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# On dense orbits in the space of subequivalence relations

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We first explain how to endow the space of subequivalence relations of any nonsingular countable equivalence relation with a Polish topology, extending the framework of Kechris' recent monograph on subequivalence relations of probability measure-preserving (p.m.p.) countable equivalence relations. We then restrict to p.m.p. equivalence relations and discuss dense orbits therein for the natural action of the full group and of the automorphism group of the relation. Our main result is a characterization of the subequivalence relations having a dense orbit in the space of subequivalence relations of the ergodic hyperfinite p.m.p. equivalence relation. We also show that in this setup, all orbits under the full group action are meager. We finally provide a few Borel complexity calculations of natural subsets in spaces of subequivalence relations using a natural metric we call the uniform metric. This answers some questions from an earlier version of Kechris' monograph.

## 1. Introduction

Measured group theory can roughly be defined as the study of countable groups through the partitions into orbits associated to their free probability measure-preserving (p.m.p.) actions on standard probability space. A fundamental notion there is that of an *orbit subgroup*: a countable group $\Lambda$ is an orbit subgroup of another countable group $\Gamma$ when $\Lambda$ admits a free p.m.p. action all of whose orbits are contained in those of a free p.m.p. action of $\Gamma$. In this more flexible framework, while the Ornstein–Weiss theorem [1980] tells us that all countable infinite amenable subgroups are orbit subgroups of one another, the Gaboriau–Lyons theorem [2009] characterizes nonamenable groups as being exactly those which admit the free group on two generators as an orbit subgroup. Their result has had many applications, making it possible to extend results which were only known for groups containing free subgroups to the optimal class of all nonamenable groups; see, e.g., [Ioana et al. 2009; Seward 2014].

This motivates the following more general question: given a partition of the space into orbits, what are the possible subpartitions into orbits? In more precise terms, let us call an equivalence relation on a standard probability space $(X, \mu)$ nonsingular when it comes from an action of a countable group which preserves $\mu$ null sets, and p.m.p. when it comes from an action of a countable group which actually preserves the measure $\mu$. We can now ask our question more precisely: given a nonsingular equivalence relation, what are its possible subequivalence relations? An essential step towards answering such a question is Kechris' remarkable result that when the ambient equivalence relation is p.m.p., the space

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of subequivalence relations can be endowed with a natural Polish topology. This allows descriptive set theory to enter the picture, and as such it is the first building block in Kechris' monograph [2024] on spaces of subequivalence relations, whose initial version appeared online in 2013. We begin by extending this fundamental result to the nonsingular case:

**Theorem A** (see Theorem 2.6). *Let $\mathcal{R}$ be a nonsingular equivalence relation on a standard probability space. Then its space* $\mathrm{Sub}(\mathcal{R})$ *of subequivalence relations is a Polish space for the topology induced by the measure algebra of $\mathcal{R}$.*

We endow $\mathcal{R}$ with the usual measure obtained by integrating cardinality of vertical fibers, allowing us to make sense of its measure algebra in the above result. In particular, we are identifying subequivalence relations which coincide on a conull set. We refer the reader to Section 2 for details. Our description of the topology on the space of subequivalence relations was already known to Kechris [2024, Section 4.4(1)] in the p.m.p. case.

We actually provide two proofs of Theorem A. The first one is very short and uses the fact that in a measure algebra, converging sequences always admit subsequences whose limit is actually equal to their lim inf (see Proposition 2.3). This observation provides a unifying viewpoint to some special cases noted by Kechris [2024, Theorems 5.1 and 18.3]. The second proof is somehow more natural, in that we directly show that the various axioms of being a subequivalence relation define closed subsets in the measure algebra. Along the way, we study a natural non-Hausdorff topology on the measure algebra that we call the lower topology. This topology has some nice continuity properties which also allow us to give a slick proof of the fact that the map which associates to a subset of an equivalence relation the equivalence relation it generates is Baire class one; see Proposition 3.19, which is a natural generalization of [Kechris 2024, Proposition 19.1].

Our next result is in the p.m.p. setup, and was motivated by the question of dense orbits in the space of subequivalence relations, asked by Kechris in his monograph. Indeed, it is very natural to try to understand subequivalence relations up to isomorphism, and this is precisely what the action of the automorphism group of $\mathcal{R}$ on subequivalence relations of $\mathcal{R}$ encodes. As it turns out, when $\mathcal{R}$ is the ergodic hyperfinite p.m.p. equivalence relation, we obtain the following complete characterization of dense orbits for the natural actions of both the full group $[\mathcal{R}]$ and the automorphism group $\mathrm{Aut}(\mathcal{R})$:

**Theorem B** (see Corollary 4.13). *For a subequivalence relation $\mathcal{S}$ of the hyperfinite ergodic p.m.p. equivalence relation $\mathcal{R}_0$, the following are equivalent*:

(i) *$\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}_0$.*

(ii) *The $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ is dense in* $\mathrm{Sub}(\mathcal{R}_0)$.

(iii) *The $\mathrm{Aut}(\mathcal{R}_0)$-orbit of $\mathcal{S}$ is dense in* $\mathrm{Sub}(\mathcal{R}_0)$.

In this theorem, $\mathcal{S}$ being everywhere of infinite index in $\mathcal{R}$ means that for every $A \subseteq X$ of positive measure, the restriction of $\mathcal{S}$ to $A$ has infinite index in the restriction of $\mathcal{R}$ to $A$ (almost every $\mathcal{R}_{\restriction A}$-class splits into infinitely many $\mathcal{S}_{\restriction A}$-classes; see the end of Section 4.1 for more on this).

**Example 1.1.** View $\mathcal{R}_0$ as the cofinite equivalence relation on $X = \{0,1\}^{\mathbb{N}}$ endowed with the probability measure $\mu = \left(\frac{1}{2}(\delta_0 + \delta_1)\right)^{\otimes \mathbb{N}}$, namely $((x_n),(y_n)) \in \mathcal{R}_0$ if and only if there is $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqslant N$. Define $\mathcal{S}$ by $((x_n),(y_n)) \in \mathcal{S}$ if and only if there is $N \in \mathbb{N}$ such that $x_{2n} = y_{2n}$ for all $n \geqslant N$. Then $\mathcal{S}$ is both aperiodic and has everywhere infinite index in $\mathcal{R}_0$ (see Proposition 4.6), so by the above theorem it has a dense orbit in $\mathrm{Sub}(\mathcal{R}_0)$.

**Example 1.2.** View $\mathcal{R}_0$ as the equivalence relation on $X = \{0,1\}^{\mathbb{Z}^2}$ endowed with the probability measure $\mu = \left(\frac{1}{2}(\delta_0 + \delta_1)\right)^{\otimes \mathbb{Z}^2}$ generated by the Bernoulli shift of $\mathbb{Z}^2$. Then since the shift is mixing, the subequivalence relation $\mathcal{S}$ generated by $\mathbb{Z} \leqslant \mathbb{Z}^2$ is ergodic. By construction, $\mathcal{S}$ is of infinite index, so since it is ergodic it is everywhere of infinite index. By the above theorem $\mathcal{S}$ has a dense orbit in $\mathrm{Sub}(\mathcal{R}_0)$. Note that by [Kechris 2024, Theorem 10.1], ergodic subequivalence relations actually form a $G_\delta$ set, so the Baire category theorem provides us with a dense $G_\delta$ subset consisting of *ergodic* subequivalence relations of $\mathcal{R}_0$ satisfying the equivalent conditions of Theorem B.

Our approach to Theorem B is very much inspired by a joint work in preparation with Fima, Mukherjee and Patri, where we similarly characterize dense orbits in the space of von Neumann subalgebras of the hyperfinite type $\mathrm{II}_1$ factor. In this setup, we have to rely on a difficult result of Popa [2019, Lemma 2.3] which characterizes the subalgebras which have a sequence of unitary conjugates converging to the trivial subalgebra. Here we similarly first need to understand which subequivalence relations have a sequence of translates converging to the trivial subequivalence relations $\Delta_X = \{(x,x) : x \in X\}$, which is the content of our next result:

**Theorem C** (see Theorem 4.7). *Let $\mathcal{R}$ be an aperiodic p.m.p. equivalence relation on a standard probability space $(X,\mu)$. Let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be a subequivalence relation. The following are equivalent*:

(i) *$\mathcal{S}$ has everywhere infinite index in $\mathcal{R}$.*

(ii) *The closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\Delta_X$.*

(iii) *The closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains $\Delta_X$.*

The proof of Theorem C relies on an inductive construction, using a countable dense subgroup of the full group and the fact that such a subgroup acts highly transitively on almost every orbit, a result due to Eisenmann and Glasner [2016, Theorem 1.19].

Once this is done, we prove Theorem B using the fact that finite equivalence relations are dense in $\mathrm{Sub}(\mathcal{R}_0)$ and that they can always be translated inside any aperiodic subequivalence relation. This allows us to apply Theorem C on a fundamental domain of the finite subequivalence relation we want to approximate. A more general statement on the closed subspace of hyperfinite subequivalence relations can actually be proved this way; see Corollary 4.12.

Using Ioana's intertwining of subequivalence relations [2012, Lemma 1.7] and the two above examples of subequivalence relations, we moreover have the following result, which again follows from a more general statement on the space of hyperfinite subequivalence relations (see Theorem 4.16).

**Theorem D** (see Corollary 4.17). *Let $\mathcal{R}_0$ be the ergodic p.m.p. hyperfinite equivalence relation. Then all $[\mathcal{R}_0]$-orbits are meager in* $\mathrm{Sub}(\mathcal{R}_0)$.

It then follows from Mycielski's theorem (see, e.g., [Gao 2009, Theorem 5.3.1]) that $\mathcal{R}_0$ contains a continuum of aperiodic subequivalence relations of infinite index which are in pairwise disjoint full group orbits. Moreover, by [Kechris 2024, Proposition 10.1] and Example 1.2, such a continuum can be found inside ergodic subequivalence relations. We leave the following question open:

**Question 1.3.** Can $\mathrm{Sub}(\mathcal{R}_0)$ contain a comeager $\mathrm{Aut}(\mathcal{R}_0)$-orbit?

This question can also be asked for equivalence relations $\mathcal{R}$ different from $\mathcal{R}_0$, although the existence of dense $\mathrm{Aut}(\mathcal{R})$- or $[\mathcal{R}]$-orbits in their space of subequivalence relations is already an open problem (see the remark preceding Corollary 4.13 for some examples where there are no dense orbits though). Also note that if $\mathcal{R}$ is a type $\mathrm{II}_\infty$ or type III ergodic nonsingular equivalence relation, it is not hard to show that there are always dense full group orbits in $\mathrm{Sub}(\mathcal{R})$ (see the remark right before Section 4.4 for a sketch of proof), but the question of comeager orbits is completely open there.

We conclude with a few complexity calculations, answering some questions from an earlier version of [Kechris 2024]. Let us highlight here a way of understanding Kechris' uniform topology on the space of subequivalence relations which we use: given two nonsingular equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$, denote by $\mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)$ the set of Borel subsets $A$ of $X$ such that $\mathcal{R}_{1\restriction A} = \mathcal{R}_{2\restriction A}$, and define

$$d_u(\mathcal{R}_1, \mathcal{R}_2) = 1 - \sup_{A \in \mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)} \mu(A).$$

This defines a metric that we propose to call the uniform metric since it induces the uniform topology defined in [Kechris 2024, Section 4.6], although it looks finer a priori (to see that the two topologies coincide, one needs to use [Ioana et al. 2009, Theorem 1]).

*Outline.* Along with basic preliminaries, Section 2 provides a first proof of Theorem A using the lim inf. Section 3 contains the second proof, which relies in a more direct way on the topology of the measure algebra, using as well the lower topology that we define and study therein. Section 4 is devoted to the proofs of Theorems B and D in their more general form, and contains in particular the proof of Theorem C in Section 4.2. Finally, Section 5 contains our complexity computations, which use the uniform metric.

## 2. The Polish topology on the space of subequivalence relations in the nonsingular setup

The main purpose of this section is to extend the topology on the space of subequivalence relations from [Kechris 2024] to the nonsingular setup. To do so, we use the framework of measure algebras, which does yield the right topology for the probability measure-preserving case, as noted in [Kechris 2024, Section 4.4(1)]. This direct approach also sheds a new light on some results from [Kechris 2024], such as Theorem 5.1 and Proposition 4.27.

**2.1.** ***The measure algebra of a standard $\sigma$-finite space.*** Let $(Y, \lambda)$ be a standard $\sigma$-finite measured space, i.e., a standard Borel space endowed with a $\sigma$-finite atomless measure. Such spaces are always

isomorphic either to $\mathbb{R}$ endowed with the Lebesgue measure or a finite-length interval, also endowed with the Lebesgue measure. The *measure algebra* of $(Y, \lambda)$ is the space $\mathrm{MAlg}(Y, \lambda)$ of all Borel subsets of $X$, where we identify $A, B \subseteq X$ Borel as soon as $\lambda(A \triangle B) = 0$.

In order to endow $\mathrm{MAlg}(Y, \lambda)$ with a complete metric, we first have to choose some finite measure $\mu$ in the same class as $\lambda$, i.e., sharing the same measure-zero sets. Note that this does not change the definition of the measure algebra. We can then equip the measure algebra $\mathrm{MAlg}(Y, \lambda) = \mathrm{MAlg}(X, \mu)$ with the metric $d_\mu(A, B) = \mu(A \triangle B)$.

**Lemma 2.1.** *Let $\mu$ and $\mu'$ be two finite measures on $Y$ in the same equivalence class. Then $d_\mu$ and $d_{\mu'}$ are uniformly equivalent: given any $\epsilon > 0$, there is $\delta > 0$ such that for all $A, B \in \mathrm{MAlg}(Y, \mu)$, we have that $d_\mu(A, B) < \delta$ implies $d_{\mu'}(A, B) < \epsilon$ and vice versa.*

*Proof.* By symmetry, given $\epsilon > 0$ it suffices to find a $\delta$ such that the direct implication holds. But this is a direct consequence of the fact that $\mu$ and $\mu'$ are in the same class: indeed by [Cohn 2013, Lemma 4.2.1], there actually is $\delta > 0$ such that whenever $\mu(C) < \delta$, we must have $\mu'(C) < \epsilon$. □

In particular, the above lemma yields that even when $\lambda$ is infinite, its measure algebra is endowed with a natural topology, independent of the choice of some $\mu$ finite in the class of $\lambda$. The fact that $d_\mu$ is complete is usually proved via the following lemma, which is important to us on its own:

**Lemma 2.2.** *Let $(A_n)$ be a sequence of elements of $\mathrm{MAlg}(Y, \mu)$ such that $\sum_n \mu(A_n \triangle A_{n+1}) < +\infty$. Let $\liminf A_n = \bigcup_N \bigcap_{n \geqslant N} A_n$. Then the sequence $(A_n)$ converges to $\liminf A_n$.*

*Proof.* By the Borel–Cantelli lemma, for almost all $x \in X$ there is $N$ such that for all $n \geqslant N$, $x \notin A_n \triangle A_{n+1}$. Letting $\epsilon > 0$, we then find $N$ and a set $X_0$ such that $\mu(X_0) > 1 - \epsilon$ and for all $x \in X$ and all $n \geqslant N$, we have $x \notin A_n \triangle A_{n+1}$. Now observe that for all $n \geqslant N$, we have $A_n \cap X_0 = \liminf A_n \cap X_0$; in particular $A_n \triangle \liminf A_n \subseteq X \setminus X_0$, which has measure less than $\epsilon$, as wanted. □

**Proposition 2.3.** *Let $(A_n)$ be a sequence of elements of $\mathrm{MAlg}(Y, \mu)$ which converges to some $A \in \mathrm{MAlg}(Y, \mu)$. Then we can find a subsequence $(A_{n_k})$ such that*

$$A = \liminf A_{n_k} = \bigcup_N \bigcap_{k \geqslant N} A_{n_k}.$$

*Proof.* Since $(A_n)$ converges, it is Cauchy; in particular it admits a subsequence $(A_{n_k})_k$ such that for all $k \in \mathbb{N}$, $d_\mu(A_{n_k}, A_{n_{k+1}}) < 1/2^k$. Thus, by the previous lemma, $A_{n_k} \to \liminf A_{n_k} = A$. □

**Proposition 2.4.** *The metric $d_\mu$ is complete, and $\mathrm{MAlg}(Y, \mu)$ is separable.*

*Proof.* Let $(A_n)$ be a Cauchy sequence, it suffices to show that it has a convergent subsequence. But because the sequence is Cauchy, we may find a subsequence $(A_{n_k})_k$ such that for all $k \in \mathbb{N}$, $\mu(A_{n_k} \triangle A_{n_{k+1}}) < 2^{-k}$. Then $\sum_k \mu(A_{n_k} \triangle A_{n_{k+1}}) < +\infty$ and we get the desired result by applying the previous lemma.

Towards proving separability, first note that since $(Y, \mu)$ is standard, we may as well assume it is an interval of finite length endowed with the Lebesgue measure. The regularity of the Lebesgue measure can then be used to show that finite unions of open intervals with rational endpoints are dense in the measure algebra, thus yielding the desired countable dense subset. □

**2.2.** ***The space of subequivalence relations.*** Given an action of a countable group $\Gamma$ by Borel bijections on a standard Borel space $X$, we obtain the associated equivalence relation $\mathcal{R}_\Gamma = \{(x, \gamma x) : x \in X,\ \gamma \in \Gamma\}$ whose equivalence classes are exactly the $\Gamma$-orbits. This equivalence relation is a Borel subset of $X \times X$, and its classes are countable: $\mathcal{R}_\Gamma$ is by definition a *countable Borel equivalence relation* (*CBER*, which can conveniently be read as "seeber"). Conversely, the Feldman–Moore theorem ensures that every CBER comes from a Borel action of a countable group; see for instance [Kechris and Miller 2004, Theorem 1.3].

Given such an equivalence relation, define its *Borel pseudofull group* as the set of all partial Borel bijections $\varphi : \operatorname{dom}\varphi \subseteq X \to \operatorname{rng}\varphi \subseteq X$ such that for all $x \in A$, we have $(x, \varphi(x)) \in \mathcal{R}$. By definition, the *Borel full group* is the group made of elements of the Borel pseudofull group whose domain and range are equal to $X$.

If $(X, \mu)$ is now a standard probability space, we say that a CBER is *nonsingular* (or that $\mu$ is quasi-invariant, or that $\mathcal{R}$ is nullpreserving) if for all $\varphi$ in its Borel pseudofull group, we have $\mu(\operatorname{dom}\varphi) = 0$ if and only if $\mu(\operatorname{rng}\varphi) = 0$. The main (and only by the Feldman–Moore theorem) source of such equivalence relations is the following:

**Definition 2.5.** An action of a countable group $\Gamma$ by Borel bijections on $X$ is called *nonsingular* (or nullpreserving, or quasipreserving $\mu$) if for all $A \subseteq X$ Borel, we have $\mu(A) = 0$ if and only if $\mu(\gamma A) = 0$.

**Remark.** By Lemma 2.1, nonsingular actions yield action by uniformly continuous homeomorphisms on the measure algebra $\mathrm{MAlg}(X, \mu)$.

Given $\Gamma \curvearrowright (X, \mu)$ nonsingular, the associated CBER $\mathcal{R}_\Gamma$ is nonsingular, and conversely any nonsingular CBER arises in this manner; see [Kechris and Miller 2004, Proposition 8.1].

Now let $\mathcal{R}$ be a nonsingular CBER. We equip $\mathcal{R}$ with the left measure $M$ defined by

$$M(A) = \int_X |A_x|\, d\mu(x)$$

for all $A \subseteq \mathcal{R}$ Borel. Let $m$ be an equivalent probability measure. Then the measure algebra of $\mathcal{R}$ with respect to $M$ is equal to that of $\mathcal{R}$ with respect to $m$. This measure is $\sigma$-finite since $\mathcal{R}$ can be covered by the graphs of the elements of a countable group $\Gamma$ such that $\mathcal{R} = \mathcal{R}_\Gamma$ (each such graph has measure 1).

It then follows from Proposition 2.4 that $\mathrm{MAlg}(\mathcal{R}, M)$ is a Polish space. Moreover, by Lemma 2.1, the topology (and actually the uniform structure) do not depend on the choice of $m$. Since the measure $m$ is not canonical, we will most of the time write our topological space as $\mathrm{MAlg}(\mathcal{R}, M)$.

A Borel subset of $\mathcal{R}$ is called a Borel subequivalence relation if it is an equivalence relation on $X$. An element of $\mathrm{MAlg}(\mathcal{R}, M)$ is called a subequivalence relation if its equivalence class *contains* an equivalence relation. Denote by $\mathrm{Sub}(\mathcal{R}) \subseteq \mathrm{MAlg}(\mathcal{R}, M)$ the space of subequivalence relations of $\mathcal{R}$. We can now easily prove the following result of Kechris (our proof is moreover motivated by [Kechris 2024, Theorem 5.1], which we have essentially recast as Proposition 2.3).

**Theorem 2.6** [Kechris 2024, Section 4.4(1)]. *The space* $\mathrm{Sub}(\mathcal{R})$ *of subequivalence relations of* $\mathcal{R}$ *is a closed subset of* $\mathrm{MAlg}(\mathcal{R}, M)$*, and hence Polish.*

*Proof.* Observe that the lim inf of any sequence of equivalence relations is an equivalence relation. By Proposition 2.3 the result follows. □

Observe that we did not even use that $\mathcal{R}$ was nonsingular in the above. However, we need nonsingularity for our space to encode the right notion of equality of subequivalence relations as follows:

**Proposition 2.7.** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two Borel subequivalence relations of a nonsingular CBER $\mathcal{R}$, and suppose $M(\mathcal{S}_1 \triangle \mathcal{S}_2) = 0$. Then there is a full measure $\mathcal{S}_1$- and $\mathcal{S}_2$-invariant set on which $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same.*

*Proof.* By definition, for almost all $x$ we have $[x]_{\mathcal{S}_1} = [x]_{\mathcal{S}_2}$. Since $\mathcal{S}_1$ is nonsingular, we find a smaller $\mathcal{S}_1$-invariant set $X_0$ for which the same conclusion holds (if $\mathcal{S}_1 = \mathcal{R}_\Gamma$ we let $X_0 = \bigcap_{\gamma\in\Gamma} \gamma X_1$, where $X_1 = \{x \in X : [x]_{\mathcal{S}_1} = [x]_{\mathcal{S}_2}\}$). □

We will also see in the next section that nonsingularity is important if we want to concretely see our space of subequivalence relations as a closed set, i.e., to say that the axioms of subequivalence relations each define closed sets. Let us conclude this section by pointing out yet another way of understanding the topology which, was pointed out to us by Stefaan Vaes.

**Remark.** Given a nonsingular equivalence relation $\mathcal{R}$ on $(X, \mu)$, consider the space of subalgebras of its von Neumann algebra $L\mathcal{R}$, which is Polish when endowed with the Maréchal topology [1973]; see also [Fima et al. 2024] for a detailed proof. Then it is a result of Aoi [2003] that subequivalence relations of $\mathcal{R}$ are in one-to-one correspondence with subalgebras of $L\mathcal{R}$ containing $\mathrm{L}^\infty(X, \mu)$. It is not hard to see that such subalgebras form a closed set, and the map which associates to $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ the subalgebra $L\mathcal{S}$ of $L\mathcal{R}$ is a homeomorphism (see Theorem E from the recent preprint by Shuoxing Zhou [2024]), yielding yet another way of understanding the Polish topology induced by the measure algebra of $\mathcal{R}$ on $\mathrm{Sub}(\mathcal{R})$. For (many!) other interpretations of this topology in the probability measure-preserving setup, the reader should consult [Kechris 2024].

## 3. Strong and lower topologies

In this section, we first study the strong topology on the space of subequivalence relations as defined in [Kechris 2024], which will be useful towards establishing our orbit density result (Theorem B). We then use a lower topology on the measure algebra so as to obtain a natural proof of the fact that the space of subequivalence relations is closed in the measure algebra, and of the fact that the map which takes a subgraph to the equivalence it generates is Baire class one [Kechris 2024, Proposition 19.1].

**3.1.** ***Topologies on measure algebras of $\sigma$-finite spaces.*** We first go back to measure algebras of $\sigma$-finite spaces, which we now write as $(Y, M)$, motivated by our previous example where $Y$ is a nonsingular CBER.

**Proposition 3.1.** *Let $(Y, M)$ be a standard $\sigma$-finite space. Let $(Y_n)$ be a countable family of finite-measure subsets of $Y$ such that $\bigcup_n Y_n = Y$. Then the map $\Phi : \mathrm{MAlg}(Y, M) \to \prod_n \mathrm{MAlg}(Y_n, M_{Y_n})$ which takes $A$ to the sequence $(A \cap Y_n)$ is a homeomorphism onto its image.*

*Proof.* Observe that the probability measure

$$m(A) := \sum_n \frac{1}{2^{n+1} M(Y_n)} M(A \cap Y_n)$$

is equivalent to $M$, and thus the associated metric $d_m$ induces the topology of $\mathrm{MAlg}(Y, M)$. Now the space $\prod_n \mathrm{MAlg}(Y_n, M_{Y_n})$ can be endowed with a compatible metric $\tilde{d}$ given by

$$\tilde{d}((A_n), (B_n)) = \sum_n \frac{1}{2^{n+1} M(Y_n)} M(A_n \triangle B_n).$$

It is then straightforward to check that $\Phi$ is an isometry for the metric $d_m$ on its domain and $\tilde{d}$ on its range. □

We now go on to obtain a more precise version of this result, following Kechris' approach to the *strong topology* on $\mathrm{Sub}(\mathcal{R})$.

Observe that every element $\varphi$ of the Borel pseudofull group of a CBER $\mathcal{R}$ is completely determined by its *graph*, namely the set $\{(x, \varphi(x)) : x \in \mathrm{dom}\, \varphi\}$, which is a Borel subset of $\mathcal{R}$. From now on, we identify elements of the Borel pseudofull group to their graph. Doing this up to measure zero when $\mathcal{R}$ is nonsingular, we arrive at the following definition:

**Definition 3.2.** The *pseudofull group* of a nonsingular equivalence relation $\mathcal{R}$ on $(X, \mu)$ is the quotient of the Borel pseudofull group by the equivalence relation which identifies $\varphi_1$ and $\varphi_2$ when $M(\varphi_1 \triangle \varphi_2) = 0$. It is denoted by $[\![\mathcal{R}]\!]$, and for ease of notation we still write its elements as $\varphi$'s. Finally the *full group* of $\mathcal{R}$ is the group $[R]$ of all elements of $[\![\mathcal{R}]\!]$ with full measure domain and range.

By the definition of $M$ we have that $M(\varphi_1 \triangle \varphi_2) = 0$ if and only if $\mu(\mathrm{dom}\, \varphi_1 \triangle \mathrm{dom}\, \varphi_2) = 0$ and for almost all $x \in \mathrm{dom}\, \varphi_1 \cap \mathrm{dom}\, \varphi_2$, we have $\varphi_1(x) = \varphi_2(x)$. The Borel pseudofull group is stable under countable increasing unions and arbitrary countable intersections, so it is stable under taking lim inf. Following the same approach as for the proof of Theorem 2.6, one obtains that $[\![\mathcal{R}]\!]$ is a closed subset of $\mathrm{MAlg}(\mathcal{R}, M)$, in particular it is Polish. Note however that $[\mathcal{R}]$ is not closed for this topology (see Example 3.6).

We now make a slight modification of Kechris' uniquely generating sequences of involutions. Let us say that an element $\varphi$ of the Borel pseudofull group is a *moving partial involution* if $\mathrm{dom}\, \varphi = \mathrm{rng}\, \varphi$ and for all $x \in \mathrm{dom}\, \varphi$, we have both $\varphi(x) \neq x$ and $\varphi(\varphi(x)) = x$.

**Definition 3.3.** A sequence $(\varphi_n)_{n \in \mathbb{N}}$ of moving partial involutions of the Borel pseudofull group of a CBER $\mathcal{R}$ is called a *uniquely generating sequence of moving partial involutions* if

$$\mathcal{R} \setminus \Delta_X = \bigsqcup_{n \in \mathbb{N}} \varphi_n,$$

where $\Delta_X = \{(x, x) : x \in X\}$ is the equality relation.

**Proposition 3.4.** *Every CBER $\mathcal{R}$ admits a uniquely generating sequence of moving partial involutions.*

*Proof.* Suppose that $(T_n)_{n \in \mathbb{N}}$ is a uniquely generating sequence of involutions for $\mathcal{R}$ as in [Kechris 2024, Proposition 4.6], and for every $n \in \mathbb{N}$ let $\varphi_n$ be the restriction of $T_n$ to its support. □

**Remark.** The disjointness of the graphs of the $\varphi_n$ simplifies a bit the arguments we carry out in this paper since it guarantees that all the $\varphi_i(x)$ are distinct whenever they are defined (see in particular the proof of (i) $\Longrightarrow$ (ii) in Theorem 4.7). The disjointness also allows us to upgrade Proposition 3.1 and obtain that the map

$$\mathrm{MAlg}(\mathcal{R}, M) \to \mathrm{MAlg}(\Delta_X, M_{\Delta_X}) \times \prod_{n\in\mathbb{N}} \mathrm{MAlg}(\varphi_n, M_{\varphi_n})$$

which takes $A$ to the sequence $(A \cap Y_n)$ is not only a homeomorphism onto its image, but actually surjective.

The following corollary is a slight reformulation of [Kechris 2024, Theorem 4.13] which identifies the strong and the weak topology on the space of subequivalence relations:

**Corollary 3.5.** *Let $\mathcal{R}$ be a nonsingular equivalence relation, and $(\varphi_n)$ be a uniquely generating sequence of moving partial involutions. Then the map*

$$\mathrm{Sub}(\mathcal{R}) \to \prod_{n\in\mathbb{N}} \mathrm{MAlg}(\varphi_n, M_{\varphi_n})$$

*which takes $\mathcal{S}$ to the sequence $(\mathcal{S} \cap \varphi_n)_{n\in\mathbb{N}}$ is a homeomorphism onto its image.*

*Proof.* This is a direct consequence of the definition of uniquely generating sequences of moving partial involutions and of Proposition 3.1, once we note that every equivalence relation contains $\Delta_X$. □

Following the above corollary and Kechris' terminology, we now define the *strong topology* to be the topology induced by $d_m$ on $\mathrm{Sub}(\mathcal{R})$, which is Polish by Theorem 2.6.

**3.2.** ***Lower topologies on measure algebras.*** Let us first work a bit more on the relationship between $M$ and $m$. Observe that $M$ defines a lower-semicontinuous function on $\mathrm{MAlg}(Y, M)$ (possibly taking $+\infty$ as a value). Indeed if $M(A) > \alpha$, we have a $\delta > 0$ such that $M(C) < M(A) - \alpha$ whenever $C$ is a subset of $A$ satisfying $m(C) < \delta$, and so as soon as $m(A \setminus B) < \delta$ we have $M(B) > \alpha$. In particular for every $k \in \mathbb{N}$ the set of elements with measure $\leqslant k$ is closed.

Let us denote by $\mathrm{MAlg}_f(\mathcal{R}, M)$ the $F_\sigma$ subset of elements of $\mathrm{MAlg}(\mathcal{R}, M)$ with finite $M$-measure. It can be endowed with the metric $d_M(A, B) = M(A \triangle B)$, which clearly refines the topology induced by $d_m$. Moreover, the proof of Proposition 2.4 can easily be adapted to show that $d_M$ is complete separable (see [Le Maître 2022, Lemma 2.1] for a complete proof). Let us see why $d_m$ and $d_M$ yield different topologies on $\mathrm{MAlg}_f(\mathcal{R}, M)$.

**Example 3.6.** Going back to the case $Y = \mathcal{R}$, observe that if $T$ in the full group of $\mathcal{R}$ is aperiodic, then $T^n \to \varnothing$ for $d_m$ while it stays in the sphere around 1 for $d_M$. In particular $d_m$ does not refine $d_M$. Let us also mention that $[\mathcal{R}]$ is $d_M$-closed, and that the metric induced by $d_M$ on $[\mathcal{R}]$ is the well-known uniform metric $d_u(S, T) = \mu(\{x \in X : S(x) \neq T(x)\})$, endowing $[\mathcal{R}]$ with a Polish group topology.

Despite the above example, $M$ and $m$ share the same *lower topologies* when restricted to $\mathrm{MAlg}_f(Y, M)$. These lower topologies are not Hausdorff; however they have nice continuity properties.

**Definition 3.7.** Let $(Y, M)$ be a $\sigma$-finite measured space. We endow its measure algebra with the *lower $M$-topology* which is defined by taking as neighborhoods of an element $A \in \mathrm{MAlg}(Y, M)$ all sets of the form

$$\{B \in \mathrm{MAlg}(Y, M) : M(A \setminus B) < \epsilon\}$$

for some $\epsilon > 0$.

This neighborhood system does satisfy the axiom that implies that it is a basic neighborhood system for the topology it induces, namely every neighborhood $V$ contains a smaller neighborhood all of whose elements admit $V$ as a neighborhood (axiom $(\mathrm{V_{IV}})$ in [Bourbaki 1989, Chapter 1, §1.2]). This is simply because if $M(A \setminus B) < \frac{1}{2}\epsilon$ and $M(B \setminus C) < \frac{1}{2}\epsilon$ then $M(A \setminus C) < \epsilon$.

Note that the only neighborhood of the empty set is the whole measure algebra, while $Y$ is contained in every neighborhood of every point. Moreover, $\{\varnothing\}$ is closed, and more generally $M$ is lower semicontinuous for the lower $M$-topology.

**Lemma 3.8.** *If $M$ is a $\sigma$-finite measure on $Y$ and $m$ is an equivalent finite measure, then the measures $M$ and $m$ induce the same lower topology on the space $\mathrm{MAlg}_f(Y, M)$ of finite $M$-measure subsets.*

*Proof.* Take $A$ such that $M(A) < +\infty$. Because $M$ and $m$, when restricted to $A$, are equivalent, we find $\delta$ such that every subset of $A$ of $m$-measure less than $\delta$ has $M$-measure less than $\epsilon$. Then $\{B : m(A \setminus B) < \delta\} \subseteq \{B : M(A \setminus B) < \epsilon\}$. So the lower $m$-topology refines the lower $M$-topology. The other inequality works the same. □

Let us note that the space of infinite measure subsets is open for the lower $M$-topology, while it is not for the lower $m$-topology. Indeed if $A$ has infinite measure, the set of $B$ such that $M(A \setminus B) < 1$ does not contain any $m$-neighborhood, because there are subsets of $A$ of very small $m$-measure but still with infinite measure.

We will now list various interesting properties of the lower topology, and then relate it to the usual topology in the last two lemmas.

**Lemma 3.9.** *Let $M$ be a $\sigma$-finite measure on $Y$. Then the intersection map $(A_1, A_2) \mapsto A_1 \cap A_2$ is continuous, where we put the $M$-lower topology everywhere.*

*Proof.* Let $\epsilon > 0$. Note that $(A_1 \cap A_2) \setminus (B_1 \cap B_2) \subseteq (A_1 \setminus B_1) \cup (A_2 \setminus B_2)$, so if $M(A_1 \setminus B_1) < \epsilon$ and $M(A_2 \setminus B_2) < \epsilon$, then $M((A_1 \cap A_2) \setminus (B_1 \cap B_2)) < 2\epsilon$ as wanted. □

**Lemma 3.10.** *Let $m$ be a finite measure on $Y$. Then the countable union map $(A_n)_{n\in\mathbb{N}} \mapsto \bigcup_{n\in\mathbb{N}} A_n$ is continuous for the $m$-lower topology everywhere.*

*Proof.* Let $(A_n)_{n\in\mathbb{N}}$ be a family of elements of the measure algebra. Then since $m$ is finite, we may find $N$ such that $\mu\big(\big(\bigcup_n A_n\big) \setminus \bigcup_{n<N} A_n\big) < \epsilon$. Now if we take $B_0, \ldots, B_{N-1}$ such that $m(A_i \setminus B_i) < \epsilon/N$, and any $B_N, B_{N+1}, \ldots$, we see that $m\big(\big(\bigcup_n A_n\big) \setminus \big(\bigcup_n B_n\big)\big) < 2\epsilon$, as wanted. □

**Remark.** The countable union map is *not* continuous if we put the usual topology everywhere (indeed $X$ can easily be obtained as a limit of unions of sequences of sets which converge coordinatewise to $\varnothing$). Here is nevertheless a positive result for the usual topology which will be useful later on.

**Lemma 3.11.** *Let $m$ be a finite measure on $Y$, and $(Y_j)_{j\in\mathbb{N}}$ be a disjoint family of measurable subsets of $Y$. Then the restriction of the countable union map to the set*

$$\{(A_n) \in \mathrm{MAlg}(Y, m)^{\mathbb{N}} : \forall n \in \mathbb{N},\, A_n \subseteq Y_n\},$$

*is continuous for the $d_m$-topology.*

*Proof.* Since the $Y_n$ are disjoint and $m$ is finite, given any $\epsilon > 0$ we can find $N$ such that $m\big(\bigcup_{n>N} Y_n\big) < \epsilon$; in particular if $(A_n)$ satisfies $A_n \subseteq Y_n$ for every $n$ then its union is, up to an $\epsilon$ error, independent of $(A_n)_{n>N}$. The conclusion thus follows from the continuity of finite unions for the $d_m$-topology. □

We now relate the Hausdorff topology associated to the metric $d_m$ and to the corresponding lower $m$-topology:

**Lemma 3.12.** *Let $m$ be a finite measure on $Y$, and denote by $\tau_m^l$ the lower $m$-topology and by $\tau_m$ the topology induced by the metric $d_m$. Then the identity map $(\mathrm{MAlg}(Y, m), \tau_l^m) \to (\mathrm{MAlg}(Y, m), \tau_m)$ is Baire class one.*

*Proof.* We begin by noting some useful continuity properties for the lower $m$-topology. By definition, for every $A \in \mathrm{MAlg}(Y, m)$ the map $B \mapsto m(A \setminus B)$ is upper semicontinuous for the lower topology. Since $m(B) = m(X) - m(X \setminus B)$, we deduce that $m$ defines a lower semicontinuous map $(\mathrm{MAlg}(Y, m), \tau_m^l) \to \mathbb{R}^+$. By Lemma 3.9, for every $A \in \mathrm{MAlg}(Y, m)$ the map $B \mapsto m(A \cap B)$ is thus lower semicontinuous as well.

We finally observe that

$$d_m(A, B) = m(A \triangle B) = m(A) + m(B) - 2m(A \cap B),$$

so $d_m(A, \cdot)$ is a Baire class one function on $(\mathrm{MAlg}(Y, \nu), \tau_m^l) \to \mathbb{R}^+$, being the sum of two Baire class one functions. In particular, $d_m$-open balls are $F_\sigma$ for $\tau_m^l$. Since every $\tau_m$-open set is a countable union of open balls by separability, the conclusion follows. □

We conclude this section by a lemma which intertwines the two topologies we have defined in presence of a finite measure:

**Lemma 3.13.** *Let $m$ be a finite measure on $Y$. The space of all $(A, B)$ such that $A \subseteq B$ is closed if we put on the first coordinate the $m$-lower topology, and on the second the usual measure algebra topology.*

*Proof.* Suppose that $A \not\subseteq B$, and let $\epsilon = m(A \setminus B)$. Then if $m(A \setminus A') < \frac{1}{2}\epsilon$ and $m(B' \triangle B) < \frac{1}{2}\epsilon$, we still have $A' \not\subseteq B'$. □

**3.3.** ***Seeing that the space of subequivalence relations is closed.*** We now provide a more concrete proof that the space of subequivalence relations is Polish by seeing that the conditions which make an equivalence relation define closed sets in $\mathrm{MAlg}(\mathcal{R}, M)$ for the topology induced by $d_m$. First note that reflexivity means containing $\Delta_X$, and hence defines a closed set by Lemma 3.13.

Let us fix a nonsingular equivalence relation $\mathcal{R}$. We recall that $m$ is some finite measure equivalent to the measure $M$ obtained by integrating the cardinalities of vertical fibers. Observe that the flip

$\sigma : (x, y) \mapsto (y, x)$ quasipreserves $M$ (this is one characterization of nonsingularity, see [Kechris and Miller 2004, Proposition 8.2]); in particular it quasipreserves $m$ and hence induces a homeomorphism on $\mathrm{MAlg}(\mathcal{R}, M)$.

This implies that the space of (Borel) *subgraphs* of $\mathcal{R}$ (symmetric subsets of $\mathcal{R} \setminus \mathrm{id}_X$) is closed in $\mathrm{MAlg}(\mathcal{R}, M)$, because it is the set of fixed points of $\sigma$ contained in $\mathcal{R} \setminus \mathrm{id}_X$. In particular, we recover the following fact from [Kechris 2024, Chapter 18]:

**Proposition 3.14.** *The space of subgraphs is Polish for the topology induced by the measure algebra* $\mathrm{MAlg}(\mathcal{R}, M)$. □

**Remark.** This could also be seen through the same approach as for Theorem 2.6, noting that the liminf of any sequence of subgraphs is a subgraph.

To see that the space of subequivalence relations is closed, we need to deal with transitivity. Observe that on the space of measurable subsets of $\mathcal{R}$, we have an operation $\circ$ defined by

$$A \circ B := \{(x, z) : \exists y \in X \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\}.$$

Note that $\circ$ extends the composition of elements of the pseudofull group of $\mathcal{R}$, and that it is well-defined only because we are in the nonsingular setup.

**Lemma 3.15.** *On the pseudofull group of $\mathcal{R}$, the composition is continuous for the $M$-lower topology, and hence for the $m$-lower topology.*

*Proof.* Note that the $M$-measure of the graph of an element of the pseudofull group is the $\mu$-measure of its domain, so by Lemma 3.8 we only need to check the continuity for the lower $M$-topology.

Now fix $\varphi_0, \psi_0 \in [\![\mathcal{R}]\!]$, let $\epsilon > 0$ and find $\delta$ such that whenever $\mu(A) < \delta$ we have $\mu(\psi_0^{-1}(A)) < \frac{1}{2}\epsilon$. Take $\varphi$ such that $M(\varphi \setminus \varphi_0) < \delta$. Then $\mu(\{x \in \mathrm{dom}(\varphi_0) : \varphi(x) \neq \varphi_0(x)\}) < \delta$ (in the definition we consider that $\varphi_0(x)$ is different from $\varphi(x)$ if $x \notin \mathrm{dom}\,\varphi$). Then, taking $\psi$ such that $M(\psi \setminus \psi_0) < \frac{1}{2}\epsilon$, the set $X_1 = \{x \in \mathrm{dom}\,\psi_0 : \psi(x) = \psi_0(x)\}$ contains $\mathrm{dom}\,\psi_0$ up to an $\frac{1}{2}\epsilon$ error, and

$$\psi_{\restriction X_1}^{-1}(\{x \in \mathrm{dom}(\varphi_0) : \varphi(x) \neq \varphi_0(x)\}) = \psi_{0\restriction X_1}^{-1}(\{x \in \mathrm{dom}(\varphi_0) : \varphi(x) \neq \varphi_0(x)\})$$

has thus measure at most $\frac{1}{2}\epsilon$. We conclude that the set of $x \in \mathrm{dom}\,\varphi_0\psi_0$ such that $\varphi\psi(x) = \varphi_0\psi_0(x)$ contains $\mathrm{dom}\,\varphi_0\psi_0$ up to an $\frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ error, as wanted. The fact that the continuity also holds for the $m$-lower topology is then a direct consequence of Lemma 3.8 since elements of $[\![\mathcal{R}]\!]$ have $M$-measure at most 1. □

**Lemma 3.16.** *The map $A, B \mapsto A \circ B$ is continuous if we put everywhere the lower $m$-topology.*

*Proof.* Fixing a countable subgroup $\Gamma \leqslant [\mathcal{R}]$ such that $\mathcal{R} = \mathcal{R}_\Gamma$, the desired continuity follows from the previous lemma along with the formula

$$A \circ B = \bigcup_{\gamma, \gamma' \in \Gamma} \gamma \cap A \circ (\gamma' \cap B)$$

and the fact that finite intersections and countable unions are continuous maps for the lower topology (Lemmas 3.9 and 3.10). □

We thus arrive at the desired result saying that the space of transitive subrelations is closed.

**Proposition 3.17.** *The space of $A \subseteq \mathcal{R}$ such that $A \circ A \subseteq A$ is closed in the Polish topology on* $\mathrm{MAlg}(\mathcal{R}, M)$.

*Proof.* This follows directly from the previous lemma along with Lemma 3.13. □

**Theorem 3.18.** $\mathrm{Sub}(\mathcal{R})$ *is a closed subset of* $\mathrm{MAlg}(\mathcal{R}, M)$ *for the $d_m$-topology.*

*Proof.* We already observed that being reflexive exactly means containing $\Delta_X$ and thus is a closed condition by Lemma 3.13. The fact that symmetry is a closed condition is a reformulation of the fact that subgraphs form a closed set (Proposition 3.14). Finally transitivity is a closed condition by the previous proposition. So $\mathrm{Sub}(\mathcal{R})$ is the intersection of three closed subsets, and hence closed itself. □

We also have the following nice consequence of our work on the lower topology, which generalizes Proposition 19.1 from [Kechris 2024].

**Proposition 3.19.** *The map which takes a measurable subset of $\mathcal{R}$ to the equivalence relation it generates is continuous in the lower topology*; *in particular it is Baire class one for the strong topology* (*induced by $d_m$*).

*Proof.* This map associates to a $A \subseteq \mathcal{R}$ the equivalence relation $\bigcup_{n\in\mathbb{N}}(\Delta_X \cup A \cup \sigma(A))^{\circ n}$, so it is a continuous map by the continuity of $\sigma$, of countable unions (Lemma 3.10) and of composition (Lemma 3.16). The statement about the strong topology then follows directly from Lemma 3.12. □

We can finally obtain the following strengthening of Proposition 4.27 from [Kechris 2024]:

**Corollary 3.20.** *The map which takes a sequence $(\mathcal{R}_n)$ of subequivalence relations of $\mathcal{R}$ to the equivalence relation $\bigvee_n \mathcal{R}_n$ they generate is continuous in the lower topology*; *in particular it is Baire class one for the strong topology.*

*Proof.* This is a direct consequence of the previous result along with the fact that taking countable unions is continuous in the lower topology (Lemma 3.10). □

## 4. Dense orbits for hyperfinite subequivalence relations in the p.m.p. case

In the present section, we exclusively work in the probability measure-preserving setup, so $\mathcal{R}$ is a p.m.p. equivalence relation on $(X, \mu)$. Recall that $\mathrm{Aut}(\mathcal{R})$ is the group of all $T \in \mathrm{Aut}(X, \mu)$ such that $T \times T\ (\mathcal{R}) = \mathcal{R}$. Observe that $\mathrm{Aut}(\mathcal{R})$ acts on $\mathrm{MAlg}(\mathcal{R}, M)$ by $T \cdot (x, y) = (T(x), T(y))$ and preserves the measure $M$. The group $\mathrm{Aut}(\mathcal{R})$ contains the full group of $\mathcal{R}$, and it is a Polish group for the topology induced by the group $\mathrm{Aut}(\mathcal{R}, M)$ of all measure-preserving bijections of $(\mathcal{R}, M)$; see [Kechris 2010, Proposition 6.3] where $\mathrm{Aut}(\mathcal{R})$ is denoted by $N[\mathcal{R}]$. The trivial subequivalence relation will play an important role; we denote it by $\Delta_X = \{(x, x) : x \in X\}$ (given our identification of full groups elements to their graphs, we could also write it as the identity map $\mathrm{id}_X$).

**4.1.** ***Preliminaries on conditional measures and index.*** In order to work in the p.m.p. setup, we have to understand which sets can be taken to other sets by (pseudo)full group elements up to measure zero. This is done here through the concept of *conditional measures*, a low-tech version of the ergodic decomposition which was already present in Dye's founding paper [1959]. We already used them in a previous paper on nonergodic p.m.p. equivalence relations (see [Le Maître 2016, Section 2]) but we recently gave a more detailed exposition for general full groups in a joint work with Slutsky, which we use here as a reference (see Appendix D in [Le Maître and Slutsky 2021]).

**Definition 4.1.** Let $\mathcal{R}$ be a p.m.p. equivalence relation on $(X, \mu)$, and denote by $M_{\mathcal{R}}$ the closed subalgebra of $(X, \mu)$ consisting of $[\mathcal{R}]$-invariant subsets and by $\mathbb{E}_{\mathcal{R}}$ the projection $\mathrm{L}^2(X, \mu) \to \mathrm{L}^2(X, M_{\mathcal{R}}, \mu)$. Given $A \in \mathrm{MAlg}(X, \mu)$, its *$M_{\mathcal{R}}$-conditional measure* $\mu_{\mathcal{R}}$ is the $M_{\mathcal{R}}$-measurable function

$$\mu_{\mathcal{R}}(A) = \mathbb{E}_{M_{\mathcal{R}}}(\chi_A).$$

It can be checked that $\mu_{\mathcal{R}}$ takes values in $[0, 1]$, satisfies the usual axioms of a measure, and that elements of $[\mathcal{R}]$ preserve $\mu_{\mathcal{R}}$; see [Le Maître and Slutsky 2021, Proposition D.6]. Say that $\mathcal{R}$ is *ergodic* when $M_{\mathcal{R}} = \{\varnothing, X\}$. Then $\mathcal{R}$ is ergodic if and only if $\mu_{\mathcal{R}} = \mu$. By [Le Maître and Slutsky 2021, Proposition D.10], we moreover have:

**Lemma 4.2.** *Let $A, B \in \mathrm{MAlg}(X, \mu)$, and let $\mathcal{R}$ be a p.m.p. equivalence relation on $(X, \mu)$. The following are equivalent*:

(i) *There is $\varphi \in [\![\mathcal{R}]\!]$ such that* $\mathrm{dom}\, \varphi = A$ *and* $\mathrm{rng}\, \varphi = B$.

(ii) $\mu_{\mathcal{R}}(A) = \mu_{\mathcal{R}}(B)$.

We finally note the following important consequence of aperiodicity (having only infinite classes) for a p.m.p. equivalence relation (it is actually a characterization, but we don't need that):

**Proposition 4.3** (Maharam's lemma; see [Le Maître and Slutsky 2021, Theorem D.12]). *Let $\mathcal{R}$ be a p.m.p. aperiodic equivalence relation, let $A \in \mathrm{MAlg}(X, \mu)$, and let $f : X \to [0, 1]$ be an $M_{\mathcal{R}}$-measurable function such that $0 \leqslant f \leqslant \mu_{\mathcal{R}}(A)$. Then there is $B \subseteq A$ such that $\mu_{\mathcal{R}}(B) = f$.*

We finish this section by recalling some important definitions on the index of subequivalence relations:

**Definition 4.4.** Let $\mathcal{R}$ be a nonsingular equivalence relation on $(X, \mu)$. We say that $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$

- has *infinite index* in $\mathcal{R}$ if for almost all $x \in X$, the $\mathcal{R}$-equivalence class of $x$ contains infinitely many distinct $\mathcal{S}$-classes,
- has *finite index* in $\mathcal{R}$ if for almost all $x \in X$, the $\mathcal{R}$-equivalence class of $x$ is the union of a finite set of $\mathcal{S}$-classes,
- has *everywhere infinite index* in $\mathcal{R}$ if for all $A \subseteq X$ such that $\mu(A) > 0$, the restriction of $\mathcal{S}$ to $A$ has infinite index in the restriction of $\mathcal{R}$ to $A$.

**Remark.** It might not be clear at first sight that having infinite index is not the same as having everywhere infinite index. Here is the simplest example of an infinite-index $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$, not everywhere of infinite index: take $\mathcal{R}$ ergodic, let $A \subseteq X$ be of positive nonfull measure, and let $\mathcal{S} = \mathcal{R}_{\restriction A} \sqcup \Delta_{X \setminus A}$.

Let us note that when $\mathcal{R}$ is ergodic, since the number of $\mathcal{S}$-classes inside the $\mathcal{R}$-class of $x \in X$ is $\mathcal{R}$-invariant, it is constant almost everywhere, and hence $\mathcal{S}$ either has finite or infinite index in $\mathcal{R}$. Similarly, if $\mathcal{S}$ is ergodic and of infinite index, then it is everywhere of infinite index because the $\mathcal{S}$-class of almost every $x \in X$ intersects $A$. We finally note the following nonergodic way of producing everywhere infinite index subequivalence relations:

**Lemma 4.5.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be nonsingular aperiodic equivalence relations on respective standard probability spaces $(X_1, \mu_1)$ and $(X_2, \mu_2)$. Then $\mathcal{R}_1 \times \Delta_{X_2}$ is everywhere of infinite index in $\mathcal{R}_1 \times \mathcal{R}_2$.*

*Proof.* Let $A \subseteq X_1 \times X_2$ be of positive measure. By Fubini's theorem, for almost all $(x_1, x_2) \in A$ the vertical section $A_{x_1} = \{x_2' \in X_2 : (x_1, x_2' \in A)\}$ has positive measure, and hence by aperiodicity the $\mathcal{R}_2$-class of $x_2$ intersects $A_{x_1}$ in an infinite set. This implies that the $(\mathcal{R}_1 \times \mathcal{R}_2)_{\restriction A}$-class of $(x_1, x_2)$ contains infinitely many $(\mathcal{R}_1 \times \Delta_{X_2})_{\restriction A}$-classes. □

**Proposition 4.6.** *The equivalence relation $\mathcal{S}$ from Example 1.1 is everywhere of infinite index in $\mathcal{R}_0$.*

*Proof.* The even–odd partition of $\mathbb{N}$ induces a bijection $\{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ under which $\mathcal{R}_0$ becomes $\mathcal{R}_0 \times \mathcal{R}_0$ and $\mathcal{S}$ becomes $\mathcal{R}_0 \times \Delta_{\{0,1\}^{\mathbb{N}}}$, so the result follows from the previous lemma. □

**4.2.** ***Approximating the diagonal.*** Recall that $\Delta_X$ denotes the equality relation on $X$. We will now characterize equivalence relations whose orbit closure contains $\Delta_X$, mirroring Popa's result [2019, Lemma 2.3] on asymptotic orthogonalization of subalgebras of a finite factor.

**Theorem 4.7.** *Let $\mathcal{R}$ be an aperiodic p.m.p. equivalence relation. Let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be a subequivalence relation. The following are equivalent*:

(i) *$\mathcal{S}$ has everywhere infinite index in $\mathcal{R}$.*

(ii) *The closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\Delta_X$.*

(iii) *The closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains $\Delta_X$.*

*Proof.* The implication (ii) $\Longrightarrow$ (iii) is clear since $[\mathcal{R}] \leqslant \mathrm{Aut}(\mathcal{R})$.

Let us then check that (iii) implies (i) by proving the contrapositive: assuming that $\mathcal{S}$ does not have infinite index everywhere, we need to show that the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ does not contain $\Delta_X$.

By assumption, we have a subset $A \subseteq X$ of positive measure such that the restriction $\mathcal{S}_{\restriction A}$ has finite index in $\mathcal{R}_{\restriction A}$. Shrinking $A$ further if necessary, we may assume the index of the restriction of $\mathcal{S}$ to $A$ in the restriction of $\mathcal{R}$ to $A$ is constant equal to $k$. Let $N \in \mathbb{N}$ such that $1/N < \mu(A)$.

By aperiodicity and Maharam's lemma with $f = 1/(2kN)$ (see Proposition 4.3), we can partition $X$ in $2kN$ pieces of equal $\mathcal{R}$-conditional measure. Using Lemma 4.2, we thus have $B \subseteq X$ and $\varphi_1, \ldots, \varphi_{2kN} \in [\![\mathcal{R}]\!]$ with domain $B$ such that $\varphi_1 = \mathrm{id}_B$ and the sets $\varphi_1(B), \ldots, \varphi_{2kN}(B)$ partition $X$.

Let $B_0$ be the set of $x \in B$ such that there are at least $k+1$ distinct indices $i \in \{1, \dots, 2kN\}$ such that $\varphi_i(x) \in A$. Write $A_0 = \{x \in A : \exists i, \varphi_i^{-1}(x) \in B_0\}$. Then since $A_0$ is covered by the disjoint translates of $B_0$, we have $\mu(A_0) \leqslant 2kN\mu(B_0)$. Letting $A_1 = A \setminus A_0$, we again have that $A_1$ is covered by the disjoint translates of $B_1 = B \setminus B_0$, but by definition for every $x \in B_1$ there are at most $k$ indices $i \in \{1, \dots, 2kN\}$ such that $\varphi(x) \in A_1$, so that

$$\mu(A_1) \leqslant k\mu(B_1) \leqslant \frac{k}{2kN}.$$

Since $A = A_0 \sqcup A_1$, we then have $\mu(A) \leqslant k/(2kN) + 2kN\mu(B_0)$. But $1/N < \mu(A)$ so $1/2N < 2kN\mu(B_0)$, and hence

$$\mu(B_0) \geqslant \frac{1}{2kN^2}.$$

Now for all $x \in B_0$, because the index of $\mathcal{S}_{\restriction A}$ in $\mathcal{R}_{\restriction A}$ is $k$, we must have two distinct $i, j$ such that $(\varphi_i(x), \varphi_j(x)) \in \mathcal{S}$. In particular,

$$\sum_{1 \leqslant i < j \leqslant 2N} M(\varphi_i^{-1}\varphi_j \cap \mathcal{S}) \geqslant \frac{1}{2kN^2}.$$

The latter estimate will actually be valid for every translate of $\mathcal{S}$ by $T \in \mathrm{Aut}(\mathcal{R})$ because $T(A)$ will still satisfy $1/N < \mu(T(A))$, so for every $T \in \mathrm{Aut}(\mathcal{R})$,

$$\sum_{1 \leqslant i < j \leqslant 2N} M(\varphi_i^{-1}\varphi_j \cap T \cdot \mathcal{S}) \geqslant \frac{1}{2kN^2}.$$

This inequality defines a closed set of subequivalence relations, but it is not satisfied by $\Delta_X$, which finishes the proof of (iii) $\Longrightarrow$ (i).

We finally prove that (i) implies (ii). We fix $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ everywhere of infinite index. Let us also fix a uniquely generating sequence $(\varphi_k)$ of moving partial involutions of $\mathcal{R}$ as provided by Proposition 3.4. Applying Proposition 3.1 to the sequence of $M$-finite measure subsets $\varphi_k$, it suffices to show that given some $k \in \mathbb{N}$ and $\epsilon > 0$, we can find $T \in [\mathcal{R}]$ such that for all $i \in \{1, \dots, k\}$,

$$M((T^{-1} \cdot \mathcal{S}) \cap \varphi_i) < \epsilon.$$

We will actually do better and construct $T \in [\mathcal{R}]$ such that for all $i \in \{1, \dots, k\}$, $M((T^{-1} \cdot \mathcal{S}) \cap \varphi_i) = 0$. Observe that it now suffices (and it is necessary) to build $T$ such that for almost all $x \in \mathrm{dom}\, \varphi_i$ and all $i \in \{1, \dots, k\}$,

$$(T(x), T\varphi_i(x)) \notin \mathcal{S}.$$

Our construction makes a crucial use of a result of Eisenmann and Glasner [2016, Proposition 1.19]: letting $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$ be a dense subgroup of $[\mathcal{R}]$, we have that $\Gamma$ acts highly transitively on the orbit of almost every $x \in X$, and we may as well restrict ourselves to the set of all such $x$. By the definition of high transitivity, this means that for all $x \in X$, every partial bijection between finite subsets of $\Gamma x$ is the restriction of some $\gamma \in \Gamma$.

We will now build $T$ as the increasing union of elements $\psi_n$ of the pseudofull group defined inductively as follows.

First, $\psi_0$ is the restriction of $\gamma_0$ to the set of all $x \in X$ such that for all $i \in \{1, \dots, k\}$, we have $(\gamma_0 x, \gamma_0 \varphi_i(x)) \notin \mathcal{S}$. Then, assuming $\psi_n$ has been built, we extend it as $\psi_{n+1}$ by letting

$$\psi_{n+1}(x) = \gamma_{n+1} x$$

if $x \notin \operatorname{dom} \psi_n$, $\gamma_{n+1} x \notin \operatorname{rng} \psi_n$, and for all $i \in \{1, \dots, k\}$

(a) if $\varphi_i(x) \in \operatorname{dom} \psi_n$ then $(\gamma_{n+1} x, \psi_n \varphi_i(x)) \notin \mathcal{S}$,

(b) if $\varphi_i(x) \notin \operatorname{dom} \psi_n$ then $(\gamma_{n+1} x, \gamma_{n+1} \varphi_i(x)) \notin \mathcal{S}$.

Letting $\psi = \bigcup_n \psi_n$, we now prove the following central claim:

**Claim.** *The element of the pseudofull group $\psi$ has full domain.*

*Proof of the claim.* Assume not. Observe first that for almost all $x$ in the complement of the domain of $\psi$, the $\mathcal{R}$-class of $x$ intersects the complement of the range of $\psi$. Indeed, otherwise we have a positive measure $\mathcal{R}$-invariant set such that $\psi^{-1}$ takes it into a subset of itself of smaller measure, contradicting that $\mathcal{R}$ is measure-preserving.

Take $x \in X \setminus \operatorname{dom} \psi$ as above. Let $i_1, \dots, i_l \in \{1, \dots, k\}$ be the indices such that $\psi\varphi_{i_1}(x), \dots, \psi\varphi_{i_l}(x)$ is defined, and denote by $j_1, \dots, j_{k-l}$ the remaining indices. Since $\mathcal{S}$ has infinite index in $X \setminus \operatorname{rng} \psi$, there are $z \in [x]_{\mathcal{R}} \cap X \setminus \operatorname{rng} \psi$ and pairwise distinct $z_1, \dots, z_{k-l} \in [x]_{\mathcal{R}} \cap X \setminus \operatorname{rng} \psi$ such that

- for all $m \in \{1, \dots, l\}$ we have $(z, \psi\varphi_{i_m}(x)) \notin \mathcal{S}$,
- for all $m \in \{1, \dots, k-l\}$ we have $(z, z_m) \notin \mathcal{S}$.

By high transitivity, there is $\gamma \in \Gamma$ such that

$$\gamma x = z, \text{ and for all } m \in \{1, \dots, k-l\} \text{ we have } \gamma \varphi_{j_m}(x) = z_m.$$

This shows that for almost all $x \in X \setminus \operatorname{dom} \psi$ there is $n \in \mathbb{N}$ such that $\gamma_{n+1} x \notin \operatorname{rng} \psi$ and for all $i \in \{1, \dots, k\}$, we have:

(a′) If $\varphi_i(x) \in \operatorname{dom} \psi$ then $(\gamma_{n+1} x, \psi\varphi_i(x)) \notin \mathcal{S}$.

(b′) If $\varphi_i(x) \notin \operatorname{dom} \psi$ then $(\gamma_{n+1} x, \gamma_{n+1}\varphi_i(x)) \notin \mathcal{S}$.

Let $n$ be the first integer such that the set of $x \in X \setminus \operatorname{dom} \varphi$ satisfying the above conditions is nonnull. By the definition of $\psi$, this set should be contained in the domain of $\psi_{n+1}$, and hence of $\psi$, a contradiction. □

So $\psi$ is everywhere defined; since $\mathcal{R}$ is measure-preserving this implies that it belongs to the full group, and we thus rather write it as $T = \psi$. We finally check that $T$ is as wanted.

Let $x \in X$. Define $n \geqslant -1$ as the least integer such that $x \in \operatorname{dom} \psi_{n+1}$, so that $T(x) = \psi_{n+1}(x) = \gamma_{n+1}(x)$ by construction. Let $i \in \{1, \dots, k\}$.

If $\varphi_i(x) \in \operatorname{dom} \psi_n$, then $(\gamma_{n+1} x, \psi_n \varphi_i(x)) \notin \mathcal{S}$ by (a), which means that $(T(x), T\varphi_i(x)) \notin \mathcal{S}$.

If $\varphi_i(x) \notin \operatorname{dom} \psi_n$, let $m$ be the least integer such that $\varphi_i(x) \in \operatorname{dom} \psi_{m+1}$. Then $m \geqslant n$ and $\psi_{m+1}\varphi_i(x) = \gamma_{m+1}\varphi_i(x)$ by construction. There are two possibilities:

- $\underline{m = n}$: Then (b) guarantees $(\gamma_{n+1}x, \gamma_{n+1}\varphi_i(x)) \notin \mathcal{S}$ so that $(\gamma_{n+1}x, \psi_{n+1}\varphi_i(x)) \notin \mathcal{S}$, and hence $(T(x), T\varphi_i(x)) \notin \mathcal{S}$.
- $\underline{m > n}$: But then $(\gamma_{m+1}\varphi_i(x), \psi_m\varphi_i\varphi_i(x)) \notin \mathcal{S}$ by (a) applied to $x' = \varphi_i(x)$ and $n' = m$. The fact that $\varphi_i$ is involutive implies that $(\psi_{m+1}\varphi_i(x), \psi_m(x)) \notin \mathcal{S}$, and hence $(T\varphi_i(x), T(x)) \notin \mathcal{S}$, which is equivalent to $(T(x), T\varphi_i(x)) \notin \mathcal{S}$.

Since in all cases we reached the desired conclusion $(T(x), T\varphi_i(x)) \notin \mathcal{S}$, the proof is finished. □

**4.3.** ***Dense orbits in the space of hyperfinite subequivalence relations.*** By definition a CBER is called *finite* when all its equivalence classes are finite, and *hyperfinite* if it can be written as an increasing union of finite Borel subequivalence relations. In our measured context, we ignore null sets and thus use the following definition:

**Definition 4.8.** A nonsingular equivalence relation $\mathcal{R}$ on $(X, \mu)$ is called *hyperfinite* when it admits a restriction to a full measure set which is hyperfinite in the above sense.

Hyperfiniteness can then be characterized in full group terms as follows: $\mathcal{R}$ is hyperfinite if and only if for all $T_1, \dots, T_n \in [\mathcal{R}]$ and $\epsilon > 0$, after throwing away a set $X'$ of measure $\epsilon$, the equivalence relation generated by the restrictions $T_{1\restriction X\setminus X'}, \dots, T_{n\restriction X\setminus X'}$ is finite.

**Remark.** The above characterization of hyperfiniteness is Dye's original notion [1959] of *approximate finiteness* for full groups (see also [Le Maître 2014, Proposition 1.57] or [Kechris and Miller 2004, Lemma 10.4] for a proof of this characterization).

**Definition 4.9.** Given a nonsingular equivalence relation $\mathcal{R}$ on $(X, \mu)$, let us denote by $\operatorname{Sub}_{\mathrm{hyp}}(\mathcal{R})$ its space of hyperfinite subequivalence relations.

Using lim inf as in the proof of Theorem 2.6 and the fact that the class of nonsingular hyperfinite equivalence relations is stable under countable increasing unions, one can show that $\operatorname{Sub}_{\mathrm{hyp}}(\mathcal{R})$ is closed in $\operatorname{Sub}(\mathcal{R})$; see [Kechris 2024, Theorem 8.1]. Let us observe that this can also be seen as a consequence of the characterization via approximate finiteness: if $\mathcal{S}$ is not hyperfinite as witnessed by some $\epsilon > 0$ and $T_1, \dots, T_n$ its in full group, any $\mathcal{S}'$ which contains the $T_i$ in its full group up to an $\frac{1}{2}\epsilon n$ error will also fail to be hyperfinite.

Before we state our main result on dense orbits in $\operatorname{Sub}_{\mathrm{hyp}}(\mathcal{R})$, we need a preparatory well-known lemma on finite subequivalence relations:

**Lemma 4.10.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Then two finite subequivalence relations $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{R}$ are in the same $[\mathcal{R}]$-orbit if and only if for all $n \in \mathbb{N}$*

$$\mu(\{x \in \mathbb{N} : |[x]_{\mathcal{S}_1}| = n\}) = \mu(\{x \in \mathbb{N} : |[x]_{\mathcal{S}_2}| = n\}). \tag{1}$$

*Proof.* The direct implication is clear. Assume conversely that (1) holds. Let us fix some notation by letting, for $i \in \{1, 2\}$ and $n \geqslant 1$,

$$X^i_n = \{x \in \mathbb{N} : |[x]_{\mathcal{S}_i}| = n\}.$$

Our assumption then becomes: the equality $\mu(X^1_n) = \mu(X^2_n)$ holds for every $n \geqslant 1$. Let $<$ be a Borel linear order on $X$. We can then define, for all $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$, the Borel set

$$Y^i_{n,j} = \{x \in X^i_n : x \text{ is the } j\text{-th element of } [x]_{\mathcal{S}_i}\}.$$

For $i \in \{1, 2\}$ and $j_1, j_2 \in \{1, \dots, n\}$, if we define $\varphi^i_{n,j_1,j_2} : Y^i_{n,j_1} \to Y^i_{n,j_2}$ by taking $x \in Y^i_{n,j_1}$ to the $j_2$-th element of its $\mathcal{S}_i$-class, then $\varphi^i_{n,j_1,j_2} \in [\![\mathcal{S}_i]\!] \subseteq [\![\mathcal{R}]\!]$. In particular since $\mathcal{R}$ is p.m.p., we have that $\mu(Y^i_{n,j_1}) = \mu(Y^i_{n,j_2})$. Since $X^i_n$ is partitioned by $(Y^i_{n,j})^n_{j=1}$, we conclude

$$n\mu(Y^1_{n,1}) = \mu(X^1_n) = \mu(X^2_n) = n\mu(Y^2_{n,1}),$$

so that $\mu(Y^1_{n,1}) = \mu(Y^2_{n,1})$.

Since $\mathcal{R}$ is ergodic, for every $n \in \mathbb{N}$ we can fix some $\varphi_n \in [\![\mathcal{R}]\!]$ such that $\operatorname{dom}\varphi_n = Y^1_{n,1}$ and $\operatorname{rng}\varphi_n = Y^2_{n,1}$. We then extend simultaneously these $\varphi_n$ as $T \in [\mathcal{R}]$ by letting, for every $n \geqslant 1$, $j \in \{1, \dots, n\}$ and $x \in X^1_{n,j}$:

$$T(x) = \varphi^2_{n,1,j}\varphi_n\varphi^1_{n,j,1}(x).$$

In other words, given $x \in X$ whose $\mathcal{S}_1$-class has cardinality $n$, we look at the first element $y$ of the $\mathcal{S}_1$-class of $x$, and then $T(x)$ is the element of the $\mathcal{S}_2$-class of $\varphi_n(y)$ which is in the same position as $x$. Then by construction $T \cdot \mathcal{S}_1 = \mathcal{S}_2$, as wanted. □

**Theorem 4.11.** *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation, and let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be aperiodic and have everywhere infinite index. Then the closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.*

*Proof.* It follows from the definition of hyperfiniteness that finite equivalence relations are dense in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$, so it suffices to approximate every finite subequivalence relation of $\mathcal{R}$ by an element of the $[\mathcal{R}]$-orbit of $S$. Let $\mathcal{R}_0$ be such a finite equivalence relation. For each $n$ let

$$X_n = \{x \in X : |[x]_{\mathcal{R}_0}| = n\}.$$

Because $\mathcal{S}$ is aperiodic, we can find an element of the $[\mathcal{R}]$-orbit of $\mathcal{R}_0$ contained in $\mathcal{S}$. Indeed, by Maharam's lemma we first have a partition of $X$ into pieces $(Y_{j,n})_{1 \leqslant j \leqslant n}$ such that $\mu_{\mathcal{S}}(Y_{j,n}) = \mu(X_n)/n$. We then have for every $2 \leqslant j \leqslant n$ some $\varphi_{j,n} \in [\![\mathcal{S}]\!]$ such that $\operatorname{dom}\varphi_{j,n} = Y_{1,n}$ and $\operatorname{rng}\varphi_{j,n} = Y_{j,n}$.

Now let $Y_1 = \bigsqcup_n Y_{1,n}$, $\psi_1 = \mathrm{id}_{Y_1}$ and for all $j \geqslant 2$, $\psi_j = \bigsqcup_{n \geqslant j} \varphi_{j,n}$. Note that the ranges of the $\psi_j$ partition $X$. Denoting by $\mathcal{S}_0$ the equivalence relation generated by all the $\psi_j$, we then have

$$\mathcal{S}_0 = \bigsqcup_n \bigsqcup_{i,j=1}^{n} \varphi_{n,i} \times \varphi_{n,j}(\Delta_{Y_{1,n}}) = \bigsqcup_{i,j} \psi_i \times \psi_j(\Delta_{Y_1} \cap (\operatorname{dom}\psi_i \times \operatorname{dom}\psi_j)).$$

In particular the $\mathcal{S}_0$-class of every $x \in \bigsqcup_{j=1}^n Y_{n,j}$ has cardinality $n$, and so by Lemma 4.10 we can fix some $T \in [\mathcal{R}]$ such that $T \cdot \mathcal{S}_0 = \mathcal{R}_0$.

By Theorem 4.7, we find a sequence $(T_k)_k$ in the full group of the restriction of $\mathcal{R}$ to $Y_1$ such that $T_k \cdot \mathcal{S}_{\restriction Y_1} \to \Delta_{Y_1}$. We then let $\widetilde{T}_k(x) = \psi_j T_k \psi_j^{-1}(x)$ for all $x \in \operatorname{rng} \psi_j$.

By construction $\widetilde{T}_k \cdot \mathcal{S} = \bigsqcup_{i,j} \psi_i \times \psi_j (T_k \cdot \mathcal{S}_{\restriction Y_1} \cap (\operatorname{dom} \psi_i \times \operatorname{dom} \psi_j))$, so by Lemma 3.11

$$\widetilde{T}_k \cdot \mathcal{S} \to \bigsqcup_{i,j} \psi_i \times \psi_j (\Delta_{Y_1} \cap (\operatorname{dom} \psi_i \times \operatorname{dom} \psi_j)) = \mathcal{S}_0,$$

which yields the desired result since we then have $T\widetilde{T}_k \cdot \mathcal{S} \to T \cdot \mathcal{S}_0 = \mathcal{R}_0$. □

**Remark.** By [Ioana et al. 2009, Corollary 5.4(ii)], if $\mathcal{R}$ is aperiodic and comes from a measure-preserving action of a property-(T) countable group, then there are no dense orbits in the space of subequivalence relations because any $\mathcal{S}$ with a dense orbit would have to have finite index in some restriction of $\mathcal{R}$, and hence cannot contain $\Delta_X$ in its orbit by Theorem 4.7. The general fact is that there cannot be dense orbits in the space of subequivalence relations of $\mathcal{R}$ as soon as $\mathcal{R}$ is not *approximable*, as defined by Gaboriau and Tucker-Drob [2016]. Indeed if $\mathcal{S}$ has a dense orbit, Lemma 2.2 provides a sequence $\mathcal{S}_n$ such that $\mathcal{R} = \liminf \mathcal{S}_n$, while nonapproximability forces some restriction of $\bigcap_{n \geqslant N} \mathcal{S}_n$ to coincide with $\mathcal{R}$ on a positive-measure subset $A$, contradicting the density of the orbit of $\mathcal{S}$ by Theorem C. For more examples of nonapproximable equivalence relations, the reader can consult the paper of Gaboriau and Tucker-Drob, where they obtain, for instance, a quantitative version of nonapproximability for some equivalence relations coming from actions of product groups; see [loc. cit., Theorem 2.4].

**Corollary 4.12.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, and let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$. The following are equivalent*:

(i) *$\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}$.*

(ii) *$\mathcal{S}$ is aperiodic and the closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\Delta_X$.*

(iii) *$\mathcal{S}$ is aperiodic and the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains $\Delta_X$.*

(iv) *The closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.*

(v) *The closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.*

*Proof.* The equivalence of (i)–(iii) is a direct consequence of Theorem 4.7. The implication (i) $\Longrightarrow$ (iv) is exactly Theorem 4.11, and (iv) clearly implies (v). Finally, let us prove (v) $\Longrightarrow$ (iii): assume the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$. Since $\Delta_X$ is hyperfinite, $\Delta_X$ belongs to the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$, so we only have to show $\mathcal{S}$ is aperiodic. Assume for the sake of contradiction that $\mathcal{S}$ is not aperiodic. Then for some $n \in \mathbb{N}$ we have $\mu(\{x \in X : |[x]_{\mathcal{S}}| \leqslant n\}) > 0$. Let $\delta = \mu(\{x \in X : |[x]_{\mathcal{S}}| \leqslant n\})$. Then $\mathcal{S}$ belongs to the $\mathrm{Aut}(\mathcal{R})$-invariant set $\boldsymbol{B}$ of all $\mathcal{S}' \in \mathrm{Sub}(\mathcal{R})$ such that

$$\sup_{T_1, \dots, T_{n+1} \in [\mathcal{R}]} M(\mathcal{S}' \cap (T_1 \cup \cdots \cup T_{n+1})) \leqslant \delta n + (1 - \delta)(n + 1).$$

Note that the intersection map is continuous for the lower $M$-topology by Lemma 3.9 and that $M$ is lower semicontinuous for the lower $M$-topology. Since arbitrary supremums of lower semicontinuous functions are lower semicontinuous, the $\mathrm{Aut}(\mathcal{R})$-invariant set $\boldsymbol{B}$ is closed for the lower $M$-topology; in particular it

is closed for the strong topology. Since the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S} \in \boldsymbol{B}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$, we have $\boldsymbol{B} \subseteq \mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.

However $\boldsymbol{B}$ is disjoint from the set of aperiodic subequivalence relations since for $\mathcal{T}$ aperiodic we can find $T_1, \ldots, T_{n+1}$ in the full group of $\mathcal{T}$ with disjoint graphs, so that $M(\mathcal{T} \cap (T_1 \sqcup T_2 \sqcup \cdots \sqcup T_{n+1})) = n+1$. Since $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ contains aperiodic subequivalence relations, we reached the desired contradiction. So $\mathcal{S}$ is aperiodic, and hence (iii) holds, as wanted. □

**Corollary 4.13.** *For a subequivalence $\mathcal{S}$ of the hyperfinite ergodic p.m.p. equivalence relation $\mathcal{R}_0$, the following are equivalent*:

(i) *$\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}_0$.*

(ii) *$\mathcal{S}$ is aperiodic and the closure of the $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ contains $\Delta_X$.*

(iii) *$\mathcal{S}$ is aperiodic and the closure of the $\mathrm{Aut}(\mathcal{R}_0)$-orbit of $\mathcal{S}$ contains $\Delta_X$.*

(iv) *The $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ is dense in $\mathrm{Sub}(\mathcal{R}_0)$.*

(v) *The $\mathrm{Aut}(\mathcal{R}_0)$-orbit of $\mathcal{S}$ is dense in $\mathrm{Sub}(\mathcal{R}_0)$.*

*Proof.* This is a direct consequence of the previous corollary since every subequivalence relation of a hyperfinite equivalence relation is hyperfinite. □

**Remark.** In the nonsingular ergodic type $\mathrm{II}_\infty$ or type III cases, one can show that there are always dense orbits in the space of subequivalence relations. Let us sketch the proof: first note that by assumption there is a sequence $(\varphi_n)$ of elements of $[\![\mathcal{R}]\!]$ such that for all $n$, $\mathrm{dom}\,\varphi_n = X$ but $X = \bigsqcup_n \mathrm{rng}\,\varphi_n$ (this actually characterizes equivalence relations of type $\mathrm{II}_\infty$ or III among ergodic nonsingular equivalence relations). We then enumerate a dense subset of $\mathrm{Sub}(\mathcal{R})$ as $(\mathcal{S}_n)$. The desired subequivalence relation with a dense orbit is $\mathcal{S} = \bigsqcup_n \varphi_n \cdot \mathcal{S}_n$, as one can see by approximating each $\varphi_n$ by a full group element in the $d_M$ metric.

**4.4.** ***Meagerness of full group orbits.*** In this section, we use Ioana's intertwining for subequivalence relations in order to show that full groups orbits are always meager in the space of hyperfinite subequivalence relations. In what follows, we use the *uniform metric* on the full group of a p.m.p. equivalence relation $\mathcal{R}$, defined by

$$d_u(T_1, T_2) = \mu(\{x \in X : T_1(x) \neq T_2(x)\}) = d_M(T_1, T_2),$$

which is biinvariant, complete and separable, thus endowing $[\mathcal{R}]$ with a Polish group topology. We will use without mention the well-known fact that if $M(\mathcal{S} \cap T) > 1 - \epsilon$, then there is $S \in [\mathcal{S}]$ such that $d_u(S, T) < \epsilon$ (to see this, note that $\mathcal{S} \cap T \in [\![\mathcal{S}]\!]$, and hence it can be extended to an element $S$ of the full group of $\mathcal{S}$ by Lemma 4.2).

**Definition 4.14** (Ioana). Let $\mathcal{S}$ and $\mathcal{T}$ be two subequivalence relations of a p.m.p. equivalence relation $\mathcal{R}$. Write $\mathcal{S} \prec \mathcal{T}$ if there is no sequence $(T_n)$ in the full group of $\mathcal{T}$ such that for all $U_1, U_2 \in [\mathcal{R}]$, we have

$$M(\mathcal{S} \cap U_1 T_n U_2) \to 0.$$

We will crucially use Ioana's version [2012, Lemma 1.7] of Popa's intertwining theorem: if $\mathcal{S} \prec \mathcal{T}$, then $\mathcal{S}$ can be translated by an element of the full group of $\mathcal{R}$ so as to have somewhere-finite index in $\mathcal{T}$. The interested reader is also referred to [Spaas 2023, Lemma 3.1] for a characterization of $\mathcal{S} \prec \mathcal{T}$ when $\mathcal{S}$ has infinite index in $\mathcal{R}$.

We leave it to the reader to check that $\prec$ is $[\mathcal{R}]$-invariant, meaning that if $\mathcal{S} \prec \mathcal{T}$ and $U_1, U_2 \in [\mathcal{R}]$, then $U_1 \cdot \mathcal{S} \prec U_2 \cdot \mathcal{T}$. Also note that we always have $\mathcal{S} \prec \mathcal{S}$.

**Lemma 4.15.** *The relation $\prec$ defines an $F_\sigma$ subset of* $\mathrm{Sub}(\mathcal{R}) \times \mathrm{Sub}(\mathcal{R})$.

*Proof.* We show that the complement of $\prec$ in $\mathrm{Sub}(\mathcal{R}) \times \mathrm{Sub}(\mathcal{R})$ is $G_\delta$. Let us fix $(U_i)$ dense in $\mathcal{R}$. It is then not hard to check that $\mathcal{S} \not\prec \mathcal{T}$ if and only if there is a sequence $(T_n)$ in the full group of $T$ such that $M(\mathcal{S} \cap U_i T_n U_j) \to 0$ for all $i, j \in \mathbb{N}$ (this is essentially the first step of the proof of [Ioana 2012, Lemma 1.7]). It follows that $\mathcal{S} \not\prec \mathcal{T}$ if and only if for every $k \in \mathbb{N}$ and $\epsilon > 0$, we can find $T \in [\mathcal{T}]$ such that

$$M(\mathcal{S} \cap U_i T U_j) < \epsilon \quad \text{for all } i, j \in \{1, \dots, k\} \tag{2}$$

Now by density of $(U_i)$, we finally have $\mathcal{S} \not\prec \mathcal{T}$ if and only if for every $k \in \mathbb{N}$ and $\epsilon > 0$, there is $l \in \mathbb{N}$ such that $M(\mathcal{T} \cap U_l) > 1 - \epsilon$ and for all $i, j \in \{1, \dots, k\}$

$$M(\mathcal{S} \cap U_i U_l U_j) < \epsilon.$$

It is now straightforward to check that this last condition defines a $G_\delta$ set, so we are done. □

**Theorem 4.16.** *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, and consider the $[\mathcal{R}]$-action on the Polish space* $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ *of hyperfinite subequivalence relations of $\mathcal{R}$. Then all $[\mathcal{R}]$-orbits in* $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ *are meager.*

*Proof.* Assume for the sake of contradiction there is a nonmeager $[\mathcal{R}]$-orbit. Since there is a dense $[\mathcal{R}]$-orbit in $\mathrm{Sub}(\mathcal{R})$, the topological 0–1 law yields that there is a comeager orbit. Denoting by $E$ the equivalence relation generated by the $[\mathcal{R}]$-action on $\mathrm{Sub}(\mathcal{R})$, we deduce that $E$ is comeager in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R}) \times \mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.

By [Dye 1959, Theorem 4], the p.m.p. equivalence relation $\mathcal{R}$ contains an ergodic hyperfinite subequivalence relation $\mathcal{S}$. Let $\mathcal{T}$ be an aperiodic subequivalence relation of $\mathcal{S}$ with diffuse ergodic decomposition, such as the one coming from Example 1.1 once we identify $\mathcal{S}$ to $\mathcal{R}_0$. The following claim will essentially finish our proof:

**Claim.** *We have $\mathcal{T} \not\prec \mathcal{S}$.*

*Proof of the claim.* Suppose $\mathcal{T} \prec \mathcal{S}$. Then by [Ioana 2012, Lemma 1.7], there is a nonzero $\varphi \in [\![\mathcal{R}]\!]$ and $k \in \mathbb{N}$ such that if $B = \mathrm{rng}\, \varphi$, then every $(\varphi \cdot \mathcal{S})_{\restriction B}$-class is contained in the union of at most $k$ classes of $\mathcal{T}_{\restriction B}$. Since $\mathcal{T}$ has diffuse ergodic decomposition, we can find $C_1, \dots, C_{k+1}$ partitioning $B$ which are all $\mathcal{T}_{\restriction B}$-invariant and have positive measure.

As $\mathcal{S}$ is ergodic, the $(\varphi \cdot \mathcal{S})_{\restriction B}$-class of almost every $x \in B$ intersects all the $C_i$, and since they are $\mathcal{T}_{\restriction B}$-invariant, we conclude that almost every $(\varphi \cdot \mathcal{S})_{\restriction B}$-class cannot be contained in less than $k+1$ classes of $\mathcal{T}_{\restriction B}$, a contradiction. □

Since $\prec$ is $[\mathcal{R}]$-invariant, its complement also is. This complement is moreover $G_\delta$ by Lemma 4.15, and by the above claim it contains the subset $[\mathcal{R}]\cdot\mathcal{T}\times[\mathcal{R}]\cdot\mathcal{S}$, which is dense in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})\times\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ by Theorem 4.11. So the complement of $\prec$ is comeager, and it should thus intersect $E$. This means that one can find a subequivalence relation $\mathcal{S}'$ and $T\in[\mathcal{R}]$ such that $\mathcal{S}'\not\prec T\cdot\mathcal{S}'$, a contradiction. □

**Remark.** Our proof is inspired by the Glasner–Weiss proof that $\mathrm{Aut}(X,\mu)$ does not have comeager conjugacy classes, replacing Rokhlin's lemma by Theorem 4.11 and Del Junco's disjointness by Ioana's intertwining relation $\not\prec$; see [Glasner and Weiss 2008, Section 3].

**Corollary 4.17.** *Let $\mathcal{R}_0$ be the hyperfinite ergodic p.m.p. equivalence relation. Then all the $[\mathcal{R}_0]$-orbits are meager in* $\mathrm{Sub}(\mathcal{R}_0)$.

*Proof.* Again this follows directly from the above theorem, since every subequivalence relation of a hyperfinite equivalence relation is hyperfinite. □

## 5. The uniform metric and complexity calculations

We begin this final section by introducing a natural metric on the space of all nonsingular equivalence relations. Given two nonsingular equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$, denote by $\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_2)$ the set of all $A\in\mathrm{MAlg}(X,\mu)$ such that $\mathcal{R}_{1\restriction A}=\mathcal{R}_{2\restriction A}$. Define

$$d_u(\mathcal{R}_1,\mathcal{R}_2)=1-\sup_{A\in\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_2)}\mu(A)=\inf_{A\in\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_2)}1-\mu(A).$$

Let us check that this is indeed a metric. Symmetry is clear. For Hausdorffness, if $d_u(\mathcal{R}_1,\mathcal{R}_2)=0$ then we find $(X_n)$ with $\mu(X_n)\geqslant 1-2^{-n}$ and $\mathcal{R}_{1\restriction X_n}=\mathcal{R}_{2\restriction X_n}$. By the Borel–Cantelli lemma, almost every $x\in X$ is in all but finitely many $X_n$, which yields a full measure set restricted to which $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide.

We finally need to show that the triangle inequality holds. Observe that if $A_1\in\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_2)$ and $A_2\in\mathfrak{C}(\mathcal{R}_2,\mathcal{R}_3)$, then $A_1\cap A_2\in\mathfrak{C}(\mathcal{R}_1,\mathcal{R}_3)$; moreover

$$X\setminus(A_1\cap A_2)\subseteq(X\setminus A_1)\cup(X\setminus A_2),$$

so by taking measures and infimums we get the desired triangle inequality.

**Lemma 5.1.** *The uniform metric refines the strong topology.*

*Proof.* The uniform metric clearly refines the uniform topology defined in [Kechris 2024, Section 4.6], which in turn refines the strong topology. □

**Remark.** The results from [Ioana et al. 2009, Theorem 1] applied to the subequivalence relation $\mathcal{R}_1\cap\mathcal{R}_2$ yield that the uniform metric actually induces the uniform topology.

In what follows, we freely identify the Cantor space $\{0,1\}^{\mathbb{N}}$ to the set of subsets of $\mathbb{N}$, and the continuous reduction alluded to is thus defined on the whole $\{0,1\}^{\mathbb{N}}$.

**Proposition 5.2.** *Suppose $\mathcal{R}$ is an aperiodic nonsingular equivalence relation. There is a continuous reduction from the set* $\mathrm{Sub}_{\mathrm{fin}}(\mathbb{N})$ *of finite subsets of $\mathbb{N}$ to the complement of the set of subequivalence*

*relations which are of infinite index, and to the set of finite index subequivalence relations, both for the uniform metric.*

*Proof.* Let $(X_n)$ be a partition of $X$ into sets which intersect almost every $\mathcal{R}$-class (e.g., obtained via Maharam's lemma as sets of $\mathcal{R}$-conditional measure constant equal to $2^{-n-1}$). Let $B \subseteq \mathbb{N}$. We associate to $B$ the set $Y_B = \bigcup_{n \in \mathbb{N} \setminus B} X_n$ and let the reduction be $B \mapsto \mathcal{R}_B$, where

$$\mathcal{R}_B = (\mathcal{R} \cap Y_B \times Y_B) \sqcup \bigcup_{n \in B} \mathcal{R} \cap (X_n \times X_n).$$

The continuity of our reduction for the uniform metric is clear from the definition (if two subsets $B$ and $B'$ coincide on the first $n$ integers, then the associated equivalence relations coincide on $\bigcup_{k<n} X_k$). Moreover, if $B$ is finite we see that $\mathcal{R}_B$ has finite index equal to $|B| + 1$, and if not it has infinite index, as wanted. □

**Corollary 5.3.** *Let $\mathcal{R}$ be a nonsingular aperiodic equivalence relation. The set of subequivalence relations of $\mathcal{R}$ with infinite index is $G_\delta$-hard in the uniform metric; in particular it is $G_\delta$-complete in the strong topology.*

The next statements are all about the strong topology.

**Proposition 5.4.** *Let $\mathcal{R}$ be a nonsingular aperiodic equivalence relation. The space of finite index subequivalence relations is $F_{\sigma\delta}$-hard if and only if $E$ has infinitely many ergodic components, otherwise it is $F_\sigma$-complete.*

*Proof.* Denote by $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ the space of finite index subequivalence relations. Note that by ergodicity, if $E$ has only finitely many ergodic components, then the space of finite index subequivalence relations is equal to the union over $k \in \mathbb{N}$ of the spaces $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ of all subequivalence relations whose index is uniformly less than $k$.

Let us show that for each $k \in \mathbb{N}$, the set $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is closed. Suppose $\mathcal{S} \notin \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$. Then there is a positive measure set of $x \in X$ such that $[x]_{\mathcal{R}}$ contains at least $k+1$ distinct $\mathcal{S}$ classes. We thus find $A \subseteq X$ of positive measure and $\varphi_1, \dots, \varphi_k \in [\![\mathcal{R}]\!]$ with common domain $A$ and such that first, $A, \operatorname{rng} \varphi_1, \dots, \operatorname{rng} \varphi_k$ are pairwise disjoint, and second, $\mathcal{S}$ is disjoint from the union of the graphs of the $\varphi_i \varphi_j^{-1}$. In other words, we find a finite subequivalence relation of $\mathcal{R}$ which is disjoint from $\mathcal{S}$ (after of course removing $\Delta_X$) and all whose nontrivial classes have cardinality $\geqslant k+1$.

Now let $\mathcal{S}_n \to \mathcal{S}$. Since

$$M(\varphi_1 \sqcup \cdots \sqcup \varphi_k \cap \mathcal{S}_n) = \int_A |\{i \in \{1, \dots, k\} : (x, \varphi_i(x)) \in \mathcal{S}_n\}| \, d\mu(x) \to 0,$$

for $n$ large enough there will be a positive measure set of $x \in A$ such that for all $i \in \{1, \dots, k\}$, $(x, \varphi_i(x)) \notin \mathcal{S}_n$; in particular, $\mathcal{S}_n$ has somewhere index $> k$. So $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is closed and hence $\mathrm{Sub}_{[<\infty]}(\mathcal{R}) = \bigcup_{k \in \mathbb{N}} \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is $F_\sigma$. Moreover, $\mathrm{Sub}_{[<\infty]}(\mathcal{R}) = \bigcup_{k \in \mathbb{N}} \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is $F_\sigma$-complete by virtue of the preceding proposition.

Now if $\mathcal{R}$ has infinitely many ergodic components, let $(X_n)$ be a partition of $X$ into $\mathcal{R}$-invariant sets of positive measure. Denote by $\mathrm{Sub}_{\mathrm{fin}}(\mathbb{N}) \subseteq \{0,1\}^{\mathbb{N}}$ the set of finite subsets of $\mathbb{N}$, viewed as finitely supported functions $\mathbb{N} \to \{0,1\}$.

For every $n \in \mathbb{N}$, let $\Phi_n : \{0,1\}^{\mathbb{N}} \to \mathrm{Sub}(\mathcal{R}_{\restriction X_n})$ be the continuous reduction from Proposition 5.2, and for a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\{0,1\}^{\mathbb{N}}$, let

$$\Phi((x_n)_{n\in\mathbb{N}}) = \bigsqcup_{n\in\mathbb{N}} \Phi_n(x_n) \in \mathrm{Sub}(\mathcal{R}).$$

By Lemma 3.11 $\Phi$ is continuous, and by construction

$$\Phi^{-1}(\mathrm{Sub}_{[<\infty]}(\mathcal{R})) = \prod_{n\in\mathbb{N}} \mathrm{Sub}_{\mathrm{fin}}(\mathbb{N}).$$

Being a (countable) infinite product of $F_\sigma$-hard sets, the latter is $F_{\sigma\delta}$-hard, so the set of finite index subequivalence relations is $F_{\sigma\delta}$-hard. □

**Remark.** It follows from [Kechris 2024, Proposition 9.4] that $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ is always $F_{\sigma\delta}$ (the proof in the nonsingular case works the same), so when $\mathcal{R}$ is aperiodic and has infinitely many ergodic components, $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ is actually $F_{\sigma\delta}$-complete.

We finally use a similar approach to show that the set of finite subequivalence relations is always $F_{\sigma\delta}$-complete (for $\mathcal{R}$ aperiodic), by first directly showing that the space of finite subequivalence relations if $F_\sigma$-hard. In the remainder of the paper, we set

$$\Gamma_0 = \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} = \{f : \mathbb{N} \to \mathbb{Z}/2\mathbb{Z} \mid f(n) = 0 \text{ for all but finitely many } n \in \mathbb{N}\}.$$

**Lemma 5.5.** *The space* $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$ *of finite subgroups of* $\Gamma_0$ *is an* $F_\sigma$*-complete subset of the space* $\mathrm{Sub}(\Gamma_0)$ *of subgroups of* $\Gamma_0$.

*Proof.* The set of finite subgroups is countable. In particular it is $F_\sigma$. It is $F_\sigma$-complete via the continuous reduction $B \subseteq \mathbb{N} \mapsto \Lambda_B := \bigoplus_{n\in B} \mathbb{Z}/2\mathbb{Z} \leqslant \Gamma$ of the $F_\sigma$-complete set of finite subsets of $\mathbb{N}$ to $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$. □

**Remark.** The above lemma can also be proven by first noting that by the Baire category theorem, the countable dense set $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$ cannot be $G_\delta$ in the perfect zero-dimensional Polish space of subgroups of $\Gamma_0$, and then applying Wadge's theorem; see [Kechris 1995, Theorem 22.10].

The following proposition was stated and proven in the p.m.p. setup in the first version of this paper (in this restricted setup, it is a consequence of [Dye 1959, Theorem 4]). We are very grateful to the referee for pointing out that the same result is true in the purely Borel setup and for indicating the proof.

**Proposition 5.6.** *For every aperiodic CBER* $\mathcal{R}$*, there is a free* $\Gamma_0$*-action which induces a subequivalence relation of* $\mathcal{R}$.

We rely on the following key lemma:

**Lemma 5.7.** *Let* $\mathcal{R}$ *be an aperiodic CBER. Then its Borel full group contains a fixed-point free involution.*

*Proof.* By [Kechris and Miller 2004, Lemma 7.4], $\mathcal{R}$ contains a Borel subequivalence relation $\mathcal{S}$, all whose classes have cardinality 2. Our involution $U$ is then defined by: for all $x \in X$, $U(x)$ is the only $y \neq x$ such that $(x, y) \in \mathcal{S}$. □

*Proof of Proposition 5.6.* We first inductively define a decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of Borel subsets of $X$ and fixed-point free partial involutions $(\varphi_n)_{n\in\mathbb{N}}$ in the Borel pseudofull group of $\mathcal{R}$ with support $A_n$ as follows. We begin with an everywhere defined involution $\varphi_0$ with support $A_0 = X$, as provided by the previous lemma and then, assuming $\varphi_n$ has been built, we let $A_{n+1}$ be a Borel fundamental domain for $\varphi_n$. Noting that the restriction of $\mathcal{R}$ to $A_{n+1}$ has to remain aperiodic, we use the above lemma to find $\varphi_{n+1}$ as a fixed-point free involution in the Borel full group of the restriction of $\mathcal{R}$ to $A_{n+1}$.

We extend these to involutions to get a free $\Gamma_0$-action. Letting $G_n = \big\{ f \in \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} : \forall m > n,\ f(m) = 0 \big\}$, we can write $\Gamma_0$ as the increasing union $\Gamma_0 = \bigcup_n G_n$, and we have $G_n = \langle s_0, \dots, s_n \rangle$, where $s_i(k) = 0$ if $i \neq k$, and $s_i(k) = 1$ if $i = k$. Our inductive construction of the action will be such that for all $n$,

$$X = \bigsqcup_{g\in G_n} g \cdot A_{n+1}. \tag{3}$$

We begin by letting $s_0 \cdot x = \varphi_0(x)$ for all $x \in X$. And then, assuming that the action of $G_n$ has been defined so that $X = \bigsqcup_{g\in G_n} g \cdot A_{n+1}$, we start defining the $s_{n+1}$ action on $A_{n+1}$ by letting $s_{n+1} \cdot x = \varphi_{n+1}(x)$ if $x \in A_{n+1}$. Since we want $s_{n+1}$ to commute with $G_n$ and we have $X = \bigsqcup_{g\in G_n} g \cdot A_{n+1}$, we are then forced to extend $s_{n+1}$ to the whole of $x$ by letting, for every $x \in A_{n+1}$ and every $g \in G_n$,

$$s_{n+1}(g \cdot x) = g \cdot s_{n+1} \cdot x.$$

It is then not hard to check that this does define a $G_{n+1}$ action with $X = \bigsqcup_{g\in G_{n+1}} g \cdot A_{n+2}$, which provides us a $\Gamma_0$-action by induction in the end. This action is free since (3) implies that the action restricted to each $G_n$ is free, which finishes the proof. □

**Remark.** It follows from [Dye 1959, Theorem 3] that when $\mathcal{R}$ is p.m.p. hyperfinite and aperiodic, one can upgrade the above result and obtain that $\mathcal{R}$ is equal to the equivalence relation generated by the $\Gamma_0$-action. This led Dye to ask whether any p.m.p. equivalence relation should come the free action of some countable group (see the paragraph right before Lemma 6.5 in [Dye 1959]). This question was answered in the negative 40 years later by Furman [1999].

Given a nonsingular equivalence relation $\mathcal{R}$, we denote by $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ the space of finite subequivalence relations of $\mathcal{R}$.

**Proposition 5.8.** *Let $\mathcal{R}$ be an aperiodic nonsingular equivalence relation. Then $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ is $F_{\sigma\delta}$-complete.*

*Proof.* Partition $X$ into $(X_n)_{n\in\mathbb{N}}$ so that each $X_n$ has positive measure and the restriction of $\mathcal{R}$ to each $X_n$ is aperiodic. By the previous lemma, each restriction of $\mathcal{R}$ to $X_n$ contains a subequivalence relation coming from a free $\Gamma_0$-action $\alpha_n$. Now consider the map $(\mathrm{Sub}(\Gamma_0))^{\mathbb{N}} \to \mathrm{Sub}(\mathcal{R})$ which takes any sequence $(\lambda_n)_{n\in\mathbb{N}}$ to the equivalence relation $\bigsqcup_{n\in\mathbb{N}} \mathcal{R}_{\alpha_n(\Lambda_n)}$. This is a topological embedding by Lemma 3.11, and

it reduces the $F_{\sigma\delta}$-complete set $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)^{\mathbb{N}} \subseteq \mathrm{Sub}(\Gamma_0)^{\mathbb{N}}$ to the set of finite equivalence relations, which is thus $F_{\sigma\delta}$-hard.

We are left with proving that $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ is $F_{\sigma\delta}$, which is due to Kechris [2024, Theorem 8.5] in the p.m.p. setup. While his proof generalizes to the nonsingular setup, we provide here another approach, relying on spaces of measurable maps. Let us fix a uniquely generating sequence of moving partial involutions $(\varphi_n)_{n\in\mathbb{N}}$. Consider for every $n \in \mathbb{N}$ the continuous map

$$\iota_n : \mathrm{Sub}(\mathcal{R}) \to \mathrm{MAlg}(X, \mu)$$

which associates to every $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ the set of $x \in \mathrm{dom}\,\varphi_n$ such that $(x, \varphi_n(x)) \in \mathcal{S}$.

Identifying subsets to characteristic functions yields a natural homeomorphism $\Phi$ between $\mathrm{MAlg}(X, \mu)$ and the Polish space of measurable maps $\mathrm{L}^0(X, \mu, \{0, 1\})$; see [Le Maître and Slutsky 2021, Appendix B] for more on $\mathrm{L}^0$ spaces. Using this homeomorphism and putting all the maps $\iota_n$ together, we get a continuous embedding

$$\iota = (\Phi \circ \iota_n)_{n\in\mathbb{N}} : \mathrm{Sub}(\mathcal{R}) \hookrightarrow \prod_{n\in\mathbb{N}} \mathrm{L}^0(X, \mu, \{0, 1\}) \simeq \mathrm{L}^0(X, \mu, \{0, 1\}^{\mathbb{N}}).$$

We will conclude the proof via a straightforward application of the claim below.

**Claim.** *Let $Y$ be a Polish space, and let $D \subseteq Y$ be $F_\sigma$. Then $\mathrm{L}^0(X, \mu, D)$ is an $F_{\sigma\delta}$ subset of $\mathrm{L}^0(X, \mu, Y)$.*

*Proof of the claim.* It follows directly from the definition of the topology of convergence in measure given in [Le Maître and Slutsky 2021, Appendix B] that for any open subset $U \subseteq Y$, the map $f \in \mathrm{L}^0(X, \mu, Y) \mapsto \mu(f^{-1}(U))$ is lower semicontinuous. Taking complements, given a closed subset $C \subseteq \{0, 1\}^{\mathbb{N}}$, the map $f \in \mathrm{L}^0(X, \mu, Y) \mapsto \mu(f^{-1}(C))$ is thus upper semicontinuous. Now write $D = \bigcup_n C_n$ as an increasing union of closed subsets $C_n$, and observe that $\mu(f^{-1}(D)) = 1$ if and only if for all $\epsilon > 0$ there is $n$ such that $\mu(f^{-1}(C_n)) \geqslant 1 - \epsilon$. □

In order to finish the proof, take $D = \mathrm{Sub}_{\mathrm{fin}}(\mathbb{N})$ to be the $F_\sigma$ set of finite subsets of $\mathbb{N}$, viewed as a subset of $Y = \{0, 1\}^{\mathbb{N}}$. Then $\mathrm{L}^0(X, \mu, \mathrm{Sub}_{\mathrm{fin}}(\mathbb{N}))$ is $F_{\sigma\delta}$ in $\mathrm{L}^0(X, \mu, \{0, 1\}^{\mathbb{N}})$ by the above claim. Finally, observe that $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R}) = \iota^{-1}(\mathrm{L}^0(X, \mu, \mathrm{Sub}_{\mathrm{fin}}(\mathbb{N})))$, which is thus $F_{\sigma\delta}$ by continuity of $\iota$. □

**Remark.** Endowing $\mathrm{Sub}(\mathcal{R})$ with its strong topology and the uniform metric, we get a Polish topometric space in the sense of Ben Yaacov [2008], and our remark preceding Corollary 4.12 can be upgraded to the fact that if a p.m.p. equivalence relation $\mathcal{R}$ has property (T), then $\mathcal{R}$ is a metrically isolated point of $\mathrm{Sub}(\mathcal{R})$; see also [Gaboriau and Tucker-Drob 2016] for more examples. I don't know whether the action of the full group or automorphism group of the hyperfinite ergodic p.m.p. equivalence relation admits a metrically generic orbit, as characterized in [Ben Yaacov and Melleray 2025].

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## References

[Aoi 2003] H. Aoi, "A construction of equivalence subrelations for intermediate subalgebras", *J. Math. Soc. Japan* **55**:3 (2003), 713–725. MR

[Ben Yaacov 2008] I. Ben Yaacov, "Topometric spaces and perturbations of metric structures", *Log. Anal.* **1**:3-4 (2008), 235–272. MR

[Ben Yaacov and Melleray 2025] I. Ben Yaacov and J. Melleray, "A topometric Effros theorem", *J. Symb. Log.* **90**:1 (2025), 310–320. MR

[Bourbaki 1989] N. Bourbaki, *General topology: chapters 1–4*, Springer, 1989. MR

[Cohn 2013] D. L. Cohn, *Measure theory*, 2nd ed., Springer, 2013. MR

[Dye 1959] H. A. Dye, "On groups of measure preserving transformations, I", *Amer. J. Math.* **81** (1959), 119–159. MR

[Eisenmann and Glasner 2016] A. Eisenmann and Y. Glasner, "Generic IRS in free groups, after Bowen", *Proc. Amer. Math. Soc.* **144**:10 (2016), 4231–4246. MR

[Fima et al. 2024] P. Fima, F. Le Maître, K. Mukherjee, and I. Patri, "Michael's selection theorem and applications to the Maréchal topology", preprint, 2024. arXiv 2407.05776

[Furman 1999] A. Furman, "Orbit equivalence rigidity", *Ann. of Math.* (2) **150**:3 (1999), 1083–1108. MR

[Gaboriau and Lyons 2009] D. Gaboriau and R. Lyons, "A measurable-group-theoretic solution to von Neumann's problem", *Invent. Math.* **177**:3 (2009), 533–540. MR

[Gaboriau and Tucker-Drob 2016] D. Gaboriau and R. Tucker-Drob, "Approximations of standard equivalence relations and Bernoulli percolation at $p_u$", *C. R. Math. Acad. Sci. Paris* **354**:11 (2016), 1114–1118. MR

[Gao 2009] S. Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics (Boca Raton) **293**, CRC, Boca Raton, FL, 2009. MR

[Glasner and Weiss 2008] E. Glasner and B. Weiss, "Topological groups with Rokhlin properties", *Colloq. Math.* **110**:1 (2008), 51–80. MR

[Ioana 2012] A. Ioana, "Uniqueness of the group measure space decomposition for Popa's $\mathcal{HT}$ factors", *Geom. Funct. Anal.* **22**:3 (2012), 699–732. MR

[Ioana et al. 2009] A. Ioana, A. S. Kechris, and T. Tsankov, "Subequivalence relations and positive-definite functions", *Groups Geom. Dyn.* **3**:4 (2009), 579–625. MR

[Kechris 1995] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics **156**, Springer, 1995. MR

[Kechris 2010] A. S. Kechris, *Global aspects of ergodic group actions*, Mathematical Surveys and Monographs **160**, Amer. Math. Soc., Providence, RI, 2010. MR

[Kechris 2024] A. S. Kechris, "The spaces of measure preserving equivalence relations and graphs", preprint, 2024, available at https://www.pma.caltech.edu/documents/6021/space_of_equivalence_relations_012book.pdf.

[Kechris and Miller 2004] A. S. Kechris and B. D. Miller, *Topics in orbit equivalence*, Lecture Notes in Mathematics **1852**, Springer, 2004. MR

[Le Maître 2014] F. Le Maître, *Sur les groupes pleins préservant une mesure de probabilité*, Ph.D. thesis, Université de Lyon, 2014, available at https://math.univ-lyon1.fr/~melleray/these_FLM.pdf.

[Le Maître 2016] F. Le Maître, "On full groups of non-ergodic probability-measure-preserving equivalence relations", *Ergodic Theory Dynam. Systems* **36**:7 (2016), 2218–2245. MR

[Le Maître 2022] F. Le Maître, "Polish topologies on groups of non-singular transformations", *J. Log. Anal.* **14** (2022), art. id. 4. MR

[Le Maître and Slutsky 2021] F. Le Maître and K. Slutsky, "$L^1$ full groups of flows", 2021. To appear in *Mem. Eur. Math. Soc.* arXiv 2108.09009

[Maréchal 1973] O. Maréchal, "Topologie et structure borélienne sur l'ensemble des algèbres de von Neumann", *C. R. Acad. Sci. Paris Sér. A-B* **276** (1973), A847–A850. MR

[Ornstein and Weiss 1980] D. S. Ornstein and B. Weiss, "Ergodic theory of amenable group actions, I: The Rohlin lemma", *Bull. Amer. Math. Soc.* (*N.S.*) **2**:1 (1980), 161–164. MR

[Popa 2019] S. Popa, "Asymptotic orthogonalization of subalgebras in $\mathrm{II}_1$ factors", *Publ. Res. Inst. Math. Sci.* **55**:4 (2019), 795–809. MR

[Seward 2014] B. Seward, "Every action of a nonamenable group is the factor of a small action", *J. Mod. Dyn.* **8**:2 (2014), 251–270. MR

[Spaas 2023] P. Spaas, "Stable decompositions and rigidity for products of countable equivalence relations", *Trans. Amer. Math. Soc.* **376**:3 (2023), 1867–1894. MR

[Zhou 2024] S. Zhou, "Noncommutative topological boundaries and amenable invariant random intermediate subalgebras", preprint, 2024. arXiv 2407.10905

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