Thom in [6]. The construction goes in three steps:

(1) Construct a triangulated category $\mathfrak{K}$ whose objects are the pairs (A, m) with $A \in \mathbf{Alg}$ and $m \in \mathbb{Z}$, and whose morphism sets are defined by

$$\hom_{\mathfrak{K}}((A, m), (B, n)) = \operatorname*{colim}_v [J^{m+v} A, B_\bullet^{S^{n+v}}];$$

see [24, Definition 3.4] for more details. There is a functor $j : \mathbf{Alg} \to \mathfrak{K}$, $j(A) = (A, 0)$, that is the universal excisive and homotopy invariant homology theory in the sense of [6, Section 6.6]; see [24, Theorem 10.16] for a proof of this statement. This category $\mathfrak{K}$ is equivalent to the one defined by Garkusha in [11, Theorem 2.6] since they both share the same universal property.

(2) Construct a triangulated category $\mathfrak{K}_f$ whose objects are those of $\mathfrak{K}$ and whose morphism sets are defined by

$$\hom_{\mathfrak{K}_f}((A, m), (B, n)) = \operatorname*{colim}_p \hom_{\mathfrak{K}}((A, m), (M_p B, n)).$$

Here, the transition maps are induced by the upper-left corner inclusions $M_p B \to M_{p+1} B$; see [22, Definition 5.1.3] for more details on the definition of $\mathfrak{K}_f$. There is a functor $j_f : \mathbf{Alg} \to \mathfrak{K}_f$, $j_f(A) = (A, 0)$, that is the universal excisive, homotopy invariant and M_n-stable homology theory; see [22, Theorem 5.1.12] for a proof of this statement. This category $\mathfrak{K}_f$ is equivalent to the one defined by Garkusha in [11, Theorem 6.5] since they both share the same universal property.

(3) Construct a triangulated category $\mathfrak{K}_s$ whose objects are those of $\mathfrak{K}_f$ and whose morphism sets are defined by

$$\hom_{\mathfrak{K}_s}((A, m), (B, n)) = \hom_{\mathfrak{K}_f}((A, m), (M_\infty B, n));$$

see [22, Definition 5.2.7] for more details. There is a functor $j_s : \mathbf{Alg} \to \mathfrak{K}_s$, $j_s(A) = (A, 0)$, that is the universal excisive, homotopy invariant and M_∞-stable homology theory; see [22, Theorem 5.2.16] for a proof of this statement. This category $\mathfrak{K}_s$ is equivalent to the category kk defined by Cortiñas and

Thom [6] and to the category defined by Garkusha in [11, Theorem 9.3] since they all share the same universal property.

Remark 4.11. The translation functors of the triangulated categories $\mathfrak{K}$, $\mathfrak{K}_f$ and $\mathfrak{K}_s$ are given on objects by $(A, n) \mapsto (A, n+1)$. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{K} \arrow[r, "t_f"] & \mathfrak{K}_f \arrow[r, "t_s"] & \mathfrak{K}_s \\ & \mathbf{Alg} \arrow[ul, bend left, "j"] \arrow[u, "j_f"'] \arrow[ur, bend right, "j_s"'] & \end{tikzcd} \tag{4.12}$$

where the horizontal functors are triangulated and are the identity on objects.

4.4. *Main theorem of the section.* We are now ready to prove Theorem 4.1 (see Theorem 4.19). For the rest of this section, we fix a category $\mathcal{C}$ with finite products and a functor $F : \mathbf{Alg} \to \mathcal{C}$ that preserves finite products and such that $F(W_H \cup W_S \cup W_E) \subseteq \mathrm{Iso}(\mathcal{C})$. We start by making precise how to identify $F(\Sigma^v J^v A) \cong F(A)$ (Definition 4.13) and $F(\Sigma^v A^{S^v}) \cong F(A)$ (Definition 4.15).

Definition 4.13. For $A \in \mathbf{Alg}$ and $n, k \geq 0$ we will define isomorphisms in $\mathcal{C}$:

$$\alpha_A^{n,k} : F(\Sigma^{k+n} J^k A) \xrightarrow{\cong} F(\Sigma^n A).$$

We proceed inductively on k. Let $\alpha_A^{n,0}$ be the identity of $F(\Sigma^n A)$. Let $\alpha_A^{n,1}$ be the following composite in $\mathcal{C}$:

$$F(\Sigma^{1+n} JA) = F(\Sigma^n(\Sigma JA)) \xrightarrow[\cong]{(c_A)_*^{-1}} F(\Sigma^n(J\Sigma A)) \xrightarrow[\cong]{(x_A)_*} F(\Sigma^n M_\infty A) \xrightarrow[\cong]{(\iota_A)_*^{-1}} F(\Sigma^n A).$$

Here, the left and middle morphisms are isomorphisms by Lemma 4.7 and the right morphism is an isomorphism by M_∞-stability. Now suppose that we have defined $\alpha_A^{n,h}$ for all $A \in \mathbf{Alg}$, all $n \geq 0$ and all $1 \leq h \leq k$. Let $\alpha_A^{n,k+1}$ be the following composite in $\mathcal{C}$:

$$F(\Sigma^{1+k+n} J^{k+1} A) = F(\Sigma^{1+k+n} J(J^k A)) \xrightarrow[\cong]{\alpha_{J^k A}^{k+n,1}} F(\Sigma^{k+n} J^k A) \xrightarrow[\cong]{\alpha_A^{n,k}} F(\Sigma^n A).$$

This defines $\alpha_A^{n,k}$ for all $A \in \mathbf{Alg}$ and all $n, k \geq 0$.

Lemma 4.14. *Let $A \in \mathbf{Alg}$ and $n, p, q \geq 0$. Then the following diagram in $\mathcal{C}$ commutes*:

$$\begin{tikzcd} F(\Sigma^{q+p+n} J^{p+q} A) \arrow[r, "\alpha_{J^p A}^{p+n,q}"] \arrow[dr, bend right, "\alpha_A^{n,p+q}"'] & F(\Sigma^{p+n} J^p A) \arrow[d, "\alpha_A^{n,p}"] \\ & F(\Sigma^n A) \end{tikzcd}$$

Proof. We proceed by induction on q. For $q = 0$ there is nothing to prove. For $q = 1$, the result holds by definition of $\alpha_A^{n,p+1}$; see Definition 4.13. Now suppose that the result holds for $q \geq 1$ and let us show that it also holds for $q + 1$. Consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
 & F(\Sigma^{1+q+p+n}J^{p+q+1}A) \arrow[ddl, bend right, "\alpha_A^{n,p+q+1}"'] \arrow[d, "\alpha_{J^{p+q}A}^{q+p+n,1}"] \arrow[ddr, bend left, "\alpha_{J^pA}^{p+n,q+1}"] & \\
 & F(\Sigma^{q+p+n}J^{p+q}A) \arrow[dl, "\alpha_A^{n,p+q}"'] \arrow[dr, "\alpha_{J^pA}^{p+n,q}"] & \\
F(\Sigma^n A) & & F(\Sigma^{p+n}J^pA) \arrow[ll, "\alpha_A^{n,p}"']
\end{tikzcd}$$

The two upper triangles commute by the case $q = 1$ and the lower triangle commutes by the inductive hypothesis. Then the outer triangle commutes as well, proving the result for $q + 1$. □

Definition 4.15. For $A \in \mathbf{Alg}$ and $n, k \geq 0$ we will define isomorphisms in $\mathcal{C}$:

$$\beta_A^{n,k} : F(\Sigma^{k+n}A^{S^k}) \xrightarrow{\cong} F(\Sigma^n A).$$

We proceed inductively on k. Let $\beta_A^{n,0}$ be the identity of $F(\Sigma^n A)$. Let $\beta_A^{n,1}$ be the following composite in $\mathcal{C}$:

$$F(\Sigma^{1+n}A^{S^1}) \xrightarrow[\cong]{(\rho_A)_*^{-1}} F(\Sigma^{1+n}JA) \xrightarrow[\cong]{\alpha_A^{n,1}} F(\Sigma^n A).$$

Here, the morphism on the left is an isomorphism by Example 4.3. Now suppose that we have defined $\beta_A^{n,h}$ for all $A \in \mathbf{Alg}$, all $n \geq 0$ and all $1 \leq h \leq k$. Let $\beta_A^{n,k+1}$ be the following composite in $\mathcal{C}$:

$$F(\Sigma^{1+k+n}A^{S^{k+1}}) \xrightarrow[\cong]{(\mu_A^{k,1})_*^{-1}} F(\Sigma^{1+k+n}(A^{S^k})^{S^1}) \xrightarrow[\cong]{\beta_{A^{S^k}}^{k+n,1}} F(\Sigma^{k+n}A^{S^k}) \xrightarrow[\cong]{\beta_A^{n,k}} F(\Sigma^n A).$$

Here, the morphism on the left is an isomorphism by Lemma 4.7. This defines $\beta_A^{n,k}$ for all $A \in \mathbf{Alg}$ and all $n, k \geq 0$.

Lemma 4.16. *Let $A \in \mathbf{Alg}$ and $n, p, q \geq 0$. Then the following diagram in $\mathcal{C}$ commutes*:

$$\begin{tikzcd}
F(\Sigma^{q+p+n}(A^{S^p})^{S^q}) \arrow[r, "(\mu_A^{p,q})_*"] \arrow[d, "\beta_{A^{S^p}}^{p+n,q}"'] & F(\Sigma^{q+p+n}A^{S^{p+q}}) \arrow[d, "\beta_A^{n,p+q}"] \\
F(\Sigma^{p+n}A^{S^p}) \arrow[r, "\beta_A^{n,p}"] & F(\Sigma^n A)
\end{tikzcd}$$

Proof. We proceed by induction on q. For $q = 0$ there is nothing to prove. For $q = 1$, the result holds by definition of $\beta_A^{n,p+1}$; see Definition 4.15. Now suppose that the result holds for $q \geq 1$ and let us show that it also holds for $q + 1$. Consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
F(\Sigma^{1+q+p+n}(A^{S^p})^{S^{q+1}}) \arrow[rrr, "(\mu_A^{p,q+1})_*"] \arrow[ddd, "\beta_{A^{S^p}}^{p+n,q+1}"'] & & & F(\Sigma^{1+q+p+n}A^{S^{p+q+1}}) \arrow[ddd, "\beta_A^{n,p+q+1}"] \\
& F(\Sigma^{1+q+p+n}((A^{S^p})^{S^q})^{S^1}) \arrow[ul, "(\mu_{A^{S^p}}^{q,1})_*"'] \arrow[r, "(\mu_A^{p,q})_*"] \arrow[d, "\beta_{(A^{S^p})^{S^q}}^{q+p+n,1}"'] & F(\Sigma^{1+q+p+n}(A^{S^{p+q}})^{S^1}) \arrow[ur, "(\mu_A^{p+q,1})_*"] \arrow[d, "\beta_{A^{S^{p+q}}}^{q+p+n,1}"] & \\
& F(\Sigma^{q+p+n}(A^{S^p})^{S^q}) \arrow[r, "(\mu_A^{p,q})_*"] \arrow[dl, "\beta_{A^{S^p}}^{p+n,q}"] & F(\Sigma^{q+p+n}A^{S^{p+q}}) \arrow[dr, "\beta_A^{n,p+q}"'] & \\
F(\Sigma^{p+n}A^{S^p}) \arrow[rrr, "\beta_A^{n,p}"] & & & F(\Sigma^n A)
\end{tikzcd}$$

The trapezoid on the top commutes by the associativity of μ, the rectangle in the center commutes by the naturality of β with respect to algebra homomorphisms, the left and right trapezoids commute by the case $q = 1$ and the trapezoid on the bottom commutes by the inductive hypothesis. Then the outer rectangle commutes as well, proving the result for $q + 1$. $\square$

Lemma 4.17. *Let $A \in$ **Alg** and $p, q, r \geq 0$. Then the following diagram in $\mathcal{C}$ commutes*:

$$\begin{tikzcd}
F(\Sigma^{p+q+r}J^p(A^{S^q})) \arrow[rr, "(-1)^{pq}(\kappa_A^{p,q})_*"] \arrow[d, "\alpha_{A^{S^q}}^{q+r,p}"'] & & F(\Sigma^{p+q+r}(J^pA)^{S^q}) \arrow[d, "\beta_{J^pA}^{p+r,q}"] \\
F(\Sigma^{q+r}A^{S^q}) \arrow[dr, "\beta_A^{r,q}"'] & & F(\Sigma^{p+r}J^pA) \arrow[dl, "\alpha_A^{r,p}"] \\
& F(\Sigma^r A) &
\end{tikzcd}$$

Note that the sign $(-1)^{pq}$ makes sense by Lemma 4.10(1).

Proof. We proceed by induction on q. For $q = 0$ there is nothing to prove. For $q = 1$, consider the following diagram, where all the morphisms are isomorphisms:

$$\begin{tikzcd}
F(\Sigma^{p+1+r}J^p(A^{S^1})) \arrow[rr, "(-1)^p(\kappa_A^{p,1})_*"] \arrow[dd, "\alpha_{A^{S^1}}^{1+r,p}"'] & & F(\Sigma^{p+1+r}(J^pA)^{S^1}) \arrow[dd, "\beta_{J^pA}^{p+r,1}"] \\
& F(\Sigma^{p+1+r}J^{p+1}A) \arrow[ul, "(\rho_A)_*"'] \arrow[ur, "(\rho_{J^pA})_*"] \arrow[dr, "\alpha_{J^pA}^{p+r,1}"] \arrow[dd, "\alpha_{JA}^{1+r,p}"] \arrow[u, phantom, "\star"] & \\
F(\Sigma^{1+r}A^{S^1}) \arrow[ur, phantom, "\star\star"] \arrow[ddr, "\beta_A^{r,1}"'] & & F(\Sigma^{p+r}J^pA) \arrow[ddl, "\alpha_A^{r,p}"] \\
& F(\Sigma^{1+r}JA) \arrow[ul, "(\rho_A)_*"'] \arrow[d, "\alpha_A^{r,1}"] \arrow[ur, phantom, "\star\star\star"] & \\
& F(\Sigma^r A) &
\end{tikzcd}$$

We have to prove that the outer diagram commutes. The triangle marked with $\star$ commutes by [24, Lemma 4.4] and Lemma 4.10(3). The square marked with $\star\star$ commutes by naturality of $\alpha_?^{1+r,p}$ and the square marked with $\star\star\star$ commutes by Lemma 4.14. The remaining two triangles commute by definition of β (Definition 4.15). This proves the result for $n = 1$.

Now suppose that the statement of the lemma holds for $q \geq 1$ and let us prove that it also holds for $q + 1$. Consider the following diagram in $\mathcal{C}$, where all the morphisms are isomorphisms. To ease notation we write $N := p + q + 1 + r$ and we omit writing the functor F, which should be applied everywhere.

$$
\begin{tikzcd}
\Sigma^N J^p(A^{S^{q+1}}) \arrow[rrrrrr, "(-1)^{p(q+1)}(\kappa_A^{p,q+1})_*"] \arrow[ddd, "\alpha^{q+1+r,p}_{A^{S^{q+1}}}"'] & & & & & & \Sigma^N (J^p A)^{S^{q+1}} \arrow[ddd, "\beta^{p+r,q+1}_{J^pA}"] \\
& \Sigma^N J^p((A^{S^q})^{S^1}) \arrow[ul, "(\mu_A^{q,1})_*"] \arrow[rr, "(-1)^p(\kappa^{p,1}_{A^{S^q}})_*"] \arrow[d, "\alpha^{q+1+r,p}_{(A^{S^q})^{S^1}}"] \arrow[drr, phantom, "\star"] & & \Sigma^N (J^p(A^{S^q}))^{S^1} \arrow[u, phantom, "\sharp\sharp"] \arrow[rr, "(-1)^{pq}(\kappa_A^{p,q})_*"] \arrow[d, "\beta^{p+q+r,1}_{J^p(A^{S^q})}"'] \arrow[drr, phantom, "\star\star"] & & \Sigma^N ((J^p A)^{S^q})^{S^1} \arrow[ur, "(\mu^{q,1}_{J^pA})_*"'] \arrow[d, "\beta^{p+q+r,1}_{(J^pA)^{S^q}}"'] & \\
& \Sigma^{q+1+r}(A^{S^q})^{S^1} \arrow[dl, "(\mu_A^{q,1})_*"] \arrow[drr, "\beta^{q+r,1}_{A^{S^q}}"] & & \Sigma^{p+q+r} J^p(A^{S^q}) \arrow[rr, "(-1)^{pq}(\kappa_A^{p,q})_*"] \arrow[d, "\alpha^{q+r,p}_{A^{S^q}}"] \arrow[drr, phantom, "\star"] \arrow[dll, phantom, "\sharp"] & & \Sigma^{p+q+r}(J^p A)^{S^q} \arrow[dr, "\beta^{p+r,q}_{J^pA}"'] & \\
\Sigma^{q+1+r} A^{S^{q+1}} \arrow[drrr, "\beta_A^{r,q+1}"] & & & \Sigma^{q+r} A^{S^q} \arrow[d, "\beta_A^{r,q}"] & & & \Sigma^{p+r} J^p A \arrow[dlll, "\alpha_A^{r,p}"'] \\
& & & \Sigma^r A & & &
\end{tikzcd}
$$

We have to prove that the outer diagram commutes. The pentagons marked with $\star$ commute by the inductive hypothesis. The square marked with $\star\star$ commutes by naturality of $\beta_?^{p+q+r,1}$. The square marked with $\sharp$ commutes by definition of β. The pentagon marked with $\sharp\sharp$ commutes by [24, Lemma 2.38]. It remains to verify that two trapezoids commute. The one on the left commutes by naturality of $\alpha_?^{q+1+r,p}$ and the one on the right commutes by Lemma 4.16. $\square$

Remark 4.18. Let $\underline{\mathfrak{K}}_s$ be the full subcategory of $\mathfrak{K}_s$ whose objects are the pairs $(A, 0)$ with $A \in \mathbf{Alg}$. We claim that the inclusion $\underline{\mathfrak{K}}_s \subseteq \mathfrak{K}_s$ is an equivalence of categories. We have to show that every object of $\mathfrak{K}_s$ is isomorphic to one of $\underline{\mathfrak{K}}_s$. For $n > 0$, we have

$$(A, n) \cong (J^n A, 0)$$

in $\mathfrak{K}$ (and thus in $\mathfrak{K}_s$) by [24, Lemma 7.2]. Moreover, we have

$$(A, -n) \cong (J^n \Sigma^n A, -n) \cong (\Sigma^n A, 0),$$

where the isomorphism on the right holds by [24, Lemma 7.2] and the one on the left is induced by $\alpha_A^{0,n} : (\Sigma^n J^n A, 0) \to (A, 0)$. It follows that $\underline{\mathfrak{K}}_s$ is equivalent to $\mathfrak{K}_s$ and, thus, to the category kk defined by Cortiñas and Thom in [6].

Theorem 4.19. *Let $\mathcal{C}$ be a category with finite products and let $F : \mathbf{Alg} \to \mathcal{C}$ be a functor that commutes with finite products and such that $F(W_H \cup W_S \cup W_E) \subseteq \mathrm{Iso}(\mathcal{C})$. Then there exists a unique functor $\tilde{F}$ making the following triangle commute*:

$$\begin{tikzcd} \mathbf{Alg} \arrow[r, "F"] \arrow[dr, bend right, "j_s"'] & \mathcal{C} \\ & \underline{\mathfrak{K}}_s \arrow[u, dashed, "\exists!\,\tilde{F}"'] \end{tikzcd} \tag{4.20}$$

Proof. Let $\underline{\mathfrak{K}}$ (respectively $\underline{\mathfrak{K}}_f$) denote the full subcategory of $\mathfrak{K}$ (respectively $\mathfrak{K}_f$) whose objects are the pairs $(A, 0)$ with $A \in \mathbf{Alg}$. To alleviate notation, we write A for an object $(A, 0)$ either of $\underline{\mathfrak{K}}$, $\underline{\mathfrak{K}}_f$ or $\underline{\mathfrak{K}}_s$. To define $\tilde{F}$ on objects, put $\tilde{F}(A) = F(A)$ for every $A \in \mathbf{Alg}$.

We will start by extending F to $\underline{\mathfrak{K}}$, that is, we will define a functor $\hat{F}$ making the following triangle commute:

$$\begin{tikzcd} \mathbf{Alg} \arrow[r, "F"] \arrow[dr, bend right, "j"'] & \mathcal{C} \\ & \underline{\mathfrak{K}} \arrow[u, dashed, "\hat{F}"'] \end{tikzcd} \tag{4.21}$$

To define $\hat{F}$ on objects, set $\hat{F}(A) := F(A)$ for every algebra A. To define $\hat{F}$ on morphisms, we must define a function

$$\hat{F} : \hom_{\mathfrak{K}}(A, B) = \operatorname*{colim}_{v}[J^v A, B_\bullet^{S^v}] \to \hom_{\mathcal{C}}(F(A), F(B)) \tag{4.22}$$

for every pair of algebras (A, B). Fix the pair (A, B). For every $v \geq 0$, let $\hat{F}_v : [J^v A, B_\bullet^{S^v}] \to \hom_{\mathcal{C}}(F(A), F(B))$ be the function that sends the class of $f : J^v A \to B_r^{S^v}$ to the following composite:

$$F(A) \xrightarrow[\cong]{(\alpha_A^{0,v})^{-1}} F(\Sigma^v J^v A) \xrightarrow{f_*} F(\Sigma^v B_r^{S^v}) \xrightarrow[\cong]{\gamma_*^{-1}} F(\Sigma^v B^{S^v}) \xrightarrow[\cong]{\beta_B^{0,v}} F(B).$$

Here, γ_* is the isomorphism induced by the last vertex map; see Lemma 4.7(3). Note that $\hat{F}_v$ is well defined on the set of homotopy classes since the functor $F(\Sigma^v -)$ is homotopy invariant. Let us show that the $\hat{F}_v$ are compatible with the transition morphisms of the colimit (4.22). The transition morphism of (4.22) sends the homotopy class of f to the homotopy class of $\mu_B^{v,1} \circ \rho_{B^{S^v}} \circ J(f)$. Consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
F(A) \arrow[r, "(\alpha_A^{0,v+1})^{-1}", "\cong"'] \arrow[dd, "\cong"', "(\alpha_A^{0,v})^{-1}"] \arrow[dr, phantom, "(1)"] & F(\Sigma^{v+1} J^{v+1} A) \arrow[r, "J(f)_*"] \arrow[ddl, "\alpha_{J^v A}^{v,1}"] \arrow[d, phantom, "(2)"] & F(\Sigma^{v+1} J(B_r^{S^v})) \arrow[r, "(\rho_{B^{S^v}})_*"] \arrow[d, "\cong"', "\gamma_*^{-1}"] \arrow[ddddll, "\alpha_{B^{S^v}}^{v,1}"] \arrow[r, phantom, shift right=4, "(4)"] & F(\Sigma^{v+1} (B_r^{S^v})^{S^1}) \arrow[d, "(\mu_B^{v,1})_*"] \arrow[dddl, "\gamma_*^{-1}", "\cong"'] \\
 & & F(\Sigma^{v+1} J(B^{S^v})) \arrow[dd, "(\rho_{B^{S^v}})_*"] \arrow[ddd, bend right=60, "\alpha_{B^{S^v}}^{v,1}"] & F(\Sigma^{v+1} B_r^{S^{v+1}}) \arrow[dd, "\cong"', "\gamma_*^{-1}"] \arrow[ddl, phantom, "(5)"] \\
F(\Sigma^v J^v A) \arrow[dd, "f_*"] & & & \\
 & \arrow[d, phantom, "(2)"] & F(\Sigma^{v+1} (B^{S^v})^{S^1}) \arrow[r, "(\mu_B^{v,1})_*"'] \arrow[d, "\beta_{B^{S^v}}^{v,1}"] \arrow[d, phantom, shift right=8, "(3)"] & F(\Sigma^{v+1} B^{S^{v+1}}) \arrow[d, "\cong"', "\beta_B^{0,v+1}"] \arrow[dl, phantom, "(6)"] \\
F(\Sigma^n B_r^{S^n}) \arrow[rr, "\gamma_*^{-1}", "\cong"'] & & F(\Sigma^v B^{S^v}) \arrow[r, "\beta_B^{0,v}", "\cong"'] & F(B)
\end{tikzcd}$$

The triangle (1) commutes by Lemma 4.14. The squares labeled (2) commute by naturality of $\alpha_?^{v,1}$. The triangle (3) commutes by definition of β. The square (4) commutes by naturality of $\rho_?$. The square (5) commutes since μ is compatible with the morphisms induced by the last vertex map. The square (6) commutes by Lemma 4.16. It follows that the outer square commutes. Composing the vertical morphisms in the left column followed by the horizontal morphisms in the bottom row we get $\hat{F}_v([f])$. Composing the horizontal morphisms in the top row followed by the vertical morphisms in the right column we get $\hat{F}_{v+1}([\mu_B^{v,1} \circ \rho_{B^{S^v}} \circ J(f)])$. This proves the desired compatibility and shows that the morphism (4.22) is indeed well defined. Let us now prove that it is compatible with composition. Let $f : J^p A \to B_r^{S^p}$ and $g : J^q B \to C_s^{S^q}$ be algebra homomorphisms. The composite $\langle g\rangle \circ \langle f\rangle$ in $\mathfrak{K}$ is defined as $\langle f \star g\rangle$, where $f \star g$ is the following composite in $[\mathbf{Alg}]^{\mathrm{ind}}$ (see [24, Definition 3.4]):

$$J^{p+q}A \xrightarrow{J^q(f)} J^q(B_\bullet^{S^p}) \xrightarrow{(-1)^{qp}\kappa_B^{q,p}} (J^q B)_\bullet^{S^p} \xrightarrow{g^{S^p}} (C_\bullet^{S^q})_\bullet^{S^p} \xrightarrow{\mu_C^{q,p}} C_\bullet^{S^{q+p}}.$$

Note that the sign makes sense since $[J^q(B_\bullet^{S^p}), (J^q B)_\bullet^{S^p}]$ is an abelian group. Consider the following commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd}
F(\Sigma^p J^p A) \arrow[d, "f_*"'] \arrow[dr, phantom, "(b)"] & & F(A) \arrow[ll, "(\alpha_A^{0,p})^{-1}", "\cong"'] \arrow[rr, "(\alpha_A^{0,p+q})^{-1}", "\cong"'] \arrow[d, phantom, "(a)"] & & F(\Sigma^{p+q} J^{p+q} A) \arrow[llll, bend left, "\alpha^{p,q}_{J^pA}"] \arrow[d, "(J^q(f))_*"] \\
F(\Sigma^p B_r^{S^p}) \arrow[d, "\gamma_*^{-1}"', "\cong"] \arrow[drr, phantom, "(b)"] & & & & F(\Sigma^{p+q} J^q(B_r^{S^p})) \arrow[llll, "\alpha^{p,q}_{B^{S^p}}"'] \arrow[dll, "\gamma_*^{-1}"', "\cong"] \arrow[d, "(-1)^{qp}(\kappa_B^{q,p})_*"] \\
F(\Sigma^p B^{S^p}) \arrow[d, "\beta_B^{0,p}"'] \arrow[ddr, phantom, "(c)"] & & F(\Sigma^{p+q} J^q(B^{S^p})) \arrow[ll, "\alpha^{p,q}_{B^{S^p}}"'] \arrow[d, "(-1)^{qp}(\kappa_B^{q,p})_*"'] & & F(\Sigma^{p+q}(J^q B)_r^{S^p}) \arrow[dll, "\gamma_*^{-1}"', "\cong"] \arrow[d, "(g^{S^p})_*"] \\
F(B) \arrow[d, "(\alpha_B^{0,q})^{-1}"', "\cong"] & & F(\Sigma^{p+q}(J^q B)^{S^p}) \arrow[dll, "\beta^{q,p}_{J^qB}"'] \arrow[d, "g_*"] & & F(\Sigma^{p+q}(C_s^{S^q})_r^{S^p}) \arrow[dll, "\gamma_*^{-1}"', "\cong"] \arrow[dd, "(\mu_C^{q,p})_*"] \\
F(\Sigma^q J^q B) \arrow[d, "g_*"'] \arrow[r, phantom, "(d)"] & {} & F(\Sigma^{p+q}(C_s^{S^q})^{S^p}) \arrow[dll, "\beta^{q,p}_{C^{S^q}}"'] \arrow[d, "\gamma_*^{-1}"', "\cong"] \arrow[dr, "(\mu_C^{q,p})_*"] & & \\
F(\Sigma^q C_s^{S^q}) \arrow[d, "\gamma_*^{-1}"', "\cong"] \arrow[r, phantom, "(d)"] & {} & F(\Sigma^{p+q}(C^{S^q})^{S^p}) \arrow[dll, "\beta^{q,p}_{C^{S^q}}"'] \arrow[drr, "(\mu_C^{q,p})_*"'] \arrow[d, phantom, "(e)"] & F(\Sigma^{p+q} C_s^{S^{p+q}}) \arrow[dr, "\gamma_*^{-1}", "\cong"'] & F(\Sigma^{p+q} C_{r+s}^{S^{p+q}}) \arrow[l, "\gamma_*^{-1}"', "\cong"] \arrow[d, "\cong"', "\gamma_*^{-1}"] \\
F(\Sigma^q C^{S^q}) \arrow[rr, "\beta_C^{0,q}"'] & & F(C) & & F(\Sigma^{p+q} C^{S^{p+q}}) \arrow[ll, "\beta_C^{0,p+q}"]
\end{tikzcd}$$

The triangle (a) commutes by Lemma 4.14. The squares marked with (b) commute by naturality of $\alpha_?^{p,q}$. The pentagon (c) commutes by Lemma 4.17. The squares marked with (d) commute by naturality of $\beta_?^{q,p}$. The square (e) commutes by Lemma 4.16. The rest of the diagram clearly commutes. Starting at $F(A)$, if we first apply $(\alpha_A^{0,p})^{-1}$, then the vertical morphisms on the left column and finally $\beta_C^{0,q}$, we get $\hat{F}(\langle g\rangle)\circ\hat{F}(\langle f\rangle)$. Again at $F(A)$, if we first apply $(\alpha_A^{0,p+q})^{-1}$, then the vertical morphisms on the right column and finally $\beta_C^{0,p+q}$, we get $\hat{F}(\langle g\star f\rangle)$. This shows that the definitions above indeed define a functor $\hat{F}$ making (4.21) commute.

Now we would like to extend $\hat{F}$ to $\underline{\mathfrak{K}}_f$, that is, we want to define a functor $\check{F}$ making the following diagram commute:

$$\begin{tikzcd} \mathbf{Alg} \arrow[rr, "F"] \arrow[dr, bend right, "j"'] & & \mathcal{C} \\ & \underline{\mathfrak{K}} \arrow[ur, "\hat{F}"] \arrow[r, "t_f"'] & \underline{\mathfrak{K}}_f \arrow[u, dashed, "\check{F}"'] \end{tikzcd} \tag{4.23}$$

Recall from [22, Definition 5.1.3] that we have

$$\hom_{\underline{\mathfrak{K}}_f}(A,B)=\operatorname*{colim}_k \hom_{\underline{\mathfrak{K}}}(A,M_kB),$$

where the transition morphisms are induced by the upper-left corner inclusions. If $f:A\to M_kB$ and $g:B\to M_lC$ are morphisms in $\underline{\mathfrak{K}}$, the composition of the corresponding morphisms in $\underline{\mathfrak{K}}_f$ is represented by the composite

$$A\xrightarrow{f}M_kB\xrightarrow{M_k(g)}M_kM_lC\overset{\theta}{\cong}M_{kl}C.$$

Here, θ stands for any bijection $\{1,\dots,k\}\times\{1,\dots,l\}\cong\{1,\dots,kl\}$. To define $\check{F}$ on objects, set $\check{F}(A):=F(A)$ for every algebra A. Let

$$\check{F}:\hom_{\underline{\mathfrak{K}}_f}(A,B)\to\hom_{\mathcal{C}}(F(A),F(B))$$

be the function that sends $f\in\hom_{\underline{\mathfrak{K}}}(A,M_kB)$ to the composite

$$F(A)\xrightarrow{\hat{F}(f)}F(M_kB)\xrightarrow[\cong]{F(\iota)^{-1}}F(B),$$

where $\iota:B\to M_kB$ is the upper-left corner inclusion. It is easily verified that these definitions indeed give rise to a functor $\check{F}$ making (4.23) commute. To prove the compatibility with the product, let $f:A\to M_kB$ and $g:B\to M_lC$ be morphisms in $\underline{\mathfrak{K}}$ and consider the following commutative diagram in $\underline{\mathfrak{K}}$:

$$\begin{tikzcd} A \arrow[r, "f"] & M_kB \arrow[r, "M_k(g)"] & M_kM_lC \arrow[r, "\theta", "\cong"'] & M_{kl}C & C \arrow[l, "\iota"'] \arrow[dll, bend left, "\iota"] \\ & B \arrow[u, "\iota"] \arrow[r, "g"] & M_lC \arrow[u, "\iota"'] & & \end{tikzcd}$$

Upon applying $\hat{F}$, the vertical morphisms become invertible since $\hat{F}(\iota)=F(\iota)$. The resulting diagram in $\mathcal{C}$ shows that $\check{F}$ preserves composition.

The final step is to extend $\check{F}$ to $\underline{\mathfrak{K}}_s$. Recall from [22, Definition 5.2.7] that we have

$$\hom_{\underline{\mathfrak{K}}_s}(A,B)=\hom_{\underline{\mathfrak{K}}_f}(A,M_\infty B).$$

Put $\tilde{F}(A):=F(A)$ for every algebra A, and let

$$\tilde{F}:\hom_{\underline{\mathfrak{K}}_s}(A,B)\to\hom_{\mathcal{C}}(F(A),F(B))$$

be the function that sends $f \in \hom_{\underline{\mathfrak{K}}_f}(A, M_\infty B)$ to the composite

$$F(A) \xrightarrow{\check{F}(f)} F(M_\infty B) \xrightarrow[\cong]{F(\iota)^{-1}} F(B),$$

where $\iota : B \to M_\infty B$ is the upper-left corner inclusion. It is straightforward to verify that this defines a functor $\tilde{F}$ making (4.20) commute.

It remains to prove the uniqueness of $\tilde{F}$. Suppose that $\tilde{F}_1, \tilde{F}_2 : \underline{\mathfrak{K}}_s \to \mathcal{C}$ are two functors making (4.20) commute. Both functors are equal on objects since they both coincide with F on objects. Let $f : J^v A \to M_k M_\infty B_r^{S^v}$ be an algebra homomorphism representing a morphism $\langle f \rangle : A \to B$ in $\underline{\mathfrak{K}}_s$. By Lemma A.17, $\tilde{F}_i(\langle f \rangle)$ equals the following composite in $\mathcal{C}$:

$$F(A) \xrightarrow{\tilde{F}_i(\alpha_A^{0,v})} F(\Sigma^v J^v A) \xrightarrow{\tilde{F}_i(\Sigma^v \tilde{f})} F(\Sigma^v B^{S^v}) \xrightarrow[\cong]{\tilde{F}_i(\beta_B^{0,v})^{-1}} F(B).$$

We claim that $\tilde{F}_1(\Sigma^v \tilde{f}) = \tilde{F}_2(\Sigma^v \tilde{f})$. Indeed, $\Sigma^v \tilde{f}$ is a zig-zag in $\mathfrak{K}_s$ of morphisms in the image of j_s — obtained upon applying $\Sigma^v(-)$ to (A.18). The functors $\tilde{F}_1$ and $\tilde{F}_2$ coincide on the image of j_s since we have $\tilde{F}_1 \circ j_s = F = \tilde{F}_2 \circ j_s$. With the same argument we can show that $\tilde{F}_1(\alpha_A^{0,v}) = \tilde{F}_2(\alpha_A^{0,v})$ and $\tilde{F}_1(\beta_B^{0,v}) = \tilde{F}_2(\beta_B^{0,v})$ — it is clear from their definitions that α and β are zig-zags of morphisms in the image of j_s. It follows that $\tilde{F}_1(\langle f \rangle) = \tilde{F}_2(\langle f \rangle)$. □

As a corollary we get the following result.

Theorem 4.24 (cf. [17, Theorem 2.5]). *The functor $j_s : \mathbf{Alg} \to \underline{\mathfrak{K}}_s$ exhibits $\underline{\mathfrak{K}}_s$ as the localization of* **Alg** *at the set of kk-equivalences. More precisely, letting W_{kk} be the set of kk-equivalences in* **Alg**, *the functor j_s induces an equivalence $\mathbf{Alg}[W_{kk}^{-1}] \cong \underline{\mathfrak{K}}_s$.*

Proof. Let $l : \mathbf{Alg} \to \mathbf{Alg}[W_{kk}^{-1}]$ be the localization functor. Since $(\mathbf{Alg}, \mathrm{Fib}, W_{kk})$ is a category of fibrant objects by Proposition 3.11, it follows from Lemma 2.7 that $\mathbf{Alg}[W_{kk}^{-1}]$ has finite products and that l preserves them. Since $l(W_H \cup W_S \cup W_E) \subseteq \mathrm{Iso}(\mathbf{Alg}[W_{kk}^{-1}])$, by Theorem 4.19 there exists a unique functor $\phi : \underline{\mathfrak{K}}_s \to \mathbf{Alg}[W_{kk}^{-1}]$ such that $l = \phi \circ j_s$. Since $j_s(W) \subseteq \mathrm{Iso}(\underline{\mathfrak{K}}_s)$, by the universal property of $\mathbf{Alg}[W_{kk}^{-1}]$ there exists a unique functor $\psi : \mathbf{Alg}[W_{kk}^{-1}] \to \underline{\mathfrak{K}}_s$ such that $j_s = \psi \circ l$. It follows from uniqueness that ϕ and ψ are mutually inverses. □

5. A stable infinity category realizing *kk*

The triangulated categories that arise naturally in mathematics are usually (but not always [21]) the homotopy categories of stable ∞-categories. In this section we show that this is the case for kk, following what was done in [17] for Kasparov's KK-theory.

5.1. *Main theorem.* We will prove that the triangulated category kk arises as the homotopy category of a stable ∞-category kk_∞. We use the language of ∞-categories as developed in [18]. We write $\mathrm{Ho}(\mathcal{C})$ for the homotopy category of an ∞-category $\mathcal{C}$. Ordinary categories will be considered as ∞-categories using the nerve, though we will not write the nerve explicitly.

We start by recalling the Dwyer–Kan localization for ∞-categories [19, Definition 1.3.4.1 and Remark 1.3.4.2]. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms in $\mathcal{C}$. Then there exists an ∞-category $\mathcal{C}[W^{-1}]$ endowed with a functor $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that, for any ∞-category $\mathcal{D}$, L induces an equivalence

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^W(\mathcal{C}, \mathcal{D}), \tag{5.1}$$

where $\mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of those functors that send morphisms in W to equivalences in $\mathcal{D}$. This universal property characterizes $\mathcal{C}[W^{-1}]$ up to equivalence of ∞-categories.

Let W_{kk} be the set of kk-equivalences in $\mathbf{Alg}$ and let

$$j_\infty : \mathbf{Alg} \to kk_\infty := \mathbf{Alg}[W_{kk}^{-1}] \tag{5.2}$$

be the Dwyer–Kan localization at W_{kk}. By the universal property of j_∞ we have an equivalence

$$\mathrm{Fun}(kk_\infty, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{W_{kk}}(\mathbf{Alg}, \mathcal{C})$$

for every ∞-category $\mathcal{C}$, induced by precomposing with j_∞. It follows from this that there exists a functor $h : kk_\infty \to kk$ making the following triangle commute:

$$\begin{tikzcd} \mathbf{Alg} \arrow[r, "j"] \arrow[dr, bend right, "j_\infty"'] & kk \\ & kk_\infty \arrow[u, "h"'] \end{tikzcd}$$

We will prove the following result:

Theorem 5.3. *The ∞-category kk_∞ is a stable infinity category and the functor* $\mathrm{Ho}(h) : \mathrm{Ho}(kk_\infty) \to \mathrm{Ho}(kk)$ *is an equivalence of triangulated categories.*

The proof of this result will be done in several steps.

Recall that a stable ∞-category is a pointed ∞-category that admits finite limits and such that the loop functor is an equivalence [19, Corollary 1.4.2.27]. To prove the existence of finite limits in kk_∞, we will calculate the localization (5.2) using a category of fibrant objects structure on $\mathbf{Alg}$ described in Section 3.

Lemma 5.4. (1) *The ∞-category kk_∞ is pointed and the algebra* 0 *is a zero object.*

(2) *The ∞-category kk_∞ has finite limits.*

(3) *The functor $j_\infty : \mathbf{Alg} \to kk_\infty$ sends Milnor squares to pullback squares.*

Proof. The assertion (1) follows from [4, Remark 7.1.15] since 0 is both final and initial in $\mathbf{Alg}$. Let Fib be the set of those surjective morphisms in $\mathbf{Alg}$ that have a linear section and let W_{kk} be the set of kk-equivalences in $\mathbf{Alg}$. Then $(\mathbf{Alg}, \mathrm{Fib}, W_{kk})$ is an ∞-category of fibrant objects in the sense of [4, Definition 7.5.7]; see also [4, Definition 7.4.12]. Indeed, this follows immediately from Proposition 3.11 — property (ii) in [4, Definition 7.4.12] holds by [1, factorization lemma]. The assertions (2) and (3) now follow from [4, Proposition 7.5.6]. $\square$

Let $\mathcal{C}$ be a pointed ∞-category with finite limits. We proceed to recall from [19, Section 1.1.2] the construction of the loop functor $\Omega_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Let $\mathcal{M}^{\Omega}$ be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ of those diagrams

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & X \end{tikzcd}$$

that are pullback squares and where 0 and $0'$ are zero objects of $\mathcal{C}$. If $\mathcal{C}$ admits finite limits, it can be shown that the evaluation at (1, 1) induces a trivial fibration $\mathcal{M}^{\Omega} \to \mathcal{C}$. Let $s : \mathcal{C} \to \mathcal{M}^{\Omega}$ be a section of this trivial fibration and let $e : \mathcal{M}^{\Omega} \to \mathcal{C}$ be the functor given by evaluation at $(0,0)$. Then $\Omega_{\mathcal{C}}$ is defined as the composite $e \circ s$. For $\mathcal{C} = kk_{\infty}$ we can give a description of this loop functor using loop algebras.

Lemma 5.5 (cf. [3, Lemma 2.6]). (1) *The functor $\Omega : \mathbf{Alg} \to \mathbf{Alg}$, $A \mapsto \Omega A$, uniquely descends to an equivalence of additive categories $\Omega : kk \to kk$ completing the square*

$$\begin{tikzcd} \mathbf{Alg} \arrow[r, "\Omega"] \arrow[d, "j"'] & \mathbf{Alg} \arrow[d, "j"] \\ kk \arrow[r, dashed, "\Omega"] & kk \end{tikzcd}$$

(2) *The functor $\Omega : \mathbf{Alg} \to \mathbf{Alg}$, $A \mapsto \Omega A$, essentially uniquely descends to a functor $\Omega : kk_{\infty} \to kk_{\infty}$ completing the square*

$$\begin{tikzcd} \mathbf{Alg} \arrow[r, "\Omega"] \arrow[d, "j_{\infty}"'] & \mathbf{Alg} \arrow[d, "j_{\infty}"] \\ kk_{\infty} \arrow[r, dashed, "\Omega"] & kk_{\infty} \end{tikzcd} \tag{5.6}$$

(3) *The loop functor $\Omega_{kk_{\infty}} : kk_{\infty} \to kk_{\infty}$ is naturally equivalent to the functor $\Omega : kk_{\infty} \to kk_{\infty}$ of (5.6).*

Proof. The assertion (1) is well known. Indeed, the functor $\Omega : kk \to kk$ is the desuspension in the triangulated category kk; see [6, Sections 6.3 and 6.4].

The functor $\Omega : \mathbf{Alg} \to \mathbf{Alg}$ preserves kk-equivalences by (1). Thus, the composite $j_{\infty} \circ \Omega : \mathbf{Alg} \to kk_{\infty}$ sends kk-equivalences to equivalences. Then (2) follows from the universal property of kk_{∞} being a Dwyer–Kan localization.

Let us prove (3). For $A \in \mathbf{Alg}$ we have a Milnor square

$$\begin{tikzcd} \Omega A \arrow[r] \arrow[d] & PA \arrow[d, "\mathrm{ev}_1"] \\ 0 \arrow[r] & A \end{tikzcd} \tag{5.7}$$

where $PA = tA[t]$ and $\Omega A = (t^2 - t)A[t]$ — note that $a \mapsto at$ is a linear section of ev_1. Since PA is contractible, it is a zero object in kk_{∞}. Thus, upon applying j_{∞} to (5.7) we get a pullback square that is equivalent to the one defining $\Omega_{kk_{\infty}}$. $\square$

Lemma 5.8 (cf. [17, Proposition 3.3]). (1) *The functor* $\mathrm{Ho}(h):\mathrm{Ho}(kk_\infty)\to\mathrm{Ho}(kk)$ *is an equivalence of ordinary categories.*

(2) *The* ∞*-category* kk_∞ *is stable.*

Proof. Let us prove (1). Let W denote the set of kk-equivalences in **Alg**. If $\mathcal{C}$ is an infinity category, write $\gamma_{\mathcal{C}}:\mathcal{C}\to\mathrm{Ho}(\mathcal{C})$ for the canonical functor. Since forming localizations is compatible with going over to homotopy categories, we have a commutative square

$$\begin{tikzcd}
\mathbf{Alg} \arrow[r, "j_\infty"] \arrow[d, "\gamma_{\mathbf{Alg}}"'] & kk_\infty \arrow[d, "\gamma_{kk_\infty}"] \\
\mathrm{Ho}(\mathbf{Alg}) \arrow[r, "\mathrm{Ho}(j_\infty)"] & \mathrm{Ho}(kk_\infty)
\end{tikzcd}$$

where $\mathrm{Ho}(j_\infty)$ presents $\mathrm{Ho}(kk_\infty)$ as the localization of $\mathrm{Ho}(\mathbf{Alg})$ at the set $\mathrm{Ho}(W)$ in the sense of ordinary categories. Since **Alg** is an ordinary category, then $\gamma_{\mathbf{Alg}}$ is an equivalence and $\mathrm{Ho}(j_\infty)\circ\gamma_{\mathbf{Alg}}$ presents its $\mathrm{Ho}(kk_\infty)$ as the localization of **Alg** at W. Now consider the following commutative diagram of ordinary categories:

$$\begin{tikzcd}
\mathbf{Alg} \arrow[r, "\gamma_{\mathbf{Alg}}"] \arrow[d, "j"'] & \mathrm{Ho}(\mathbf{Alg}) \arrow[r, "\mathrm{Ho}(j_\infty)"] \arrow[dr, "\mathrm{Ho}(j)"'] & \mathrm{Ho}(kk_\infty) \arrow[d, "\mathrm{Ho}(h)"] \\
kk \arrow[rr, "\gamma_{kk}"'] & & \mathrm{Ho}(kk)
\end{tikzcd}$$

On one hand, we have seen that $\mathrm{Ho}(j_\infty)\circ\gamma_{\mathbf{Alg}}$ presents $\mathrm{Ho}(kk_\infty)$ as the localization of **Alg** at W. On the other, the functor $\gamma_{kk}\circ j$ presents $\mathrm{Ho}(kk)$ as the localization of **Alg** at W. Indeed, the latter follows from Theorem 4.24 and the fact that kk is an ordinary category, and thus γ_{kk} is an equivalence. Since both composites share the same universal property, it follows that $\mathrm{Ho}(h)$ is an equivalence.

To prove (2) we have to show that the loop functor Ω_{kk_∞} on kk_∞ is an equivalence. By [17, Lemma 3.4(2)] it suffices to prove that Ω_{kk_∞} induces an equivalence on $\mathrm{Ho}(kk_\infty)$. By Lemma 5.5(3), it is enough to show that $\Omega:kk_\infty\to kk_\infty$ induces an equivalence on $\mathrm{Ho}(kk_\infty)$. In view of (1), the latter is identified with $\Omega:kk\to kk$, which is an equivalence by Lemma 5.5(1). □

Lemma 5.9 (cf. [3, Lemma 2.17]). (1) *Every morphism in* kk_∞ *is equivalent to a morphism* $j_\infty(f)$ *for some morphism* f *in* **Alg**.

(2) *Every product in* kk_∞ *is equivalent to the image under* j_∞ *of a product in* **Alg**.

(3) *Every distinguished triangle in* $\mathrm{Ho}(kk_\infty)$ *is isomorphic to one of the form*

$$\Omega B \xrightarrow{[j_\infty(\iota_f)]} P_f \xrightarrow{[j_\infty(\pi_f)]} A \xrightarrow{[j_\infty(f)]} B$$

for some algebra homomorphism $f:A\to B$; *see (2.11) for the definitions of* ι_f *and* π_f. *Here, we are identifying* $\Omega_{kk_\infty}(j_\infty(B))\simeq j_\infty(\Omega B)$ *as explained in Lemma 5.5.*

Proof. To prove (1), let g be a morphism in kk_∞. Since kk_∞ is the ∞-category associated to the ∞-category of fibrant objects (**Alg**, Fib, W_{kk}), the class $[g]$ in $\mathrm{Ho}(kk_\infty)$ equals $[j_\infty(f)]\circ[j_\infty(s)]^{-1}$ for some morphisms f and s in **Alg**, with s a kk-equivalence; see, for example, the proof of [4, Theorem 7.5.6]. Then $j_\infty(f)$ is a composition of the morphisms $j_\infty(s)$ and g in kk_∞ and therefore the morphisms g and $j_\infty(f)$ are equivalent.

Let us prove (2). First note that any object of kk_∞ is equivalent to one in the image of j_∞. Let $j_\infty(A)$ and $j_\infty(B)$ be two objects of kk_∞. Since the square of algebras

$$\begin{tikzcd} A\times B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & 0 \end{tikzcd}$$

is a Milnor square, it becomes a pullback upon applying j_∞ by Lemma 5.4(3). This implies that $j_\infty(A)\times j_\infty(B)\simeq j_\infty(A\times B)$ in kk_∞.

Let us prove (3). Since $\mathrm{Ho}(kk_\infty)$ is a triangulated category, every morphism in $\mathrm{Ho}(kk_\infty)$ can be extended to a distinguished triangle that is uniquely determined up to isomorphism, and every distinguished triangle arises in this way. By (1), it suffices to consider those distinguished triangles that extend a morphism of the form $[j_\infty(f)]$ for some morphism f in **Alg**. Let $f : A \to B$ be an algebra homomorphism and consider the following diagram of algebras:

$$\begin{tikzcd} \Omega B \arrow[r, "\iota_f"] \arrow[d] & P_f \arrow[r] \arrow[d, "\pi_f"] & PB \arrow[d, "\mathrm{ev}_1"] \\ 0 \arrow[r] & A \arrow[r, "f"] & B \end{tikzcd}$$

Since the vertical morphisms belong to Fib, both squares are Milnor squares and thus become pullbacks in kk_∞. The assertion (3) now follows from (the dual of) [19, Definition 1.1.2.11]. □

Proposition 5.10 (cf. [3, Proposition 2.18]). *The functor* $\mathrm{Ho}(h) : \mathrm{Ho}(kk_\infty) \to \mathrm{Ho}(kk)$ *is an equivalence of triangulated categories.*

Proof. By Lemma 5.8(1), the functor $\mathrm{Ho}(h)$ is an equivalence of ordinary categories. Since binary products in both $\mathrm{Ho}(kk_\infty)$ and kk are represented by the cartesian product of algebras, it follows that $\mathrm{Ho}(h)$ preserves binary products and that it is an additive functor. By Lemma 5.5, it follows that $\mathrm{Ho}(h)$ is compatible with the desuspension functors. Finally, it follows from Section 2.4 and Lemma 5.9(3) that $\mathrm{Ho}(h)$ sends distinguished triangles in $\mathrm{Ho}(kk_\infty)$ to distinguished triangles in kk. □

This finishes the proof of Theorem 5.3.

5.2. *Inverting polynomial homotopy equivalences.* A stable ∞-category version of KK-theory is constructed in [2] by making a sequence of localizations of the category of C^***Alg** to enforce the universal properties of KK. An alternative approach to constructing kk_∞ would be to mimic the steps described in [2, p. 3] in our algebraic context. We will show that this can be done to some extent in our setting, though it is not clear for the authors how to go further on. The first of these steps consists of inverting homotopy equivalences.

Definition 5.11. Let $L_h : \mathbf{Alg} \to \mathbf{Alg}_h$ be the Dwyer–Kan localization of **Alg** at the set W_H of polynomial homotopy equivalences.

Let $\mathbf{Alg}_\infty$ be the ∞-category that results from applying the homotopy coherent nerve to the enrichment of **Alg** over Kan complexes described in Section 2.2. We have a functor $\mathbf{Alg} \to \mathbf{Alg}_\infty$ induced by the inclusions of the zero skeletons of the mapping spaces.

Proposition 5.12 (cf. [2, Proposition 3.5]). *The functor* $\mathbf{Alg} \to \mathbf{Alg}_\infty$ *presents the Dwyer–Kan localization of* **Alg** *at the set* W_H *of polynomial homotopy equivalences.*

Proof. Let $\mathcal{C}$ be the opposite category of the simplicially enriched category of algebras and let $\mathcal{C}'$ be the opposite category of the Kan complex enriched category of algebras. Explicitly, $\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{C}') = \mathrm{ob}(\mathbf{Alg})$ and for algebras A and B we have

$$\mathrm{Map}_{\mathcal{C}}(A, B) = \underline{\mathrm{hom}}(B, A), \quad \mathrm{Map}_{\mathcal{C}'}(A, B) = \underline{\mathrm{HOM}}(B, A).$$

Note that the underlying ordinary category of $\mathcal{C}$ is $\mathcal{C}_0 = \mathbf{Alg}^{\mathrm{op}}$ and that $\mathcal{C}'$ is a fibrant replacement of $\mathcal{C}$ as a simplicially enriched category. Let W be the collection of morphisms in $\mathcal{C}$ corresponding to polynomial homotopy equivalences. Following the proof of [2, Proposition 3.5], we would like to invoke [19, Proposition 1.3.4.7] but this is not possible because the exponential law fails in this algebraic context; see [6, Remark 3.1.4]. Nevertheless, we will show that the proof of [19, Proposition 1.3.4.7] carries on with some changes. More precisely, we will show that the canonical morphism $\theta : (N(\mathcal{C}_0), W) \to (N(\mathcal{C}'), W)$ is a weak equivalence of marked simplicial sets.

Let $f : A \to B$ be a morphism in $\mathcal{C}$ that belongs to W, that is, a zero simplex $f \in \mathrm{Map}_{\mathcal{C}}(A, B)_0 = \mathrm{hom}_{\mathbf{Alg}}(B, A)$ that is a polynomial homotopy equivalence. It is clear that if $g : A \to B$ is any morphism that belongs to the same connected component of $\mathrm{Map}_{\mathcal{C}}(A, B)$ as f, then g belongs to W. Indeed, unravelling the definitions, f and g are connected by an edge of $\mathrm{Map}_{\mathcal{C}}(A, B)$ if and only if they are elementarily homotopic.

Fix $n \geq 0$ and let $\mathcal{C}_n$ be the ordinary category whose objects are those of $\mathcal{C}$ and whose morphisms are given by

$$\mathrm{hom}_{\mathcal{C}_n}(A, B) = \mathrm{hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{Map}_{\mathcal{C}}(A, B)) = \mathrm{hom}_{\mathbf{Alg}}(B, A^{\Delta^n}).$$

Let W_n be the collection of morphisms in $\mathcal{C}_n$ corresponding to morphisms $\Delta^n \to \mathrm{Map}_{\mathcal{C}}(A, B)$ that carry each vertex of Δ^n to an element of W. As explained in the proof of [19, Proposition 1.3.4.7], we must show that the morphism $F : (N(\mathcal{C}_0), W_0) \to (N(\mathcal{C}_n), W_n)$ induced by $[n] \to [0]$ is a weak equivalence of marked simplicial sets. Let $G : (N(\mathcal{C}_n), W_n) \to (N(\mathcal{C}_0), W_0)$ be induced by the inclusion $[0] \to [n]$. Then $G \circ F = \mathrm{id}$. We will show that $F \circ G$ is homotopic to the identity of $\mathcal{C}_n$. Define a functor $U : \mathcal{C}_n \to \mathcal{C}_n$ as follows:

- On objects, U is given by $U(A) = A^{\Delta^1}$.

- On morphisms, we should define compatible functions:

$$\begin{array}{c}\hom_{\mathcal{C}_n}(A, B) = \hom_{\mathbf{Alg}}(B, A^{\Delta^n}) \\ {\scriptstyle U}\big\downarrow \\ \hom_{\mathcal{C}_n}(U(A), U(B)) = \hom_{\mathbf{Alg}}(B^{\Delta^1}, (A^{\Delta^1})^{\Delta^n})\end{array}$$

Recall that we have isomorphisms

$$\iota_p : \ell^{\Delta^p} = \frac{\ell[t_0, \dots, t_p]}{\langle 1 - \sum t_i \rangle} \to \ell[s_1, \dots, s_p], \qquad \iota_p(t_i) = s_i \text{ for } i > 0.$$

Define $\rho : (\ell^{\Delta^n})^{\Delta^1} \to (\ell^{\Delta^n})^{\Delta^1}$ as the composite

$$(\ell^{\Delta^n})^{\Delta^1} \xrightarrow[\cong]{\iota_n \otimes \iota_1} \ell[s_1, \dots, s_n][\sigma] \xrightarrow{r} \ell[s_1, \dots, s_n][\sigma] \xleftarrow[\cong]{\iota_n \otimes \iota_1} (\ell^{\Delta^n})^{\Delta^1},$$

where r is determined by $r(\sigma) = \sigma$ and $r(s_i) = \sigma s_i$ for $1 \le i \le n$. Upon tensoring ρ with A we get an algebra homomorphism $\rho : (A^{\Delta^n})^{\Delta^1} \to (A^{\Delta^n})^{\Delta^1}$. Let $f \in \hom_{\mathcal{C}_n}(A, B)$ be represented by $f : B \to A^{\Delta^n}$. Define $U(f)$ as the morphism represented by the composite

$$B^{\Delta^1} \xrightarrow{f^{\Delta^1}} (A^{\Delta^n})^{\Delta^1} \xrightarrow{\rho} (A^{\Delta^n})^{\Delta^1} \cong (A^{\Delta^1})^{\Delta^n}.$$

It is straightforward but tedious to verify that the latter defines a functor U. For $i = 0, 1$, let $\mathrm{const}_i : [n] \to [1]$ denote the constant function taking the value i and let $\tau_i \in \hom_{\mathcal{C}_n}(A, U(A)) = \hom_{\mathbf{Alg}}(A^{\Delta^1}, A^{\Delta^n})$ be represented by

$$(\mathrm{const}_i)^* : A^{\Delta^1} \to A^{\Delta^n}.$$

Another straightforward verification shows that the τ_i assemble into natural transformations $\tau_0 : F \circ G \to U$ and $\tau_1 : \mathrm{id}_{\mathcal{C}_n} \to U$. Note that both τ_0 and τ_1 are given by morphisms in W_n (any morphism $A^{\Delta^1} \to A$ induced by a morphism $\Delta^0 \to \Delta^1$ is a polynomial homotopy equivalence). It follows that τ_0 and τ_1 determine the desired homotopy from $F \circ G$ to the identity of $\mathcal{C}_n$. □

Corollary 5.13 (cf. [2, Corollary 3.7]). *For any two algebras A and B we have a natural equivalence of spaces*:

$$\mathrm{Map}_{\mathbf{Alg}_h}(A, B) \simeq \underline{\mathrm{HOM}}(A, B).$$

Proposition 5.14 (cf. [2, Proposition 3.8]). *The category $\mathbf{Alg}_h$ admits finite products and finite coproducts, and L_h preserves them.*

Proof. It follows from Corollary 5.13 and the isomorphisms of simplicial sets

$$\underline{\mathrm{HOM}}\Big(A, \prod_{i \in I} B_i\Big) \cong \prod_{i \in I} \underline{\mathrm{HOM}}(A, B_i), \quad \underline{\mathrm{HOM}}\Big(\mathop{*}_{i \in I} A_i, B\Big) \cong \prod_{i \in I} \underline{\mathrm{HOM}}(A_i, B)$$

for finite I. □

Lemma 5.15 (cf. [2, Lemma 3.10]). *The category $\mathbf{Alg}_h$ is pointed and the functor $L_h : \mathbf{Alg} \to \mathbf{Alg}_h$ is reduced, i.e., it preserves zero objects.*

Proof. The zero algebra represents the initial and final object of $\mathbf{Alg}_h$ since

$$\underline{\mathrm{HOM}}(0, A) \cong \Delta^0 \cong \underline{\mathrm{HOM}}(A, 0)$$

for every algebra A. $\square$

At this point, a difference arises between the topological and the algebraic contexts. In [2, Definition 3.2], Bunke defines $L_h : C^*\mathbf{Alg} \to C^*\mathbf{Alg}_h$ as the Dwyer–Kan localization of the category of C^*-algebras at the homotopy equivalences. Later on, he uses the category of fibrant objects structure on $(C^*\mathbf{Alg}, \mathcal{F}_{\mathrm{Sch}}, W_h)$ to prove that $C^*\mathbf{Alg}_h$ has finite limits and that L_h sends Schochet fibrant cartesian squares in $C^*\mathbf{Alg}$ to pullbacks in $C^*\mathbf{Alg}_h$; see [2, Proposition 3.17]. This is a key point in Bunke's construction of the ∞-category version of KK-theory since the existence of finite limits in $C^*\mathbf{Alg}_h$ is carried over by formal arguments to the successive steps that enforce the universal properties of KK. In our algebraic context, the authors do not know whether $\mathbf{Alg}_h$ has finite limits, though it is easily verified that the Milnor squares

$$\begin{tikzcd} \Omega_i \arrow[r] \arrow[d] & P_i \arrow[d] \\ 0 \arrow[r] & \ell \end{tikzcd} \qquad (i = 0, 1)$$

do not give pullbacks in $\mathbf{Alg}_h$ simultaneously. Indeed, if they did, they would induce a morphism of homotopy pullbacks of Kan complexes as in (3.9) and the morphism $\beta : \Omega_0 \to \Omega_1$ would be an homotopy equivalence. We have seen in Lemma 3.1 that this is not the case if ℓ is a domain with $KH_0(\ell) \neq 0$.

Appendix: Algebraic kk-theory

Let $A \in \mathbf{Alg}$. We proceed to recall some extensions associated to A.

A.1. ***The cone extension.*** Recall from [6, Section 4.7] that we have an extension of rings

$$M_\infty \to \Gamma \to \Sigma$$

that splits as a sequence of abelian groups. For the rest of the paper, we fix a splitting $\tau : \Sigma \to \Gamma$. Upon tensoring with A, we get an extension of algebras:

$$\mathscr{C}_A : M_\infty A \to \Gamma A \xrightarrow{q_A} \Sigma A. \tag{A.1}$$

We will always consider $\tau_A = \tau \otimes A$ as a splitting for (A.1). We call (A.1) the *cone extension* of A. It is clear that an algebra homomorphism $A \to B$ induces a strong morphism of extensions $\mathscr{C}_A \to \mathscr{C}_B$. We write $x_A : J\Sigma A \to M_\infty A$ for the classifying map of $\mathscr{C}_A$.

A.2. ***The universal extension.*** Recall from [24, Section 2.22] that we have the *universal extension* of A

$$\mathscr{U}_A : JA \to TA \xrightarrow{\eta_A} A,$$

with splitting σ_A. An algebra homomorphism $A \to B$ induces a strong morphism of extensions $\mathscr{U}_A \to \mathscr{U}_B$.

A.3. ***The path extension.*** Let $n, r \geq 0$. Recall from [24, Section 2.28] we have the *path extension* of A:

$$\mathscr{P}_{n,A} : A_r^{S^{n+1}} \to P(n, A)_r \to A_r^{S^n}. \tag{A.2}$$

An algebra homomorphism $A \to B$ induces a strong morphism of extensions $\mathscr{P}_{n,A} \to \mathscr{P}_{n,B}$. Moreover, the last vertex map induces strong morphisms of extensions $\mathscr{P}_{n,A,r} \to \mathscr{P}_{n,A,r+1}$. Explicit descriptions of these extensions for $n = 0$ and $r = 0, 1$ are given in [24, Example 2.32]. We write $\rho_A : JA \to A_r^{S^1}$ for the classifying map of $\mathscr{P}_{0,A}$.

A.4. *Every morphism in kk is a zig-zag of algebra homomorphisms.*

Lemma A.3. *Let $A \in$ **Alg**. Then the following square in $\mathfrak{K}$ anticommutes (i.e., one way equals minus the other way):*

$$\begin{tikzcd}
(J\Sigma A, 0) \arrow[r, "\langle J(\rho_{\Sigma A})\rangle"] \arrow[d, "(x_A)_*"'] & (J(\Sigma A^{S^1}), -1) \arrow[d, "(x_{A^{S^1}})_*"] \\
(M_\infty A, 0) \arrow[r, "\langle \rho_{M_\infty A}\rangle"] & (M_\infty A^{S^1}, -1)
\end{tikzcd}$$

Proof. It is easily verified that the composite of the top morphism followed by the right morphism is represented by $x_{A_1^{S^1}} \circ J(\rho_{\Sigma A}) : J^2\Sigma A \to M_\infty A_1^{S^1}$. Moreover, the composite of the left morphism followed by the bottom one is represented by $\rho_{M_\infty A} \circ J(x_A) : J^2\Sigma A \to M_\infty A_1^{S^1}$. Thus, to prove that the square anticommutes it suffices to show that $x_{A_1^{S^1}} \circ J(\rho_{\Sigma A})$ and $\rho_{M_\infty A} \circ J(x_A)$ are mutually inverses in the group $[J^2\Sigma A, (M_\infty A)_\bullet^{S^1}]$. The idea is the same as in [24, Lemma 4.3]. Define

$$I = \ker\big(t(\Gamma A)[t] \xrightarrow{\mathrm{ev}_1} \Gamma A \xrightarrow{q_A} \Sigma A\big)$$

and let $s : \Sigma A \to t(\Gamma A)[t]$ be given by $s(a) = \tau_A(a)t$ — recall that τ_A is the section of the cone extension $\mathscr{C}_A$ (A.1). We have an extension

$$(\mathscr{E}, s) : I \to t(\Gamma A)[t] \xrightarrow{q_A \circ \mathrm{ev}_1} \Sigma A.$$

Now put

$$E = \big\{(u, v) \in t(\Gamma A)[t] \times t(M_\infty A)[t] \mid u(1) = v(1)\big\}$$

and let $s' : I \to E$ be given by $s'(u) = (u, u(1)t)$ — here we use that $\mathrm{ev}_1 : I \to \Gamma A$ factors through $M_\infty A$. We have an extension

$$(\mathscr{E}', s') : (t^2 - t)(M_\infty A)[t] \xrightarrow{(0,\mathrm{incl})} E \xrightarrow{\mathrm{pr}_1} I.$$

Let $\chi : J\Sigma A \to I$ be the classifying map of $(\mathscr{E}, s)$. The following diagram shows a strong morphism of extensions $\mathscr{U}_{\Sigma A} \to \mathscr{C}_A$ extending the identity of ΣA:

$$\begin{tikzcd}
\mathscr{U}_{\Sigma A} \arrow[d] & J\Sigma A \arrow[r] \arrow[d, "\chi"'] & T\Sigma A \arrow[r] \arrow[d] & \Sigma A \arrow[d, equal] \\
(\mathscr{E}, s) \arrow[d] & I \arrow[r] \arrow[d, "\mathrm{ev}_1"'] & t(\Gamma A)[t] \arrow[r] \arrow[d, "\mathrm{ev}_1"] & \Sigma A \arrow[d, equal] \\
\mathscr{C}_A & M_\infty A \arrow[r] & \Gamma A \arrow[r] & \Sigma A
\end{tikzcd}$$

It follows that $\mathrm{ev}_1 \circ \chi = x_A$. Now let ω be the automorphism of $(M_\infty A)[t]$ given by $\omega(t) = 1-t$ and consider the following strong morphism of extensions $\mathscr{U}_{J\Sigma A} \to \mathscr{P}_{0,M_\infty A}$ extending x_A:

$$\begin{tikzcd}
\mathscr{U}_{J\Sigma A} \arrow[d] & J^2\Sigma A \arrow[r] \arrow[d] & TJ\Sigma A \arrow[r] \arrow[d] & J\Sigma A \arrow[d, "\chi"] \arrow[dd, bend left, "x_A"] \\
(\mathscr{E}', s') \arrow[d] & (t^2-t)(M_\infty A)[t] \arrow[r, "{(0,\mathrm{incl})}"] \arrow[d, equal] & E \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & I \arrow[d, "\mathrm{ev}_1"] \\
\mathscr{E}'' \arrow[d] & (t^2-t)(M_\infty A)[t] \arrow[r] \arrow[d, "\omega"'] & t(M_\infty A)[t] \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\omega"] & M_\infty A \arrow[d, equal] \\
\mathscr{P}_{0,M_\infty A} & (t^2-t)(M_\infty A)[t] \arrow[r] & (1-t)(M_\infty A)[t] \arrow[r, "\mathrm{ev}_0"] & M_\infty A
\end{tikzcd}$$

Since $\omega^{-1} = \omega$, it follows from [24, Proposition 2.26] that the classifying map of χ with respect to $(\mathscr{E}', s')$ equals the composite

$$J^2\Sigma A \xrightarrow{J(x_A)} J(M_\infty A) \xrightarrow{\rho_{M_\infty A}} M_\infty A_0^{S^1} \xrightarrow{\omega} M_\infty A_0^{S^1}.$$

Define $\theta : E \to (\Gamma A)_1^{S^1} = t(\Gamma A)[t] \,{}_{\mathrm{ev}_1}\!\times_{\mathrm{ev}_1} t(\Gamma A)[t]$ by $\theta(u, v) = (v, u)$ and consider the following morphisms of extensions:

$$\begin{tikzcd}
\mathscr{U}_{J\Sigma A} \arrow[d] & J^2\Sigma A \arrow[r] \arrow[d, "\omega\circ\rho_{M_\infty A}\circ J(x_A)"'] & TJ\Sigma A \arrow[r] \arrow[d] & J\Sigma A \arrow[d, "\chi"] \\
\mathscr{E}' \arrow[d] & (M_\infty A)_0^{S^1} \arrow[r] \arrow[d, "{(\mathrm{incl},0)}"'] & E \arrow[r] \arrow[d, "\theta"] & I \arrow[d, "{(0,q_A)}"] \\
(\mathscr{C}_A)_1^{S^1} & (M_\infty A)_1^{S^1} \arrow[r] & (\Gamma A)_1^{S^1} \arrow[r] & (\Sigma A)_1^{S^1}
\end{tikzcd} \tag{A.4}$$

Note that the morphism $\mathscr{E}' \to (\mathscr{C}_A)_1^{S^1}$ is not strong and that $(\mathscr{C}_A)_1^{S^1} \cong \mathscr{C}_{A_1^{S^1}}$. Put $\psi := (0, q_A) \circ \chi : J\Sigma A \to (\Sigma A)_1^{S^1}$. It follows from [24, Proposition 2.26] applied to (A.4) that the following diagram commutes in [**Alg**]:

$$\begin{tikzcd}
J^2\Sigma A \arrow[rrrr, "J(\psi)"] \arrow[dd, equal] & & & & J[(\Sigma A)_1^{S^1}] \arrow[r, equal] & J\Sigma(A_1^{S^1}) \arrow[d, "x_{A_1^{S^1}}"] \\
& & & & & M_\infty(A_1^{S^1}) \arrow[d, equal] \\
J^2\Sigma A \arrow[r, "J(x_A)"] & J(M_\infty A) \arrow[r, "\rho_{M_\infty A}"] & (M_\infty A)_0^{S^1} \arrow[r, "\omega"] & (M_\infty A)_0^{S^1} \arrow[rr, "\gamma"] & & (M_\infty A)_1^{S^1}
\end{tikzcd} \tag{A.5}$$

Here $\gamma = (\mathrm{incl}, 0) : (t^2 - t)(M_\infty A)[t] \to t(M_\infty A)[t] \,{}_{\mathrm{ev}_1}\!\times_{\mathrm{ev}_1} t(M_\infty A)[t]$ is the morphism induced by the last vertex map; see [24, Example 2.32]. We claim that $J(\psi) = J(\rho_{\Sigma A})$ in [**Alg**]. By [24, Lemma 2.27] it suffices to show that $\psi = \rho_{\Sigma A}$ in [**Alg**]. Define $\beta : t(\Gamma A)[t] \to (\Sigma A)[t]\,{}_{\mathrm{ev}_1}\!\times_{\mathrm{ev}_1} t(\Sigma A)[t]$ by $\beta(u) = (q_A(u)(1-t), 0)$. The following diagram exhibits a strong morphism of extensions $\mathscr{U}_{\Sigma A} \to \mathscr{P}_{\Sigma A}$ extending the identity of ΣA:

$$\begin{tikzcd}
\mathscr{U}_{\Sigma A} \arrow[d] & J\Sigma A \arrow[r] \arrow[d, "\chi"'] & T\Sigma A \arrow[r] \arrow[d] & \Sigma A \arrow[d, equal] \\
(\mathscr{E}, s) \arrow[d] & I \arrow[r] \arrow[d, "\beta"'] & t(\Gamma A)[t] \arrow[r] \arrow[d, "\beta"] & \Sigma A \arrow[d, equal] \\
\mathscr{P}_{\Sigma A} & (\Sigma A)_1^{S^1} \arrow[r] & P(0, \Sigma A)_1 \arrow[r] & \Sigma A
\end{tikzcd}$$

It follows that $\rho_{\Sigma A}$ equals the composite $J\Sigma A \xrightarrow{\chi} I \xrightarrow{\beta} (\Sigma A)_1^{S^1}$. To prove that $\rho_{\Sigma A}$ is homotopic to $\psi = (0, q_A) \circ \chi$, it suffices to show that $\beta, (0, q_A) : I \to (\Sigma A)_1^{S^1}$ are homotopic, and the latter is easily verified (see, for example, the last part of the proof of [24, Lemma 4.3]). This proves our claim that $J(\psi) = J(\rho_{\Sigma A})$ in [**Alg**]. Now, the commutativity of (A.5) implies that we have

$$x_{A_1^{S^1}} \circ J(\rho_{\Sigma A}) = [\rho_{M_\infty A} \circ J(x_A)]^{-1} \in [J^2\Sigma A, (M_\infty A)_\bullet^{S^1}].$$

Here, the superscript -1 is the inverse for the group structure on $[J^2\Sigma A, (M_\infty A)_\bullet^{S^1}]$; see Example 3.12 of [23]. $\square$

Lemma A.6. *Let $A \in$ **Alg**, $n \geq 0$ and $k \in \mathbb{Z}$. Then the identity of $J^n A$ induces an isomorphism*

$$\langle \mathrm{id}_{J^n A} \rangle : (A, k) \to (J^n A, k - n) \tag{A.7}$$

in $\mathfrak{K}_s$ that is natural in A with respect to algebra homomorphisms.

Proof. By [24, Lemma 7.2] we have a natural isomorphism

$$\langle \mathrm{id}_{J^n A} \rangle : (A, k) \to (J^n A, k - n)$$

in $\mathfrak{K}$ that is natural in A with respect to algebra homomorphisms. Upon applying $t_s \circ t_f$ we obtain the desired isomorphism (A.7). $\square$

Lemma A.8. *Let $A \in$ **Alg**, $n, r \geq 0$ and $k \in \mathbb{Z}$. Then the identity of $A_r^{S^n}$ induces an isomorphism*

$$\langle \mathrm{id}_{A_r^{S^n}} \rangle : (A_r^{S^n}, k - n) \to (A, k) \tag{A.9}$$

in $\mathfrak{K}_s$ that is natural in A with respect to algebra homomorphisms.

Proof. By [24, Lemma 7.8] we have a natural isomorphism

$$\langle \mathrm{id}_{A_r^{S^n}} \rangle : (A_r^{S^n}, k - n) \to (A, k)$$

in $\mathfrak{K}$ that is natural in A with respect to algebra homomorphisms. Upon applying $t_s \circ t_f$ we obtain the desired isomorphism (A.9). $\square$

Definition A.10. Let $A \in \mathbf{Alg}$. For $n \geq 0$ we will define an isomorphism

$$\epsilon_A^n : (\Sigma^n A, k) \to (A, k-n)$$

in $\mathfrak{K}_s$ that is natural in A with respect to algebra homomorphisms. Let ϵ_A^0 be the identity of (A,k) and let ϵ_A^1 be the following composite in $\mathfrak{K}_s$:

$$(\Sigma A, k) \xrightarrow{\langle \mathrm{id}_{J\Sigma A}\rangle} (J\Sigma A, k-1) \xrightarrow{(x_A)_*} (M_\infty A, k-1) \xrightarrow{(\iota_A)_*^{-1}} (A, k-1).$$

It is clear that ϵ_A^1 is a natural isomorphism since each of the morphisms in the above composition are. Suppose now that we have defined ϵ_A^n for $n \geq 1$. Let ϵ_A^{n+1} be the following composite in $\mathfrak{K}_s$:

$$(\Sigma^{n+1} A, k) \xrightarrow{\epsilon^1_{\Sigma^n A}} (\Sigma^n A, k-1) \xrightarrow{\epsilon_A^n} (A, k-1-n).$$

This defines ϵ_A^n for every $n \geq 0$.

Lemma A.11. *Let $A \in \mathbf{Alg}$. For any $p, q \geq 0$ we have $\epsilon_A^{p+q} = \epsilon_A^p \circ \epsilon^q_{\Sigma^p A}$.*

Proof. It follows from an easy induction on q. The base case $q = 1$ holds by definition of ϵ_A^{p+1}. ☐

Lemma A.12. *Let $A \in \mathbf{Alg}$. Recall from Definition 4.13 that we have isomorphisms $\alpha_A^{n,1} : (\Sigma^{1+n} JA, 0) \to (\Sigma^n A, 0)$ in $\mathfrak{K}_s$ for any $n \geq 0$. Then the following square in $\mathfrak{K}_s$ commutes for all $n \geq 0$:*

$$\begin{tikzcd}
(\Sigma^{1+n} JA, 0) \arrow[r, "\alpha_A^{n,1}"] \arrow[d, "\epsilon^n_{\Sigma JA}"'] & (\Sigma^n A, 0) \arrow[d, "\epsilon_A^n"] \\
(\Sigma JA, -n) \arrow[r, "\alpha_A^{0,1}"] & (A, -n)
\end{tikzcd}$$

Proof. The following diagram in $\mathfrak{K}_s$ is commutative by naturality of $\epsilon^n_?$:

$$\begin{tikzcd}
(\Sigma^{1+n} JA, 0) \arrow[r, "(c_A)_*^{-1}", "\cong"'] \arrow[d, "\epsilon^n_{\Sigma JA}"'] & (\Sigma^n J\Sigma A, 0) \arrow[r, "(x_A)_*"] \arrow[d, "\epsilon^n_{J\Sigma A}"'] & (\Sigma^n M_\infty A, 0) \arrow[r, "(\iota_A)_*^{-1}", "\cong"'] \arrow[d, "\epsilon^n_{M_\infty A}"] & (\Sigma^n A, 0) \arrow[d, "\epsilon^n_A"] \\
(\Sigma JA, -n) \arrow[r, "(c_A)_*^{-1}", "\cong"'] & (J\Sigma A, -n) \arrow[r, "(x_A)_*"] & (M_\infty A, -n) \arrow[r, "(\iota_A)_*^{-1}", "\cong"'] & (A, -n)
\end{tikzcd}$$

By Definition 4.13, the composite of the morphisms in the top row is $\alpha_A^{n,1}$ and the composite of the morphisms in the bottom row is $\alpha_A^{0,1}$. ☐

Lemma A.13. *The following diagram in $\mathfrak{K}_s$ commutes for all $A \in \mathbf{Alg}$ and $n \geq 0$:*

$$\begin{tikzcd}
(\Sigma^n A^{S^n}, 0) \arrow[r, "(-1)^n \epsilon^n_{A^{S^n}}"] \arrow[d, "\beta_A^{0,n}"'] & (A^{S^n}, -n) \arrow[dl, bend left, "\langle \mathrm{id}_{A^{S^n}}\rangle"] \\
(A, 0) &
\end{tikzcd} \tag{A.14}$$

Proof. We proceed by induction on n. For $n = 0$ there is nothing to prove. Let us prove that (A.14) commutes for $n = 1$. The following diagram shows a strong morphism of extensions $\mathscr{U}_{\Sigma A} \to \mathscr{P}_{\Sigma A}$ extending the identity of ΣA:

$$
\begin{tikzcd}
\mathscr{U}_{\Sigma A} \arrow[d] & J\Sigma A \arrow[r] \arrow[d, "c_A"'] & T\Sigma A \arrow[r] \arrow[d] & \Sigma A \arrow[d, equal] \\
\Sigma(\mathscr{U}_A) \arrow[d] & \Sigma JA \arrow[r] \arrow[d, "\Sigma(\rho_A)"'] & \Sigma TA \arrow[r] \arrow[d] & \Sigma A \arrow[d, equal] \\
\Sigma(\mathscr{P}_A) & \Sigma A^{S^1} \arrow[r] & \Sigma PA \arrow[r] & \Sigma A
\end{tikzcd}
$$

It follows that $\rho_{\Sigma A} = \Sigma(\rho_A) \circ c_A : J\Sigma A \to \Sigma A^{S^1}$. Now consider the following diagram in $\mathfrak{K}_s$, where all the morphisms are isomorphisms:

$$
\begin{tikzcd}
(\Sigma A^{S^1}, 0) \arrow[r, "\langle \mathrm{id}_{J(\Sigma A^{S^1})} \rangle"] \arrow[d, "(\rho_{\Sigma A})_*^{-1}"'] & (J(\Sigma A^{S^1}), -1) \arrow[r, "-(x_{A^{S^1}})_*"] & (M_\infty A^{S^1}, -1) \arrow[r, "(\iota_{A^{S^1}})_*^{-1}"] \arrow[ddll, bend left, "\langle \mathrm{id}_{M_\infty A^{S^1}} \rangle"] & (A^{S^1}, -1) \arrow[dddlll, bend left, "\langle \mathrm{id}_{A^{S^1}} \rangle"] \\
(J\Sigma A, 0) \arrow[ur, bend right, "\langle J(\rho_{\Sigma A}) \rangle"'] \arrow[d, "(x_A)_*"'] & & & \\
(M_\infty A, 0) \arrow[d, "(\iota_A)_*^{-1}"'] & & & \\
(A, 0) & & &
\end{tikzcd}
$$

The small triangle in the upper-left corner commutes by definition of the composition law in $\mathfrak{K}$. The lower rectangle commutes by naturality of the isomorphism $(A, 0) \cong (A^{S^1}, -1)$. The rectangle in the middle commutes by Lemma A.3 (recall from [24, Lemma 7.4] that $\langle \mathrm{id}_{M_\infty A^{S^1}} \rangle$ is the inverse of $\langle \rho_{M_\infty A} \rangle$). It follows that the outer square commutes, and this proves the result for $n = 1$. Indeed, the composite of the morphisms in the top row equals $-\epsilon^1_{A^{S^1}}$ and the composite of the morphisms in the left column equals $\beta^{0,1}_A$.

Suppose that (A.14) commutes for $n \geq 1$ and let us prove that it also commutes for $n + 1$. Consider the following diagram in $\mathfrak{K}_s$ where all the morphisms are isomorphisms:

$$
\begin{tikzcd}
(\Sigma^{n+1} A^{S^{n+1}}, 0) \arrow[r, "(-1)^n \epsilon^n_{\Sigma A^{S^{n+1}}}"] \arrow[d, "(\mu^{n,1}_A)_*^{-1}"'] & (\Sigma A^{S^{n+1}}, -n) \arrow[r, "-\epsilon^1_{A^{S^{n+1}}}"] \arrow[d, "(\mu^{n,1}_A)_*^{-1}"'] & (A^{S^{n+1}}, -n-1) \arrow[d, "(\mu^{n,1}_A)_*^{-1}"] \\
(\Sigma^{n+1} (A^{S^n})^{S^1}, 0) \arrow[r, "(-1)^n \epsilon^n_{\Sigma (A^{S^n})^{S^1}}"] \arrow[d, "\beta^{n,1}_{A^{S^n}}"'] & (\Sigma (A^{S^n})^{S^1}, -n) \arrow[r, "-\epsilon^1_{(A^{S^n})^{S^1}}"] \arrow[d, "\beta^{0,1}_{A^{S^n}}"'] & ((A^{S^n})^{S^1}, -n-1) \arrow[dl, bend left, "\langle \mathrm{id}_{(A^{S^n})^{S^1}} \rangle"] \\
(\Sigma^n A^{S^n}, 0) \arrow[r, "(-1)^n \epsilon^n_{A^{S^n}}"] \arrow[d, "\beta^{0,n}_A"'] & (A^{S^n}, -n) \arrow[dl, bend left, "\langle \mathrm{id}_{A^{S^n}} \rangle"] & \\
(A, 0) & &
\end{tikzcd}
$$

The three squares commute by naturality of $\epsilon^n_?$ and the two triangles commute by the inductive hypothesis. Is follows that the outer diagram commutes, proving that (A.14) commutes for $n+1$. Indeed, the composite of the morphisms in the top row equals $(-1)^{n+1}\epsilon^{n+1}_{S^{S^{n+1}}}$ by Lemma A.11, the composite of the morphisms in the left column equals $\beta^{0,n+1}_A$ by Lemma 4.16 and the composite of the rightmost vertical morphism followed by the diagonal ones equals $\langle \mathrm{id}_{A^{S^{n+1}}}\rangle$ by definition of the composition law in $\mathfrak{K}$. □

Lemma A.15. *The following diagram in $\mathfrak{K}_s$ commutes for all $A \in$ **Alg** and $n \geq 0$:*

$$\begin{tikzcd}
(\Sigma^n J^n A, 0) \arrow[r, "(-1)^n \epsilon^n_{J^n A}"] \arrow[d, "\alpha^{0,n}_A"'] & (J^n A, -n) \\
(A, 0) \arrow[ur, bend right, "\langle \mathrm{id}_{J^n A}\rangle"'] &
\end{tikzcd} \tag{A.16}$$

Proof. We proceed by induction on n. For $n = 0$ there is nothing to prove. For $n = 1$, consider the following diagram in $\mathfrak{K}_s$ where all the morphisms are isomorphisms:

$$\begin{tikzcd}
(\Sigma JA) \arrow[r, "-\epsilon^1_{JA}"] \arrow[d, "(\rho_A)_*"'] & (JA, -1) \arrow[d, "(\rho_A)_*"] \\
(\Sigma A^{S^1}) \arrow[r, "-\epsilon^1_{A^{S^1}}"] \arrow[d, "\beta^{0,1}_A"'] & (A^{S^1}, -1) \arrow[dl, bend left, "\langle \mathrm{id}_{A^{S^1}}\rangle"] \\
(A, 0) &
\end{tikzcd}$$

The square commutes by naturality of $\epsilon^1_?$ and the triangle commutes by Lemma A.13. It follows that the outer diagram commutes. The composite of the morphisms in the left column equals $\alpha^{0,1}_A$ by definition of $\beta^{0,1}_A$ and $\langle \mathrm{id}_{A^{S^1}}\rangle \circ (\rho_A)_* = \langle \rho_A \rangle$ is the inverse of $\langle \mathrm{id}_{JA}\rangle$ by [24, Lemma 7.2]. This proves that (A.16) commutes for $n = 1$.

Suppose (A.16) commutes for $n \geq 1$ and let us prove that it also commutes for $n+1$. Consider the following diagram in $\mathfrak{K}_s$:

$$\begin{tikzcd}
(\Sigma^{1+n} J^{n+1} A, 0) \arrow[r, "(-1)^n \epsilon^n_{\Sigma J^{n+1} A}"] \arrow[d, "\alpha^{n,1}_{J^n A}"'] & (\Sigma J^{n+1} A, -n) \arrow[r, "-\epsilon^1_{J^{n+1} A}"] \arrow[d, "\alpha^{0,1}_{J^n A}"] & (J^{n+1} A, -1-n) \\
(\Sigma^n J^n A, 0) \arrow[r, "(-1)^n \epsilon^n_{J^n A}"] \arrow[d, "\alpha^{0,n}_A"'] & (J^n A, -n) \arrow[ur, bend right, "\langle \mathrm{id}_{J^{n+1} A}\rangle"'] & \\
(A, 0) \arrow[ur, bend right, "\langle \mathrm{id}_{J^n A}\rangle"'] & &
\end{tikzcd}$$

Both triangles commute by the inductive hypothesis and the square commutes by Lemma A.12. The composite of the morphisms in the top row is $(-1)^{n+1}\epsilon^{n+1}_{J^{n+1}A}$ by Lemma A.11 and the composite of the morphisms in the left column is $\alpha^{0,n+1}_A$ by Lemma 4.14. The result follows. □

Lemma A.17. *Let $\alpha \in \mathfrak{K}_s(A, B)$ and let $f : J^n A \to M_p M_\infty B_r^{S^n}$ be an algebra homomorphism representing α. Let $\tilde{f}$ be the following composite in $\mathfrak{K}_s$:*

$$J^n A \xrightarrow{f_*} M_p M_\infty B_r^{S^n} \xrightarrow[\cong]{\iota_*^{-1}} M_\infty B_r^{S^n} \xrightarrow[\cong]{\iota_*^{-1}} B_r^{S^n} \xrightarrow[\cong]{\gamma_*^{-1}} B^{S^n}. \tag{A.18}$$

Here, the isomorphism labeled γ is induced by the last vertex map and the isomorphisms labeled ι are induced by upper-left corner inclusions into matrix algebras. Then the following diagram in $\mathfrak{K}_s$ commutes:

$$\begin{tikzcd}
\Sigma^n J^n A \arrow[r, "\alpha_A^{0,n}", "\cong"'] \arrow[d, "\Sigma^n \tilde{f}"'] & A \arrow[d, "\alpha"] \\
\Sigma^n B^{S^n} \arrow[r, "\beta_B^{0,n}", "\cong"'] & B
\end{tikzcd}$$

Proof. By Lemmas A.13 and A.15 we may replace $\alpha_A^{0,n}$ by the composite

$$(\Sigma^n J^n A, 0) \xrightarrow[\cong]{\epsilon^n_{J^n A}} (J^n A, -n) \xrightarrow[\cong]{\langle \mathrm{id}_{J^n A}\rangle^{-1}} (A, 0)$$

and $\beta_B^{0,n}$ by the composite

$$(\Sigma^n B^{S^n}, 0) \xrightarrow[\cong]{\epsilon^n_{B^{S^n}}} (B^{S^n}, -n) \xrightarrow[\cong]{\langle \mathrm{id}_{B^{S^n}}\rangle} (B, 0)$$

note that the signs $(-1)^n$ cancel out. Consider the following diagram in $\mathfrak{K}_s$:

$$\begin{tikzcd}
(\Sigma^n J^n A, 0) \arrow[r, "\epsilon^n", "\cong"'] \arrow[d, "f_*"'] & (J^n A, -n) \arrow[r, "\langle \mathrm{id}_{J^n A}\rangle^{-1}", "\cong"'] \arrow[d, "f_*"'] & (A, 0) \arrow[d, "f_*"] \\
(\Sigma^n M_p M_\infty B_r^{S^n}, 0) \arrow[r, "\epsilon^n", "\cong"'] \arrow[d, "\iota_*^{-1}"'] & (M_p M_\infty B_r^{S^n}, -n) \arrow[r, "\langle \mathrm{id}_{M_p M_\infty B^{S^n}}\rangle", "\cong"'] \arrow[d, "\iota_*^{-1}"'] & (M_p M_\infty B, 0) \arrow[d, "\iota_*^{-1}"] \\
(\Sigma^n M_\infty B_r^{S^n}, 0) \arrow[r, "\epsilon^n", "\cong"'] \arrow[d, "\iota_*^{-1}"'] & (M_\infty B_r^{S^n}, -n) \arrow[r, "\langle \mathrm{id}_{M_\infty B^{S^n}}\rangle", "\cong"'] \arrow[d, "\iota_*^{-1}"'] & (M_\infty B, 0) \arrow[d, "\iota_*^{-1}"] \\
(\Sigma^n B_r^{S^n}, 0) \arrow[r, "\epsilon^n", "\cong"'] \arrow[d, "\gamma_*^{-1}"'] & (B_r^{S^n}, -n) \arrow[r, "\langle \mathrm{id}_{B^{S^n}}\rangle", "\cong"'] \arrow[d, "\gamma_*^{-1}"'] & (B, 0) \arrow[d, equal] \\
(\Sigma^n B^{S^n}, 0) \arrow[r, "\epsilon^n", "\cong"'] & (B^{S^n}, -n) \arrow[r, "\langle \mathrm{id}_{B^{S^n}}\rangle", "\cong"'] & (B, 0)
\end{tikzcd}$$

Since the composite of the vertical morphisms on the left column is $\Sigma^n \tilde{f}$ and the composite of the morphisms on the right column is α, the result will follow if we prove that the outer square commutes. The squares on the left commute bu naturality of ϵ^n. The upper-right square commutes by [24, Lemma 7.10]. The remaining squares commute by the naturality stated in Lemma A.8. ☐

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