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Estimation of persistence diagrams via the three-gap theorem

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The time delay (or sliding-window) embedding is a technique from dynamical systems to reconstruct attractors from time-series data. Recently, descriptors from topological data analysis (TDA) — specifically, persistence diagrams — have been used to measure the shape of said reconstructed attractors in applications including periodicity and quasiperiodicity quantification. Despite their utility, the fast computation of persistence diagrams of sliding-window embeddings is still poorly understood. We present theoretical and computational schemes to approximate the persistence diagrams of sliding-window embeddings from quasiperiodic functions. We do so by combining the three-gap theorem from number theory with the persistent Künneth formula from TDA, and derive fast and provably correct persistent homology approximations. The input to our procedure is the spectrum of the signal, and we provide numerical as well as theoretical evidence of its utility to capture the shape of toroidal attractors.

1. Introduction

The sliding-window embedding is a powerful technique for reconstructing attractors of a dynamical system from observed time-series data. When combined with tools from topological data analysis (TDA), such as persistence diagrams, one can measure the topological properties of the state space to detect patterns in the original signal. In general, combining sliding windows with persistence can be used to probe for the presence of nontrivial recurrent behavior. For example, Figure 1 depicts a system with behavior more complex than periodicity.

Indeed, leveraging measurements of the system, such as the horizontal position $x(t)$ of the block or the angular position $\theta(t)$ of the pendulum, one can determine (as we will explain below) that the system exhibits a type of recurrent behavior known as *quasiperiodicity* (Figure 3 and Section 2.2). In dynamical systems this behavior emerges when oscillators are superimposed, also in the transition between stable and chaotic dynamics [Weixing et al. 1993], and has been observed, for instance, in fMRI scans obtained from mice [Belloy et al. 2017], in the formation of quasicrystals [Levine and Steinhardt 1984] and in the study of biphonation in mammal vocalization [Wilden et al. 1998].

Mathematically, we say that $f : \mathbb{R} \to \mathbb{C}$ is quasiperiodic (see Definition 2.29) with frequency vector $\omega = (\omega_1, \ldots, \omega_N)$ if $\{\omega_i\}_{i=1}^{N}$ are positive real numbers which are linearly independent over $\mathbb{Q}$ (i.e., incommensurate) and

$$f(t) = F(\omega_1 t, \ldots, \omega_N t),$$

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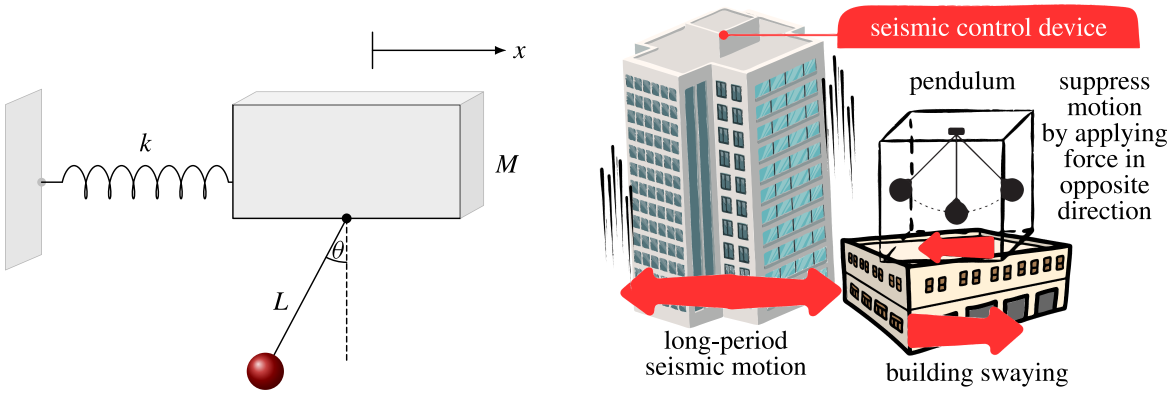


Figure 1. Left: schematic model for a pendulum on a sliding block. Right: illustration of antiearthquake technology (see also [Yirka 2013]), whose dynamics are captured by the model on the left.

where $F : \mathbb{T}^N \to \mathbb{C}$ is a complex-valued continuous function on the N-torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$, called the parent function of f [Gakhar and Perea 2024]. Note that in the case of a single frequency, f is just a periodic function.

Periodicity or quasiperiodicity in a signal with frequency vector of length N corresponds to observing the traversal of a (topological) circle or an N-torus attractor, respectively, in the phase space of the underlying dynamics [Skraba et al. 2012]. Takens' embedding theorem [1981], on the other hand, under appropriate conditions [Xu et al. 2019], ensures that the sliding-window embedding of the signal (Section 2.2 and Figure 2, center) provides a topological reconstruction of said attractors. Finally, periodicity/quasiperiodicity can be quantified with shape descriptors from TDA: the Rips persistence diagrams (defined in Section 2.1 and shown in Figure 2, right) of the reconstructed attractors via sliding-window embeddings. In short, they measure the prominence and dimension of holes in a point cloud, providing a convenient summary of its multiscale geometry and topology.

Much work has been done to establish the usefulness of sliding windows and persistence, with applications including parameter estimation in damping mechanisms [Myers and Khasawneh 2022], stability of stochastic delay equations [Khasawneh and Munch 2016], quantifying the presence of biphonation in vibrating vocal folds [Tralie and Perea 2018] and periodicity detection in gene expression data [Perea et al. 2015]. Furthermore, there are well-defined methodologies for optimizing the embedding parameters of the sliding-window embedding, in such a way that the topological features in the persistence diagrams are maximally amplified [Gakhar and Perea 2024]. However, the algorithmic complexity of computing the persistent homology of Rips filtrations remains a challenge in real-world applications. This is the case even with highly optimized libraries like Ripser [Bauer 2021], which exploits cohomology duality, apparent pairs and clearing optimizations to achieve substantial speed-ups. The main difficulty lies in that the worst-case complexity of the matrix-reduction algorithm [Zomorodian and Carlsson 2005] for computing persistent homology in degrees $j = 0, \dots, J$ is cubic in the number of simplices of dimensions $0, \dots, J+1$ [Morozov 2005]; the case relevant to us is $J = N$, i.e., the number of $\mathbb{Q}$-linearly

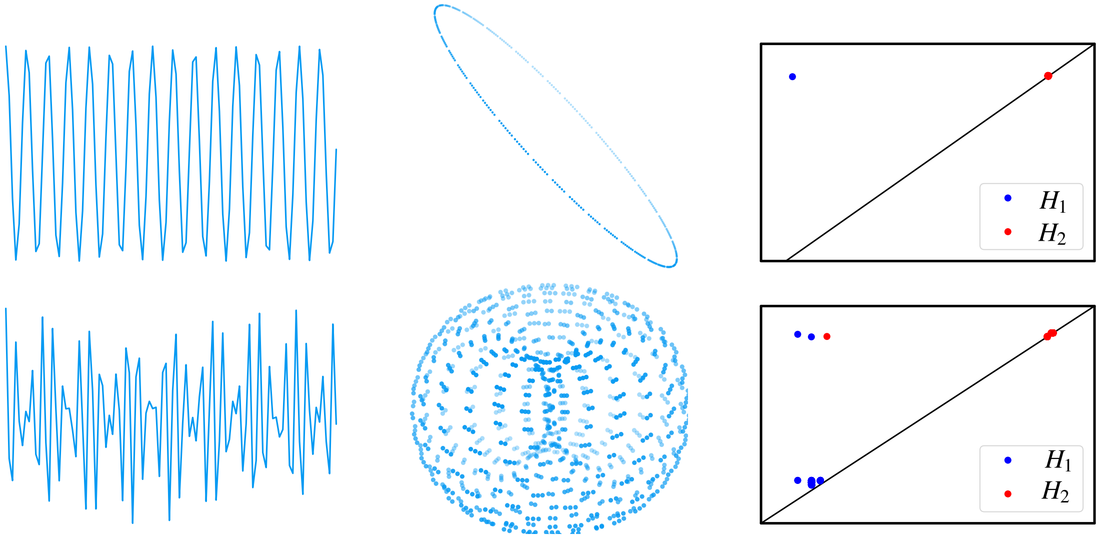


Figure 2. Given a time series (left column), we reconstruct the underlying attractor via its sliding-window embedding (middle column), and then compute its persistent homology (right column). Top row: the signal from the periodic motion of an ideal pendulum (left). The sliding-window point cloud parametrizes a circle (middle), so its persistence diagram exhibits exactly one nontrivial 1-dimensional homology class (right). Bottom row: A quasiperiodic function with two incommensurate frequencies (left); its sliding-window embedding (center) recovers a 2-torus. The corresponding persistence diagrams (right) confirm this topology by showing two independent 1-dimensional classes and one 2-dimensional class.

independent frequencies in the signal f. Moreover, the number of j-dimensional simplices for Ripser grows as $\binom{n_{\text{SW}}}{j+1}$, where n_{SW} is the number of points in the sliding-window embedding. In practice, this proves to be computationally taxing even when $N = 2$ and $n_{\text{SW}} = 1000$, and motivates our efforts to develop approximation methods with known error bounds (see Figure 3) that would provide a faster and computationally accessible alternative (see Figure 11 and Table 2). Our method relies on the three-gap theorem, which uses continued-fraction expansions from number theory to describe how points — sampled at irrational rotation angles — are distributed around a circle. We also use the Künneth formula in persistent homology. Thanks to the results in Section 2.3, our pipeline applies to a general quasiperiodic function f. It suffices to compute the spectrum of f — e.g., via the fast Fourier transform (FFT) — to obtain the input to our algorithm (Section 2.5). Specifically, we replace the standard Ripser-based step 3 by our two-part alternative (3b) shown below (we keep the original procedure as step (3a) for comparison).

(1) Start with a time series f.

(2) Reconstruct the phase space by computing $\{\text{SW}_{d,\tau} f(t)\}_{t=0}^{T}$.

(3a) Compute $\text{dgm}_j^R(\{\text{SW}_{d,\tau} f(t)\}_{t=0}^{T}, d_2)$ using Ripser.

(3b) (i) Use the FFT to retrieve the frequencies of f and then compute their continued-fraction expansion (CFE).

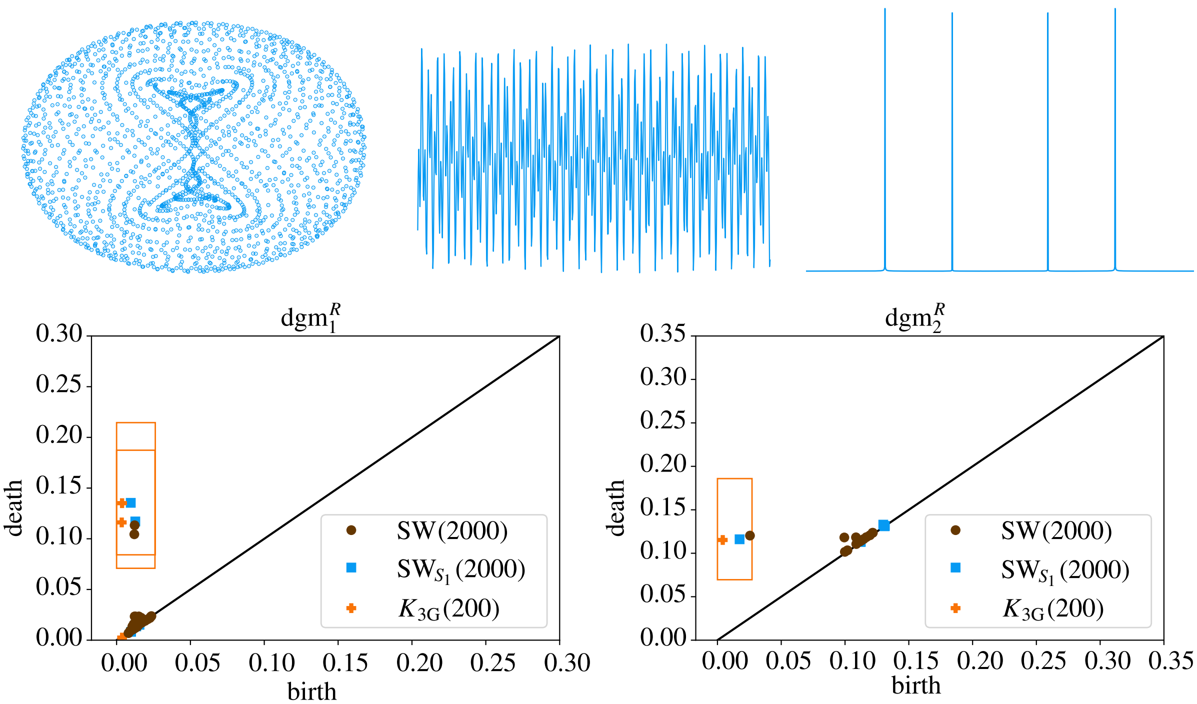


Figure 3. Top row: the plot on the left is the phase space $(x, \dot{x})$ from the pendulum attached to a sliding block shown in Figure 1. The solution $x(t)$ we used for the sliding-window embedding is plotted in the middle, followed by the modulus of its discrete Fourier transform on the right. Bottom row: persistence diagrams of the sliding-window embedding of $x(t)$ (brown circles) of a truncated Fourier series of $x(t)$ (blue squares), and our proposed approximation $K_{3\mathrm{G}}$ (orange crosses). The orange rectangles depict the theoretical approximation bound.

(ii) Use the CFE as shown in Section 3 and then apply the results from Section 4 to approximate $\mathrm{dgm}_j^R(\{\mathrm{SW}_{d,\tau} f(t)\}_{t=0}^T, d_2)$ for $j = 1, 2$.

In step (3b)(ii), the continued-fraction expansions allow us to compute exact persistence diagrams on each frequency circle (via the three-gap theorem; see Section 3), which we then assemble — using the Künneth formula as detailed in Section 4 — to approximate the persistence diagrams of interest.

Remark 1. The idea of using Fourier analysis together with the Künneth formula to compute persistent homology is not unique to our work. Indeed, in [Kim and Jung 2026], these tools are combined to construct a multiparameter filtration whose persistent homology can be computed exactly. As the authors state:

> Our main idea is to transform time-series data into a barcode through the Liouville torus without utilizing sliding-window embedding.

Thus their method depends on reconstructing the full Liouville torus.

feature	our 3G method	method of [Kim and Jung 2026]
FFT-based decomposition	yes	yes
Künneth-type product formula	yes	yes
Takens' theorem	yes	yes
sliding-window embedding	yes	no
Liouville torus	no	yes
continued fraction expansion	yes	no
filtration type	1-parameter	multiparameter
type of signal	quasiperiodic	periodic
exactness vs. approximation	approximate (error bound)	exact

Table 1. At-a-glance comparison of our 3G pipeline vs. the exact multiparameter method of [Kim and Jung 2026].

By contrast, our work approximates the persistence diagram via a single-parameter sliding-window Rips filtration

$$\underset{\text{(time-series data)}}{f} \longrightarrow \underset{\text{(Rips complex of sliding-window embedding)}}{R_\epsilon(\mathrm{SW}_{d,\tau} f)} \longrightarrow \underset{\text{(persistence diagram)}}{\mathrm{dgm}_j^R(\mathrm{SW}_{d,\tau} f).}$$

Meanwhile, [Kim and Jung 2026] provides exact formulas for the persistence diagram of the Liouville torus Ψ_f, obtained from the multiparameter filtration

$$\underset{\text{(time-series data)}}{f} \longrightarrow \underset{\text{(multiparameter filtration of the Liouville torus)}}{R_{\epsilon_1}(\pi_1 \Psi_f) \times \cdots \times R_{\epsilon_N}(\pi_N \Psi_f)} \longrightarrow \underset{\text{(persistence diagram)}}{\mathrm{dgm}_j^{R,l}(\Psi_f).}$$

We summarize the distinguishing features of these two methods in Table 1.

2. Background and definitions

In this section we cover the mathematical concepts needed for our three-gap (3G) method. We first define persistence modules, their barcodes and stability theory, and the special case of persistent homology for filtered simplicial complexes. Next we introduce dynamical systems and highlight Takens' reconstruction theorem, which underlies the sliding-window embedding. After that, we show how the Fourier transform recovers the frequency vector of a general quasiperiodic signal. We then recall continued-fraction expansions and their fundamental properties, leading directly to the three-gap theorem — the number-theoretic pillar of our work. We conclude with an overview of our 3G method (named for its use of the three-gap theorem), emphasizing how these components fit together.

2.1. ***Persistent homology.*** Our presentation of persistence modules and their interleaving distance follows [Chazal et al. 2016, Chapters 1, 3 and 4]. These concepts provide an algebraic account of stability, used herein.

2.1.1. *Persistence modules.*

Definition 2.1. A persistence module $\mathbb{V}$ over the real numbers $\mathbb{R}$ is an indexed family of vector spaces

$$\{V_t \mid t \in \mathbb{R}\},$$

together with a doubly indexed family of linear maps

$$\{v_s^t : V_s \to V_t \mid s \leq t\}$$

which satisfy the composition law

$$v_s^t \circ v_r^s = v_r^t$$

whenever $r \leq s \leq t$, and where v_t^t is the identity map on V_t for every $t \in \mathbb{R}$.

Definition 2.2. A morphism Φ between two persistence modules $\mathbb{U}$ and $\mathbb{V}$ is a collection of linear maps

$$\varphi_t : U_t \to V_t$$

such that the diagram

$$\begin{array}{ccc} U_s & \xrightarrow{u_s^t} & U_t \\ {\scriptstyle \varphi_s}\downarrow & & \downarrow{\scriptstyle \varphi_t} \\ V_s & \xrightarrow{v_s^t} & V_t \end{array}$$

commutes for all $s \leq t$. Composition is defined in the obvious way, as are identity morphisms. This makes the collection of persistence modules into a category. The set of all morphisms from $\mathbb{U}$ to $\mathbb{V}$ is denoted by

$$\mathrm{Hom}(\mathbb{U}, \mathbb{V}).$$

When $\mathbb{U} = \mathbb{V}$, these are called *endomorphisms*, and we write

$$\mathrm{End}(\mathbb{V}) = \mathrm{Hom}(\mathbb{V}, \mathbb{V}).$$

We have the following generalization of a morphism:

Definition 2.3. Let $\mathbb{U} = \{U_t, u_s^t\}$ and $\mathbb{V} = \{V_t, v_s^t\}$ be persistence modules indexed by $t \in \mathbb{R}$. A *morphism of degree* $\delta \geq 0$ is a family of linear maps

$$\varphi_t : U_t \to V_{t+\delta} \quad \text{for } t \in \mathbb{R},$$

such that for all $s \leq t$ the square

$$\begin{array}{ccc} U_s & \xrightarrow{u_s^t} & U_t \\ {\scriptstyle \varphi_s}\downarrow & & \downarrow{\scriptstyle \varphi_t} \\ V_{s+\delta} & \xrightarrow{v_{s+\delta}^{t+\delta}} & V_{t+\delta} \end{array}$$

commutes. We write

$$\mathrm{Hom}^\delta(\mathbb{U}, \mathbb{V}) = \{\varphi \text{ of degree } \delta\}.$$

When $\mathbb{U} = \mathbb{V}$, such a map is called a *degree-δ endomorphism*, and we write

$$\mathrm{End}^{\delta}(\mathbb{V}) = \mathrm{Hom}^{\delta}(\mathbb{V}, \mathbb{V}).$$

Definition 2.4. For any persistence module $\mathbb{V} = \{V_t, v_s^t\}$ and $\delta \geq 0$, the *shift map*

$$1_{\mathbb{V}}^{\delta} = \{v_t^{t+\delta} \mid V_t \to V_{t+\delta}\}_{t\in\mathbb{R}}$$

is a degree-δ endomorphism in $\mathrm{End}^{\delta}(\mathbb{V})$.

Definition 2.5. Let $\mathbb{U}$ and $\mathbb{V}$ be persistence modules and fix $\delta \geq 0$. A *δ-interleaving* consists of

$$\Phi \in \mathrm{Hom}^{\delta}(\mathbb{U}, \mathbb{V}) \quad \text{and} \quad \Psi \in \mathrm{Hom}^{\delta}(\mathbb{V}, \mathbb{U}),$$

such that

$$\Psi\Phi = 1_{\mathbb{U}}^{2\delta} \quad \text{and} \quad \Phi\Psi = 1_{\mathbb{V}}^{2\delta}.$$

Definition 2.6. The *interleaving distance* between two persistence modules $\mathbb{U}$ and $\mathbb{V}$ is

$$d_I(\mathbb{U}, \mathbb{V}) = \inf\{\delta \geq 0 \mid \mathbb{U} \text{ and } \mathbb{V} \text{ admit a } \delta\text{-interleaving}\}.$$

If no finite δ exists, set $d_I(\mathbb{U}, \mathbb{V}) = \infty$.

Under appropriate finiteness conditions, which we describe below, a persistence module can be uniquely determined up to isomorphism by an invariant called the *barcode*. Indeed, first note that the direct sum of persistence modules can be defined componentwise; that is, $\mathbb{U} \oplus \mathbb{V}$ has vector spaces $U_t \oplus V_t$ (direct sum of vector spaces) and linear maps $v_s^t \oplus u_s^t$.

Definition 2.7. Let $\mathbb{V}$ be a persistence module such that V_t is finite dimensional for each $t \in \mathbb{R}$ (i.e., $\mathbb{V}$ is pointwise finite). By [Crawley-Boevey 2015], there is a unique multiset $\mathrm{bcd}(\mathbb{V})$ of intervals $I \subseteq \mathbb{R}$, called the barcode of $\mathbb{V}$, such that

$$\mathbb{V} \cong \bigoplus_{I \in \mathrm{bcd}(\mathbb{V})} \mathbb{I}_I,$$

where $\mathbb{I}_I$ is the persistence module with vector spaces the field $\mathbb{F}$ for each $t \in I$ and the zero vector space otherwise, and linear maps the identity whenever possible and the zero map when not. By multiset we mean that intervals may appear with finite repetition (multiplicity). The persistence diagram of $\mathbb{V}$ is the following multiset of points in the extended plane $\overline{\mathbb{R}}^2 = [-\infty, \infty] \times [-\infty, \infty]$:

$$\mathrm{dgm}(\mathbb{V}) = \{(\inf(I), \sup(I)) \mid I \in \mathrm{bcd}(\mathbb{V})\}.$$

The space of persistence diagrams can be endowed with an extended pseudometric called the bottleneck distance d_B [Cohen-Steiner et al. 2007; Chazal et al. 2016]. In the following definition we consider the points in the diagonal $y = x$ as part of the persistence diagrams; moreover, each point (x, x) is included with countable infinite multiplicity.

Definition 2.8. Let dgm, $\mathrm{dgm}' \subseteq \overline{\mathbb{R}}^2$ be two persistence diagrams. Their bottleneck distance is

$$d_B(\mathrm{dgm}, \mathrm{dgm}') = \inf_{\phi:\mathrm{dgm}\to\mathrm{dgm}'} \sup_{y\in\mathrm{dgm}} \{\|y - \phi(y)\|_\infty\},$$

where ϕ is a bijection of multisets and $\|(a, b)\|_\infty = \max\{|a|, |b|\}$ is the extended sup norm on $\overline{\mathbb{R}}^2$.

Conceptually, given two persistence diagrams we consider all multiset bijections that pair points between them. The distortion of a given pairing can be quantified using the largest infinity norm over all paired points. The bottleneck distance is then the smallest possible distortion over all bijective pairings. In essence, this captures how similar two persistence diagrams are to each other in the plane. We note that including all points (x, x) along the diagonal ensures that any two diagrams admit a bijective matching (so that unmatched off-diagonal points may be paired with diagonal points).

The interleaving distance between persistence modules and the bottleneck distance between their persistence diagrams satisfy the celebrated isometry theorem [Lesnick 2015],

Theorem 2.9. *Let $\mathbb{U}$ and $\mathbb{V}$ be pointwise finite persistence modules. Then*

$$d_I(\mathbb{U}, \mathbb{V}) = d_B(\mathrm{dgm}(\mathbb{U}), \mathrm{dgm}(\mathbb{V})).$$

2.1.2. *Persistent homology of filtered simplicial complexes.* In this section we describe a formal notion of "shape" for discrete data sets. Unless otherwise noted, definitions follow [Hatcher 2002; Gakhar and Perea 2024].

Definition 2.10. Given a set X, an abstract simplicial complex (or a simplicial complex, for short) with vertices in X is a subset $K \subseteq \mathcal{P}(X)$ of the power set of X, such that

(1) every $\sigma \in K$ is finite and nonempty,

(2) for any $\sigma \in K$, if $\tau \neq \varnothing$ and $\tau \subseteq \sigma$, then $\tau \in K$.

The elements of K are called simplices and the dimension of a simplex is one less than its cardinality: $\dim(\sigma) = |\sigma| - 1$. We also define $\dim(K) = \max\{\dim(\sigma) \mid \sigma \in K\}$. A face of σ is any nonempty proper subset of it. The n-th skeleton of K is denoted by $K^{(n)} := \{\sigma \in K \mid \dim(\sigma) \leq n\}$ for $n \in \mathbb{N} = \{0, 1, \ldots\}$; in particular, $K^{(0)}$ is called the set of vertices of K and $K^{(1)} \smallsetminus K^{(0)}$ is its set of edges. A subcomplex of K is a subset $L \subseteq K$ which is also an abstract simplicial complex.

We can construct an abstract simplicial complex from any set of points in a metric space. Indeed, this is achieved via the *Rips complex*, which treats each point in a metric space (X, d), say $x \in X$, as a vertex; edges $\{x_0, x_1\}$ are created between two vertices if their distance is at most some fixed $\epsilon \geq 0$, i.e., $d(x_0, x_1) \leq \epsilon$; connecting three vertices creates a triangle $\{x_0, x_1, x_2\}$, four a tetrahedron $\{x_0, x_1, x_2, x_3\}$, and so on. Formally:

Definition 2.11. Given a metric space (X, d) and a real number ϵ, the Rips complex $R_\epsilon(X, d)$ is the simplicial complex

$$R_\epsilon(X, d) = \{\{x_0, \ldots, x_n\} \subseteq X \mid d(x_i, x_j) \leq \epsilon, \ \forall 0 \leq i, j \leq n\}.$$

With this construction we can leverage a powerful shape descriptor from algebraic topology — simplicial homology — which assigns algebraic objects to an abstract simplicial complex in a deformation-invariant manner. In particular, the 0-th dimensional homology measures the number of connected components, the 1-st dimensional homology detects loops, and the 2-nd dimensional homology detects cavities. In general, the n-th dimensional homology detects n-dimensional holes. The relevant definitions are below.

Definition 2.12. Let K be a simplicial complex and $\mathbb{F}$ a field. For $n \in \mathbb{N}$, let $C_n(K;\mathbb{F})$ denote the $\mathbb{F}$-vector space with basis the set of n-simplices of K:

$$C_n(K;\mathbb{F}) = \left\{ \sum_{\sigma \in K^{(n)} \smallsetminus K^{(n-1)}} a_\sigma \sigma \;\middle|\; a_\sigma \in \mathbb{F},\ a_\sigma \neq 0 \text{ for only finitely many } \sigma \right\}.$$

For $n < 0$, we let $C_n(K;\mathbb{F})$ be the zero vector space.

We call an element $\tau \in C_n(K;\mathbb{F})$ an *n-chain*. If

$$\tau = \sum_{\sigma \in K^{(n)} \smallsetminus K^{(n-1)}} a_\sigma \sigma$$

and σ_0 is one of the simplices in the sum (i.e., $a_{\sigma_0} \neq 0$), we write $\sigma_0 \in \tau$.

Definition 2.13. Fix a partial (e.g., a total) order $\preceq$ on the vertices of K so that each simplex $\sigma \in K$ is totally ordered. The n-th boundary map $\partial_n : C_n(K;\mathbb{F}) \to C_{n-1}(K;\mathbb{F})$ is the linear transformation defined for any $\sigma = \{v_0, \dots, v_n\} \in K^{(n)} \smallsetminus K^{(n-1)}$ as

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^n \{v_0, \dots, \hat{v_i}, \dots, v_n\},$$

where $v_i \preceq v_{i+1}$ for $i = 0, \dots, n-1$, and $\{v_0, \dots, \hat{v_i}, \dots, v_n\}$ denotes the $(n-1)$-th face of σ obtained by removing the i-th vertex v_i from the totally ordered set $\{v_0, \dots, v_n\}$.

One can show that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{N}$.

Definition 2.14. For $n \in \mathbb{N}$, a field $\mathbb{F}$ and a simplicial complex K, let

$$\ker(\partial_n) = \{\tau \in C_n(K;\mathbb{F}) \mid \partial_n(\tau) = 0\} \quad \text{and} \quad \operatorname{im}(\partial_{n+1}) = \{\partial_{n+1}(\beta) \mid \beta \in C_{n+1}(K;\mathbb{F})\} \subseteq \ker(\partial_n).$$

The n-th homology of K with coefficients in $\mathbb{F}$ is the quotient vector space

$$H_n(K;\mathbb{F}) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}.$$

We note that the isomorphism type of $H_n(K;\mathbb{F})$ is independent of the partial order on the vertices of K. Moreover, when $K = R_\epsilon(X,d)$ and $\epsilon \geq 0$, the nonzero elements of $H_n(R_\epsilon(X,d);\mathbb{F})$ correspond to nontrivial n-cycles (i.e., elements of $\ker(\partial_n)$) not bounding any $(n+1)$-chain in the Rips complex $R_\epsilon(X,d)$, and thus measure n-dimensional holes in (X,d) at scale ϵ. We note that

$$R_\epsilon(X,d) \subseteq R_{\epsilon'}(X,d) \quad \text{whenever } \epsilon \leq \epsilon',$$

which is a particular case of the following definition:

Definition 2.15. A filtered simplicial complex $\mathcal{K} = \{K_\epsilon\}_{\epsilon\in\mathbb{R}}$ is a collection of simplicial complexes K_ϵ such that

$$K_\epsilon \subseteq K_{\epsilon'}$$

whenever $\epsilon \le \epsilon'$. We refer to $\mathcal{K}$ as a filtration.

We note that the family

$$\mathcal{R}(X, d) = \{R_\epsilon(X, d)\}_{\epsilon\in\mathbb{R}}$$

is a filtered complex in the sense of Definition 2.15, called the *Rips filtration* of the metric space (X, d). The main idea behind persistent homology is to track the evolution of homology classes in a filtration as ϵ changes. For instance, as ϵ increases one may see several connected components (0-cycles) merge into one, or an empty spherical cavity (a 2-cycle) become filled. The resulting multiscale summary is recorded in the persistence diagram of the following persistence module.

Definition 2.16. Let $\mathcal{K} = \{K_\epsilon\}_{\epsilon\in\mathbb{R}}$ be a filtration. For each $n \in \mathbb{N}$, define the n-th persistence module

$$H_n(\mathcal{K}; \mathbb{F}) = \{H_n(K_\epsilon; \mathbb{F}),\ T_{\epsilon,\epsilon'} : H_n(K_\epsilon; \mathbb{F}) \to H_n(K_{\epsilon'}; \mathbb{F}), \epsilon \le \epsilon'\}$$

as the family of $\mathbb{F}$-vector spaces $H_n(K_\epsilon; \mathbb{F})$ and linear transformations $T_{\epsilon,\epsilon'}$ induced by the inclusion maps $K_\epsilon \hookrightarrow K_{\epsilon'}$, for $\epsilon \le \epsilon'$. The n-th persistent homology groups are

$$H_n^{\epsilon,\epsilon'}(\mathcal{K}; \mathbb{F}) = \mathrm{im}(T_{\epsilon,\epsilon'})$$

and their dimensions over $\mathbb{F}$ are the persistent Betti numbers

$$\beta_n^{\epsilon,\epsilon'}(\mathcal{K}; \mathbb{F}) := \mathrm{rank}(T_{\epsilon,\epsilon'}) = \dim_{\mathbb{F}}(H_n^{\epsilon,\epsilon'}(\mathcal{K}; \mathbb{F})).$$

If $\mathcal{K} = \{K_\epsilon\}_{\epsilon\in\mathbb{R}}$ and $n \in \mathbb{N}$ are such that $H_n(\mathcal{K}; \mathbb{F})$ is pointwise finite dimensional, then Definition 2.7 ensures the existence of the barcode

$$\mathrm{bcd}_n(\mathcal{K}) = \mathrm{bcd}(H_n(\mathcal{K}; \mathbb{F}))$$

and the persistence diagram $\mathrm{dgm}_n(\mathcal{K}) = \mathrm{dgm}(H_n(\mathcal{K}; \mathbb{F}))$.

Remark 2. When the n-th homology of a filtration changes at only finitely many filtration values, the resulting persistence diagram has a particularly simple description in terms of persistent Betti numbers; see [Cohen-Steiner et al. 2007, p. 106]. Indeed, let $\mathcal{K} = \{K_\epsilon\}_{\epsilon\in\mathbb{R}}$ be a filtration whose n-th persistence module is pointwise finite and changes at only finitely many values of ϵ, namely $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_J$ (i.e., finitely many homological critical values). Then given interleaved values $\epsilon_j < r_j < \epsilon_{j+1}$ with $r_0 = \epsilon_{-1} = -\infty$ and $r_{J+1} = \epsilon_{J+1} = \infty$, the persistence diagram $\mathrm{dgm}_n(\mathcal{K})$ is the collection of points (ϵ_i, ϵ_j) that have *positive* multiplicity

$$\beta_n^{r_{i-1},r_j}(\mathcal{K}; \mathbb{F}) - \beta_n^{r_i,r_j}(\mathcal{K}; \mathbb{F}) + \beta_n^{r_i,r_{j-1}}(\mathcal{K}; \mathbb{F}) - \beta_n^{r_{i-1},r_{j-1}}(\mathcal{K}; \mathbb{F}). \tag{1}$$

Unless otherwise stated, we assume that every metric space (X, d) is finite. Hence for each $\epsilon \in \mathbb{R}$ the Rips complex $R_\epsilon(X, d)$ is a finite simplicial complex and the persistence module

$$H_n(\mathcal{R}(X, d); \mathbb{F})$$

is pointwise finite. The corresponding barcode and persistence diagram are denoted by $\mathrm{bcd}_n^R(X, d)$ and $\mathrm{dgm}_n^R(X, d)$, respectively, and the intervals $I \in \mathrm{bcd}_n^R(X, d)$ are closed on the left and open on the right. An interval $[a, b) \in \mathrm{bcd}_n^R(X, d)$ indicates that an n-cycle is born at $\epsilon = a$ and becomes trivial at $\epsilon = b$; its persistence (lifetime) $b - a$ provides a measure of robustness for this homological feature. Plotting $\mathrm{dgm}_n^R(X, d)$ in the extended plane $\overline{\mathbb{R}}^2$ yields a visualization of the persistent homology of the Rips filtration $\mathcal{R}(X, d)$. Points $(a, b) \in \mathrm{dgm}_n^R(X, d)$ that lie far from the diagonal $y = x$ represent topological features of (X, d) with greater robustness, as the stability theorem (Theorem 2.19, which we describe next) implies.

When working with metric spaces and the Rips filtration, we can connect the bottleneck distance of the corresponding persistence diagrams with the Gromov–Hausdorff distance. The latter measures the similarity between bounded metric spaces and is defined below.

Definition 2.17. Given two nonempty bounded subsets A and B of a metric space (X, d), the *Hausdorff distance* $d_H(A, B)$ is defined as

$$d_H(A, B) = \max\big\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(b, a)\big\},$$

where $d(a, b)$ denotes the distance between points a and b in X.

Definition 2.18. For two bounded metric spaces (X_1, d_1) and (X_2, d_2), the Gromov–Hausdorff distance

$$d_{GH}(X_1, X_2)$$

is defined as follows. Take any common metric space Z and any pair of isometric embeddings $\phi_1 : X_1 \to Z$ and $\phi_2 : X_2 \to Z$; compute the Hausdorff distance between $\phi_1(X_1)$ and $\phi_2(X_2)$ in Z. Then $d_{GH}(X_1, X_2)$ is the infimum of all such Hausdorff distances:

$$d_{GH}(X_1, X_2) = \inf_{Z,\phi_1,\phi_2} d_H(\phi_1(X_1), \phi_2(X_2)).$$

The aforementioned connection is the content of the well-known Rips stability theorem [Cohen-Steiner et al. 2007; Chazal et al. 2016].

Theorem 2.19. *Let X_1 and X_2 be finite metric spaces. Then*

$$d_B(\mathrm{dgm}_n^R(X_1, d_1), \mathrm{dgm}_n^R(X_2, d_2)) \le 2d_{GH}(X_1, X_2).$$

In addition to stability, one can also describe the Rips persistence of metric products.

Definition 2.20. Let (X, d_X) and (Y, d_Y) be metric spaces. The maximum metric d_∞ is given by

$$d_\infty((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\},$$

where $(x, y), (x', y') \in X \times Y$.

We note that $(X \times Y, d_\infty)$ is a metric space. Further, its barcodes are given by [Gakhar and Perea 2019]:

Theorem 2.21 (persistent Künneth formula). *Let (X, d_X) and (Y, d_Y) be finite metric spaces. Then,*

$$\mathrm{bcd}_n^R(X \times Y, d_\infty) = \bigcup_{i+j=n} \{I \cap J \mid I \in \mathrm{bcd}_i^R(X, d_X),\ J \in \mathrm{bcd}_j^R(Y, d_Y)\},$$

for all $n \in \mathbb{N}$.

Finally, we present an observation which we use repeatedly:

Proposition 2.22. *Let (X, d_X) and (Y, d_Y) be two finite metric spaces. Suppose there exists a bijection $f : X \to Y$ such that for some $\lambda > 0$*

$$d_X(t_1, t_2) = \lambda d_Y(f(t_1), f(t_2)).$$

Then

$$\mathrm{dgm}_n^R(X, d_X) = \{(\lambda a, \lambda b) \mid (a, b) \in \mathrm{dgm}_n^R(Y, d_Y)\}.$$

Proof. Let us define the metric $d_Y^\lambda = \lambda d_Y$ on Y. By assumption, f is an isometry between (X, d_X) and (Y, d_Y^λ). This implies that f induces simplicial isomorphisms between $R_\epsilon(X, d_X)$ and $R_\epsilon(Y, d_Y^\lambda)$ for all $\epsilon \geq 0$. This, in turn, induces isomorphisms on the corresponding persistence modules, which implies they have the same persistence diagrams. □

2.2. *Dynamical systems.* Having presented persistent homology as a tool to measure shape in discrete data, we now introduce the framework that models the source of said data. Indeed, several real-world scientific measurements arise from underlying deterministic mechanisms. Although said mechanism may be unknown or highly complex, the theory of dynamical systems enables its mathematical description. The framework consists of a phase space M that represents all of the relevant states of the system and a function Φ that keeps track of the evolution of the system. The following definitions are taken from [Perea 2019].

Definition 2.23. A global continuous time dynamical system is a pair (M, Φ), where M is a topological space and $\Phi : \mathbb{R} \times M \to M$ is a continuous map such that $\Phi(0, p) = p$, and $\Phi(s, \Phi(t, p)) = \Phi(s + t, p)$ for all $p \in M$ and all $t, s \in \mathbb{R}$.

Note that any autonomous system of ordinary differential equations

$$\dot{x}(t) = \Lambda(x(t)), \quad x(t) \in M,$$

on a smooth manifold M defines a dynamical system. Its flow $\Phi_t : M \to M$ is given by following the integral curves (solutions) of the ODE from each initial condition. Hence the evolution of the system is fully determined by the vector field Λ and the chosen initial state.

Representing a model in this fashion allows for a topological understanding of the system. This is done by looking at trajectories in the phase space M; indeed, although most systems can only be solved numerically, given equations, a general understanding of the shape of trajectories in M provides qualitative information of the system [Skraba et al. 2012]. In particular, knowing the topology of an *attractor* — i.e., a subset of M that pulls nearby trajectories into it — is of great significance. Formally:

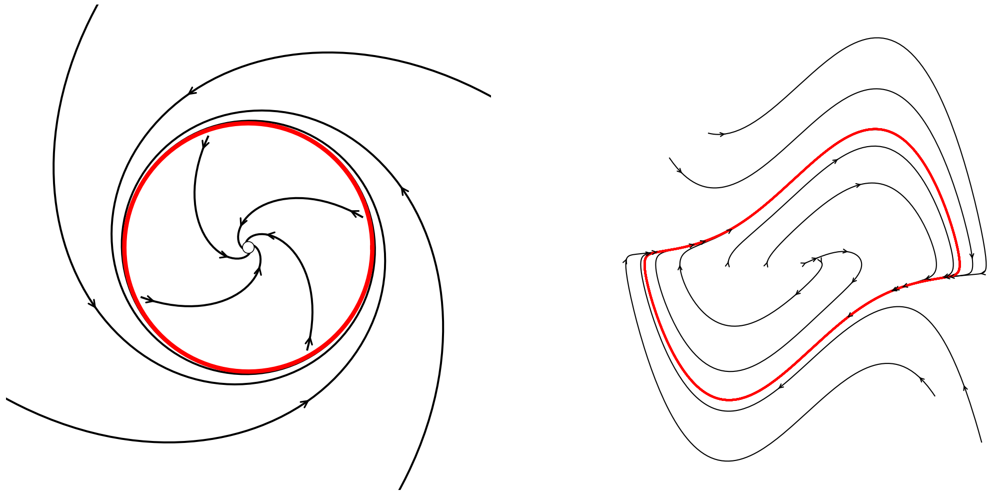


Figure 4. Trajectories of the system from Example 2.25, plotted in the xy-plane (left) and from Example 2.26 in the $x\dot{x}$ plane (right). In each case, the attractor is a topological circle, highlighted in red.

Definition 2.24. A set $A \subseteq M$ is called an attractor if

(1) it is compact,

(2) it is an invariant set, i.e., if $a \in A$ then $\Phi(t, a) \in A$ for all $t \geq 0$,

(3) there is an invariant open neighborhood U of A such that $A = \bigcap_{t \geq 0} \{\Phi(t, p) \mid p \in U\}$.

Example 2.25. Consider the dynamical system given by the radial equation (in polar coordinates)

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1,$$

where $r \geq 0$. This system has a circle as an attractor in the xy-plane, shown in red in Figure 4, left.

Example 2.26. Consider the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0,$$

where $\mu \geq 0$ is a parameter. This system also has an attractor in the $x\dot{x}$-plane as shown in Figure 4, right. In this case, it is not a round unit circle, yet it is topologically equivalent to it.

An attractor that is homeomorphic to a circle corresponds to a system exhibiting periodicity. In general, an attractor that is homeomorphic to an N-dimensional torus, a toroidal attractor, comes from a quasiperiodic system [Gakhar and Perea 2024]. In practice, we often only have a measurement of the dynamical system in the form of time-series data. The underlying equations governing the system are unknown — i.e., neither M nor Φ are given. Nevertheless, there is a way to reconstruct attractors of this unknown system while preserving qualitative information. Concretely, Takens' landmark theorem [1981] shows that a single time series measurement — sampled from a generic observation function — suffices to reconstruct the topology of the underlying attractor.

Theorem 2.27 (Takens' embedding). *Let M be an m-dimensional compact Riemannian manifold. For pairs $(\phi, \bar{f})$, where $\phi \in C^2(M, M)$ and $\bar{f} \in C^2(M, \mathbb{R})$, it is a generic property that the following map is an embedding*:

$$\Psi_{\phi,\bar{f}} : M \to \mathbb{R}^{2m+1}, \quad \Psi_{\phi,\bar{f}}(p) = \big(\bar{f}(p), \bar{f}(\phi(p)), \bar{f}(\phi^2(p)), \ldots, \bar{f}(\phi^{2m}(p))\big)$$

The reconstruction given by the image of $\Psi_{\phi,\bar{f}}$ may look different from the original state space M, but we are assured it will be topologically equivalent. This is a powerful guarantee that validates the *sliding-window embedding* (defined below) as a means to reconstruct observed dynamics from time-series data.

Definition 2.28. For a function $f : \mathbb{R} \to \mathbb{C}$, an integer $d > 0$ called the embedding dimension, and a real number $\tau > 0$ called the time delay, the sliding-window embedding of f at t is given by

$$\mathrm{SW}_{d,\tau} f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ \vdots \\ f(t+d\tau) \end{bmatrix} \in \mathbb{C}^{d+1}.$$

For $T \subseteq \mathbb{R}$ a set of time points, the sliding-window point cloud of f sampled at T is the set

$$\mathbb{SW}_{d,\tau}(T) = \{\mathrm{SW}_{d,\tau} f(t) \mid t \in T\}.$$

In the context of a dynamical system $\Phi : \mathbb{R} \times M \to M$, an observation function $\bar{f} : M \to \mathbb{R}$, and an initial condition $p_0 \in M$ such that the trajectory $t \mapsto \Phi(t, p_0)$ densely samples an attractor $A \subseteq M$, Takens' theorem implies that the sliding window of the time series $t \mapsto \bar{f} \circ \Phi(t, p_0)$ — with appropriate parameters $\tau > 0$ and $d \geq \dim(M)$ — produces a point cloud whose Rips persistence diagrams can be used to infer the homology of A.

In particular, for toroidal attractors and selecting (d, τ) according to [Gakhar and Perea 2024], we are detecting homological features of an N-dimensional torus, as depicted in Figures 3 and 5. In summary, here are steps for detecting the presence of toroidal attractors given time-series data:

(1) Start with a time series pertaining to a measurement of a system.

(2) Construct the sliding-window point cloud from it.

(3) Compute its persistence diagrams to determine if the point cloud samples an N-torus.

Although sliding windows and persistence are well established and have found multiple applications [Tralie and Perea 2018; Perea et al. 2015; Gakhar and Perea 2019], computing persistence diagrams remains computationally taxing. We now move onto the tools that will enable the fast approximation of Rips persistence diagrams of sliding-window point clouds from quasiperiodic functions.

2.3. *Fourier analysis of quasiperiodic signals.*

Definition 2.29. We say that $f : \mathbb{R} \to \mathbb{C}$ is quasiperiodic with frequency vector $\omega = (\omega_1, \ldots, \omega_N)$ if $\{\omega_i\}_{i=1}^N$ are positive real numbers which are linearly independent over $\mathbb{Q}$ (i.e., incommensurate) and

$$f(t) = F(\omega_1 t, \ldots, \omega_N t),$$

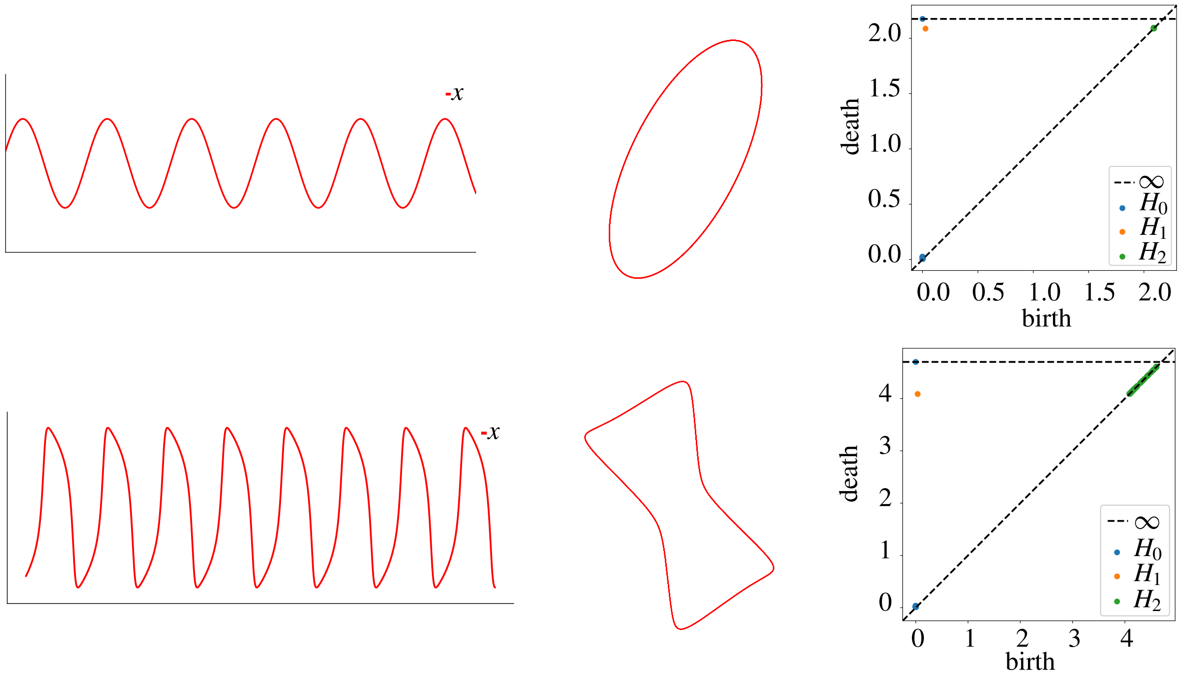


Figure 5. Left column: the $x(t)$ coordinate of the solution to the radial system from Example 2.25 (top row) and the van der Pol system from Example 2.26 (bottom row). Middle column: the sliding-window embedding of $x(t)$ with appropriate parameters d and τ. Right column: the Rips persistence diagrams in dimensions 0, 1 and 2 for the sliding-window point cloud.

where $F : \mathbb{T}^N \to \mathbb{C}$ is a complex-valued continuous function on the N-torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$, called the parent function of f.

As we will see below, one can leverage the spectrum of a quasiperiodic function to infer the underlying frequency vector, as well as to approximate the Rips persistence diagrams of the sliding-window point cloud. This is enabled by the following result [Gakhar and Perea 2024, Theorem 1.7 and Corollary 1.13].

Theorem 2.30. *Let $f : \mathbb{R} \to \mathbb{C}$ be a quasiperiodic function with frequency vector $\boldsymbol{\omega}$ and parent function F. For each $\boldsymbol{k} \in \mathbb{Z}^N$, the $\boldsymbol{k}$-th Fourier coefficient of F can be computed as*

$$\widehat{F}(\boldsymbol{k}) = \lim_{\lambda\to\infty} \frac{2\pi}{\lambda} \int_0^\lambda f(t) e^{-2\pi i \langle \boldsymbol{k}, t\boldsymbol{\omega}\rangle}\, dt,$$

where $i = \sqrt{-1}$. Furthermore, if

$$S_K f(t) = \sum_{\|\boldsymbol{k}\|_\infty \le K} \widehat{F}(\boldsymbol{k}) e^{2\pi i \langle \boldsymbol{k}, t\boldsymbol{\omega}\rangle} \quad \textit{for } K \in \mathbb{N}$$

then for each $j, T \in \mathbb{N}$, $\mathrm{dgm}_j^R(\mathbb{SW}_{d,\tau} f(T))$ can be approximated in bottleneck distance by

$$\mathrm{dgm}_j^R(\mathbb{SW}_{d,\tau} S_K f(T))$$

as $K \to \infty$. The approximation is of order $O(K^{N/2-r})$ when $|\widehat{F}(\boldsymbol{k})| = O(\|\boldsymbol{k}\|_2^{-r})$ and $r > \frac{1}{2}N$.

Theorem 2.30 shows that, in bottleneck distance, the Rips persistence diagrams of a quasiperiodic signal f can be arbitrarily closely approximated by those of its truncated Fourier series $S_K f(t)$. Moreover, the coefficients of $S_K f$ are determined by the spectrum of f, which can be estimated via the discrete Fourier transform (implemented in the FFT algorithm).

This result justifies reducing general quasiperiodic functions to finite exponential sums of the form

$$f(t) = \sum_{j=1}^{M} c_j e^{2\pi i \omega_j t} \quad \text{for } M \in \mathbb{N},\ c_j \in \mathbb{C} \setminus \{0\} \text{ and } \omega_j \in \mathbb{R},$$

where $\omega_1, \ldots, \omega_M$ are not necessarily $\mathbb{Q}$-linearly independent. Set $\tilde{c}_j = \sqrt{d+1}c_j$, choose an embedding dimension $d \in \mathbb{N}$ and a delay $\tau > 0$, and let

$$A = \frac{1}{\sqrt{d+1}} \begin{pmatrix} 1 & \cdots & 1 \\ e^{2\pi i \omega_1 \tau} & \cdots & e^{2\pi i \omega_M \tau} \\ \vdots & \ddots & \vdots \\ e^{2\pi i \omega_1 d\tau} & \cdots & e^{2\pi i \omega_M d\tau} \end{pmatrix} \in \mathbb{C}^{(d+1)\times M} \quad \text{and} \quad v_t = \begin{pmatrix} \tilde{c}_1 e^{2\pi i \omega_1 t} \\ \vdots \\ \tilde{c}_M e^{2\pi i \omega_M t} \end{pmatrix} \in \mathbb{C}^M.$$

In this setting, the sliding-window embedding can be written succinctly as

$$\mathrm{SW}_{d,\tau} f(t) = A v_t,$$

and according to [Gakhar and Perea 2024, Corollary 1.3], the set $\{v_t\}_{t\in\mathbb{R}}$ is a dense subset of a space homeomorphic to the N-torus, for $N \le M$ equal to the dimension of $\mathrm{span}_{\mathbb{Q}}\{\omega_1, \ldots, \omega_M\}$ as a $\mathbb{Q}$-vector space. The multidimensional version of Kronecker's approximation theorem, as presented in [Hlawka et al. 1991, Chapter 2, Theorem 2], allows one to take $t \in \mathbb{N}$ and retain density. Furthermore, d and τ can be chosen so that the same holds true for $\{\mathrm{SW}_{d,\tau} f(t)\}_{t\in\mathbb{N}}$ [Gakhar and Perea 2024, Theorem 1.15].

With this in mind, the Cartesian product

$$G_T = \prod_{j=1}^{M} \{\tilde{c}_j e^{2\pi i \omega_j t} \mid t = 0, 1, \ldots, T\} \tag{2}$$

offers a way to approximate $\{v_t\}_{t=0}^{T}$ in Hausdorff distance, and hence the resulting Rips persistence diagrams in bottleneck distance, by Theorem 2.19, as $T \to \infty$. See also Figure 10. This is useful since the persistence diagrams of G_T can be readily computed using the persistent Künneth formula (Theorem 2.21). We will show in Theorem 4.3 that this approach can be used to approximate the Rips persistence diagrams of the sliding-window embedding of f.

2.4. ***Continued fraction expansion.*** The continued-fraction expansion of a real number lies at the heart of our work. All definitions, propositions, and theorems in this section are classical and can be found in standard references — most notably [Olds 1963, Chapter 3], and for Theorem 2.38 see, e.g., [Rockett and Szüsz 1992, p. 22]. As we will show, continued-fraction data enables us to infer persistent-homology information of sliding-window point clouds once the frequencies of the signal are estimated — e.g., with the FFT.

Definition 2.31. Let ω be a real number. The continued-fraction expansion of ω is given by

$$\omega = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots}}} := [a_1, a_2, a_3, \dots],$$

where $a_1 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$, for $k > 1$.

Note that the expansion, which is obtained by repeated applications of the division algorithm, is infinite if and only if ω is irrational. From here onward, we let ω denote a positive irrational number.

Definition 2.32. Let $k \in \mathbb{N}$. The k-th convergent of ω is given by

$$\frac{p_k}{q_k} = [a_1, a_2, \dots, a_k] \in \mathbb{Q}.$$

The terms p_k and q_k play a special role in the theory, and can be obtained recursively as follows.

Proposition 2.33. *The numerator p_k and the denominator q_k of the k-th convergent of ω satisfy*

$$p_k = a_k p_{k-1} + p_{k-2} \quad \textit{and} \quad q_k = a_k q_{k-1} + q_{k-2},$$

for $k \geq 1$, where

$$p_0 = 1, \quad q_0 = 0, \quad p_{-1} = 0, \quad q_{-1} = 1.$$

Proposition 2.33 provides a convenient computational method for obtaining the k-th convergent of ω, and implies the following result.

Proposition 2.34. *The terms p_k and q_k are relatively prime.*

We can quantify the quality of the convergent p_k/q_k as an approximation to ω by examining the absolute error

$$\left| \omega - \frac{p_k}{q_k} \right|.$$

Proposition 2.35. *Let $k \geq 1$. Then*

$$\frac{1}{2q_k q_{k+1}} < \left| \omega - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} < \frac{1}{q_k^2}.$$

Furthermore, this result allows us to establish the existence and uniqueness of infinite continued-fraction expansions. Indeed, this follows by noting that $q_k < q_{k+1}$ for all $k \geq 1$, and justifies the equality sign in Definition 2.31. The next result further characterizes their convergence behavior.

Proposition 2.36. *Let $k \geq 1$ be odd. Then*

$$\frac{p_k}{q_k} < \omega < \frac{p_{k+1}}{q_{k+1}}.$$

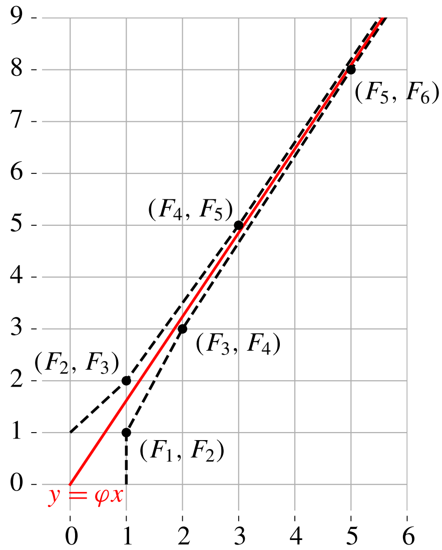


Figure 6. The Klein diagram for the golden ratio $\varphi = \frac{1}{2}(1 + \sqrt{5})$. Here F_k denotes the k-th Fibonacci number, which can be used to compute the k-th convergent p_k/q_k of φ; specifically, $p_k/q_k = F_k/F_{k-1}$.

Example 2.37. The results presented here can be used to argue that the golden ration φ is the "most" irrational number. Indeed, by computing

$$\varphi = \frac{1}{2}(1 + \sqrt{5}) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = [1, 1, 1, \cdots],$$

one can show that this expression corresponds to the slowest possible rate of convergence. The Klein diagram shown in Figure 6 provides a geometric picture of this observation.

We end this section by stating a critical property of the k-th convergent of ω attributed to Lagrange. Namely, that the k-th convergent provides a best rational approximation with bounded denominator to ω; see [Rockett and Szüsz 1992, p. 22].

Theorem 2.38. *Let a/b be different from p_{k+1}/q_{k+1} with $0 < b \leq q_{k+1}$. Then*

$$|b\omega - a| \geq |q_k\omega - p_k| > |q_{k+1}\omega - p_{k+1}|.$$

2.5. ***The three-gap theorem method.*** Thus far we have shown that exponential functions are fundamental in the study of quasiperiodic signals, and that their frequencies can be optimally approximated with their k-th convergents. If we let

$$f(t) = e^{2\pi i \omega t},$$

then viewed as a map $f : \mathbb{R} \to S^1 \subseteq \mathbb{C}$, $f(t)$ parametrizes the unit circle. Fix a natural number $T \in \mathbb{N}$ and consider the times $t = 0, 1, \ldots, T$. This yields the sample

$$\{f(t) \mid t = 0, 1, \ldots, T\} = \{e^{2\pi i \omega t} \mid t = 0, 1, \ldots, T\},$$

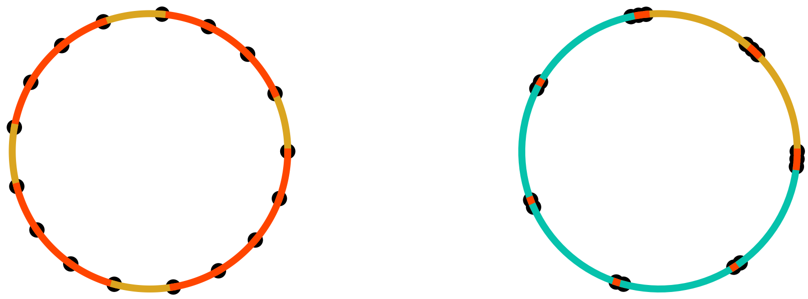


Figure 7. On the left, we depict $S_{\pi-1,17}$ in black. Each gap size is shown in a different color; in this case, there are only two distinct gaps (red and yellow). On the right, we show $S_{\sqrt{5},17}$ in black, which has three distinct gap sizes (green, red and yellow).

which in flat coordinates (i.e., modulo 1), corresponds to

$$\{t\omega \mod 1 \mid t = 0, 1, \dots, T\} \subseteq [0, 1].$$

Definition 2.39. For any real number $x \in \mathbb{R}$, write

$$[x] = x \mod 1$$

for its equivalence class on the flat circle $[0, 1]/0 \sim 1$. For $T \in \mathbb{N}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ we let

$$S_{\omega,T} := \{[0], [\omega], [2\omega], \dots, [T\omega]\}.$$

By identifying the endpoints of $[0, 1]$, we view $S_{\omega,T}$ as a finite sequence of points on the circle (see Figure 7). To measure distances in this space we use the quotient metric

$$\bar{d}([x], [y]) = \min\{|x - y|, |1 - (y - x)|, |1 - (x - y)|\},$$

which turns $(S_{\omega,T}, \bar{d})$ into a metric space. When we write $S_{\omega,T}$ we will require $\omega \in \mathbb{R} \setminus \mathbb{Q}$ unless otherwise stated. The asymptotic sampling properties of this set are well documented [Zhigljavsky and Aliev 1999] and follow in large part from the three-gap theorem (Steinhaus conjecture) [van Ravenstein 1988], which we state below.

Theorem 2.40 (three-gap theorem). *Let $T \in \mathbb{N}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$. The points in $S_{\omega,T}$ partition* $[0, 1]/0 \sim 1$ *into $T + 1$ intervals, such that their lengths take at least two and at most three different values. If three distinct lengths occur — denoted in increasing order by δ_A, δ_B and δ_C — then*

$$\delta_C = \delta_A + \delta_B.$$

Importantly, the conclusion of the three-gap theorem — that there are at most three distinct interval lengths — holds for every irrational ω and every sample size T. We illustrate the content of the three gap theorem in Figure 7 for $\omega = \pi - 1$ (left), $\omega = \sqrt{5}$ (right) and $T = 17$.

The lengths δ_A, δ_B and δ_C in the three gap theorem are in fact governed by the convergents of ω [Leong 2017]; see Proposition 3.2 for explicit formulas for the gaps and their multiplicities. This observation yields

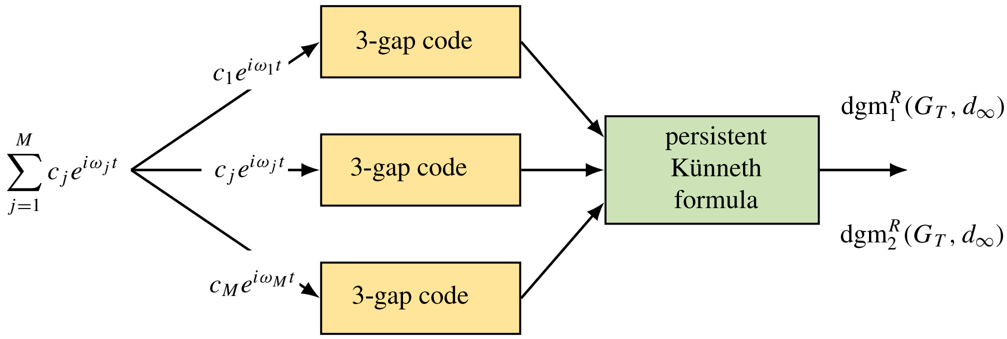


Figure 8. A schematic representation of the 3G method. Starting from a sum of exponentials approximating a quasiperiodic function, we extract its frequency parameters via the FFT. We then apply the three gap construction independently to each frequency circle and compute the resulting persistence diagrams. Finally, we assemble these individual diagrams using the persistent Künneth formula, yielding an approximation to the Rips persistence of the sliding-window point cloud of the original signal.

a complete description of pairwise distances between neighboring points in $S_{\omega,T}$, and we will show in the next section how they can be used to compute the persistence diagrams of the Rips filtration $\mathcal{R}(S_\omega, \bar{d})$; see Figure 9 for an example. This leads to our approximation scheme for the persistence diagrams of sliding-window point clouds from quasiperiodic functions: The three-gap theorem method (3G), depicted schematically in Figure 8.

In short, we first extract the independent frequencies ω_j of the input quasiperiodic signal; then, for each frequency ω_j, we compute its continued-fraction expansion and use its convergents to find the lengths δ_A and δ_B (and $\delta_C = \delta_A + \delta_B$ when appropriate) in $(S_{\omega_j,T}, \bar{d})$, as well as their multiplicities (see Proposition 3.2); the lengths and multiplicities of these gaps are used to derive the barcodes $\mathrm{bcd}_n^R(S_{\omega_j,T}, \bar{d})$ and persistence diagrams $\mathrm{dgm}_n^R(S_{\omega_j,T}, \bar{d})$ — this is the content of Corollaries 3.4 and 3.8, and the output of Algorithm 1 below; finally, we use the persistent Künneth theorem to compute $\mathrm{dgm}_n^R(G_T, d_\infty)$ — see (2) for the definition of G_T and Theorem 4.5 for the results. The quality of the approximation of $\mathrm{dgm}_n^R(\mathbb{SW}_{d,\tau} f(T), d_2)$ via $\mathrm{dgm}_n^R(G_T, d_\infty)$ is discussed in detail in Section 4, for d_2 the standard Euclidean distance in $\mathbb{C}^{d+1}$. The relevant approximation results are given in Theorem 4.3 and further clarified in Remark 3.

1: Input frequency ω_j
2: Scale frequency $\omega_j = \omega_j/(2\pi)$
3: Obtain continued-fraction expansion, C.F.E., of ω_j
4: Use C.F.E. as shown in Proposition 3.2 to obtain gaps and their multiplicities
5: Scale gaps as detailed in Theorem 4.5 to obtain the barcode of $\mathcal{R}(S_{\omega_j,T}, \bar{d})$

Algorithm 1. 3-Gap code.

3. Main results

We now move on to detailing the connection between the three-gap theorem and persistence diagrams. The theorem allows us to compute persistence diagrams by leveraging information from the continued-fraction expansion of the underlying frequencies estimated with the FFT of a quasiperiodic signal. With this at hand, we show how, using the persistent Künneth formula, we can establish an approximation method for the persistence diagrams of sliding-window embeddings from quasiperiodic functions. Finally, we show how to obtain error bounds for our method.

3.1. ***Persistence diagrams.*** Computing $\mathrm{dgm}_0^R(S_{\omega,T}, \bar{d})$ via the three-gap theorem reduces to relating the lengths of the gaps to the convergents of ω. Although the connection between gap sizes and convergents is classical (see [van Ravenstein 1988; Beresnevich and Leong 2017]), we include an alternative proof of the three-gap theorem that highlights how each continued-fraction coefficient contributes to the 0-th persistence diagram. Throughout, let $\{0, 1, \dots, T\} \subseteq \mathbb{N}$ denote the (discrete) set of time values, and let $\omega \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational frequency. We begin by finding the distance from a point in $S_{\omega,T}$ to its nearest neighbor:

$$\mathrm{Cl}(j) = \min\bigl\{\bar{d}([j\omega], x) \mid x \in S_{\omega,T} \setminus \{[j\omega]\}\bigr\} \quad \text{for } 0 \le j \le T.$$

Lemma 3.1. *Let $T \in \mathbb{N}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ with i-th convergent p_i/q_i. Fix $0 \le j_0 \le T$ and let*

$$\bar{k}(j_0) = \max\{j \mid q_j \le \max\{j_0, T - j_0\}\}.$$

If $0 \le j_1 \le T$ satisfies

$$|j_0 - j_1| = q_{\bar{k}(j_0)},$$

then

$$\mathrm{Cl}(j_0) = \bar{d}([j_0\omega], [j_1\omega]).$$

Proof. Let us first show the result when $j_0 = 0$. In this case, $\mathrm{Cl}(j_0) = \bar{d}([j_0\omega], [j_1\omega])$ if and only if

$$[j_1\omega] = \min\{[j\omega], 1 - [j\omega] \mid 1 \le j \le T\}.$$

The latter is equivalent to finding $p \in \mathbb{N}$ such that for all $p_0, q_0 \in \mathbb{N}$ with $q_0 \le T$,

$$|j_1\omega - p| \le |q_0\omega - p_0|.$$

By Theorem 2.38 this is satisfied by the convergents of ω, namely by $j_1 = q_{\bar{k}(0)}$ and $p = p_{\bar{k}(0)}$. This shows the result for $j_0 = 0$. For the general case, we note that since

$$\bar{d}([j_0\omega], [j_1\omega]) = \bar{d}(0, [|j_0 - j_1|\omega]),$$

we can apply the previous argument, now restricting $p_0 \le \max\{j_0, T - j_0\}$ since $|j_0 - j_1| \le \max\{j_0, T - j_0\}$. Similarly, the minimum is achieved when $|j_0 - j_1| = q_{\bar{k}_j(j_0)}$, and the result follows. □

Proposition 3.2. *Let $T \in \mathbb{N}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ with continued-fraction expansion $[a_1, a_2, a_3, \dots]$ and i-th convergent p_i/q_i, and arrange the elements of $S_{\omega,T} = \{[0], [\omega], \dots, [T\omega]\}$ in increasing order as*

$$[n_i\omega] < [n_{i+1}\omega] \quad \textit{for } 0 \le i < T.$$

Let k, r and s be the unique integers for which

$$q_k + q_{k-1} \le T < q_k + q_{k+1} \qquad \textit{and} \qquad T = rq_k + q_{k-1} + s \quad \textit{for } 1 \le r \le a_{k+1} \textit{ and } 0 \le s \le q_k - 1,$$

and let $D_i = q_i\omega - p_i$ for each i. If

$$\delta_A = |D_k|, \quad \delta_B = |D_{k+1}| + (a_{k+1} - r)|D_k|, \quad \delta_C = \delta_A + \delta_B,$$

and $\oplus$ denotes addition modulo $T + 1$, then for $0 \le i \le T$,

$$\{\bar{d}([n_i\omega], [n_{i\oplus 1}\omega])\} \subseteq \{\delta_A, \delta_B, \delta_C\}.$$

Furthermore, denoting the multiplicity of gaps by $N_I = \#\{i \mid \bar{d}([n_i\omega], [n_{i\oplus 1}\omega]) = \delta_I\}$, we have

$$N_A = T + 1 - q_k, \quad N_B = s + 1, \quad N_C = q_k - s - 1.$$

Proof. We first establish the existence and uniqueness of k, r and s. For all $i \ge 0$,

$$q_i + q_{i-1} < q_i + q_{i+1}.$$

Thus $T \in [q_k + q_{k-1}, q_k + q_{k+1})$ for some k; that this k is unique follows by the monotonicity of $\{q_i + q_{i-1}\}_i$. We also note that by assumption $T - q_{k-1} \ge q_k$; thus by the division algorithm there exist unique r and s such that

$$T - q_{k-1} = rq_k + s,$$

where $1 \le r$ and $0 \le s \le q_k$. On the other hand, since $T - q_{k-1} < q_k + q_{k+1} - q_{k-1} = (a_{k+1} + 1)q_k$, by Proposition 2.33, we conclude $r \le a_{k+1}$. This shows the existence and uniqueness of k, r and s.

For convenience of presentation in what follows, we define the notion of the "right" and "left" of a point. For $x, y \in S_{\omega,T}$, we say that y is to the right of x if $\bar{d}(x, y)$ is achieved by traversing from x clockwise to y in this representation. Similarly, achieving the distance while traversing counterclockwise is used to define left. Since translation is an isometry under $\bar{d}$ we note $[i_1\omega]$ is to the right or left of $[i_2\omega]$ if and only if for all $j \ge 0$, $[(i_1 + j)\omega]$ is to the right or left, respectively, of $[(i_2 + j)\omega]$. We make use of this fact repeatedly.

Let us now show that

$$\{\bar{d}(0, [n_1\omega]), \bar{d}(0, [n_T\omega])\} = \{\delta_A, \delta_B\}.$$

This equality will apply to general points in $S_{\omega,T}$ by the isometry property of the translation map. In particular, it will allow us to compute N_A and N_B, and find the explicit pairs that achieve the corresponding gap. We assume without loss of generality that

$$[q_k\omega] \le \tfrac{1}{2},$$

i.e., $[q_k\omega]$ is to the right of 0. By Proposition 2.36, $[q_{k-1}\omega]$ and $[q_{k+1}\omega]$ are to the left of 0, i.e., $[q_k\omega] < [q_{k-1}\omega], [q_{k+1}\omega]$. Thus by Lemma 3.1 $n_1 = q_k$, and thus

$$\bar{d}(0, [n_1\omega]) = [q_k\omega] = \delta_A.$$

In fact, Lemma 3.1 also shows

$$[q_k\omega] < [q_{k-1}\omega] < [q_{k+1}\omega].$$

Moreover, since

$$\bar{d}([q_{k-1}\omega], [(q_{k-1}+q_k)\omega]) = \bar{d}([q_k\omega], 0) = [q_k\omega]$$

and, by Theorem 2.38,

$$\bar{d}([q_k\omega], 0) < \bar{d}([q_{k-1}\omega], 0),$$

we conclude

$$[q_{k-1}\omega] < [(q_{k-1}+q_k)\omega].$$

Repeating this argument and noting $q_{k+1} = a_{k+1}q_k + q_{k-1}$ we can say that for $0 \le i \le a_{k+1}$,

$$[q_{k-1}\omega] \le [(q_{k-1}+iq_k)\omega] \le [(q_{k-1}+(i+1)q_k)\omega] \le [q_{k+1}\omega].$$

Thus

$$\begin{aligned}\bar{d}([(q_{k-1}+iq_k)\omega], 0) &= \bar{d}([q_{k+1}\omega], 0) + \sum_{j=0}^{a_{k+1}-i-1} \bar{d}([(q_{k-1}+jq_k)\omega], [(q_{k-1}+(j+1)q_k)\omega]) \\ &= |D_{k+1}| + (a_{k+1}-i)|D_k|.\end{aligned}$$

By Lemma 3.1 we can conclude $n_T = q_{k-1} + rq_k$ and the above computation shows

$$\bar{d}([(q_{k-1}+rq_k)\omega], 0) = |D_{k+1}| + (a_{k+1}-r)|D_k| = \delta_B.$$

Thus we have shown

$$\{\bar{d}(0, [n_1\omega]), \bar{d}(0, [n_T\omega])\} = \{\delta_A, \delta_B\}.$$

Now consider $n_j \in [0, (r-1)q_k + q_{k-1} + s]$. Since $n_j + q_k \le T$ and

$$\bar{d}([n_j\omega], [(n_j+q_k)\omega]) = \bar{d}([q_k\omega], 0) = [q_k\omega] = \delta_A,$$

we conclude, by the same reasoning as with $n_j = 0$, that $[(n_j+q_k)\omega]$ is the closest point to the right of $[n_j\omega]$, i.e., $n_{j\oplus 1} = n_j + q_k$. This shows there are

$$(r-1)q_k + q_{k-1} + s + 1 = (rq_k + q_{k-1} + s) + 1 - q_k = T + 1 - q_k$$

different gaps of length δ_A, i.e., $N_A = T + 1 - q_k$.

Similarly, for $n_j \in [rq_k + q_{k-1}, T]$ we note

$$n_j - (rq_k + q_{k-1}) \in [0, s],$$

and thus

$$\bar{d}([n_j\omega], [(n_j - (rq_k + q_{k-1}))\omega]) = \bar{d}([(rq_k + q_{k-1})\omega], 0) = |D_{k+1}| + (a_{k+1}-r)|D_k| = \delta_B.$$

Carrying over the arguments for the case $n_j = rq_k + q_{k-1}$, we conclude $[(n_j - (rq_k + q_{k-1}))\omega]$ is the closest point to the left of $[n_j\omega]$, i.e., $n_{j-1} = n_j - (rq_k + q_{k-1})$. Thus there are $s+1$ different gaps of length δ_B, i.e., $N_B = s+1$.

What remains to consider is the case $n_j \in ((r-1)q_k + q_{k-1} + s, rq_k + q_{k-1})$. If $r > 1$, we note

$$\bar{d}([n_j\omega], [(n_j - q_k)\omega]) = \bar{d}([q_k\omega], 0) = [q_k\omega] = \delta_A.$$

Thus $n_{j-1} = n_j - q_k$. Since $n_j - q_k < (r-1)q_k + q_{k-1} + s$, we have already accounted for this gap. On the other hand,

$$\bar{d}([n_j\omega], [n_{j\oplus 1}\omega]) \neq \delta_A, \delta_B$$

since $n_j + q_k, n_j + (rq_k + q_{k-1}) > T$. As a consequence of the inequality shown before,

$$[q_{k-1}\omega] \leq [(q_{k-1} + iq_k)\omega] \leq [(q_{k-1} + (i+1)q_k)\omega] \leq [q_{k+1}\omega],$$

we conclude the next smallest gap length is

$$|D_{k+1}| + (a_{k+1} - (r-1))|D_k| = |D_{k+1}| + (a_{k+1} - r + 1)|D_k| = \delta_A + \delta_B = \delta_C.$$

This gap is achieved when $n_{j+1} = n_j - ((r-1)q_k + q_{k-1})$. Thus there are $q_k - s - 1$ different gaps of length δ_C, i.e., $N_C = q_k - s - 1$.

We have shown there are distinct gaps, distances between adjacent points, of lengths δ_A, δ_B and δ_C. Furthermore, since

$$N_A + N_B + N_C = T + 1,$$

we conclude all gaps have been accounted for, i.e.,

$$\{\bar{d}([n_i\omega], [n_{i\oplus 1}\omega])\} \subseteq \{\delta_A, \delta_B, \delta_C\}. \qquad \square$$

It is clear that the three-gap theorem follows from Proposition 3.2. Furthermore, Proposition 3.2 details the relation between the length of the gaps and the k-th convergents of ω. The next theorem translates what this means in terms of the 0-th dimensional persistence diagram of $S_{\omega,T}$. We assume $r < a_{k+1}$ and $s+1 < p_k$ to streamline the exposition; the other parameter regimes lead to analogous conclusions:

- Case: $r \geq a_{k+1}$. In this regime one shows that $\delta_A > \delta_B$. In Theorem 3.3 below, one then swaps the labels δ_A and δ_B so that the combinatorial pattern of gap lengths remains identical.
- Case: $s+1 \geq p_k$. Here $N_C = 0$, meaning no third gap appears. Equivalently, the final merge of adjacent points occurs at $\epsilon = \max\{\delta_A, \delta_B\}$, which then plays the role of δ_C in the theorem's description.

In each scenario, relabeling or omitting the appropriate gap lengths shows that the proof of Theorem 3.3 carries over without change.

Theorem 3.3. *Let $T \in \mathbb{N}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ with continued-fraction expansion $[a_1, a_2, a_3, \dots]$ and i-th convergent p_i/q_i, and let k, r and s be the unique integers for which*

$$q_k + q_{k-1} \le T < q_k + q_{k+1} \qquad \text{and} \qquad T = rq_k + q_{k-1} + s \quad \text{for } 1 \le r \le a_{k+1} \text{ and } 0 \le s \le q_k - 1.$$

Let

$$D_i = q_i\omega - p_i,$$

and suppose

$$r < a_{k+1} \quad \text{and} \quad s + 1 < p_k.$$

Then

$$H_0(R_\epsilon(S_{\omega,T}, \bar{d}); \mathbb{F}) = \begin{cases} \mathbb{F}^{T+1} & \text{if } 0 \le \epsilon < |D_k|, \\ \mathbb{F}^{q_k} & \text{if } |D_k| \le \epsilon < |D_{k+1}| + (a_{k+1} - r)|D_k|, \\ \mathbb{F}^{q_k - s - 1} & \text{if } |D_{k+1}| + (a_{k+1} - r)|D_k| \le \epsilon < |D_{k+1}| + (a_{k+1} - r + 1)|D_k|, \\ \mathbb{F} & \text{if } |D_{k+1}| + (a_{k+1} - r + 1)|D_k| \le \epsilon. \end{cases}$$

Proof. Let δ_I and N_I, where $I \in \{A, B, C\}$, be as defined in Proposition 3.2. Since $r < a_{k+1}$ we note $\bar{k}(T) = k$, and thus by Lemma 3.1,

$$\delta_A < \delta_B.$$

Furthermore, since $s + 1 < q_k$, there are three possible lengths for the gaps, i.e., $N_C > 0$.

At $\epsilon = 0$,

$$H_0(R_0(S_{\omega,T}, \bar{d}); \mathbb{F}) = \mathbb{F}^{T+1}$$

by definition. We can interpret this as connected components that are separated by $N_A + N_B + N_C = T + 1$ gaps. Since the Rips complex connects two points when ϵ is greater or equal to their distance, the first time points will be connected is at $\epsilon = \delta_A$. Now, the newly formed connected components will be separated from each other only at the gaps of length δ_B and δ_C, i.e.,

$$H_0(R_{\delta_A}(S_{\omega,T}, \bar{d}); \mathbb{F}) = \mathbb{F}^{N_B + N_C} = \mathbb{F}^{q_k}.$$

Similarly, the next time we connect points is at $\epsilon = \delta_B$; now the remaining connected components will be separated by only N_C gaps, and thus

$$H_0(R_{\delta_B}(S_{\omega,T}, \bar{d}); \mathbb{F}) = \mathbb{F}^{N_C} = \mathbb{F}^{q_k - s - 1}.$$

The last time we connected points was at δ_C; since this means all adjacent points are connected we conclude there is only one connected component, i.e.,

$$H_0(R_{\delta_C}(S_{\omega,T}, \bar{d}); \mathbb{F}) = \mathbb{F}.$$

□

Corollary 3.4. *Let $T \in \mathbb{N}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ with continued-fraction expansion $[a_1, a_2, a_3, \dots]$ and i-th convergent p_i/q_i, and let k, r and s be the unique integers for which*

$$q_k + q_{k-1} \le T < q_k + q_{k+1} \qquad \text{and} \qquad T = rq_k + q_{k-1} + s \quad \text{for } 1 \le r \le a_{k+1} \text{ and } 0 \le s \le q_k - 1.$$

Let

$$D_i = q_i\omega - p_i,$$

and suppose

$$r < a_{k+1} \quad \textit{and} \quad s+1 < p_k.$$

If

$$\delta_A = |D_k|, \quad \delta_B = |D_{k+1}| + (a_{k+1} - r)|D_k|, \quad \delta_C = \delta_A + \delta_B,$$

then

$$\mathrm{dgm}_0^R(S_{\omega,T}, \bar{d}) = \{(0, \delta_A)^{T+1-q_k}, (0, \delta_B)^{s+1}, (0, \delta_C)^{q_k-s-2}, (0, \infty)\},$$

where the upper indices indicate the multiplicity of each point in the diagram.

Proof. For $\epsilon \leq \epsilon'$ let

$$T_{\epsilon,\epsilon'} : H_0(R_\epsilon(S_{\omega,T}, \bar{d}); \mathbb{F}) \to H_0(R_{\epsilon'}(S_{\omega,T}, \bar{d}); \mathbb{F})$$

denote the linear transformation induced by the inclusion of Rips complexes, and let $\beta_0^{\epsilon,\epsilon'}$ be its rank. Since $S_{\omega,T}$ is the vertex set of $R_\epsilon(S_{\omega,T}, \bar{d})$ for all $\epsilon \geq 0$, and a basis for $H_0(K;\mathbb{F})$ is given by choosing a vertex in each path-connected component of the simplicial complex K, then $T_{\epsilon,\epsilon'}$ is surjective whenever $0 \leq \epsilon \leq \epsilon'$, and the zero map for $\epsilon < 0$. Hence $\beta_0^{\epsilon,\epsilon'}$ is equal to 0 if $\epsilon < 0$, and equal to $\dim_{\mathbb{F}}(H_0(R_{\epsilon'}(S_{\omega,T}, \bar{d}); \mathbb{F}))$ if $0 \leq \epsilon \leq \epsilon'$. The result follows from applying Corollary 3.4 to compute $\dim_{\mathbb{F}}(H_0(R_{\epsilon'}(S_{\omega,T}, \bar{d}); \mathbb{F}))$, and combining the result with the formula from (1); see Remark 2. □

Example 3.5. Consider the parameter value $\sqrt{5}$ shown in Figure 9, top left. To compute the lengths of the gaps in $S_{\sqrt{5},16}$, we follow Proposition 3.2. We note that

$$\sqrt{5} = [2, 4, 4, 4, 4, 4, \dots],$$

and thus using Proposition 2.33 we obtain $q_1 = 1$, $q_2 = 4$, $q_3 = 17$ and $q_4 = 72$. Since

$$q_1 + q_2 \leq 16 < q_2 + q_3$$

we conclude $k = 2$. One can readily check that

$$16 = 3q_2 + q_1 + 3$$

is the desired decomposition, i.e., $r = 3$ and $s = 3$. Hence we have $N_A = 16 + 1 - 4 = 13$ gaps of length $\delta_A = |4\sqrt{5} - 9| \approx 0.05573$, $N_B = 3 + 1 = 4$ gaps of length $\delta_B = |17\sqrt{5} - 38| + (4-3)|4\sqrt{5} - 9| \approx 0.06888$, and $N_C = 4 - 3 - 1 = 0$ gaps of length $\delta_C = |17\sqrt{5} - 38| + (4 - 3 + 1)|4\sqrt{5} - 9| \approx 0.12461$. Thus, by using Corollary 3.4 in the case of two gaps, we have

$$\mathrm{dgm}_0^R(S_{\sqrt{5},16}, \bar{d}) = \{(0, 0.05573)^{13}, (0, 0.06888)^3, (0, \infty)\}.$$

Example 3.6. Similarly, we consider the set $S_{\pi-1,16}$ shown in Figure 9, bottom left. In this case

$$\pi - 1 = [2, 7, 15, 292, 1, 1, \dots],$$

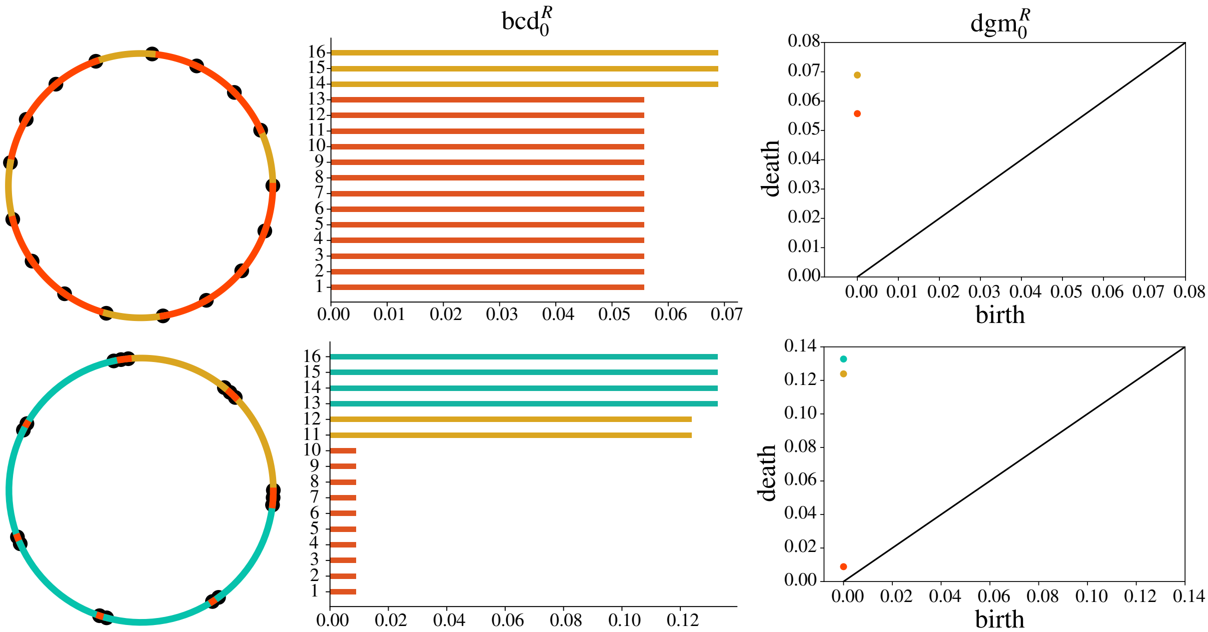


Figure 9. We illustrate above how the 0-th dimensional persistent homology can be computed using the three-gap theorem. At $\epsilon = 0$, all points in $(S_{\omega,T}, \bar{d})$ count as a basis in $H_0(R_\epsilon(S_{\omega,T}, \bar{d}); \mathbb{F})$ (birth time). At $\epsilon = \delta_A$, points get connected for the first time. The three possible gaps are the only instances when components are connected (death time). Top row: the set $S_{\sqrt{5},16}$ is illustrated with the two gaps it creates. These can be traced in the barcodes and persistent diagram as shown. Bottom row: the set $S_{\pi-1,16}$ creates three gaps.

$p_1 = 1$, $p_2 = 7$, $p_3 = 106$ and $p_4 = 113$. Thus $k = 2$ since

$$p_1 + p_2 \leq 16 < p_2 + p_3.$$

Furthermore, the desired decomposition is given by

$$16 = 2p_2 + p_1 + 1,$$

i.e., $r = 2$ and $s = 1$. Thus we have $N_A = 16 + 1 - 7 = 10$ gaps of length $\delta_A = |7(\pi - 1) - 15| \approx 0.00885$, $N_B = 2$ gaps of length $\delta_B = |106(\pi - 1) - 227| + (15 - 2)|7(\pi - 1) - 15| \approx 0.12389$, and $N_C = 7 - 1 - 1 = 5$ gaps of length $\delta_C = |106(\pi - 1) - 227| + (15 - 2 + 1)|7(\pi - 1) - 15| \approx 0.13274$. Moreover, by Corollary 3.4

$$\mathrm{dgm}_0^R(S_{\pi-1,16}, \bar{d}) = \{(0, 0.00885)^{10}, (0, 0.12389)^2, (0, 0.13274)^4, (0, \infty)\}.$$

We also compute $\mathrm{dgm}_1^R(S_{\omega,T}, \bar{d})$. In Theorem 3.7 we assume

$$s + 1 < q_k \quad \text{and} \quad \delta_C < \tfrac{1}{3}$$

to simplify the presentation. If instead $s + 1 \geq q_k$, then $N_C = 0$ and only the two gap lengths δ_A and δ_B occur — so one may simply set $\max\{\delta_A, \delta_B\}$ in place of δ_C. Likewise, if the largest gap δ_M satisfies

$\delta_M \geq \frac{1}{3}$, then by Lemma A.5 we have $\lambda \leq \delta_M$, forcing every potential 1-cycle to be 0 in H_1, i.e., a trivial case. We denote addition modulo $T+1$ by $\oplus$.

Theorem 3.7. *Let $T \in \mathbb{N}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ with continued-fraction expansion $[a_1, a_2, a_3, \dots]$ and i-th convergent p_i/q_i, and let k, r and s be the unique integers for which*

$$q_k + q_{k-1} \leq T < q_k + q_{k+1} \qquad \text{and} \qquad T = rq_k + q_{k-1} + s \quad \text{for } 1 \leq r \leq a_{k+1} \text{ and } 0 \leq s \leq q_k - 1.$$

Let

$$D_i = q_i\omega - p_i,$$

and assume

$$s + 1 < q_k \quad \text{and} \quad |D_{k+1}| + (a_{k+1} - r + 1)|D_k| < \tfrac{1}{3}.$$

Let Γ be the set containing the pairs (x, y) for $x, y \in S_{\omega,T}$ and $x < y$, for which there exists a $z \in ([0, x) \cup (y, 1)) \cap S_{\omega,T}$ such that $1 - \bar{d}(x, y) \leq 2\max\{\bar{d}(y, z), \bar{d}(x, z)\} \leq 2\bar{d}(x, y)$. If

$$\lambda = \min\{\bar{d}(x, y) \mid (x, y) \in \Gamma\},$$

then

$$H_1(R_\epsilon(S_{\omega,T}, \bar{d}); \mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } |D_{k+1}| + (a_{k+1} - r + 1)|D_k| \leq \epsilon < \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\{x_i\}_{i=0}^T$ be the points of $S_{\omega,T}$ in ascending order, and consider the 1-chain

$$\sigma = \sum_{i=0}^{T} \{x_i, x_{i\oplus 1}\},$$

where $i \oplus 1$ denotes $i + 1 \mod (T+1)$. By Proposition 3.2, the maximum distance between consecutive points is

$$\delta_C = |D_{k+1}| + (a_{k+1} - r + 1)|D_k|.$$

Hence

$$\sigma \in C_1(R_\epsilon(S_{\omega,T}, \bar{d})) \iff \epsilon \geq \delta_C.$$

One checks immediately that $\partial\sigma = 0$. Lemma A.4 then shows any nonboundary 1-cycle in

$$C_1(R_{\delta_C}(S_{\omega,T}, \bar{d}))$$

is homologous to σ. Consequently $[\sigma]$ generates $H_1(R_{\delta_C}(S_{\omega,T}, \bar{d}); \mathbb{F})$.

To show that σ is not a boundary, it suffices to prove there is no 2-simplex $\tau \in C_2(R_{\delta_C}(S_{\omega,T}, \bar{d}))$ with $\partial\tau = \sigma$. We argue by contradiction. Suppose there exists $\tau \in C_2(R_{\delta_C}(S_{\omega,T}, \bar{d}))$ such that $\partial\tau = \sigma$. Let us first show we may assume, without loss of generality, that

$$\bar{d}(0, x_1) = \delta_C.$$

To see this, we construct an explicit isometry on $(S_{\omega,T}, \bar{d})$ which reindexes the points accordingly. By hypothesis, there is an index i_0 with

$$\bar{d}(x_{i_0}, x_{i_0 \oplus 1}) = \delta_C.$$

Define

$$\phi(x_i) = \begin{cases} x_i - x_{i_0} & \text{if } i \geq i_0, \\ 1 - (x_{i_0} - x_i) & \text{if } i < i_0. \end{cases}$$

A direct check shows

$$\bar{d}(\phi(x_i), \phi(x_j)) = \bar{d}(x_i, x_j) \quad \text{for all } \ i, j,$$

so ϕ is an isometry. By Proposition 2.22, it induces identical persistence diagrams. Finally, set

$$y_j = \phi(x_{i_0 \oplus j}) \quad \text{for } \ j = 0, 1, \ldots, T,$$

so that

$$0 = y_0 < y_1 < \cdots < y_T < 1 \quad \text{and} \quad \bar{d}(y_0, y_1) = \delta_C.$$

This relabeled sequence $\{y_j\}$ therefore has the desired property; we thereafter write $x_j := y_j$ for all j.

Now, since $\partial\tau = \sigma$, τ must contain a 2-simplex of the form $\{0, x_1, x_{i_0}\}$ for some $i_0 > 1$. If $x_{i_0} \leq \frac{1}{2}$, then

$$\bar{d}(0, x_{i_0}) = \bar{d}(0, x_1) + \bar{d}(x_1, x_{i_0}) > \delta_C.$$

If instead $x_{i_0} > \frac{1}{2}$ and $\bar{d}(x_1, x_{i_0}) \leq \delta_C$, then

$$\bar{d}(0, x_{i_0}) = \min\{\bar{d}(0, x_1) + \bar{d}(x_1, x_{i_0}), 1 - [\bar{d}(0, x_1) + \bar{d}(x_1, x_{i_0})]\} > \delta_C,$$

since $\delta_C < \frac{1}{3}$. In either case $\{0, x_1, x_{i_0}\} \notin \tau$, contradicting $\partial\tau = \sigma$. Therefore no such τ exists, and σ is not a boundary.

To conclude the proof, it suffices to show that for any ϵ with $\delta_C < \epsilon \leq \lambda$, a 2-simplex

$$\tau \in C_2(R_\epsilon(S_{\omega,T}, \bar{d}))$$

satisfying $\partial\tau = \sigma$ appears precisely at $\epsilon = \lambda$. In particular, no such τ exists when $\epsilon < \lambda$, but at $\epsilon = \lambda$ one does. Hence for all $\epsilon \geq \lambda$, the class $[\sigma]$ is trivial in $H_1(R_\epsilon(S_{\omega,T}, \bar{d}); \mathbb{F})$.

We now demonstrate by explicit construction that at $\epsilon = \lambda$ there is a 2-chain $\tau \in C_2(R_\lambda(S_{\omega,T}, \bar{d}))$ with $\partial\tau = \sigma$. Choose $(x_{n_1}, x_{n_2}) \in \Gamma$ so that $\lambda = \bar{d}(x_{n_1}, x_{n_2})$, and let x_{n_3} be the third point paired with (x_{n_1}, x_{n_2}). By assumption

$$\max\{\bar{d}(x_{n_2}, x_{n_3}), \bar{d}(x_{n_1}, x_{n_3})\} \leq \bar{d}(x_{n_2}, x_{n_3}) = \lambda.$$

Hence $\{x_{n_1}, x_{n_2}, x_{n_3}\}$ lies in $R_\lambda(S_{\omega,T}, \bar{d})$.

Next define the 2-chain

$$\tau_1 = \sum_{i=n_1}^{n_2-1} \{x_i, x_{i+1}, x_{n_2}\}.$$

Because for $n_1 \leq i < n_2$,

$$\bar{d}(x_i, x_{i+1}) < \bar{d}(x_{n_1}, x_{n_2}),$$

$\tau_1 \in C_2(R_\lambda(S_{\omega,T}, \bar{d}))$.

To define the analogous chains τ_2 and τ_3, we use the fact that $S_{\omega,T}$ can be identified with the circle. Let $\{x_i^2\}_{i=1}^{A_2}$ be the sequence of points obtained by starting at x_{n_2} and moving clockwise until x_{n_3}, inclusive — so $x_1^2 = x_{n_2}$ and $x_{A_2}^2 = x_{n_3}$. We then set

$$\tau_2 = \sum_{i=1}^{A_2-1} \{x_i^2, x_{i+1}^2, x_{n_3}\},$$

interpreting $\tau_2 = 0$ if $A_2 = 2$.

Similarly, let $\{x_i^3\}_{i=1}^{A_3}$ be the sequence of points starting at x_{n_3} and moving clockwise until x_{n_1}, with $x_1^3 = x_{n_3}$ and $x_{A_3}^3 = x_{n_1}$. We define

$$\tau_3 = \sum_{i=1}^{A_3-1} \{x_i^3, x_{i+1}^3, x_{n_1}\},$$

again taking $\tau_3 = 0$ if $A_3 = 2$. In all cases, each τ_j lies in $C_2(R_\lambda(S_{\omega,T}, \bar{d}))$.

Finally set

$$\tau = \tau_1 + \tau_2 + \tau_3 + \{x_{n_1}, x_{n_2}, x_{n_3}\}.$$

By construction $\tau \in C_2(R_\lambda(S_{\omega,T}, \bar{d}))$, and one checks directly that $\partial\tau = \sigma$. This completes the existence argument at $\epsilon = \lambda$.

To complete the proof, let $\epsilon \in (\delta_C, \lambda)$. If one assumes the existence of a 2-simplex $\tau_0 \in C_2(R_\epsilon(S_{\omega,T}, \bar{d}))$ with $\partial\tau_0 = \sigma$, then by considering

$$\max\{\bar{d}(x, y), \bar{d}(y, z), \bar{d}(z, x) \mid \{x, y, z\} \in \tau_0\}$$

one obtains a contradiction to the minimality of λ; see Lemma A.5. ☐

Corollary 3.8. *Let $T \in \mathbb{N}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ with continued-fraction expansion $[a_1, a_2, a_3, \dots]$ and i-th convergent p_i/q_i, and let k, r and s be the unique integers for which*

$$q_k + q_{k-1} \le T < q_k + q_{k+1} \qquad \text{and} \qquad T = rq_k + q_{k-1} + s \quad \text{for } 1 \le r \le a_{k+1} \text{ and } 0 \le s \le q_k - 1.$$

Let

$$D_i = q_i\omega - p_i,$$

and assume

$$s + 1 < q_k \quad \text{and} \quad |D_{k+1}| + (a_{k+1} - r + 1)|D_k| < \tfrac{1}{3}.$$

Let Γ be the set containing the pairs (x, y) for $x, y \in S_{\omega,T}$ and $x < y$, for which there exists a $z \in ([0, x) \cup (y, 1)) \cap S_{\omega,T}$ such that $1 - \bar{d}(x, y) \le 2\max\{\bar{d}(y, z), \bar{d}(x, z)\} \le 2\bar{d}(x, y)$. If

$$\lambda = \min\{\bar{d}(x, y) \mid (x, y) \in \Gamma\},$$

then

$$\mathrm{dgm}_1^R(S_{\omega,T}, \bar{d}) = \{(|D_{k+1}| + (a_{k+1} - r + 1)|D_k|, \lambda)\}.$$

Proof. The 1-cycle σ constructed in the proof of Theorem 3.7 provides a persistent generator for

$$H_1(R_\epsilon(S_{\omega,T}, \bar{d}); \mathbb{F})$$

whenever $|D_{k+1}| + (a_{k+1} - r + 1)|D_k| = \delta_C \leq \epsilon < \lambda$. This shows that the persistent Betti numbers $\beta_1^{\epsilon,\epsilon'}$ are zero for $\epsilon < \delta_C$ or $\lambda \leq \epsilon'$, and equal to one for $\delta_C \leq \epsilon \leq \epsilon' < \lambda$. The result follows from Remark 2. □

4. Approximation method

We now detail our approximation method for the sliding-window embedding of quasiperiodic functions. To handle general quasiperiodic functions, it is sufficient to consider those that are sums of exponentials, as discussed in Section 2.3. When f takes this form, its sliding-window embedding admits the matrix representation

$$\mathrm{SW}_{d,\tau} f(t) = A v_t.$$

Recall that

$$\mathbb{SW}_{d,\tau} f(T) = \{\mathrm{SW}_{d,\tau} f(t) \mid t = 0, \ldots, T\}$$

denotes the corresponding sliding-window point cloud, and let ϕ_T be the trajectory $\{v_t \mid t = 0, \ldots, T\}$; we assume $T' \leq T$, and let G_T be as defined in (2). Throughout this section let $l_0, M, N \in \mathbb{N}$, we use j and l as dummy variables, and we let $i = \sqrt{-1} \in \mathbb{C}$. Our approximation method relates the following metric spaces, and their corresponding persistence diagrams:

$$(\mathbb{SW}_{d,\tau} f(T), d_2) \sim (\phi_T, d_\infty) \sim (G_{T'}, d_\infty).$$

The first relation is obtained using an eigenvalue argument. Let $\sigma_{\min}$ and $\sigma_{\max}$ be the smallest and largest singular values of A, respectively.

Lemma 4.1. *Let*

$$f(t) = \sum_{j=1}^{M} c_j e^{2\pi i \omega_j t} \quad \text{for } c_j \in \mathbb{C} \setminus \{0\} \text{ and } \omega_j \in \mathbb{R}.$$

Define $k = \max\{\sigma_{\min}^{-1}, \sigma_{\max}\sqrt{M}\}$. *Suppose* $(a_1, b_1) \in \mathrm{dgm}_{l_0}^R(\phi_T, d_\infty)$. *If*

$$\frac{b_1}{a_1} > k^2,$$

then there exists a unique $(a_0, b_0) \in \mathrm{dgm}_{l_0}^R(\mathbb{SW}_{d,\tau} f(T), d_2)$ *such that*

$$\frac{1}{k} < \frac{\max\{b_0, b_1\}}{\min\{b_0, b_1\}}, \frac{\max\{a_0, a_1\}}{\min\{a_0, a_1\}} < k.$$

Proof. Note that

$$\sigma_{\min} d_2(v_{t_1}, v_{t_2}) \leq d_2(Av_{t_1}, Av_{t_2}) \leq \sigma_{\max} d_2(v_{t_1}, v_{t_2}) \quad \text{and} \quad d_\infty(v_{t_1}, v_{t_2}) \leq d_2(v_{t_1}, v_{t_2}) \leq \sqrt{M} d_\infty(v_{t_1}, v_{t_2}).$$

We conclude that

$$d_\infty(v_{t_1}, v_{t_2}) \le d_2(v_{t_1}, v_{t_2}) \le \frac{1}{\sigma_{\min}} d_2(Av_{t_1}, Av_{t_2}) \le \frac{\sigma_{\max}}{\sigma_{\min}} d_2(Av_{t_1}, Av_{t_2}) \le \sqrt{M} d_\infty(v_{t_1}, v_{t_2}).$$

Since $\mathrm{SW}_{d,\tau} f(t) = Av_t$ and A is full-rank (provided d and τ are chosen according to [Gakhar and Perea 2024]), then

$$R_\epsilon(\mathbb{SW}_{d,\tau} f(T), d_2) \hookrightarrow R_{\epsilon/\sigma_{\min}}(\phi_T, d_\infty) \hookrightarrow R_{\epsilon(\sigma_{\max}/\sigma_{\min})}(\mathbb{SW}_{d,\tau} f(T), d_2)$$

under the induced simplicial maps obtained from $\mu : \mathbb{SW}_{d,\tau} f(T) \to \phi_T$, $\mu(Av_t) = v_t$ and $\nu : \phi_T \to \mathbb{SW}_{d,\tau} f(T)$, $\nu(v_t) = Av_t$. We have the diagram

$$\begin{array}{ccc} R_\epsilon(\mathbb{SW}_{d,\tau} f(T), d_2) & \xrightarrow{\iota} & R_{k^2\epsilon}(\mathbb{SW}_{d,\tau} f(T), d_2) \\ & {}_{\mu}\searrow \quad \nearrow^{\nu} & \\ & R_{k\epsilon}(\phi_T, d_\infty) & \end{array}$$

where ι is the inclusion $R_\epsilon(\mathbb{SW}_{d,\tau} f(T), d_2) \hookrightarrow R_{k^2\epsilon}(\mathbb{SW}_{d,\tau} f(T), d_2)$. Furthermore ι is contiguous to $\nu \circ \mu$; that is, for $\sigma \in R_\epsilon(\mathbb{SW}_{d,\tau} f(T), d_2)$ we have $\iota(\sigma) \cup \nu \circ \mu(\sigma) \in R_{k^2\epsilon}(\mathbb{SW}_{d,\tau} f(T), d_2)$. Hence we obtain a commutative diagram at the level of homology, which we reparametrize in logarithmic scale as

$$\begin{array}{ccc} H_n(R_{\ln(\epsilon)}(\mathbb{SW}_{d,\tau} f(T), d_2); \mathbb{F}) & \xrightarrow{\iota_*} & H_n(R_{2\ln(k)+\ln(\epsilon)}(\mathbb{SW}_{d,\tau} f(T), d_2); \mathbb{F}) \\ & {}_{\mu_*}\searrow \quad \nearrow^{\nu_*} & \\ & H_n(R_{\ln(k)+\ln(\epsilon)}(\phi_T, d_\infty); \mathbb{F}) & \end{array}$$

This gives us a $\ln(k)$ interleaving which implies by Theorem 2.9 that, if for $(a_1, b_1) \in \mathrm{dgm}^R_{l_0}(\phi_T, d_\infty)$,

$$\ln(b_1) - \ln(a_1) > 2\ln(k),$$

i.e., $b_1/a_1 > k^2$, there exists a unique $(a_0, b_0) \in \mathrm{dgm}^R_{l_0}(\mathbb{SW}_{d,\tau} f(T), d_2)$ such that

$$|\ln(b_1) - \ln(b_0)| < \ln(k) \quad \text{and} \quad |\ln(a_1) - \ln(a_0)| < \ln(k).$$

This implies

$$-\ln(k) < \ln\left(\frac{\max\{b_0, b_1\}}{\min\{b_0, b_1\}}\right), \ln\left(\frac{\max\{a_0, a_1\}}{\min\{a_0, a_1\}}\right) < \ln(k),$$

from which the result follows. □

To relate the two metric spaces

$$(\phi_T, d_\infty) \quad \text{and} \quad (G_{T'}, d_\infty),$$

we apply Theorem 2.19 together with the definition of the bottleneck distance.

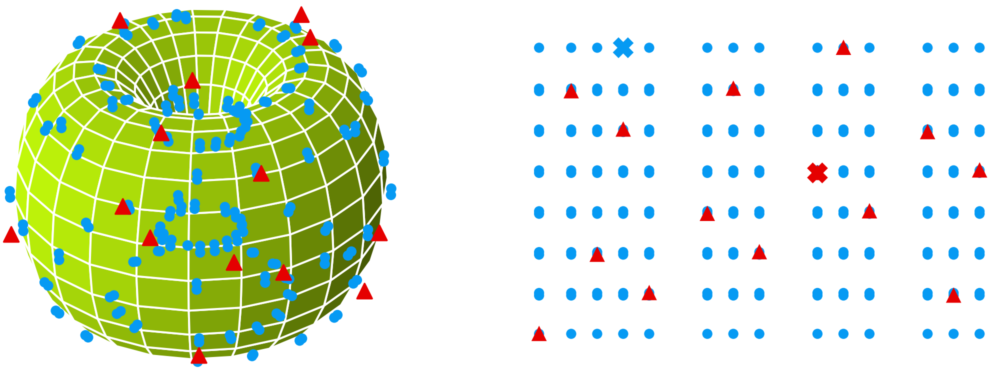


Figure 10. Comparison of ϕ_{13} (red triangles) and G_{13} (blue dots) for a two-frequency torus with incommensurate ω_1 and ω_2. Left: points plotted on the torus $T^2 \subset \mathbb{C}^2$. Right: same points in the rectangular fundamental domain $[0, 1)^2$. The pair marked $\times$ realizes the Hausdorff distance between ϕ_{13} and G_{13}.

Lemma 4.2. *Let* $\lambda = d_{GH}(\phi_T, G_{T'})$ *and* $(a_2, b_2) \in \mathrm{dgm}^R_{l_0}(G_{T'}, d_\infty)$. *If*

$$b_2 - a_2 > 4\lambda,$$

then there exists a unique $(a_1, b_1) \in \mathrm{dgm}^R_{l_0}(\phi_T, d_\infty)$ *such that*

$$|b_1 - b_2| < 2\lambda \quad \textit{and} \quad |a_1 - a_2| < 2\lambda.$$

Proof. Let $B = d_B(\mathrm{dgm}^R_{l_0}(G_{T'}, d_\infty), \mathrm{dgm}^R_{l_0}(\phi_T, d_\infty))$. By Theorem 2.19, $B < 2\lambda$. Thus

$$b_2 - a_2 > 4\lambda > 2B,$$

and hence by the definition of the bottleneck distance, Definition 2.8, there exists a unique $(a_1, b_1) \in \mathrm{dgm}^R_{l_0}(\phi_T, d_\infty)$ such that

$$|b_1 - b_2| < B < 2\lambda \quad \text{and} \quad |a_1 - a_2| < B < 2\lambda, \qquad \square$$

Using the same notation as before, we combine the results to obtain a direct relation between $\mathrm{dgm}^R_{l_0}(\mathbb{SW}_{d,\tau} f(T), d_2)$ and $\mathrm{dgm}^R_{l_0}(G_{T'}, d_\infty)$:

Theorem 4.3. *If* $(a_2, b_2) \in \mathrm{dgm}^R_{l_0}(G_{T'}, d_\infty)$ *satisfies*

$$\frac{b_2 - 2\lambda}{a_2 + 2\lambda} > \max\{k^2, 1\},$$

then there exists a unique $(a_0, b_0) \in \mathrm{dgm}^R_{l_0}(\{\mathrm{SW}_{d,\tau} f(t)\}_{t=0}^T, d_2)$ *satisfying*

$$\max\left\{0, \frac{b_2 - 2\lambda}{k}\right\} \le b_0 \le k(b_2 + 2\lambda) \quad \textit{and} \quad \max\left\{0, \frac{a_2 - 2\lambda}{k}\right\} \le a_0 \le k(a_2 + 2\lambda).$$

Proof. Let us note that the assumption implies

$$b_2 - a_2 > b_2 - a_2 \max\{k^2, 1\} > 2\lambda + 2\lambda \max\{k^2, 1\} > 4\lambda,$$

and thus by Lemma 4.2 there exists a unique $(a_1, b_1) \in \mathrm{dgm}^R_{l_0}(\phi_T, d_\infty)$ such that

$$b_2 - 2\lambda < b_1 < b_2 + 2\lambda \quad \text{and} \quad a_2 - 2\lambda < a_1 < a_2 + 2\lambda.$$

We conclude

$$\frac{b1}{a_1} > \frac{b_2 - 2\lambda}{a_2 + 2\lambda} > k^2,$$

and thus by Lemma 4.1 there exists a unique $(a_0, b_0) \in \mathrm{dgm}^R_{l_0}(\mathbb{SW}_{d,\tau} f(T), d_2)$ such that

$$\frac{1}{k} < \frac{\max\{b_0, b_1\}}{\min\{b_0, b_1\}}, \frac{\max\{a_0, a_1\}}{\min\{a_0, a_1\}} < k.$$

without loss of generality we assume $b_0 = \max\{b_0, b_1\}$. Combining our inequalities, we get

$$\frac{b_2 - 2\lambda}{k} < \frac{b_1}{k} < b_0 < kb_1 < k(b_2 + 2\lambda).$$

An analogous inequality can be obtained for a_0, completing the proof. □

A similar result can be obtained for "symmetric" signals. The following shows it is sufficient to scale our coefficients and consider only half of the frequencies.

Corollary 4.4. *Let*

$$f(t) = \sum_{j=1}^{N} c_j (e^{2\pi i \omega_j t} + e^{-2\pi i \omega_j t}) \quad \textit{for } c_j \in \mathbb{C} \setminus \{0\} \textit{ and } \omega_j \in \mathbb{R}.$$

Let ϕ_T denote the trajectory

$$\phi_T = \{(\tilde{c}_1 e^{2\pi i \omega_1 t}, \dots, \tilde{c}_N e^{2\pi i \omega_N t})^\top \mid t = 0, 1, \dots, T\},$$

and let $G_{T'}$ denote the grid

$$G_{T'} = \{(\tilde{c}_1 e^{2\pi i \omega_1 t_1}, \dots, \tilde{c}_N e^{2\pi i \omega_N t_N})^\top \mid t_n = 0, 1, \dots, T', \ n = 1, \dots, N\}.$$

Let $k = \max\{\sigma_{\min}^{-1}, \sigma_{\max}\sqrt{2N}\}$ and $\lambda = d_{GH}(\phi_T, G_{T'})$. If $(a_2, b_2) \in \mathrm{dgm}^R_{l_0}(G_{T'}, d_\infty)$ satisfies

$$\frac{b_2 - 2\lambda}{a_2 + 2\lambda} > \max\{k^2, 1\},$$

then there exists a unique $(a_0, b_0) \in \mathrm{dgm}^R_{l_0}(\mathbb{SW}_{d,\tau} f(T), d_2)$ satisfying

$$\max\left\{0, \frac{b_2 - 2\lambda}{k}\right\} \le b_0 \le k(b_2 + 2\lambda) \quad \textit{and} \quad \max\left\{0, \frac{a_2 - 2\lambda}{k}\right\} \le a_0 \le k(a_2 + 2\lambda).$$

Proof. Let

$$\bar{\phi}_T = \{2^{-\frac{1}{2}}(\tilde{c}_1 e^{2\pi i\omega_1 t}, \dots, \tilde{c}_N e^{2\pi i\omega_N t}, \tilde{c}_1 e^{-2\pi i\omega_1 t}, \dots, \tilde{c}_N e^{-2\pi i\omega_N t})^{\intercal}\}_{0\le t\le T}.$$

By Lemma 4.1, if $(\bar{a}_1, \bar{b}_1) \in \mathrm{dgm}_{l_0}^R(\bar{\phi}_T, d_\infty)$ and

$$\frac{\bar{b}_1}{\bar{a}_1} > k^2,$$

then there exists a unique $(a_0, b_0) \in \mathrm{dgm}_{l_0}^R(\mathbb{SW}_{d,\tau} f(T), d_2)$ such that

$$\frac{1}{k} < \frac{\max\{b_0, \bar{b}_1\}}{\min\{b_0, \bar{b}_1\}}, \frac{\max\{a_0, \bar{a}_1\}}{\min\{a_0, \bar{a}_1\}} < k.$$

On the other hand, since

$$d_2(c_l e^{i\omega_l t_1}, c_l e^{i\omega_l t_2}) = d_2(c_l e^{-i\omega_l t_1}, c_l e^{-i\omega_l t_2}),$$

the maps $\mu : \bar{\phi}_T \to \phi_T$, $\mu(v_t) = \sqrt{2} v_t$ and $\nu : \phi_T \to \bar{\phi}_T$, $\nu(v_t) = 2^{-\frac{1}{2}} v_t$ are isometries. Thus, by Proposition 2.22, $(\bar{a}_1, \bar{b}_1) \in \mathrm{dgm}_{l_0}^R(\bar{\phi}_T, d_\infty)$ if and only if there exist a unique $(a_1, b_1) \in \mathrm{dgm}_{l_0}^R(\phi_T, d_\infty)$ such that

$$a_1 = \bar{a}_1 \quad \text{and} \quad b_1 = \bar{b}_1.$$

Combining this fact with Lemma 4.2, we can say that if $(a_2, b_2) \in \mathrm{dgm}_{l_0}^R(G_{T'}, d_\infty)$ and

$$b_2 - a_2 > 4\lambda,$$

then there exists a unique $(\bar{a}_1, \bar{b}_1) \in \mathrm{dgm}_{l_0}^R(\bar{\phi}_T, d_\infty)$ such that

$$|\bar{b}_1 - b_2| < 2\lambda \quad \text{and} \quad |\bar{a}_1 - a_2| < 2\lambda.$$

By the equivalences that have now been shown, we can replicate the arguments in Theorem 4.3 to conclude the result. □

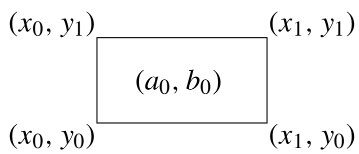
Remark 3 (error-bound confidence region). Let

$$(a_2, b_2) \in \mathrm{dgm}_{l_0}^R(G_{T'}, d_\infty).$$

Under the hypotheses of Theorem 4.3 there is a corresponding $(a_0, b_0) \in \mathrm{dgm}_{l_0}^R(\mathbb{SW}_{d,\tau} f(T), d_2)$ satisfying

$$\max\left\{0, \frac{b_2 - 2\lambda}{k}\right\} \le b_0 \le k(b_2 + 2\lambda) \quad \text{and} \quad \max\left\{0, \frac{a_2 - 2\lambda}{k}\right\} \le a_0 \le k(a_2 + 2\lambda).$$

It follows that (a_0, b_0) lies in the rectangle



where $x_0 = \max\{0, (a_2 - 2\lambda)/k\}$, $x_1 = k(a_2 + 2\lambda)$, $y_0 = \max\{0, (b_2 - 2\lambda)/k\}$ and $y_1 = k(b_2 + 2\lambda)$. This rectangle serves as a confidence region for the approximation.

We have shown explicitly how $\mathrm{dgm}^R_{l_0}(G_T, d_\infty)$ approximates $\mathrm{dgm}^R_{l_0}(\mathbb{SW}_{d,\tau} f(T), d_2)$. In order to leverage our results from Section 3 we use the Künneth formula; see Theorem 2.21. It allow us to apply Theorems 3.3 and 3.7 to compute $\mathrm{dgm}^R_{l_0}(G_T, d_\infty)$ for $l_0 = 1, 2$. This approach is detailed in the next result. For ease of presentation, we shall express persistence information using the barcode notation bcd^R_i in the following result; these barcodes correspond exactly to the persistence diagrams dgm^R_i as detailed in Section 2.

Theorem 4.5. *Let $c_j \in \mathbb{C}$ and $\omega_j \in \mathbb{R}$. For $a, b, s \in \mathbb{R}$, let $\bar{a} = 2\sin(\pi a)$, $\omega'_l = \omega_l/(2\pi)$ and, for $I = [a, b)$, we let $\bar{I} = [\bar{a}, \bar{b})$ and write sI to denote $[sa, sb)$. Let*

$$G_T = \{(c_1 e^{i\omega_1 t_1}, \ldots, c_N e^{i\omega_N t_N})^\mathsf{T}\}_{0 \le t_i \le T}.$$

Then,

$$\mathrm{bcd}^R_{l_0}(G_T, d_\infty) = \bigcup_{\sum_l r_l = l_0} \{|c_1|\bar{I}_1 \cap \cdots \cap |c_N|\bar{I}_N \mid I_l \in \mathrm{bcd}^R_{r_l}(S_{\omega'_l, T}, \bar{d})\}.$$

Proof. For $0 \le l \le N$ let

$$X_l = \{c_l e^{i\omega_l t}\}_{t=0}^T.$$

By Theorem 2.21,

$$\mathrm{bcd}^R_{l_0}(G_T, d_\infty) = \bigcup_{\sum_l r_l = l_0} \{J_1 \cap \cdots \cap J_N \mid J_l \in \mathrm{bcd}^R_{r_l}(X_l, d_2)\}.$$

Thus it is sufficient to show that $J_l \in \mathrm{bcd}^R_{r_l}(X_l, d_2)$ if and only if there exists a unique $I_l \in \mathrm{bcd}^R_{r_l}(S_{\omega'_l, T'}, \bar{d})$ such that

$$J_l = |c_l|\bar{I}_l.$$

To do so, define $x_j = \omega_l t_j \mod 2\pi$ and consider the metric d_l on X_l given by

$$d_l(c_l e^{i\omega_l t_1}, c_l e^{i\omega_l t_2}) = |c_l| \min\{|x_1 - x_2|, |2\pi - x_2 + x_1|, |2\pi - x_1 + x_2|\}.$$

This metric captures the geodesic distance between points along the circle of radius $|c_l|$ centered at $(0, 0)$. By trigonometric identities,

$$d_2(c_l e^{i\omega_l t_1}, c_l e^{i\omega_l t_2}) = \frac{|c_l| \sin(|c_l|^{-1} d_l(c_l e^{i\omega_l t_1}, c_l e^{i\omega_l t_2}))}{\sin\left(\frac{1}{2}(\pi - |c_l|^{-1} d_l(c_l e^{i\omega_l t_1}, c_l e^{i\omega_l t_2}))\right)} = 2|c_l| \sin\left(\tfrac{1}{2}(|c_l|^{-1} d_l(c_l e^{i\omega_l t_1}, c_l e^{i\omega_l t_2}))\right).$$

Thus, as a consequence of Theorem 2.9, $J_l \in \mathrm{bcd}^R_{r_l}(X_l, d_2)$ if and only if there exists a unique $[a, b) \in \mathrm{bcd}^R_{r_l}(X_l, d_l)$ such that

$$J_l = \left[2|c_l| \sin\left(\tfrac{1}{2}|c_l|^{-1} a\right), 2|c_l| \sin\left(\tfrac{1}{2}|c_l|^{-1} b\right)\right).$$

Furthermore, noting

$$d_l(c_l e^{i\omega_l t_1}, c_l e^{i\omega_l t_2}) = 2\pi |c_l| \bar{d}([\omega'_l t_1], [\omega'_l t_2]),$$

we can similarly say $[a, b) \in \mathrm{bcd}^R_{r_l}(X_l, d_l)$ if and only if there exists a unique $[a_0, b_0) \in \mathrm{bcd}^R_{r_l}(S_{\omega'_l,T}, \bar{d})$ such that

$$[a, b) = [2\pi|c_l|a_0, 2\pi|c_l|b_0).$$

Combining these equivalences gives the desired result. □

4.1. *Example.* We execute the workflow illustrated in Figure 8, which takes as input a quasiperiodic function and produces the diagrams, for $T' \le T$,

$$\mathrm{dgm}^R_l(G_{T'}, d_\infty) \quad \text{for } l = 1, 2.$$

Concretely, we set

$$f(t) = \frac{1}{\sqrt{2}}e^{i\sqrt{3}t} + \frac{1}{\sqrt{2}}e^{i\sqrt{5}t}.$$

We then carry out three stages:

(1) Algorithm 1 (three-gap code): For each incommensurate frequency $\omega_j \in \{\sqrt{3}, \sqrt{5}\}$, run Algorithm 1 using the results from Section 3 to obtain the exact diagrams

$$\mathrm{dgm}^R_l(S_{\omega'_j,T'}, \bar{d}) \quad \text{for } l = 1, 2 \text{ and } \omega'_j = \frac{\omega_j}{2\pi}.$$

(2) Persistent Künneth formula: Assemble these single-frequency outputs via Theorem 4.5 to obtain the diagrams

$$\mathrm{dgm}^R_l(G_{T'}, d_\infty) \quad \text{for } l = 1, 2.$$

(3) Error bounds: Finally, Theorem 4.3 provides an explicit bottleneck-distance bound showing that $\mathrm{dgm}^R_l(G_{T'}, d_\infty)$ approximates the true sliding-window diagram $\mathrm{dgm}^R_l(\mathrm{SW}_{d,\tau} f(T), d_2)$ within the claimed error.

Remark 4. In practical applications to real-world data, the frequencies ω_j are first estimated via the FFT of the observed time series (see Section 5).

With the roadmap in place, we begin by constructing the sliding-window embedding matrix as detailed at the end of Section 2.3. In our example one obtains

$$\mathrm{SW}_{d,\tau} f(t) = \frac{1}{\sqrt{d+1}} \begin{pmatrix} 1 & 1 \\ e^{i\sqrt{3}\tau} & e^{i\sqrt{5}\tau} \\ \vdots & \vdots \\ e^{i\sqrt{3}d\tau} & e^{i\sqrt{5}d\tau} \end{pmatrix} \begin{pmatrix} \tilde{c}_1 e^{i\sqrt{3}t} \\ \tilde{c}_2 e^{i\sqrt{5}t} \end{pmatrix},$$

where $\tilde{c}_1 = \tilde{c}_2 = \sqrt{d+1}/\sqrt{2}$.

We take

$$d = 1 \quad \text{and} \quad \tau = \frac{\pi}{\sqrt{3}-\sqrt{5}}.$$

With these parameters A is orthonormal. Hence

$$k = \max\{\sigma^{-1}_{\min}, \sigma_{\max}\sqrt{M}\} = \max\{1, \sqrt{2}\} = \sqrt{2}, \quad \text{where } M = 2.$$

Finally, we approximate the sliding-window point cloud

$$\mathbb{SW}_{d,\tau} f(2000)$$

using $T' = 500$. This sets the stage for fleshing out our 3G approximation method. We next apply Algorithm 1 to each frequency component in turn.

First, the scaled frequencies admit the continued-fraction expansions

$$\omega_1' = \frac{\sqrt{3}}{2\pi} = [0; 3, 1, 1, 1, 2, \dots] \quad \text{and} \quad \omega_2' = \frac{\sqrt{5}}{2\pi} = [0; 2, 1, 4, 3, 1, \dots].$$

By Corollaries 3.4 and 3.8, the resulting persistence diagrams are

$$\begin{aligned}
&\mathrm{dgm}_0^R(S_{\omega_1',500}, \bar{d}) = \{(0, 1.577\times10^{-3})^{160}, (0, 2.077\times10^{-3})^{316}, (0, 3.654\times10^{-3})^{24}, (0,\infty)\},\\
&\mathrm{dgm}_1^R(S_{\omega_1',500}, \bar{d}) = \{(3.654\times10^{-3}, 334.551\times10^{-3})\},\\
&\mathrm{dgm}_0^R(S_{\omega_2',500}, \bar{d}) = \{(0, 0.368\times10^{-3})^{161}, (0, 2.637\times10^{-3})^{220}, (0, 3.005\times10^{-3})^{119}, (0,\infty)\},\\
&\mathrm{dgm}_1^R(S_{\omega_2',500}, \bar{d}) = \{(3.005\times10^{-3}, 334.846\times10^{-3})\}.
\end{aligned}$$

With our single-frequency persistence diagrams in hand, we now apply the persistent Künneth formula (Theorem 4.5). Recall

$$G_{500} = \{(e^{i\omega_1 t_1}, e^{i\omega_2 t_2})\}_{0\le t_1,t_2\le 500}.$$

To verify the hypotheses of Theorem 4.3, we identify the diagrams of maximal persistence. From $\mathrm{dgm}_1^R(S_{\omega_1',500}, \bar{d})$ the interval $(a_1', b_1') = (3.654\times10^{-3}, 334.551\times10^{-3})$ induces in $\mathrm{dgm}_1^R(G_{500}, d_\infty)$ the interval

$$(a_1'', b_1'') = (2\sin(a_1'\pi), 2\sin(b_1'\pi)) = (0.02296, 1.73586).$$

Similarly, from $\mathrm{dgm}_1^R(S_{\omega_2',500}, \bar{d})$ with $(a_2', b_2') = (3.005\times10^{-3}, 334.846\times10^{-3})$ we obtain

$$(a_2'', b_2'') = (2\sin(a_2'\pi), 2\sin(b_2'\pi)) = (0.01888, 1.73678).$$

In $\mathrm{dgm}_2^R(G_{500}, d_\infty)$ the most persistent interval is $(a_3'', b_3'') = (0.02296, 1.73586)$.

Since $\lambda \le d_H(\phi_{2000}, G_{500}) = 0.20975$ and $\max\{k^2, 1\} = 2$, we compute

$$\frac{b_1''-2\lambda}{a_1''+2\lambda} = \frac{1.73586-2\cdot0.20975}{0.02296+2\cdot0.20975} = 2.9751 > 2 \quad \text{and} \quad \frac{b_2''-2\lambda}{a_2''+2\lambda} = \frac{1.73678-2\cdot0.20975}{0.01888+2\cdot0.20975} = 3.0049 > 2.$$

Thus the conditions of Theorem 4.3 are met. Define, for any x,

$$X(x) = \max\left\{0, \frac{x-2\lambda}{k}\right\} \quad \text{and} \quad Y(x) = k(x+2\lambda).$$

We compute the error bounds to be

$$\begin{aligned}
(X(a_1''), Y(a_1'')) &= (0, 0.6257), \quad (X(b_1''), Y(b_1'')) = (0.9308, 3.0481),\\
(X(a_2''), Y(a_2'')) &= (0, 0.6200), \quad (X(b_2''), Y(b_2'')) = (0.9315, 3.0494),\\
(X(a_3''), Y(a_3'')) &= (0, 0.6257), \quad (X(b_3''), Y(b_3'')) = (0.9308, 3.0481).
\end{aligned}$$

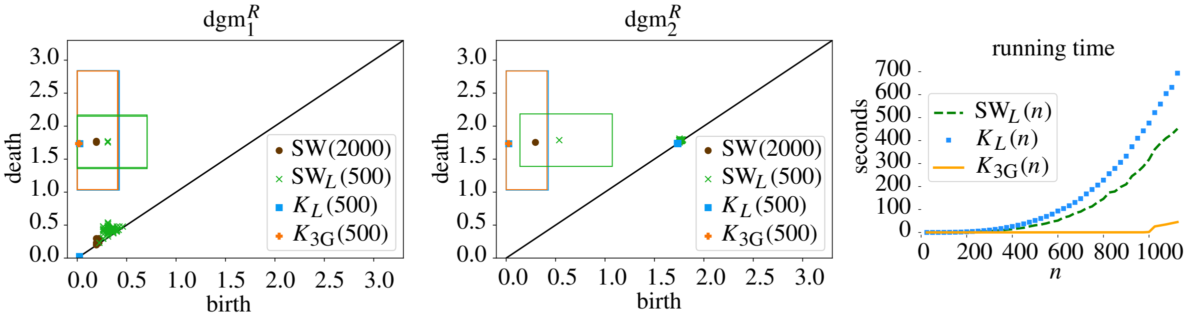


Figure 11. Persistence diagrams from Section 4.1. We also report the computation times for Ripser [Bauer 2016] on the sliding-window embedding (SW) and the method (K_L). Our implementation (K_{3G}) does not invoke Ripser at any stage.

Therefore Theorem 4.3 guarantees unique diagrams

$$(a_1, b_1), (a_2, b_2) \in \mathrm{dgm}_1^R(\mathbb{SW}_{d,\tau} f(2000), d_2) \quad \text{and} \quad (a_3, b_3) \in \mathrm{dgm}_2^R(\mathbb{SW}_{d,\tau} f(2000), d_2)$$

satisfying

$$\begin{aligned} 0 \le a_1 \le 0.6257, &\quad 0.9308 \le b_1 \le 3.0481, \\ 0 \le a_2 \le 0.6200, &\quad 0.9315 \le b_2 \le 3.0494, \\ 0 \le a_3 \le 0.6257, &\quad 0.9308 \le b_3 \le 3.0481. \end{aligned}$$

Finally, we compute the actual persistence diagrams of the sliding-window embedding for verification. By evaluating

$$\mathrm{dgm}_i^R(\mathbb{SW}_{d,\tau} f(2000), d_2) \quad \text{for } i = 1, 2,$$

one finds that the diagrams predicted by our 3G method are

$$(a_1, b_1) = (0.193, 1.771), \quad (a_2, b_2) = (0.194, 1.753) \quad \text{and} \quad (a_3, b_3) = (0.296, 1.753).$$

Each (a_i, b_i) lies within its respective error-bound rectangle, as shown in Figure 11. For comparison, we also include two alternative approximations — the sliding-window landmark method $\mathrm{SW}_L(n)$ and the landmark-based Künneth method $\mathrm{K}_L(n)$ [Gakhar and Perea 2019] — together with their corresponding error-bound rectangles and measured computation times for varying sample sizes n.

5. Applications

In this section, we will apply our pipeline, 3G, to the following examples: synthetic tremor signals in nonlinear tuned vibration absorbers as indicators of safe or unsafe operations [Detroux et al. 2015]; neuroscience, where quasiperiodicity is associated with task-specific functions in certain brain areas [Kaslik et al. 2022]; and celestial mechanics, where quasiperiodicity translates to favorable trajectories for missions [Gómez and Mondelo 2001]. We contribute to these fields by providing a faster alternative (see Figure 11 and Table 2) to analyze the reconstructed signals of potential quasiperiodic systems.

5.1. ***Methodology.*** We focus on dynamical systems whose flow Φ arises from a known system of ordinary differential equations. To generate a corresponding time series "observation", we solve these ODEs from a chosen initial condition and record the resulting solution values as the signal $f(t)$. We reconstruct the system by computing the sliding-window point cloud $\{\mathrm{SW}_{d,\tau} f(\epsilon t)\}_{t=0}^{T}$, where ϵ is chosen according to the Nyquist–Shannon sampling theorem [Shannon 1949]. The framework detailed in [Gakhar and Perea 2024] provides a method for selecting d and τ and approximating $\mathrm{dgm}_j^R(\{\mathrm{SW}_{d,\tau} f(\epsilon t)\}_{t=0}^{T}, d_2)$ with $\mathrm{dgm}_j^R(\{\mathrm{SW}_{d,\tau} S_1 f(\epsilon t)\}_{t=0}^{T}, d_2)$. Since $S_1 f$ is a sum of the form shown in Corollary 4.4, we can apply our result to approximate the latter, thus closely approximating the former. Indeed, in the examples we consider,

$$S_1 f(\epsilon t) = \sum_{j=1}^{2} c_j (e^{i\epsilon\omega_j t} + e^{-i\epsilon\omega_j t}).$$

Thus 3G can be used to approximate $\mathrm{dgm}_j^R(\{\mathrm{SW}_{d,\tau} S_1 f(\epsilon t)\}_{t=0}^{T}, d_2)$ with a known error bound as detailed in Remark 3.

5.2. ***Results.*** We consider systems with known quasiperiodic behavior, specifically those with two incommensurate frequencies. Hence the sliding-window point cloud $\{\mathrm{SW}_{d,\tau} f(\epsilon t)\}_{t=0}^{T}$ samples a 2-torus. We label the corresponding persistence diagrams as $\mathrm{SW}(T)$ (shown as brown dots). By $\mathrm{SW}_{S_1}(T)$, we denote the persistence diagrams of the sliding-window embedding $S_1(f)$ (shown as blue squares). The approximation obtained from 3G is labeled $K_{3G}(T)$ (shown as orange crosses).

5.2.1. *Double gyre.* The driven double gyre system provides a model for patterns occurring in geophysical flows [Shadden et al. 2005]. A topological approach to it has shown great success. Indeed, as shown in [Charó et al. 2020], there are multiple topological classifications that can be observed in the system representing the motion of a fluid particle. In this work, they noted that the motion of fluid can be sparse, meaning some initial conditions will imply a particle will be contained in a small region. Different initial conditions have shown trajectories with the topology of a standard strip, five-handle structure with torsion, torus, and möbius strip. This result was achieved by calculating homologies in a branched manifold while keeping track of the orientability chains, allowing for the identification of the branches and the localization of twists or torsions. This approach is called branched manifold analysis through homologies (BraMAH). At its core, it follows the principle of reconstructing a dynamical system from a time series. Yet the details at each step are completely different from SW. Nevertheless, for the corresponding initial conditions, we independently corroborated the presence of a torus, i.e., quasiperiodicity in the motion of the fluid particle; see Figure 12. This highlights the robustness offered by a topological approach to dynamical systems.

Concretely, the driven double gyre is given by the equation $\dot{x} = (-\partial\psi/\partial x_2, \partial\psi/\partial x_1)$, where

$$\psi(x_1, x_2, t) = A\sin(\pi g(x_1, t))\sin(\pi x_2)$$

with

$$g(x_1, t) = \eta\sin(\lambda t)(x_1^2 - 2x_1) + x_1,$$

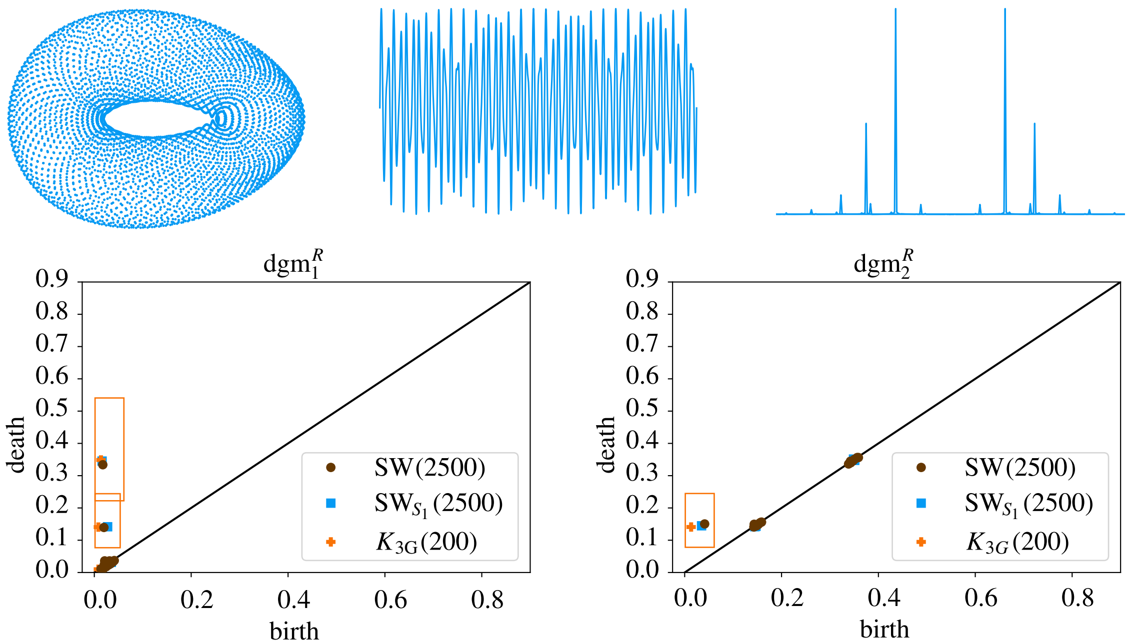


Figure 12. Top row: phase portrait of the driven double gyre system with initial condition x_0 (left), the corresponding trajectory $x_1(t)$ used for sliding-window embedding (center), and its Fourier power spectrum via FFT (right). Bottom row: the persistence diagrams of the resulting Rips filtrations for $x_1(t)$.

and parameter values A, $\mu = 0.1$ and $\lambda = \frac{1}{5}\pi$. The system has a toroidal attractor for the initial condition $x_0 = (0.5, 0.625)$ [Charó et al. 2020]. We solve the system using $dt = 0.1$ up to $t = 800$. Furthermore, SW is done using $f = x_1$, $d = 4$, $\tau = 119.03$ and $\epsilon = 0.1$.

5.2.2. *The torus.* We now consider a toroidal attractor in $\mathbb{R}^4$ with coordinates (x, y, z, r). The dynamical system is given by the set of equations

$$\dot{x} = -y + x(1 - \sqrt{x^2 + y^2}), \quad \dot{y} = x + y(1 - \sqrt{x^2 + y^2}),$$
$$\dot{z} = -kr + z(4 - \sqrt{z^2 + r^2}), \quad \dot{r} = kz + r(4 - \sqrt{z^2 + r^2}).$$

When k is irrational, the solutions span a torus [Penalva Vadell 2018]. We consider the case $k = \sqrt{2}$ with the initial condition $(x_0, y_0, z_0, r_0) = (1, 0, 4, 0)$. We solve the system with $dt = 0.1$ up to $t = 800$. The sliding window is done using $f = x + z$, $d = 4$, $\tau = 87.56$ and $\epsilon = 0.1$; see Figure 13.

5.2.3. *Pendulum attached to a sliding block.* We now consider a particular type of tuned mass damper (TMD). A TMD is a device used to suppress vibration by moving a mass attached to the main structure through springs and dampers [Wen et al. 2019]. This model has been successfully used to dampen the effect of long-duration earthquake ground motions [Murudi and Mane 2004]. Indeed, one can consider the main structure being a skyscraper and the incoming wave being generated by the earthquake. Thus a better understanding of this system translates to earthquake-resistant technologies.

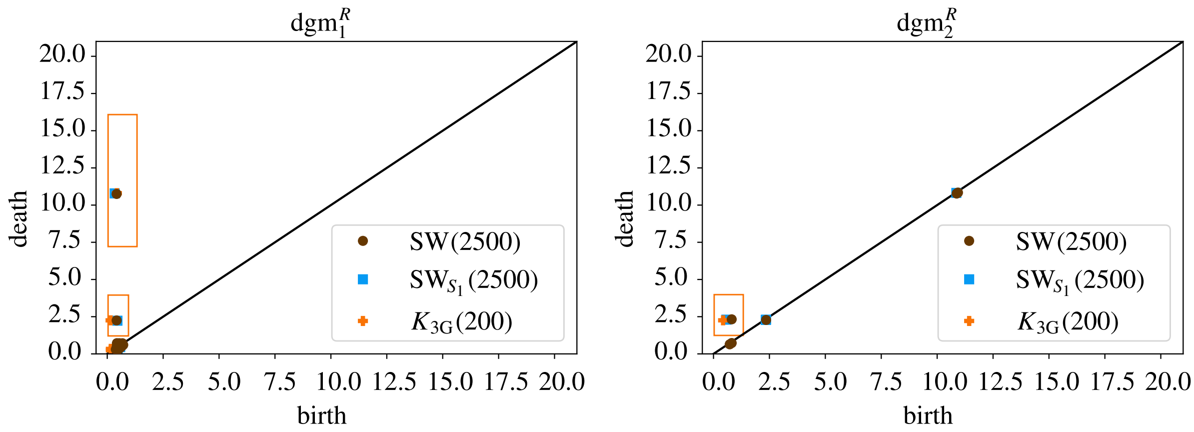


Figure 13. Diagrams obtained for the torus in $\mathbb{R}^4$.

We consider the case of a pendulum attached to a sliding block; see Figure 14. It was shown to exhibit quasiperiodicity in [Wen et al. 2019]. The governing equations are

$$\ddot{x} + \alpha^2 x - \bar{\epsilon} g\theta - \bar{\epsilon} L\dot{\theta}^2\theta = 0 \quad \text{and} \quad \ddot{\theta} + (1+\bar{\epsilon})\beta^2\theta - \bar{\epsilon} h\alpha^2 x + \bar{\epsilon}\dot{\theta}^2\theta = 0,$$

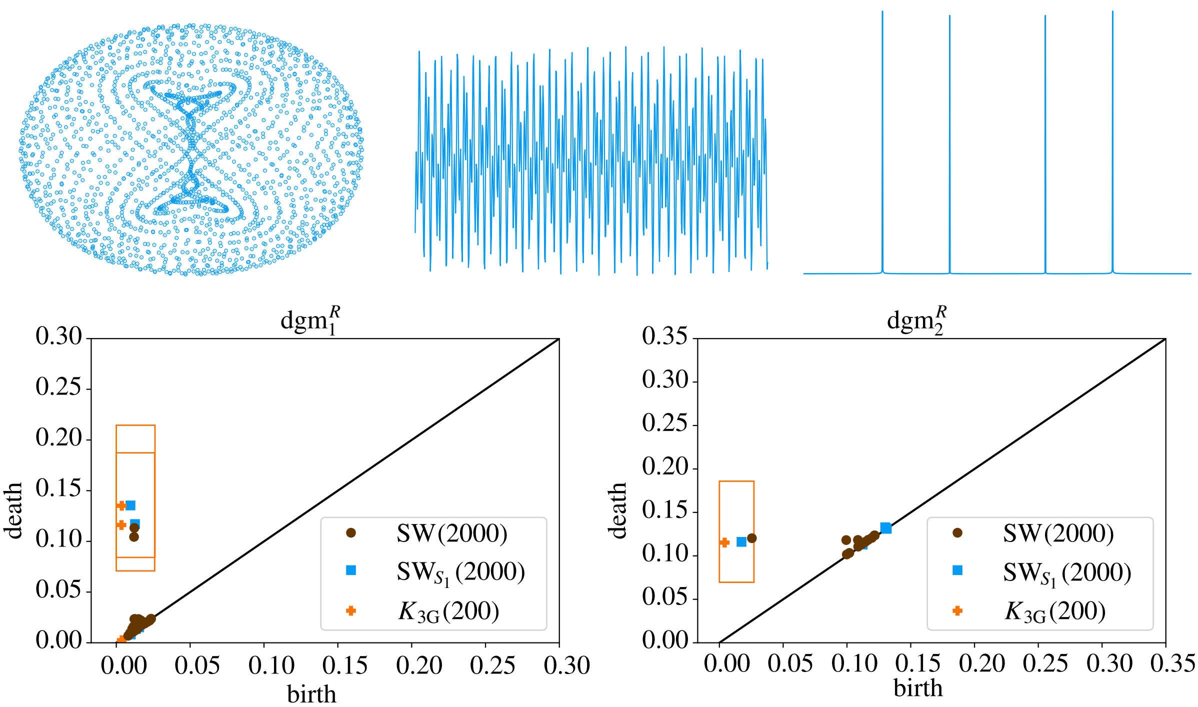


Figure 14. Top row: phase portrait $(x, \dot{x})$ of the pendulum on a sliding-block system (left), the scalar trajectory $x(t)$ used for sliding-window embedding (center), and its Fourier spectrum via FFT (right). Bottom row: persistence diagrams of the Rips filtration on the sliding-window point cloud derived from $x(t)$.

in which

$$\bar{\epsilon} = \frac{m}{M}, \quad \alpha = \sqrt{\frac{k}{M}}, \quad \beta = \sqrt{\frac{g}{L}}, \quad h = \frac{1}{\epsilon L},$$

and g is the acceleration of gravity. The parameter values are $m = 0.5$, $M = 1$, $L = 1$ and $k = 5$. We solve the system with the initial condition $(x_0, \dot{x}_0, \theta_0, \dot{\theta}_0) = (0.1, 0, -0.1, 0)$ and $dt = 0.27$ up to $t = 540$. The sliding window is done using $f = x$, $d = 4$, $\tau = 108.05$ and $\epsilon = 0.027$.

5.2.4. *Generalized Wilson–Cowan equations.* We consider a generalized version of the Wilson–Cowan equations. Traditionally, these equations are derived via a time-coarse graining technique that averages the response. They also restrict to a weak gamma distribution of time delays. The extended model has been treated in [Kaslik et al. 2022], and is given by

$$\dot{u}(t) = -u(t) + f_1\left(\theta_u + \int_{-\infty}^{t} h(t-s)(au(s) + bv(s))\, ds\right),$$
$$\dot{v}(t) = -v(t) + f_2\left(\theta_v + \int_{-\infty}^{t} h(t-s)(cu(s) + dv(s))\, ds\right),$$

where $u(t)$ and $v(t)$ model the firing activity in two neuronal populations, a, b, c and d are the connection weights, and θ_u and θ_v are background drives. The activation functions f_1 and f_2 are smooth and increasing on the real line. We consider three types of delay kernels $h : [0, \infty) \to [0, \infty)$, namely, weak gamma, strong gamma and Dirac kernels. Their analysis indicates a preferable kernel based on the function of the model populations.

One can readily verify that this system recovers the Wilson–Cowan equations in the case of a weak gamma kernel. We consider the case of a Dirac kernel; see Figure 15. This system has been shown to exhibit quasiperiodicity [Kaslik et al. 2022; Coombes and Laing 2009]. Furthermore, it has applications to the subthalamic nucleus–globus pallidus network involved in Parkinson's disease (guided by anatomical and electrophysiological research) [Holgado et al. 2010]. The system of interest becomes

$$\dot{u}(t) = -u(t) + f_1(\theta_u + au(t-\tau_1) + bv(t-\tau_2)), \quad \dot{v}(t) = -v(t) + f_2(\theta_v + cu(t-\tau_2) + dv(t-\tau_1)),$$

where

$$f_1(x) = f_2(x) = \frac{1}{1 + e^{-\delta x}}.$$

The parameter values are $\theta_u = 0.1$, $\theta_v = 0.2$, $\tau_1 = \tau_2 = 0.152$, $a = d = -19$, $b = c = 10$ and $\delta = 10$. We solve the system with initial conditions $(u_0, v_0) = (0.05, 0.05)$ and $dt = 0.001$ up to $t = 50$. The sliding window is done using $f = u$, $d = 4$, $\tau = 1.712$ and $\epsilon = 0.01$.

5.2.5. *Electromagnetic radiation on a Wilson neuron model.* Wilson [1999] introduced a simplified model for a neocortical neuron by making assumptions on the Hodgkin–Huxley model. The biophysics of these neurons is governed by the interplay of about a dozen ion currents. His model showed the need for only four ion currents to accurately replicate known spiking behavior. We analyze an extension of his model that incorporates the presence of electromagnetic radiation (EMR).

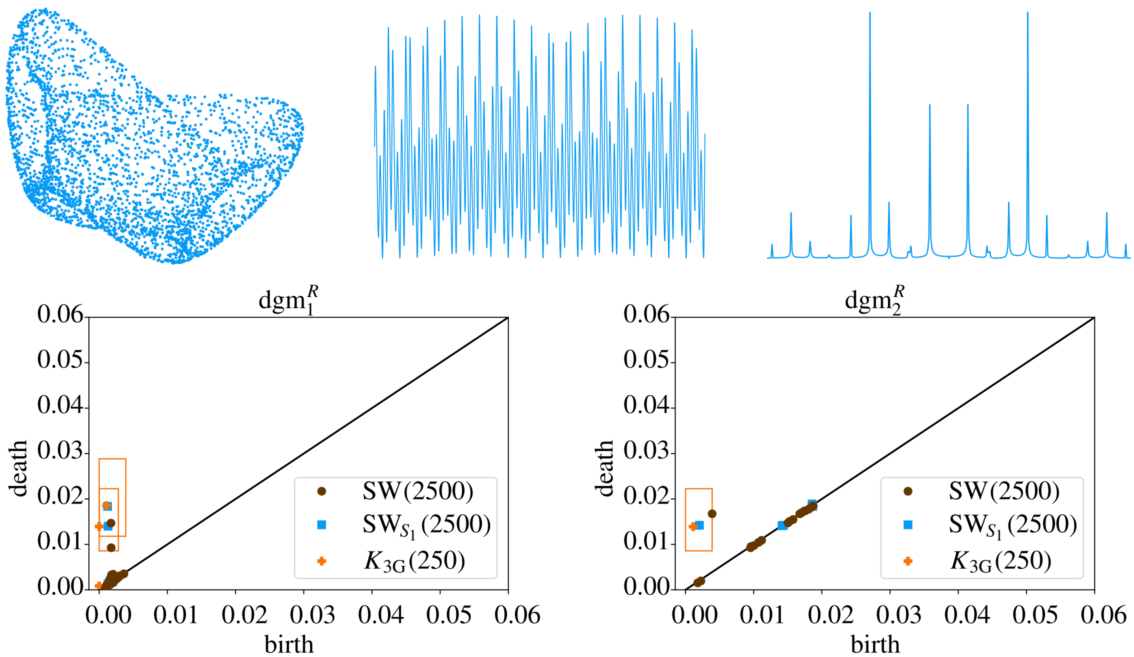


Figure 15. Top row: phase portrait (u, v) of the generalized Wilson–Cowan equations (left), the scalar trajectory $u(t)$ used for sliding-window embedding (center), and its Fourier power spectrum via FFT (right). Bottom row: persistence diagrams of the Rips filtration on the sliding-window point cloud derived from $u(t)$.

Commonplace presence of electronic devices introduces EMR exposure to neurons. The effects EMR has was imitated with the presence of a flux-controlled memristor in [Ju et al. 2022]. Their proposed model was shown to exhibit quasiperiodicity and their results were successfully replicated using a microcontroller unit-based hardware platform. Their mathematical model describes membrane potential v, recovery variable r and inner state of the memristor ϕ,

$$C_m \frac{dv}{dt} = -m_\infty(v)(v - E_{Na}) - g_K r(v - E_K) + I_{\text{ext}} - k_1 W(\phi)v,$$
$$\frac{dr}{dt} = \frac{1}{\tau_r}(-r + r_\infty(v)), \qquad \frac{d\phi}{dt} = v - k_2\phi + \phi_{\text{ext}},$$

where

$$m_\infty(v) = 17.8 + 47.6v + 33.8v^2 \quad \text{and} \quad r_\infty(v) = 1.24 + 3.7v + 3.2v^2.$$

We use typical model parameters for the membrane capacitor, $C_m = 1$, reverse potential for sodium and potassium, $E_{Na} = 0.5$ and $E_K = -0.95$, respectively, maximal conductance of potassium, $g_K = 26$, potassium ion channel activation time constant, $\tau_r = 5$, and external stimulus current $I_{\text{ext}} = 1$ [Xu et al. 2021]. The EMR external contribution is given by $\phi_{\text{ext}} = A\sin(2\pi Ft)$, and the terms $k_1 W(\phi)v$ and $k_2\phi$ denote the induction current caused by variation of magnetic flux and the leakage of magnetic flux. The memductance of the memristor is given by $W(\phi) = a - b\tanh(\phi)$ [Bao et al. 2017]. The remaining

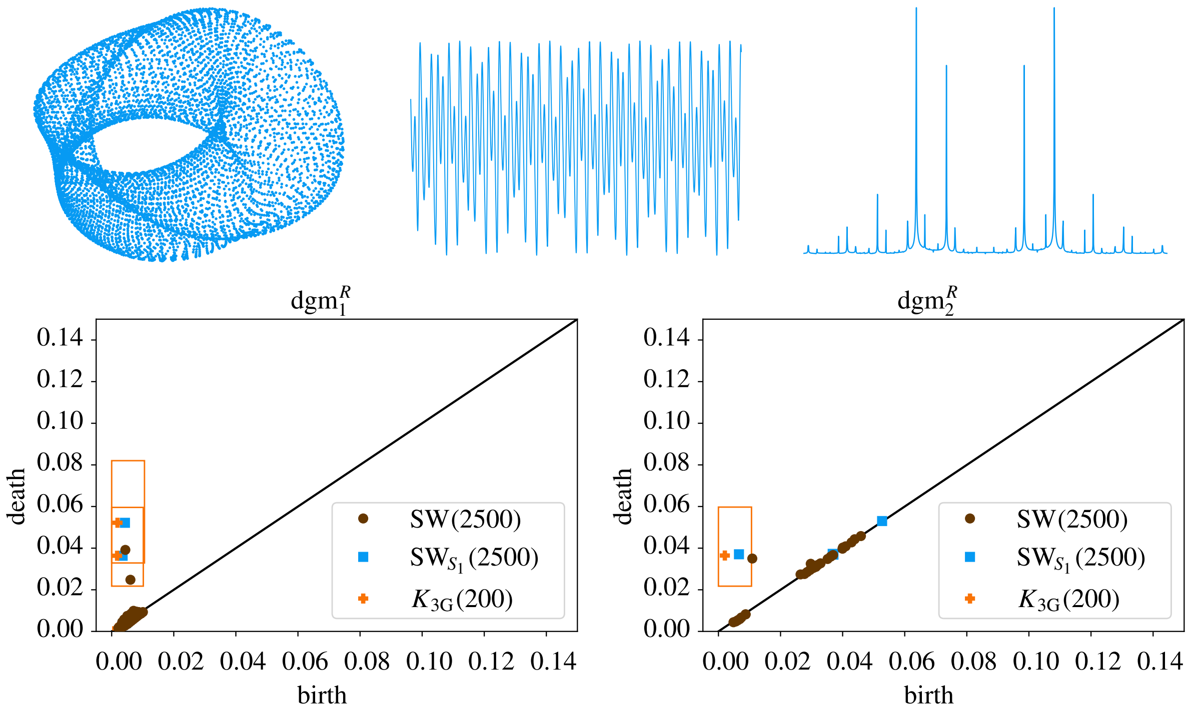


Figure 16. Top row: phase portrait (v, r, ϕ) of the system (left), the radial coordinate trajectory $r(t)$ used for sliding-window embedding (center), and its Fourier power spectrum via FFT (right). Bottom row: persistence diagrams of the Rips filtration on the sliding-window point cloud derived from $r(t)$.

parameter values are $a = 1$, $b = 3$, $k_1 = 1.2$, $k_2 = 0.5$, $A = 0.35$ and $F = 0.22$. We solve the system using the initial conditions $(v_0, r_0, \phi_0) = (0, 0, 0)$ and $dt = 0.01$ up to $t = 500$. The sliding window was done with $f = r$, $d = 4$, $\tau = 72.21$ and $\epsilon = 0.07$; see Figure 16.

5.2.6. *Competitive threshold-linear network.* Threshold-linear networks (TLNs) provide an accessible model incorporating a threshold nonlinearity [Morrison et al. 2024]. TLNs refine linear approximations of networks. The TLN model can be described by the equations

$$\frac{dx_i}{dt} = -x_i + \left[\sum_{j=1}^{n} W_{ij} x_j + b_i \right]_+$$

where $i = 1, \ldots, n$ and x_i represents the activity level of node i. The term W_{ij} is the directed connection strength, b_i is the external drive and $[y]_+ = \max\{y, 0\}$ is the threshold nonlinearity.

We consider competitive TLNs, where $W_{ij} \leq 0$, $W_{ii} = 0$ and $b_i \geq 1$. Furthermore, we consider the connection strength:

$$W_{ij} = \begin{cases} 0 & \text{if } i = j, \\ -1 + \lambda & \text{if node } j \text{ is connected to node } i, \\ -1 - \delta & \text{otherwise.} \end{cases}$$

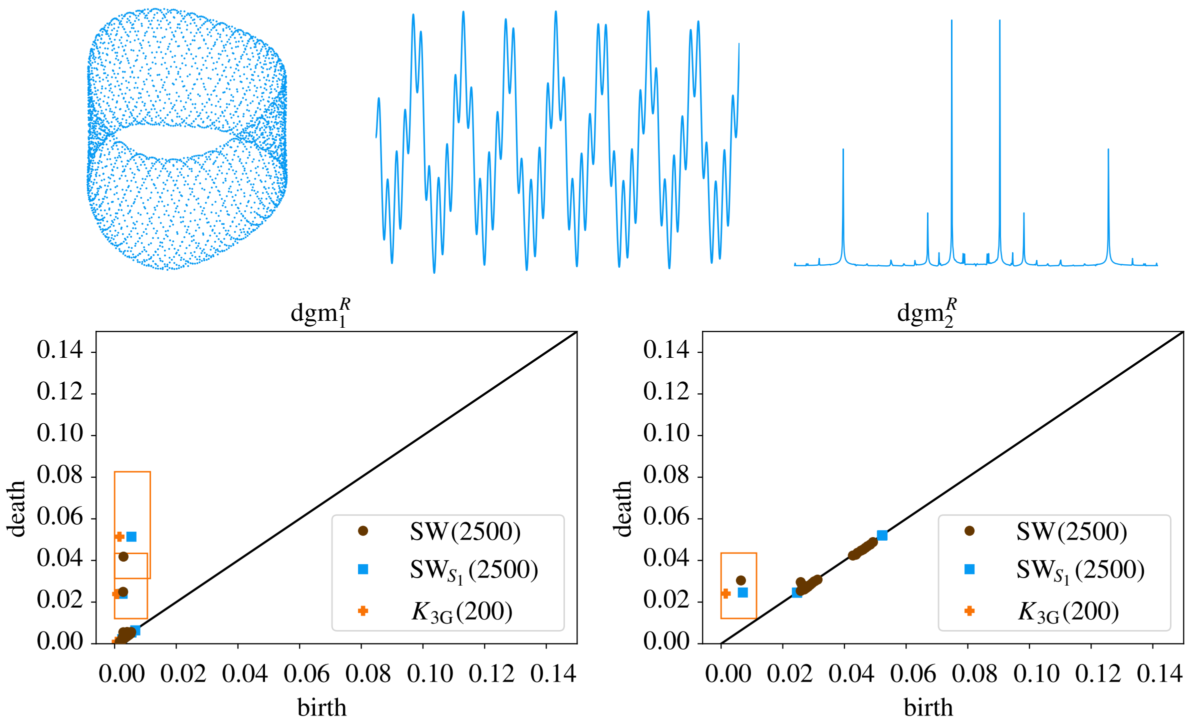


Figure 17. Top row: three-dimensional PCA projection of trajectories from the competitive TLN model (left), the scalar trajectory $x_4(t)$ used for sliding-window embedding (center), and its Fourier power spectrum via FFT (right). Bottom row: persistence diagrams of the Rips filtration on the sliding-window point cloud derived from $x_4(t)$.

This model has shown complex behavior, including quasiperiodicity. We replicate the quasiperiodic behavior of [Morrison et al. 2024, Figure 2] and refer to it for the initial condition and connection matrix. The remaining parameter values are $n = 25$, $b_i = 1$, $\lambda = 0.25$ and $\delta = 0.5$. We solve up to $t = 600$ and perform the sliding window with $f = x_4$, $d = 4$, $\tau = 27.53$ and $\epsilon = 0.07$; see Figure 17.

5.2.7. *Restricted three-body problem.* In celestial mechanics, the restricted three-body problem (RTBP) offers an accessible model with known equilibrium points [Koon et al. 2000]. The equations describe the progression of three celestial bodies in which one of them is considered a massless particle. The other two bodies, called primaries, are assumed to move in circular orbits around their center of mass. By taking a coordinate system that rotates with the primaries and under the appropriate scaling, it can be assumed the primaries have masses $1 - \mu$ and μ, $\mu \in \left[0, \frac{1}{2}\right]$, are fixed at $(\mu, 0, 0)$ and $(\mu - 1, 0, 0)$, respectively, and complete one revolution in 2π [Gómez and Mondelo 2001]. This framework allows us to express the motion of the massless particle by the equations

$$\ddot{x} - 2\dot{y} = \Omega_x, \quad \ddot{y} + 2\dot{x} = \Omega_y, \quad \ddot{z} = \Omega_z,$$

where

$$\Omega = \tfrac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \tfrac{1}{2}\mu(1-\mu),$$

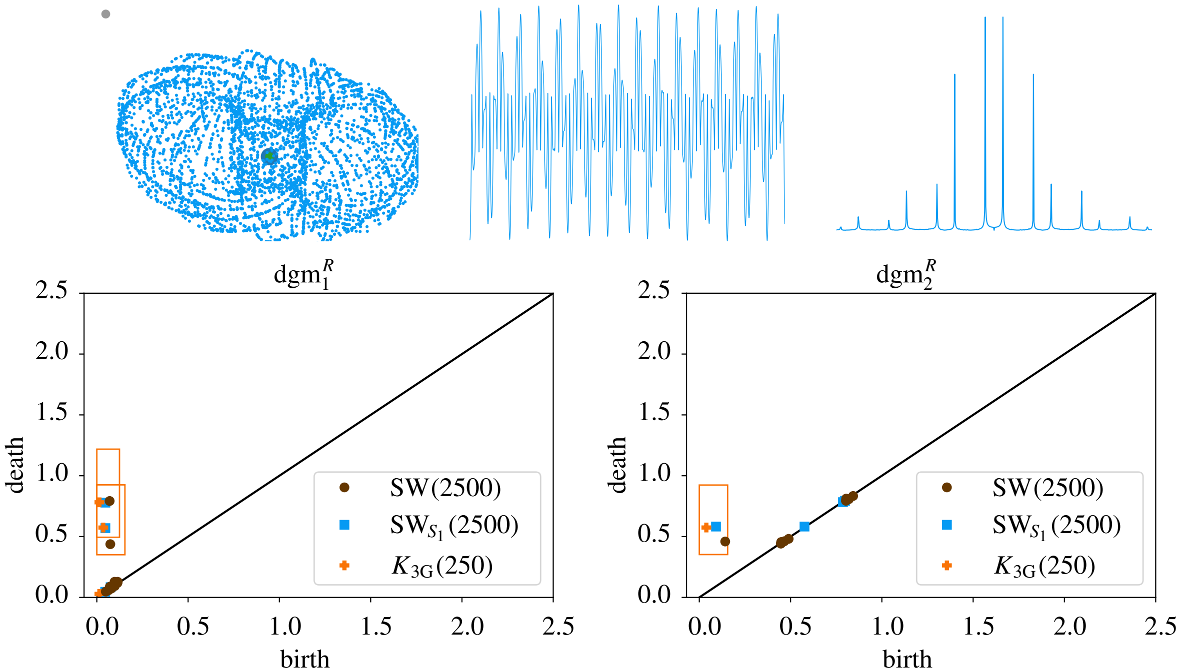


Figure 18. Top row: phase portrait of the restricted three-body problem in (x, y, z)-space, showing the Earth (blue/green) and the Moon (gray) as fixed primaries (left), the scalar trajectory $x(t)$ used for sliding-window embedding (center), and its Fourier power spectrum via FFT (right). Bottom row: persistence diagrams of the Rips filtrations on the sliding-window point cloud derived from $x(t)$.

and

$$r_1 = \sqrt{(x-\mu)^2 + y^2 + z^2} \quad \text{and} \quad r_2 = \sqrt{(x-\mu+1)^2 + y^2 + z^2}$$

are the distances from the particles to the primaries.

The case of the Earth–Moon system is of particular interest since it can aid in spacecraft missions interested in the Sun and the magnetosphere of the Earth [Gómez and Mondelo 2001]. Indeed, near the equilibrium points of the system, quasiperiodicity is present which translates to nice trajectories for a mission. We replicate this behavior for the Earth–Moon system; see Figure 18. In this case, $\mu = 0.0121506$. We solve the system with the initial condition $(x_0, y_0, z_0, \dot{x}, \dot{y}, \dot{z}) = (-0.5, 0, 0, 0, 0, 0.73)$ up to $t = 100$. The sliding window was done with $f = x$, $d = 4$, $\tau = 4.37$ and $\epsilon = 0.03$.

5.2.8. *Bicircular restricted four-body problem.* Considering the effects of the sun in the RTBP model gives rise to the bicircular restricted four-body problem (BCR4BP). The model is of importance for exploiting the force from the sun in trajectory designs for lunar missions and has found applications for ballistic lunar transfers to the lunar region [McCarthy and Howell 2023]. The new model uses the same coordinate axes as the RTBP and the same circular motion assumptions of the Earth and Moon, but it now includes terms pertaining to the influence of the Sun. The Sun is assumed to lie in the x–y plane at a fixed distance to the origin, a_4, and move with a constant angular velocity, $\dot{\theta}_S$, in circular motion.

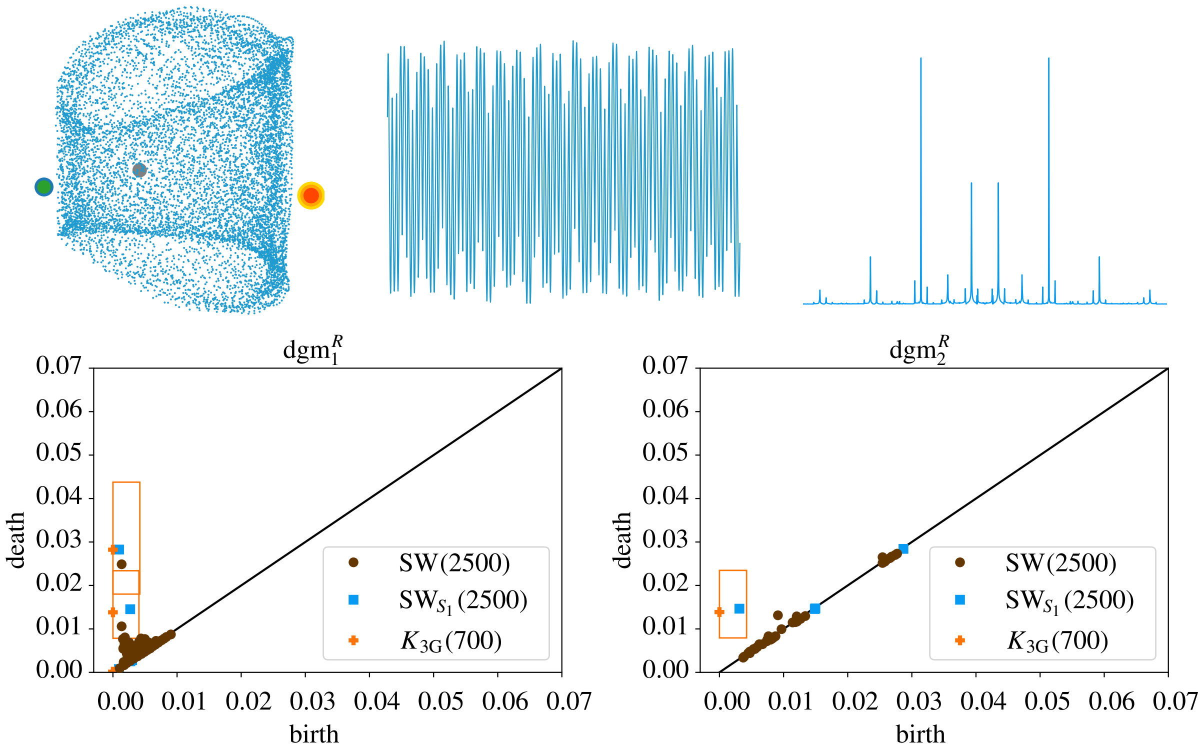


Figure 19. Top row: phase portrait of the bicircular restricted four-body problem in (x, y, z)-space, showing Earth (blue/green), Moon (gray), and Sun (red/yellow) as fixed primaries (positions not to scale). Center: the scalar trajectory $z(t)$ used for sliding-window embedding. Right: its Fourier power spectrum via FFT. Bottom row: persistence diagrams of the Rips filtrations on the sliding-window point cloud derived from $z(t)$.

Furthermore, the model assumes the Earth and Moon are not perturbed by solar gravity. Under these assumptions, the equations are [McCarthy and Howell 2023]

$$\ddot{x} = 2\dot{y} + \frac{\partial \Upsilon}{\partial x}, \quad \ddot{y} = -2\dot{x} + \frac{\partial \Upsilon}{\partial y}, \quad \ddot{z} = \frac{\partial \Upsilon}{\partial z},$$

where

$$\Upsilon = \frac{1-\mu}{r_{13}} + \frac{\mu}{r_{23}} + \tfrac{1}{2}(x^2 + y^2) + \lambda\left(\frac{m_4}{r_{43}} - \frac{m_4}{a_4^3}(x_4 x + y_4 y + z_4 z)\right),$$

and (x_4, y_4, z_4) denotes the position of the sun, r_{13}, r_{23} and r_{43} the distance of the Earth to the massless object, the Moon to the massless object, and the Sun to the massless object, respectively, and m_4 is the nondimensional mass of the Sun. We note that the case $\lambda = 0$ reduces to the RTBP. A study of the system as λ increases to 1 is done in [McCarthy and Howell 2023], where they showed the model exhibited quasiperiodicity. We replicate the quasiperiodic behavior (see Figure 19) for parameter values $\lambda = 1$, $\mu = 0.012155$, $a_4 = 388.84$, $m_4 = 328950.69$ and $\dot{\theta}_S = -0.9251986$. The system was solved with the initial condition $(x_0, y_0, z_0, \dot{x}, \dot{y}, \dot{z}, \theta_S) = (1.09, 0, 0, 0, 0.19, 0.05, 2)$ up to $t = 200$. The sliding window was done with $f = z$, $d = 4$, $\tau = 39.99$ and $\epsilon = 0.009$.

example	SW	SW(S1)	K_{3G}
5.2.1	7008.66	7672.85	0.87
5.2.2	4351.15	5351.15	0.42
5.2.3	3126.81	3218.73	0.50
5.2.4	5556.86	7628.48	0.98
5.2.5	5137.54	6505.54	0.66
5.2.6	5805.28	7193.97	0.22
5.2.7	6900.29	7937.03	2.09
5.2.8	6615.33	8231.25	0.59

Table 2. Running times, in seconds.

5.3. ***Concluding remarks.*** In this section, we detailed how our 3G method can be implemented on general quasiperiodic functions. Our approximation method was used on dynamical systems known to exhibit quasiperiodicity. As our figures show, we successfully approximated the diagrams obtained from SW. Error bounds can also be computed for our method, illustrated in the plots as rectangles. Furthermore, our approach significantly reduces computational time, as shown in Table 2. We hope our work contributes to the implementation of the sliding-window embedding technique on large data sets.

6. Conclusions and future work

We detailed how SW is a vital method used for better understanding of quasiperiodic signals. The correspondence between a quasiperiodic function of N incommensurate frequencies and an N-dimensional torus makes persistent homology an ideal component. However, the exponential computational cost of the latter limits the datasets that can be impacted by SW. Our contribution is providing an approximation to the persistence diagrams of interest (Section 4). Our method, 3G, achieves this within known error bounds. On average, our computation took less than 1 second to run for the examples depicted in Section 5. This contrasts with an average computation time of 5,563 seconds when using a standard library, Ripser [Bauer 2016], to compute the persistence diagrams of interest (Table 2).

Future work for our project would involve improving our approximation by tightening the error bounds. A more granular approach would be to inspect the specific transformation matrix instead of relying on a singular value argument. Another direction is to apply our work to real-world data, such as electrocardiogram datasets.

Appendix

Proposition A.1. *Let* $\sigma \in C_1(R_\epsilon(S_{\omega,T}))$ *and* $\sigma \neq 0$. *If* $\partial(\sigma) = 0$, *then there exist* $N \in \mathbb{N}$ *and* $\sigma_i \in C_1(R_\epsilon(S_{\omega,T}))$ *with* $\sigma_i \neq 0$, *such that*

$$\sigma = \sum_{i=0}^{N-1} \sigma_i,$$

$\partial(\sigma_i) = 0$ *and if* $\tau \subsetneq \sigma_i$,

$$\partial(\tau) \neq 0.$$

Moreover, σ_i *can be expressed as a sum of the form*

$$\sigma_i = \sum_{j=0}^{N_i-1} [x_j, x_{j+1}],$$

where $x_i \in S_{\omega,T}$, $N_i \in \mathbb{N}$ *and* $x_{N_i} = x_0$.

Proof. Let $\tau_1 \subsetneq \sigma$ be such that $\partial(\tau_1) = 0$ and there are no $\bar{\tau} \subsetneq \sigma$ such that $\partial(\bar{\tau}) = 0$ and $\tau_1 \subsetneq \bar{\tau}$. If no such τ_1 exists, then let $\sigma_1 = \sigma$ and the result follows. Otherwise, let $\sigma_1 = \sigma/\tau_1$. By the maximality of τ_1, σ_1 satisfies the stated conditions. Now let $\tau_2 \subsetneq \tau_1$ be such that $\partial(\tau_2) = 0$ and there are no $\bar{\tau} \subsetneq \tau_1$ such that $\partial(\bar{\tau}) = 0$ and $\tau_2 \subsetneq \bar{\tau}$. If no such τ_2 exists, then let $\sigma_2 = \tau_1$ and the result follows. Otherwise, let $\sigma_2 = \tau_1/\tau_2$. As before, one can check σ_2 has the desired conditions. After finitely many steps, the process will stop resulting in the desired sum.

We now show σ_i can be expressed as stated. Note that since $\sigma_i \neq 0$, there exists $[x_0, x_1] \in \sigma_i$. Furthermore, since $\partial(\sigma_i) = 0$, there must exist a $[x_1, x_2] \in \sigma_i$ to cancel the term x_1. Similarly, there must also exist a term $[x_2, x_3] \in \sigma_i$ to cancel the term x_2. After repeating this process finitely many times, say N_i, we must reach a point with a term $[x_{N_i-1}, x_0]$ to cancel the term x_0. Clearly

$$\partial\left(\sum_{j=0}^{N_i-2} [x_j, x_{j+1}] + [x_{N_i-1}, x_0]\right) = 0;$$

thus by the properties of σ_i, all of its terms must have been accounted for. □

For $[x, y] \in C_1(R_\epsilon(S_{\omega,T}))$, we let

$$L_{x,y} = 1$$

if in the circle representation of $S_{\omega,T}$, $\bar{d}(x, y)$ is the length of the arc starting at x and ending at y obtained by traversing clockwise. Similarly, we let

$$L_{x,y} = -1$$

if $\bar{d}(x, y)$ is the length of the arc traversing counterclockwise. Since $\bar{d}(x, y) \neq \frac{1}{2}$, for $x, y \in S_{\omega,T}$, $L_{x,y}$ is well defined. We now define

$$f([x, y]) = L_{x,y}\,\bar{d}(x, y).$$

Using linearity, we extend this function to all of $C_1(R_\epsilon(S_{\omega,T}))$, i.e., if $\sigma = \sum_i [x_i, y_i] \in C_1(R_\epsilon(S_{\omega,T}))$,

$$f(\sigma) = \sum_i f([x_i, y_i]) = \sum_i L_{x_i,y_i}\,\bar{d}(x_i, y_i).$$

Proposition A.2. *Let* $\sigma \in C_1(R_\epsilon(S_{\omega,T}))$, $f(\sigma) \neq 0$ *and* $N \in \mathbb{N}$ *such that*

$$\sigma = \sum_{i=0}^{N} [x_i, x_{i+1}].$$

If there exists an i_0 *for which*

$$L_{x_i,x_{i+1}} = \pm 1$$

for $0 \le i \le i_0$ *and*

$$L_{x_i,x_{i+1}} = \mp 1$$

for $i_0 < i \le N$, *then* σ *is homologous to a*

$$\bar{\sigma} = \sum_{i=0}^{\overline{N}} [\bar{x}_i, \bar{x}_{i+1}] \in C_1(R_\epsilon(S_{\omega,T})),$$

where $\overline{N} \in \mathbb{N}$, $\bar{x}_{i+1}$ *is adjacent to* $\bar{x}_i$ *and*

$$L_{\bar{x}_i,\bar{x}_{i+1}} = \operatorname{sgn}(f(\sigma)),$$

for $0 \le i \le \overline{N}$.

Proof. Without loss of generality we let

$$L_{x_i,x_{i+1}} = 1$$

for $0 \le i \le i_0$. Consider the circle representation of $S_{\omega,T}$. Let $\{x_j^0\}_{j=0}^{N_0}$ denote the points in the arc starting at x_0 and ending at x_1 traversing clockwise contained in $S_{\omega,T}$. We note $x_0^0 = x_0$ and $x_{N_0}^0 = x_1$. If $N_0 > 1$, one can verify

$$\sum_{j=0}^{N_0-1} [x_j^0, x_{j+1}^0] - [x_0, x_1] = \partial\left(\sum_{j=0}^{N_0-2} [x_j^0, x_{j+1}^0, x_{N_0}^0]\right),$$

i.e., $[x_0, x_1]$ is homologous to

$$\sigma_0 = \sum_{j=0}^{N_0-1} [x_j^0, x_{j+1}^0].$$

If $N_0 = 1$ we let $\sigma_0 = [x_0^0, x_{N_0}^0] = [x_0, x_1]$. By repeating this process, we construct σ_i for $0 \le i \le i_0$ such that

$$\sum_{i=0}^{i_0} \sigma_i = \sum_{i=0}^{i_0} \sum_{j=0}^{N_i-1} [x_j^i, x_{j+1}^i] = \sum_{i=0}^{A_1} [y_i, y_{i+1}],$$

where $y_0 = x_0$ and $y_{A_1+1} = x_{i_0+1}$, is homologous to

$$\sum_{i=0}^{i_0} [x_i, x_{i+1}].$$

Similarly, by considering the points in the arc starting at x_i and ending at x_{i+1}, for $i_0 < i \le N$, traversing counterclockwise contained in $S_{\omega,T}$, we can construct σ_i as before such that

$$\sum_{i=i_0+1}^{N} \sigma_i = \sum_{i=i_0+1}^{N} \sum_{j=0}^{N_i-1} [x_j^i, x_{j+1}^i] = \sum_{i=0}^{A_2} [\bar{y}_i, \bar{y}_{i+1}],$$

where $\bar{y}_0 = x_{i_0+1}$ and $\bar{y}_{A_2+1} = x_{N+1}$, is homologous to

$$\sum_{i=i_0+1}^{N} [x_i, x_{i+1}].$$

We note $y_{A_1+1} = \bar{y}_0 = x_{i_0+1}$ and $y_{A_1} = \bar{y}_1$ since the points of the clockwise arc, σ_{i_0}, end where the points of the counterclockwise arc, σ_{i_0+1}, start. Thus $[y_{A_1}, y_{A_1-1}] = -[\bar{y}_0, \bar{y}_1]$ and

$$\sum_{i=0}^{N} \sigma_i = \sum_{i=0}^{A_1-1} [y_i, y_{i+1}] + \sum_{i=1}^{A_2} [\bar{y}_i, \bar{y}_{i+1}].$$

Repeating this argument $A = \min\{A_1+1, A_2+1\}$ times, we conclude $[y_{A_1-i}, y_{A_1-i+1}] = -[\bar{y}_i, \bar{y}_{i+1}]$ for $0 \leq i < A$. We note that by construction

$$f\left(\sum_{i=0}^{A_1} [y_i, y_{i+1}] + \sum_{i=0}^{A_2} [\bar{y}_i, \bar{y}_{i+1}]\right) = f(\sigma) \neq 0.$$

Thus $A_1 \neq A_2$; i.e.,

$$\sum_{i=0}^{N} \sigma_i = \sum_{i=0}^{A_1-A_2-1} [y_i, y_{i+1}] \quad \text{or} \quad \sum_{i=0}^{N} \sigma_i = \sum_{i=A_1+1}^{A_2} [\bar{y}_i, \bar{y}_{i+1}].$$ □

Corollary A.3. *Let $\sigma \in C_1(R_\epsilon(S_{\omega,T}))$ and $\partial(\sigma) = 0$. Then σ is homologous to 0 if and only if $f(\sigma) = 0$.*

Proof. By Proposition A.1, σ can be expressed as

$$\sum_{i=0}^{N-1} [x_i, x_{i+1}],$$

where $x_i \in S_{\omega,T}$, $N \in \mathbb{N}$ and $x_N = x_0$. The result follows by noting that in Proposition A.2, $f(\sigma) = 0$ if and only if $A_1 = A_2$, which happens if and only if σ is homologous to 0. □

For the following result we let $\{x_i\}_{i=0}^{T}$ denote the elements of $S_{\omega,T}$ in ascending order, $\oplus$ denote mod $T+1$ addition, and $k \in \mathbb{N}$.

Lemma A.4. *Let $\sigma \in C_1(R_\epsilon(S_{\omega,T}))$ be such that $\partial(\sigma) = 0$ and σ is not homologous to 0. Then σ is homologous to*

$$k \sum_{i=0}^{T} [x_i, x_{i\oplus 1}].$$

Proof. By Proposition A.1, we can express σ as

$$\sum_{i=0}^{N-1} [\bar{x}_i, \bar{x}_{i+1}],$$

where $\bar{x}_i \in S_{\omega,T}$, $N \in \mathbb{N}$, and $\bar{x}_N = \bar{x}_0$. Furthermore, by repeated application of Proposition A.2, σ is homologous to

$$\bar{\sigma} = \sum_{i=0}^{N_0} [y_i, y_{i+1}],$$

where $y_i \in S_{\omega,T}$, $N_0 \in \mathbb{N}$, $y_{N_0+1} = y_0$, y_{i+1} is adjacent to y_i and

$$L_{y_i, y_{i+1}} = \operatorname{sgn}(f(\sigma))$$

for $0 \leq i \leq N_0$. We note that a necessary condition to cancel a term $[y_i, y_{i+1}]$ is the existence of term $[y_{n_i}, y_{n_i+1}]$ for which

$$L_{y_{n_i}, y_{n_i+1}} = -\operatorname{sgn}(f(\sigma)).$$

Since this can't happen, we conclude the expression for $\bar{\sigma}$ can't be reduced. Furthermore, we note that by construction $T \leq N_0$. Indeed, the minimal number j_0 of terms needed to obtain

$$\partial\left(\sum_{i=0}^{j_0} [y_i, y_{i+1}]\right) = 0$$

is $j_0 = T$ since this covers all of the points in $S_{\omega,T}$. We conclude

$$\sum_{i=0}^{T} [y_i, y_{i+1}] = \sum_{i=0}^{T} [x_i, x_{i\oplus 1}].$$

In fact, an application of Proposition A.1 to $\bar{\sigma}$ shows

$$\bar{\sigma} = \sum_{i=0}^{N_2-1} \sigma_i$$

for some $N_2 \in \mathbb{N}$. By the properties of σ_i and what has been shown, we conclude

$$\sigma_i = \sum_{j=0}^{T} [x_j, x_{j\oplus 1}]$$

for all $0 \leq i \leq N_2 - 1$. □

Lemma A.5. *Let $T \in \mathbb{N}$ and $\omega \in \mathbb{R}\setminus\mathbb{Q}$ with continued-fraction expansion $[a_1, a_2, a_3, \dots]$ and i-th convergent p_i/q_i, and k, r and s be the unique numbers for which*

$$q_k + q_{k-1} \leq T < q_k + q_{k+1}$$

and

$$T = rq_k + q_{k-1} + s \quad \text{for } 1 \leq r \leq a_{k+1} \text{ and } 0 \leq s \leq q_k - 1.$$

Let

$$D_i = q_i\omega - p_i.$$

We assume

$$s+1<q_k \quad \textit{and} \quad |D_{k+1}|+(a_{k+1}-r+1)|D_k|<\tfrac{1}{3}.$$

Let Γ *be the set containing the pairs* (x, y) *for* $x, y \in S_{\omega,T}$ *and* $x<y$, *for which there exists a* $z \in ([0, x) \cup (y, 1)) \cap S_{\omega,T}$ *such that* $1-\bar{d}(x, y) \leq 2 \max\{\bar{d}(y, z), \bar{d}(x, z)\} \leq 2\bar{d}(x, y)$. *If*

$$\lambda=\min\{\bar{d}(x, y) \mid (x, y) \in \Gamma\},$$

then for $\epsilon \in (|D_{k+1}|+(a_{k+1}-r+1)|D_k|, \lambda)$,

$$[\sigma]=0$$

in $H_1(R_\epsilon(S_{\omega,T}, \bar{d}); \mathbb{F})$.

Proof. Let us write $\delta_C=|D_{k+1}|+(a_{k+1}-r+1)|D_k|$. Since $\delta_C<\frac{1}{3}$ and ω is irrational, $\lambda>\delta_C$; thus ϵ is well defined. To show the result, it is sufficient to show there does not exists a $\tau \in C_2(R_\epsilon(S_{\omega,T}))$ such that

$$\sigma=\partial(\tau).$$

We will assume such τ exists and arrive at a contradiction. Thus suppose there exists a $\tau \in C_2(R_\epsilon(S_{\omega,T}))$ such that

$$\sigma=\partial(\tau).$$

Let

$$\lambda_0=\max\{\bar{d}(x, y), \bar{d}(y, z), \bar{d}(z, y)\} \mid \text{there exists } [x, y, z] \in \tau\}.$$

Let $y_1, y_2 \in S_{\omega,T}$ for $y_1<y_2$ be such that $\bar{d}(y_1, y_2)=\lambda_0$. By assumption there exists $y_3 \in S_{\omega,T}$ such that $[y_1, y_2, y_3] \in \tau$ or $[y_1, y_3, y_2] \in \tau$. We note that by construction

$$\max\{\bar{d}(y_2, y_3), \bar{d}(y_1, y_3)\} \leq \bar{d}(y_1, y_2).$$

Furthermore, without loss of generality we can assume $y_3 \in ([0, y_1) \cup (y_2, 1))$. Indeed, let us suppose $y_3 \in (y_1, y_2)$. There must exist a term of the form $[y_1, y_2, z_0]$ to cancel the term $-[y_1, y_2]$ in the case of the term $[y_1, y_3, y_2]$. In the case of the term $[y_1, y_2, y_3]$ take $z_0=y_3$. If $z_0 \in ([0, y_1) \cup (y_2, 1))$ we are done. Thus we assume $z_0 \in (y_1, y_2)$. This implies there must exists a term $[y_1, z_0, z_1] \in \tau$ to cancel the term $-[y_1, z_0]$ from $\partial([y_1, y_2, z_0])$. If $z_1 \in (y_1, y_2)$, there must be a term of the form $[y_1, z_1, z_2] \in \tau$ to cancel the term $-[y_1, z_1]$ from $\partial([y_1, z_0, z_1])$. On the other hand, if $z_1 \in ([0, y_1) \cup (y_2, 1))$ we note

$$L_{y_1,z_1}=-L_{y_1,y_2},$$

or otherwise

$$\bar{d}(y_1, z_1)>\bar{d}(y_1, y_2),$$

a contradiction. As before, there must exists a $[z_0, z_2, z_1]$ to cancel the term $[z_0, z_1]$ from $\partial([y_1, z_0, z_1])$. Since in both scenarios we end up in a case analogous to the term $[y_1, y_2, z_0]$, we conclude this process won't end. Thus without loss of generality we assume $y_3 \in ([0, y_1) \cup (y_2, 1))$, as stated. We note

$$L_{y_1,y_2}=L_{y_2,y_3}$$

since otherwise $\bar{d}(y_2, y_3) > \bar{d}(y_1, y_2) = \lambda_0$, a contradiction. Similarly, it must be the case that

$$L_{y_1,y_2} = L_{y_3,y_1}$$

since otherwise $\bar{d}(y_3, y_1) > \bar{d}(y_1, y_2) = \lambda_0$. These two conditions imply

$$1 - \bar{d}(y_1, y_2) = \bar{d}(y_2, y_3) + \bar{d}(y_3, y_1) \le 2\max\{\bar{d}(y_2, y_3), \bar{d}(y_1, y_3)\},$$

i.e., $(y_1, y_2) \in \Gamma$, but since

$$\lambda_0 \le \epsilon < \lambda$$

this is impossible by the minimality of λ. Thus no such τ exists. □

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On dense orbits in the space of subequivalence relations

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We first explain how to endow the space of subequivalence relations of any nonsingular countable equivalence relation with a Polish topology, extending the framework of Kechris' recent monograph on subequivalence relations of probability measure-preserving (p.m.p.) countable equivalence relations. We then restrict to p.m.p. equivalence relations and discuss dense orbits therein for the natural action of the full group and of the automorphism group of the relation. Our main result is a characterization of the subequivalence relations having a dense orbit in the space of subequivalence relations of the ergodic hyperfinite p.m.p. equivalence relation. We also show that in this setup, all orbits under the full group action are meager. We finally provide a few Borel complexity calculations of natural subsets in spaces of subequivalence relations using a natural metric we call the uniform metric. This answers some questions from an earlier version of Kechris' monograph.

1. Introduction

Measured group theory can roughly be defined as the study of countable groups through the partitions into orbits associated to their free probability measure-preserving (p.m.p.) actions on standard probability space. A fundamental notion there is that of an *orbit subgroup*: a countable group Λ is an orbit subgroup of another countable group Γ when Λ admits a free p.m.p. action all of whose orbits are contained in those of a free p.m.p. action of Γ. In this more flexible framework, while the Ornstein–Weiss theorem [1980] tells us that all countable infinite amenable subgroups are orbit subgroups of one another, the Gaboriau–Lyons theorem [2009] characterizes nonamenable groups as being exactly those which admit the free group on two generators as an orbit subgroup. Their result has had many applications, making it possible to extend results which were only known for groups containing free subgroups to the optimal class of all nonamenable groups; see, e.g., [Ioana et al. 2009; Seward 2014].

This motivates the following more general question: given a partition of the space into orbits, what are the possible subpartitions into orbits? In more precise terms, let us call an equivalence relation on a standard probability space (X, μ) nonsingular when it comes from an action of a countable group which preserves μ null sets, and p.m.p. when it comes from an action of a countable group which actually preserves the measure μ. We can now ask our question more precisely: given a nonsingular equivalence relation, what are its possible subequivalence relations? An essential step towards answering such a question is Kechris' remarkable result that when the ambient equivalence relation is p.m.p., the space

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of subequivalence relations can be endowed with a natural Polish topology. This allows descriptive set theory to enter the picture, and as such it is the first building block in Kechris' monograph [2024] on spaces of subequivalence relations, whose initial version appeared online in 2013. We begin by extending this fundamental result to the nonsingular case:

Theorem A (see Theorem 2.6). *Let $\mathcal{R}$ be a nonsingular equivalence relation on a standard probability space. Then its space* $\mathrm{Sub}(\mathcal{R})$ *of subequivalence relations is a Polish space for the topology induced by the measure algebra of $\mathcal{R}$.*

We endow $\mathcal{R}$ with the usual measure obtained by integrating cardinality of vertical fibers, allowing us to make sense of its measure algebra in the above result. In particular, we are identifying subequivalence relations which coincide on a conull set. We refer the reader to Section 2 for details. Our description of the topology on the space of subequivalence relations was already known to Kechris [2024, Section 4.4(1)] in the p.m.p. case.

We actually provide two proofs of Theorem A. The first one is very short and uses the fact that in a measure algebra, converging sequences always admit subsequences whose limit is actually equal to their lim inf (see Proposition 2.3). This observation provides a unifying viewpoint to some special cases noted by Kechris [2024, Theorems 5.1 and 18.3]. The second proof is somehow more natural, in that we directly show that the various axioms of being a subequivalence relation define closed subsets in the measure algebra. Along the way, we study a natural non-Hausdorff topology on the measure algebra that we call the lower topology. This topology has some nice continuity properties which also allow us to give a slick proof of the fact that the map which associates to a subset of an equivalence relation the equivalence relation it generates is Baire class one; see Proposition 3.19, which is a natural generalization of [Kechris 2024, Proposition 19.1].

Our next result is in the p.m.p. setup, and was motivated by the question of dense orbits in the space of subequivalence relations, asked by Kechris in his monograph. Indeed, it is very natural to try to understand subequivalence relations up to isomorphism, and this is precisely what the action of the automorphism group of $\mathcal{R}$ on subequivalence relations of $\mathcal{R}$ encodes. As it turns out, when $\mathcal{R}$ is the ergodic hyperfinite p.m.p. equivalence relation, we obtain the following complete characterization of dense orbits for the natural actions of both the full group $[\mathcal{R}]$ and the automorphism group $\mathrm{Aut}(\mathcal{R})$:

Theorem B (see Corollary 4.13). *For a subequivalence relation $\mathcal{S}$ of the hyperfinite ergodic p.m.p. equivalence relation $\mathcal{R}_0$, the following are equivalent*:

(i) *$\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}_0$.*

(ii) *The $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ is dense in* $\mathrm{Sub}(\mathcal{R}_0)$.

(iii) *The* $\mathrm{Aut}(\mathcal{R}_0)$*-orbit of $\mathcal{S}$ is dense in* $\mathrm{Sub}(\mathcal{R}_0)$.

In this theorem, $\mathcal{S}$ being everywhere of infinite index in $\mathcal{R}$ means that for every $A \subseteq X$ of positive measure, the restriction of $\mathcal{S}$ to A has infinite index in the restriction of $\mathcal{R}$ to A (almost every $\mathcal{R}_{\restriction A}$-class splits into infinitely many $\mathcal{S}_{\restriction A}$-classes; see the end of Section 4.1 for more on this).

Example 1.1. View $\mathcal{R}_0$ as the cofinite equivalence relation on $X = \{0, 1\}^{\mathbb{N}}$ endowed with the probability measure $\mu = \big(\frac{1}{2}(\delta_0 + \delta_1)\big)^{\otimes\mathbb{N}}$, namely $((x_n), (y_n)) \in \mathcal{R}_0$ if and only if there is $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqslant N$. Define $\mathcal{S}$ by $((x_n), (y_n)) \in \mathcal{S}$ if and only if there is $N \in \mathbb{N}$ such that $x_{2n} = y_{2n}$ for all $n \geqslant N$. Then $\mathcal{S}$ is both aperiodic and has everywhere infinite index in $\mathcal{R}_0$ (see Proposition 4.6), so by the above theorem it has a dense orbit in $\mathrm{Sub}(\mathcal{R}_0)$.

Example 1.2. View $\mathcal{R}_0$ as the equivalence relation on $X = \{0, 1\}^{\mathbb{Z}^2}$ endowed with the probability measure $\mu = \big(\frac{1}{2}(\delta_0 + \delta_1)\big)^{\otimes\mathbb{Z}^2}$ generated by the Bernoulli shift of $\mathbb{Z}^2$. Then since the shift is mixing, the subequivalence relation $\mathcal{S}$ generated by $\mathbb{Z} \leqslant \mathbb{Z}^2$ is ergodic. By construction, $\mathcal{S}$ is of infinite index, so since it is ergodic it is everywhere of infinite index. By the above theorem $\mathcal{S}$ has a dense orbit in $\mathrm{Sub}(\mathcal{R}_0)$. Note that by [Kechris 2024, Theorem 10.1], ergodic subequivalence relations actually form a G_δ set, so the Baire category theorem provides us with a dense G_δ subset consisting of *ergodic* subequivalence relations of $\mathcal{R}_0$ satisfying the equivalent conditions of Theorem B.

Our approach to Theorem B is very much inspired by a joint work in preparation with Fima, Mukherjee and Patri, where we similarly characterize dense orbits in the space of von Neumann subalgebras of the hyperfinite type II_1 factor. In this setup, we have to rely on a difficult result of Popa [2019, Lemma 2.3] which characterizes the subalgebras which have a sequence of unitary conjugates converging to the trivial subalgebra. Here we similarly first need to understand which subequivalence relations have a sequence of translates converging to the trivial subequivalence relations $\Delta_X = \{(x, x) : x \in X\}$, which is the content of our next result:

Theorem C (see Theorem 4.7). *Let $\mathcal{R}$ be an aperiodic p.m.p. equivalence relation on a standard probability space (X, μ). Let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be a subequivalence relation. The following are equivalent*:

(i) *$\mathcal{S}$ has everywhere infinite index in $\mathcal{R}$.*

(ii) *The closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains Δ_X.*

(iii) *The closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains Δ_X.*

The proof of Theorem C relies on an inductive construction, using a countable dense subgroup of the full group and the fact that such a subgroup acts highly transitively on almost every orbit, a result due to Eisenmann and Glasner [2016, Theorem 1.19].

Once this is done, we prove Theorem B using the fact that finite equivalence relations are dense in $\mathrm{Sub}(\mathcal{R}_0)$ and that they can always be translated inside any aperiodic subequivalence relation. This allows us to apply Theorem C on a fundamental domain of the finite subequivalence relation we want to approximate. A more general statement on the closed subspace of hyperfinite subequivalence relations can actually be proved this way; see Corollary 4.12.

Using Ioana's intertwining of subequivalence relations [2012, Lemma 1.7] and the two above examples of subequivalence relations, we moreover have the following result, which again follows from a more general statement on the space of hyperfinite subequivalence relations (see Theorem 4.16).

Theorem D (see Corollary 4.17). *Let $\mathcal{R}_0$ be the ergodic p.m.p. hyperfinite equivalence relation. Then all $[\mathcal{R}_0]$-orbits are meager in* $\mathrm{Sub}(\mathcal{R}_0)$.

It then follows from Mycielski's theorem (see, e.g., [Gao 2009, Theorem 5.3.1]) that $\mathcal{R}_0$ contains a continuum of aperiodic subequivalence relations of infinite index which are in pairwise disjoint full group orbits. Moreover, by [Kechris 2024, Proposition 10.1] and Example 1.2, such a continuum can be found inside ergodic subequivalence relations. We leave the following question open:

Question 1.3. Can $\mathrm{Sub}(\mathcal{R}_0)$ contain a comeager $\mathrm{Aut}(\mathcal{R}_0)$-orbit?

This question can also be asked for equivalence relations $\mathcal{R}$ different from $\mathcal{R}_0$, although the existence of dense $\mathrm{Aut}(\mathcal{R})$- or $[\mathcal{R}]$-orbits in their space of subequivalence relations is already an open problem (see the remark preceding Corollary 4.13 for some examples where there are no dense orbits though). Also note that if $\mathcal{R}$ is a type II_∞ or type III ergodic nonsingular equivalence relation, it is not hard to show that there are always dense full group orbits in $\mathrm{Sub}(\mathcal{R})$ (see the remark right before Section 4.4 for a sketch of proof), but the question of comeager orbits is completely open there.

We conclude with a few complexity calculations, answering some questions from an earlier version of [Kechris 2024]. Let us highlight here a way of understanding Kechris' uniform topology on the space of subequivalence relations which we use: given two nonsingular equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$, denote by $\mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)$ the set of Borel subsets A of X such that $\mathcal{R}_{1\restriction A} = \mathcal{R}_{2\restriction A}$, and define

$$d_u(\mathcal{R}_1, \mathcal{R}_2) = 1 - \sup_{A \in \mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)} \mu(A).$$

This defines a metric that we propose to call the uniform metric since it induces the uniform topology defined in [Kechris 2024, Section 4.6], although it looks finer a priori (to see that the two topologies coincide, one needs to use [Ioana et al. 2009, Theorem 1]).

Outline. Along with basic preliminaries, Section 2 provides a first proof of Theorem A using the lim inf. Section 3 contains the second proof, which relies in a more direct way on the topology of the measure algebra, using as well the lower topology that we define and study therein. Section 4 is devoted to the proofs of Theorems B and D in their more general form, and contains in particular the proof of Theorem C in Section 4.2. Finally, Section 5 contains our complexity computations, which use the uniform metric.

2. The Polish topology on the space of subequivalence relations in the nonsingular setup

The main purpose of this section is to extend the topology on the space of subequivalence relations from [Kechris 2024] to the nonsingular setup. To do so, we use the framework of measure algebras, which does yield the right topology for the probability measure-preserving case, as noted in [Kechris 2024, Section 4.4(1)]. This direct approach also sheds a new light on some results from [Kechris 2024], such as Theorem 5.1 and Proposition 4.27.

2.1. ***The measure algebra of a standard σ-finite space.*** Let (Y, λ) be a standard σ-finite measured space, i.e., a standard Borel space endowed with a σ-finite atomless measure. Such spaces are always isomorphic either to $\mathbb{R}$ endowed with the Lebesgue measure or a finite-length interval, also endowed with the Lebesgue measure. The *measure algebra* of (Y, λ) is the space $\mathrm{MAlg}(Y, \lambda)$ of all Borel subsets of X, where we identify $A, B \subseteq X$ Borel as soon as $\lambda(A \triangle B) = 0$.

In order to endow $\mathrm{MAlg}(Y, \lambda)$ with a complete metric, we first have to choose some finite measure μ in the same class as λ, i.e., sharing the same measure-zero sets. Note that this does not change the definition of the measure algebra. We can then equip the measure algebra $\mathrm{MAlg}(Y, \lambda) = \mathrm{MAlg}(X, \mu)$ with the metric $d_\mu(A, B) = \mu(A \triangle B)$.

Lemma 2.1. *Let μ and μ' be two finite measures on Y in the same equivalence class. Then d_μ and $d_{\mu'}$ are uniformly equivalent: given any $\epsilon > 0$, there is $\delta > 0$ such that for all $A, B \in \mathrm{MAlg}(Y, \mu)$, we have that $d_\mu(A, B) < \delta$ implies $d_{\mu'}(A, B) < \epsilon$ and vice versa.*

Proof. By symmetry, given $\epsilon > 0$ it suffices to find a δ such that the direct implication holds. But this is a direct consequence of the fact that μ and μ' are in the same class: indeed by [Cohn 2013, Lemma 4.2.1], there actually is $\delta > 0$ such that whenever $\mu(C) < \delta$, we must have $\mu'(C) < \epsilon$. □

In particular, the above lemma yields that even when λ is infinite, its measure algebra is endowed with a natural topology, independent of the choice of some μ finite in the class of λ. The fact that d_μ is complete is usually proved via the following lemma, which is important to us on its own:

Lemma 2.2. *Let (A_n) be a sequence of elements of $\mathrm{MAlg}(Y, \mu)$ such that $\sum_n \mu(A_n \triangle A_{n+1}) < +\infty$. Let $\liminf A_n = \bigcup_N \bigcap_{n \geqslant N} A_n$. Then the sequence (A_n) converges to $\liminf A_n$.*

Proof. By the Borel–Cantelli lemma, for almost all $x \in X$ there is N such that for all $n \geqslant N$, $x \notin A_n \triangle A_{n+1}$. Letting $\epsilon > 0$, we then find N and a set X_0 such that $\mu(X_0) > 1 - \epsilon$ and for all $x \in X$ and all $n \geqslant N$, we have $x \notin A_n \triangle A_{n+1}$. Now observe that for all $n \geqslant N$, we have $A_n \cap X_0 = \liminf A_n \cap X_0$; in particular $A_n \triangle \liminf A_n \subseteq X \setminus X_0$, which has measure less than ϵ, as wanted. □

Proposition 2.3. *Let (A_n) be a sequence of elements of $\mathrm{MAlg}(Y, \mu)$ which converges to some $A \in \mathrm{MAlg}(Y, \mu)$. Then we can find a subsequence (A_{n_k}) such that*

$$A = \liminf A_{n_k} = \bigcup_N \bigcap_{k \geqslant N} A_{n_k}.$$

Proof. Since (A_n) converges, it is Cauchy; in particular it admits a subsequence $(A_{n_k})_k$ such that for all $k \in \mathbb{N}$, $d_\mu(A_{n_k}, A_{n_{k+1}}) < 1/2^k$. Thus, by the previous lemma, $A_{n_k} \to \liminf A_{n_k} = A$. □

Proposition 2.4. *The metric d_μ is complete, and $\mathrm{MAlg}(Y, \mu)$ is separable.*

Proof. Let (A_n) be a Cauchy sequence, it suffices to show that it has a convergent subsequence. But because the sequence is Cauchy, we may find a subsequence $(A_{n_k})_k$ such that for all $k \in \mathbb{N}$, $\mu(A_{n_k} \triangle A_{n_{k+1}}) < 2^{-k}$. Then $\sum_k \mu(A_{n_k} \triangle A_{n_{k+1}}) < +\infty$ and we get the desired result by applying the previous lemma.

Towards proving separability, first note that since (Y, μ) is standard, we may as well assume it is an interval of finite length endowed with the Lebesgue measure. The regularity of the Lebesgue measure can then be used to show that finite unions of open intervals with rational endpoints are dense in the measure algebra, thus yielding the desired countable dense subset. □

2.2. ***The space of subequivalence relations.*** Given an action of a countable group Γ by Borel bijections on a standard Borel space X, we obtain the associated equivalence relation $\mathcal{R}_\Gamma = \{(x, \gamma x) : x \in X,\ \gamma \in \Gamma\}$ whose equivalence classes are exactly the Γ-orbits. This equivalence relation is a Borel subset of $X \times X$, and its classes are countable: $\mathcal{R}_\Gamma$ is by definition a *countable Borel equivalence relation* (*CBER*, which can conveniently be read as "seeber"). Conversely, the Feldman–Moore theorem ensures that every CBER comes from a Borel action of a countable group; see for instance [Kechris and Miller 2004, Theorem 1.3].

Given such an equivalence relation, define its *Borel pseudofull group* as the set of all partial Borel bijections $\varphi : \operatorname{dom}\varphi \subseteq X \to \operatorname{rng}\varphi \subseteq X$ such that for all $x \in A$, we have $(x, \varphi(x)) \in \mathcal{R}$. By definition, the *Borel full group* is the group made of elements of the Borel pseudofull group whose domain and range are equal to X.

If (X, μ) is now a standard probability space, we say that a CBER is *nonsingular* (or that μ is quasi-invariant, or that $\mathcal{R}$ is nullpreserving) if for all φ in its Borel pseudofull group, we have $\mu(\operatorname{dom}\varphi) = 0$ if and only if $\mu(\operatorname{rng}\varphi) = 0$. The main (and only by the Feldman–Moore theorem) source of such equivalence relations is the following:

Definition 2.5. An action of a countable group Γ by Borel bijections on X is called *nonsingular* (or nullpreserving, or quasipreserving μ) if for all $A \subseteq X$ Borel, we have $\mu(A) = 0$ if and only if $\mu(\gamma A) = 0$.

Remark. By Lemma 2.1, nonsingular actions yield action by uniformly continuous homeomorphisms on the measure algebra $\operatorname{MAlg}(X, \mu)$.

Given $\Gamma \curvearrowright (X, \mu)$ nonsingular, the associated CBER $\mathcal{R}_\Gamma$ is nonsingular, and conversely any nonsingular CBER arises in this manner; see [Kechris and Miller 2004, Proposition 8.1].

Now let $\mathcal{R}$ be a nonsingular CBER. We equip $\mathcal{R}$ with the left measure M defined by

$$M(A) = \int_X |A_x| \, d\mu(x)$$

for all $A \subseteq \mathcal{R}$ Borel. Let m be an equivalent probability measure. Then the measure algebra of $\mathcal{R}$ with respect to M is equal to that of $\mathcal{R}$ with respect to m. This measure is σ-finite since $\mathcal{R}$ can be covered by the graphs of the elements of a countable group Γ such that $\mathcal{R} = \mathcal{R}_\Gamma$ (each such graph has measure 1).

It then follows from Proposition 2.4 that $\operatorname{MAlg}(\mathcal{R}, M)$ is a Polish space. Moreover, by Lemma 2.1, the topology (and actually the uniform structure) do not depend on the choice of m. Since the measure m is not canonical, we will most of the time write our topological space as $\operatorname{MAlg}(\mathcal{R}, M)$.

A Borel subset of $\mathcal{R}$ is called a Borel subequivalence relation if it is an equivalence relation on X. An element of $\operatorname{MAlg}(\mathcal{R}, M)$ is called a subequivalence relation if its equivalence class *contains* an equivalence relation. Denote by $\operatorname{Sub}(\mathcal{R}) \subseteq \operatorname{MAlg}(\mathcal{R}, M)$ the space of subequivalence relations of $\mathcal{R}$. We can now easily prove the following result of Kechris (our proof is moreover motivated by [Kechris 2024, Theorem 5.1], which we have essentially recast as Proposition 2.3).

Theorem 2.6 [Kechris 2024, Section 4.4(1)]. *The space* $\operatorname{Sub}(\mathcal{R})$ *of subequivalence relations of* $\mathcal{R}$ *is a closed subset of* $\operatorname{MAlg}(\mathcal{R}, M)$, *and hence Polish.*

Proof. Observe that the lim inf of any sequence of equivalence relations is an equivalence relation. By Proposition 2.3 the result follows. □

Observe that we did not even use that $\mathcal{R}$ was nonsingular in the above. However, we need nonsingularity for our space to encode the right notion of equality of subequivalence relations as follows:

Proposition 2.7. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two Borel subequivalence relations of a nonsingular CBER $\mathcal{R}$, and suppose $M(\mathcal{S}_1 \triangle \mathcal{S}_2) = 0$. Then there is a full measure $\mathcal{S}_1$- and $\mathcal{S}_2$-invariant set on which $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same.*

Proof. By definition, for almost all x we have $[x]_{\mathcal{S}_1} = [x]_{\mathcal{S}_2}$. Since $\mathcal{S}_1$ is nonsingular, we find a smaller $\mathcal{S}_1$-invariant set X_0 for which the same conclusion holds (if $\mathcal{S}_1 = \mathcal{R}_\Gamma$ we let $X_0 = \bigcap_{\gamma \in \Gamma} \gamma X_1$, where $X_1 = \{x \in X : [x]_{\mathcal{S}_1} = [x]_{\mathcal{S}_2}\}$). □

We will also see in the next section that nonsingularity is important if we want to concretely see our space of subequivalence relations as a closed set, i.e., to say that the axioms of subequivalence relations each define closed sets. Let us conclude this section by pointing out yet another way of understanding the topology which, was pointed out to us by Stefaan Vaes.

Remark. Given a nonsingular equivalence relation $\mathcal{R}$ on (X, μ), consider the space of subalgebras of its von Neumann algebra $L\mathcal{R}$, which is Polish when endowed with the Maréchal topology [1973]; see also [Fima et al. 2024] for a detailed proof. Then it is a result of Aoi [2003] that subequivalence relations of $\mathcal{R}$ are in one-to-one correspondence with subalgebras of $L\mathcal{R}$ containing $\mathrm{L}^\infty(X, \mu)$. It is not hard to see that such subalgebras form a closed set, and the map which associates to $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ the subalgebra $L\mathcal{S}$ of $L\mathcal{R}$ is a homeomorphism (see Theorem E from the recent preprint by Shuoxing Zhou [2024]), yielding yet another way of understanding the Polish topology induced by the measure algebra of $\mathcal{R}$ on $\mathrm{Sub}(\mathcal{R})$. For (many!) other interpretations of this topology in the probability measure-preserving setup, the reader should consult [Kechris 2024].

3. Strong and lower topologies

In this section, we first study the strong topology on the space of subequivalence relations as defined in [Kechris 2024], which will be useful towards establishing our orbit density result (Theorem B). We then use a lower topology on the measure algebra so as to obtain a natural proof of the fact that the space of subequivalence relations is closed in the measure algebra, and of the fact that the map which takes a subgraph to the equivalence it generates is Baire class one [Kechris 2024, Proposition 19.1].

3.1. ***Topologies on measure algebras of σ-finite spaces.*** We first go back to measure algebras of σ-finite spaces, which we now write as (Y, M), motivated by our previous example where Y is a nonsingular CBER.

Proposition 3.1. *Let (Y, M) be a standard σ-finite space. Let (Y_n) be a countable family of finite-measure subsets of Y such that $\bigcup_n Y_n = Y$. Then the map $\Phi : \mathrm{MAlg}(Y, M) \to \prod_n \mathrm{MAlg}(Y_n, M_{Y_n})$ which takes A to the sequence $(A \cap Y_n)$ is a homeomorphism onto its image.*

Proof. Observe that the probability measure

$$m(A) := \sum_n \frac{1}{2^{n+1} M(Y_n)} M(A \cap Y_n)$$

is equivalent to M, and thus the associated metric d_m induces the topology of $\mathrm{MAlg}(Y, M)$. Now the space $\prod_n \mathrm{MAlg}(Y_n, M_{Y_n})$ can be endowed with a compatible metric $\tilde{d}$ given by

$$\tilde{d}((A_n), (B_n)) = \sum_n \frac{1}{2^{n+1} M(Y_n)} M(A_n \triangle B_n).$$

It is then straightforward to check that Φ is an isometry for the metric d_m on its domain and $\tilde{d}$ on its range. □

We now go on to obtain a more precise version of this result, following Kechris' approach to the *strong topology* on $\mathrm{Sub}(\mathcal{R})$.

Observe that every element φ of the Borel pseudofull group of a CBER $\mathcal{R}$ is completely determined by its *graph*, namely the set $\{(x, \varphi(x)) : x \in \operatorname{dom}\varphi\}$, which is a Borel subset of $\mathcal{R}$. From now on, we identify elements of the Borel pseudofull group to their graph. Doing this up to measure zero when $\mathcal{R}$ is nonsingular, we arrive at the following definition:

Definition 3.2. The *pseudofull group* of a nonsingular equivalence relation $\mathcal{R}$ on (X, μ) is the quotient of the Borel pseudofull group by the equivalence relation which identifies φ_1 and φ_2 when $M(\varphi_1 \triangle \varphi_2) = 0$. It is denoted by $[\![\mathcal{R}]\!]$, and for ease of notation we still write its elements as φ's. Finally the *full group* of $\mathcal{R}$ is the group $[R]$ of all elements of $[\![\mathcal{R}]\!]$ with full measure domain and range.

By the definition of M we have that $M(\varphi_1 \triangle \varphi_2) = 0$ if and only if $\mu(\operatorname{dom}\varphi_1 \triangle \operatorname{dom}\varphi_2) = 0$ and for almost all $x \in \operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2$, we have $\varphi_1(x) = \varphi_2(x)$. The Borel pseudofull group is stable under countable increasing unions and arbitrary countable intersections, so it is stable under taking $\liminf$. Following the same approach as for the proof of Theorem 2.6, one obtains that $[\![\mathcal{R}]\!]$ is a closed subset of $\mathrm{MAlg}(\mathcal{R}, M)$, in particular it is Polish. Note however that $[\mathcal{R}]$ is not closed for this topology (see Example 3.6).

We now make a slight modification of Kechris' uniquely generating sequences of involutions. Let us say that an element φ of the Borel pseudofull group is a *moving partial involution* if $\operatorname{dom}\varphi = \operatorname{rng}\varphi$ and for all $x \in \operatorname{dom}\varphi$, we have both $\varphi(x) \neq x$ and $\varphi(\varphi(x)) = x$.

Definition 3.3. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ of moving partial involutions of the Borel pseudofull group of a CBER $\mathcal{R}$ is called a *uniquely generating sequence of moving partial involutions* if

$$\mathcal{R} \setminus \Delta_X = \bigsqcup_{n\in\mathbb{N}} \varphi_n,$$

where $\Delta_X = \{(x, x) : x \in X\}$ is the equality relation.

Proposition 3.4. *Every CBER $\mathcal{R}$ admits a uniquely generating sequence of moving partial involutions.*

Proof. Suppose that $(T_n)_{n\in\mathbb{N}}$ is a uniquely generating sequence of involutions for $\mathcal{R}$ as in [Kechris 2024, Proposition 4.6], and for every $n \in \mathbb{N}$ let φ_n be the restriction of T_n to its support. □

Remark. The disjointness of the graphs of the φ_n simplifies a bit the arguments we carry out in this paper since it guarantees that all the $\varphi_i(x)$ are distinct whenever they are defined (see in particular the proof of (i) $\Longrightarrow$ (ii) in Theorem 4.7). The disjointness also allows us to upgrade Proposition 3.1 and obtain that the map

$$\mathrm{MAlg}(\mathcal{R}, M) \to \mathrm{MAlg}(\Delta_X, M_{\Delta_X}) \times \prod_{n\in\mathbb{N}} \mathrm{MAlg}(\varphi_n, M_{\varphi_n})$$

which takes A to the sequence $(A \cap Y_n)$ is not only a homeomorphism onto its image, but actually surjective.

The following corollary is a slight reformulation of [Kechris 2024, Theorem 4.13] which identifies the strong and the weak topology on the space of subequivalence relations:

Corollary 3.5. *Let $\mathcal{R}$ be a nonsingular equivalence relation, and (φ_n) be a uniquely generating sequence of moving partial involutions. Then the map*

$$\mathrm{Sub}(\mathcal{R}) \to \prod_{n\in\mathbb{N}} \mathrm{MAlg}(\varphi_n, M_{\varphi_n})$$

which takes $\mathcal{S}$ to the sequence $(\mathcal{S} \cap \varphi_n)_{n\in\mathbb{N}}$ is a homeomorphism onto its image.

Proof. This is a direct consequence of the definition of uniquely generating sequences of moving partial involutions and of Proposition 3.1, once we note that every equivalence relation contains Δ_X. ☐

Following the above corollary and Kechris' terminology, we now define the *strong topology* to be the topology induced by d_m on $\mathrm{Sub}(\mathcal{R})$, which is Polish by Theorem 2.6.

3.2. ***Lower topologies on measure algebras.*** Let us first work a bit more on the relationship between M and m. Observe that M defines a lower-semicontinuous function on $\mathrm{MAlg}(Y, M)$ (possibly taking $+\infty$ as a value). Indeed if $M(A) > \alpha$, we have a $\delta > 0$ such that $M(C) < M(A) - \alpha$ whenever C is a subset of A satisfying $m(C) < \delta$, and so as soon as $m(A \setminus B) < \delta$ we have $M(B) > \alpha$. In particular for every $k \in \mathbb{N}$ the set of elements with measure $\leqslant k$ is closed.

Let us denote by $\mathrm{MAlg}_f(\mathcal{R}, M)$ the F_σ subset of elements of $\mathrm{MAlg}(\mathcal{R}, M)$ with finite M-measure. It can be endowed with the metric $d_M(A, B) = M(A \triangle B)$, which clearly refines the topology induced by d_m. Moreover, the proof of Proposition 2.4 can easily be adapted to show that d_M is complete separable (see [Le Maître 2022, Lemma 2.1] for a complete proof). Let us see why d_m and d_M yield different topologies on $\mathrm{MAlg}_f(\mathcal{R}, M)$.

Example 3.6. Going back to the case $Y = \mathcal{R}$, observe that if T in the full group of $\mathcal{R}$ is aperiodic, then $T^n \to \varnothing$ for d_m while it stays in the sphere around 1 for d_M. In particular d_m does not refine d_M. Let us also mention that $[\mathcal{R}]$ is d_M-closed, and that the metric induced by d_M on $[\mathcal{R}]$ is the well-known uniform metric $d_u(S, T) = \mu(\{x \in X : S(x) \neq T(x)\})$, endowing $[\mathcal{R}]$ with a Polish group topology.

Despite the above example, M and m share the same *lower topologies* when restricted to $\mathrm{MAlg}_f(Y, M)$. These lower topologies are not Hausdorff; however they have nice continuity properties.

Definition 3.7. Let (Y, M) be a σ-finite measured space. We endow its measure algebra with the *lower M-topology* which is defined by taking as neighborhoods of an element $A \in \mathrm{MAlg}(Y, M)$ all sets of the form

$$\{B \in \mathrm{MAlg}(Y, M) : M(A \setminus B) < \epsilon\}$$

for some $\epsilon > 0$.

This neighborhood system does satisfy the axiom that implies that it is a basic neighborhood system for the topology it induces, namely every neighborhood V contains a smaller neighborhood all of whose elements admit V as a neighborhood (axiom (V_{IV}) in [Bourbaki 1989, Chapter 1, §1.2]). This is simply because if $M(A \setminus B) < \frac{1}{2}\epsilon$ and $M(B \setminus C) < \frac{1}{2}\epsilon$ then $M(A \setminus C) < \epsilon$.

Note that the only neighborhood of the empty set is the whole measure algebra, while Y is contained in every neighborhood of every point. Moreover, $\{\varnothing\}$ is closed, and more generally M is lower semicontinuous for the lower M-topology.

Lemma 3.8. *If M is a σ-finite measure on Y and m is an equivalent finite measure, then the measures M and m induce the same lower topology on the space $\mathrm{MAlg}_f(Y, M)$ of finite M-measure subsets.*

Proof. Take A such that $M(A) < +\infty$. Because M and m, when restricted to A, are equivalent, we find δ such that every subset of A of m-measure less than δ has M-measure less than ϵ. Then $\{B : m(A \setminus B) < \delta\} \subseteq \{B : M(A \setminus B) < \epsilon\}$. So the lower m-topology refines the lower M-topology. The other inequality works the same. □

Let us note that the space of infinite measure subsets is open for the lower M-topology, while it is not for the lower m-topology. Indeed if A has infinite measure, the set of B such that $M(A \setminus B) < 1$ does not contain any m-neighborhood, because there are subsets of A of very small m-measure but still with infinite measure.

We will now list various interesting properties of the lower topology, and then relate it to the usual topology in the last two lemmas.

Lemma 3.9. *Let M be a σ-finite measure on Y. Then the intersection map $(A_1, A_2) \mapsto A_1 \cap A_2$ is continuous, where we put the M-lower topology everywhere.*

Proof. Let $\epsilon > 0$. Note that $(A_1 \cap A_2) \setminus (B_1 \cap B_2) \subseteq (A_1 \setminus B_1) \cup (A_2 \setminus B_2)$, so if $M(A_1 \setminus B_1) < \epsilon$ and $M(A_2 \setminus B_2) < \epsilon$, then $M((A_1 \cap A_2) \setminus (B_1 \cap B_2)) < 2\epsilon$ as wanted. □

Lemma 3.10. *Let m be a finite measure on Y. Then the countable union map $(A_n)_{n\in\mathbb{N}} \mapsto \bigcup_{n\in\mathbb{N}} A_n$ is continuous for the m-lower topology everywhere.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a family of elements of the measure algebra. Then since m is finite, we may find N such that $\mu\big(\big(\bigcup_n A_n\big) \setminus \bigcup_{n<N} A_n\big) < \epsilon$. Now if we take $B_0, \dots, B_{N-1}$ such that $m(A_i \setminus B_i) < \epsilon/N$, and any $B_N, B_{N+1}, \dots$, we see that $m\big(\big(\bigcup_n A_n\big) \setminus \big(\bigcup_n B_n\big)\big) < 2\epsilon$, as wanted. □

Remark. The countable union map is *not* continuous if we put the usual topology everywhere (indeed X can easily be obtained as a limit of unions of sequences of sets which converge coordinatewise to $\varnothing$). Here is nevertheless a positive result for the usual topology which will be useful later on.

Lemma 3.11. *Let m be a finite measure on Y, and $(Y_j)_{j\in\mathbb{N}}$ be a disjoint family of measurable subsets of Y. Then the restriction of the countable union map to the set*

$$\{(A_n) \in \mathrm{MAlg}(Y, m)^{\mathbb{N}} : \forall n \in \mathbb{N}, A_n \subseteq Y_n\},$$

is continuous for the d_m-topology.

Proof. Since the Y_n are disjoint and m is finite, given any $\epsilon > 0$ we can find N such that $m\big(\bigcup_{n>N} Y_n\big) < \epsilon$; in particular if (A_n) satisfies $A_n \subseteq Y_n$ for every n then its union is, up to an ϵ error, independent of $(A_n)_{n>N}$. The conclusion thus follows from the continuity of finite unions for the d_m-topology. □

We now relate the Hausdorff topology associated to the metric d_m and to the corresponding lower m-topology:

Lemma 3.12. *Let m be a finite measure on Y, and denote by τ_m^l the lower m-topology and by τ_m the topology induced by the metric d_m. Then the identity map $(\mathrm{MAlg}(Y, m), \tau_l^m) \to (\mathrm{MAlg}(Y, m), \tau_m)$ is Baire class one.*

Proof. We begin by noting some useful continuity properties for the lower m-topology. By definition, for every $A \in \mathrm{MAlg}(Y, m)$ the map $B \mapsto m(A \setminus B)$ is upper semicontinuous for the lower topology. Since $m(B) = m(X) - m(X \setminus B)$, we deduce that m defines a lower semicontinuous map $(\mathrm{MAlg}(Y, m), \tau_m^l) \to \mathbb{R}^+$. By Lemma 3.9, for every $A \in \mathrm{MAlg}(Y, m)$ the map $B \mapsto m(A \cap B)$ is thus lower semicontinuous as well.

We finally observe that

$$d_m(A, B) = m(A \triangle B) = m(A) + m(B) - 2m(A \cap B),$$

so $d_m(A, \cdot)$ is a Baire class one function on $(\mathrm{MAlg}(Y, \nu), \tau_m^l) \to \mathbb{R}^+$, being the sum of two Baire class one functions. In particular, d_m-open balls are F_σ for τ_m^l. Since every τ_m-open set is a countable union of open balls by separability, the conclusion follows. □

We conclude this section by a lemma which intertwines the two topologies we have defined in presence of a finite measure:

Lemma 3.13. *Let m be a finite measure on Y. The space of all (A, B) such that $A \subseteq B$ is closed if we put on the first coordinate the m-lower topology, and on the second the usual measure algebra topology.*

Proof. Suppose that $A \nsubseteq B$, and let $\epsilon = m(A \setminus B)$. Then if $m(A \setminus A') < \frac{1}{2}\epsilon$ and $m(B' \triangle B) < \frac{1}{2}\epsilon$, we still have $A' \nsubseteq B'$. □

3.3. ***Seeing that the space of subequivalence relations is closed.*** We now provide a more concrete proof that the space of subequivalence relations is Polish by seeing that the conditions which make an equivalence relation define closed sets in $\mathrm{MAlg}(\mathcal{R}, M)$ for the topology induced by d_m. First note that reflexivity means containing Δ_X, and hence defines a closed set by Lemma 3.13.

Let us fix a nonsingular equivalence relation $\mathcal{R}$. We recall that m is some finite measure equivalent to the measure M obtained by integrating the cardinalities of vertical fibers. Observe that the flip

$\sigma : (x, y) \mapsto (y, x)$ quasipreserves M (this is one characterization of nonsingularity, see [Kechris and Miller 2004, Proposition 8.2]); in particular it quasipreserves m and hence induces a homeomorphism on $\mathrm{MAlg}(\mathcal{R}, M)$.

This implies that the space of (Borel) *subgraphs* of $\mathcal{R}$ (symmetric subsets of $\mathcal{R} \setminus \mathrm{id}_X$) is closed in $\mathrm{MAlg}(\mathcal{R}, M)$, because it is the set of fixed points of σ contained in $\mathcal{R} \setminus \mathrm{id}_X$. In particular, we recover the following fact from [Kechris 2024, Chapter 18]:

Proposition 3.14. *The space of subgraphs is Polish for the topology induced by the measure algebra* $\mathrm{MAlg}(\mathcal{R}, M)$. □

Remark. This could also be seen through the same approach as for Theorem 2.6, noting that the liminf of any sequence of subgraphs is a subgraph.

To see that the space of subequivalence relations is closed, we need to deal with transitivity. Observe that on the space of measurable subsets of $\mathcal{R}$, we have an operation $\circ$ defined by

$$A \circ B := \{(x, z) : \exists y \in X \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\}.$$

Note that $\circ$ extends the composition of elements of the pseudofull group of $\mathcal{R}$, and that it is well-defined only because we are in the nonsingular setup.

Lemma 3.15. *On the pseudofull group of $\mathcal{R}$, the composition is continuous for the M-lower topology, and hence for the m-lower topology.*

Proof. Note that the M-measure of the graph of an element of the pseudofull group is the μ-measure of its domain, so by Lemma 3.8 we only need to check the continuity for the lower M-topology.

Now fix $\varphi_0, \psi_0 \in [\![\mathcal{R}]\!]$, let $\epsilon > 0$ and find δ such that whenever $\mu(A) < \delta$ we have $\mu(\psi_0^{-1}(A)) < \frac{1}{2}\epsilon$. Take φ such that $M(\varphi \setminus \varphi_0) < \delta$. Then $\mu(\{x \in \mathrm{dom}(\varphi_0) : \varphi(x) \neq \varphi_0(x)\}) < \delta$ (in the definition we consider that $\varphi_0(x)$ is different from $\varphi(x)$ if $x \notin \mathrm{dom}\,\varphi$). Then, taking ψ such that $M(\psi \setminus \psi_0) < \frac{1}{2}\epsilon$, the set $X_1 = \{x \in \mathrm{dom}\,\psi_0 : \psi(x) = \psi_0(x)\}$ contains $\mathrm{dom}\,\psi_0$ up to an $\frac{1}{2}\epsilon$ error, and

$$\psi_{\restriction X_1}^{-1}(\{x \in \mathrm{dom}(\varphi_0) : \varphi(x) \neq \varphi_0(x)\}) = \psi_{0\restriction X_1}^{-1}(\{x \in \mathrm{dom}(\varphi_0) : \varphi(x) \neq \varphi_0(x)\})$$

has thus measure at most $\frac{1}{2}\epsilon$. We conclude that the set of $x \in \mathrm{dom}\,\varphi_0\psi_0$ such that $\varphi\psi(x) = \varphi_0\psi_0(x)$ contains $\mathrm{dom}\,\varphi_0\psi_0$ up to an $\frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ error, as wanted. The fact that the continuity also holds for the m-lower topology is then a direct consequence of Lemma 3.8 since elements of $[\![\mathcal{R}]\!]$ have M-measure at most 1. □

Lemma 3.16. *The map $A, B \mapsto A \circ B$ is continuous if we put everywhere the lower m-topology.*

Proof. Fixing a countable subgroup $\Gamma \leqslant [\mathcal{R}]$ such that $\mathcal{R} = \mathcal{R}_\Gamma$, the desired continuity follows from the previous lemma along with the formula

$$A \circ B = \bigcup_{\gamma, \gamma' \in \Gamma} \gamma \cap A \circ (\gamma' \cap B)$$

and the fact that finite intersections and countable unions are continuous maps for the lower topology (Lemmas 3.9 and 3.10). □

We thus arrive at the desired result saying that the space of transitive subrelations is closed.

Proposition 3.17. *The space of* $A \subseteq \mathcal{R}$ *such that* $A \circ A \subseteq A$ *is closed in the Polish topology on* $\mathrm{MAlg}(\mathcal{R}, M)$.

Proof. This follows directly from the previous lemma along with Lemma 3.13. □

Theorem 3.18. $\mathrm{Sub}(\mathcal{R})$ *is a closed subset of* $\mathrm{MAlg}(\mathcal{R}, M)$ *for the* d_m*-topology.*

Proof. We already observed that being reflexive exactly means containing Δ_X and thus is a closed condition by Lemma 3.13. The fact that symmetry is a closed condition is a reformulation of the fact that subgraphs form a closed set (Proposition 3.14). Finally transitivity is a closed condition by the previous proposition. So $\mathrm{Sub}(\mathcal{R})$ is the intersection of three closed subsets, and hence closed itself. □

We also have the following nice consequence of our work on the lower topology, which generalizes Proposition 19.1 from [Kechris 2024].

Proposition 3.19. *The map which takes a measurable subset of* $\mathcal{R}$ *to the equivalence relation it generates is continuous in the lower topology*; *in particular it is Baire class one for the strong topology* (*induced by* d_m).

Proof. This map associates to a $A \subseteq \mathcal{R}$ the equivalence relation $\bigcup_{n\in\mathbb{N}}(\Delta_X \cup A \cup \sigma(A))^{\circ n}$, so it is a continuous map by the continuity of σ, of countable unions (Lemma 3.10) and of composition (Lemma 3.16). The statement about the strong topology then follows directly from Lemma 3.12. □

We can finally obtain the following strengthening of Proposition 4.27 from [Kechris 2024]:

Corollary 3.20. *The map which takes a sequence* $(\mathcal{R}_n)$ *of subequivalence relations of* $\mathcal{R}$ *to the equivalence relation* $\bigvee_n \mathcal{R}_n$ *they generate is continuous in the lower topology*; *in particular it is Baire class one for the strong topology.*

Proof. This is a direct consequence of the previous result along with the fact that taking countable unions is continuous in the lower topology (Lemma 3.10). □

4. Dense orbits for hyperfinite subequivalence relations in the p.m.p. case

In the present section, we exclusively work in the probability measure-preserving setup, so $\mathcal{R}$ is a p.m.p. equivalence relation on (X, μ). Recall that $\mathrm{Aut}(\mathcal{R})$ is the group of all $T \in \mathrm{Aut}(X, \mu)$ such that $T \times T\,(\mathcal{R}) = \mathcal{R}$. Observe that $\mathrm{Aut}(\mathcal{R})$ acts on $\mathrm{MAlg}(\mathcal{R}, M)$ by $T \cdot (x, y) = (T(x), T(y))$ and preserves the measure M. The group $\mathrm{Aut}(\mathcal{R})$ contains the full group of $\mathcal{R}$, and it is a Polish group for the topology induced by the group $\mathrm{Aut}(\mathcal{R}, M)$ of all measure-preserving bijections of $(\mathcal{R}, M)$; see [Kechris 2010, Proposition 6.3] where $\mathrm{Aut}(\mathcal{R})$ is denoted by $N[\mathcal{R}]$. The trivial subequivalence relation will play an important role; we denote it by $\Delta_X = \{(x, x) : x \in X\}$ (given our identification of full groups elements to their graphs, we could also write it as the identity map id_X).

4.1. ***Preliminaries on conditional measures and index.*** In order to work in the p.m.p. setup, we have to understand which sets can be taken to other sets by (pseudo)full group elements up to measure zero. This is done here through the concept of *conditional measures*, a low-tech version of the ergodic decomposition which was already present in Dye's founding paper [1959]. We already used them in a previous paper on nonergodic p.m.p. equivalence relations (see [Le Maître 2016, Section 2]) but we recently gave a more detailed exposition for general full groups in a joint work with Slutsky, which we use here as a reference (see Appendix D in [Le Maître and Slutsky 2021]).

Definition 4.1. Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ), and denote by $M_{\mathcal{R}}$ the closed subalgebra of (X, μ) consisting of $[\mathcal{R}]$-invariant subsets and by $\mathbb{E}_{\mathcal{R}}$ the projection $\mathrm{L}^2(X, \mu) \to \mathrm{L}^2(X, M_{\mathcal{R}}, \mu)$. Given $A \in \mathrm{MAlg}(X, \mu)$, its *$M_{\mathcal{R}}$-conditional measure $\mu_{\mathcal{R}}$* is the $M_{\mathcal{R}}$-measurable function

$$\mu_{\mathcal{R}}(A) = \mathbb{E}_{M_{\mathcal{R}}}(\chi_A).$$

It can be checked that $\mu_{\mathcal{R}}$ takes values in [0, 1], satisfies the usual axioms of a measure, and that elements of $[\mathcal{R}]$ preserve $\mu_{\mathcal{R}}$; see [Le Maître and Slutsky 2021, Proposition D.6]. Say that $\mathcal{R}$ is *ergodic* when $M_{\mathcal{R}} = \{\varnothing, X\}$. Then $\mathcal{R}$ is ergodic if and only if $\mu_{\mathcal{R}} = \mu$. By [Le Maître and Slutsky 2021, Proposition D.10], we moreover have:

Lemma 4.2. *Let $A, B \in \mathrm{MAlg}(X, \mu)$, and let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ). The following are equivalent*:

(i) *There is $\varphi \in [\![\mathcal{R}]\!]$ such that* $\operatorname{dom}\varphi = A$ *and* $\operatorname{rng}\varphi = B$.

(ii) $\mu_{\mathcal{R}}(A) = \mu_{\mathcal{R}}(B)$.

We finally note the following important consequence of aperiodicity (having only infinite classes) for a p.m.p. equivalence relation (it is actually a characterization, but we don't need that):

Proposition 4.3 (Maharam's lemma; see [Le Maître and Slutsky 2021, Theorem D.12]). *Let $\mathcal{R}$ be a p.m.p. aperiodic equivalence relation, let $A \in \mathrm{MAlg}(X, \mu)$, and let $f : X \to [0, 1]$ be an $M_{\mathcal{R}}$-measurable function such that $0 \leqslant f \leqslant \mu_{\mathcal{R}}(A)$. Then there is $B \subseteq A$ such that $\mu_{\mathcal{R}}(B) = f$.*

We finish this section by recalling some important definitions on the index of subequivalence relations:

Definition 4.4. Let $\mathcal{R}$ be a nonsingular equivalence relation on (X, μ). We say that $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$

- has *infinite index* in $\mathcal{R}$ if for almost all $x \in X$, the $\mathcal{R}$-equivalence class of x contains infinitely many distinct $\mathcal{S}$-classes,
- has *finite index* in $\mathcal{R}$ if for almost all $x \in X$, the $\mathcal{R}$-equivalence class of x is the union of a finite set of $\mathcal{S}$-classes,
- has *everywhere infinite index* in $\mathcal{R}$ if for all $A \subseteq X$ such that $\mu(A) > 0$, the restriction of $\mathcal{S}$ to A has infinite index in the restriction of $\mathcal{R}$ to A.

Remark. It might not be clear at first sight that having infinite index is not the same as having everywhere infinite index. Here is the simplest example of an infinite-index $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$, not everywhere of infinite index: take $\mathcal{R}$ ergodic, let $A \subseteq X$ be of positive nonfull measure, and let $\mathcal{S} = \mathcal{R}_{\restriction A} \sqcup \Delta_{X \setminus A}$.

Let us note that when $\mathcal{R}$ is ergodic, since the number of $\mathcal{S}$-classes inside the $\mathcal{R}$-class of $x \in X$ is $\mathcal{R}$-invariant, it is constant almost everywhere, and hence $\mathcal{S}$ either has finite or infinite index in $\mathcal{R}$. Similarly, if $\mathcal{S}$ is ergodic and of infinite index, then it is everywhere of infinite index because the $\mathcal{S}$-class of almost every $x \in X$ intersects A. We finally note the following nonergodic way of producing everywhere infinite index subequivalence relations:

Lemma 4.5. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be nonsingular aperiodic equivalence relations on respective standard probability spaces (X_1, μ_1) and (X_2, μ_2). Then $\mathcal{R}_1 \times \Delta_{X_2}$ is everywhere of infinite index in $\mathcal{R}_1 \times \mathcal{R}_2$.*

Proof. Let $A \subseteq X_1 \times X_2$ be of positive measure. By Fubini's theorem, for almost all $(x_1, x_2) \in A$ the vertical section $A_{x_1} = \{x_2' \in X_2 : (x_1, x_2' \in A)\}$ has positive measure, and hence by aperiodicity the $\mathcal{R}_2$-class of x_2 intersects A_{x_1} in an infinite set. This implies that the $(\mathcal{R}_1 \times \mathcal{R}_2)_{\restriction A}$-class of (x_1, x_2) contains infinitely many $(\mathcal{R}_1 \times \Delta_{X_2})_{\restriction A}$-classes. □

Proposition 4.6. *The equivalence relation $\mathcal{S}$ from Example 1.1 is everywhere of infinite index in $\mathcal{R}_0$.*

Proof. The even–odd partition of $\mathbb{N}$ induces a bijection $\{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ under which $\mathcal{R}_0$ becomes $\mathcal{R}_0 \times \mathcal{R}_0$ and $\mathcal{S}$ becomes $\mathcal{R}_0 \times \Delta_{\{0,1\}^{\mathbb{N}}}$, so the result follows from the previous lemma. □

4.2. *Approximating the diagonal.* Recall that Δ_X denotes the equality relation on X. We will now characterize equivalence relations whose orbit closure contains Δ_X, mirroring Popa's result [2019, Lemma 2.3] on asymptotic orthogonalization of subalgebras of a finite factor.

Theorem 4.7. *Let $\mathcal{R}$ be an aperiodic p.m.p. equivalence relation. Let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be a subequivalence relation. The following are equivalent*:

(i) *$\mathcal{S}$ has everywhere infinite index in $\mathcal{R}$.*

(ii) *The closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains Δ_X.*

(iii) *The closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains Δ_X.*

Proof. The implication (ii) $\Longrightarrow$ (iii) is clear since $[\mathcal{R}] \leqslant \mathrm{Aut}(\mathcal{R})$.

Let us then check that (iii) implies (i) by proving the contrapositive: assuming that $\mathcal{S}$ does not have infinite index everywhere, we need to show that the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ does not contain Δ_X.

By assumption, we have a subset $A \subseteq X$ of positive measure such that the restriction $\mathcal{S}_{\restriction A}$ has finite index in $\mathcal{R}_{\restriction A}$. Shrinking A further if necessary, we may assume the index of the restriction of $\mathcal{S}$ to A in the restriction of $\mathcal{R}$ to A is constant equal to k. Let $N \in \mathbb{N}$ such that $1/N < \mu(A)$.

By aperiodicity and Maharam's lemma with $f = 1/(2kN)$ (see Proposition 4.3), we can partition X in $2kN$ pieces of equal $\mathcal{R}$-conditional measure. Using Lemma 4.2, we thus have $B \subseteq X$ and $\varphi_1, \dots, \varphi_{2kN} \in [\![\mathcal{R}]\!]$ with domain B such that $\varphi_1 = \mathrm{id}_B$ and the sets $\varphi_1(B), \dots, \varphi_{2kN}(B)$ partition X.

Let B_0 be the set of $x \in B$ such that there are at least $k+1$ distinct indices $i \in \{1, \dots, 2kN\}$ such that $\varphi_i(x) \in A$. Write $A_0 = \{x \in A : \exists i, \varphi_i^{-1}(x) \in B_0\}$. Then since A_0 is covered by the disjoint translates of B_0, we have $\mu(A_0) \leqslant 2kN\mu(B_0)$. Letting $A_1 = A \setminus A_0$, we again have that A_1 is covered by the disjoint translates of $B_1 = B \setminus B_0$, but by definition for every $x \in B_1$ there are at most k indices $i \in \{1, \dots, 2kN\}$ such that $\varphi(x) \in A_1$, so that

$$\mu(A_1) \leqslant k\mu(B_1) \leqslant \frac{k}{2kN}.$$

Since $A = A_0 \sqcup A_1$, we then have $\mu(A) \leqslant k/(2kN) + 2kN\mu(B_0)$. But $1/N < \mu(A)$ so $1/2N < 2kN\mu(B_0)$, and hence

$$\mu(B_0) \geqslant \frac{1}{2kN^2}.$$

Now for all $x \in B_0$, because the index of $\mathcal{S}_{\restriction A}$ in $\mathcal{R}_{\restriction A}$ is k, we must have two distinct i, j such that $(\varphi_i(x), \varphi_j(x)) \in \mathcal{S}$. In particular,

$$\sum_{1 \leqslant i < j \leqslant 2N} M(\varphi_i^{-1}\varphi_j \cap \mathcal{S}) \geqslant \frac{1}{2kN^2}.$$

The latter estimate will actually be valid for every translate of $\mathcal{S}$ by $T \in \mathrm{Aut}(\mathcal{R})$ because $T(A)$ will still satisfy $1/N < \mu(T(A))$, so for every $T \in \mathrm{Aut}(\mathcal{R})$,

$$\sum_{1 \leqslant i < j \leqslant 2N} M(\varphi_i^{-1}\varphi_j \cap T \cdot \mathcal{S}) \geqslant \frac{1}{2kN^2}.$$

This inequality defines a closed set of subequivalence relations, but it is not satisfied by Δ_X, which finishes the proof of (iii) $\Longrightarrow$ (i).

We finally prove that (i) implies (ii). We fix $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ everywhere of infinite index. Let us also fix a uniquely generating sequence (φ_k) of moving partial involutions of $\mathcal{R}$ as provided by Proposition 3.4. Applying Proposition 3.1 to the sequence of M-finite measure subsets φ_k, it suffices to show that given some $k \in \mathbb{N}$ and $\epsilon > 0$, we can find $T \in [\mathcal{R}]$ such that for all $i \in \{1, \dots, k\}$,

$$M((T^{-1} \cdot \mathcal{S}) \cap \varphi_i) < \epsilon.$$

We will actually do better and construct $T \in [\mathcal{R}]$ such that for all $i \in \{1, \dots, k\}$, $M((T^{-1} \cdot \mathcal{S}) \cap \varphi_i) = 0$. Observe that it now suffices (and it is necessary) to build T such that for almost all $x \in \operatorname{dom} \varphi_i$ and all $i \in \{1, \dots, k\}$,

$$(T(x), T\varphi_i(x)) \notin \mathcal{S}.$$

Our construction makes a crucial use of a result of Eisenmann and Glasner [2016, Proposition 1.19]: letting $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$ be a dense subgroup of $[\mathcal{R}]$, we have that Γ acts highly transitively on the orbit of almost every $x \in X$, and we may as well restrict ourselves to the set of all such x. By the definition of high transitivity, this means that for all $x \in X$, every partial bijection between finite subsets of Γx is the restriction of some $\gamma \in \Gamma$.

We will now build T as the increasing union of elements ψ_n of the pseudofull group defined inductively as follows.

First, ψ_0 is the restriction of γ_0 to the set of all $x \in X$ such that for all $i \in \{1, \dots, k\}$, we have $(\gamma_0 x, \gamma_0\varphi_i(x)) \notin \mathcal{S}$. Then, assuming ψ_n has been built, we extend it as ψ_{n+1} by letting

$$\psi_{n+1}(x) = \gamma_{n+1}x$$

if $x \notin \operatorname{dom} \psi_n,\ \gamma_{n+1}x \notin \operatorname{rng} \psi_n$, and for all $i \in \{1, \dots, k\}$

(a) if $\varphi_i(x) \in \operatorname{dom} \psi_n$ then $(\gamma_{n+1}x, \psi_n\varphi_i(x)) \notin \mathcal{S}$,

(b) if $\varphi_i(x) \notin \operatorname{dom} \psi_n$ then $(\gamma_{n+1}x, \gamma_{n+1}\varphi_i(x)) \notin \mathcal{S}$.

Letting $\psi = \bigcup_n \psi_n$, we now prove the following central claim:

Claim. *The element of the pseudofull group ψ has full domain.*

Proof of the claim. Assume not. Observe first that for almost all x in the complement of the domain of ψ, the $\mathcal{R}$-class of x intersects the complement of the range of ψ. Indeed, otherwise we have a positive measure $\mathcal{R}$-invariant set such that ψ^{-1} takes it into a subset of itself of smaller measure, contradicting that $\mathcal{R}$ is measure-preserving.

Take $x \in X \setminus \operatorname{dom} \psi$ as above. Let $i_1, \dots, i_l \in \{1, \dots, k\}$ be the indices such that $\psi\varphi_{i_1}(x), \dots, \psi\varphi_{i_l}(x)$ is defined, and denote by $j_1, \dots, j_{k-l}$ the remaining indices. Since $\mathcal{S}$ has infinite index in $X \setminus \operatorname{rng} \psi$, there are $z \in [x]_{\mathcal{R}} \cap X \setminus \operatorname{rng} \psi$ and pairwise distinct $z_1, \dots, z_{k-l} \in [x]_{\mathcal{R}} \cap X \setminus \operatorname{rng} \psi$ such that

- for all $m \in \{1, \dots, l\}$ we have $(z, \psi\varphi_{i_m}(x)) \notin \mathcal{S}$,
- for all $m \in \{1, \dots, k-l\}$ we have $(z, z_m) \notin \mathcal{S}$.

By high transitivity, there is $\gamma \in \Gamma$ such that

$$\gamma x = z, \text{ and for all } m \in \{1, \dots, k-l\} \text{ we have } \gamma\varphi_{j_m}(x) = z_m.$$

This shows that for almost all $x \in X \setminus \operatorname{dom} \psi$ there is $n \in \mathbb{N}$ such that $\gamma_{n+1}x \notin \operatorname{rng} \psi$ and for all $i \in \{1, \dots, k\}$, we have:

(a′) If $\varphi_i(x) \in \operatorname{dom} \psi$ then $(\gamma_{n+1}x, \psi\varphi_i(x)) \notin \mathcal{S}$.

(b′) If $\varphi_i(x) \notin \operatorname{dom} \psi$ then $(\gamma_{n+1}x, \gamma_{n+1}\varphi_i(x)) \notin \mathcal{S}$.

Let n be the first integer such that the set of $x \in X \setminus \operatorname{dom} \varphi$ satisfying the above conditions is nonnull. By the definition of ψ, this set should be contained in the domain of ψ_{n+1}, and hence of ψ, a contradiction. □

So ψ is everywhere defined; since $\mathcal{R}$ is measure-preserving this implies that it belongs to the full group, and we thus rather write it as $T = \psi$. We finally check that T is as wanted.

Let $x \in X$. Define $n \geqslant -1$ as the least integer such that $x \in \operatorname{dom} \psi_{n+1}$, so that $T(x) = \psi_{n+1}(x) = \gamma_{n+1}(x)$ by construction. Let $i \in \{1, \dots, k\}$.

If $\varphi_i(x) \in \operatorname{dom} \psi_n$, then $(\gamma_{n+1}x, \psi_n\varphi_i(x)) \notin \mathcal{S}$ by (a), which means that $(T(x), T\varphi_i(x)) \notin \mathcal{S}$.

If $\varphi_i(x) \notin \operatorname{dom} \psi_n$, let m be the least integer such that $\varphi_i(x) \in \operatorname{dom} \psi_{m+1}$. Then $m \geqslant n$ and $\psi_{m+1}\varphi_i(x) = \gamma_{m+1}\varphi_i(x)$ by construction. There are two possibilities:

- $\underline{m = n}$: Then (b) guarantees $(\gamma_{n+1}x, \gamma_{n+1}\varphi_i(x)) \notin \mathcal{S}$ so that $(\gamma_{n+1}x, \psi_{n+1}\varphi_i(x)) \notin \mathcal{S}$, and hence $(T(x), T\varphi_i(x)) \notin \mathcal{S}$.
- $\underline{m > n}$: But then $(\gamma_{m+1}\varphi_i(x), \psi_m\varphi_i\varphi_i(x)) \notin \mathcal{S}$ by (a) applied to $x' = \varphi_i(x)$ and $n' = m$. The fact that φ_i is involutive implies that $(\psi_{m+1}\varphi_i(x), \psi_m(x)) \notin \mathcal{S}$, and hence $(T\varphi_i(x), T(x)) \notin \mathcal{S}$, which is equivalent to $(T(x), T\varphi_i(x)) \notin \mathcal{S}$.

Since in all cases we reached the desired conclusion $(T(x), T\varphi_i(x)) \notin \mathcal{S}$, the proof is finished. □

4.3. ***Dense orbits in the space of hyperfinite subequivalence relations.*** By definition a CBER is called *finite* when all its equivalence classes are finite, and *hyperfinite* if it can be written as an increasing union of finite Borel subequivalence relations. In our measured context, we ignore null sets and thus use the following definition:

Definition 4.8. A nonsingular equivalence relation $\mathcal{R}$ on (X, μ) is called *hyperfinite* when it admits a restriction to a full measure set which is hyperfinite in the above sense.

Hyperfiniteness can then be characterized in full group terms as follows: $\mathcal{R}$ is hyperfinite if and only if for all $T_1, \dots, T_n \in [\mathcal{R}]$ and $\epsilon > 0$, after throwing away a set X' of measure ϵ, the equivalence relation generated by the restrictions $T_{1\restriction X\setminus X'}, \dots, T_{n\restriction X\setminus X'}$ is finite.

Remark. The above characterization of hyperfiniteness is Dye's original notion [1959] of *approximate finiteness* for full groups (see also [Le Maître 2014, Proposition 1.57] or [Kechris and Miller 2004, Lemma 10.4] for a proof of this characterization).

Definition 4.9. Given a nonsingular equivalence relation $\mathcal{R}$ on (X, μ), let us denote by $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ its space of hyperfinite subequivalence relations.

Using lim inf as in the proof of Theorem 2.6 and the fact that the class of nonsingular hyperfinite equivalence relations is stable under countable increasing unions, one can show that $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ is closed in $\mathrm{Sub}(\mathcal{R})$; see [Kechris 2024, Theorem 8.1]. Let us observe that this can also be seen as a consequence of the characterization via approximate finiteness: if $\mathcal{S}$ is not hyperfinite as witnessed by some $\epsilon > 0$ and $T_1, \dots, T_n$ its in full group, any $\mathcal{S}'$ which contains the T_i in its full group up to an $\frac{1}{2}\epsilon n$ error will also fail to be hyperfinite.

Before we state our main result on dense orbits in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$, we need a preparatory well-known lemma on finite subequivalence relations:

Lemma 4.10. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Then two finite subequivalence relations $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{R}$ are in the same $[\mathcal{R}]$-orbit if and only if for all $n \in \mathbb{N}$*

$$\mu(\{x \in \mathbb{N} : |[x]_{\mathcal{S}_1}| = n\}) = \mu(\{x \in \mathbb{N} : |[x]_{\mathcal{S}_2}| = n\}). \tag{1}$$

Proof. The direct implication is clear. Assume conversely that (1) holds. Let us fix some notation by letting, for $i \in \{1, 2\}$ and $n \geqslant 1$,

$$X_n^i = \{x \in \mathbb{N} : |[x]_{\mathcal{S}_i}| = n\}.$$

Our assumption then becomes: the equality $\mu(X_n^1) = \mu(X_n^2)$ holds for every $n \geqslant 1$. Let $<$ be a Borel linear order on X. We can then define, for all $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$, the Borel set

$$Y_{n,j}^i = \{x \in X_n^i : x \text{ is the } j\text{-th element of } [x]_{\mathcal{S}_i}\}.$$

For $i \in \{1, 2\}$ and $j_1, j_2 \in \{1, \dots, n\}$, if we define $\varphi_{n,j_1,j_2}^i : Y_{n,j_1}^i \to Y_{n,j_2}^i$ by taking $x \in Y_{n,j_1}^i$ to the j_2-th element of its $\mathcal{S}_i$-class, then $\varphi_{n,j_1,j_2}^i \in [\![\mathcal{S}_i]\!] \subseteq [\![\mathcal{R}]\!]$. In particular since $\mathcal{R}$ is p.m.p., we have that $\mu(Y_{n,j_1}^i) = \mu(Y_{n,j_2}^i)$. Since X_n^i is partitioned by $(Y_{n,j}^i)_{j=1}^n$, we conclude

$$n\mu(Y_{n,1}^1) = \mu(X_n^1) = \mu(X_n^2) = n\mu(Y_{n,1}^2),$$

so that $\mu(Y_{n,1}^1) = \mu(Y_{n,1}^2)$.

Since $\mathcal{R}$ is ergodic, for every $n \in \mathbb{N}$ we can fix some $\varphi_n \in [\![\mathcal{R}]\!]$ such that $\operatorname{dom} \varphi_n = Y_{n,1}^1$ and $\operatorname{rng} \varphi_n = Y_{n,1}^2$. We then extend simultaneously these φ_n as $T \in [\mathcal{R}]$ by letting, for every $n \geqslant 1$, $j \in \{1, \dots, n\}$ and $x \in X_{n,j}^1$:

$$T(x) = \varphi_{n,1,j}^2 \varphi_n \varphi_{n,j,1}^1(x).$$

In other words, given $x \in X$ whose $\mathcal{S}_1$-class has cardinality n, we look at the first element y of the $\mathcal{S}_1$-class of x, and then $T(x)$ is the element of the $\mathcal{S}_2$-class of $\varphi_n(y)$ which is in the same position as x. Then by construction $T \cdot \mathcal{S}_1 = \mathcal{S}_2$, as wanted. □

Theorem 4.11. *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation, and let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be aperiodic and have everywhere infinite index. Then the closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.*

Proof. It follows from the definition of hyperfiniteness that finite equivalence relations are dense in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$, so it suffices to approximate every finite subequivalence relation of $\mathcal{R}$ by an element of the $[\mathcal{R}]$-orbit of S. Let $\mathcal{R}_0$ be such a finite equivalence relation. For each n let

$$X_n = \{x \in X : |[x]_{\mathcal{R}_0}| = n\}.$$

Because $\mathcal{S}$ is aperiodic, we can find an element of the $[\mathcal{R}]$-orbit of $\mathcal{R}_0$ contained in $\mathcal{S}$. Indeed, by Maharam's lemma we first have a partition of X into pieces $(Y_{j,n})_{1 \leqslant j \leqslant n}$ such that $\mu_{\mathcal{S}}(Y_{j,n}) = \mu(X_n)/n$. We then have for every $2 \leqslant j \leqslant n$ some $\varphi_{j,n} \in [\![\mathcal{S}]\!]$ such that $\operatorname{dom} \varphi_{j,n} = Y_{1,n}$ and $\operatorname{rng} \varphi_{j,n} = Y_{j,n}$.

Now let $Y_1 = \bigsqcup_n Y_{1,n}$, $\psi_1 = \mathrm{id}_{Y_1}$ and for all $j \geqslant 2$, $\psi_j = \bigsqcup_{n \geqslant j} \varphi_{j,n}$. Note that the ranges of the ψ_j partition X. Denoting by $\mathcal{S}_0$ the equivalence relation generated by all the ψ_j, we then have

$$\mathcal{S}_0 = \bigsqcup_n \bigsqcup_{i,j=1}^n \varphi_{n,i} \times \varphi_{n,j}(\Delta_{Y_{1,n}}) = \bigsqcup_{i,j} \psi_i \times \psi_j(\Delta_{Y_1} \cap (\operatorname{dom} \psi_i \times \operatorname{dom} \psi_j)).$$

In particular the $\mathcal{S}_0$-class of every $x \in \bigsqcup_{j=1}^n Y_{n,j}$ has cardinality n, and so by Lemma 4.10 we can fix some $T \in [\mathcal{R}]$ such that $T \cdot \mathcal{S}_0 = \mathcal{R}_0$.

By Theorem 4.7, we find a sequence $(T_k)_k$ in the full group of the restriction of $\mathcal{R}$ to Y_1 such that $T_k \cdot \mathcal{S}_{\restriction Y_1} \to \Delta_{Y_1}$. We then let $\widetilde{T}_k(x) = \psi_j T_k \psi_j^{-1}(x)$ for all $x \in \operatorname{rng} \psi_j$.

By construction $\widetilde{T}_k \cdot \mathcal{S} = \bigsqcup_{i,j} \psi_i \times \psi_j (T_k \cdot \mathcal{S}_{\restriction Y_1} \cap (\operatorname{dom} \psi_i \times \operatorname{dom} \psi_j))$, so by Lemma 3.11

$$\widetilde{T}_k \cdot \mathcal{S} \to \bigsqcup_{i,j} \psi_i \times \psi_j (\Delta_{Y_1} \cap (\operatorname{dom} \psi_i \times \operatorname{dom} \psi_j)) = \mathcal{S}_0,$$

which yields the desired result since we then have $T\widetilde{T}_k \cdot \mathcal{S} \to T \cdot \mathcal{S}_0 = \mathcal{R}_0$. □

Remark. By [Ioana et al. 2009, Corollary 5.4(ii)], if $\mathcal{R}$ is aperiodic and comes from a measure-preserving action of a property-(T) countable group, then there are no dense orbits in the space of subequivalence relations because any $\mathcal{S}$ with a dense orbit would have to have finite index in some restriction of $\mathcal{R}$, and hence cannot contain Δ_X in its orbit by Theorem 4.7. The general fact is that there cannot be dense orbits in the space of subequivalence relations of $\mathcal{R}$ as soon as $\mathcal{R}$ is not *approximable*, as defined by Gaboriau and Tucker-Drob [2016]. Indeed if $\mathcal{S}$ has a dense orbit, Lemma 2.2 provides a sequence $\mathcal{S}_n$ such that $\mathcal{R} = \liminf \mathcal{S}_n$, while nonapproximability forces some restriction of $\bigcap_{n \geqslant N} \mathcal{S}_n$ to coincide with $\mathcal{R}$ on a positive-measure subset A, contradicting the density of the orbit of $\mathcal{S}$ by Theorem C. For more examples of nonapproximable equivalence relations, the reader can consult the paper of Gaboriau and Tucker-Drob, where they obtain, for instance, a quantitative version of nonapproximability for some equivalence relations coming from actions of product groups; see [loc. cit., Theorem 2.4].

Corollary 4.12. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, and let $\mathcal{S} \in \operatorname{Sub}(\mathcal{R})$. The following are equivalent*:

(i) *$\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}$.*

(ii) *$\mathcal{S}$ is aperiodic and the closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains Δ_X.*

(iii) *$\mathcal{S}$ is aperiodic and the closure of the $\operatorname{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains Δ_X.*

(iv) *The closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\operatorname{Sub}_{\text{hyp}}(\mathcal{R})$.*

(v) *The closure of the $\operatorname{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains $\operatorname{Sub}_{\text{hyp}}(\mathcal{R})$.*

Proof. The equivalence of (i)–(iii) is a direct consequence of Theorem 4.7. The implication (i) $\Longrightarrow$ (iv) is exactly Theorem 4.11, and (iv) clearly implies (v). Finally, let us prove (v) $\Longrightarrow$ (iii): assume the closure of the $\operatorname{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains $\operatorname{Sub}_{\text{hyp}}(\mathcal{R})$. Since Δ_X is hyperfinite, Δ_X belongs to the closure of the $\operatorname{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$, so we only have to show $\mathcal{S}$ is aperiodic. Assume for the sake of contradiction that $\mathcal{S}$ is not aperiodic. Then for some $n \in \mathbb{N}$ we have $\mu(\{x \in X : |[x]_{\mathcal{S}}| \leqslant n\}) > 0$. Let $\delta = \mu(\{x \in X : |[x]_{\mathcal{S}}| \leqslant n\})$. Then $\mathcal{S}$ belongs to the $\operatorname{Aut}(\mathcal{R})$-invariant set $\boldsymbol{B}$ of all $\mathcal{S}' \in \operatorname{Sub}(\mathcal{R})$ such that

$$\sup_{T_1,\dots,T_{n+1} \in [\mathcal{R}]} M(\mathcal{S}' \cap (T_1 \cup \cdots \cup T_{n+1})) \leqslant \delta n + (1-\delta)(n+1).$$

Note that the intersection map is continuous for the lower M-topology by Lemma 3.9 and that M is lower semicontinuous for the lower M-topology. Since arbitrary supremums of lower semicontinuous functions are lower semicontinuous, the $\operatorname{Aut}(\mathcal{R})$-invariant set $\boldsymbol{B}$ is closed for the lower M-topology; in particular it is closed for the strong topology. Since the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S} \in \boldsymbol{B}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$, we have $\boldsymbol{B} \subseteq \mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.

However $\boldsymbol{B}$ is disjoint from the set of aperiodic subequivalence relations since for $\mathcal{T}$ aperiodic we can find $T_1, \dots, T_{n+1}$ in the full group of $\mathcal{T}$ with disjoint graphs, so that $M(\mathcal{T} \cap (T_1 \sqcup T_2 \sqcup \cdots \sqcup T_{n+1})) = n+1$. Since $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ contains aperiodic subequivalence relations, we reached the desired contradiction. So $\mathcal{S}$ is aperiodic, and hence (iii) holds, as wanted. □

Corollary 4.13. *For a subequivalence $\mathcal{S}$ of the hyperfinite ergodic p.m.p. equivalence relation $\mathcal{R}_0$, the following are equivalent*:

(i) *$\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}_0$.*

(ii) *$\mathcal{S}$ is aperiodic and the closure of the $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ contains Δ_X.*

(iii) *$\mathcal{S}$ is aperiodic and the closure of the $\mathrm{Aut}(\mathcal{R}_0)$-orbit of $\mathcal{S}$ contains Δ_X.*

(iv) *The $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ is dense in $\mathrm{Sub}(\mathcal{R}_0)$.*

(v) *The $\mathrm{Aut}(\mathcal{R}_0)$-orbit of $\mathcal{S}$ is dense in $\mathrm{Sub}(\mathcal{R}_0)$.*

Proof. This is a direct consequence of the previous corollary since every subequivalence relation of a hyperfinite equivalence relation is hyperfinite. □

Remark. In the nonsingular ergodic type II_∞ or type III cases, one can show that there are always dense orbits in the space of subequivalence relations. Let us sketch the proof: first note that by assumption there is a sequence (φ_n) of elements of $[\![\mathcal{R}]\!]$ such that for all n, $\operatorname{dom}\varphi_n = X$ but $X = \bigsqcup_n \operatorname{rng}\varphi_n$ (this actually characterizes equivalence relations of type II_∞ or III among ergodic nonsingular equivalence relations). We then enumerate a dense subset of $\mathrm{Sub}(\mathcal{R})$ as $(\mathcal{S}_n)$. The desired subequivalence relation with a dense orbit is $\mathcal{S} = \bigsqcup_n \varphi_n \cdot \mathcal{S}_n$, as one can see by approximating each φ_n by a full group element in the d_M metric.

4.4. *Meagerness of full group orbits.* In this section, we use Ioana's intertwining for subequivalence relations in order to show that full groups orbits are always meager in the space of hyperfinite subequivalence relations. In what follows, we use the *uniform metric* on the full group of a p.m.p. equivalence relation $\mathcal{R}$, defined by

$$d_u(T_1, T_2) = \mu(\{x \in X : T_1(x) \neq T_2(x)\}) = d_M(T_1, T_2),$$

which is biinvariant, complete and separable, thus endowing $[\mathcal{R}]$ with a Polish group topology. We will use without mention the well-known fact that if $M(\mathcal{S} \cap T) > 1 - \epsilon$, then there is $S \in [\mathcal{S}]$ such that $d_u(S, T) < \epsilon$ (to see this, note that $\mathcal{S} \cap T \in [\![\mathcal{S}]\!]$, and hence it can be extended to an element S of the full group of $\mathcal{S}$ by Lemma 4.2).

Definition 4.14 (Ioana). Let $\mathcal{S}$ and $\mathcal{T}$ be two subequivalence relations of a p.m.p. equivalence relation $\mathcal{R}$. Write $\mathcal{S} \prec \mathcal{T}$ if there is no sequence (T_n) in the full group of $\mathcal{T}$ such that for all $U_1, U_2 \in [\mathcal{R}]$, we have

$$M(\mathcal{S} \cap U_1 T_n U_2) \to 0.$$

We will crucially use Ioana's version [2012, Lemma 1.7] of Popa's intertwining theorem: if $\mathcal{S} \prec \mathcal{T}$, then $\mathcal{S}$ can be translated by an element of the full group of $\mathcal{R}$ so as to have somewhere-finite index in $\mathcal{T}$. The interested reader is also referred to [Spaas 2023, Lemma 3.1] for a characterization of $\mathcal{S} \prec \mathcal{T}$ when $\mathcal{S}$ has infinite index in $\mathcal{R}$.

We leave it to the reader to check that $\prec$ is $[\mathcal{R}]$-invariant, meaning that if $\mathcal{S} \prec \mathcal{T}$ and $U_1, U_2 \in [\mathcal{R}]$, then $U_1 \cdot \mathcal{S} \prec U_2 \cdot \mathcal{T}$. Also note that we always have $\mathcal{S} \prec \mathcal{S}$.

Lemma 4.15. *The relation $\prec$ defines an F_σ subset of* $\mathrm{Sub}(\mathcal{R}) \times \mathrm{Sub}(\mathcal{R})$.

Proof. We show that the complement of $\prec$ in $\mathrm{Sub}(\mathcal{R}) \times \mathrm{Sub}(\mathcal{R})$ is G_δ. Let us fix (U_i) dense in $\mathcal{R}$. It is then not hard to check that $\mathcal{S} \not\prec \mathcal{T}$ if and only if there is a sequence (T_n) in the full group of T such that $M(\mathcal{S} \cap U_i T_n U_j) \to 0$ for all $i, j \in \mathbb{N}$ (this is essentially the first step of the proof of [Ioana 2012, Lemma 1.7]). It follows that $\mathcal{S} \not\prec \mathcal{T}$ if and only if for every $k \in \mathbb{N}$ and $\epsilon > 0$, we can find $T \in [\mathcal{T}]$ such that

$$M(\mathcal{S} \cap U_i T U_j) < \epsilon \quad \text{for all } i, j \in \{1, \dots, k\} \tag{2}$$

Now by density of (U_i), we finally have $\mathcal{S} \not\prec \mathcal{T}$ if and only if for every $k \in \mathbb{N}$ and $\epsilon > 0$, there is $l \in \mathbb{N}$ such that $M(\mathcal{T} \cap U_l) > 1 - \epsilon$ and for all $i, j \in \{1, \dots, k\}$

$$M(\mathcal{S} \cap U_i U_l U_j) < \epsilon.$$

It is now straightforward to check that this last condition defines a G_δ set, so we are done. □

Theorem 4.16. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, and consider the $[\mathcal{R}]$-action on the Polish space $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ of hyperfinite subequivalence relations of $\mathcal{R}$. Then all $[\mathcal{R}]$-orbits in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ are meager.*

Proof. Assume for the sake of contradiction there is a nonmeager $[\mathcal{R}]$-orbit. Since there is a dense $[\mathcal{R}]$-orbit in $\mathrm{Sub}(\mathcal{R})$, the topological 0–1 law yields that there is a comeager orbit. Denoting by E the equivalence relation generated by the $[\mathcal{R}]$-action on $\mathrm{Sub}(\mathcal{R})$, we deduce that E is comeager in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R}) \times \mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.

By [Dye 1959, Theorem 4], the p.m.p. equivalence relation $\mathcal{R}$ contains an ergodic hyperfinite subequivalence relation $\mathcal{S}$. Let $\mathcal{T}$ be an aperiodic subequivalence relation of $\mathcal{S}$ with diffuse ergodic decomposition, such as the one coming from Example 1.1 once we identify $\mathcal{S}$ to $\mathcal{R}_0$. The following claim will essentially finish our proof:

Claim. *We have $\mathcal{T} \not\prec \mathcal{S}$.*

Proof of the claim. Suppose $\mathcal{T} \prec \mathcal{S}$. Then by [Ioana 2012, Lemma 1.7], there is a nonzero $\varphi \in [\![\mathcal{R}]\!]$ and $k \in \mathbb{N}$ such that if $B = \operatorname{rng} \varphi$, then every $(\varphi \cdot \mathcal{S})_{\restriction B}$-class is contained in the union of at most k classes of $\mathcal{T}_{\restriction B}$. Since $\mathcal{T}$ has diffuse ergodic decomposition, we can find $C_1, \dots, C_{k+1}$ partitioning B which are all $\mathcal{T}_{\restriction B}$-invariant and have positive measure.

As $\mathcal{S}$ is ergodic, the $(\varphi \cdot \mathcal{S})_{\restriction B}$-class of almost every $x \in B$ intersects all the C_i, and since they are $\mathcal{T}_{\restriction B}$-invariant, we conclude that almost every $(\varphi \cdot \mathcal{S})_{\restriction B}$-class cannot be contained in less than $k+1$ classes of $\mathcal{T}_{\restriction B}$, a contradiction. □

Since $\prec$ is $[\mathcal{R}]$-invariant, its complement also is. This complement is moreover G_δ by Lemma 4.15, and by the above claim it contains the subset $[\mathcal{R}]\cdot\mathcal{T}\times[\mathcal{R}]\cdot\mathcal{S}$, which is dense in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})\times\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ by Theorem 4.11. So the complement of $\prec$ is comeager, and it should thus intersect E. This means that one can find a subequivalence relation $\mathcal{S}'$ and $T\in[\mathcal{R}]$ such that $\mathcal{S}'\not\prec T\cdot\mathcal{S}'$, a contradiction. □

Remark. Our proof is inspired by the Glasner–Weiss proof that $\mathrm{Aut}(X,\mu)$ does not have comeager conjugacy classes, replacing Rokhlin's lemma by Theorem 4.11 and Del Junco's disjointness by Ioana's intertwining relation $\not\prec$; see [Glasner and Weiss 2008, Section 3].

Corollary 4.17. *Let $\mathcal{R}_0$ be the hyperfinite ergodic p.m.p. equivalence relation. Then all the $[\mathcal{R}_0]$-orbits are meager in $\mathrm{Sub}(\mathcal{R}_0)$.*

Proof. Again this follows directly from the above theorem, since every subequivalence relation of a hyperfinite equivalence relation is hyperfinite. □

5. The uniform metric and complexity calculations

We begin this final section by introducing a natural metric on the space of all nonsingular equivalence relations. Given two nonsingular equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$, denote by $\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_2)$ the set of all $A\in\mathrm{MAlg}(X,\mu)$ such that $\mathcal{R}_{1\restriction A}=\mathcal{R}_{2\restriction A}$. Define

$$d_u(\mathcal{R}_1,\mathcal{R}_2)=1-\sup_{A\in\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_2)}\mu(A)=\inf_{A\in\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_2)}1-\mu(A).$$

Let us check that this is indeed a metric. Symmetry is clear. For Hausdorffness, if $d_u(\mathcal{R}_1,\mathcal{R}_2)=0$ then we find (X_n) with $\mu(X_n)\geqslant 1-2^{-n}$ and $\mathcal{R}_{1\restriction X_n}=\mathcal{R}_{2\restriction X_n}$. By the Borel–Cantelli lemma, almost every $x\in X$ is in all but finitely many X_n, which yields a full measure set restricted to which $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide.

We finally need to show that the triangle inequality holds. Observe that if $A_1\in\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_2)$ and $A_2\in\mathfrak{C}(\mathcal{R}_2,\mathcal{R}_3)$, then $A_1\cap A_2\in\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_3)$; moreover

$$X\setminus(A_1\cap A_2)\subseteq(X\setminus A_1)\cup(X\setminus A_2),$$

so by taking measures and infimums we get the desired triangle inequality.

Lemma 5.1. *The uniform metric refines the strong topology.*

Proof. The uniform metric clearly refines the uniform topology defined in [Kechris 2024, Section 4.6], which in turn refines the strong topology. □

Remark. The results from [Ioana et al. 2009, Theorem 1] applied to the subequivalence relation $\mathcal{R}_1\cap\mathcal{R}_2$ yield that the uniform metric actually induces the uniform topology.

In what follows, we freely identify the Cantor space $\{0,1\}^{\mathbb{N}}$ to the set of subsets of $\mathbb{N}$, and the continuous reduction alluded to is thus defined on the whole $\{0,1\}^{\mathbb{N}}$.

Proposition 5.2. *Suppose $\mathcal{R}$ is an aperiodic nonsingular equivalence relation. There is a continuous reduction from the set $\mathrm{Sub}_{\mathrm{fin}}(\mathbb{N})$ of finite subsets of $\mathbb{N}$ to the complement of the set of subequivalence* *relations which are of infinite index, and to the set of finite index subequivalence relations, both for the uniform metric.*

Proof. Let (X_n) be a partition of X into sets which intersect almost every $\mathcal{R}$-class (e.g., obtained via Maharam's lemma as sets of $\mathcal{R}$-conditional measure constant equal to 2^{-n-1}). Let $B \subseteq \mathbb{N}$. We associate to B the set $Y_B = \bigcup_{n\in\mathbb{N}\setminus B} X_n$ and let the reduction be $B \mapsto \mathcal{R}_B$, where

$$\mathcal{R}_B = (\mathcal{R} \cap Y_B \times Y_B) \sqcup \bigcup_{n\in B} \mathcal{R} \cap (X_n \times X_n).$$

The continuity of our reduction for the uniform metric is clear from the definition (if two subsets B and B' coincide on the first n integers, then the associated equivalence relations coincide on $\bigcup_{k<n} X_k$). Moreover, if B is finite we see that $\mathcal{R}_B$ has finite index equal to $|B|+1$, and if not it has infinite index, as wanted. □

Corollary 5.3. *Let $\mathcal{R}$ be a nonsingular aperiodic equivalence relation. The set of subequivalence relations of $\mathcal{R}$ with infinite index is G_δ-hard in the uniform metric; in particular it is G_δ-complete in the strong topology.*

The next statements are all about the strong topology.

Proposition 5.4. *Let $\mathcal{R}$ be a nonsingular aperiodic equivalence relation. The space of finite index subequivalence relations is $F_{\sigma\delta}$-hard if and only if E has infinitely many ergodic components, otherwise it is F_σ-complete.*

Proof. Denote by $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ the space of finite index subequivalence relations. Note that by ergodicity, if E has only finitely many ergodic components, then the space of finite index subequivalence relations is equal to the union over $k \in \mathbb{N}$ of the spaces $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ of all subequivalence relations whose index is uniformly less than k.

Let us show that for each $k \in \mathbb{N}$, the set $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is closed. Suppose $\mathcal{S} \notin \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$. Then there is a positive measure set of $x \in X$ such that $[x]_\mathcal{R}$ contains at least $k+1$ distinct $\mathcal{S}$ classes. We thus find $A \subseteq X$ of positive measure and $\varphi_1, \dots, \varphi_k \in [[\mathcal{R}]]$ with common domain A and such that first, $A, \operatorname{rng}\varphi_1, \dots, \operatorname{rng}\varphi_k$ are pairwise disjoint, and second, $\mathcal{S}$ is disjoint from the union of the graphs of the $\varphi_i\varphi_j^{-1}$. In other words, we find a finite subequivalence relation of $\mathcal{R}$ which is disjoint from $\mathcal{S}$ (after of course removing Δ_X) and all whose nontrivial classes have cardinality $\geqslant k+1$.

Now let $\mathcal{S}_n \to \mathcal{S}$. Since

$$M(\varphi_1 \sqcup \cdots \sqcup \varphi_k \cap \mathcal{S}_n) = \int_A |\{i \in \{1, \dots, k\} : (x, \varphi_i(x)) \in \mathcal{S}_n\}|\, d\mu(x) \to 0,$$

for n large enough there will be a positive measure set of $x \in A$ such that for all $i \in \{1, \dots, k\}$, $(x, \varphi_i(x)) \notin \mathcal{S}_n$; in particular, $\mathcal{S}_n$ has somewhere index $> k$. So $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is closed and hence $\mathrm{Sub}_{[<\infty]}(\mathcal{R}) = \bigcup_{k\in\mathbb{N}} \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is F_σ. Moreover, $\mathrm{Sub}_{[<\infty]}(\mathcal{R}) = \bigcup_{k\in\mathbb{N}} \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is F_σ-complete by virtue of the preceding proposition.

Now if $\mathcal{R}$ has infinitely many ergodic components, let (X_n) be a partition of X into $\mathcal{R}$-invariant sets of positive measure. Denote by $\mathrm{Sub}_{\mathrm{fin}}(\mathbb{N}) \subseteq \{0,1\}^{\mathbb{N}}$ the set of finite subsets of $\mathbb{N}$, viewed as finitely supported functions $\mathbb{N} \to \{0,1\}$.

For every $n \in \mathbb{N}$, let $\Phi_n : \{0,1\}^{\mathbb{N}} \to \mathrm{Sub}(\mathcal{R}_{\restriction X_n})$ be the continuous reduction from Proposition 5.2, and for a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\{0,1\}^{\mathbb{N}}$, let

$$\Phi((x_n)_{n\in\mathbb{N}}) = \bigsqcup_{n\in\mathbb{N}} \Phi_n(x_n) \in \mathrm{Sub}(\mathcal{R}).$$

By Lemma 3.11 Φ is continuous, and by construction

$$\Phi^{-1}(\mathrm{Sub}_{[<\infty]}(\mathcal{R})) = \prod_{n\in\mathbb{N}} \mathrm{Sub}_{\mathrm{fin}}(\mathbb{N}).$$

Being a (countable) infinite product of F_σ-hard sets, the latter is $F_{\sigma\delta}$-hard, so the set of finite index subequivalence relations is $F_{\sigma\delta}$-hard. □

Remark. It follows from [Kechris 2024, Proposition 9.4] that $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ is always $F_{\sigma\delta}$ (the proof in the nonsingular case works the same), so when $\mathcal{R}$ is aperiodic and has infinitely many ergodic components, $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ is actually $F_{\sigma\delta}$-complete.

We finally use a similar approach to show that the set of finite subequivalence relations is always $F_{\sigma\delta}$-complete (for $\mathcal{R}$ aperiodic), by first directly showing that the space of finite subequivalence relations if F_σ-hard. In the remainder of the paper, we set

$$\Gamma_0 = \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} = \{f : \mathbb{N} \to \mathbb{Z}/2\mathbb{Z} \mid f(n) = 0 \text{ for all but finitely many } n \in \mathbb{N}\}.$$

Lemma 5.5. *The space* $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$ *of finite subgroups of* Γ_0 *is an* F_σ*-complete subset of the space* $\mathrm{Sub}(\Gamma_0)$ *of subgroups of* Γ_0.

Proof. The set of finite subgroups is countable. In particular it is F_σ. It is F_σ-complete via the continuous reduction $B \subseteq \mathbb{N} \mapsto \Lambda_B := \bigoplus_{n\in B} \mathbb{Z}/2\mathbb{Z} \leqslant \Gamma$ of the F_σ-complete set of finite subsets of $\mathbb{N}$ to $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$. □

Remark. The above lemma can also be proven by first noting that by the Baire category theorem, the countable dense set $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$ cannot be G_δ in the perfect zero-dimensional Polish space of subgroups of Γ_0, and then applying Wadge's theorem; see [Kechris 1995, Theorem 22.10].

The following proposition was stated and proven in the p.m.p. setup in the first version of this paper (in this restricted setup, it is a consequence of [Dye 1959, Theorem 4]). We are very grateful to the referee for pointing out that the same result is true in the purely Borel setup and for indicating the proof.

Proposition 5.6. *For every aperiodic CBER* $\mathcal{R}$, *there is a free* Γ_0*-action which induces a subequivalence relation of* $\mathcal{R}$.

We rely on the following key lemma:

Lemma 5.7. *Let* $\mathcal{R}$ *be an aperiodic CBER. Then its Borel full group contains a fixed-point free involution.*

Proof. By [Kechris and Miller 2004, Lemma 7.4], $\mathcal{R}$ contains a Borel subequivalence relation $\mathcal{S}$, all whose classes have cardinality 2. Our involution U is then defined by: for all $x \in X$, $U(x)$ is the only $y \neq x$ such that $(x, y) \in \mathcal{S}$. □

Proof of Proposition 5.6. We first inductively define a decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of Borel subsets of X and fixed-point free partial involutions $(\varphi_n)_{n\in\mathbb{N}}$ in the Borel pseudofull group of $\mathcal{R}$ with support A_n as follows. We begin with an everywhere defined involution φ_0 with support $A_0 = X$, as provided by the previous lemma and then, assuming φ_n has been built, we let A_{n+1} be a Borel fundamental domain for φ_n. Noting that the restriction of $\mathcal{R}$ to A_{n+1} has to remain aperiodic, we use the above lemma to find φ_{n+1} as a fixed-point free involution in the Borel full group of the restriction of $\mathcal{R}$ to A_{n+1}.

We extend these to involutions to get a free Γ_0-action. Letting $G_n = \{f \in \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} : \forall m > n,\ f(m) = 0\}$, we can write Γ_0 as the increasing union $\Gamma_0 = \bigcup_n G_n$, and we have $G_n = \langle s_0, \dots, s_n\rangle$, where $s_i(k) = 0$ if $i \neq k$, and $s_i(k) = 1$ if $i = k$. Our inductive construction of the action will be such that for all n,

$$X = \bigsqcup_{g\in G_n} g \cdot A_{n+1}. \tag{3}$$

We begin by letting $s_0 \cdot x = \varphi_0(x)$ for all $x \in X$. And then, assuming that the action of G_n has been defined so that $X = \bigsqcup_{g\in G_n} g \cdot A_{n+1}$, we start defining the s_{n+1} action on A_{n+1} by letting $s_{n+1} \cdot x = \varphi_{n+1}(x)$ if $x \in A_{n+1}$. Since we want s_{n+1} to commute with G_n and we have $X = \bigsqcup_{g\in G_n} g \cdot A_{n+1}$, we are then forced to extend s_{n+1} to the whole of x by letting, for every $x \in A_{n+1}$ and every $g \in G_n$,

$$s_{n+1}(g \cdot x) = g \cdot s_{n+1} \cdot x.$$

It is then not hard to check that this does define a G_{n+1} action with $X = \bigsqcup_{g\in G_{n+1}} g \cdot A_{n+2}$, which provides us a Γ_0-action by induction in the end. This action is free since (3) implies that the action restricted to each G_n is free, which finishes the proof. □

Remark. It follows from [Dye 1959, Theorem 3] that when $\mathcal{R}$ is p.m.p. hyperfinite and aperiodic, one can upgrade the above result and obtain that $\mathcal{R}$ is equal to the equivalence relation generated by the Γ_0-action. This led Dye to ask whether any p.m.p. equivalence relation should come the free action of some countable group (see the paragraph right before Lemma 6.5 in [Dye 1959]). This question was answered in the negative 40 years later by Furman [1999].

Given a nonsingular equivalence relation $\mathcal{R}$, we denote by $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ the space of finite subequivalence relations of $\mathcal{R}$.

Proposition 5.8. *Let $\mathcal{R}$ be an aperiodic nonsingular equivalence relation. Then $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ is $F_{\sigma\delta}$-complete.*

Proof. Partition X into $(X_n)_{n\in\mathbb{N}}$ so that each X_n has positive measure and the restriction of $\mathcal{R}$ to each X_n is aperiodic. By the previous lemma, each restriction of $\mathcal{R}$ to X_n contains a subequivalence relation coming from a free Γ_0-action α_n. Now consider the map $(\mathrm{Sub}(\Gamma_0))^{\mathbb{N}} \to \mathrm{Sub}(\mathcal{R})$ which takes any sequence $(\lambda_n)_{n\in\mathbb{N}}$ to the equivalence relation $\bigsqcup_{n\in\mathbb{N}} \mathcal{R}_{\alpha_n(\Lambda_n)}$. This is a topological embedding by Lemma 3.11, and it reduces the $F_{\sigma\delta}$-complete set $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)^{\mathbb{N}} \subseteq \mathrm{Sub}(\Gamma_0)^{\mathbb{N}}$ to the set of finite equivalence relations, which is thus $F_{\sigma\delta}$-hard.

We are left with proving that $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ is $F_{\sigma\delta}$, which is due to Kechris [2024, Theorem 8.5] in the p.m.p. setup. While his proof generalizes to the nonsingular setup, we provide here another approach, relying on spaces of measurable maps. Let us fix a uniquely generating sequence of moving partial involutions $(\varphi_n)_{n\in\mathbb{N}}$. Consider for every $n \in \mathbb{N}$ the continuous map

$$\iota_n : \mathrm{Sub}(\mathcal{R}) \to \mathrm{MAlg}(X, \mu)$$

which associates to every $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ the set of $x \in \operatorname{dom}\varphi_n$ such that $(x, \varphi_n(x)) \in \mathcal{S}$.

Identifying subsets to characteristic functions yields a natural homeomorphism Φ between $\mathrm{MAlg}(X, \mu)$ and the Polish space of measurable maps $\mathrm{L}^0(X, \mu, \{0, 1\})$; see [Le Maître and Slutsky 2021, Appendix B] for more on L^0 spaces. Using this homeomorphism and putting all the maps ι_n together, we get a continuous embedding

$$\iota = (\Phi \circ \iota_n)_{n\in\mathbb{N}} : \mathrm{Sub}(\mathcal{R}) \hookrightarrow \prod_{n\in\mathbb{N}} \mathrm{L}^0(X, \mu, \{0, 1\}) \simeq \mathrm{L}^0(X, \mu, \{0, 1\}^{\mathbb{N}}).$$

We will conclude the proof via a straightforward application of the claim below.

Claim. *Let Y be a Polish space, and let $D \subseteq Y$ be F_σ. Then $\mathrm{L}^0(X, \mu, D)$ is an $F_{\sigma\delta}$ subset of $\mathrm{L}^0(X, \mu, Y)$.*

Proof of the claim. It follows directly from the definition of the topology of convergence in measure given in [Le Maître and Slutsky 2021, Appendix B] that for any open subset $U \subseteq Y$, the map $f \in \mathrm{L}^0(X, \mu, Y) \mapsto \mu(f^{-1}(U))$ is lower semicontinuous. Taking complements, given a closed subset $C \subseteq \{0, 1\}^{\mathbb{N}}$, the map $f \in \mathrm{L}^0(X, \mu, Y) \mapsto \mu(f^{-1}(C))$ is thus upper semicontinuous. Now write $D = \bigcup_n C_n$ as an increasing union of closed subsets C_n, and observe that $\mu(f^{-1}(D)) = 1$ if and only if for all $\epsilon > 0$ there is n such that $\mu(f^{-1}(C_n)) \geqslant 1 - \epsilon$. □

In order to finish the proof, take $D = \mathrm{Sub}_{\mathrm{fin}}(\mathbb{N})$ to be the F_σ set of finite subsets of $\mathbb{N}$, viewed as a subset of $Y = \{0, 1\}^{\mathbb{N}}$. Then $\mathrm{L}^0(X, \mu, \mathrm{Sub}_{\mathrm{fin}}(\mathbb{N}))$ is $F_{\sigma\delta}$ in $\mathrm{L}^0(X, \mu, \{0, 1\}^{\mathbb{N}})$ by the above claim. Finally, observe that $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R}) = \iota^{-1}(\mathrm{L}^0(X, \mu, \mathrm{Sub}_{\mathrm{fin}}(\mathbb{N})))$, which is thus $F_{\sigma\delta}$ by continuity of ι. □

Remark. Endowing $\mathrm{Sub}(\mathcal{R})$ with its strong topology and the uniform metric, we get a Polish topometric space in the sense of Ben Yaacov [2008], and our remark preceding Corollary 4.12 can be upgraded to the fact that if a p.m.p. equivalence relation $\mathcal{R}$ has property (T), then $\mathcal{R}$ is a metrically isolated point of $\mathrm{Sub}(\mathcal{R})$; see also [Gaboriau and Tucker-Drob 2016] for more examples. I don't know whether the action of the full group or automorphism group of the hyperfinite ergodic p.m.p. equivalence relation admits a metrically generic orbit, as characterized in [Ben Yaacov and Melleray 2025].

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Coincidence of critical points for directed polymers for general environments and random walks

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For the directed polymer in a random environment (DPRE), two critical inverse-temperatures can be defined. The first one, β_c, separates the strong disorder regime (in which the normalized partition function W_n^β tends to zero) from the weak disorder regime (in which W_n^β converges to a nontrivial limit). The other, $\bar{\beta}_c$, delimits the very strong disorder regime (in which W_n^β converges to zero exponentially fast). It was proved in earlier work that $\beta_c = \bar{\beta}_c$ when the random environment is bounded above for the DPRE based on the simple random walk. We extend this result to general environments and an arbitrary reference walk. We also prove that $\beta_c = 0$ if and only if the L^2 critical point is trivial.

1. Introduction

We start by defining the directed polymer in a random environment with a general reference random walk. For the simple random walk case, an overview of known results can be found in the lecture notes [Comets 2017] and the survey [Zygouras 2024]. We let $X = (X_k)_{k\geq 0}$ denote a random walk on $\mathbb{Z}^d$ starting from the origin and with independent identically distributed (i.i.d.) increments. The associated probability is denoted by P. We let $\eta \in [0, \infty]$ denote the exponent associated with the tail-decay of $|X_1|$, defined as

$$-\limsup_{u\to\infty} \frac{\log P(|X_1| \geq u)}{\log u} =: \eta \in [0, \infty], \tag{1}$$

where here and in what follows $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^d$. Our main results are proved under the assumption that $\eta > 0$.

Additionally, we consider a collection $\omega = (\omega_{k,x})_{k\geq 1, x\in\mathbb{Z}^d}$ of i.i.d. real-valued random variables (we let $\mathbb{P}$ denote the associated probability) and make the assumption (throughout the whole paper except for Theorem 2.9) that $\omega_{k,x}$ has exponential moments of all order, that is

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_{1,0}}] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{2}$$

The above assumption implies that $\omega_{1,0}$ has finite mean and variance, and without loss of generality we may assume that

$$\mathbb{E}[\omega_{1,0}] = 0 \quad \text{and} \quad \mathbb{E}[\omega_{1,0}^2] = 1. \tag{3}$$

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Given a realization of ω (the random environment), $\beta \geq 0$ (the inverse temperature) and $n \geq 1$ (the polymer length), we define the polymer measure $P^{\beta}_{\omega,n}$ as a modification of the distribution P which favors trajectories that visit sites where ω is large, namely

$$P^{\beta}_{\omega,n}(dX) := \frac{1}{W^{\beta}_n} e^{\sum_{k=1}^{n}(\beta\omega_{k,X_k}-\lambda(\beta))} P(dX), \quad \text{where } W^{\beta}_n := E[e^{\sum_{k=1}^{n}(\beta\omega_{k,X_k}-\lambda(\beta))}].$$

The quantity W^{β}_n (the dependence in ω is omitted in the notation for better readability) is referred to as the (normalized) partition function of the model. A direct application of Fubini yields $\mathbb{E}[W^{\beta}_n] = 1$. Using Fubini for the conditional expectation, it was observed in [Bolthausen 1989] that the process $(W^{\beta}_n)_{n\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)$ defined by

$$\mathcal{F}_n := \sigma(\omega_{k,x} : k \leq n,\ x \in \mathbb{Z}^d).$$

As a consequence, $(W^{\beta}_n)_{n\in\mathbb{N}}$ converges almost surely as $n \to \infty$, and we let W^{β}_{∞} denote its limit. By Kolmogorov's 0–1 law, $\mathbb{P}(W^{\beta}_{\infty} = 0) \in \{0, 1\}$. Using terminology established in [Comets et al. 2003], we say that *strong disorder* holds if $\mathbb{P}(W^{\beta}_{\infty} = 0) = 1$ and that *weak disorder* holds if $\mathbb{P}(W^{\beta}_{\infty} > 0) = 1$. Finally (relying on [Comets et al. 2003, Proposition 2.5] for the existence of the limit, which easily generalizes to the case of general random walks) we define the free energy of the directed polymer by setting

$$\mathfrak{f}(\beta) := \lim_{n\to\infty} \frac{1}{n} \log W^{\beta}_n = \frac{1}{n}\mathbb{E}[\log W^{\beta}_n].$$

We say that *very strong disorder* holds when $\mathfrak{f}(\beta) < 0$ (that is to say when W^{β}_n converges exponentially fast to zero). It was established in [Comets and Yoshida 2006] that the "strength of disorder" is monotone in β in the sense that there exist β_c and $\bar{\beta}_c$ in $[0, \infty]$ such that:

(a) Weak disorder holds when $\beta < \beta_c$ and strong disorder holds when $\beta > \beta_c$.

(b) Very strong disorder holds if and only if $\beta > \bar{\beta}_c$.

The weak and strong disorder regimes correspond to different asymptotic behavior of X under $P^{\beta}_{\omega,n}$ as $n \to \infty$. When weak disorder holds, in the simple random walk case (see [Comets and Yoshida 2006], and see also [Lacoin 2025] for a recent short proof) the scaling limit of $(X_k)_{k=1}^{n}$ under $P^{\beta}_{\omega,\beta}$ on the diffusive scale is the same as under P, i.e., Brownian motion. Analogous results have been proved in the case when X is in the domain of attraction of an α-stable law [Wei 2016].

On the contrary, when strong disorder holds, the polymer is believed to exhibit a different behavior: the trajectories should *localize* around a favorite corridor along which the environment ω is particularly favorable. This conjectured localized behavior has been corroborated by several mathematical results [Carmona and Hu 2002; 2006; Comets et al. 2003; Bates and Chatterjee 2020], the stronger results being obtained under the assumption of *very strong disorder* and additional technical restrictions.

This distinction between strong and very strong disorder is however not crucial: it was conjectured in [Carmona and Hu 2006; Comets and Yoshida 2006] that the two critical points β_c and $\bar{\beta}_c$ coincide, and this conjecture was proved to hold true in [Junk and Lacoin 2024] under the assumption that the disorder is bounded above. We remove this assumption and extend the result to arbitrary random walks.

We refer to the introduction of [Junk and Lacoin 2024] and to the recent survey [Zygouras 2024] for a more detailed discussion on the localization transition.

Let us also mention — although the present paper does not bring any new perspective on the topic — that, in the case where X is the nearest-neighbor random walk in $\mathbb{Z}^d$, beyond the change from delocalization to localization, the critical point $\bar{\beta}_c$ is also expected to mark a transition from diffusive to superdiffusive behavior. This is captured by the transversal fluctuation exponent ξ, defined informally through the relation $E_n^{\omega,\beta}[|X_n|^2] = n^{2\xi+o(1)}$ under the polymer measure $P_{\omega,n}^{\beta}$. In the weak disorder phase, the invariance principle implies $\xi = \frac{1}{2}$ while in the strong disorder phase it is expected that the polymer becomes superdiffusive, with $\xi > \frac{1}{2}$. Proving this for the standard model is a major open problem in this field.

To our knowledge, superdiffusivity results have only been obtained for models for which it is known that $\beta_c = 0$. For the DPRE introduced above, it is predicted that $\xi = \frac{2}{3}$ for all $\beta > 0$ when $d = 1$, a conjecture which has proved to hold true in the specific case log-Γ distributed environment in [Seppäläinen 2012]. Upper and lower bounds for ξ have been achieved in [Petermann 2000; Mejane 2004] for a polymer model in which both the environment and the random walk are Gaussian. In addition, results have been obtained for DPRE with heavy-tailed environments [Auffinger and Louidor 2011; Dey and Zygouras 2016; Berger and Torri 2019] as well as related models where the environments displays long-range correlations [Lacoin 2011; 2012a; 2012b]. The question of finding a directed polymer model for which $\xi = \frac{1}{2}$ for small values of β and $\xi > \frac{1}{2}$ for large values of β remains widely open.

To facilitate the discussion of the results, let us finally introduce the L^2 critical point

$$\beta_2 := \sup\left\{\beta \geq 0 : (e^{\lambda(2\beta)-2\lambda(\beta)} - 1)\sum_{k\geq 1} P^{\otimes 2}(X_k^{(1)} = X_k^{(2)}) \leq 1\right\}, \tag{4}$$

where $P^{\otimes 2}$ is the law of two independent copies $X^{(1)}$ and $X^{(2)}$ of the random walk X. It is not difficult to check that $\sup_{n\geq 0} \mathbb{E}[(W_n^{\beta})^2] < \infty$ if and only if $\beta \in [0, \beta_2) \cup \{0\}$, which implies that $(W_n^{\beta})_{n\in\mathbb{N}}$ converges in L^2 for $\beta < \beta_2$, and hence $\beta_c \geq \beta_2$. We also note the equivalence

$$\beta_2 = 0 \iff (X_k^{(1)} - X_k^{(2)})_{k\geq 0} \text{ is recurrent.} \tag{5}$$

Historically, the β_2 critical point was used as a sufficient criterion to ensure $\beta_c > 0$ in the case when $X^{(1)} - X^{(2)}$ is transient (in particular in the simple random walk case in $d \geq 3$; see [Imbrie and Spencer 1988; Bolthausen 1989]). Here we show that this condition is also necessary in the sense that $\beta_c = 0$ whenever $X^{(1)} - X^{(2)}$ is recurrent. We refer the reader to [Cosco et al. 2021] for a previous work exploring the relation between $\beta_2 = 0$ and $\beta_c = 0$ considering a more general setup in which X is only assumed to be a Markov chain on a countable state space.

2. Results

2.1. *Coincidence of the critical points.* As mentioned earlier, it had been conjectured in [Carmona and Hu 2006; Comets and Yoshida 2006] that there is a sharp transition from weak to very strong disorder, or in other words that $\beta_c = \bar{\beta}_c$. This conjecture was formulated in the case where the reference walk P is the simple random walk on $\mathbb{Z}^d$, and it was recently proved to hold true under the additional assumption that the environment ω is bounded above [Junk and Lacoin 2024]. We extend the validity of this result by relaxing the assumptions on the random walk and on the environment; recall (1).

Theorem 2.1. *If $\eta > 0$, then $\beta_c = \bar{\beta}_c$. Furthermore, if $\beta_c > \beta_2$ then weak disorder holds at β_c.*

Remark 2.2. The assumption $\eta > 0$ is necessary and the result may fail to hold when the assumption is violated. For instance, it was proved in [Viveros 2023] that for a very heavy-tailed one-dimensional random walk, one may have $\bar{\beta}_c = \infty$ and $\beta_c \in (0, \infty)$. To illustrate this point further, we show in Proposition 2.11 that it is even possible to have $\beta_c = 0$ and $\bar{\beta}_c = \infty$.

In the case of the simple random walk on $\mathbb{Z}^d$ with $d \geq 3$, we have $\beta_c > \beta_2$ (see [Birkner and Sun 2010, Section 1.4] for $d \geq 4$ and [Junk and Lacoin 2024, Theorem B] for the full details concerning the case $d = 3$). This strict inequality is however not always valid. A simple counterexample is that of the simple random walk when $d = 1$ or 2 (for which $\beta_c = \beta_2 = 0$, and weak disorder trivially holds at β_c). We later show that the equality $\beta_c = \beta_2$ is in fact valid whenever $\beta_2 = 0$; see Theorem 2.9. We may also have $\beta_c = \beta_2$ when $\beta_2 > 0$, for instance if either $\eta = d = 1$ or $\eta = d = 2$; see [Junk and Lacoin 2025, Corollary 2.22]. It is an interesting question whether weak or strong disorder holds at β_c in that case.

To complement our result and to highlight the gap between what we can prove and what we believe to be true, we present a sufficient condition for $\beta_c > \beta_2$. This criterion is related to the tail distribution of the first intersection time of two independent walks

$$T := \inf\{n > 0 : X_n^{(1)} = X_n^{(2)}\},$$

with the convention $\inf \varnothing = \infty$. We then define the exponent α by

$$\alpha := -\limsup_{n\to\infty} \frac{\log P^{\otimes 2}(T \in [n, \infty))}{\log n} \in [0, \infty]. \tag{6}$$

Proposition 2.3. *If $\beta_2 > 0$ and $\alpha > \frac{1}{2}$, then $\beta_c > \beta_2$.*

Remark 2.4. The example with $\beta_c = \beta_2$ from [Junk and Lacoin 2025, Corollary 2.22] has $\alpha = 0$. We believe that one should have $\beta_c > \beta_2$ for all $\alpha > 0$, but the ideas used in our proof — which combine the observations made in [Birkner and Sun 2010, Section 1.4] with pinning model estimates adapted from [Derrida et al. 2009] — clearly stop working whenever $\alpha < \frac{1}{2}$. Note that $\alpha = \frac{1}{2}d - 1$ for the simple random walk and thus the assumption $\alpha > \frac{1}{2}$ is satisfied when $d \geq 4$ but not for $d = 3$.

2.2. *Integrability of W_n^β at criticality.*

We defined the *integrability threshold* exponent as

$$p^*(\beta) := \sup\Big\{p \geq 1 : \sup_{n\geq 1} \mathbb{E}[(W_n^\beta)^p] < \infty\Big\}.$$

It was introduced in [Junk 2025a] and provides detailed information concerning the tail behavior of the partition function. When strong disorder holds, we have $p^*(\beta) = 1$. On the other hand, by [Junk and Lacoin 2025, Corollaries 2.8 and 2.20], if $\eta > 0$ we have $p^*(\beta) > 1$ in the weak disorder phase and in that case we have $\mathbb{P}[W_\infty^\beta \geq u] \asymp u^{-p^*(\beta)}$ (where $f(u) \asymp g(u)$ if $f(u)/g(u)$ is bounded away from 0 and ∞ as $u \to \infty$).

As observed in [Junk and Lacoin 2024], the fact that weak disorder holds at β_c combined with results from [Junk 2025b] allows us to deduce the value of p^* at the critical point. The proof of the following result is identical to the one found in [Junk and Lacoin 2024]. It relies on an extension of [Junk 2025a, Corollary 1.3] to the case of unbounded ω which is proved in [Junk and Lacoin 2025].

Corollary 2.5. *Assuming $(X_k)_{k\geq 0}$ is the simple random walk on $\mathbb{Z}^d$ and $d \geq 3$, we have $p^*(\beta_c) = 1+2/d$.*

Proof. From [Junk 2025b, Theorem 1.2] it is known that $\beta \mapsto p^*(\beta)$ is right-continuous at points where $p^*(\beta) \in (1+2/d, 2]$. Since p^* jumps from $p^*(\beta_c) > 1$ to 1 at β_c and since $\beta_c > \beta_2$ implies that $p^*(\beta_c) \leq 2$, we necessarily have $p^*(\beta_c) \leq 1+2/d$. On the other hand, from [Junk and Lacoin 2025, Corollary 2.8] we have $p^*(\beta_c) \geq 1+2/d$. □

Corollary 2.5 can be extended beyond the case of the simple random walk. We let D denote *the transpose of* the transition matrix of X (defined by $D(x, y) := P(X_1 = x - y)$). With a small abuse of notation we set

$$\|D^k\|_\infty := \max_{x\in\mathbb{Z}^d} D^k(0, x) = \max_{x\in\mathbb{Z}^d} D^k(x, 0) = \max_{x\in\mathbb{Z}^d} P(X_k = x), \quad \nu := -\limsup_{k\to\infty} \frac{\log \|D^k\|_\infty}{\log k}. \tag{7}$$

Let us note that if $X^{(1)} - X^{(2)}$ is transient, η, α and ν (recall (1) and (6)) satisfy the following inequality (we provide a proof in Appendix B for completeness):

$$\nu \leq \alpha + 1 \leq \frac{d}{2\wedge\eta}. \tag{8}$$

Proposition 2.6. *If $\beta_2 > 0$ and weak disorder holds at β_c, then*

$$p^*(\beta_c) \in \Big[1 + \frac{2\wedge\eta}{d}, 1 + \frac{1}{\nu\vee 1}\Big].$$

Remark 2.7. From Theorem 2.1, the assumption made in Proposition 2.6 is satisfied when $\beta_c > \beta_2$, but we do not require that latter condition. At the moment it is not known whether weak disorder holds at β_2 in general. Moreover, the interval $[1 + (2\wedge\eta)/d, 1 + 1/(\nu \vee 1)]$ is ill-defined if $d = 1$ and $\eta > 1$, but in that case the walk $X^{(1)} - X^{(2)}$ is recurrent and $\beta_2 = 0$; recall (5).

Remark 2.8. There are plenty of examples for which $\nu = d/(2\wedge\eta) = \alpha+1$. This includes all d-dimensional random walks where X_1 has a finite second moment and a support that generates $\mathbb{Z}^d$ (or a subgroup of finite index), and symmetric 1-dimensional walks satisfying $P(X_1 = x) = |x|^{-1-\eta+o(1)}$ as $x \to \infty$. In these cases, Proposition 2.6 allows one to identify the value of $p^*(\beta_c)$ if one assumes that weak disorder holds at β_c.

2.3. ***Absence of phase transition in the recurrent case.*** It was established in [Carmona and Hu 2002; Comets et al. 2003] that $\beta_c = 0$ in dimensions 1 and 2 (see also [Comets and Vargas 2006] and [Lacoin 2010] for proofs that $\bar{\beta}_c = 0$ when $d = 1$ and $d = 2$, respectively). These results — and their proofs — suggest that there is no weak disorder phase as soon as $\beta_2 = 0$. We provide a proof of this statement under the minimal assumption that ω admits some finite exponential moments,

$$\{\beta > 0 : \mathbb{E}[e^{\beta\omega_{1,0}}] < \infty\} \neq \varnothing. \tag{9}$$

Note that W_n^β is only defined for β such that $\lambda(\beta) < \infty$ in that case.

Theorem 2.9. *Assume that* (9) *holds. If* $\beta_2 = 0$, *then* $\beta_c = 0$.

For the rest of this section, we return to assuming (2). Combining the above and Theorem 2.1, we obtain as a corollary that very strong disorder holds for all β, provided that the power-tail assumption is satisfied.

Corollary 2.10. *If* $\eta > 0$ *and* $\beta_2 = 0$, *then* $\bar{\beta}_c = 0$.

To illustrate the necessity of the assumption $\eta > 0$ for this last result, we present an example of a polymer model for which $\beta_2 = 0$ and $\bar{\beta}_c = \infty$. We define the tower sequence $(a_k)_{k\geq 0}$ by $a_0 = 1$ and $a_{k+1} := 2^{a_k}$, and a function $f(x)$ on $\mathbb{Z}$ by $f(x) = 1$ if $|x| \leq 1$ and

$$f(x) = \frac{1}{(2a_k + 1)a_{k-1}^4} \quad \text{if } |x| \in (a_{k-1}, a_k] \text{ for } k \geq 1.$$

With the above definition, we have $\sum_{x\in\mathbb{Z}} f(x) \leq 5$. We can thus define $g(x) = f(x)/\sum_{y\in\mathbb{Z}} f(y)$ and consider a simple random walk on $\mathbb{Z}$ whose increment distribution satisfies $P(X_1 = x) = g(x)$.

Proposition 2.11. *For a directed polymer based on the above random walk, we have* $\beta_2 = \beta_c = 0$ *but* $\mathfrak{f}(\beta) = 0$ *for every* $\beta \geq 0$.

2.4. ***Organization.*** In Section 3, we present three technical results. These results are adapted from [Junk and Lacoin 2024] and are used to prove Theorems 2.1 and 2.9, and Proposition 2.3.

In Sections 4 and 5, we prove Theorem 2.1. The proof largely follows the reasoning used in [loc. cit.] to treat the case of an upper-bounded environment, but a couple of technical innovations are required to deal with an unbounded environment and general random walks. While we shortly recap some of the main ideas, we direct the interested reader to [loc. cit.] for more in-depth explanation of the proof mechanism.

In Section 6, we prove Theorem 2.9 using the material from Section 3.

The proof of Proposition 2.3 is based on an observation made in [Birkner and Sun 2010, Section 1.4] and on an adaption of the methods used in [Derrida et al. 2009], which is developed in Appendix A.

One of the bounds in Proposition 2.6 can be derived directly from [Junk and Lacoin 2025, Corollary 2.20] but the other one requires an extension of [Junk 2025b, Theorem 1.1] to the case of an arbitrary random walk. This is done in Appendix B.

Finally, the proof of Proposition 2.11, which is a result of illustrative value, is detailed in Appendix C.

3. A toolbox of preliminary results

We present here a couple of technical results required for our proof of Theorems 2.1 and 2.9. These results can be found in [Junk and Lacoin 2024, Section 3]. We present them here with a couple of key modifications to fit our setup.

3.1. ***A finite volume criterion relying on fractional moments.*** By Jensen's inequality, for any $\theta \in (0, 1)$,

$$\mathbb{E}[\log W_n] = \theta^{-1}\mathbb{E}[\log(W_n)^\theta] \leq \theta^{-1} \log \mathbb{E}[(W_n)^\theta]. \tag{10}$$

Hence, to show that very strong disorder holds, it is sufficient to show that $\mathbb{E}[(W_n)^\theta]$ decays exponentially fast in n. Adapting the argument used to prove [Comets and Vargas 2006, Theorem 3.3], we show that such an exponential decay holds if there exists some n such that $\mathbb{E}[W_n^\theta]$ is smaller than a large power of n. To make this statement precise, we need to fix the value of θ, so, recalling (1), we set[1]

$$\bar{\eta} := \eta \wedge 1, \quad \theta := 1 - \frac{\bar{\eta}}{4d} \quad \text{and} \quad K = \frac{20d}{\bar{\eta}}. \tag{11}$$

Proposition 3.1. *Assume that $\eta > 0$. There exists n_0 such that, if for some $n \geq n_0$ we have*

$$\mathbb{E}[(W_n)^\theta] < 2n^{-K}, \tag{12}$$

then very strong disorder holds.

Proof. From (10), we have

$$\mathfrak{f}(\beta) \leq \liminf_{m\to\infty} \frac{1}{\theta n m} \log \mathbb{E}[(W_{nm})^\theta]. \tag{13}$$

Given $x_1, \dots, x_m \in \mathbb{Z}^d$, we let $\widehat{W}_{nm}(x_1, x_2, \dots, x_m)$ denote the contribution to the partition function of trajectories that go through $x_1, x_2, \dots, x_m$ at times $n, 2n, \dots, nm$, that is,

$$\widehat{W}_{nm}(x_1, x_2, \dots, x_m) := E[e^{\beta \sum_{k=1}^{nm} \omega_{k,X_k} - nm\lambda(\beta)} \mathbb{1}_{\{\forall i \in [\![1,m]\!],\, X_{ni}=x_i\}}].$$

The case $m = 1$ defines the point-to-point partition function,

$$\widehat{W}_n^\beta(x) = E[e^{\beta \sum_{k=1}^{n} \omega_{k,X_k} - n\lambda(\beta)} \mathbb{1}_{\{X_n = x\}}], \tag{14}$$

which will play an important role later. Now, using the inequality

$$\Big(\sum_{i\in I} a_i\Big)^\theta \leq \sum_{i\in I} a_i^\theta, \tag{15}$$

which is valid for an arbitrary collection of nonnegative numbers $(a_i)_{i\in I}$ and $\theta \in (0, 1)$ — in the remainder of the paper we simply say *by subadditivity* when using (15) — we obtain

$$\begin{aligned}\mathbb{E}[(W_{nm})^\theta] &\leq \sum_{(x_1,\dots,x_m)\in(\mathbb{Z}^d)^m} \mathbb{E}[(\widehat{W}_{nm}(x_1, x_2, \dots, x_m))^\theta] = \sum_{(x_1,\dots,x_m)\in(\mathbb{Z}^d)^m} \prod_{i=1}^m \mathbb{E}[(\widehat{W}_n(x_i - x_{i-1}))^\theta] \\ &= \Big(\sum_{x\in\mathbb{Z}^d} \mathbb{E}[(\widehat{W}_n(x))^\theta]\Big)^m,\end{aligned}$$

where the factorization for the first equality is obtained by combining the Markov property for the random walk with the independence of the environment. In order to conclude using (13), we need to show that under the assumption (12) we have $\sum_{x\in\mathbb{Z}^d} \mathbb{E}[(\widehat{W}_n(x))^\theta] < 1$.

We set $R = R_n := n^{16/\bar{\eta}}$. Using the inequality $\widehat{W}_n(x) \leq W_n$ for $|x| \leq R$ and Jensen's inequality for $|x| > R$, we obtain

$$\sum_{x\in\mathbb{Z}^d} \mathbb{E}[(\widehat{W}_n(x))^\theta] \leq (2R+1)^d \mathbb{E}[(W_n)^\theta] + \sum_{|x|>R} P(X_n = x)^\theta. \tag{16}$$

[1]Throughout the paper we use the notation $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$ for $a, b \in \mathbb{R}$.

Let us show that both terms on the right side of (16) are smaller that $\frac{1}{3}$ for n sufficiently large. For the first term, it is simply a consequence of the assumption (12) and our choice for R. To bound the second term, for any $k \geq 0$, we use Jensen's inequality for the uniform measure on the annulus $\{x \in \mathbb{Z}^d : |x| \in (2^k R, 2^{k+1} R]\}$ to obtain

$$\sum_{|x| \in (2^k R, 2^{k+1} R]} P(X_n = x)^\theta \leq (2^{k+2} R)^{d(1-\theta)} P(|X_n| \in (2^k R, 2^{k+1} R])^\theta. \tag{17}$$

Recalling (1), we have $\mathbb{P}(|X_1| \geq u) \leq u^{-\bar{\eta}/2}$ for u sufficiently large. Hence

$$P(|X_n| \in (2^k R, 2^{k+1} R]) \leq P(|X_n| > 2^k R) \leq n P(|X_1| > (2^k R/n)) \leq (2^k R)^{-\bar{\eta}/2} n^{1+\bar{\eta}/2}. \tag{18}$$

Since by choice $d(1-\theta) - \theta\bar{\eta}/2 = -\bar{\eta}/4 + \bar{\eta}^2/8d \leq -\bar{\eta}/8 < 0$, by combining (17) and (18) and summing over k we obtain that there exists $C > 0$ such that

$$\sum_{|x| > R} P(X_n = x)^\theta \leq C n^{\theta(1+\bar{\eta}/2)} R^{-\bar{\eta}/8} = C n^{\theta(\bar{\eta}/2+1)-2}.$$

Since $\bar{\eta}, \theta \leq 1$, this concludes the proof. □

3.2. ***Bounding the fractional moment using the size-biased measure.*** We introduce the size-biased measure for the environment defined by $\widetilde{\mathbb{P}}_n(d\omega) := W_n^\beta \mathbb{P}(d\omega)$. Intuitively, strong disorder holds when $\widetilde{\mathbb{P}}_n$ and $\mathbb{P}$ are "asymptotically singular" (meaning that there exist typical events under $\mathbb{P}$ that become increasingly untypical under $\widetilde{\mathbb{P}}_n$ as n grows). The following result (which is a reformulation of [Junk and Lacoin 2024, Lemma 3.2]) is a quantitative version of this statement:

Lemma 3.2. *For any measurable event A and $\theta \in (0, 1)$,*

$$\mathbb{E}[(W_n^\beta)^\theta] \leq \mathbb{P}(A)^{(1-\theta)} + \widetilde{\mathbb{P}}_n(A^{\mathrm{c}})^\theta.$$

Proof. We adapt the proof found in [Berger et al. 2025, Lemma 2.2]. We split the expectation in two and then bound the first part using Hölder's inequality and the second part using Jensen's inequality,

$$\mathbb{E}[(W_n^\beta)^\theta] = \mathbb{E}[(W_n^\beta)^\theta \mathbb{1}_A] + \mathbb{E}[(W_n^\beta)^\theta \mathbb{1}_{A^{\mathrm{c}}}] \leq \mathbb{E}[W_n^\beta]^\theta \mathbb{P}(A)^{(1-\theta)} + \mathbb{E}[W_n^\beta \mathbb{1}_{A^{\mathrm{c}}}]^\theta,$$

which gives the desired result. □

To prove that $\mathfrak{f}(\beta) < 0$, our strategy is to combine Proposition 3.1 with Lemma 3.2 and find an event A_n which is unlikely under the original measure $\mathbb{P}$ and typical under the size-biased measure $\widetilde{\mathbb{P}}_n$.

3.3. ***Spine representation for the size-biased measure.*** In this section, we recall a well-known representation for the size-biased measure. We define $(\hat{\omega}_i)_{i\geq 1}$ as a sequence of i.i.d. random variables — whose distribution is denoted by $\widehat{\mathbb{P}}$ — with marginal distribution given by

$$\widehat{\mathbb{P}}[\hat{\omega}_1 \in \cdot\,] = \mathbb{E}[e^{\beta\omega_{1,0}-\lambda(\beta)} \mathbb{1}_{\{\omega_{1,0} \in \cdot\}}], \tag{19}$$

and X a random walk with distribution P. Note that $\widehat{\mathbb{P}}$ and $\hat{\omega}$ are unrelated to the notation $\widehat{W}_n^{\beta}(x)$ for the point-to-point partition function introduced in (14). Given ω, $\hat{\omega}$ and X, all sampled independently, we define a new environment $\tilde{\omega} = \tilde{\omega}(X, \omega, \hat{\omega})$ by

$$\tilde{\omega}_{i,x} := \begin{cases} \omega_{i,x} & \text{if } x \neq X_i, \\ \hat{\omega}_i & \text{if } x = X_i. \end{cases}$$

In words, $\tilde{\omega}$ is obtained by tilting the distribution of the environment on the graph of $(i, X_i)_{i=1}^{\infty}$. The following result states that the distribution of $\tilde{\omega}$ corresponds to the size-biased measure. We refer to [Junk and Lacoin 2024, Lemma 3.3] for comments on and a proof of the following classical statement:

Lemma 3.3. $$\widetilde{\mathbb{P}}_n((\omega_{i,x})_{i\in[\![1,n]\!],x\in\mathbb{Z}^d} \in \cdot\,) = P \otimes \mathbb{P} \otimes \widehat{\mathbb{P}}((\tilde{\omega}_{i,x})_{i\in[\![1,n]\!],x\in\mathbb{Z}^d} \in \cdot\,).$$

In the course of our proof we refer to the above as the *spine representation* of the size-biased measure. Under $\widetilde{\mathbb{P}}_n$, the distribution of the environment has been tilted along a random trajectory X, which we refer to as the *spine*.

4. Organization of the proof of Theorem 2.1

Let us start by reformulating the result. We want to show that the following implication holds:

$$\mathbb{P}(W_{\infty}^{\beta} = 0) = 1 \ \text{ and } \ \beta > \beta_2 \implies \mathfrak{f}(\beta) < 0. \tag{20}$$

It is a simple task to check that (20) implies both statements in Theorem 2.1.

4.1. ***Identifying the right event.*** Recalling (11), we introduce a new family of parameters. We set

$$K_2 := \frac{2}{\eta}\Big(\frac{K}{1-\theta} + 1\Big) + 1 \quad \text{and} \quad K_3 := 1 + dK_2 + \frac{2K}{1-\theta}. \tag{21}$$

Using a union bound, we have, for n sufficiently large,

$$P\Big(\max_{k\in[\![1,n]\!]} |X_k| \geq n^{K_2}\Big) \leq nP(|X_1| \geq n^{K_2-1}) \leq \tfrac{1}{2}n^{-K/(1-\theta)}. \tag{22}$$

We introduce the shifted environment $\theta_{n,z}\omega$ by setting

$$(\theta_{n,z}\omega)_{k,x} := \omega_{n+k,z+x}, \tag{23}$$

and let $\theta_{n,z}$ act on functions of ω by setting $\theta_{n,k} f(\omega) = f(\theta_{n,z}\omega)$.

Proposition 4.1. *Assume that strong disorder holds and $\beta > \beta_2$. Then there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exists $s = s_n \in [\![0, C\log n]\!]$ such that, setting*

$$A_n := \{\exists (y, m) \in [\![0, n-s]\!] \times [\![-n^{K_2}, n^{K_2}]\!]^d : \theta_{m,y} W_s^{\beta} \geq n^{K_3}\},$$

we have

$$\mathbb{P}(A_n) \leq n^{-K/(1-\theta)} \quad \text{and} \quad \widetilde{\mathbb{P}}_n(A_n^{\mathrm{c}}) \leq n^{-K/\theta}.$$

Proof of Theorem 2.1 assuming Proposition 4.1. We assume that strong disorder holds and that $\beta > \beta_2$. Combining Lemma 3.2 and Proposition 4.1, for θ and K given by (11), we have, for any $n \geq n_0$,

$$\mathbb{E}[(W_n^\beta)^\theta] \leq (n^{-K/(1-\theta)})^{1-\theta} + (n^{-K/\theta})^\theta = 2n^{-K},$$

and we can conclude the proof of (20) using Proposition 3.1 for n sufficiently large. □

Bounding $\mathbb{P}(A_n)$ is not difficult (and the bound is valid for any choice of s). Indeed, by translation invariance for ω, a union bound and Markov's inequality, we have

$$\mathbb{P}(A_n) \leq n(2n^{K_2}+1)^d \mathbb{P}[W_s^\beta \geq n^{K_3}] \leq n(2n^{K_2}+1)^d n^{-K_3} \leq n^{-K/\theta}, \tag{24}$$

where the last inequality is valid for n sufficiently large, due to our choice of parameters (21).

The harder problem is to estimate $\widetilde{\mathbb{P}}_n(A_n^{\mathrm{c}})$. The strategy we use for this is the same as in [Junk and Lacoin 2024]. The main task is to obtain a good lower bound on $\mathbb{P}(W_s^\beta \geq n^{K_3})$ — and hence on $\widetilde{\mathbb{P}}_s(W_s^\beta \geq n^{K_3})$ — in the strong disorder regime. For this, we extend the validity of [loc. cit., Proposition 4.2] (which is proved only for upper-bounded disorder and the simple random walk):

Proposition 4.2. *If strong disorder holds and $\beta > \beta_2$, then for any $\varepsilon > 0$ there exist $C = C(\beta, \varepsilon) > 0$ and $u_0 = u_0(\beta, \varepsilon) > 1$ such that for every $u \geq u_0$,*

$$\mathbb{P}[W_s^\beta \geq u] \geq u^{-(1+\varepsilon)} \text{ for some } s \in [\![0, C\log u]\!]. \tag{25}$$

Note that $\mathbb{P}[W_s^\beta \geq u] \geq u^{-(1+\varepsilon)}$ immediately implies that

$$\widetilde{\mathbb{P}}_s[W_s^\beta \geq u] = \mathbb{E}[W_s^\beta \mathbb{1}_{\{W_s^\beta \geq u\}}] \geq u^{-\varepsilon}. \tag{26}$$

The proof follows the same road map as in [Junk and Lacoin 2024], but several technical adaptions are required. This is detailed in the next subsections.

4.2. ***Tail distribution of the maximum of the point-to-point partition function.*** The starting point in proving Proposition 4.2 is to establish that the tail distribution of $\max_{n\geq 0} W_n^\beta$ decreases like u^{-1} in the strong disorder regime. In the case of upper-bounded disorder, this is an easy consequence of the martingale stopping theorem; see [Junk and Lacoin 2024, Lemma 4.3]. Without this assumption, the proof is more subtle, but it has already been obtained in [Junk and Lacoin 2025, Corollary 2.6 and Proposition 2.19] (we consider the special case when $p^*(\beta) = 1$).

Lemma 4.3. *If strong disorder holds, then there exists $c > 0$ such that, for any $u \geq 1$,*

$$\mathbb{P}(\exists n \geq 0 : W_n^\beta \geq u) \in \left[\frac{c}{u}, \frac{1}{u}\right].$$

Proof. The lower bound is due to [Junk and Lacoin 2025, Corollary 2.6] and the upper bound holds for any nonnegative martingale due to Doob's martingale inequality. □

To prove Proposition 4.2, we show that a similar lower bound holds for the maximum of *point-to-point* partition functions introduced in (14).

Theorem 4.4. *If strong disorder holds and $\beta > \beta_2$, then there exists $c > 0$ such that for all $u \geq 1$*

$$\mathbb{P}\Big[\sup_{n\geq 0,\, x\in\mathbb{Z}^d} \widehat{W}_n^{\beta}(x) \geq u\Big] \geq \frac{c}{u}. \tag{27}$$

Proof of Proposition 4.2. The inequality in (25) is deduced from (27) in [Junk and Lacoin 2024, Section 6], without relying on the assumption that the environment is bounded above. That argument also does not rely on the specifics of the simple random walk, so we merely sketch the proof. We set

$$A_{v,M} := \Big\{\sup_{n\in[\![0,M]\!],\, x\in[\![-M,M]\!]^d} \widehat{W}_n^{\beta}(x) \geq v\Big\}$$

to be a truncated analog of the event considered in (27). For fixed $v > 1$, using (27) we can find $M(v)$ such that $\mathbb{P}(A_{v,M(v)}) \geq c/2v$. On $A := A_{v,M(v)}$, we let (T, Y) be the minimal element (for the lexicographical order) in $[\![1, M]\!] \times [\![-M, M]\!]^d$ such that $\widehat{W}_T^{\beta}(Y) \geq v$. Choosing (T, Y) minimal guarantees that the shifted environment $\theta_{T,Y}\omega$ is independent of the past and distributed like ω. Note that on the event $A \cap \theta_{T,Y} A$ we have $\max_{(t,y)\in[\![1,2M]\!]\times[\![-2M,2M]\!]^d} \widehat{W}_t(y) \geq v^2$, and by independence $\mathbb{P}(A \cap \theta_{T,Y} A) = \mathbb{P}(A)^2 \geq (c/2v)^2$. Repeating this argument, we find that

$$\mathbb{P}(\exists n \leq kM,\ W_n^{\beta} \geq v^k) \geq \Big(\frac{c}{2v}\Big)^k \geq v^{-k(1+\varepsilon/2)},$$

where the last inequality holds for v large enough (depending on ε). From this lower bound, (25) is deduced easily (with $u_0 = v$ and $C = 2M/\log v$) by replacing u by v^k with $k = \lceil \log_v u \rceil$ and observing that

$$\max_{s\in[\![1,C\log u]\!]} \mathbb{P}[W_s^{\beta} \geq u] \geq \frac{1}{C\log u}\mathbb{P}[\exists s \in [\![1, C\log u]\!] : W_s^{\beta} \geq u]. \qquad \square$$

To conclude, we need to show that Proposition 4.1 follows from Proposition 4.2, and we need to prove Theorem 4.4. The first task is accomplished in the next subsection. The proof of Theorem 4.4 is technically more demanding and is detailed in Section 5.

4.3. ***Proof of Proposition 4.1.*** The following adapts the argument presented in [Junk and Lacoin 2024, Section 6] to the setup of a generic random walk.

Recall from (24) that the required bound on $\mathbb{P}(A_n)$ is valid for any choice of s. We can thus focus on bounding $\widetilde{\mathbb{P}}(A_n^{\mathrm{c}})$, assuming that Proposition 4.2 holds. We apply Proposition 4.2 for $u = n^{K_3}$ and $\varepsilon = 1/(3K_3)$ and we consider $s \in [\![0, C' \log n]\!]$, which is such that (recall (26))

$$\widetilde{\mathbb{P}}_s[W_s^{\beta} \geq n^{K_3}] \geq n^{-K_3\varepsilon} = n^{-1/3}. \tag{28}$$

Using the spine representation, we wish to bound $P \otimes \widehat{\mathbb{P}} \otimes \mathbb{P}[\tilde{\omega} \in A_n^{\mathrm{c}}]$. Clearly $\{\tilde{\omega} \in A_n^{\mathrm{c}}\} \subset B_n^{(1)} \cup B_n^{(2)}$, where

$$B_n^{(1)} := \Big\{\max_{k\in[\![1,n]\!]} |X_k| > n^{K_2}\Big\}, \quad B_n^{(2)} := \{\forall i \in [\![0, \lfloor n/s \rfloor - 1]\!] : \theta_{is,X_{is}} \widetilde{W}_s^{\beta} < n^{K_3}\},$$

and where we recall that $(X_k)_{k\geq 0}$ denotes the spine. Now from (22) we have

$$P \otimes \widehat{\mathbb{P}} \otimes \mathbb{P}(B_n^{(1)}) = P(B_n^{(1)}) \leq \tfrac{1}{2} n^{-K/(1-\theta)}.$$

To estimate the probability of $B_n^{(2)}$, we observe that, by construction, the variables $\theta_{is,X_{is}}\widetilde{W}_s^\beta$ are i.i.d. under $P\otimes\widehat{\mathbb{P}}\otimes\mathbb{P}$ (see [Junk and Lacoin 2024, Lemma 5.1] for a proof of this claim). Hence

$$P\otimes\widehat{\mathbb{P}}\otimes\mathbb{P}[B_n^{(2)}]=P\otimes\widehat{\mathbb{P}}\otimes\mathbb{P}[\widetilde{W}_s^\beta<n^{K_3}]^{\lfloor n/s\rfloor}=\widetilde{\mathbb{P}}_s[W_s^\beta<n^{K_3}]^{\lfloor n/s\rfloor}\le(1-n^{-1/3})^{\lfloor n/s\rfloor}\le e^{-\sqrt{n}},$$

where the second equality is a consequence of Lemma 3.3, the first inequality comes from (28) and the last (valid for n sufficiently large) from the fact that $s=s_n$ is $O(\log n)$. Overall, we obtain that, for n sufficiently large,

$$\widetilde{\mathbb{P}}_n(A_n^{\mathrm{c}})\le P\otimes\widehat{\mathbb{P}}\otimes\mathbb{P}[B_n^{(1)}]+P\otimes\widehat{\mathbb{P}}\otimes\mathbb{P}[B_n^{(2)}]\le\tfrac{1}{2}n^{-K/(1-\theta)}+e^{-\sqrt{n}}\le n^{-K/(1-\theta)}.\qquad\square$$

5. Proof of Theorem 4.4

5.1. *Overview.* The reasoning we use to prove Theorem 4.4 is analogous to that in [Junk and Lacoin 2024]. We explain here how it can be decomposed in three separate steps. For this we require some notation. We let μ_n denote the endpoint measure for the polymer and I_n the probability that two independent polymers share the same endpoint. More precisely, recalling that D is the transpose of the transition matrix of X (see above (7)), we set

$$\mu_n^\beta(x):=P_{\omega,n}^\beta(X_n=x)=\frac{\widehat{W}_n^\beta(x)}{W_n^\beta},\quad I_n:=\sum_{x\in\mathbb{Z}^d}(D\mu_{n-1}^\beta(x))^2. \tag{29}$$

We have $D\mu_{n-1}^\beta(x)=P_{\omega,n-1}^\beta(X_n=x)$ — the only reason why we introduce D as the transpose of the transition matrix is for the convenience of multiplying on the left by D rather than on the right. To justify the expression of I_n, let us mention that the quantity naturally appears when computing the bracket increment of W_n,

$$\mathbb{E}[(W_n^\beta-W_{n-1}^\beta)^2\mid\mathcal{F}_{n-1}]=\chi(\beta)(W_{n-1}^\beta)^2I_n,\quad\text{where }\ \chi(\beta)=e^{\lambda(2\beta)-2\lambda(\beta)}-1. \tag{30}$$

We introduce the notation $I_{(a,b]}:=\sum_{n=a+1}^b I_n$ and set $\tau_u:=\inf\{n:W_n\ge u\}$. The first step is now to show that if W_n increases from level u to uK, the accumulated sum of I_n increases by an amount proportional to $\log K$.

Proposition 5.1. *If strong disorder holds then there exist $\zeta>0$ and $K_0>0$ such that for all $u\ge1$ and $K>K_0$ we have*

$$\mathbb{P}(\tau_{Ku}<\infty\,;\,I_{(\tau_u,\tau_{Ku}]}\le\zeta\log K)\le u^{-1}K^{-2}.$$

The second step establishes that with large probability, if $I_{(\tau_u,\tau_{Ku}]}$ is large then I_n cannot be small on the whole interval $(\tau_u,\tau_{Ku}]$.

Proposition 5.2. *If strong disorder holds and $\beta>\beta_2$, then for any $\zeta>0$ there exists $\delta>0$ such that, for all $u>1$ and $K\ge K_0$ large enough,*

$$\mathbb{P}\Big(\tau_{Ku}<\infty\,;\,I_{(\tau_u,\tau_{Ku}]}\ge\zeta\log K\,;\,\max_{n\in(\tau_u,\tau_{Ku}]}I_n\le\delta\Big)\le u^{-1}K^{-2}.$$

Finally, in a third very short step we guarantee that with large probability there are no significant dips of W_n^β between τ_u and τ_{Ku}. We set $\sigma_{K,u} := \min\{n \geq \tau_u : W_n^\beta \leq u/K\}$.

Lemma 5.3. $$\mathbb{P}(\sigma_{K,u} < \tau_{Ku} < \infty) \leq u^{-1}K^{-2}.$$

Proof. Note that $\{\sigma_{K,u} < \tau_{Ku} < \infty\} \subset \{\tau_u < \infty\,;\, \exists n \geq 0,\ W^\beta_{n+\sigma_{K,u}} \geq K^2 W^\beta_{\sigma_{K,u}}\}$. Applying the optional stopping theorem to the martingale $(W^\beta_{n+\sigma_{K,u}})$ (considering the filtration $\mathcal{F}_{n+\sigma_{K,u}}$), we obtain that on the event $\{\tau_u < \infty\}$ we have

$$\mathbb{P}[\exists n \geq 0 : W^\beta_{n+\sigma_{K,u}} \geq K^2 W^\beta_{\sigma_{K,u}} \mid \mathcal{F}_{\sigma_{K,u}}] \leq K^{-2}.$$

We conclude by taking expectation on the event $\{\tau_u < \infty\}$ and using Lemma 4.3. □

We can now deduce Theorem 4.4. Before presenting the proof, let us explain how Propositions 5.1 and 5.2 are proved. Proposition 5.1 is the analog of [Junk and Lacoin 2024, Proposition 8.1] but it requires a different proof since several of the arguments used in [loc. cit.] are specific to upper-bounded disorder. All the details are provided in Section 5.2. On the other hand, the proof of Proposition 5.2 is identical to that of [loc. cit., Proposition 8.2], and we only provide a sketch of the argument in Section 5.3.

Proof of Theorem 4.4. We have

$$\Big\{\tau_{Ku} < \sigma_{K,u}\,;\, \max_{n\in(\tau_u,\tau_{Ku}]} I_n > \delta\Big\} \subseteq \Big\{\max_{n\geq 0,\, x\in\mathbb{Z}^d} \widehat{W}_n^\beta(x) \geq \frac{\delta u}{K}\Big\}. \tag{31}$$

Indeed, if $n \in (\tau_u, \tau_{Ku}]$ is such that $I_n \geq \delta$, and $\tau_{Ku} < \sigma_{K,u}$, we have $W^\beta_{n-1} \geq u/K$ and thus

$$\max_{x\in\mathbb{Z}^d} \widehat{W}^\beta_{n-1}(x) \geq \max_{x\in\mathbb{Z}^d} D\widehat{W}^\beta_{n-1}(x) = W^\beta_{n-1} \max_{x\in\mathbb{Z}^d} D\mu_{n-1}(x) \geq W^\beta_{n-1} I_n \geq \frac{\delta u}{K}.$$

We estimate the probability of the left side event in (31) as follows: first we observe that

$$\begin{aligned}\mathbb{P}\big(\tau_{Ku} < \sigma_{u,K}\,;\, &\max_{n\in(\tau_u,\tau_{Ku}]} I_n > \delta\big) \\ &\geq \mathbb{P}(\tau_{Ku} < \infty) - \mathbb{P}(\sigma_{K,u} < \tau_{Ku} < \infty) - \mathbb{P}\big(\tau_{Ku} < \infty\,;\, \max_{n\in(\tau_u,\tau_{Ku}]} I_n \leq \delta\big),\end{aligned} \tag{32}$$

and then that

$$\begin{aligned}\mathbb{P}\big(\tau_{Ku} < \infty\,;\, &\max_{n\in(\tau_u,\tau_{Ku}]} I_n \leq \delta\big) \\ &\leq \mathbb{P}(\tau_{Ku} < \infty\,;\, I_{(\tau_u,\tau_{Ku}]} \leq \zeta \log K) + \mathbb{P}\big(\tau_{Ku} < \infty\,;\, I_{(\tau_u,\tau_{Ku}]} > \zeta \log K;\, \max_{n\in(\tau_u,\tau_{Ku}]} I_n \leq \delta\big).\end{aligned} \tag{33}$$

Using Propositions 5.1 and 5.2 in (33), we obtain that

$$\mathbb{P}\big(\tau_{Ku} < \infty\,;\, \max_{n\in(\tau_u,\tau_{Ku}]} I_n \leq \delta\big) \leq 2u^{-1}K^{-2}.$$

Then, combining this with Lemmas 4.3 and 5.3 in (32), we obtain that

$$\mathbb{P}\big(\tau_{Ku} < \sigma_{u,K}\,;\, \max_{n\in(\tau_u,\tau_{Ku}]} I_n > \delta\big) \geq \frac{c}{Ku} - \frac{3}{K^2 u},$$

which allows us to conclude using (31) by taking $K \geq 6/c$. □

5.2. ***Proof of Proposition 5.1.*** Note that in an unbounded environment, we have to deal with the possibility that $\tau_{uK} = \tau_u$ and thus $I_{(\tau_u,\tau_{uK}]} = 0$ (for upper-bounded ω we a priori have $W^\beta_{\tau_u} \le Lu$ for some fixed L, and one may choose $K \ge L$). There is another more serious reason why the argument used in [Junk and Lacoin 2024] requires modification, but to explain it we need to recall it. The argument relies on the martingale $\widetilde{M}_n$ defined by $\widetilde{M}_0 = 0$ and

$$\widetilde{M}_{n+1} = \widetilde{M}_n + \frac{W^\beta_{n+1} - W^\beta_n}{W^\beta_n}.$$

We have, by construction, $\widetilde{M}_n - \widetilde{M}_m \ge \log W^\beta_n - \log W^\beta_m$ for any $n \ge m$, and the quadratic variation of this martingale is directly related to I_n via the relation (recall (30))

$$\langle \widetilde{M} \rangle_b - \langle \widetilde{M} \rangle_a = \chi(\beta) I_{(a,b]}.$$

We observe that if W^β_n increases by a multiplicative amount K, then $\widetilde{M}$ has to increase by at least $\log K$. Such an increase can occur only if the increase of the bracket $\langle \widetilde{M} \rangle$ is large. In order to derive satisfactory quantitative estimates from this line of reasoning, we rely on Azuma-like concentration results for martingales that require boundedness of the increments of $\widetilde{M}$; see [Junk and Lacoin 2024, Lemma 8.4]. In the present setup, the assumption (2) only guarantees that the moments of $(\widetilde{M}_{n+1} - \widetilde{M}_n)_{n\ge 1}$ are all finite, and thus such an exponential concentration result cannot hold with full generality.

For this reason, we consider, instead of $(\widetilde{M}_n)$, the martingale (M_n) defined as the martingale part in the Doob decomposition of $(\log W_n)_{n\in\mathbb{N}}$, that is to say

$$M_n = \sum_{k=1}^{n} \Big(\log(W^\beta_k / W^\beta_{k-1}) - \mathbb{E}[\log(W^\beta_k / W^\beta_{k-1}) \mid \mathcal{F}_{k-1}]\Big).$$

This martingale enjoys similar properties as $\widetilde{M}$. For instance, since the conditional expectations in the previous display are nonpositive by Jensen's inequality, we have for $n \ge m$

$$M_n - M_m \ge \log W^\beta_n - \log W^\beta_m, \tag{34}$$

and the increments of the bracket of M_n are also proportional to I_n (recall the meaning of $\asymp$ introduced at the beginning of Section 2.2),

$$\langle M \rangle_n - \langle M \rangle_{n-1} := \mathbb{E}[(M_n - M_{n-1})^2 \mid \mathcal{F}_{n-1}] \asymp I_n. \tag{35}$$

One of the two inequalities implied by the $\asymp$ symbol is proved in [Comets et al. 2003] and can be deduced directly from (37) in Lemma 5.7 below. The other one is left as an exercise to the interested reader and is not used in the proof.

Our strategy is thus to adapt the argument used in [Junk and Lacoin 2024] by applying it to the martingale $(M_n)_{n\in\mathbb{N}}$. In addition to (34) and (35), we also need a concentration result similar to [Junk and Lacoin 2024, Lemma 8.4]. To prove it, we require an overshoot estimate which guarantees that $W^\beta_{\tau_u}$ is much smaller than uK with large probability, which has been proved in [Junk and Lacoin 2025].

Proposition 5.4 [Junk and Lacoin 2025, Proposition 2.3]. *For any $p \geq 1$ there exists C_p such that, for all $u > 1$,*

$$\mathbb{E}[(W^\beta_{\tau_u})^p \mid \tau_u < \infty] \leq C_p u^p.$$

Now let us turn to our replacement for [Junk and Lacoin 2024, Lemma 8.4], which we derive from the following technical estimate:

Lemma 5.5. *Let $(U_i)_{i\geq 1}$ be a sequence of positive, i.i.d. random variables with moments of all orders (positive and negative) and $\mathbb{E}[U_i] = 1$. There exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $r \geq 0$ and every nonnegative sequence $(\alpha_i)_{i\geq 0}$ such that $\sum_{i\geq 1} \alpha_i = 1$,*

$$\mathbb{E}[e^{rV}] \leq e^{\varphi(r) \sum_{i\geq 1} \alpha_i^2},$$

where $V := \log U - \mathbb{E}[\log U]$ with $U = \sum_{i\geq 1} \alpha_i U_i$.

We postpone the proof of Lemma 5.5 to the end of this section. Applying Lemma 5.5 to the increments of M_n, we obtain the following estimate, which is going to yield the desired concentration property:

Lemma 5.6. *Let φ be given by Lemma 5.5 in the case where U_i has the same law as $e^{\beta\omega_{1,0}-\lambda(\beta)}$. For every $v \geq 0$ and $u > 1$, we have*

$$\mathbb{E}[e^{v(M_{\tau_{Ku}} - M_{\tau_u}) - \varphi(v) I_{(\tau_u,\tau_{Ku}]}} \mathbb{1}_{\{\tau_{Ku}<\infty\}} \mid \tau_u < \infty] \leq 1.$$

Proof. Recalling (29), we have

$$\frac{W^\beta_n}{W^\beta_{n-1}} = \sum_{x\in\mathbb{Z}^d} D\mu^\beta_{n-1}(x) e^{\beta\omega_{n,x}-\lambda(\beta)}.$$

Since $D\mu_{n-1}(x)$ is $\mathcal{F}_{n-1}$ measurable and $\omega_{x,n}$ is independent of $\mathcal{F}_{n-1}$, we can apply Lemma 5.5 with $(\alpha_i)_{i\geq 1}$ replaced by $(D\mu_{n-1}(x))_{x\in\mathbb{Z}^d}$ and $(U_i)_{i\geq 1}$ by $(e^{\beta\omega_{x,n}-\lambda(\beta)})_{x\in\mathbb{Z}^d}$. This yields

$$\mathbb{E}[e^{v(M_n - M_{n-1})} \mid \mathcal{F}_{n-1}] \leq e^{\varphi(v) I_n}.$$

Applying this observation at times $\tau_u + n - 1$ and $\tau_u + n$ yields, with $\mathcal{G}_n := \mathcal{F}_{\tau_u+n}$,

$$\mathbb{E}[e^{v(M_{n+\tau_u} - M_{n-1+\tau_u}) - \varphi(v) I_{\tau_u+n}} \mid \mathcal{G}_{n-1}] \leq 1,$$

and hence $(e^{vM_{n+\tau_u} - \varphi(v) I_{(\tau_u,\tau_u+n]}})_{n\geq 0}$ is a supermartingale for the filtration $(\mathcal{G}_n)$. Combining the optional stopping theorem (at time $n \wedge (\tau_{Ku} - \tau_u)$) and conditional Fatou (to let $n \to \infty$), we obtain

$$\mathbb{E}[e^{v(M_{\tau_{Ku}} - M_{\tau_u}) - \varphi(v) I_{(\tau_u,\tau_{Ku}]}} \mathbb{1}_{\{\tau_{Ku}<\infty\}} \mid \mathcal{G}_0] \leq \mathbb{E}[\liminf_{m\to\infty} e^{v(M_{\tau_{Ku}\wedge m} - M_{\tau_u}) - \varphi(v) I_{(\tau_u,\tau_{Ku}\wedge m]}} \mid \mathcal{G}_0] \leq 1,$$

which yields the result after taking expectation over the $\mathcal{G}_0$-measurable event $\{\tau_u < \infty\}$. □

Proof of Proposition 5.1. For this proof only, we set $\overline{\mathbb{P}}_u := \mathbb{P}(\,\cdot \mid \tau_u < \infty)$. From Lemma 4.3, it is sufficient to prove that

$$\overline{\mathbb{P}}_u(\tau_{Ku} < \infty\,;\, I_{(\tau_u,\tau_{Ku}]} \leq \zeta \log K) \leq K^{-2}.$$

We have

$$\overline{\mathbb{P}}_u(\tau_{Ku}<\infty\,;\,I_{(\tau_u,\tau_{Ku}]}\le\zeta\log K)\le\overline{\mathbb{P}}_u(W^\beta_{\tau_u}>\sqrt{K}u)+\overline{\mathbb{P}}_u(\tau_{Ku}<\infty\,;\,I_{(\tau_u,\tau_{Ku}]}\le\zeta\log K;\,W^\beta_{\tau_u}\le\sqrt{K}u).$$

We are going to prove that both terms in the right-hand side are smaller than $\frac{1}{2}K^{-2}$. Using Markov's inequality and Proposition 5.4 , there exists a constant $C>0$ such that

$$\overline{\mathbb{P}}_u(W^\beta_{\tau_u}>\sqrt{K}u)\le\frac{\bar{\mathbb{E}}_u[(W_{\tau_u})^6]}{K^3u^6}\le CK^{-3},$$

yielding the right result if $K_0>2C$. For the second term, if $\tau_{Ku}<\infty$ and $W^\beta_{\tau_u}\le\sqrt{K}u$ then (34) implies that $M_{\tau_{Ku}}-M_{\tau_u}\ge\frac{1}{2}\log K$. Hence Lemma 5.6 for $v=6$ implies that

$$\begin{aligned}1&\ge\bar{\mathbb{E}}_u[e^{6(M_{\tau_{Ku}}-M_{\tau_u})-\varphi(6)I_{(\tau_u,\tau_{Ku}]}}\mathbb{1}_{\{\tau_{Ku}<\infty;\,W^\beta_{\tau_u}\le\sqrt{K}u;\,I_{(\tau_u,\tau_{Ku}]}\le\zeta\log K\}}]\\&\ge K^{3-\varphi(6)\zeta}\,\overline{\mathbb{P}}_u(\tau_{Ku}<\infty\,;\,I_{(\tau_u,\tau_{Ku}]}\le\zeta\log K\,;\,W^\beta_{\tau_u}\le\sqrt{K}u).\end{aligned}$$

Setting $\zeta=1/(2\varphi(6))$, this implies that, for $K>4$,

$$\overline{\mathbb{P}}_u(\tau_{Ku}<\infty\,;\,I_{(\tau_u,\tau_{Ku}]}\le\zeta\log K;\,W^\beta_{\tau_u}\le\sqrt{K}u)\le K^{-5/2}\le\tfrac{1}{2}K^{-2},$$

and thus the desired result. □

It remains to prove Lemma 5.5. We rely on an estimate for the two first moments of $\log U$ (as defined in Lemma 5.5) proved in [Comets et al. 2003]. More precisely, in [loc. cit.] the result is stated in the case of convex combination of finitely many variables, but the proof extends immediately to countable sums by passing to the limit. Let us display it here:

Lemma 5.7 [Comets et al. 2003, Lemma 3.1]. *Let $(U_i)_{i\ge1}$ be a sequence of positive i.i.d. random variables such that $\mathbb{E}[U_1^3+(\log U_1)^2]<\infty$ and $\mathbb{E}[U_i]=1$. There exist $c,C>0$ such that, for any nonnegative sequence $(\alpha_i)_{i\ge0}$ such that $\sum_{i\ge1}\alpha_i=1$ and setting $U:=\sum_{i\ge1}\alpha_iU_i$,*

$$c\sum_{i\ge1}\alpha_i^2\le\mathbb{E}[\log(1/U)]\le C\sum_{i\ge1}\alpha_i^2,\tag{36}$$

and

$$\mathbb{E}[(\log U)^2]\le C\sum_{i\ge1}\alpha_i^2.\tag{37}$$

Proof of Lemma 5.5. Without loss of generality we may assume that $r\ge\frac{1}{4}$. Setting $\overline{V}=\log U$, we have

$$\mathbb{E}[e^{rV}]=\mathbb{E}[e^{r\overline{V}}]e^{-r\mathbb{E}[\log U]}$$

and we use (36) to control the second term on the right-hand side. For the first term, note that Taylor's formula implies that

$$\mathbb{E}[e^{r\overline{V}}]\le\mathbb{E}\big[1+r\overline{V}+\tfrac{1}{2}r^2(\overline{V})^2e^{r\overline{V}_+}\big]\le1+\mathbb{E}\big[\tfrac{1}{2}r^2(\overline{V})^2e^{r\overline{V}_+}\big],$$

where $x_+ := \max(x, 0)$ denotes the positive part of x. Hence, using the inequality $1 + x \le e^x$, we can conclude, if we can define φ which does not depend on the α_i and satisfies

$$\mathbb{E}\Big[\tfrac{1}{2} r^2 \overline{V}^2 e^{r\overline{V}_+}\Big] \le \varphi(r) \sum_{i\ge 1} \alpha_i^2.$$

We have

$$\mathbb{E}\Big[\tfrac{1}{2} r^2 \overline{V}^2 e^{r\overline{V}_+}\Big] \le \tfrac{1}{2} e^r r^2 \mathbb{E}[\overline{V}^2 \mathbb{1}_{\{\overline{V}\le 1\}}] + \mathbb{E}[\mathbb{1}_{\{\overline{V}>1\}} e^{2r\overline{V}}].$$

To bound the first term we simply observe that by (37) we have

$$\mathbb{E}[\overline{V}^2 \mathbb{1}_{\{\overline{V}\le 1\}}] \le \mathbb{E}[(\log U)^2] \le C \sum_{i\ge 1} \alpha_i^2.$$

As for the second one, we have

$$\mathbb{E}[\mathbb{1}_{\{\overline{V}\ge 1\}} e^{2r\overline{V}}] \le \mathbb{P}(\overline{V} \ge 1)^{1/2} \mathbb{E}[e^{4r\overline{V}}]^{1/2}.$$

To bound the second factor in a way that does not depend on the α_i, we observe that by Jensen's inequality (recall that we assume $4r \ge 1$) we have

$$\mathbb{E}[e^{4r\overline{V}}] = \mathbb{E}[U^{4r}] \le \mathbb{E}[U_1^{4r}].$$

To conclude the proof it is sufficient to show that

$$\mathbb{P}(\overline{V} \ge 1) \le C \sum_{i\ge 1} \alpha_i^4, \tag{38}$$

and to observe that $\big(\sum_{i\ge 1} \alpha_i^4\big)^{1/2} \le \sum_{i\ge 1} \alpha_i^2$ by subadditivity. We have

$$\mathbb{P}(\overline{V} \ge 1) = \mathbb{P}[U \ge e] \le \mathbb{P}[\exists i : U_i \ge \alpha_i^{-1/2}] + \mathbb{P}\Big[\sum_{i\ge 1} \alpha_i U_i \mathbb{1}_{\{U_i \le \alpha_i^{-1/2}\}} \ge e\Big].$$

The first term can be bounded by $\mathbb{E}[U_1^8] \sum_{i\ge 1} \alpha_i^4$ using union bound and Markov's inequality for U_i^8. The second term can be controlled using Chernoff's inequality: for any $a > 0$,

$$\begin{aligned}\mathbb{P}\Big[\sum_{i\ge 1} \alpha_i U_i \mathbb{1}_{\{U_i \le \alpha_i^{-1/2}\}} \ge e\Big] &\le \mathbb{P}\Big[\sum_i \alpha_i (U_i - 1) \mathbb{1}_{\{U_i \le \alpha_i^{-1/2}\}} \ge e - 1\Big] \\ &\le \mathbb{E}\Big[\exp\Big(a\Big(\sum_{i\ge 1} \alpha_i (U_i - 1) \mathbb{1}_{\{U_i \le \alpha_i^{-1/2}\}} - (e-1)\Big)\Big)\Big].\end{aligned}$$

Using it with $a = (\alpha_{\max})^{-1/2}$ we obtain (using the inequality $e^x \le 1 + x + x^2$, valid for $|x| \le 1$)

$$\begin{aligned}\mathbb{E}[\exp(\alpha_{\max}^{-1/2} \alpha_i (U_i - 1) \mathbb{1}_{\{U_i \le \alpha_i^{-1/2}\}})] &\le 1 + \mathbb{E}[\alpha_{\max}^{-1/2} \alpha_i (U_1 - 1) \mathbb{1}_{\{U_1 \le \alpha_i^{-1/2}\}}] + \mathbb{E}[\alpha_{\max}^{-1} \alpha_i^2 (U_1 - 1)^2] \\ &\le 1 + C\alpha_{\max}^{-1} \alpha_i^2 \le 1 + C\alpha_i \le e^{C\alpha_i}.\end{aligned}$$

The penultimate term in the first line above is nonpositive due to the FKG inequality. Multiplying over all $i \ge 1$, we obtain

$$\mathbb{P}\Big[\sum_i \alpha_i U_i \mathbb{1}_{\{U_i \le \alpha_i^{-1/2}\}} \ge e\Big] \le \exp(C - (\alpha_{\max})^{-1/2}(e-1)) \le \frac{e^C 8!}{(e-1)^4} \alpha_{\max}^4,$$

concluding the proof of (38) (in the last inequality we use $e^{-\sqrt{x}} \le 8! x^{-4}$ for $x \ge 0$). ☐

5.3. ***Proof of Proposition 5.2.*** Our proof closely follows the one presented in [Junk and Lacoin 2024]. However, since some technical adjustments are necessary, we provide a sketch of it. In this section we are going to condition on the event $\{\tau_u < \infty\}$ and consider the conditional probability $\mathbb{P}_u := \mathbb{P}[\,\cdot \mid \mathcal{F}_{\tau_u}]$. We let $\mathcal{T} = \mathcal{T}(u, \zeta, K) := \inf\{m \geq \tau_u : I_{(\tau_u, m]} \geq \zeta \log K\}$ and we are going to prove that if $\beta > \beta_2$ then there exists a $\delta > 0$ such that

$$\mathbb{P}_u\Big(\mathcal{T} < \infty\,;\ \max_{n \in (\tau_u, \mathcal{T}]} I_n \leq \delta\Big) \leq K^{-2}. \tag{39}$$

Since $\{I_{(\tau_u, \tau_{Ku}]} \geq \zeta \log K\} \subset \{\mathcal{T} < \infty\}$, the bound (39) directly implies that

$$\mathbb{P}_u\Big(\max_{n \in (\tau_u, \tau_{Ku}]} I_n \leq \delta\,;\ I_{(\tau_u, \tau_{Ku}]} \geq \zeta \log K\Big) \leq K^{-2}, \tag{40}$$

and we obtain Proposition 5.2 by taking expectation in (40) on the event $\{\tau_u < \infty\}$, which has probability smaller than $1/u$ (see Lemma 4.3).

Let us now describe the main idea we use to prove (39). Recalling from (30) that $\chi(\beta) := e^{\lambda(2\beta) - 2\lambda(\beta)} - 1$, we introduce a parameter $\varepsilon > 0$ defined by (recall (4) and the assumption $\beta > 0$)

$$\chi(\beta) \sum_{n=1}^{\infty} P^{\otimes 2}(X_n^{(1)} = X_n^{(2)}) =: 1 + 4\varepsilon. \tag{41}$$

We show that $I_n \leq \delta$, for δ sufficiently small, implies that a certain bounded stochastic process (J_n) — which is obtained by considering a positive quadratic form based on the Green function evaluated at μ_n and is defined below — has a "positive drift" proportional to I_n in the sense that

$$I_n \leq \delta \implies \mathbb{E}[J_n - J_{n-1} \mid \mathcal{F}_{n-1}] \geq 2\varepsilon I_n. \tag{42}$$

The above implies that on the event $\{\mathcal{T} < \infty\,;\ \max_{n \in (\tau_u, \mathcal{T}]} I_n \leq \delta\}$, the accumulated drift of J on the interval $(\tau_u, \mathcal{T}]$ is large (at least $2\varepsilon\zeta \log K$) and since J is bounded, it must necessarily be compensated by the martingale part of J taking a large negative value. We then show that the latter is unlikely using martingale concentration estimates.

Let us now introduce the functional J. Recalling (41), we fix $n_0 > 0$ such that

$$\chi(\beta) \sum_{n=1}^{n_0} P^{\otimes 2}(X_n^{(1)} = X_n^{(2)}) \geq 1 + 3\varepsilon \tag{43}$$

and introduce G_0, the Green function associated with $X^{(1)} - X^{(2)}$ truncated at time n_0,

$$g_0(x) := \sum_{n=1}^{n_0} P^{\otimes 2}(X_n^{(1)} - X_n^{(2)} = x), \qquad G_0(x, y) := g_0(y - x).$$

The transition matrix of $X^{(1)} - X^{(2)}$ is given by $T := DD^* = D^*D$ (note that since we are only considering convolution operators on $\mathbb{Z}^d$ they all commute) and $G_0 = (T - T^{(n_0+1)})/(1 - T)$. Next we define

$$J_n := (\mu_n, G_0\mu_n) = \frac{(\widehat{W}_n^\beta, G_0 \widehat{W}_n^\beta)}{(W_n^\beta)^2},$$

where, if a and b are such that $\sum_{x\in\mathbb{Z}^d} |a(x)||b(x)| < \infty$, we use the notation

$$(a, b) = \sum_{x\in\mathbb{Z}^d} a(x)b(x).$$

At this point, it is maybe difficult to see the intuition for choosing J_n in this way, and we refer to the discussion following [Junk and Lacoin 2024, Proposition 7.7] as well as its proof.

The truncation to level n_0 in the definition of g_0 is necessary to treat the case when the random walk $X^{(1)} - X^{(2)}$ is recurrent, but even in the transient case there is a technical reason why we prefer to consider G_0 rather than $G := T/(1-T)$. This is because it is convenient to have $\|g_0\|_1 < \infty$. Indeed, using first the Cauchy–Schwarz inequality, then $\|g * f\|_2 \le \|g\|_1\|f\|_2$ and finally $\|g_0\|_1 = n_0$, we have

$$0 \le J_n \le \|G_0\mu_n^\beta\|_2\|\mu_n^\beta\|_2 \le \|g_0\|_1\|\mu_n^\beta\|_2^2 = n_0\|\mu_n^\beta\|_2^2. \tag{44}$$

While the above inequality is not required to establish that J_n is bounded (we have $J_n \le \|g_0\|_\infty$), it plays a crucial role in the proof of Lemma 5.9. We consider Doob's decomposition of J_n,

$$J_n - J_0 = N_n + A_n,$$

where A_n and N_n are defined by $A_0 = 0$, $N_0 = 0$ and

$$A_{n+1} = A_n + \mathbb{E}[J_{n+1} - J_n \mid \mathcal{F}_n], \quad N_{n+1} = N_n + (J_{n+1} - J_n) - \mathbb{E}[J_{n+1} - J_n \mid \mathcal{F}_n].$$

The implication (42) is proved using the following lower bound on the increments of (A_n) (recall that $\chi(\beta)g_0(0) - 1 > 0$ by construction):

Lemma 5.8. *We have*

$$A_n - A_{n-1} \ge (\chi(\beta)g_0(0) - 1)I_n - 4\chi(\beta)((D\mu_{n-1}^\beta)^2, G_0\mu_{n-1}^\beta) - 2\chi_3(\beta)\sum_{x\in\mathbb{Z}^d} D\mu_{n-1}^\beta(x)^3, \tag{45}$$

where $\chi_3(\beta) := \mathbb{E}[(e^{\beta\omega_{1,0}-\lambda(\beta)} - 1)^3] = e^{\lambda(3\beta)-3\lambda(\beta)} - 3e^{\lambda(2\beta)-2\lambda(\beta)} + 2$. *As a consequence, there exists a constant* C (*which depends on* β) *such that*

$$A_n - A_{n-1} \ge (\chi(\beta)g_0(0) - 1)I_n - CI_n^{3/2}. \tag{46}$$

Since $J_n \ge 0$, the above bound can be used to control the increments of (N_n). Next, we obtain the following:

Lemma 5.9. *We have*, *almost surely for every* n,

$$\mathbb{E}[(N_n - N_{n-1})^2 \mid \mathcal{F}_{n-1}] \le \kappa I_n^2,$$

where (recall (7)) $\kappa := n_0^2(\|D\|_\infty^{-4} + e^{(\lambda(8\beta)+\lambda(-8\beta))/2})$.

The proofs of Lemmas 5.8 and 5.9 replicate those of the analogous statements proved for the simple random walk [Junk and Lacoin 2024, Lemmas 8.5 and 8.6]. At the end of the section we provide indications about the changes that are required. The last input needed for the proof of Proposition 5.2 is a concentration result for bounded martingales from [loc. cit.].

Lemma 5.10 [Junk and Lacoin 2024, Lemma 8.4]. *Let* (N_n) *be a discrete-time martingale starting at* 0, *with increments that are bounded in absolute value by* $A > 0$. *For* $v > 0$, *let* T_v *be the first time that* (N_n) *hits* $[v, \infty)$. *For any* $a > 0$, *we have*

$$P[\langle N\rangle_{T_v} \le a\,;\, T_v < \infty] \le e^{-v/(A+1)(\log(v/a(A+1))-1)}.$$

As a consequence, for any stopping time T *we have*

$$P[\langle N\rangle_T \le a\,;\, T < \infty\,;\, N_T \ge v] \le e^{-v/(A+1)(\log(v/a(A+1))-1)}.$$

Proof of (39) *assuming Lemmas 5.8 and 5.9.* If $I_n \le \delta$, recalling (43), we have from (46),

$$A_n - A_{n-1} \ge I_n(\chi(\beta)g_0(0) - 1 - C\delta^{1/2}) \ge 2\varepsilon I_n,$$

where the second inequality is valid for $\delta \le \delta_0(\varepsilon)$ sufficiently small. Thus, if $\mathcal{T} < \infty$ and $\max_{n\in(\tau_u,\mathcal{T}]} I_n \le \delta$, we have

$$A_{\mathcal{T}} - A_{\tau_u} \ge 2\zeta\varepsilon \log K. \tag{47}$$

Using (44) and the trivial bound $\|\mu_n^\beta\|_2 \le 1$, we have

$$N_{\mathcal{T}} - N_{\tau_u} = A_{\tau_u} - A_{\mathcal{T}} + J_{\mathcal{T}} - J_{\tau_u} \le A_{\tau_u} - A_{\mathcal{T}} + n_0. \tag{48}$$

Hence (48) and (47) imply that, for $K \ge K_0(n_0, \zeta, \varepsilon)$ sufficiently large,

$$N_{\mathcal{T}} - N_{\tau_u} \le -\zeta\varepsilon \log K.$$

On the other hand, if $\max_{n\in(\tau_u,\mathcal{T}]} I_n \le \delta$, we have, from Lemma 5.9 and the definition of $\mathcal{T}$,

$$\langle N\rangle_{\mathcal{T}} - \langle N\rangle_{\tau_u} \le \kappa \sum_{n=\tau_u+1}^{\mathcal{T}} I_n^2 \le \kappa\left(\delta \sum_{n=\tau_u+1}^{\mathcal{T}-1} I_n + \delta^2\right) \le \kappa\delta(\zeta \log K + \delta) \le 2\kappa\delta\zeta \log K.$$

Thus we obtain

$$\mathbb{P}_u\Big(\max_{n\in(\tau_u,\mathcal{T}]} I_n \le \delta\,;\, \mathcal{T} < \infty\Big) \le \mathbb{P}_u(\mathcal{T} < \infty\,;\, N_{\mathcal{T}} - N_{\tau_u} \le -\zeta\varepsilon \log K\,;\, \langle N\rangle_{\mathcal{T}} - \langle N\rangle_{\tau_u} \le 2\kappa\delta\zeta \log K).$$

Using Lemma 5.10 for the martingale $(N_{\tau_u} - N_{\tau_u+m})_{m\ge 0}$, stopping time $\mathcal{T} - \tau_u$ with $A = n_0$ (which is a bound for the increments of N; see (44)), $v = \zeta\varepsilon \log K$ and $a = 2\kappa\zeta\delta \log K$, we have

$$\mathbb{P}_u(N_{\tau_u} - N_{\mathcal{T}} \ge v\,;\, \mathcal{T} < \infty\,;\, \langle N\rangle_{\mathcal{T}} - \langle N\rangle_{\tau_u} \le a) \le e^{-v/(n_0+1)(\log(v/a(n_0+1))-1)}.$$

To conclude the proof of (39), and hence of Proposition 5.2, we need to check that the exponent is smaller than $-2\log K$, which is immediate if one chooses δ sufficiently small (since ε is fixed). □

Proof of Lemma 5.8. Let us first explain how (46) is deduced from (45). We have

$$\sum_{x\in\mathbb{Z}^d} D\mu_{n-1}^{\beta}(x)^3 \le I_n \|D\mu_{n-1}^{\beta}\|_\infty \le I_n^{3/2},$$
$$((D\mu_{n-1}^{\beta})^2, G_0\mu_{n-1}^{\beta}) \le I_n\|G_0\mu_{n-1}^{\beta}\|_\infty \le I_n\|g_0\|_1\|\mu_{n-1}^{\beta}\|_\infty \le \frac{n_0}{\|D\delta_0\|_\infty} I_n^{3/2},$$

where in the last inequality above we used the fact that $\|\mu_{n-1}^{\beta}\|_\infty \le \|D\mu_{n-1}^{\beta}\|_\infty/\|D\delta_0\|_\infty$. These bounds replace those given in the display following [Junk and Lacoin 2024, (61)]. The proof of (45) is now identical to the one presented in the simple random walk case, namely [loc. cit., Lemma 8.5]. The only modification needed is to replace D^2 by $T = DD^*$ in [loc. cit., (67)]. ☐

Proof of Lemma 5.9. Using (44), we have

$$\begin{aligned}\mathbb{E}[(N_n - N_{n-1})^2 \mid \mathcal{F}_{n-1}] &= \mathbb{E}[(J_n - J_{n-1})^2 \mid \mathcal{F}_{n-1}] - \mathbb{E}[J_n - J_{n-1} \mid \mathcal{F}_{n-1}]^2 \le \mathbb{E}[J_n^2 + J_{n-1}^2 \mid \mathcal{F}_{n-1}]\\ &\le n_0^2(\mathbb{E}[\|\mu_n^{\beta}\|_2^4 \mid \mathcal{F}_{n-1}] + \|\mu_{n-1}^{\beta}\|_2^4).\end{aligned}$$

To bound the second term we observe that $(D\mu)(x) \ge \|D\|_\infty \mu(x + z_{\max})$, where $z_{\max} \in \mathbb{Z}^d$ satisfies $D(0, z_{\max}) = \|D\|_\infty$, and thus $I_n = \|D\mu_{n-1}^{\beta}\|_2^2 \ge \|D\|_\infty^2 \|\delta_{z_{\max}} * \mu_{n-1}^{\beta}\|_2^2 = \|D\|_\infty^2 \|\mu_{n-1}^{\beta}\|_2^2$. To conclude, we need to prove that

$$\mathbb{E}[\|\mu_n^{\beta}\|_2^4 \mid \mathcal{F}_{n-1}] \le e^{(\lambda(8\beta)+\lambda(-8\beta))/2} I_n^2,$$

which can be done by replicating the argument in the proof of [loc. cit., Lemma 8.6]. ☐

6. Proof of Theorem 2.9

6.1. ***Proof strategy.*** Let us assume that $\beta_2 = 0$ and fix $\beta > 0$ such that $\lambda(2\beta) < \infty$. We are going to identify a sequence of events A_n which is such that

$$\lim_{n\to\infty} \mathbb{P}(A_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \widetilde{\mathbb{P}}_n(A_n^{\mathrm{c}}) = 0. \tag{49}$$

From Lemma 3.2, this implies that $\lim_{n\to\infty} \mathbb{E}[(W_n^{\beta})^{\theta}] = 0$ for $\theta \in \left[\frac{1}{2}, 1\right)$ and hence W_n^{β} converges to zero in probability. More precisely, we are going to prove that (49) holds along a diverging sequence of integers n (which is of course sufficient for our purpose).

The key point is to find an observable R_n (a function of ω) whose value is greatly affected by the size biasing. Aiming for the simplest possible choice, we take a linear combination of the $(\omega_{k,x})$. We set $p_k(x) := P(X_k = x)$ and define

$$R_n := \sum_{k=1}^{n} \sum_{x\in\mathbb{Z}^d} p_k(x)\omega_{k,x}.$$

The choice of the coefficients is made with the spine representation in mind (recall Section 3.3), since $p_k(x)$ is the probability that the site (k, x) is visited by the spine. By (3), we have $\mathbb{E}[R_n] = 0$ and

$$\mathbb{E}[R_n^2] = \sum_{k=1}^{n} \sum_{x\in\mathbb{Z}^d} p_k(x)^2 =: \Sigma_n. \tag{50}$$

On the other hand, from Lemma 3.3 we have

$$\widetilde{\mathbb{E}}_n[R_n] = \hat{\mathbb{E}} \otimes E\left[\sum_{k=1}^{n} p_k(X_k)\hat{\omega}_k\right] = \lambda'(\beta)\Sigma_n, \tag{51}$$

where in the last identity we used that $\hat{\mathbb{E}}[\hat{\omega}_k] = \lambda'(\beta)$, (which is an immediate consequence of the definition (19)). Recalling (5), since $\beta_2 = 0$ and Σ_n is the expected number of return to zero before time n of $(X_k^{(1)} - X_k^{(2)})_{k\geq 0}$, we have $\lim_{n\to\infty} \Sigma_n = \infty$. Now (50) implies that the typical order of magnitude of R_n under $\mathbb{P}$ is at most $\sqrt{\Sigma_n}$. On the other hand, (51) indicates that R_n may be typically much larger — of order Σ_n — under $\widetilde{\mathbb{P}}_n$. This motivates our definition for the event

$$A_n := \{R_n \geq \Sigma_n^{3/4}\}. \tag{52}$$

We are going to prove the following:

Proposition 6.1. *For any $\beta > 0$ such that $\lambda(2\beta) < \infty$ and A_n defined as in* (52), *we have*

$$\lim_{n\to\infty} \mathbb{P}(A_n) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} \widetilde{\mathbb{P}}_n(A_n^{\mathrm{c}}) = 0.$$

A consequence of the above and Lemma 3.2 is that $\liminf_{n\to\infty} \mathbb{E}[(W_n^\beta)^\theta] = 0$ for $\theta \in \big[\frac{1}{2}, 1\big)$ and thus that strong disorder holds, proving Theorem 2.9.

6.2. *Proof of Proposition 6.1.* The first equation directly follows from (50), since $\lim_{n\to\infty} \Sigma_n = \infty$ and

$$\mathbb{P}(A_n) \leq \mathbb{E}[R_n^2](\Sigma_n)^{-3/2} = (\Sigma_n)^{-1/2}.$$

For the second we use Lemma 3.2. We have

$$\widetilde{\mathbb{P}}_n(A_n^{\mathrm{c}}) = \mathbb{P} \otimes \widehat{\mathbb{P}} \otimes P(\tilde{\omega}(\omega, \hat{\omega}, X) \in A_n^{\mathrm{c}}).$$

To estimate the above, we split $R_n(\tilde{\omega})$ into two parts:

$$R_n(\tilde{\omega}) = \lambda'(\beta)\sum_{k=1}^{n} p_k(X_k) + \sum_{k=1}^{n}\sum_{x\in\mathbb{Z}^d} p_k(x)(\tilde{\omega}_{k,x} - \lambda'(\beta)\mathbb{1}_{\{X_k=x\}}) =: R_n^{(1)}(X) + R_n^{(2)}(\omega, \hat{\omega}, X).$$

Hence

$$\mathbb{P} \otimes \widehat{\mathbb{P}} \otimes P(R_n(\tilde{\omega}) < \Sigma_n^{3/4}) \leq P(R_n^{(2)}(\omega, \hat{\omega}, X) \leq -\Sigma_n^{3/4}) + \mathbb{P} \otimes \widehat{\mathbb{P}} \otimes P(R_n^{(1)}(X) \leq 2\Sigma_n^{3/4}).$$

We are going to show that along some subsequence, both terms on the right-hand side converge to zero. Starting with $R_n^{(2)}(\omega, \hat{\omega}, S)$, we have for every realization of X

$$\mathbb{E} \otimes \hat{\mathbb{E}}[R_n^{(2)}(\omega, \hat{\omega}, X)^2] \leq \Sigma_n + (\lambda''(\beta) - 1)\sum_{k=1}^{n} p_k(X_k)^2 \leq \max(1, \lambda''(\beta))\Sigma_n$$

(we have used that $\hat{\mathbb{E}}[(\omega - \lambda'(\beta))^2] = \lambda''(\beta)$), so that by Chebyshev's inequality

$$\mathbb{P} \otimes \widehat{\mathbb{P}} \otimes P(R_n^{(2)}(\omega, \hat{\omega}, X) \leq -\Sigma_n^{3/4}) \leq \max(1, \lambda''(\beta))\Sigma_n^{-1/2}.$$

Hence we have the desired convergence to zero. To conclude, we need to show that

$$\liminf_{n\to\infty} P(R_n^{(1)}(X) \le 2\Sigma_n^{3/4}) = 0. \tag{53}$$

Contrary to $R_n^{(2)}$, the variable $R_n^{(1)}$ is not — in full generality — concentrated around its mean. This is the reason why we do not employ the second moment method to prove (53). To add symmetry to the problem, it is convenient to consider, rather than $R_n^{(1)}(X)$, a random variable with an extra layer of randomness whose conditional mean is equal to $R_n^{(1)}(X)$. More specifically, we let X' denote a simple random walk which is independent of X and has the same distribution (we let $P \otimes P'$ denote the distribution of (X, X')), and set

$$Q_n(X, X') := \lambda'(\beta) \sum_{k=1}^{n} \mathbb{1}_{\{X_k = X'_k\}}.$$

We have $R_n^{(1)}(X) = E'[Q_n(X, X')]$, and thus

$$P(R_n^{(1)}(X) \le 2\Sigma_n^{3/4}) \le 2P \otimes P'(Q_n(X, X') \le 4\Sigma_n^{3/4}). \tag{54}$$

Let us quickly justify (54). If $f(X, X')$ is a nonnegative function and $u \ge 0$, we have by Markov inequality

$$E'[f(X, X')] \le u \implies P'(f(X, X') \le 2u) \ge \tfrac{1}{2},$$

and thus, using Markov inequality again for the variable $Y = P'(f(X, X') \le 2u)$,

$$P(E'[f(X, X')] \le u) \le P\big(Y \ge \tfrac{1}{2}\big) \le 2E[Y] = 2P \otimes P'(f(X, X') \le 2u).$$

The remaining step is to bound the right-hand side in (54). For notational simplicity, we set $\boldsymbol{P} := P \otimes P'$ and introduce the renewal process

$$\tau := \{k \ge 0 : X_k = X'_k\}.$$

With this new notation, we have

$$P \otimes P'(Q_n(X, X') \le 4\Sigma_n^{3/4}) = \boldsymbol{P}\big(\lambda'(\beta)|\tau \cap [1, n]| \le 4\boldsymbol{E}[|\tau \cap [1, n]|]^{3/4}\big).$$

We conclude the proof of (53), and hence of Proposition 6.1, by applying the following technical result (proved in the next subsection) to the renewal τ. □

Lemma 6.2. *Given $c > 0$ and τ a recurrent renewal process, we have*

$$\liminf_{n\to\infty} \boldsymbol{P}\big(|\tau \cap [1, n]| \le c\boldsymbol{E}[|\tau \cap [1, n]|]^{3/4}\big) = 0. \tag{55}$$

6.3. *Proof of Lemma 6.2.* With a small abuse of notation, we identify the set τ with an increasing sequence $(\tau_i)_{i\ge 0}$. We set $\alpha_n := \boldsymbol{E}[\tau_1 \wedge n]$ and $\overline{K}(n) := \boldsymbol{P}(\tau_1 > n)$. Setting $\bar{\tau}_k := \sum_{i=1}^{k} (\tau_i - \tau_{i-1}) \wedge n$, we have for any $k \ge 1$

$$\boldsymbol{P}(|\tau \cap [1, n]| \le k) = \boldsymbol{P}(\tau_{k+1} > n) = \boldsymbol{P}(\bar{\tau}_{k+1} > n) \le \frac{\boldsymbol{E}[\bar{\tau}_{k+1}]}{n} = \frac{(k+1)\alpha_n}{n}. \tag{56}$$

On the other hand,

$$\boldsymbol{P}(|\tau\cap[1,n]|\geq i)\leq \mathbb{P}(\forall j\in[1,i]:\tau_{i+1}-\tau_i\leq n)=(1-\overline{K}(n))^i,$$

which implies after summing over i that

$$\boldsymbol{E}[|\tau\cap[1,n]|]\leq(\overline{K}(n))^{-1}\wedge n.$$

Replacing k in (56) by $c((\overline{K}(n))^{-1}\wedge n)^{3/4}$, we obtain that (55) holds if

$$\liminf_{n\to\infty}((\overline{K}(n))^{-1}\wedge n)^{3/4}\frac{\alpha_n}{n}=0. \tag{57}$$

If $\liminf_{n\to\infty}\alpha_n n^{-1/4}=0$, then there is nothing to prove. Otherwise we consider n such that

$$\alpha_{n+1}(n+1)^{-1/5}\geq\alpha_n n^{-1/5} \tag{58}$$

(the fact that $\alpha_n n^{-1/5}$ diverges implies in particular that it is not eventually decreasing, so that one can find an infinite sequence satisfying the above). Since $\alpha_{n+1}=\alpha_n+\overline{K}(n)$, (58) implies that for n sufficiently large

$$\overline{K}(n)\geq\left(\left(1+\frac{1}{n}\right)^{1/5}-1\right)\alpha_n\geq\frac{\alpha_n}{6n},$$

and we obtain that, along the subsequence satisfying (58),

$$((\overline{K}(n))^{-1}\wedge n)^{3/4}\frac{\alpha_n}{n}\leq 6(\alpha_n/n)^{1/4}.$$

Since $\alpha_n=o(n)$ by dominated convergence, this is sufficient to conclude that (57) holds. □

Appendix A: Proof of Proposition 2.3

Assume that $\beta_2>0$ and $\alpha>\frac{1}{2}$ (recall (6)). We will show that there exist $\beta>\beta_2$ and $\gamma\in(0,1)$ such that

$$\sup_{n\geq 0}\mathbb{E}[(W_n^\beta)^{1+\gamma}]=\sup_{n\geq 0}\tilde{\mathbb{E}}_n[(W_n^\beta)^\gamma]<\infty. \tag{59}$$

This implies uniform integrability of W_n^β, and hence that weak disorder holds at $\beta>\beta_2$. The first step of the proof, inspired from [Birkner and Sun 2010, Section 1.4], is to reduce the problem to the control of the partition function of another model, *the disordered pinning model*. The argument is based on the size-biased representation from Lemma 3.3. Let $\widetilde{W}_n^\beta:=E'[e^{\sum_{k=1}^n(\beta\tilde{\omega}_{k,X'_k}-\lambda(\beta))}]$ (where X' with law P' has the same distribution as X), and observe that

$$\tilde{\mathbb{E}}_n[(W_n^\beta)^\gamma]=\hat{\mathbb{E}}\otimes\mathbb{E}\otimes E[(\widetilde{W}_n^\beta)^\gamma]\leq\hat{\mathbb{E}}[(\mathbb{E}\otimes E[\widetilde{W}_n^\beta])^\gamma]. \tag{60}$$

Now, using the notation $\delta_n:=\mathbb{1}_{\{X_n=X'_n\}}$ and $\boldsymbol{P}=P\otimes P'$, we introduce notation $(\widehat{Z}_n)$ for the partition function appearing in the right-hand side of (60), as well as a counterpart $\widehat{Z}_n^c$ with constrained endpoint:

$$\widehat{Z}_n:=\mathbb{E}\otimes E[\widetilde{W}_n^\beta]=\boldsymbol{E}[e^{\sum_{k=1}^n(\beta\hat{\omega}_k-\lambda(\beta))\delta_k}],\quad \widehat{Z}_n^c:=\boldsymbol{E}[e^{\sum_{k=1}^n(\beta\hat{\omega}_k-\lambda(\beta))\delta_k}\delta_n], \tag{61}$$

with the convention $\widehat{Z}_0 = \widehat{Z}_0^c = 1$. With (59) and (60), the proof of Proposition 2.3 is reduced to proving that

$$\sup_{n\geq 0} \hat{\mathbb{E}}[(\widehat{Z}_n)^\gamma] < \infty.$$

To show the above, we rely on a method developed in [Derrida et al. 2009]. The notable differences with [loc. cit.] are that we are trying to control the unconstrained partition function rather than the constrained one and that we make no regularity assumptions on the interarrival law $K(n) := \boldsymbol{P}(\tau_1 = n)$, but these differences only result in minor modifications in the argument. We refer to [loc. cit.] for more insight concerning the proof.

We fix an integer $m \geq 1$ and set $\gamma = 1 - 1/\log m$ and $\beta = \beta_2 + 1/m^2$. We define the shifted environment $\theta_j \hat{\omega}$ by setting $(\theta_j \hat{\omega})_k = \hat{\omega}_{j+k}$ and the shifted partition functions $\theta_j \widehat{Z}_n$ by replacing $\hat{\omega}$ by $\theta_j \hat{\omega}$ in (61). Decomposing the partition function according to the value of the last renewal point before m (variable a) and first renewal point between m and n (variable b), we obtain that

$$\widehat{Z}_n = \sum_{a=0}^{m-1} \widehat{Z}_a^c \Big(\sum_{b=m}^{n} K(b-a) e^{\beta\hat{\omega}_b - \lambda(\beta)} \theta_b \widehat{Z}_{n-b} + \boldsymbol{P}(\tau > n-a) \Big).$$

Thus, setting $A_n = \hat{\mathbb{E}}[(\widehat{Z}_n)^\gamma]$ and $B_n = \hat{\mathbb{E}}[(\widehat{Z}_n^c)^\gamma]^{1/\gamma}$ and using the subadditivity of $x \mapsto x^\gamma$, we obtain

$$A_n \leq \sum_{a=0}^{m-1} B_a^\gamma \Big(\sum_{b=m}^{n} K(b-a)^\gamma \hat{\mathbb{E}}[e^{\gamma\beta\hat{\omega}_b - \gamma\lambda(\beta)}] A_{n-b} + \boldsymbol{P}(\tau > n-a)^\gamma \Big). \tag{62}$$

Let us now set $\bar{A}_n := \max(A_0, \ldots, A_n)$ and observe that with our choice for β and γ,

$$\hat{\mathbb{E}}[e^{\gamma\beta\hat{\omega}_1 - \gamma\lambda(\beta)}] = \mathbb{E}[e^{(1+\gamma)\beta\hat{\omega}_1 - (1+\gamma)\lambda(\beta)}] \leq e^{\lambda(2(\beta_2+1)) - 2\lambda(\beta_2+1)} =: \rho.$$

The inequality (62) implies that

$$A_n \leq \rho \Big(\sum_{a=0}^{m-1} \sum_{b=m}^{n} B_a^\gamma K(b-a)^\gamma \Big) \bar{A}_{n-m} + \Big(\sum_{a=0}^{m-1} B_a^\gamma \Big) \leq 2\rho \Big(\sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a^\gamma K(b-a)^\gamma \Big) \bar{A}_{n-m} + m. \tag{63}$$

To obtain the first line we used that $\boldsymbol{P}(\tau > n-a)^\gamma \leq 1$, and in the second line we used the bound $B_a^\gamma \leq 2$, which will be proved below in (67). From (63), we deduce that $(A_n)_{n\in\mathbb{N}}$ is bounded if

$$2\rho \Big(\sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a^\gamma K(b-a)^\gamma \Big) < 1. \tag{64}$$

We are going to prove that (64) holds for m sufficiently large. We do so by combining two arguments. The first is the observation that by taking γ sufficiently close to one, we can, at the cost of some small error term, drop the power of γ in our sum.

Lemma A.1. *For $\gamma = 1 - 1/\log m$ and $\beta = \beta_2 + 1/m^2$, there exists $C > 0$ such that for all m sufficiently large we have*

$$\sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a^\gamma K(b-a)^\gamma \leq C \Big(\sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a K(b-a) \Big) + Cm^{-1}.$$

The second point is to demonstrate, via a change of measure argument, that for most n we have $B_n \ll \hat{\mathbb{E}}[\widehat{Z}_n^c]$, and that, as a consequence, the sum we wish to control is small.

Lemma A.2. *For $\gamma = 1 - 1/\log m$ and $\beta = \beta_2 + 1/m^2$, we have*

$$\lim_{m\to\infty} \sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a K(b-a) = 0.$$

The combination of Lemmas A.1 and A.2 allows us to conclude that (64) holds.

Proof of Lemma A.1. The idea is that $B_a^\gamma K(b-a)^\gamma \le C B_a K(b-a)$ for the values of a and b that most contribute to the sum and bound the remainder of the contribution by Cm^{-1}. Letting κ be a large integer and using that $\gamma = 1 - 1/\log m$, we have

$$\sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a^\gamma K(b-a)^\gamma \\ \le e^{\kappa/\gamma} \sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a K(b-a) \mathbb{1}_{\{B_a^\gamma K(b-a)^\gamma \ge m^{-\kappa}\}} + \sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a^\gamma K(b-a)^\gamma \mathbb{1}_{\{B_a^\gamma K(b-a)^\gamma < m^{-\kappa}\}}.$$

To control the second term, we split it into two. As the double sum below contains $m^{\kappa-1}$ terms, we have

$$\sum_{a=0}^{m-1} \sum_{b=m}^{m+m^{\kappa-2}-1} B_a^\gamma K(b-a)^\gamma \mathbb{1}_{\{B_a^\gamma K(b-a)^\gamma < m^{-\kappa}\}} \le m^{-1}.$$

On the other hand, using again $B_a \le 1$, we have

$$\sum_{a=0}^{m-1} \sum_{b>m+m^{\kappa-2}} B_a^\gamma K(b-a)^\gamma \le m \sum_{r \ge m^{\kappa-2}} K(r)^\gamma.$$

Finally, we observe that for m sufficiently large,

$$\sum_{r\ge m^{\kappa-2}} K(r)^\gamma \le \sum_{r\ge m^{\kappa-2}} \max(r^{\alpha/3} K(r), r^{-2}) \le C(m^{-(\kappa-2)\alpha/3} + m^{2-\kappa}) \le m^{-2}. \tag{65}$$

The first inequality above boils down considering separately the cases $K(r) \ge r^{-3}$ (for which the first bound holds provided that $\log m \ge 9/\alpha$) and $K(r) < r^{-3}$ (for which the second bound holds provided that m is not too small). For the second inequality we recall the definition (6) of α and observe that, grouping terms, we have for n sufficiently large,

$$\sum_{r\ge n} r^{\alpha/3} K(r) \le \sum_{k\ge 1} (2^k n)^{\alpha/3} \boldsymbol{P}(\tau_1 \in [n2^{k-1}, n2^k)) \le \sum_{k\ge 1} (2^k n)^{\alpha/3} (2^k n)^{-3\alpha/4} \le n^{-\alpha/3}.$$

The last inequality in (65) is valid provided κ is chosen sufficiently large (we have not chosen an explicit κ to underline that this step of the proof only requires $\alpha > 0$). □

Proof of Lemma A.2. Given $X^{(1)}$ and $X^{(2)}$ two independent random walks with distribution P, we let $L(X^{(1)}, X^{(2)}) := \sum_{k=1}^{\infty} \mathbb{1}_{\{X_k^{(1)}=X_k^{(2)}\}}$ denote the number of times they meet. By the strong Markov property, $L(X^{(1)}, X^{(2)})$ is a geometric random variables with parameter

$$P^{\otimes 2}(L(X^{(1)}, X^{(2)}) = 0) = 1 - \sum_{n\geq 1} K(n).$$

Hence by (4) we have

$$1 = (e^{\lambda(2\beta_2)-2\lambda(\beta_2)} - 1)E^{\otimes 2}[L(X^{(1)}, X^{(2)})] = (e^{\lambda(2\beta_2)-2\lambda(\beta_2)} - 1)\frac{\sum_{n\geq 1} K(n)}{1-\sum_{n\geq 1} K(n)}. \tag{66}$$

Let us set $K''(n) = e^{\lambda(2\beta_2)-2\lambda(\beta_2)}K(n)$. By (66) we have $\sum_{n\geq 1} K''(n) = 1$. We let τ'' denote a renewal process with interarrival law K'' (let $\boldsymbol{P}''$ denote the distribution). We are going to show that

$$\lim_{m\to\infty} \sum_{a=0}^{m-1} \sum_{b=m}^{\infty} B_a K''(b-a) = 0.$$

A first idea might be to note that, by Jensen's inequality, we have

$$B_a \leq \hat{\mathbb{E}}[\widehat{Z}_a^c] = \boldsymbol{E}[e^{(\lambda(2\beta)-2\lambda(\beta))\sum_{k=1}^a \delta_k}\delta_a] \leq e^{Ca/m^2}\boldsymbol{E}[e^{(\lambda(2\beta_2)-2\lambda(\beta_2))\sum_{k=1}^a \delta_k}\delta_a] \leq 2\boldsymbol{P}''(a\in\tau''), \tag{67}$$

where in the second inequality we used the fact that with our choice of β,

$$(\lambda(2\beta) - 2\lambda(\beta)) \leq (\lambda(2\beta_2) - 2\lambda(\beta_2)) + Cm^{-2},$$

and the last inequality is valid for m sufficiently large since $a \leq m$. Hence

$$\sum_{a=0}^{m-1}\sum_{b=m}^{\infty} B_a K''(b-a) \leq 2\sum_{a=0}^{m-1} \boldsymbol{P}''(a\in\tau'')\boldsymbol{P}''(\tau_1'' \geq m-a) = 2$$

(the summand in the right-hand side is the probability that a is the last renewal point before m). Since this is not sufficient for our purposes, the idea in our proof is to use something sharper than Jensen's inequality in (67). We use Hölder's inequality instead, and observe, for any positive function $g(\hat{\omega})$,

$$B_a \leq \hat{\mathbb{E}}[g(\hat{\omega})\widehat{Z}_a^c]\hat{\mathbb{E}}[g(\hat{\omega})^{-\gamma/(1-\gamma)}]^{(1-\gamma)/\gamma}. \tag{68}$$

We set $g(\hat{\omega}) = e^{-\varepsilon_m \sum_{i=1}^m \hat{\omega}_i + m[\lambda(\beta)-\lambda(\beta-\varepsilon_m)]}$ with $\varepsilon_m := (m\log m)^{-1/2}$. We have

$$\frac{1-\gamma}{\gamma}\log\hat{\mathbb{E}}[g(\hat{\omega})^{-\gamma/(1-\gamma)}] = m\left(\frac{1-\gamma}{\gamma}\lambda\left(\beta+\frac{\gamma\varepsilon_m}{1-\gamma}\right)+\lambda(\beta-\varepsilon_m)-\frac{1}{\gamma}\lambda(\beta)\right) \leq C\frac{m\varepsilon_m^2}{1-\gamma}.$$

The last inequality is Taylor's formula at order 2 with C being a bound for λ'' in an interval around β (say $[0, \beta_2+1]$). With our choice of ε_m, we obtain that $\hat{\mathbb{E}}[g(\hat{\omega})^{-\gamma/(1-\gamma)}]^{(1-\gamma)/\gamma}$ is uniformly bounded in m. Summing over a and b in (68), we can conclude, if we can show that

$$\lim_{m\to\infty}\sum_{a=0}^{m-1}\sum_{b=m}^{\infty}\hat{\mathbb{E}}[g(\hat{\omega})\widehat{Z}_a^c]K''(b-a) = 0. \tag{69}$$

Note that, since $\hat{\mathbb{E}}[g(\hat{\omega})] = 1$, we can interpret $g(\hat{\omega})$ as a probability density. It has the effect of tilting down the values of $\hat{\omega}$ while keeping them independent. We have, for any $i \le m$,

$$\hat{\mathbb{E}}[g(\hat{\omega})e^{\beta\hat{\omega}_i - \lambda(\beta)}] = e^{-\zeta_m + \lambda(2\beta_2) - 2\lambda(\beta_2)},$$

where

$$\zeta_m := \lambda(2\beta_2) - \lambda(2\beta - \varepsilon_m) - 2\lambda(\beta_2) + \lambda(\beta) + \lambda(\beta - \varepsilon_m) \overset{m\to\infty}{\sim} \varepsilon_m(\lambda'(2\beta_2) - \lambda'(\beta_2)).$$

Recall that $\beta = \beta(m) = \beta_2 + 1/m^2$, and in particular that $|\beta - \beta_2| \ll \varepsilon_m$, and hence β can be replaced by β_2 when computing the asymptotic equivalence. Setting $\delta''_n = \mathbb{1}_{\{n\in\tau''\}}$ and using Fubini as in (67) as well as the definition of K'', we obtain

$$\hat{\mathbb{E}}[g(\hat{\omega})\widehat{Z}^c_a] = \boldsymbol{E}[e^{\sum_{k=1}^{a}(\lambda(2\beta_2) - 2\lambda(\beta_2) - \zeta_m)\delta_k}\delta_a] = \boldsymbol{E}''[e^{-\zeta_m \sum_{k=1}^{a}\delta''_k}\delta''_a]. \tag{70}$$

Considering a decomposition over the last renewal point in $[0, m-1]$, we deduce from (70) that

$$\sum_{a=0}^{m-1}\sum_{b=m}^{\infty}\hat{\mathbb{E}}[g(\hat{\omega})\widehat{Z}^c_a] = \boldsymbol{E}''[e^{-\zeta_m \sum_{k=1}^{m-1}\delta''_k}].$$

Furthermore,

$$\boldsymbol{E}''[e^{-\zeta_m \sum_{k=1}^{m-1}\mathbb{1}_{\{k\in\tau''\}}}] \le \frac{1}{m} + \boldsymbol{P}''(|\tau'' \cap [1, m-1]| \le (\zeta_m)^{-1}\log m).$$

Using the same truncation method as in (56), we have

$$\boldsymbol{P}(\tau''_k \ge (m-1)) \le \frac{k\boldsymbol{E}[\tau''_1 \wedge (m-1)]}{m-1} \le k m^{-(\alpha\wedge 1) + o(1)},$$

where the last inequality follows from the definition of α. Replacing k by $(\zeta_m)^{-1}\log m$ (which is $m^{1/2+o(1)}$) and recalling that $\alpha > \frac{1}{2}$, we conclude the proof of (69). $\square$

Appendix B: Proof of Proposition 2.6

Recall that $X^{(1)}$ and $X^{(2)}$ are two independent random walks with distribution P (starting from the origin). We assume that $(X^{(1)}_n - X^{(2)}_n)_{n\ge 2}$ is transient (which is equivalent to $\beta_2 > 0$).

B.1. *Preliminaries.* We first prove the inequality (8). Let us denote by $N_n := \sum_{k\ge n}\mathbb{1}_{\{X^{(1)}_k = X^{(2)}_k\}}$ the number of collisions after time n. The first thing we want to show is that (recall (6))

$$-\alpha = \limsup_{n\to\infty}\frac{\log P^{\otimes 2}(N_n \ge 1)}{\log n}. \tag{71}$$

The inequality $\le$ is an immediate consequence of $P^{\otimes 2}(T \in [n, \infty)) \le P^{\otimes 2}(N_n \ge 1))$. For the other direction, observe that if $N_{n(\log n)^2} \ge 1$ then either $N_1 > (\log n)^2$ or there exists a gap larger than n between two consecutive collisions, and thus by the strong Markov property,

$$P^{\otimes 2}(N_{n(\log n)^2} \ge 1) \le P^{\otimes 2}(N_1 > (\log n)^2) + \log(n)^2 P^{\otimes 2}(T \in [n, \infty)).$$

Since N_1 is a geometric random variable, we obtain

$$P^{\otimes 2}(T \in [n, \infty)) \geq \frac{1}{(\log n)^2}\big(P^{\otimes 2}(N_{n(\log n)^2} \geq 1) - P(T < \infty)^{(\log n)^2}\big).$$

Since $P(T < \infty) < 1$, this allows us to conclude the proof of (71). Next we observe that by the strong Markov property applied at the first intersection time after n,

$$P^{\otimes 2}(N_n \geq 1) = \frac{E^{\otimes 2}[N_n]}{E^{\otimes 2}[N_0]} = \frac{\sum_{k \geq n} P^{\otimes 2}(X_k^{(1)} = X_k^{(2)})}{E^{\otimes 2}[N_0]}. \tag{72}$$

Since the denominator is positive and does not depend on n, we can relate the "tail exponent" for $P^{\otimes 2}(N_n \geq 1)$ to that of a the sequence $P^{\otimes 2}(X_n^{(1)} = X_n^{(2)})$ using the following elementary lemma (the proof is left to the reader):

Lemma B.1. *If $(f_n)_{n \geq 1}$ is a sequence of positive numbers such that $\sum_{n \geq 1} f_n < \infty$ and $F_n := \sum_{k \geq n} f_k$, we have*

$$\overline{\lim_{n\to\infty}} \frac{\log F_n}{\log n} \leq 1 + \overline{\lim_{n\to\infty}} \frac{\log f_n}{\log n} \quad \text{and} \quad \underline{\lim_{n\to\infty}} \frac{\log F_n}{\log n} \geq 1 + \underline{\lim_{n\to\infty}} \frac{\log f_n}{\log n}.$$

A particular consequence of the above and of (72) is that

$$1 + \underline{\lim_{k\to\infty}} \frac{P^{\otimes 2}(X_k^{(1)} = X_k^{(2)})}{\log k} \leq \overline{\lim_{n\to\infty}} \frac{\log P^{\otimes 2}(N_n \geq 1)}{\log n} \leq 1 + \overline{\lim_{k\to\infty}} \frac{P^{\otimes 2}(X_k^{(1)} = X_k^{(2)})}{\log k}.$$

Proving (8) then reduces to proving that

$$\liminf_{k\to\infty} \frac{P^{\otimes 2}(X_k^{(1)} = X_k^{(2)})}{\log k} \geq -\frac{d}{\eta \wedge 2} \quad \text{and} \quad \limsup_{k\to\infty} \frac{P^{\otimes 2}(X_k^{(1)} = X_k^{(2)})}{\log k} \leq -\nu. \tag{73}$$

The second inequality in (73) follows from the fact that $P^{\otimes 2}(X_k^{(1)} = X_k^{(2)}) = \|D^k\|_2^2 \leq \|D^k\|_\infty$. To prove the first inequality in (73) we first use Cauchy–Schwarz to observe that, for any $x \in \mathbb{Z}^d$,

$$P^{\otimes 2}(X_k^{(1)} - X_k^{(2)} = x) = \sum_{y \in \mathbb{Z}^d} D^k(y + x, 0) D^k(y, 0) \leq \sum_{y \in \mathbb{Z}^d} D^k(y, 0)^2 = P^{\otimes 2}(X_k^{(1)} = X_k^{(2)}).$$

As a consequence, for every $L \geq 1$,

$$P^{\otimes 2}(X_k^{(1)} = X_k^{(2)}) \geq \frac{1}{(2L+1)^d} P^{\otimes 2}(|X_k^{(1)} - X_k^{(2)}| \leq L).$$

Taking $L = k^{1/(2\wedge\eta)+\varepsilon/d}$ for $\varepsilon > 0$ arbitrary, we prove below that

$$\lim_{k\to\infty} P^{\otimes 2}(|X_k^{(1)} - X_k^{(2)}| \leq k^{1/(2\wedge\eta)+\varepsilon/d}) = 1, \tag{74}$$

which implies that for k sufficiently large $P^{\otimes 2}(X_k^{(1)} = X_k^{(2)}) \geq (1/2^{d+1}) k^{-d/(2\wedge\eta)-\varepsilon}$ and hence, by sending $\varepsilon \to 0$, that the first inequality in (73) holds.

We can restrict the proof of (74) to the case $d = 1$. Moreover, the inequality follows from Chebyshev's inequality if $\eta > 2$. If $\eta \le 2$, we truncate the increments of $X^{(1)}$ and $X^{(2)}$ at level $L = k^{1/\eta+\varepsilon}$ before applying Chebyshev's inequality: given $\delta > 0$ and recalling (1), this yields, for $k \ge k_0(\delta)$ sufficiently large,

$$P^{\otimes 2}(|X_k^{(1)} - X_k^{(2)}| \ge L) \le 2kP(|X_1| \ge L) + L^{-2}2kE[X_1^2 \mathbb{1}_{|X_1| \le L}] \le 2kL^{-\eta+\delta} + L^{-2}2kL^{2-\eta+\delta} \le 4kL^{-\eta+\delta},$$

from which (74) follows by taking, for instance, $\delta = \eta^2\varepsilon$. □

B.2. ***Proof of Proposition 2.6.*** The bound $p^*(\beta) \ge 1 + \frac{1}{2}(\eta \wedge 2)$ is proved as [Junk and Lacoin 2025, Corollary 2.20] under the assumption that weak disorder holds, so we only have to prove the upper bound $p^*(\beta_c) \le 1 + 1/(\nu \vee 1)$. Moreover, if $\beta_2 > 0$ then by [loc. cit., Corollary 2.11] we have $p^*(\beta_2) = 2$; hence $p^*(\beta_c) \le 2$. Therefore we only have to prove $p^*(\beta_c) \le 1 + 1/\nu$ in the case $\nu > 1$, which we assume in the following. This claim is a consequence of the following result, which partly generalizes [Junk 2025b, Theorem 1.2] to the case of an arbitrary reference random walk.

Proposition B.2. *If* $p^*(\beta) \in (1 + 1/\nu, 2]$ *then* $\beta < \beta_c$ *and* $\lim_{u \to 0+} p^*(\beta + u) = p^*(\beta)$.

Given $T \ge 0$, we set $\mathcal{T}_1 := \inf\{k \ge T : X_k^{(1)} = X_k^{(2)}\}$. Given this value $T > 0$, we introduce a new partition function by setting, for $t \ge T$ and $n \in \mathbb{N}$,

$$\mathcal{Z}_n^\beta(t, x) := E^{\otimes 2}\left[\exp\left(\sum_{k=1}^{n}[\beta(\omega_{k,X_k^{(1)}} + \omega_{k,X_k^{(2)}}) - 2\lambda(\beta)]\right)\mathbb{1}_{\{\mathcal{T}_1 = t \text{ and } X_t^{(1)} = x\}}\right],$$

and $\mathcal{Z}_n^\beta(t, x) = 0$ if $t \le T - 1$. Proposition B.2 follows from the combination of two technical results. The first one is a bound on $\mathbb{E}[(W_n^\beta)^p]$ which is uniform in n and follows from a decomposition of the squared partition function $(W_n^\beta)^2$. We recall the definition $\chi(\beta) := e^{\lambda(2\beta) - 2\lambda(\beta)} - 1$.

Lemma B.3. *We have, for any* $\beta > 0$, $n \ge 0$ *and* $p \le 2$,

$$\mathbb{E}[(W_n^\beta)^p] \le \mathbb{E}[(W_{T-1}^\beta)^p] \sum_{j \ge 0}\left((1 + \chi(\beta)) \sum_{t \ge T} \sum_{x \in \mathbb{Z}^d} \mathbb{E}[\mathcal{Z}_{T-1}^\beta(t, x)^{p/2}]\right)^j.$$

Of course, the above result is meaningful only if

$$\sum_{t \ge T} \sum_{x \in \mathbb{Z}^d} \mathbb{E}[\mathcal{Z}_{T-1}^\beta(t, x)^{p/2}] < \frac{1}{1 + \chi(\beta)}.$$

As a consequence of our second technical result, this condition can be satisfied by taking T sufficiently large, if p lies in a certain range,

Lemma B.4. (i) *If* $p \in (1 + 1/\nu, p^*(\beta))$, *then*

$$\lim_{T \to \infty} \sum_{t \ge T} \sum_{x \in \mathbb{Z}^d} \mathbb{E}[\mathcal{Z}_{T-1}^\beta(t, x)^{p/2}] = 0.$$

(ii) *If the value of T is fixed and $p > 1 + 1/\nu$, then the function*

$$\beta \mapsto \sum_{t\geq T}\sum_{x\in\mathbb{Z}^d} \mathbb{E}[\mathcal{Z}^{\beta}_{T-1}(t,x)^{p/2}]$$

is continuous in β.

Proof of Proposition B.2. Assume $p^*(\beta) \in (1 + 1/\nu, 2]$. Since $p^*(\beta)$ is nonincreasing in β, we need to show that for any $p \in (1 + 1/\nu, p^*(\beta))$ there exists $u > 0$ such that $p^*(\beta + u) > p$. First, using Lemma B.4(i), we choose T in such a way that

$$\sum_{t\geq T}\sum_{x\in\mathbb{Z}^d} \mathbb{E}[\mathcal{Z}^{\beta}_{T-1}(t,x)^{p/2}] \leq \frac{1}{4(1+\chi(\beta+1))}.$$

Then, using (ii) and the monotonicity of $\chi(\beta)$, we find $u \in (0, 1)$ such that

$$\sum_{t\geq T}\sum_{x\in\mathbb{Z}^d} \mathbb{E}[\mathcal{Z}^{\beta+u}_{T-1}(t,x)^{p/2}] \leq \frac{1}{2(1+\chi(\beta+u))}.$$

Finally, we use Lemma B.3 to conclude that

$$\sup_{n\geq 0} \mathbb{E}[(W^{\beta+u}_n)^p] \leq 2\mathbb{E}[(W^{\beta+u}_{T-1})^p] < \infty. \qquad \square$$

B.3. ***Proof of Lemma B.3.*** We introduce, for $i \geq 1$,

$$\mathcal{T}_{i+1} := \inf\{k \geq \mathcal{T}_i + T : X^{(1)}_k = X^{(2)}_k\}.$$

Given $n, j \geq 1$, $\boldsymbol{t} := (t_1, \dots, t_j) \in \mathbb{N}^j$ and $\boldsymbol{x} := (x_1, \dots, x_j) \in (\mathbb{Z}^d)^j$, we define the event

$$A_{n,j}(\boldsymbol{t},\boldsymbol{x}) = \{\forall i \in [\![1, j]\!] : \mathcal{T}_i = t_i \text{ and } X^{(1)}_{t_i} = x_i\} \cap \{t_j \leq n\}.$$

For $j = 0$ we set $A_{n,0} = \{\mathcal{T}_1 > n\}$ and define, for $j \geq 0$,

$$\mathcal{Z}^{\beta}_{n,j}(\boldsymbol{t},\boldsymbol{x}) := E^{\otimes 2}\left[\exp\left(\sum_{k=1}^{n}[\beta(\omega_{k,X^{(1)}_k} + \omega_{k,X^{(2)}_k}) - 2\lambda(\beta)]\right)\mathbb{1}_{A_{n,j}(\boldsymbol{t},\boldsymbol{x})}\right].$$

With this notation we have $\mathcal{Z}^{\beta}_t(t,x) = \mathcal{Z}^{\beta}_{t,1}(t,x)$. Since the events $(A_{n,j}(\boldsymbol{t},\boldsymbol{x}))_{j,\boldsymbol{t},\boldsymbol{x}}$ partition the probability space, we have

$$(W^{\beta}_n)^2 = \sum_{j\geq 0}\sum_{(\boldsymbol{t},\boldsymbol{x})\in\mathbb{N}^j\times(\mathbb{Z}^d)^j} \mathcal{Z}^{\beta}_{n,j}(\boldsymbol{t},\boldsymbol{x}).$$

Note now that $\mathcal{Z}^{\beta}_{n,j}(\boldsymbol{t},\boldsymbol{x})$ can be factorized using the Markov property for the random walks $X^{(1)}$ and $X^{(2)}$ at time $\mathcal{T}_j$ and iterating. Recalling the definition of (23), for $j \geq 0$ and $t_j \leq n$ we obtain, setting $(t_0, x_0) = (0, 0)$ as well as $\Delta_i \boldsymbol{t} = t_i - t_{i-1}$ and $\Delta_i \boldsymbol{x} = x_i - x_{i-1}$,

$$\mathcal{Z}^{\beta}_{n,j}(\boldsymbol{t},\boldsymbol{x}) = \left(\prod_{i=1}^{j} \theta_{t_{i-1},x_{i-1}} \mathcal{Z}^{\beta}_{\Delta_i \boldsymbol{t}}(\Delta_i \boldsymbol{t}, \Delta_i \boldsymbol{x})\right)\theta_{t_j,x_j}\mathcal{Z}^{\beta}_{n-t_j,0}.$$

Assuming that $p \leq 2$ and combining subadditivity with shift invariance, we obtain that

$$\mathbb{E}[(W_n^\beta)^p] \leq \sum_{j\geq 0}\sum_{(\boldsymbol{t},\boldsymbol{x})} \mathbb{1}_{\{t_j\leq n\}} \prod_{i=1}^{j} \mathbb{E}[\mathcal{Z}^\beta_{\Delta_i \boldsymbol{t}}(\Delta_i \boldsymbol{t}, \Delta_i \boldsymbol{x})^{p/2}]\mathbb{E}[(\mathcal{Z}^\beta_{n-t_j,0})^{p/2}]. \tag{75}$$

For the last factor, we now observe that

$$\mathbb{E}[(\mathcal{Z}^\beta_{n,0})^{p/2}] \leq \mathbb{E}[\mathbb{E}[\mathcal{Z}^\beta_{n,0} \mid \mathcal{F}_{T-1}]^{p/2}] \leq \mathbb{E}[(W^\beta_{T-1})^p],$$

where for the last inequality we simply used Fubini as follows:

$$\mathbb{E}[\mathcal{Z}^\beta_{n,0} \mid \mathcal{F}_{T-1}] = E^{\otimes 2}\Bigg[\exp\Bigg(\sum_{k=1}^{T-1}[\beta(\omega_{k,X_k^{(1)}} + \omega_{k,X_k^{(2)}}) - 2\lambda(\beta)]\Bigg)\mathbb{1}_{\{\forall i\geq T: X_i^{(1)}\neq X_i^{(2)}\}}\Bigg] \leq (W^\beta_{T-1})^2.$$

Similarly, for each of the factors in the product in (75), we estimate

$$\mathbb{E}[\mathcal{Z}^\beta_{\Delta_i \boldsymbol{t}}(\Delta_i \boldsymbol{t}, \Delta_i \boldsymbol{x})^{p/2}] \leq \mathbb{E}[\mathbb{E}[\mathcal{Z}^\beta_{\Delta_i \boldsymbol{t}}(\Delta_i \boldsymbol{t}) \mid \mathcal{F}_{T-1}]^{p/2}] = \mathbb{E}[\mathcal{Z}^\beta_{T-1}(\Delta_i \boldsymbol{t}, \Delta_i \boldsymbol{x})^{p/2}](1+\chi)^{p/2},$$

where the factor $1+\chi$ comes from the fact that $X^{(1)}$ and $X^{(2)}$ collide at time $\Delta_i \boldsymbol{t}$. Taking the sum over all ordered sequences of $(t_1, \dots, t_j)$ instead of restricting to $t_j \leq n$, we obtain

$$\sum_{j\geq 0}\sum_{(\boldsymbol{t},\boldsymbol{x})}\prod_{i=1}^{j} \mathbb{E}[\mathcal{Z}^\beta_{\Delta_i \boldsymbol{t}}(\Delta_i \boldsymbol{t}, \Delta_i \boldsymbol{x})^{p/2}] \leq \sum_{j\geq 0}\Bigg((1+\chi)\sum_{t\geq T, x\in\mathbb{Z}^d} \mathbb{E}[\mathcal{Z}^\beta_{T-1}(t,x)^{p/2}]\Bigg)^j, \tag{76}$$

and hence the desired result follows from the combination of (75)–(76).

B.4. ***Proof of Lemma B.4.*** Recall that D is the transpose of the transition matrix of X, and also recall the definition (29) of μ_n. We have

$$\begin{aligned}\mathcal{Z}^\beta_{T-1}(T-1+s, x)^{p/2} &= (W^\beta_{T-1})^p P^{\otimes 2}_{\mu_{T-1}}(X_s^{(1)} = X_s^{(2)} = x, \forall r \in [\![1, s-1]\!], X_r^{(1)} \neq X_r^{(2)})^{p/2} \\ &\leq (W^\beta_{T-1})^p (D^s \mu_{T-1}(x))^p = (D^s \widehat{W}_{T-1}(x))^p. \end{aligned}\tag{77}$$

Setting $Y_T := \sum_{s\geq 1}\sum_{x\in\mathbb{Z}^d}(D^s\mu_{T-1}(x))^p$, the above yields

$$\sum_{t\geq T}\sum_{x\in\mathbb{Z}^d} \mathcal{Z}^\beta_{T-1}(t,x)^{p/2} \leq (W^\beta_{T-1})^p Y_T \leq \big(\sup_{n\geq 0} W_n\big)^p Y_T.$$

To prove item (i) of the lemma, it is thus sufficient to show that

$$\lim_{T\to\infty} \mathbb{E}\Big[\sup_{n\geq 1}(W_n^\beta)^p Y_T\Big] = 0. \tag{78}$$

We are going to show that there exists some $C > 0$ such that almost surely

$$Y_T \leq C \quad \text{and} \quad \lim_{T\to\infty} Y_T = 0. \tag{79}$$

The convergence (78) follows from (79) and dominated convergence for $\mathbb{P}$. To ensure domination, we use the fact that, by [Junk and Lacoin 2025, Theorem 2.1], $\sup_{n\geq 1}(W_n^\beta) \in L^p$ for $p < p^*(\beta)$. To prove (79), we observe that

$$\sum_{x\in\mathbb{Z}^d} (D^s \mu_{T-1}(x))^p \leq \min(\|\mu_{T-1}\|_p^p, \|D^s\|_\infty^{p-1}). \tag{80}$$

The bound $\|\mu_{T-1}\|_p^p$ comes from the fact that convolution contracts ℓ^p norms and the other bound comes from the fact that $(D^s\mu_{T-1}(x))^p \leq \|D^s\|_\infty^{p-1} D^s\mu_{T-1}(x)$. Since $p > 1+1/\nu$, the definition of (7) implies that $\|D^s\|_\infty^{p-1}$ is summable, which implies the first part of (79).

We then note that weak disorder implies $\lim_{T\to\infty} \|\mu_{T-1}\|_p = 0$ (this is, for instance, a consequence of [Comets et al. 2003, Theorem 2.1], which proves that $\lim_{n\to\infty} I_n = 0$). Hence, using the dominated convergence theorem for the sum over s ($\|D^s\|_\infty^{p-1}$ is used for the domination), we obtain the second part of (80).

Now let us prove item (ii), which is an exercise in applying the dominated convergence theorem. For $\beta' < \beta''$ and every k and x, we have the bound

$$(\beta\omega_{k,x} - \lambda(\beta)) \leq \beta''(\omega_{k,x})_+, \tag{81}$$

valid on $[\beta', \beta'']$. For fixed $t \geq T-1$ and $x \in \mathbb{Z}^d$, and almost every ω, applying dominated convergence to P and using (81) as the domination, we obtain that $\beta \mapsto \mathcal{Z}_{T-1}^\beta(t, x)$ is continuous. Next we apply the dominated convergence theorem to the counting measure on $\mathbb{N}\times\mathbb{Z}^d$. From (77) we have

$$\sup_{\beta\in[\beta',\beta'']} \mathcal{Z}_{T-1}^\beta(t, x)^{p/2} \leq \sup_{\beta\in[\beta',\beta'']} (D^{t-T}\widehat{W}_{T-1}^\beta(x))^p,$$

and we have, in the same way as (80),

$$\sum_{x\in\mathbb{Z}^d} \sup_{\beta\in[\beta',\beta'']} (D^{t-T}\widehat{W}_{T-1}^\beta(x))^p \leq \sup_{\beta\in[\beta',\beta'']} (W_{T-1}^\beta)^p \|D^{t-T}\|_\infty^{p-1}.$$

The first term is finite by (81) and the second is summable in t; hence $\beta \mapsto \sum_{t\geq T}\sum_{x\in\mathbb{Z}^d} \mathcal{Z}_{T-1}^\beta(t, x)^{p/2}$ is continuous for almost every ω. Finally, using the combination of the above inequalities, we have

$$\sup_{\beta\in[\beta',\beta'']} \sum_{x\in\mathbb{Z}^d}\sum_{t\geq T} \mathcal{Z}_{T-1}^\beta(t, x)^{p/2} \leq \sup_{\beta\in[\beta',\beta'']} (W_{T-1}^\beta)^p \sum_{t\geq T} \|D^{t-T}\|_\infty^{p-1}.$$

Since the right side is in L^1 (again by (81)), we apply dominated convergence with respect to $\mathbb{P}$ to obtain the claim of (ii). □

Appendix C: Proof of Proposition 2.11

Before starting the proof let us explain its mechanism. Our choice for the distribution of X_1 was made so that it has a *varying tail exponent*, namely

$$\liminf_{n\to\infty} \frac{\log P(|X_1| > n)}{\log n} = -4 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log P(|X_1| > n)}{\log n} = 0.$$

The statement about the lim inf ensures that $(X_k)_{k\geq 0}$ is recurrent, which will be proved in Section C.1, and thus by (5) this implies that $\beta_2 = 0$ (the fact that $\beta_c = 0$ is then implied by Theorem 2.9). On the other hand, the information about the lim sup allows us to replicate the argument given proof given in [Viveros 2023, Theorem 1.1], which proves that $\mathfrak{f}(\beta) = 0$ for all β if the tail distribution of X_1 is sufficiently fat. This is the content of Section C.2.

C.1. ***Proving the recurrence of $(X_k)_{k\geq 0}$.*** Note that $g(x)$ is symmetric and has a unique local maximum at zero, and thus $P(X_k = x) = g^{*k}(x)$ has the same property for every k. This implies that

$$\max_{x\in\mathbb{Z}^d} P(X_k = x) = P(X_k = 0) \quad \text{and} \quad P(X_k = 0) \geq P(X_{k+1} = 0).$$

Using these properties, we have for any choice of N and M

$$\sum_{i=N+1}^{2N} P(X_i = 0) \geq N P(X_{2N} = 0) \geq \frac{N}{2N+1} P(|X_{2N}| \leq N).$$

In view of the above, to prove recurrence of X, it is sufficient to show that

$$\limsup_{N\to\infty} P(|X_{2N}| \leq N) > 0, \tag{82}$$

which we will do by showing that the lim sup is equal to one. In order to estimate $P(|X_{2N}| \leq N)$, we truncate the increments and apply the second moment methods. Given $M \geq 1$, we set

$$\overline{X}_N := \sum_{i=1}^{N} (X_i - X_{i-1}) \mathbb{1}_{\{|X_i - X_{i-1}| \leq M\}}.$$

Using the union bound and Chebyshev (note that $\mathbb{E}[|\overline{X}_1^2|] \leq M^2$),

$$\begin{aligned} P(|X_{2N}| \leq N) \geq P(|\overline{X}_{2N}| \leq N;\, X_{2N} = \overline{X}_{2N}) &\geq 1 - P(|\overline{X}_{2N}| > N) - 2N P(|X_1| > M) \\ &\geq 1 - \frac{2M^2}{N} - 2N \sum_{|x|>M} g(x). \end{aligned}$$

We take $N = M^3$ and $M = a_k$ for $k \geq 1$. In that case $2M^2/N = 2a_k^{-1}$, which converge to zero as $k \to \infty$. As for the other term, it is bounded as follows:

$$N \sum_{|x|>a_k} f(x) \leq N \sum_{i\geq k} \frac{1}{a_i^4} \leq 4N a_k^{-4} \leq 4 a_k^{-1}.$$

Taking $k \to \infty$ this proves (82) and concludes the proof that $(X_k)_{k\geq 0}$ is recurrent. □

C.2. ***Proving that $\mathfrak{f}(\beta) \equiv 0$.*** Now we replicate the proof of [Viveros 2023, Theorem 1.1] (proved under a slightly different assumption). Let us consider a large N (whose value will be determined later) and set

$$\begin{aligned} V_N(x) &:= E[e^{\sum_{k=0}^{N-1}(\beta\omega_{k+1,X_k+x} - \lambda(\beta))} \mathbb{1}_{\{\forall k\in[0,N-1]:|X_k - x|\leq N^2\}}], \\ \overline{V}_N(x) &:= \frac{V_N(x)}{P(\forall k \in [0, N-1] : |X_k| \leq N^2)} = \frac{V_N(x)}{\mathbb{E}[V_N(x)]}. \end{aligned}$$

In words, $\overline{V}_N(x)$ is the partition function started at time 1 from x whose underlying random walk is conditioned to remain within distance N^2 from the starting point. We have, for any $k \geq 1$,

$$W_N^\beta = \sum_{x\in\mathbb{Z}} g(x) e^{\beta\omega_{1,x}-\lambda(\beta)} \theta_{1,x} W_{N-1}^\beta \geq \sum_{|x|\leq a_k} g(x) V_N(x) \geq g(|a_k|) \sum_{|x|\leq a_k} V_N(x). \tag{83}$$

Setting $h(N) = P(\forall k \in [0, N-1] : |X_k| \leq N^2)$, we can rewrite the sum above as

$$g(|a_k|) \sum_{|x|\leq a_k} V_N(x) = (2a_k+1) g(|a_k|) h(N) \left(\frac{1}{2a_k+1} \sum_{|x|\leq a_k} \overline{V}_N(x) \right) =: (2a_k+1) g(|a_k|) h(N) U_{N,k}.$$

Now we set $N = N_k := \lceil \sqrt{a_{k-1}} \rceil$. With this setup, we are going to show that

$$\lim_{k\to\infty} \frac{1}{N_k} \log[(2a_k+1) g(|a_k|) h(N_k)] = 0, \qquad \lim_{k\to\infty} U_{N_k,k} = 1, \tag{84}$$

where the second convergence holds in probability. The combination of the two limits in (84) ensures that

$$\lim_{k\to\infty} \frac{1}{N_k} \log\left(g(|a_k|) \sum_{|x|\leq a_k} V_{N_k}(x) \right) = 0$$

holds in probability, and hence from (83) we deduce that $\mathfrak{f}(\beta) = 0$ (we obtain $\geq$ from (83) but the other inequality is trivial). To prove the first line in (84), we simply observe that $(2a_k+1)g(|a_k|)$ is of order a_{k-1}^{-4} (or N_k^{-8}) and that $h(N) \geq P(|X_1| \leq N)^N$ does not decay exponentially fast in N.

For the second line, since by construction $\mathbb{E}[U_{N_k,k}] = 1$, we only need to show that the variance goes to zero. Using the fact that

$$\mathbb{E}[\overline{V}_N^2(x)] \leq e^{N(\lambda(2\beta)-2\lambda(\beta))}$$

and that $V_N^2(x)$ and $V_N^2(y)$ are independent if $|x-y| > 2N^2$, we obtain

$$\mathrm{Var}(U_{N,k}^\beta) \leq e^{N(\lambda(2\beta)-2\lambda(\beta))} \frac{4N^2+1}{2a_k+1},$$

and the result follows from our choice for N_k. □

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Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles are usually in English or French, but articles written in other languages are welcome.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not refer to bibliography keys. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and a Mathematics Subject Classification for the article, and, for each author, affiliation (if appropriate) and email address.

Format. Authors are encouraged to use LaTeX and the standard amsart class, but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should normally be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages — Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc. — allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with as many details as you can about how your graphics were generated.

Bundle your figure files into a single archive (using zip, tar, rar or other format of your choice) and upload on the link you been provided at acceptance time. Each figure should be captioned and numbered so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript with all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables.

White Space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

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