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The Segre determinant

Elizabeth Pratt

The Segre determinant is a polynomial which encodes the condition for points to lie on a bilinear hypersurface in the product of projective spaces. We study Segre determinants and compute them in various coordinate systems. We show that the Segre determinant represents the Chow–Lam form of a generic torus orbit in the Grassmannian. These Chow–Lam forms were introduced as a generalization of Chow forms for projective varieties, and enjoy many similar properties. We also present applications to algebraic vision and to Chow quotients of Grassmannians.

1. Introduction

Fix vector spaces V and W over $\mathbb{C}$ of dimensions k and l. Consider r points $A_1 \times B_1, \dots, A_r \times B_r$ in $\mathbb{P}(V) \times \mathbb{P}(W)$. The *Segre matrix* of this point configuration is the $kl \times r$ matrix

$$\begin{bmatrix} \vdots & & \vdots \\ A_1 \otimes B_1 & \cdots & A_r \otimes B_r \\ \vdots & & \vdots \end{bmatrix}. \tag{1}$$

It is a flattening of the $k \times l \times r$ tensor with slices $A_i \otimes B_i$. When $r = kl$, the Segre matrix is square and we call its determinant the *Segre determinant*. This determinant vanishes whenever the kl points lie on a hyperplane section of $\mathbb{P}(V) \times \mathbb{P}(W)$ in the Segre embedding; hence the name. Indeed, the equation of such a hyperplane section is in the left kernel of the Segre matrix. We will use $\mathrm{Seg}_{k,l}$ to denote the Segre determinant as a polynomial in the coordinates of the points A_i and B_i.

Our aim is to introduce the Segre determinant and give various applications. After establishing its basic properties (Section 2), we show how it appears in various guises: in algebraic vision (Section 3), as a Chow–Lam form (Sections 4–5), and in the theory of Chow quotients of Grassmannians (Section 6).

The Segre determinant is a generalization of the classical determinant for $k = 1$. The special case $k = l = 2$ encodes the condition for two configurations of four ordered points in $\mathbb{P}^1$ to be the same up to automorphism; see Example 1.1 below. The case $k = l = 3$ appears in algebraic vision as a necessary condition for two configurations of nine ordered points in $\mathbb{P}^2$ to be linear projections of a common configuration of ordered points in $\mathbb{P}^3$ [Agarwal et al. 2017].

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Example 1.1. Consider four points

$$\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \times \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}, \dots, \begin{bmatrix} a_{1,4} \\ a_{2,4} \end{bmatrix} \times \begin{bmatrix} b_{1,4} \\ b_{2,4} \end{bmatrix}$$

in the surface $\mathbb{P}^1 \times \mathbb{P}^1$. The Segre matrix looks like

$$\begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & a_{1,4}b_{1,4} \\ a_{1,1}b_{2,1} & a_{1,2}b_{2,2} & a_{1,3}b_{2,3} & a_{1,4}b_{2,4} \\ a_{2,1}b_{1,1} & a_{2,2}b_{1,2} & a_{2,3}b_{1,3} & a_{2,4}b_{1,4} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & a_{2,3}b_{2,3} & a_{2,4}b_{2,4} \end{bmatrix}. \tag{2}$$

This expression vanishes whenever the four points in $\mathbb{P}^1 \times \mathbb{P}^1$ lie on a common (1, 1)-curve. We collect our eight vectors into two matrices

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix}, \qquad B := \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \end{bmatrix}.$$

Let $[ij]$ denote the six maximal minors of A and $\langle ij \rangle$ denote the six maximal minors of B. Then we compute

$$\mathrm{Seg}_{2,2} = [12][34]\langle 13\rangle\langle 24\rangle - [13][24]\langle 12\rangle\langle 34\rangle.$$

It vanishes whenever the cross-ratio $[12][34]/[13][24]$ of the four points A_i in $\mathbb{P}^1$ is equal to the cross-ratio of the four points B_i in $\mathbb{P}^1$. The cross-ratio is the fundamental invariant for PGL_2 acting on $(\mathbb{P}^1)^4$.

The first two sections focus on computations and examples. In Section 2 we prove that the Segre determinant may be written in terms of the maximal minors of the $k \times kl$ matrix A and the $l \times kl$ matrix B, generalizing the computation in Example 1.1. These minors generate the invariant rings of SL_k and SL_l acting on $(\mathbb{P}^{k-1})^n$ and $(\mathbb{P}^{l-1})^n$, respectively [Sturmfels 1993, Theorem 3.2.1].

We also discuss how to compute such expressions, which is difficult in practice. This flavor of question has a long history in classical algebraic geometry, and is related to the problem of synthetic (ruler-and-compass) constructions. The analogous problem for the Veronese embedding is to figure out when $\binom{n+k-1}{k}$ points in $\mathbb{P}^{n-1}$ lie on a hypersurface of degree k. Pascal's theorem for $k = 2$ and $n = 3$ is the earliest such result, giving a condition for six points in the plane to lie on a conic. The *Bruxelles problem*, posed in 1825 by l'Académie de Bruxelles, asks for a synthetic construction to determine when ten points in $\mathbb{P}^3$ lie on a quadric surface ($k = 2$ and $n = 4$). This is a significant leap in difficulty, and a construction was only recently obtained by Traves [2024].

Such a property of a point configuration is SL_k-invariant, so we may ask how to write it in terms of minors of A and B, perhaps thus gleaning some geometric insight. Indeed, Turnbull and Young [1927] gave an algebraic expression in terms of the 4×4 minors of the 4×10 matrix parametrizing the points. In this spirit, we present the analogous computation for eight points in $\mathbb{P}^1 \times \mathbb{P}^3$ and nine points in $\mathbb{P}^2 \times \mathbb{P}^2$ in (9) and (10), respectively.

Section 3 gives an application of the Segre determinant $\mathrm{Seg}_{3,3}$ to algebraic vision. In this setting, the Segre determinant is a necessary condition for two distinct configurations of nine ordered points in $\mathbb{P}^2$ to be linear projections of a common set of nine ordered points in $\mathbb{P}^3$. Our computation answers a question of Rekha Thomas about writing this condition in terms of SL_3-invariants. This section is completely independent from Sections 4–6.

Sections 4 and 5 give a geometric interpretation of the Segre determinant for general k and l as the Chow–Lam form of a torus orbit in $\mathrm{Gr}(k, kl)$. The Chow–Lam form was introduced in [Pratt and Sturmfels 2025] as a generalization of the classical Chow form in algebraic geometry. It associates to a subvariety of $\mathrm{Gr}(k, n)$ a single polynomial which in most cases completely encodes the variety [Pratt and Ranestad 2025]. These forms first arose in calculations of scattering amplitudes in high-energy physics. Our main theorem in Section 5 is the following:

Theorem 5.2 (Segre determinant). *Suppose $k, l \geq 2$ and let $n = kl$. Fix a point A in $\mathrm{Gr}(k, n)$ with nonzero Plücker coordinates. Then the Chow–Lam form of the torus orbit of A in primal Plücker coordinates B on $\mathrm{Gr}(n-l, n)$ equals the Segre determinant $\mathrm{Seg}_{k,l}(A, B)$.*

In Section 6 we define the *Segre coefficient variety*. It parametrizes Segre determinants as polynomials in the B variables as A varies. We show in Theorem 6.2 that every monomial of the form $[I_1]\cdots[I_l]$ with $I_1 \cup \cdots \cup I_l = [kl]$ appears in the linear span of the coefficients of the Segre polynomial. This gives the following result:

Corollary 6.3. *The variety $\mathrm{Coeff}(\mathrm{Seg}_{2,l})$ is isomorphic to the GIT quotient $(\mathbb{P}^1)^{2l} /\!/_w \mathrm{SL}_2$, where $w = 1^{2l}$ is the linearization.*

Thus the Segre determinant uniquely determines torus orbit closures of generic points in $\mathrm{Gr}(2, n)$. However, this fails for $k = 3$. We give two torus orbit closures with the same Segre determinant in Example 6.4.

2. Properties and computation

We now discuss the computation of the Segre determinant in various coordinate systems. As in the introduction, fix vector spaces V and W of dimensions k and l, and let $n := kl$.

Consider the polynomial rings $\mathbb{C}[a_{ij}]_{1 \leq i \leq k, 1 \leq j \leq n}$ and $\mathbb{C}[b_{ij}]_{1 \leq i \leq l, 1 \leq j \leq n}$. As in the introduction, we may collect the indeterminates a_{ij} and b_{ij} into a $k \times n$ matrix A and an $l \times n$ matrix B. Given a subset $I \in \binom{[n]}{k}$, we define the *bracket* $[I]$ to be the determinant of the submatrix of A with columns indexed by I. Similarly, given $J \in \binom{[n]}{l}$, we define the bracket $\langle J \rangle$ to be the determinant of the submatrix of B with columns J. These rings have actions of SL_k and SL_l, respectively, given by left multiplication of the matrices A and B. The following is sometimes called the *first fundamental theorem of invariant theory*:

Theorem 2.1 [Sturmfels 1993, Theorem 3.2.1]. *The brackets $[I]$ generate the invariant ring $\mathbb{C}[a_{ij}]^{\mathrm{SL}_k}$.*

We denote the invariant ring $\mathbb{C}[a_{ij}]^{\mathrm{SL}_k}$ by $\mathcal{B}_{k,n}$ and call it the *bracket algebra*. The Segre determinant lives in the tensor product $\mathbb{C}[a_{ij}] \otimes \mathbb{C}[b_{ij}]$ of $\mathbb{C}$-algebras. It is separately invariant under the SL_k and SL_l

actions on $V \otimes W$, and thus may be written either in brackets $[I]$ and indeterminates b_{ij}, or in brackets $\langle J\rangle$ and indeterminates a_{ij}. These can be realized concretely as two different block Laplace expansions of the Segre matrix. The following result states that $\mathrm{Seg}_{k,l}$ may be written *simultaneously* in the two systems of brackets:

Proposition 2.2. *The Segre determinant* $\mathrm{Seg}_{k,l}$ *is a polynomial of bidegree* (l, k) *in brackets* $[I]$ *and* $\langle J\rangle$.

The main ingredient in proving Proposition 2.2 is the following:

Lemma 2.3. *Suppose that* V *and* W *are finite-dimensional representations of groups* G *and* H, *respectively. Then*

$$V^G \otimes W^H = (V \otimes W)^{G\times H}. \tag{3}$$

Proof. The inclusion $\subseteq$ is immediate. For the other, choose a basis $\alpha_1, \dots, \alpha_r$ for V^G and add vectors v_j to extend it to a basis for V. Similarly, choose a basis $\beta_1, \dots, \beta_s$ for W^H and add vectors w_j to extend it to a basis for W. We observe that the right-hand side of (3) is contained in the vector space

$$(V^G \otimes W) \cap (V \otimes W^G) \subseteq V \otimes W.$$

An element f of the intersection may be written uniquely in each basis as

$$f = \sum_{i\le r,\, j\le s,} c_{ij}\alpha_i \otimes \beta_j + \sum_{i\le r,\, j>s}^{k} c_{ij}\alpha_i \otimes w_j \tag{4}$$

$$= \sum_{i\le r,\, j\le s} d_{ij}\alpha_i \otimes \beta_j + \sum_{i>r,\, j\le s}^{k} d_{ij}v_i \otimes \beta_j. \tag{5}$$

Since the expressions are unique, $f = \sum_{i\le r, j\le s} c_{ij}\alpha_i \otimes \beta_j$. □

Proof of Proposition 2.2. The proof is mostly type-checking. For clarity, within this proof we use $\boxtimes$ when forming the representation of a direct product of groups. The Segre determinant lives in the coordinate ring $\mathrm{Sym}^\bullet(V \boxtimes W) \otimes \cdots \otimes \mathrm{Sym}^\bullet(V \boxtimes W)$ of the product $\mathbb{P}(V \boxtimes W) \times \cdots \times \mathbb{P}(V \boxtimes W)$. This ring has a multigrading given by taking the grading in each tensor factor, and the Segre determinant lives in the component with grading $1^n = (1, \dots, 1)$. Consider the map of $(\mathrm{SL}_k \times \mathrm{SL}_l)$-representations

$$\begin{gathered}\psi : (V \boxtimes W)^{\otimes n} \to V^{\otimes n} \boxtimes W^{\otimes n},\\ (x_1 \boxtimes y_1) \otimes \cdots \otimes (x_n \boxtimes y_n) \mapsto (x_1 \otimes \cdots \otimes x_n) \boxtimes (y_1 \otimes \cdots \otimes y_n).\end{gathered} \tag{6}$$

There is a map $\mathrm{SL}_k \times \mathrm{SL}_l \to \mathrm{GL}(V \boxtimes W)$ given by $(A, B) \cdot x_i \otimes y_j = Ax_i \otimes By_j$. The image lies in $\mathrm{SL}_{kl} = \mathrm{SL}(V \boxtimes W)$. Thus there is an inclusion of invariant vector spaces

$$\iota : ((V \boxtimes W)^{\otimes n})^{\mathrm{SL}_{kl}} \hookrightarrow (V^{\otimes n} \boxtimes W^{\otimes n})^{\mathrm{SL}_k\times \mathrm{SL}_l}. \tag{7}$$

Finally, by Lemma 2.3,

$$(V^{\otimes n} \boxtimes W^{\otimes n})^{\mathrm{SL}_k\times \mathrm{SL}_l} = (V^{\otimes n})^{\mathrm{SL}_k} \boxtimes (W^{\otimes n})^{\mathrm{SL}_l}. \tag{8}$$

By the first fundamental theorem of invariant theory, the right-hand side of (8) is the algebra generated by the brackets $[I]$ and $\langle J\rangle$. By (7), the Segre determinant $\mathrm{Seg}_{k,l}$ lies in $(V^{\otimes n}\boxtimes W^{\otimes n})^{\mathrm{SL}_k\times\mathrm{SL}_l}$, and applying ι gives its expansion into brackets $[I]$ and $\langle J\rangle$. Note that the total degree of $\mathrm{Seg}_{k,l}$ in each set of variables a_{ij} and b_{ij} is n. Since $[I]$ has degree k and $\langle J\rangle$ has degree l when expanded into these variables, the bidegree of the Segre determinant is $(n/k, n/l)=(l,k)$. □

Remark 2.4. The Segre determinant has further symmetries. It transforms equivariantly under the actions of $(\mathbb{C}^*)^n$ on $\mathcal{B}_{k,n}$ and $\mathcal{B}_{l,n}$ obtained by scaling columns of matrix representatives. It is also invariant (up to sign) under permuting the points $A_i\times B_i$ and, when $k=l$, exchanging the roles of A_i and B_i. The equivariance condition and the degree considerations in Proposition 2.2 are enough to deduce, for example, that the polynomial $\mathrm{Seg}_{2,2}$ in Example 1.1 is a linear combination of $[12][34]\langle 12\rangle\langle 34\rangle$, $[13][24]\langle 12\rangle\langle 34\rangle$, $[12][34]\langle 13\rangle\langle 24\rangle$, and $[13][24]\langle 13\rangle\langle 24\rangle$. The subvector space spanned by invariants of the aforementioned S_4 and S_2 actions is spanned by

$$[12][34]\langle 12\rangle\langle 34\rangle+[13][24]\langle 13\rangle\langle 24\rangle \quad\text{and}\quad [12][34]\langle 13\rangle\langle 24\rangle-[13][24]\langle 12\rangle\langle 34\rangle.$$

It follows that the Segre determinant is a linear combination of these.

The brackets in $\mathcal{B}_{k,n}$ satisfy certain relations called the Plücker relations, coming from the relations between maximal minors of a $k\times n$ matrix. As the name implies, these are the same relations defining the Grassmannian in its Plücker embedding. Indeed, $\mathrm{Proj}\,\mathcal{B}_{k,n}$ equals $\mathrm{Gr}(k,n)$. It is a fact, sometimes called the *second fundamental theorem of invariant theory*, that the Plücker relations generate all relations between brackets [Sturmfels 1993, Theorem 3.1.7]

The bracket algebra comes with a convenient basis in each graded component, which we explain as follows. A *Young tableau* is the filling of a partition diagram with entries in $[n]$, allowing repeats. A *semistandard Young tableau* has the additional property that the numbers are nondecreasing within each row and strictly increasing within each column. We call a monomial $[I_1]\cdots[I_r]$ in $\mathcal{B}_{k,n}$ *standard* if the $k\times r$ rectangular Young tableau obtained by stacking $I_1,\dots,I_r$ vertically and then transposing is a semistandard Young tableau. For example, in $\mathcal{B}_{2,4}$ the monomials [12][12] and [13][24] are standard, but the monomial [14][23] is not. There are 21 degree-2 monomials in total in $\mathcal{B}_{2,4}$, 20 of which are standard. This use of the term "standard monomial" is consistent with the theory of Gröbner bases: the Plücker relations form a Gröbner basis for the ideal of relations in $\mathcal{B}_{k,n}$ under tableau order, and these are the standard monomials [Sturmfels 1993, Theorem 3.1.7].

The semistandard monomials of a fixed degree r form a basis for the degree r part of the bracket algebra [Sturmfels 1993, Corollary 3.1.9]. The *straightening algorithm* applies the Plücker relations to put a polynomial in $\mathcal{B}_{k,n}$ into the unique representation such that every monomial within it is standard. Note that the Segre determinant is an eigenvector of the torus $(\mathbb{C}^*)^{kl}$ acting on either $\mathcal{B}_{k,kl}$ or $\mathcal{B}_{l,kl}$. Thus each bracket monomial appearing within it has the same multiset of indices, namely $[n]$. From here on when we speak of the "standard basis" we mean the basis of semistandard Young tableaux for the multilinear component, in which the multiset of the indices is $[n]$.

Example 2.5. The Segre determinant $\mathrm{Seg}_{2,4}$ has bidegree (4, 2) and a total of 22 monomials in the standard basis. Its expansion in standard brackets equals

$$\begin{aligned}\mathrm{Seg}_{2,4} = &-(\langle 1235\rangle\langle 4678\rangle - \langle 1245\rangle\langle 3678\rangle + \langle 1257\rangle\langle 3468\rangle)[13][24][56][78]\\ &-(\langle 1237\rangle\langle 4568\rangle - \langle 1357\rangle\langle 2468\rangle + \langle 1345\rangle\langle 2678\rangle)[12][34][56][78]\\ &-(\langle 1235\rangle\langle 4678\rangle - \langle 1236\rangle\langle 4578\rangle + \langle 1356\rangle\langle 2478\rangle)[12][34][57][68]\\ &+\langle 1234\rangle\langle 5678\rangle[15][26][37][48] + (\langle 1234\rangle\langle 5678\rangle + \langle 1346\rangle\langle 2578\rangle)[12][35][47][68]\\ &-\langle 1347\rangle\langle 2568\rangle[12][35][46][78] - \langle 1345\rangle\langle 2678\rangle[12][36][47][58]\\ &+\langle 1256\rangle\langle 3478\rangle[13][24][57][68] - \langle 1246\rangle\langle 3578\rangle[13][25][47][68]\\ &+\langle 1245\rangle\langle 3678\rangle[13][26][47][58] - \langle 1237\rangle\langle 4568\rangle[14][25][36][78]\\ &+\langle 1236\rangle\langle 4578\rangle[14][25][37][68] - \langle 1235\rangle\langle 4678\rangle[14][26][37][58]\\ &+(\langle 1234\rangle\langle 5678\rangle + \langle 1247\rangle\langle 3568\rangle)[13][25][46][78].\end{aligned} \tag{9}$$

One may in principle compute expressions such as Example 2.5 by intersecting the expressions in a_{ij} and b_{ij} with the subring of invariants in $\mathbb{C}[a_{ij}] \otimes \mathbb{C}[b_{ij}]$. This intersection is typically computed via elimination algorithms which use Gröbner bases, such as that in [Eisenbud 1995, Section 15.10.4]. However, these are unlikely to terminate. It is far more efficient to leverage the standard basis and use linear algebra. For instance, to do the above computation we performed the following steps:

(1) Do a block Laplace expansion of the Segre matrix (1) in the $[I]$ brackets and then straighten to the 14 standard ones.

(2) For each standard monomial T in the $[I]$ brackets:
  - (i) Define f_T to be the coefficient of T in the variables b_{ij}.
  - (ii) Write $f_T = c_1\langle 1234\rangle\langle 5678\rangle + \cdots + c_{14}\langle 1347\rangle\langle 2568\rangle$, where the coefficients c_i are unknown.
  - (iii) Choose a random degree-8 monomial m in the b_{ij} appearing in f_T. Its incidence vector with the brackets $\langle J_1\rangle\langle J_2\rangle$ imposes a linear constraint on the c_i.
  - (iv) Repeat until the coefficients c_i are determined.

The final step is linear algebra in a 14-dimensional vector space, so it runs very quickly.

3. An application to algebraic vision

Computer vision is the study of "cameras", namely linear projections from $\mathbb{P}^3_{\mathbb{R}}$ to $\mathbb{P}^2_{\mathbb{R}}$, and how a computer gains information from them. A typical computer vision problem is to reconstruct an object in 3-space from a set of 2-dimensional snapshots. In algebraic vision, the object one is taking a picture of is an algebraic variety in $\mathbb{P}^3_{\mathbb{R}}$, such as a curve or a collection of points. The survey [Kileel and Kohn 2025] gives a nice overview of this research area. For the rest of this section we work over the field $\mathbb{R}$.

Fix two configurations A and B of eight ordered points in $\mathbb{P}^2$. One natural question in computer vision is: when are they linear projections of a common configuration in $\mathbb{P}^3$? To answer this, consider the 9×8 Segre matrix with columns $A_1 \otimes B_1, \ldots, A_8 \otimes B_8$. This matrix has a 1-dimensional kernel, provided that the point configurations are sufficiently generic (see [Longuet-Higgins 1981] for the exact conditions).

The kernel of the Segre matrix lies in $\mathbb{P}(\mathbb{R}^3 \times \mathbb{R}^3)$. It may be viewed as the span of a 3×3 matrix, which we denote by F. Then a necessary and sufficient condition for A and B to have a common recovery is that F has rank 2 [Longuet-Higgins 1981]. In algebraic vision F is called the *fundamental matrix*.

If one has instead two configurations of *nine* points, there is an extra condition, namely, the 9×9 Segre matrix must have a kernel. In (10) we express the Segre determinant as a polynomial in brackets $[I]$ and $\langle J \rangle$. This answers a question of Rekha Thomas about how to write the condition using SL_3-invariants. The expression is relatively sparse, with 110 terms total in the standard basis, which has cardinality $42^2 = 1764$. It is computed in the same manner as $\mathrm{Seg}_{2,4}$, described in the discussion after Example 2.5. A short script reproducing the computation may be found at https://github.com/lizziepratt/SegreDeterminant.

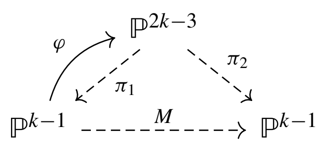
$$
\begin{aligned}
\mathrm{Seg}_{3,3} = {} & [123][456][789]\big(3\langle 123\rangle\langle 457\rangle\langle 689\rangle - \langle 123\rangle\langle 467\rangle\langle 589\rangle + 3\langle 124\rangle\langle 356\rangle\langle 789\rangle - 3\langle 124\rangle\langle 357\rangle\langle 689\rangle \\
& \quad + \langle 124\rangle\langle 367\rangle\langle 589\rangle - \langle 124\rangle\langle 368\rangle\langle 579\rangle - \langle 125\rangle\langle 346\rangle\langle 789\rangle + \langle 125\rangle\langle 347\rangle\langle 689\rangle + \langle 127\rangle\langle 348\rangle\langle 569\rangle \\
& \quad - \langle 134\rangle\langle 258\rangle\langle 679\rangle - \langle 135\rangle\langle 247\rangle\langle 689\rangle + \langle 145\rangle\langle 267\rangle\langle 389\rangle + \langle 147\rangle\langle 258\rangle\langle 369\rangle\big) \\
& + [123][457][689](-3\langle 123\rangle\langle 456\rangle\langle 789\rangle + \langle 124\rangle\langle 368\rangle\langle 579\rangle - \langle 126\rangle\langle 348\rangle\langle 579\rangle + \langle 135\rangle\langle 246\rangle\langle 789\rangle - \langle 146\rangle\langle 258\rangle\langle 379\rangle) \\
& + [123][458][679](-\langle 124\rangle\langle 367\rangle\langle 589\rangle - \langle 125\rangle\langle 346\rangle\langle 789\rangle + \langle 126\rangle\langle 347\rangle\langle 589\rangle + \langle 146\rangle\langle 257\rangle\langle 389\rangle) \\
& + [123][467][589]\big(\langle 123\rangle\langle 456\rangle\langle 789\rangle - \langle 124\rangle\langle 358\rangle\langle 679\rangle + \langle 125\rangle\langle 348\rangle\langle 679\rangle + \langle 134\rangle\langle 256\rangle\langle 789\rangle \\
& \quad - \langle 135\rangle\langle 246\rangle\langle 789\rangle + \langle 145\rangle\langle 268\rangle\langle 379\rangle\big) \\
& + [124][356][789](-3\langle 123\rangle\langle 456\rangle\langle 789\rangle + \langle 123\rangle\langle 468\rangle\langle 579\rangle + \langle 135\rangle\langle 247\rangle\langle 689\rangle - \langle 135\rangle\langle 267\rangle\langle 489\rangle - \langle 137\rangle\langle 258\rangle\langle 469\rangle) \\
& + [124][357][689](3\langle 123\rangle\langle 456\rangle\langle 789\rangle - \langle 123\rangle\langle 468\rangle\langle 579\rangle - \langle 135\rangle\langle 246\rangle\langle 789\rangle + \langle 136\rangle\langle 258\rangle\langle 479\rangle) \\
& + [124][358][679](\langle 123\rangle\langle 467\rangle\langle 589\rangle - \langle 136\rangle\langle 257\rangle\langle 489\rangle) \\
& + [124][367][589](-\langle 123\rangle\langle 456\rangle\langle 789\rangle + \langle 123\rangle\langle 458\rangle\langle 679\rangle + \langle 135\rangle\langle 246\rangle\langle 789\rangle - \langle 135\rangle\langle 268\rangle\langle 479\rangle) \\
& + [124][368][579](\langle 123\rangle\langle 456\rangle\langle 789\rangle - \langle 123\rangle\langle 457\rangle\langle 689\rangle + \langle 135\rangle\langle 267\rangle\langle 489\rangle) \\
& + [125][346][789]\big(\langle 123\rangle\langle 456\rangle\langle 789\rangle + \langle 123\rangle\langle 458\rangle\langle 679\rangle - \langle 123\rangle\langle 468\rangle\langle 579\rangle - \langle 134\rangle\langle 257\rangle\langle 689\rangle \\
& \quad + \langle 134\rangle\langle 267\rangle\langle 589\rangle + \langle 137\rangle\langle 248\rangle\langle 569\rangle\big) \\
& + [125][347][689](-\langle 123\rangle\langle 456\rangle\langle 789\rangle + \langle 123\rangle\langle 468\rangle\langle 579\rangle + \langle 134\rangle\langle 256\rangle\langle 789\rangle - \langle 136\rangle\langle 248\rangle\langle 579\rangle) \\
& + [125][348][679](-\langle 123\rangle\langle 467\rangle\langle 589\rangle + \langle 136\rangle\langle 247\rangle\langle 589\rangle) + [125][367][489](-\langle 134\rangle\langle 256\rangle\langle 789\rangle + \langle 134\rangle\langle 268\rangle\langle 579\rangle) \\
& - [125][368][479]\langle 134\rangle\langle 267\rangle\langle 589\rangle + [126][347][589](-\langle 123\rangle\langle 458\rangle\langle 679\rangle + \langle 135\rangle\langle 248\rangle\langle 679\rangle) \\
& + [126][348][579](\langle 123\rangle\langle 457\rangle\langle 689\rangle - \langle 135\rangle\langle 247\rangle\langle 689\rangle) - [126][357][489]\langle 134\rangle\langle 258\rangle\langle 679\rangle \\
& + [126][358][479]\langle 134\rangle\langle 257\rangle\langle 689\rangle + [127][348][569](-\langle 123\rangle\langle 456\rangle\langle 789\rangle + \langle 135\rangle\langle 246\rangle\langle 789\rangle) \\
& - [127][358][469]\langle 134\rangle\langle 256\rangle\langle 789\rangle + [134][256][789]\big(-\langle 123\rangle\langle 467\rangle\langle 589\rangle - \langle 125\rangle\langle 347\rangle\langle 689\rangle \\
& \quad + \langle 125\rangle\langle 367\rangle\langle 489\rangle + \langle 127\rangle\langle 358\rangle\langle 469\rangle\big) \\
& + [134][257][689](\langle 125\rangle\langle 346\rangle\langle 789\rangle - \langle 126\rangle\langle 358\rangle\langle 479\rangle) + [134][258][679](\langle 123\rangle\langle 456\rangle\langle 789\rangle + \langle 126\rangle\langle 357\rangle\langle 489\rangle) \\
& + [134][267][589](-\langle 125\rangle\langle 346\rangle\langle 789\rangle + \langle 125\rangle\langle 368\rangle\langle 479\rangle) \\
& - [134][268][579]\langle 125\rangle\langle 367\rangle\langle 489\rangle + [135][246][789]\big(-\langle 123\rangle\langle 457\rangle\langle 689\rangle + \langle 123\rangle\langle 467\rangle\langle 589\rangle \\
& \quad + \langle 124\rangle\langle 357\rangle\langle 689\rangle - \langle 124\rangle\langle 367\rangle\langle 589\rangle - \langle 127\rangle\langle 348\rangle\langle 569\rangle\big) \\
& + [135][247][689](\langle 123\rangle\langle 456\rangle\langle 789\rangle - \langle 124\rangle\langle 356\rangle\langle 789\rangle + \langle 126\rangle\langle 348\rangle\langle 579\rangle) \\
& - [135][248][679]\langle 126\rangle\langle 347\rangle\langle 589\rangle + [135][267][489](\langle 124\rangle\langle 356\rangle\langle 789\rangle - \langle 124\rangle\langle 368\rangle\langle 579\rangle) \\
& + [135][268][479]\langle 124\rangle\langle 367\rangle\langle 589\rangle - [136][247][589]\langle 125\rangle\langle 348\rangle\langle 679\rangle + [136][248][579]\langle 125\rangle\langle 347\rangle\langle 689\rangle \\
& + [136][257][489]\langle 124\rangle\langle 358\rangle\langle 679\rangle - [136][258][479]\langle 124\rangle\langle 357\rangle\langle 689\rangle - [137][248][569]\langle 125\rangle\langle 346\rangle\langle 789\rangle \\
& + [137][258][469]\langle 124\rangle\langle 356\rangle\langle 789\rangle + [145][267][389](-\langle 123\rangle\langle 456\rangle\langle 789\rangle + \langle 123\rangle\langle 468\rangle\langle 579\rangle) \\
& - [145][268][379]\langle 123\rangle\langle 467\rangle\langle 589\rangle - [146][257][389]\langle 123\rangle\langle 458\rangle\langle 679\rangle + [146][258][379]\langle 123\rangle\langle 457\rangle\langle 689\rangle \\
& - [147][258][369]\langle 123\rangle\langle 456\rangle\langle 789\rangle.
\end{aligned}
\tag{10}
$$

Remark 3.1. In principle, one may also compute the rank condition on F in terms of brackets. However, the degree in brackets gets quite large. For eight points in $\mathbb{P}^2$, by Cramer's rule each entry of F is an 8×8 minor of the Segre matrix of the eight points. Thus one may express the rank condition as a polynomial of bidegree (8, 8) in the brackets $[I]$ and $\langle J \rangle$ on two copies of $\mathcal{B}_{3,8}$.

We close this section with a theorem which states that in general, existence of a common recovery implies that the fundamental matrix has rank at most 2. When there are k^2 or more points, there is an additional condition that the Segre matrix must drop rank. Theorem 3.2 is a consequence of the more general Theorem 2 in [Bertolini et al. 2017] (thanks to Timothy Duff for pointing this out). However, we include a self-contained proof for our setting:

Theorem 3.2. *Let A and B be distinct configurations of $k^2 - 1$ ordered points in $\mathbb{P}^{k-1}$. Suppose that there exists a configuration of ordered points C in $\mathbb{P}^{2k-3}$ and linear projections $\pi_1, \pi_2 : \mathbb{P}^{2k-3} \dashrightarrow \mathbb{P}^{k-1}$ such that $\pi_1(C) = A$ and $\pi_2(C) = B$. Furthermore, suppose that the points are sufficiently generic for the kernel of the $k^2 \times (k^2 - 1)$ Segre matrix to be 1-dimensional. Then the kernel F, when viewed as a $k \times k$ matrix, has rank at most 2.*

Proof. Our strategy is to construct a matrix F of rank at most 2 such that for each pair (a, b) of points in $\mathbb{P}^{k-1}$ which are projections of a common point in $\mathbb{P}^{2k-3}$, the product $b^T F a$ vanishes. Since $b^T F a = \sum_{i,j} F_{ij} a_i b_j$, this would imply that the $1 \times k^2$ flattening of F is in the left kernel of the $k^2 \times (k^2 - 1)$ Segre matrix. We first fix a partial inverse φ to π_1. Define $V := \widehat{\pi_2(\ker \pi_1)}$, which is a $(k-2)$-dimensional subspace of $\mathbb{R}^k$. Let M be the map $\pi_2 \circ \varphi$. In summary, we have the data in the commutative diagram



Then consider the map

$$\widetilde{F} : \mathrm{Gr}(1, k) \dashrightarrow \mathrm{Gr}(k-1, k), \quad x \mapsto V \oplus M(x).$$

This map is linear in x in the sense that it is componentwise given by linear forms. More precisely, representing V with a $(k-2) \times k$ matrix and $M(x)$ with a $1 \times k$ matrix of the same names, the dual Plücker coordinates of $\widetilde{F}(x)$ are the $(k-1)$-minors of the $(k-1) \times k$ matrix

$$\left[\begin{array}{c} M(x) \\ \hline V \end{array} \right] \tag{11}$$

The map $\widetilde{F}$ is thus represented by a matrix, which we will call F. Its base locus is $\mathbb{P}(V)$. Thus the rank of F is 2 and the image is all hyperplanes in $\mathbb{P}^{k-1}$ containing $\mathbb{P}(V) = \mathbb{P}(\ker F)$. Now, suppose that there exists c such that $\pi_1(c) = a$ and $\pi_2(c) = b$. Then c is in the projectivization of the vector space $\varphi(a) \oplus \ker \pi_1$. So $\pi_2(c) = b$ is on the line $F(a)$. In terms of matrices, we then have $b^T F a = 0$. ☐

Remark 3.3. Theorem 3.2 gives a necessary condition for two point configurations to be projections of a common configuration. Theorem 2 of [Bertolini et al. 2017] proves this is also sufficient, and Section 5.1.2 gives an algorithm for reconstructing the points and projection matrices.

4. The Chow–Lam form

The Grassmannian $\mathrm{Gr}(k, n)$ is a smooth projective variety parametrizing k-dimensional subspaces of an n-dimensional vector space. Each $k \times n$ matrix A determines a point of $\mathrm{Gr}(k, n)$ via its rowspan. The Grassmannian is embedded into projective space by taking the maximal minors of matrix representatives; these are known as (*dual*) *Plücker coordinates* of the point.

The rest of this section will explain the Chow–Lam form for subvarieties of the Grassmannian, building towards Theorem 5.2. The Chow–Lam form was introduced in [Pratt and Sturmfels 2025] as a generalization of the Chow form from classical algebraic geometry. We briefly recall the setup here. The input data is a projective variety $\mathcal{V} \subset \mathrm{Gr}(k, n)$ of dimension $k(r-k)-1$ for some $r \leq n$. Then one obtains the projections in Figure 1, where $\mathrm{Fl}(k, n-r+k, n)$ is the partial flag variety parametrizing pairs of a k-space contained in an $(n-r+k)$-space in $\mathbb{C}^n$. The *Chow–Lam locus* $\mathcal{CL}_{\mathcal{V}}$ is defined as $\pi_2(\pi_1^{-1}(\mathcal{V}))$ in $\mathrm{Gr}(n-r+k, n)$. Note that X is closed, and thus so is its preimage in $\mathrm{Fl}(k, n-r+k, n)$. Then since π_2 is a proper map, the Chow–Lam locus is closed in $\mathrm{Gr}(n-r+k, n)$. By Lemma 3.2 of [Pratt and Sturmfels 2025], if $\mathcal{V}$ is irreducible then $\mathcal{CL}_{\mathcal{V}}$ is a proper irreducible subvariety of $\mathrm{Gr}(n-r+k, n)$. When $\mathcal{CL}_{\mathcal{V}}$ is a hypersurface, it is defined by a single equation in Plücker coordinates, which we call the *Chow–Lam form* and denote by $\mathrm{CL}_{\mathcal{V}}$. If $\mathcal{CL}_{\mathcal{V}}$ is not a hypersurface, we set $\mathrm{CL}_{\mathcal{V}} := 1$.

Example 4.1 (curve in $\mathrm{Gr}(2, 4)$). Let $\mathcal{V}$ be a curve in the Grassmannian $\mathrm{Gr}(2, 4)$. Here $k = 2$, $n = 4$, and $r = 3$. Thus the Chow–Lam locus lives in $\mathrm{Gr}(3, 4) = (\mathbb{P}^3)^\vee$. It consists of planes P containing a line Q in $\mathbb{P}^3$ such that Q is a point of $\mathcal{V}$ in $\mathrm{Gr}(2, 4)$. Figure 2 displays the incidences in $\mathbb{P}^3$ and $\mathrm{Gr}(2, 4)$.

Let $X_{\mathcal{V}}$ be the ruled surface in $\mathbb{P}^3$ swept out by all of the lines in $\mathcal{V}$. Then any tangent plane to $X_{\mathcal{V}}$ contains the two lines in $X_{\mathcal{V}}$ through the point of tangency; thus the dual surface $X_{\mathcal{V}}^\vee$ is contained in $\mathcal{CL}_{\mathcal{V}}$. In fact, since they are irreducible varieties of the same dimension, the Chow–Lam locus is precisely $X_{\mathcal{V}}^\vee$.

The case $k = 1$ recovers the classical Chow form for projective varieties. It was introduced by Chow and van der Waerden [1937] as a way to encode any projective variety by a single polynomial. That is, given a Chow form of $\mathcal{V}$, one may recover the radical ideal of $\mathcal{V}$. They used this to construct a space known as the *Chow variety* parametrizing cycles of a fixed degree and dimension in a projective space. For an introduction to Chow forms see [Dalbec and Sturmfels 1995].

$$\mathcal{V} \subset \mathrm{Gr}(k, n) \xleftarrow{\ \pi_1\ } \mathrm{Fl}(k, n-r+k, n) \xrightarrow{\ \pi_2\ } \mathrm{Gr}(n-r+k, n)$$

Figure 1. Definition of the Chow–Lam locus.

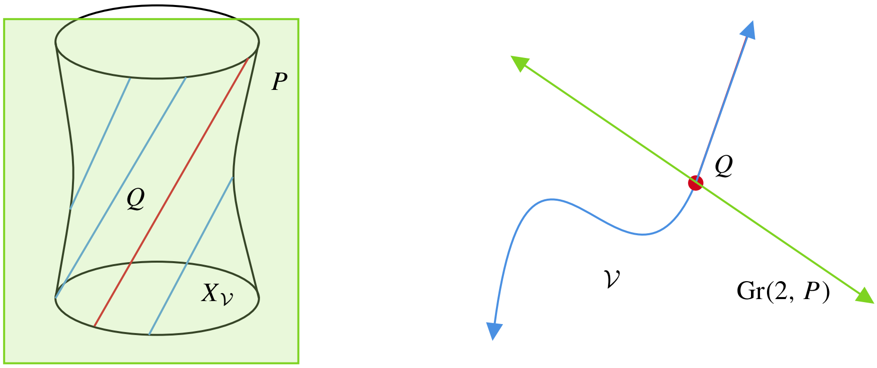


Figure 2. Geometry in $\mathbb{P}^3$ (left) and Gr(2, 4) (right).

Unique recovery of a variety fails for the Chow–Lam form even for curves in Gr(2, 4), as one can see in Example 4.1. The two rulings of $X_{\mathcal{V}}$ give two different curves in Gr(2, 4) with the same Chow–Lam locus. The extent to which one may recover a subvariety of $\mathrm{Gr}(k, n)$ from its Chow–Lam form is explored in [Pratt and Ranestad 2025]. In particular, for curves whose corresponding ruled surface is not a cone, it turns out that ruling the same surface is equivalent to having the same Chow–Lam form.

To compute Chow–Lam forms, it is convenient to use multiple different coordinate systems on the Grassmannian. One may parametrize a k-subspace A of $\mathbb{C}^n$ as the rowspan of a matrix. Then the entries of this matrix are called the *dual Stiefel coordinates* and the maximal minors are the *dual Plücker coordinates*. Alternatively, one may represent A as the kernel of an $(n-k)\times n$ matrix. In this case the entries are called the *primal Stiefel coordinates* and the maximal minors are called the *primal Plücker coordinates* of A.

We denote the dual Plücker coordinates by $q_I(A)$ and the primal Plücker coordinates by $p_J(A)$, where

$$I \in \binom{[n]}{k} \quad \text{and} \quad J \in \binom{[n]}{n-k}.$$

They are related up to sign by taking complements of indices: specifically, if $I = [i_1 \cdots i_k]$ we define

$$\varepsilon(I) := i_1 + \cdots + i_k - (1 + \cdots + k).$$

Then $p_I = (-1)^{\varepsilon(I)} q_{[n]\setminus I}$. If k is close to n then it is more convenient to use primal coordinates. For example, the ten Plücker coordinates on Gr(3, 5) are

$$\begin{matrix} p_{12} & p_{13} & p_{14} & p_{15} & p_{23} & p_{24} & p_{25} & p_{34} & p_{35} & p_{45}, \\ q_{345} & -q_{245} & q_{235} & -q_{234} & q_{145} & -q_{135} & q_{134} & q_{125} & -q_{124} & q_{123}. \end{matrix} \tag{12}$$

An important statistic of a subvariety of a Grassmannian is its class in $\mathrm{H}^*(\mathrm{Gr}(k, n), \mathbb{Z})$. As an abelian group, this is isomorphic to $\mathbb{Z}^{\binom{n}{k}}$ with basis given by the *Schubert cycles*. We will define these cycles with respect to intersections with the standard flag and index them using partitions.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$, and let $E_i := \mathrm{span}(e_1, \ldots, e_i)$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ fitting inside a $k \times (n-k)$ box, define the Schubert variety Ω_λ to be

$$\Omega_\lambda = \{L \in \mathrm{Gr}(k, n) : \dim L \cap E_{n-k+\lambda_i - i} \geq i\}. \tag{13}$$

The Schubert variety Ω_λ is a closed irreducible subvariety of codimension $\sum_i \lambda_i$. In particular, $\Omega_\varnothing$ is the class of $\mathrm{Gr}(k, n)$ and $\Omega_{k\times(n-k)}$ is the class of a point. The class of each subvariety $\mathcal{V}$ of $\mathrm{Gr}(k, n)$ has a unique expansion into Schubert classes:

$$[\mathcal{V}] = \sum_{\lambda\subseteq k\times(n-k)} \delta_\lambda(\mathcal{V})\cdot[\Omega_\lambda].$$

Let $\mathcal{V}$ be a subvariety of $\mathrm{Gr}(k, n)$ of dimension $k(r-k)-1$. Let $\alpha = (n-r+1, n-r, \ldots, n-r)$. We take the convention that our parts are weakly decreasing in size. Then any partition with total size equal to the codimension of $\mathcal{V}$ satisfies $\alpha_1 \geq n-r+1$, and α is the unique partition for which we have equality. We set $\alpha(\mathcal{V}) := \delta_\alpha(\mathcal{V})$ and call this the *Chow–Lam degree* of $\mathcal{V}$. Indeed, the name is consistent:

Lemma 4.2 [Pratt and Sturmfels 2025, Theorem 3.5]. *Let $\mathcal{V}$ be a subvariety of dimension $k(r-k)-1$ in the Grassmannian $\mathrm{Gr}(k, n)$. The Chow–Lam form $\mathrm{CL}_\mathcal{V}$ is a polynomial of degree $\alpha(\mathcal{V})$ in the Plücker coordinates on $\mathrm{Gr}(k+n-r, n)$.*

5. Torus orbits in the Grassmannian

In this section we study torus orbits in the Grassmannian and their Chow–Lam forms. The Grassmannian $\mathrm{Gr}(k, n)$ is equipped with an action of $T := (\mathbb{C}^*)^n$. In terms of a matrix parametrization, this may be seen as scaling the columns of a $k\times n$ matrix representative. Given any point $A \in \mathrm{Gr}(k, n)$, we write $\mathcal{T}_A$ for the Zariski closure of the orbit $T\cdot A$ in $\mathrm{Gr}(k, n)$. This is a toric variety of dimension at most $n-1$. It has dimension exactly $n-1$ if A is general, in particular if the Plücker coordinates of A are all nonzero. As in the introduction, we fix positive integers k and l and let $n := kl$.

Example 5.1 (torus orbit in $\mathrm{Gr}(2, 6)$). Let $k = 2$ and $l = 3$, and let A be a generic point in $\mathrm{Gr}(2, 6)$ whose Plücker coordinates are nonzero. Then $\mathcal{T}_A$ has dimension $6-1 = 5$. The Chow–Lam locus lives in $\mathrm{Gr}(3, 6)$. Following [Pratt and Sturmfels 2025, Example 4.3], it is parametrized in dual Stiefel coordinates by the matrices

$$\begin{bmatrix} a_{11}t_1 & a_{12}t_2 & a_{13}t_3 & a_{14}t_4 & a_{15}t_5 & a_{16}t_6 \\ a_{21}t_1 & a_{22}t_2 & a_{23}t_3 & a_{24}t_4 & a_{25}t_5 & a_{26}t_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{bmatrix}.$$

Here $(t_1, \ldots, t_6)$ varies over elements of $(\mathbb{C}^*)^6$. However, we could also parametrize it in primal Stiefel coordinates as 3×6 matrices B such that, for some $t \in (\mathbb{C}^*)^6$, we have

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \end{bmatrix} \cdot \mathrm{diag}(t_1, \ldots, t_6) \cdot \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \\ a_{15} & a_{25} \\ a_{16} & a_{26} \end{bmatrix} = 0.$$

Rearranging, we obtain the expression

$$t_1\begin{bmatrix}a_{11}b_{11}\\a_{11}b_{21}\\a_{11}b_{31}\\a_{21}b_{11}\\a_{21}b_{21}\\a_{21}b_{31}\end{bmatrix}+t_2\begin{bmatrix}a_{12}b_{12}\\a_{12}b_{22}\\a_{12}b_{32}\\a_{22}b_{12}\\a_{22}b_{22}\\a_{22}b_{32}\end{bmatrix}+t_3\begin{bmatrix}a_{13}b_{13}\\a_{13}b_{23}\\a_{13}b_{33}\\a_{23}b_{13}\\a_{23}b_{23}\\a_{23}b_{33}\end{bmatrix}+t_4\begin{bmatrix}a_{14}b_{14}\\a_{14}b_{24}\\a_{14}b_{34}\\a_{24}b_{14}\\a_{24}b_{24}\\a_{24}b_{34}\end{bmatrix}+t_5\begin{bmatrix}a_{15}b_{15}\\a_{15}b_{25}\\a_{15}b_{35}\\a_{25}b_{15}\\a_{25}b_{25}\\a_{25}b_{35}\end{bmatrix}+t_6\begin{bmatrix}a_{16}b_{16}\\a_{16}b_{26}\\a_{16}b_{36}\\a_{26}b_{16}\\a_{26}b_{26}\\a_{26}b_{36}\end{bmatrix}=0.$$

Thus B is in the Chow–Lam locus of $\mathcal{T}_A$ in $\mathrm{Gr}(3, 6)$ if and only if the Segre determinant $\mathrm{Seg}_{2,3}(A, B)$ vanishes. In dual Plücker coordinates on $\mathrm{Gr}(2, 6)$ and primal Plücker coordinates on $\mathrm{Gr}(3, 6)$, we obtain the expression

$$\begin{aligned}\mathrm{Seg}_{2,3} = ([12][34][56]+[14][25][36])\langle 123\rangle\langle 456\rangle - [13][25][46]\langle 124\rangle\langle 356\rangle + [12][35][46]\langle 134\rangle\langle 256\rangle \\ - [12][34][56]\langle 135\rangle\langle 246\rangle + [13][24][56]\langle 125\rangle\langle 346\rangle.\end{aligned}$$

Suppose that $n = kl$ for some $l \geq 2$. Let A be a general point in $\mathrm{Gr}(k, n)$. Then the Chow–Lam locus is a subvariety of $\mathrm{Gr}(n-l, n)$. The analysis in Example 5.1 extends to the following result:

Theorem 5.2 (Segre determinant). *Suppose $k, l \geq 2$ and let $n = kl$. Fix a point A in $\mathrm{Gr}(k, n)$ with nonzero Plücker coordinates. Then the Chow–Lam form of $\mathcal{T}_A$ in primal Plücker coordinates B on $\mathrm{Gr}(n-l, n)$ equals the Segre determinant $\mathrm{Seg}_{k,l}(A, B)$.*

We will need tools from later in this section to prove Theorem 5.2 completely. For now, we establish the following lemma:

Lemma 5.3 (factor of Segre determinant). *Fix $k, l \geq 2$ and let $n = kl$. Fix a point A in $\mathrm{Gr}(k, n)$ such that $\dim \mathcal{T}_A = n-1$. Then the Chow–Lam form of $\mathcal{T}_A$ in primal Plücker coordinates B on $\mathrm{Gr}(n-l, n)$ divides the Segre determinant $\mathrm{Seg}_{k,l}(A, B)$.*

Proof. The Chow–Lam locus is the Zariski closure of the set of points B in $\mathrm{Gr}(n-l, n)$ which contain a subspace $t \cdot A$ for some $t \in T$. Representing B in primal Stiefel coordinates and $t \cdot A$ in dual Stiefel coordinates, we are seeking $l \times n$ matrices B such that $B \cdot \mathrm{diag}(t_1, \ldots, t_n) \cdot A = 0$. Rearranging, we obtain the condition that for some $t \in (\mathbb{C}^*)^n$,

$$\sum_{i=1}^{n} t_i (A_i \otimes B_i) = 0. \tag{14}$$

If (14) holds, then $A_1 \otimes B_1, \ldots, A_n \otimes B_n$ are linearly dependent. Thus the Chow–Lam form is an irreducible factor of the Segre determinant. □

Example 5.4. For special matrices A, the Segre determinant becomes reducible and the Chow–Lam form is one of the irreducible factors. For instance, let A in $\mathrm{Gr}(2, 4)$ be any point whose Plücker coordinate $q_{12}(A)$ is zero. Then $\mathcal{T}_A$ is the 3-dimensional variety with ideal generated by q_{12} and the Plücker relation for Gr(2, 4). By Example 1.1,

$$\mathrm{Seg}_{2,2}(A, B) = [13][24]\langle 12\rangle\langle 34\rangle.$$

However, the Chow–Lam form of $\mathcal{T}_A$ in primal coordinates is $\langle 34\rangle$. Indeed, the vanishing of $p_{34} = q_{12}$ exactly cuts out the original variety. The extra factor of $\langle 12\rangle$ represents the lines passing through the singular point $A' := \mathrm{span}(e_3, e_4)$ in $\mathcal{T}_A$. Indeed, any matrix with $p_{34} = 0$ satisfies $B \cdot A' = 0$. In terms of the proof of Lemma 5.3, we have a linear dependence where the coefficients $(0, 0, t_3, t_4)$ are not in $(\mathbb{C}^*)^4$.

To better predict situations like Example 5.4, we will introduce some combinatorial tools to compute the Chow–Lam degree of $\mathcal{T}_A$. The *matroid* of a point A in the Grassmannian $\mathrm{Gr}(k, n)$ is defined by its *bases*, namely the collection of indices $I \in \binom{[n]}{k}$ such that the corresponding Plücker coordinate q_I is nonzero. In general, the variety $\mathcal{T}_A$ will (up to isomorphism) only depend on the underlying matroid of the point A; see, e.g., [Michałek and Sturmfels 2021, Proposition 13.12]. Thus we may denote $\mathcal{T}_A$ by $\mathcal{T}_M$, where M is the matroid of A. We let $\delta_\lambda(M)$ denote the Schubert coefficient $\delta_\lambda(\mathcal{T}_M)$ and call the collection of these the *Schubert coefficients of the matroid M*. In particular, $\alpha(M)$ denotes the Chow–Lam degree of $\mathcal{T}_M$.

The *uniform matroid* $U_{k,n}$ has as its bases all size-k subsets of $[n]$. It arises as the matroid of a point whose Plücker coordinates are all nonzero. Klyachko [1985] computed the Schubert coefficients of the uniform matroid in terms of dimensions of irreducible SL_n-representations. His formula involves a count $\#\,\mathrm{SSYT}(\lambda, n)$ of the number of semistandard Young tableaux of shape λ with entries in $[n]$. This formula comes from representation theory; in that context, the number $\mathrm{SSYT}(\lambda, n)$ is the dimension of the irreducible SL_n-representation obtained by applying the Schur functor $\mathbb{S}_\lambda$ to the standard representation of SL_n. The partition complement λ^c is obtained by removing λ from a $k \times (n-k)$ rectangle and rotating by 180 degrees.

Proposition 5.5 [Klyachko 1985, Theorem 6]. *Let λ be a partition fitting in a $k \times (n-k)$ rectangle. Then the coefficient $\delta_\lambda(U_{k,n})$ is*

$$\delta_\lambda(U_{k,n}) = \sum_{i=0}^{k} (-1)^i \binom{n}{i} \#\,\mathrm{SSYT}(\lambda^c, k-i). \tag{15}$$

Proof of Theorem 5.2. By Lemma 5.3, the Chow–Lam form divides the Segre determinant. Thus it suffices to show that the Chow–Lam degree of $U_{k,n}$ is k. We do this using Klyachko's formula. The complement of $\alpha = (n-r+1, n-r, \dots, n-r)$ in the $k \times (n-k)$ rectangle is $\alpha^c = (r-k, \dots, r-k, r-k-1)$. Because α^c has k parts, the contribution to the sum in (15) is nonzero only when $i = 0$. The semistandard condition fixes all but the last column of each tableau of shape α^c, giving that $\alpha(U_{k,n}) = k$. □

6. The Segre coefficient variety

In this section, we introduce the *Segre coefficient variety*, which parametrizes Segre determinants of points in $\mathrm{Gr}(k, kl)$. The ambient space of the Segre coefficient variety is the projectivization of the vector space $H^0(\mathrm{Gr}(l, kl), \mathcal{O}_{\mathrm{Gr}(l,kl)}(k))$ spanned by degree-k monomials in the B-variables. We prove in Theorem 6.2

that the linear span of these Segre determinants is as large as possible, namely that it equals the *multilinear component*. This results in Corollary 6.3, which states that for $k = 2$, the Segre coefficient variety recovers the GIT quotient $(\mathbb{P}^1)^{2l} /\!/ \mathrm{SL}(2)$ parametrizing configurations of points on the projective line.

Let $\mathrm{Gr}(k, kl)^\circ \subset \mathrm{Gr}(k, kl)$ be the Zariski open subset of points whose matroid is uniform. Consider the map

$$\pi : \mathrm{Gr}(k, kl)^\circ \to \mathbb{P}H^0(\mathrm{Gr}(l, kl), \mathcal{O}_{\mathrm{Gr}(l,kl)}(k)), \quad A \mapsto \mathrm{Seg}(A, B). \tag{16}$$

The map π sends a point A to $\mathrm{Seg}(A, B)$, viewed as a polynomial in the B-coordinates. While π is defined without choosing a basis for the target space, it is often convenient to take a basis of standard B-monomials of degree k for $\mathcal{O}_{\mathrm{Gr}(l,kl)}(k)$, as in Example 6.1. Then π sends A to the vector of A-coefficients of the standard B-monomials appearing in $\mathrm{Seg}_{k,l}(A, B)$. We define the *Segre coefficient variety* $\mathrm{Coeff}(\mathrm{Seg}_{k,l})$ as the Zariski closure of the image of π.

Example 6.1 (Segre cubic). From Example 5.1 we get the map

$$\begin{gathered} \pi : \mathrm{Gr}(2, 6)^\circ \to \mathbb{P}^4 \\ A \mapsto ([12][34][56] + [14][25][36], -[13][25][46], [12][35][46], -[12][34][56], [13][24][56]). \end{gathered} \tag{17}$$

Its image is cut out by the following degree-3 polynomial, which is known as the *Segre cubic*:

$$x_0x_1x_3 - x_1x_2x_3 - x_0x_2x_4 - x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4.$$

The corresponding variety is the *Segre cubic threefold*. It is the unique (up to isomorphism) cubic hypersurface in $\mathbb{P}^4$ with ten ordinary double points, the maximum possible [Kalker 1986].

The Segre coefficient variety recovers a known construction for $k = 2$, namely the GIT quotient $(\mathbb{P}^1)^{2l} /\!/_{1^{2l}} \mathrm{SL}_2$ parametrizing configurations of $2l$ distinct points on a projective line. This is defined as $\mathrm{Proj}\, R$, where R is the graded ring of SL_2-invariants of $2l$ ordered points in $\mathbb{P}^1$. This ring is studied by Howard, Millson, Snowden, and Vakil in [Howard et al. 2012], and they give generators for the ideal of relations between these invariants. The equivalence of that construction with our construction rests on Theorem 6.2, which says that the monomials appearing as coefficients of Segre determinants of points in $\mathrm{Gr}(k, kl)$ span the whole multilinear component.

Theorem 6.2. *Every monomial of the form $[I_1] \cdots [I_l]$ with $I_1 \cup \cdots \cup I_l = [kl]$ appears in the linear span of the coefficients of the B-monomials in $\mathrm{Seg}_{k,l}(A, B)$.*

Proof. Let V be the vector space spanned by monomials $[I_1] \cdots [I_l]$ such that $I_1 \cup \cdots \cup I_l = [kl]$. Suppose for the sake of contradiction that the coefficients of the Segre determinant do not span V. Then there is a linear relation among them when $\mathrm{Seg}_{k,l}(A, B)$ is written in the basis of standard monomials. Thus the Segre coefficient variety lies in a hyperplane in $\mathbb{P}(V)$.

We show that the Segre coefficient variety does not lie in a hyperplane, so that the assumption is false. To do this, we will produce a collection of points in $\mathrm{Coeff}(\mathrm{Seg}_{k,l})$ whose linear span is all of $\mathbb{P}(V)$.

Consider the point A in $\mathrm{Gr}(k, kl)$ given by the rowspan of a matrix where columns $(i-1)l+1$ through il are equal to e_i, for $1 \le i \le k$. The Segre matrix in (1) develops a block form in this case. Explicitly, if we choose the basis $e_1 \otimes e_1, e_1 \otimes e_2, \dots, e_1 \otimes e_l, \dots, e_k \otimes e_l$ for $V \otimes W$, then the Segre matrix will have k blocks of size $l \times l$ along its diagonal and be zero elsewhere:

$$\mathrm{Seg}_{k,l}(A, B) = \det \begin{bmatrix} b_{11} & \dots & b_{1l} & & & & \\ \vdots & \ddots & \vdots & & 0 & & \\ b_{l1} & \dots & b_{l,l} & & & & \\ & & & \ddots & & & \\ & & & & b_{1,(k-1)l+1} & \dots & b_{1,kl} \\ & 0 & & & \vdots & \ddots & \vdots \\ & & & & b_{l,(k-1)l+1} & \dots & b_{l,kl} \end{bmatrix}.$$

The determinant (up to scalar) factors as $\langle 1 \cdots l \rangle \langle l+1 \cdots 2l \rangle \cdots \langle (k-1)l+1 \cdots kl \rangle$, where as before the angle brackets are minors of the B-matrix. Thus the Segre determinant of $\mathcal{T}_A$ is nonzero, so the map π extends from $\mathrm{Gr}(k, kl)^\circ$ to A. By symmetry, we may take any point A given as the rowspan of a matrix whose columns are partitioned into k classes of l pairwise parallel elements. By continuity, the image $\pi(A)$ will be in the closure of the Segre coefficient variety. The collection of these points together spans all of $\mathbb{P}(V)$. □

In the case $k = 2$, another spanning set of the multilinear component is given by Kempe's *matchings*. These are directed planar graphs with n labeled vertices embedded in a disc, where vertices are in clockwise order and each vertex has degree 1; see Figure 3.

To each such graph Γ we associate the SL_2-invariant

$$X_\Gamma := \prod_{(i \to j) \in \Gamma} x_i y_j - x_j y_i.$$

Kempe proved that the set of X_Γ span the space of invariants of SL_2 acting on $V^{\otimes n}$, where V is 2-dimensional and SL_2 takes the standard action on each component of the tensor product. Furthermore, the noncrossing matchings, i.e., the planar graphs, form a basis [Kempe 1894]. Denoting the multilinear component by W, the map

$$\psi : (V^{\otimes n})^{\mathrm{SL}_2} \to W, \quad X_\Gamma \mapsto \prod_{(i \to j) \in \Gamma} [ij] \tag{18}$$

is an isomorphism of vector spaces. That is, consider the expansion of each Plücker monomial in the 2×2 minors of a $2 \times n$ matrix with entries $x_{11}, \dots, x_{2n}$. Then a set of Plücker monomials is dependent if and only if the expansions are dependent in $\mathbb{C}[x_{11}, \dots, x_{2n}]$. This identification of vector spaces leads to the following corollary:

Corollary 6.3. *The variety* $\mathrm{Coeff}(\mathrm{Seg}_{2,l})$ *is isomorphic to the GIT quotient* $(\mathbb{P}^1)^{2l} /\!\!/_w \mathrm{SL}_2$, *where* $w = 1^{2l}$ *is the linearization.*

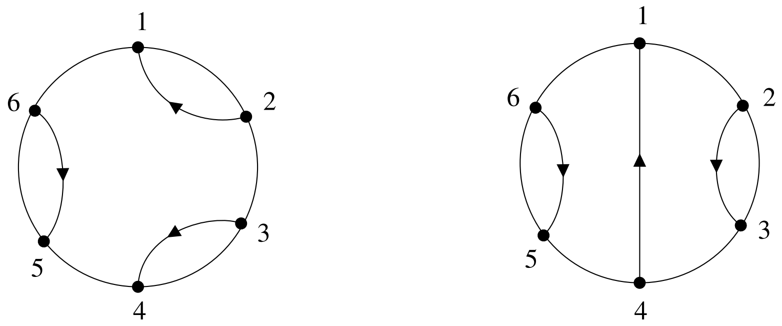


Figure 3. Two Kempe diagrams for $n = 6$.

Proof. Let $n := 2l$. Let R denote of the ring of invariants of SL_2 acting on $(\mathbb{P}^1)^n$, where we choose the linearization $(1, \ldots, 1)$ of the action. That is,

$$R := \bigoplus_{d=0}^{\infty} H^0((\mathbb{P}^1)^n, \mathcal{O}(d, \ldots, d)).$$

By Kempe's theorem [1894], R is generated in degree 1; for a more modern proof see [Howard et al. 2012, Theorem 2.1]. Let $R^{(1)}$ denote the degree-1 piece of R. The quotient map $\mathrm{Sym}(R^{(1)}) \to R$ gives a map $\mathrm{Proj}\, R \hookrightarrow \mathrm{Proj}\, \mathrm{Sym}(R^{(1)})$. Thus we obtain an embedding of the GIT quotient $\mathrm{Proj}\, R$ into the projectivization $\mathbb{P}(R^{(1)})$ of the vector space $R^{(1)}$. By φ we denote the composition

$$\varphi : (\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2 \hookrightarrow \mathbb{P}(R^{(1)}).$$

The stable locus of φ contains the dense open subset U of tuples of n distinct points in $(\mathbb{P}^1)^n$; see, e.g., [Dolgachev and Ortland 1988, Section 7.1]. Because U is dense and φ is continuous, the Zariski closure $\overline{\varphi(U)}$ is equal to the Zariski closure of the image of φ.

Let $\mathrm{Mat}(2, n)^\circ$ denote $2 \times n$ matrices whose maximal minors are nonzero. By composing with the projection map from $\mathrm{Mat}(2, n)^\circ$ to U, we obtain

$$G : \mathrm{Mat}(2, n)^\circ \to \mathbb{P}(R^{(1)}).$$

This map is invariant with respect to the action of SL_2 on $\mathrm{Mat}(2, n)^\circ$ by left multiplication. Thus it factors through the Grassmannian $\mathrm{Gr}(2, n)$, and in particular through the open subset $\mathrm{Gr}(2, n)^\circ$ of points whose Plücker coordinates are nonzero. Let $H : \mathrm{Gr}(2, n)^\circ \to \mathbb{P}(R^{(1)})$ denote the resulting map, of which the closure of the image is the embedded GIT quotient.

We will show that the closure of the image of H is linearly equivalent to the image of our map π defining the Segre coefficient variety, which will prove the theorem. Let $\Gamma_1, \ldots, \Gamma_r$ be the noncrossing matchings, whose corresponding invariants span $R^{(1)}$. By composing with the isomorphism ψ from (18), we may write H in our graphical basis as

$$\mathrm{Gr}(2, n)^\circ \to \mathbb{P}(W), \quad A \mapsto \Bigg[\prod_{(i \to j) \in \Gamma_1} [ij] : \cdots : \prod_{(i \to j) \in \Gamma_r} [ij] \Bigg],$$

where the $[ij]$ denote the Plücker coordinates of A. These are certain monomials $[I_1]\cdots[I_l]$ where $I_1\cup\cdots\cup I_l=[2l]$. Theorem 6.2 tells us that every such monomial lies in the linear span of the coefficients of the Segre polynomial, so the image is linearly isomorphic to the Segre coefficient variety. □

If we drop the assumption that $I_1\cup\cdots\cup I_l=[kl]$ and instead range over all sets $\{I_1,\ldots,I_l\}$ with entries in $[kl]$, the collection of monomials $[I_1]\cdots[I_l]$ uniquely determines the torus orbit closure; see, e.g., [Michałek and Sturmfels 2021, Chapter 13]. However, the Segre coefficient map is not injective on generic torus orbit closures in general.

Example 6.4 (six points in $\mathbb{P}^2$). From Example 5.1 the Segre coefficient map is linearly equivalent to

$$\pi' : \mathrm{Gr}(3,6)^\circ \to \mathbb{P}^4, \quad A \mapsto ([123][456],[124][356],[125][346],[134][256],[135][246]). \tag{19}$$

If we parametrize the Grassmannian by 3×6 matrices, then by taking products of 3×3 minors as in (19) the following two matrices have nonzero Plücker coordinates and each go to the point $(-2,2,9,2,8)$ under the map π':

$$p=\begin{bmatrix}1&0&0&1&1&1\\0&1&0&1&2&4\\0&0&1&1&3&5\end{bmatrix}, \qquad q:=\begin{bmatrix}1&0&0&1&1&1\\0&1&0&1&-4&-2\\0&0&1&1&-3&-1\end{bmatrix}.$$

However, their torus orbit closures are different. We may see this by noting that $[123][145][246][356]$ evaluates to -8 for the first point and 4 for the second in the affine chart given by $[123]=1$, and is constant on torus orbits.

Recall that by Theorem 5.2, the Segre determinant computes the Chow–Lam form of a torus orbit closure. From the point of view of the Chow–Lam form, this gives us a new family of varieties which are distinct but have the same Chow–Lam form, thus building on the work of [Pratt and Ranestad 2025]. We hope to explore the case of $k=3$ in future work.

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Groupoid models for relative Cuntz–Pimsner algebras of groupoid correspondences

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A groupoid correspondence on an étale, locally compact groupoid induces a C^*-correspondence on its groupoid C^*-algebra. We show that the Cuntz–Pimsner algebra for this C^*-correspondence relative to an ideal associated to an open invariant subset of the groupoid is again a groupoid C^*-algebra for a certain groupoid. We describe this groupoid explicitly and characterise it by a universal property that specifies its actions on topological spaces. Our construction unifies the construction of groupoids underlying the C^*-algebras of topological graphs and self-similar groups.

1. Introduction

We show that the Cuntz–Pimsner algebra of the C^*-correspondence defined by a groupoid correspondence is a groupoid C^*-algebra for a canonical groupoid, which is characterised by a universal property that specifies its actions on topological spaces. Our groupoid construction unifies several existing ones, starting with the groupoids underlying Cuntz algebras (see [Renault 1980]) and graph C^*-algebras (see [Kumjian et al. 1998]), and continuing with topological graphs (see [Katsura 2004]), self-similar groups (see [Nekrashevych 2009]), self-similar graphs (see [Exel and Pardo 2017]), or self-similar actions of groupoids (see [Laca et al. 2018]). In fact, the Cuntz–Pimsner algebras of groupoid correspondences may be viewed as being the C^*-algebras of self-similar actions of general étale groupoids as opposed to the discrete groupoids studied in [Laca et al. 2018].

Let $\mathcal{G}$ and $\mathcal{H}$ be étale, locally compact groupoids, possibly non-Hausdorff. A groupoid correspondence $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ is a topological space $\mathcal{X}$ with commuting actions of $\mathcal{H}$ on the left and $\mathcal{G}$ on the right, such that the right action is free and proper and its anchor map $s : \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism; this forces $\mathcal{X}$ to be locally Hausdorff and locally quasicompact. Such a groupoid correspondence induces a $C^*(\mathcal{H})$–$C^*(\mathcal{G})$-correspondence, that is, a Hilbert $C^*(\mathcal{G})$-module with a nondegenerate action of $C^*(\mathcal{H})$ by adjointable operators (see [Antunes et al. 2022]).

The anchor map $r : \mathcal{X} \to \mathcal{H}^0$ of the left action on $\mathcal{X}$ induces a continuous map $r_* : \mathcal{X}/\mathcal{G} \to \mathcal{H}^0$ because it is $\mathcal{G}$-invariant. Let $\mathcal{R} \subseteq \mathcal{H}^0$ be an open $\mathcal{H}$-invariant subset. Then $\mathcal{G}_{\mathcal{R}} = s^{-1}(\mathcal{R}) = r^{-1}(\mathcal{R})$ is an open subgroupoid of $\mathcal{G}$ and $C^*(\mathcal{G}_{\mathcal{R}})$ is an ideal in $C^*(\mathcal{G})$. We call $\mathcal{X}$ *relatively proper* on $\mathcal{R}$ if r_* restricts to

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a proper map $r_*^{-1}(\mathcal{R}) \to \mathcal{R}$. This implies that $C^*(\mathcal{G}_\mathcal{R})$ acts by compact operators on $C^*(\mathcal{X})$. If $\mathcal{G} = \mathcal{H}$, then this allows us to define a Cuntz–Pimsner algebra $\mathcal{O}_{C^*(\mathcal{X}),C^*(\mathcal{G}_\mathcal{R})}$ for $C^*(\mathcal{X})$ relative to the ideal $C^*(\mathcal{G}_\mathcal{R})$ in $C^*(\mathcal{G})$. If $\mathcal{R}$ is empty, then this is the Toeplitz C^*-algebra of $C^*(\mathcal{X})$. We are going to write $\mathcal{O}_{C^*(\mathcal{X}),C^*(\mathcal{G}_\mathcal{R})}$ as the C^*-algebra of an étale groupoid, which we call the *groupoid model* of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$.

A groupoid model as above was first constructed in the case $\mathcal{R} = \mathcal{G}^0$ by Albandik [2015]. It is described in a better way in [Meyer 2022], and our description will use the same idea. These two articles also consider diagrams of groupoid correspondences, which correspond roughly to higher-rank graphs. The generalisation to relative Cuntz–Pimsner algebras has been worked out by Antunes [2023]. The special case of a groupoid correspondence of a discrete groupoid is also treated in [Miller and Steinberg 2024]. Here we mostly follow [Antunes 2023], correcting one important inaccuracy (see Remark 3.5). Currently, there is no treatment that allows relative Cuntz–Pimsner algebras of diagrams of groupoid correspondences that fail to be proper. Indeed, this theory is more difficult because extra concepts such as compact alignment become relevant on the C^*-algebraic side.

To define the groupoid model, we first specify what an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ on a topological space is. The groupoid model $\mathcal{M}$ is defined as a groupoid whose actions are in a natural bijection with actions of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$. We show that there is a groupoid with this universal property and that it is unique up to isomorphism. A key step here is to rewrite an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ as an action of a certain inverse semigroup $\mathcal{I}$ with certain extra properties. Thus our groupoid model arises as an inverse semigroup transformation groupoid $\Omega \rtimes \mathcal{I}$. We describe *-homomorphisms from both $\mathcal{O}_{C^*(\mathcal{X}),C^*(\mathcal{G}_\mathcal{R})}$ and $C^*(\Omega \rtimes \mathcal{I})$ to $\mathbb{K}(\mathcal{E})$ for a Hilbert module $\mathcal{E}$ over a C^*-algebra in terms of the original diagram. These universal properties are reasonably close to each other, and then we do some technical work to identify the data that appears in both universal properties. This implies the desired isomorphism $\mathcal{O}_{C^*(\mathcal{X}),C^*(\mathcal{G}_\mathcal{R})} \cong C^*(\Omega \rtimes \mathcal{I})$.

The groupoid model may be used to characterise when the relative Cuntz–Pimsner algebra is simple, by checking whether it is Hausdorff, effective, amenable, and minimal. Criteria for this are developed in [Antunes 2023]. In ongoing work with de Castro [de Castro and Meyer 2026] we use the universal property that characterises the groupoid model to study morphisms between graph C^*-algebras through certain morphisms between the underlying groupoids.

2. Relatively proper groupoid correspondences

This article depends on [Antunes et al. 2022], which defines groupoid correspondences and the associated C^*-correspondences. In addition, a composition of groupoid correspondences is defined and shown to be compatible with the composition of C^*-correspondences in the sense that the passage from groupoid to C^*-correspondences is a homomorphism of bicategories. We shall not repeat the theory developed in [Antunes et al. 2022].

Throughout this article, we fix an (étale, locally compact) groupoid $\mathcal{G}$, an (étale, locally compact) groupoid correspondence $\mathcal{X} : \mathcal{G} \leftarrow \mathcal{G}$, and an open $\mathcal{G}$-invariant subset $\mathcal{R} \subseteq \mathcal{G}^0$ such that $r_* : \mathcal{X}/\mathcal{G} \to \mathcal{G}^0$ restricts to a proper map $r_*^{-1}(\mathcal{R}) \to \mathcal{R}$; then we briefly call $\mathcal{X}$ *proper on* $\mathcal{R}$. Our notation differs from [Antunes 2023] because we do *not* assume $\mathcal{R} \subseteq r(\mathcal{X})$ as in [Antunes 2023].

The case $\mathcal{R} = \varnothing$ is allowed, and then $\mathcal{X}$ may be any groupoid correspondence. The assumptions imply that $\mathcal{G}^0$ and the orbit space $\mathcal{X}/\mathcal{G}$ of the right $\mathcal{G}$-action on $\mathcal{X}$ are locally compact Hausdorff spaces. In contrast, the arrow space $\mathcal{G}$ and the space $\mathcal{X}$ itself need not be Hausdorff.

We will discuss self-similar groupoid actions as an example in Section 9. Throughout the paper, we use the case of directed graphs and their C^*-algebras to illustrate our theory.

Example 2.1. Let $\mathcal{G} = V$ be just a discrete set with only identity arrows. So $C^*(V) = C_0(V)$. A correspondence $\mathcal{X} : \mathcal{G} \leftarrow \mathcal{G}$ is just a discrete set E with two maps $r, s : E \rightrightarrows V$ or, in other words, a directed graph; (see [Antunes et al. 2022, Example 4.1]). Recall that a vertex $v \in V$ is *regular* if $r^{-1}(v) \subseteq E$ is nonempty and finite. Let $\mathcal{R} \subseteq V$ be the subset of all regular vertices. Then the Cuntz–Pimsner algebra of $C^*(E)$ relative to the ideal $C_0(\mathcal{R}) \subseteq C_0(V)$ is the usual graph C^*-algebra of the graph $r, s : E \rightrightarrows V$ (see, for instance, [Raeburn 2005, Section 8]).

The next lemma describes all possible choices for $\mathcal{R}$:

Lemma 2.2. *Let $f : X \to Y$ be a continuous map between Hausdorff locally compact spaces. There is an open subset $Y_{\max} \subseteq Y$ such that for an open subset $W \subseteq X$ the restriction $f^{-1}(W) \to W$ of f is proper if and only if $W \subseteq Y_{\max}$.*

Proof. The map f induces a nondegenerate *-homomorphism

$$f^* : C_0(Y) \to \mathcal{M}(C_0(X)) = C_b(X).$$

The restriction of f to $f^{-1}(W) \to W$ is proper for an open subset $W \subseteq Y$ if and only if f^* maps $C_0(W)$ to $C_0(f^{-1}(W))$. Since $f^*(h)$ for $h \in C_0(W)$ vanishes outside $f^{-1}(W)$, this happens if and only if $f^*(C_0(W)) \subseteq C_0(X)$. The preimage $(f^*)^{-1}(C_0(X))$ is an ideal in $C_0(Y)$. It must be of the form $C_0(Y_{\max})$ for an open subset $Y_{\max} \subseteq Y$. The argument above shows that the restriction of f to $f^{-1}(W) \to W$ is proper if and only if $W \subseteq Y_{\max}$. $\square$

The following theorem would have fitted well into [Antunes et al. 2022]. We include it here because it was used in a previous version of the proof of the main theorem. Muhly, Renault, and Williams [Muhly et al. 1987] show that a Morita equivalence between two Hausdorff locally compact groupoids induces a Morita equivalence of their C^*-algebras. This is extended by Tu [2004] to non-Hausdorff groupoids, but using a different, more complicated construction of the bimodule.

Theorem 2.3. *Let $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ be a Morita equivalence between two étale groupoids. Then $C^*(\mathcal{X})$ is a Morita–Rieffel equivalence bimodule between $C^*(\mathcal{H})$ and $C^*(\mathcal{G})$.*

Proof. Let $\mathcal{X}$ be a Morita equivalence between two étale groupoids $\mathcal{H}$ and $\mathcal{G}$ and let $\mathcal{X}^*$ be $\mathcal{X}$ with the left and right actions exchanged. This is also a Morita equivalence. Both $\mathcal{X}$ and $\mathcal{X}^*$ are groupoid correspondences: both the left and the right actions on $\mathcal{X}$ must be free and proper, and the anchor maps induce homeomorphisms $\mathcal{X}/\mathcal{G} \cong \mathcal{H}^0$ and $\mathcal{H}\backslash\mathcal{X} \cong \mathcal{G}^0$. Orbit space projections are local homeomorphisms by [Antunes et al. 2022, Lemma 2.10]. It follows that the source and range anchor maps on $\mathcal{X}$ are local homeomorphisms. In addition, $\mathcal{X} \circ \mathcal{X}^*$ and $\mathcal{X}^* \circ \mathcal{X}$ are isomorphic to the identity correspondences on

$\mathcal{G}$ and $\mathcal{H}$, respectively. So $\mathcal{X}$ is an equivalence in the bicategory of groupoid correspondences. The map $\mathcal{X} \mapsto C^*(\mathcal{X})$ is part of a homomorphism of bicategories by [Antunes et al. 2022, Theorem 7.13]. Therefore $C^*(\mathcal{X})$ is an equivalence in the bicategory of C^*-correspondences. These equivalences are precisely the Morita–Rieffel equivalences (see [Buss et al. 2013, Proposition 2.11]). So $C^*(\mathcal{X})$ is a Morita–Rieffel equivalence bimodule between $C^*(\mathcal{H})$ and $C^*(\mathcal{G})$. □

Remark 2.4. The left inner product on $C^*(\mathcal{X})$ is defined by

$$\langle\langle \xi \mid \eta \rangle\rangle(h) = \sum_{\{x \in \mathcal{X} : r(x) = s(h)\}} \xi(h \cdot x)\overline{\eta(x)}$$

for all $\xi, \eta \in \mathfrak{S}(\mathcal{X})$ and $h \in \mathcal{H}$. This is the formula we get by transferring the right inner product on $C^*(\mathcal{X}^*)$ defined in [Antunes et al. 2022, (7.2)] to a left inner product on $C^*(\mathcal{X})$.

3. Actions of groupoid correspondences

Let Y be a topological space. We are going to define actions of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ and then use these to define the groupoid model of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$.

Definition 3.1 (compare [Antunes 2023, Definition 4.1]). An *action* of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ on Y consists of a groupoid action of $\mathcal{G}$ on Y and a continuous, open map μ from the fibre product $\mathcal{X} \times_{s,\mathcal{G}^0,r} Y$ to Y, written multiplicatively as $(x, y) \mapsto x \cdot y$, such that:

(3.1.1) $r(x \cdot y) = r(x)$ and $g \cdot (x \cdot y) = (g \cdot x) \cdot y$ for $g \in \mathcal{G}$, $x \in \mathcal{X}$, and $y \in Y$, with $s(g) = r(x)$ and $s(x) = r(y)$.

(3.1.2) Let $x, x' \in \mathcal{X}$ and $y, y' \in Y$ satisfy $s(x) = r(y)$ and $s(x') = r(y')$. Then $x \cdot y = x' \cdot y'$ if and only if there is $g \in \mathcal{G}$ with $x' = x \cdot g$ and $y = g \cdot y'$.

(3.1.3) $r^{-1}(\mathcal{R}) \subseteq Y$ is contained in the image $\mathcal{X} \cdot Y$ of μ.

(3.1.4) If $K \subseteq \mathcal{X}/\mathcal{G}$ is compact, then the set of all $x \cdot y$ with $x \in \mathcal{X}$, $y \in Y$, $[x] \in K$, and $s(x) = r(y)$ is closed in Y.

A map $\varphi : Y_1 \to Y_2$ between two $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-actions is $(\mathcal{G}, \mathcal{X})$-*equivariant* if

- $r_{Y_2} \circ \varphi = r_{Y_1}$;
- $\varphi(g \cdot y) = g \cdot \varphi(y)$ for all $g \in \mathcal{G}$ and $y \in Y_1$ with $s(g) = r(y)$;
- $\varphi(x \cdot y) = x \cdot \varphi(y)$ for all $x \in \mathcal{X}$ and $y \in Y_1$ with $s(x) = r(y)$; and
- $\varphi^{-1}(\mathcal{X} \cdot Y_2) = \mathcal{X} \cdot Y_1$.

Let $\mathcal{X} \circ Y$ be the orbit space of $\mathcal{X} \times_{s,\mathcal{G}^0,r} Y$ for the diagonal $\mathcal{G}$-action $(x, y) \cdot g := (x \cdot g, g^{-1} \cdot y)$. This is how the composition of groupoid correspondences is defined, and it has similar properties here although nothing acts on Y on the right:

Lemma 3.2. *Let $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence and let Y be a $\mathcal{G}$-space.*

(3.2.1) *The orbit space projection $\mathcal{X} \times_{s,\mathcal{G}^0,r} Y \to \mathcal{X} \circ Y$ is a surjective local homeomorphism.*

(3.2.2) *The formula* $h \cdot [x, y] = [h \cdot x, y]$ *defines a continuous* $\mathcal{H}$*-action on* $\mathcal{X} \circ Y$.

(3.2.3) *There is a well defined* $\mathcal{H}$*-equivariant open and continuous map*

$$\pi_1 : \mathcal{X} \circ Y \to \mathcal{X}/\mathcal{G}, \quad [x, y] \mapsto [x].$$

Proof. The diagonal $\mathcal{G}$-action on $\mathcal{X} \times_{s,\mathcal{G}^0,r} Y$ is basic because the action on $\mathcal{X}$ is. Then the first statement follows from [Antunes et al. 2022, Lemma 2.10]. The second statement is easy. The map π_1 is induced on orbit spaces by the coordinate projection $\mathrm{pr}_1 : \mathcal{X} \times_{s,\mathcal{G}^0,r} Y \to \mathcal{X}$, which is equivariant for $\mathcal{G}$ and $\mathcal{H}$. As such, it is well defined, $\mathcal{H}$-equivariant, and continuous. It is also open because all orbit space projections are continuous, open, and surjective and pr_1 is open. □

Remark 3.3. Conditions (3.1.1) and (3.1.2) say exactly that μ induces a well-defined injective $\mathcal{G}$-equivariant map $\mu' : \mathcal{X} \circ Y \to Y$. By Lemma 3.2, μ is continuous and open if and only if μ' is, if and only if μ' is a homeomorphism onto its image. The condition (3.1.3) says that the image of μ' contains $r^{-1}(\mathcal{R}) \subseteq Y$. The condition (3.1.4) says that $\mu'(\pi_1^{-1}(K)) \subseteq Y$ is closed in Y. Since $\mathcal{X}/\mathcal{G}$ is Hausdorff, any compact subset $K \subseteq \mathcal{X}/\mathcal{G}$ is closed, and then $\mu'(\pi_1^{-1}(K))$ is relatively closed in $\mathcal{X} \cdot Y = \mu'(\mathcal{X} \circ Y)$ because π_1 is continuous and μ' is a homeomorphism onto its image $\mathcal{X} \cdot Y$.

Remark 3.4. If $\mathcal{R} = \mathcal{G}^0$, then (3.1.3) says that μ' is a homeomorphism $\mathcal{X} \circ Y \xrightarrow{\sim} Y$. So (3.1.4) is automatic by the previous remark. Thus the definition of a groupoid correspondence action in [Meyer 2022, Definition 4.1] is the special case $\mathcal{R} = \mathcal{G}^0$ of our definition.

The condition (3.1.4) in our definition corrects a subtle error in [Antunes 2023] and deserves further discussion. We first discuss the problem in [Antunes 2023]. Then we show by an example that this condition is not automatic, and finally we prove that it follows if the anchor map $Y \to \mathcal{G}^0$ is proper or, more generally, if there is an equivariant map to another action with proper anchor map.

Remark 3.5. In [Antunes 2023], it is assumed instead of (3.1.4) that the anchor map $r : Y \to \mathcal{G}^0$ should be proper. We will show in Lemma 3.7 that this is a stronger condition. The universal action built in [Antunes 2023] has this extra property. However, a general action of the groupoid model $\Omega \rtimes \mathcal{I}$ that is constructed in [Antunes 2023] has this extra property if and only if its anchor map to Ω is proper. So [Antunes 2023, Theorem 5.17] is not correct as stated.

The following example shows that (3.1.4) is necessary to make the theory work when we allow actions whose anchor map fails to be proper.

Example 3.6. Let $\mathcal{G}$ be the one-point groupoid, let $\mathcal{X} : \mathcal{G} \leftarrow \mathcal{G}$ be the two-point correspondence, and let $\mathcal{R} = \varnothing$ be empty. The resulting Toeplitz algebra is the Cuntz–Toeplitz algebra $\mathcal{TO}_2$. This is known to be a groupoid C^*-algebra of an étale groupoid with a totally disconnected object space. Now let Y be the open interval $(0, 1)$ and let $\mu : Y \sqcup Y \cong \mathcal{X} \times Y \to Y$ be the map that identifies the two copies of $(0, 1)$ with $\left(0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$ by piecewise linear maps and then embeds $\left(0, \frac{1}{2}\right) \sqcup \left(\frac{1}{2}, 1\right)$ into $(0, 1)$. This is a homeomorphism from $\mathcal{X} \times Y$ onto an open subset of Y. It has all the properties of an action in Definition 3.1 except (3.1.4). It should not be an action of $(\mathcal{G}, \mathcal{X}, \varnothing)$ because there is no equivariant continuous map to the object space of the groupoid model of $\mathcal{TO}_2$.

Lemma 3.7. *Let Y be a locally compact, Hausdorff $\mathcal{G}$-space with a $\mathcal{G}$-equivariant map $\mathcal{X} \circ Y \to Y$ that is a homeomorphism onto an open subset of Y. If the anchor map $Y \to \mathcal{G}^0$ is proper or if there is an equivariant map to another action with proper anchor map, then the data above also satisfies (3.1.4), making it an action of $(\mathcal{G}, \mathcal{X}, \varnothing)$.*

Proof. By [Antunes et al. 2022, Lemma 5.1], pullbacks of proper maps are again proper. So the first coordinate projection $\mathrm{pr}_1 : \mathcal{X} \times_{s,\mathcal{G}^0,r} Y \to \mathcal{X}$ is proper. This implies that the induced map on orbit spaces $\mathcal{X} \circ Y \to \mathcal{X}/\mathcal{G}$ is proper (see [Antunes et al. 2022, Lemma 5.5]). As a consequence, the preimage of a compact subset $K \subseteq \mathcal{X}/\mathcal{G}$ in $\mathcal{X} \circ Y$ is compact. Then it must be closed in the Hausdorff space Y.

Let $\varphi : Y_1 \to Y_2$ be an equivariant map of $(\mathcal{G}, \mathcal{X}, \varnothing)$-actions and assume that the anchor map of Y_2 is proper. Then the preimage of K in $\mathcal{X} \circ Y_2$ is compact and hence closed in Y_2. Therefore its φ-preimage in Y_1 is also closed because φ is continuous. □

In the following, the indices in nets belong to some directed sets, which we do not care to name.

Lemma 3.8. *Let Y be a locally compact Hausdorff $\mathcal{G}$-space with a $\mathcal{G}$-equivariant map $\mu : \mathcal{X} \circ Y \to Y$ that is a homeomorphism onto an open subset of Y. The following statements are equivalent to* (3.1.4):

(3.8.1) *if (x_n, y_n) is a net in $\mathcal{X} \times_{s,\mathcal{G}^0,r} Y$ such that the nets (x_n) in $\mathcal{X}$ and $x_n \cdot y_n$ in Y converge, then y_n also converges in Y;*

(3.8.2) *for all $h \in C_0(\mathcal{X}/\mathcal{G})$, the function $\pi_1^* h : Y \to \mathbb{C}$ defined by $\pi_1^*(y) := 0$ if $y \notin \mathcal{X} \cdot Y$ and $(\pi_1^* h)(x \cdot y) := h([x])$ otherwise is continuous.*

Proof. We first assume (3.1.4) and show that (3.8.1) follows. Let (x_n) and (y_n) be as in (3.8.1). Since $\mathcal{X}/\mathcal{G}$ is locally compact, we may assume without loss of generality that all (x_n) belong to a compact subset $K \subseteq \mathcal{X}/\mathcal{G}$. Then (3.1.4) implies that $z_\infty := \lim x_n \cdot y_n$ also belongs to $\mathcal{X} \cdot Y$. So $z_\infty = x \cdot y$ for some $(x, y) \in \mathcal{X} \times_{s,\mathcal{G}^0,r} Y$. Then $\lim [x_n, y_n] = [x, y]$ in $\mathcal{X} \circ Y$ because the multiplication map μ' is a homeomorphism onto its image by Remark 3.3. Since the map $\pi_1 : \mathcal{X} \circ Y \to \mathcal{X}/\mathcal{G}$ is continuous, $[x] = \pi_1(z_\infty) = \lim \pi_1(x_n \cdot y_n) = \lim [x_n]$. This means that $\lim x_n = x \cdot g$ for some $g \in \mathcal{G}$. Replacing (x, y) by $(x \cdot g, g^{-1} \cdot y)$, we may arrange that $\lim x_n = x$. The orbit space projection $\mathcal{X} \times_{s,r} Y \to \mathcal{X} \circ Y$ is a local homeomorphism by Lemma 3.2. Therefore, since (x_n, y_n) converges in the orbit space $\mathcal{X} \circ Y = (\mathcal{X} \times_{s,r} Y)/\mathcal{G}$, there are elements $h_n \in \mathcal{G}$ such that $(x_n \cdot h_n, h_n^{-1} \cdot y_n)$ is defined and converges in $\mathcal{X} \times_{s,r} Y$. Since $\mathcal{G}$ is étale, its right action on $\mathcal{X}$ is free and proper, and (x_n) itself already converges, (h_n) must be eventually constant. Then (y_n) converges, which is what we had to show.

Now assume that (3.1.4) fails. We prove that (3.8.1) fails. By assumption, there is a compact subset $K \subseteq \mathcal{X}/\mathcal{G}$ whose preimage under $\pi_1 : \mathcal{X} \cdot Y \to \mathcal{X}/\mathcal{G}$ is not closed in Y. Since the preimage of K is relatively closed in $\mathcal{X} \cdot Y$, this set must have an accumulation point outside $\mathcal{X} \cdot Y$. So there is a net of the form $x_n \cdot y_n$ with $[x_n] \in K$, $y_n \in Y$, and $s(x_n) = r(y_n)$, which converges towards some point $z \notin \mathcal{X} \cdot Y$. Since K is compact, we may pass to a subnet such that $\lim [x_n]$ exists. Since the orbit space projection $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ is a local homeomorphism, we may find a net $g_n \in \mathcal{G}$ such that $\lim x_n \cdot g_n$ exists in $\mathcal{X}$. Replacing (x_n, y_n) by $(x_n \cdot g_n, g_n^{-1} \cdot y_n)$, we may arrange that already $x = \lim x_n$ exists in $\mathcal{X}$. If the net y_n would converge towards some $y \in Y$, then $\lim x_n \cdot y_n = x \cdot y$ would belong to $\mathcal{X} \cdot Y$, which is a contradiction. So we have found a net (x_n, y_n) in $\mathcal{X} \times_{s,\mathcal{G}^0,r} Y$ such that the nets (x_n) and $(x_n \cdot y_n)$ converge, but (y_n) does not converge.

Next we prove that (3.1.4) implies (3.8.2). Let $h \in C_0(\mathcal{X}/\mathcal{G})$. Since $\pi_1 : \mathcal{X} \cdot Y \to \mathcal{X}/\mathcal{G}$ is continuous, the function $\pi_1^* h$ is continuous on $\mathcal{X} \cdot Y$. By construction, it vanishes outside. So it remains to prove that the set of points with $|(\pi_1^* h)(y)| < \varepsilon$ is open for all $\varepsilon > 0$. The complement of this is the preimage of the set of all $[x] \in \mathcal{X}/\mathcal{G}$ with $|h([x])| \geq \varepsilon$. Since the latter set is compact, the preimage is closed in Y by (3.1.4). So its complement is open in Y, as needed. Conversely, assume that there is a compact subset $K \subseteq \mathcal{X}/\mathcal{G}$ whose preimage in Y is not closed. There is a function $h \in C_0(\mathcal{X}/\mathcal{G})$ with $h|_K \geq 1$. Then the function $\pi_1^* h$ is not continuous at any accumulation point of the preimage of K outside $\mathcal{X} \cdot Y$, and such accumulation points exist by assumption. □

Definition 3.9. A *groupoid model* for $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ is an étale groupoid $\mathcal{M}$ such that for all topological spaces Y, there is a natural bijection between $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-actions and $\mathcal{M}$-actions on Y. Here naturality means that a continuous map between two spaces is $(\mathcal{G}, \mathcal{X})$-equivariant if and only if it is $\mathcal{M}$-equivariant for the associated $\mathcal{M}$-actions.

Example 3.10. Let Y carry an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ with $\mathcal{X} \cdot Y \subsetneq Y$. The subset $\mathcal{X} \cdot Y \subsetneq Y$ is invariant for the $\mathcal{G}$-action, and the multiplication restricts to a map $\mathcal{X} \circ (\mathcal{X} \cdot Y) \to (\mathcal{X} \cdot Y)$. This defines an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ on $\mathcal{X} \cdot Y$; the proof of Lemma 5.1 shows that the restricted action inherits (3.1.4). It is remarkable that the inclusion map $\iota : \mathcal{X} \cdot Y \hookrightarrow Y$ is *not* $(\mathcal{G}, \mathcal{X})$-equivariant because $\iota^{-1}(\mathcal{X} \cdot (\mathcal{X} \cdot Y)) \neq \mathcal{X} \cdot Y$. Therefore the subset $\mathcal{X} \cdot Y \subseteq Y$ cannot be invariant for the induced action of the groupoid model. In fact, this will be visible from the construction of that action below. Any slice of $\mathcal{X}$ will map $Y \setminus \mathcal{X} \cdot Y$ to $\mathcal{X} \cdot Y \setminus \mathcal{X} \cdot \mathcal{X} \cdot Y$, so that its adjoint maps some points of $\mathcal{X} \cdot Y$ into $Y \setminus \mathcal{X} \cdot Y$.

Next we are going to prove that the groupoid model is unique up to isomorphism. Later we will construct such a groupoid and then show that its groupoid C^*-algebra is canonically isomorphic to the Cuntz–Pimsner algebra of $C^*(\mathcal{X})$ relative to the ideal $C^*(\mathcal{G}_\mathcal{R})$.

Definition 3.11. A $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action Ω is *universal* if for any $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action Y, there is a unique $(\mathcal{G}, \mathcal{X})$-equivariant map $Y \to \Omega$, that is, it is a terminal object in the category of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-actions.

Proposition 3.12. *Let $\mathcal{M}$ be a groupoid model for $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-actions. Its object space $\mathcal{M}^0$ carries a canonical $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action, which is universal.*

Proof. Give $\mathcal{M}^0$ its canonical $\mathcal{M}$-action, where an arrow $g \in \mathcal{M}$ maps $s(g)$ to $r(g)$. We claim that $\mathcal{M}^0$ is terminal in the category of $\mathcal{M}$-actions. Let Y be an $\mathcal{M}$-space. The anchor map $s : Y \to \mathcal{M}^0$ is $\mathcal{M}$-equivariant. It is the only $\mathcal{M}$-equivariant map $Y \to \mathcal{M}^0$ because any $\mathcal{M}$-equivariant map $f : Y \to \mathcal{M}^0$ must intertwine the anchor maps to $\mathcal{M}^0$. Thus $\mathcal{M}^0$ is terminal in the category of $\mathcal{M}$-actions. By the definition of a groupoid model, we may turn an $\mathcal{M}$-action on a space Y into a $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action on Y, such that this defines an isomorphism of categories. Therefore it preserves terminal objects. □

Proposition 3.13. *Let $\mathcal{M}$ and $\mathcal{M}'$ be groupoid models for $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-actions. There is a unique groupoid isomorphism $\mathcal{M} \cong \mathcal{M}'$ that is compatible with the equivalence between actions of $\mathcal{M}$, $\mathcal{M}'$, and $(\mathcal{G}, \mathcal{X}, \mathcal{R})$.*

Proof. This may be deduced quickly from [Meyer and Zhu 2015, Propositions 4.23 and 4.24] together with a description of the invariant maps for étale groupoid actions in terms of equivariant maps. We give a more direct argument here. The canonical actions of $\mathcal{M}$ and $\mathcal{M}'$ on their object spaces correspond to actions of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$, which are both universal. Since any two terminal objects are isomorphic with a unique isomorphism, there is a unique $(\mathcal{G}, \mathcal{X})$-equivariant homeomorphism $\mathcal{M}^0 \cong (\mathcal{M}')^0$. To simplify notation, we identify $\mathcal{M}^0$ and $(\mathcal{M}')^0$ through this isomorphism.

An $\mathcal{M}$-action on a space Y contains an anchor map $Y \to \mathcal{M}^0$. Forgetting the rest of the action gives a forgetful functor from the category of $\mathcal{M}$-actions to the category of spaces over $\mathcal{M}^0$. This functor has a left adjoint, namely, the map that sends a space Z with a continuous map $\varrho : Z \to \mathcal{M}^0$ to $\mathcal{M} \times_{s,\mathcal{M}^0,\varrho} Z$ with the obvious left $\mathcal{M}$-action, $\gamma_1 \cdot (\gamma_2, z) := (\gamma_1 \cdot \gamma_2, z)$. Being left adjoint to the forgetful functor means that $\mathcal{M}$-equivariant continuous maps $\psi : \mathcal{M} \times_{s,\mathcal{M}^0,\varrho} Z \to Y$ for an $\mathcal{M}$-space Y are in natural bijection with maps $\varphi : Z \to Y$ that satisfy $r \circ \varphi = \varrho$. Under this natural bijection, ψ corresponds to the map $\psi^\flat : Z \to Y$, $z \mapsto \psi(1_{\varrho(z)}, y)$, whereas $\varphi : Z \to Y$ corresponds to the map $\varphi^\# : \mathcal{M} \times_{s,\mathcal{M}^0,\varrho} Z \to Y$ defined by $\varphi^\#(\gamma, z) := \gamma \cdot \varphi(z)$. We give $\mathcal{M}$ the left multiplication action of $\mathcal{M}$, whose anchor map is r. The adjunct of the unit map $u : \mathcal{M}^0 \to \mathcal{M}$, $x \mapsto 1_x$, is the canonical $\mathcal{M}$-equivariant isomorphism $u^\# : \mathcal{M} \times_{s,\mathcal{M}^0,\mathrm{id}} \mathcal{M}^0 \xrightarrow{\sim} \mathcal{M}$. The identity map $\mathcal{M} \to \mathcal{M}$ and the map $ur : \mathcal{M} \to \mathcal{M}$, $\gamma \mapsto 1_{r(\gamma)}$, are maps over $\mathcal{M}^0$. Their adjuncts are the multiplication map $\mathrm{id}^\# : \mathcal{M} \times_{s,\mathcal{M}^0,r} \mathcal{M} \to \mathcal{M}$, $(\gamma, \eta) \mapsto \gamma \cdot \eta$, and the first coordinate projection $(ur)^\# : \mathcal{M} \times_{s,\mathcal{M}^0,r} \mathcal{M} \to \mathcal{M}$, $(\gamma, \eta) \mapsto \gamma \cdot 1_{r(\eta)} = \gamma$, respectively.

Since $\mathcal{M}$ and $\mathcal{M}'$ are both groupoid models for $(\mathcal{G}, \mathcal{X}, \mathcal{R})$, the categories of $\mathcal{M}$-actions and $\mathcal{M}'$-actions are isomorphic to the category of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-actions. It follows that there is a unique isomorphism between the left adjoint functors $\mathcal{M} \times_{s,\mathcal{M}^0,\varrho} \sqcup$ and $\mathcal{M}' \times_{s,\mathcal{M}^0,\varrho} \sqcup$ that preserves the adjunction $\varphi \mapsto \varphi^\#$. Plugging in $\mathcal{M}^0$, this gives a $(\mathcal{G}, \mathcal{X})$-equivariant homeomorphism $h : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$. Since h is $(\mathcal{G}, \mathcal{X})$-equivariant, it intertwines the range maps on $\mathcal{M}$ and $\mathcal{M}'$. If $x \in \mathcal{M}^0$, then the inclusion map makes $\{x\}$ a space over $\mathcal{M}^0$, and the inclusion $\{x\} \hookrightarrow \mathcal{M}^0$ is a map of spaces over $\mathcal{M}^0$. By naturality, the homeomorphism h intertwines the inclusion of $\mathcal{M} \times_{s,\mathcal{M}^0,x} \{x\} = \{\gamma \in \mathcal{M} : s(\gamma) = x\}$ into $\mathcal{M}$ and the corresponding map for $\mathcal{M}'$. Therefore h intertwines the source maps to $\mathcal{M}^0$.

Next the natural isomorphism between $\mathcal{M} \times_{s,\mathcal{M}^0,\varrho} \sqcup$ and $\mathcal{M}' \times_{s,\mathcal{M}^0,\varrho} \sqcup$ specialises to a $(\mathcal{G}, \mathcal{X})$-equivariant homeomorphism $\mathcal{M} \times_{s,\mathcal{M}^0,r} \mathcal{M} \xrightarrow{\sim} \mathcal{M}' \times_{s,\mathcal{M}^0,r} \mathcal{M}$. Composing with the homeomorphism $\mathcal{M} \cong \mathcal{M}'$, we get a $(\mathcal{G}, \mathcal{X})$-equivariant homeomorphism $h_2 : \mathcal{M} \times_{s,\mathcal{M}^0,r} \mathcal{M} \xrightarrow{\sim} \mathcal{M}' \times_{s,\mathcal{M}^0,r} \mathcal{M}'$. This map must be compatible with the adjunction $\varphi \mapsto \varphi^\#$ for the maps id and ur described above. So h_2 intertwines the multiplication maps and the first coordinate projections. Since the inclusion map $\{\eta\} \hookrightarrow \mathcal{M}$ is a map of spaces over $\mathcal{M}^0$, naturality implies that $h_2(\gamma, \eta)$ has the form $(\gamma', h(\eta))$. Since the first coordinate projection is intertwined, we get $h_2(\gamma, \eta) = (h(\gamma), h(\eta))$. Thus h is a groupoid isomorphism. □

Example 3.14. Let $\mathcal{G} = V$ be a discrete set. So $\mathcal{X}$ is a directed graph $r, s : E \rightrightarrows V$ (see Example 2.1). Let $\mathcal{R} \subseteq V$ be a set of regular vertices. An action of $\mathcal{X}$ on a space Y is given by a continuous map $r : Y \to V$

and a continuous, open map $\mu : E \times_{s,V,r} Y \to Y$, $(e, y) \mapsto e \cdot y$. The conditions in Definition 3.1 simplify to $r(e \cdot y) = r(e)$, μ being injective, $r^{-1}(\mathcal{R}) \subseteq E \cdot Y$, and $F \cdot Y \subseteq Y$ being closed in Y for all finite $F \subseteq E$. It suffices, of course, to require this when F is a singleton. The map $r : Y \to V$ is equivalent to a disjoint union decomposition $Y = \bigsqcup_{v \in V} Y_v$ with $Y_v := r^{-1}(v)$. Then $E \times_{s,V,r} Y \cong \bigsqcup_{e \in E} Y_{s(e)}$. So the injective continuous open map μ is equivalent to a family of homeomorphisms μ_e from $Y_{s(e)}$ to clopen subsets $e \cdot Y_{s(e)} \subseteq Y_{r(e)}$ for all $e \in E$, such that the subsets $e \cdot Y_{s(e)}$ for $e \in E$ are mutually disjoint. In addition, (3.1.3) is equivalent to $Y_v = \bigsqcup_{r(e)=v} e \cdot Y_{s(e)}$ for all $v \in \mathcal{R}$.

4. Encoding actions through inverse semigroups

In this section, we encode an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ on Y in terms of certain partial homeomorphisms of Y. Partial homeomorphisms generate pseudogroups or inverse semigroups, and we then rewrite an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ as an action of a certain inverse semigroup with certain extra properties. Like a group action, an inverse semigroup action on a space has a transformation groupoid. We will later construct the groupoid model as the transformation groupoid for the action of this inverse semigroup induced by the universal action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$. This is why it is crucial to rewrite the action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ as an inverse semigroup action.

Let $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence. A *slice* of $\mathcal{X}$ is an open subset $\mathcal{U} \subseteq \mathcal{X}$ such that both $s : \mathcal{X} \to \mathcal{G}^0$ and the orbit space projection $p : \mathcal{X} \twoheadrightarrow \mathcal{X}/\mathcal{G}$ are injective on $\mathcal{U}$ (see [Antunes et al. 2022, Definition 7.2]). It is shown in [Antunes et al. 2022] that any point in $\mathcal{X}$ has an open neighbourhood that is a slice. Let $\mathcal{B}(\mathcal{X})$ be the set of all slices of $\mathcal{X}$ and let

$$\mathcal{B} := \mathcal{B}(\mathcal{G}) \sqcup \mathcal{B}(\mathcal{X}).$$

If $\mathcal{X} = \mathcal{G}$ is the arrow space of a groupoid, then our slices are the usual ones, which are also often called bisections. If $\mathcal{U}$ and $\mathcal{V}$ are slices in composable groupoid correspondences $\mathcal{X}$ and $\mathcal{Y}$, then $\mathcal{U}\mathcal{V} = \{x \cdot y : x \in \mathcal{U},\ y \in \mathcal{V}\}$ is a slice in $\mathcal{X} \circ \mathcal{Y}$ (see [Antunes et al. 2022, Lemma 7.11]). In particular, since $\mathcal{H} \circ \mathcal{X} = \mathcal{X} = \mathcal{X} \circ \mathcal{G}$, the product of a slice in $\mathcal{X}$ with slices in $\mathcal{H}$ and $\mathcal{G}$ is again a slice. If $\mathcal{U}$ and $\mathcal{V}$ are two slices in $\mathcal{X}$, let $\langle \mathcal{U} \mid \mathcal{V} \rangle$ be the set of all $g \in \mathcal{G}$ for which there are $x \in \mathcal{U}$ and $y \in \mathcal{V}$ with $x \cdot g = y$. This is a slice in $\mathcal{G}$ (see [Antunes et al. 2022, Lemma 7.4]). The bracket notation is justified by [Antunes et al. 2022, Proposition 3.5].

Lemma 4.1. *Let $\mathcal{X} : \mathcal{H} \leftarrow \mathcal{G}$ be a groupoid correspondence and let Y be a $\mathcal{G}$-space. Let $\mathcal{U} \subseteq \mathcal{X}$ be a slice. Write $[\gamma, y]$ instead of $p(\gamma, y)$ for the image of $(\gamma, y) \in \mathcal{X} \times_{s,\mathcal{G}^0,r} Y$ in $\mathcal{X} \circ_{\mathcal{G}} Y$. There is a homeomorphism $\mathcal{U}_*$ between the open subsets $r^{-1}(s(\mathcal{U})) \subseteq Y$ and $p(\mathcal{U} \times_{s,\mathcal{G}^0,r} Y) \subseteq \mathcal{X} \circ_{\mathcal{G}} Y$ which maps $y \in Y$ to $[\gamma, y]$ for the unique $\gamma \in \mathcal{U}$ with $s(\gamma) = r(y)$.*

Proof. Since $s|_{\mathcal{U}}$ is a homeomorphism $\mathcal{U} \to s(\mathcal{U}) \subseteq \mathcal{G}^0$, the map $\mathcal{U}_*$ is well defined and continuous. Its image is $p(\mathcal{U} \times_{s,\mathcal{G}^0,r} Y) \subseteq \mathcal{X} \circ_{\mathcal{G}} Y$. Suppose $\mathcal{U}_*(y_1) = \mathcal{U}_*(y_2)$ for $y_1, y_2 \in r^{-1}(s(\mathcal{U}))$. That is, there are $\gamma_i \in \mathcal{U}$ with $s(\gamma_i) = r(y_i)$ for $i = 1, 2$ and $[\gamma_1, y_1] = [\gamma_2, y_2]$ in $\mathcal{X} \circ_{\mathcal{G}} Y$. Then there is $g \in \mathcal{G}$ with $\gamma_2 = \gamma_1 g$

and $y_1 = gy_2$. Then $p(\gamma_1) = p(\gamma_2)$. This implies $\gamma_1 = \gamma_2$ because p is injective on $\mathcal{U}$. Since the right $\mathcal{G}$-action on $\mathcal{X}$ is free, g is a unit and hence $y_1 = y_2$. Thus $\mathcal{U}_*$ is injective.

The quotient map $p : \mathcal{X} \times_{s,\mathcal{G}^0,r} Y \twoheadrightarrow \mathcal{X} \circ_{\mathcal{G}} Y$ is a local homeomorphism by Lemma 3.2. So is the coordinate projection $\mathrm{pr}_2 : \mathcal{X} \times_{s,\mathcal{G}^0,r} Y \twoheadrightarrow Y$ by [Antunes et al. 2022, Lemma 2.9] because $s : \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism. The map $\mathcal{U}_*$ is the composite of the local section $y \mapsto (\gamma, y)$ of pr_2 and the map p. Hence it is a local homeomorphism. Being injective, it is a homeomorphism onto an open subset of $\mathcal{X} \circ_{\mathcal{G}} Y$. □

An action of an étale groupoid induces an action of its slices by partial homeomorphisms (see [Exel 2008]). The following construction generalises this to an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ on Y. A slice $\mathcal{U} \subseteq \mathcal{X}$ induces a partial homeomorphism $\mathcal{U}_*$ from Y to $\mathcal{X} \circ Y$ by Lemma 4.1. Composing with $\mu' : \mathcal{X} \circ Y \to Y$ gives a partial homeomorphism

$$\vartheta(\mathcal{U}) := \mu' \circ \mathcal{U}_* : Y \supseteq r^{-1}(s(\mathcal{U})) \to Y;$$

it maps y to $\gamma \cdot y$ for the unique $\gamma \in \mathcal{U}$ with $s(\gamma) = r(y)$. We also view $\vartheta(\mathcal{U})$ as a partial homeomorphism of Y. Similarly, the $\mathcal{G}$-action induces a homeomorphism $\mathcal{G} \circ Y \to Y$, and so a slice $\mathcal{U}$ in $\mathcal{G}$ induces a partial homeomorphism of Y as well.

Lemma 4.2. *Define $\vartheta(\mathcal{U})$ for slices of $\mathcal{G}$ and $\mathcal{X}$ as above, and let $\vartheta(\mathcal{U})^*$ be the partial inverse of $\vartheta(\mathcal{U})$. Then $\vartheta(\mathcal{U}\mathcal{V}) = \vartheta(\mathcal{U})\vartheta(\mathcal{V})$ for all $\mathcal{U}, \mathcal{V} \in \mathcal{B}$ with $\mathcal{U} \in \mathcal{B}(\mathcal{G})$ or $\mathcal{V} \in \mathcal{B}(\mathcal{G})$, and $\vartheta(\mathcal{U}_1)^*\vartheta(\mathcal{U}_2) = \vartheta(\langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle)$ if $\mathcal{U}_1$ and $\mathcal{U}_2$ are slices in $\mathcal{X}$.*

Proof. We prove the relation $\vartheta(\mathcal{U}_1)^*\vartheta(\mathcal{U}_2) = \vartheta(\langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle)$. The relation $\vartheta(\mathcal{U}\mathcal{V}) = \vartheta(\mathcal{U})\vartheta(\mathcal{V})$ for $\mathcal{U}, \mathcal{V} \in \mathcal{B}$ with $\mathcal{U} \in \mathcal{B}(\mathcal{G})$ or $\mathcal{V} \in \mathcal{B}(\mathcal{G})$ follows similarly from (3.1.1) and (3.1.2). Recall that $\vartheta(\mathcal{U}_i)$ for $i = 1, 2$ is the partial homeomorphism that maps $y_i \in Y$ with $r(y_i) \in s(\mathcal{U}_i)$ to $\gamma_i \cdot y_i$ for the unique $\gamma_i \in \mathcal{U}_i$ with $s(\gamma_i) = r(y_i)$. Thus the composite partial homeomorphism $\vartheta(\mathcal{U}_1)^*\vartheta(\mathcal{U}_2)$ maps y_2 to y_1 if and only if $\gamma_1 \cdot y_1 = \gamma_2 \cdot y_2$. By (3.1.2), this happens if and only if there is $g \in \mathcal{G}$ with $\gamma_1 g = \gamma_2$ and $gy_2 = y_1$. If such a g exists, then it is unique and belongs to $\vartheta(\langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle)$, and we get $y_1 = \vartheta(\langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle)y_2$. Thus $\vartheta(\mathcal{U}_1)^*\vartheta(\mathcal{U}_2) = \vartheta(\langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle)$. □

Lemma 4.3. *Let $r_Y : Y \to \mathcal{G}^0$ be the anchor map and let $\pi_1 : \mathcal{X} \cdot Y \to \mathcal{X}/\mathcal{G}$ be as in Lemma 3.2. If $\mathcal{U} \in \mathcal{B}(\mathcal{G})$, then $\vartheta(\mathcal{U})$ is a homeomorphism from $r_Y^{-1}(s(\mathcal{U})) \subseteq Y$ to $r_Y^{-1}(r(\mathcal{U})) \subseteq Y$. If $\mathcal{U} \in \mathcal{B}(\mathcal{X})$, then $\vartheta(\mathcal{U})$ is a homeomorphism from $r_Y^{-1}(s(\mathcal{U})) \subseteq Y$ to $\pi_1^{-1}(p(\mathcal{U})) \subseteq Y$.*

Proof. We prove the claim for $\mathcal{U} \in \mathcal{B}(\mathcal{X})$. The claims for the source and range of $\vartheta(\mathcal{U})$ for $\mathcal{U} \in \mathcal{B}(\mathcal{G})$ are proven similarly. By construction, the image of $\vartheta(\mathcal{U})$ is the set of all $x \cdot y$ for $x \in \mathcal{X}$ and $y \in Y$ with $s(x) = r(y)$ and $x \in \mathcal{U}$. Since $x \cdot y = (x \cdot g) \cdot (g^{-1} \cdot y)$, we may replace the condition $x \in \mathcal{U}$ by $p(x) \in p(\mathcal{U})$. So the range of $\vartheta(\mathcal{U})$ is equal to $\pi_1^{-1}(p(\mathcal{U})) \subseteq \mathcal{X} \cdot Y$. □

For a space Y, let $I(Y)$ denote the inverse semigroup of all partial homeomorphisms on Y.

Lemma 4.4. *Let Y be a space and let $\vartheta : \mathcal{B} \to I(Y)$ be a map. It comes from a $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action on Y if and only if*

(4.4.1) *$\vartheta(\mathcal{U}\mathcal{V}) = \vartheta(\mathcal{U})\vartheta(\mathcal{V})$ if $\mathcal{U}, \mathcal{V} \in \mathcal{B}$ and $\mathcal{U} \in \mathcal{B}(\mathcal{G})$ or $\mathcal{V} \in \mathcal{B}(\mathcal{G})$;*

(4.4.2) $\vartheta(\mathcal{U}_1)^*\vartheta(\mathcal{U}_2) = \vartheta(\langle\mathcal{U}_1 \mid \mathcal{U}_2\rangle)$ *for all* $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{B}(\mathcal{X})$;

(4.4.3) $\vartheta(\mathcal{G}^0)$ *is the identity map on all of* Y *and* $\vartheta(\varnothing)$ *is the empty partial map*;

(4.4.4) *the restriction of* ϑ *to open subsets of* $\mathcal{G}^0$ *commutes with arbitrary unions*;

(4.4.5) *the images of* $\vartheta(\mathcal{U})$ *for* $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ *cover the domain of* $\vartheta(\mathcal{R})$;

(4.4.6) *if* $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ *is precompact in* $\mathcal{X}$, *then the closure of the codomain of* $\vartheta(\mathcal{U})$ *is contained in the union of the codomains of* $\vartheta(W)$ *for* $W \in \mathcal{B}(\mathcal{X})$.

The corresponding $(\mathcal{G}, \mathcal{X}, \mathcal{R})$*-action on* Y *is unique if it exists.*

Proof. Assume first that the maps ϑ come from an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$. Then (4.4.1) and (4.4.2) follow from Lemma 4.2. If $W \subseteq \mathcal{G}^0$ is open, then $\vartheta(W)$ acts on Y by the identity map on $r^{-1}(W)$. This implies the statements in (4.4.3) and (4.4.4). Lemma 4.3 implies that the ranges of $\vartheta(\mathcal{U})$ for $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ cover $r^{-1}(\mathcal{R})$ if and only if $r^{-1}(\mathcal{R})$ is contained in $\mathcal{X} \cdot Y$. So (4.4.5) holds.

We claim that (4.4.6) is equivalent to (3.1.4). In one direction, the closure of the image of $\mathcal{U}$ in $\mathcal{X}/\mathcal{G}$ is compact, and so $\vartheta(\mathcal{U}) \cdot Y = \mathcal{U} \cdot Y$ is contained in a relatively closed subset in $\mathcal{X} \cdot Y \subseteq Y$ by (3.1.4). In the other direction, if $K \subseteq \mathcal{X}/\mathcal{G}$, there are finitely many slices $\mathcal{U}_i$ in $\mathcal{X}$ whose images in $\mathcal{X}/\mathcal{G}$ cover K. By (4.4.6), the codomains of $\mathcal{U}_i$ have closure contained in $\mathcal{X} \cdot Y$, and then this remains so for their union. This implies that the preimage of K in $\mathcal{X} \cdot Y \subseteq Y$ is relatively closed as required in (3.1.4).

So far, we have seen that all the conditions in the lemma are necessary for ϑ to come from an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$. Next, we prove that they are also sufficient. Any open subset of $\mathcal{G}^0$ is an idempotent slice in $\mathcal{G}$. The condition (4.4.1) for such slices implies that their image is again idempotent, so that $\vartheta(W)$ for $W \subseteq \mathcal{G}^0$ is the identity map on some open subset of Y. So ϑ restricts to a map from the lattice of open subsets of $\mathcal{G}^0$ to the lattice of open subsets of Y. The conditions (4.4.1), (4.4.3), and (4.4.4) for subsets of $\mathcal{G}^0$ say that the latter map commutes with finite intersections and arbitrary unions. Since the two spaces involved are locally compact, they are "sober". Therefore, by [Meyer and Nest 2009, Lemma 2.25], any map between their lattices of open subsets that commutes with arbitrary unions and finite intersections is of the form $W \mapsto r^{-1}(W)$ for a continuous map $r : Y \to \mathcal{G}^0$. That is, $\vartheta(W)$ is the identity map on $r^{-1}(W)$ for any open subset $W \subseteq \mathcal{G}^0$.

Let $\mathcal{U} \in \mathcal{B}(\mathcal{X})$, $x \in \mathcal{U}$, and $y \in Y$ satisfy $s(x) = r(y)$. The construction of r says that y is in the domain of $\mathcal{U}^*\mathcal{U} = s(\mathcal{U})$. So $\vartheta(\mathcal{U})$ is defined at y. We want to put $x \cdot y := \vartheta(\mathcal{U})(y)$. This is the only possibility for a $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action that induces the map $\vartheta(\mathcal{U})$. We must show that this formula for all (x, y) gives a well-defined map $\mathcal{X} \times_{s,\mathcal{G}^0,r} Y \to Y$. If $\mathcal{V} \in \mathcal{B}(\mathcal{X})$ is another slice with $x \in \mathcal{V}$, then $x \in \mathcal{U} \cap \mathcal{V}$. Let $W = s(\mathcal{U} \cap \mathcal{V}) \subseteq \mathcal{G}^0$, viewed as a slice of $\mathcal{G}$. Then $\mathcal{U} \cdot W = \mathcal{U} \cap \mathcal{V} = \mathcal{V} \cdot W$ and

$$\vartheta(\mathcal{U})y = \vartheta(\mathcal{U})\vartheta(W)y = \vartheta(\mathcal{V})\vartheta(W)y = \vartheta(\mathcal{V})y.$$

Thus $x \cdot y$ is well defined. The resulting map $\mathcal{X} \times_{s,r} Y \to Y$ is continuous and open because the maps $\vartheta(\mathcal{U})$ are partial homeomorphisms. Its image contains the domain of $\vartheta(\mathcal{R})$ by (4.4.5); this is $r^{-1}(\mathcal{R})$ by the construction of r.

A similar construction for slices in $\mathcal{G}$ gives a continuous multiplication map $\mathcal{G} \times_{s,\mathcal{G}^0,r} Y \to Y$. The condition (4.4.3) implies $1_{r(y)} \cdot y = y$ for all $y \in Y$, and (4.4.1) for two slices in $\mathcal{G}$ implies that this is a $\mathcal{G}$-action. The multiplicativity of ϑ for slices in $\mathcal{G}^0$ and $\mathcal{X}$ implies $r(x \cdot y) = r(x)$ for all $x \in \mathcal{U}$, $y \in Y$ with $s(x) = r(y)$ because both sides are in the domains of the same $\vartheta(W)$ for $W \subseteq \mathcal{G}^0$ open. When applied to slices in $\mathcal{G}$ and $\mathcal{X}$, it implies $g \cdot (x \cdot y) = (g \cdot x) \cdot y$ for all $g \in \mathcal{G}$ with $s(g) = r(x)$. When applied to slices in $\mathcal{X}$ and $\mathcal{G}$, it implies $(x \cdot g) \cdot (g^{-1} \cdot y) = x \cdot y$ for all $g \in \mathcal{G}$ with $r(g) = s(x)$. Assume now that $x_1 \cdot y_1 = x_2 \cdot y_2$ for $x_1, x_2 \in \mathcal{X}$ and $y_1, y_2 \in Y$. Choose $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{B}(\mathcal{X})$ with $x_j \in \mathcal{U}_j$ for $j = 1, 2$. Then

$$\vartheta(\mathcal{U}_1) y_1 = x_1 \cdot y_1 = x_2 \cdot y_2 = \vartheta(\mathcal{U}_2) y_2,$$

and so $y_1 = \vartheta(\mathcal{U}_1)^* \vartheta(\mathcal{U}_2) y_2$, which is equal to $\vartheta(\langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle) y_2$ by (4.4.5). Equivalently, $y_1 = g \cdot y_2$ for the unique $g \in \langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle$ with $s(g) = r(y_2)$. Now $r(g) = r(y_1) = s(x_1)$ and $g \in \langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle$ imply $x_1 \cdot g = x_2$. So there is $g \in \mathcal{G}$ with $x_2 = x_1 \cdot g$ and $y_1 = g \cdot y_2$, as required in (3.1.2). We have already shown that (4.4.6) is equivalent to (3.1.4). So we get the data and all the conditions for a $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action. □

The next lemma describes $(\mathcal{G}, \mathcal{X})$-equivariant maps through the partial homeomorphisms $\vartheta(\mathcal{U})$ for $\mathcal{U} \in \mathcal{B}$. This is important to characterise the groupoid model.

Lemma 4.5. *Let Y and Y' be $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-actions. A continuous map $\varphi : Y \to Y'$ is $(\mathcal{G}, \mathcal{X})$-equivariant if and only if $\vartheta'(\mathcal{U}) \circ \varphi = \varphi \circ \vartheta(\mathcal{U})$ and $\vartheta'(\mathcal{U})^* \circ \varphi = \varphi \circ \vartheta(\mathcal{U})^*$ as partial maps for all $\mathcal{U} \in \mathcal{B}$.*

Proof. Our construction of the maps $\vartheta(\mathcal{U})$ from an action implies $\vartheta'(\mathcal{U}) \circ \varphi = \varphi \circ \vartheta(\mathcal{U})$ and $\vartheta'(\mathcal{U})^* \circ \varphi = \varphi \circ \vartheta(\mathcal{U})^*$ as partial maps if φ is equivariant. A subtle point here is the equality of the domains. By Lemma 4.3, the domains of $\vartheta'(\mathcal{U}) \circ \varphi$ and $\varphi \circ \vartheta(\mathcal{U})$ are $\varphi^{-1}\big(r_{Y'}^{-1}(s(\mathcal{U}))\big)$ and $r_Y^{-1}(s(\mathcal{U}))$, respectively, and the domains of $\vartheta'(\mathcal{U})^* \circ \varphi$ and $\varphi \circ \vartheta(\mathcal{U})^*$ for $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ are $\varphi^{-1}\big(\pi_1^{-1}(p(\mathcal{U}))\big)$ and $\pi_1^{-1}(p(\mathcal{U}))$, respectively, for the canonical map $\pi_1 : \mathcal{X} \circ Y \to \mathcal{X}/\mathcal{G}$. The equality of the first two domains is equivalent to $r \circ \varphi = r$. The equality of the latter domains for all $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ is proven using $\varphi^{-1}(\mathcal{X} \cdot Y') = \mathcal{X} \cdot Y$. Conversely, if $\vartheta'(\mathcal{U}) \circ \varphi = \varphi \circ \vartheta(\mathcal{U})$ and $\vartheta'(\mathcal{U})^* \circ \varphi = \varphi \circ \vartheta(\mathcal{U})^*$ holds for all bisections $\mathcal{U}$ in $\mathcal{G}$ and $\mathcal{X}$, then the map φ must be equivariant; to prove this, use the construction of r in the proof of Lemma 4.4 and that any element x of $\mathcal{G}$ or $\mathcal{X}$ belongs to a slice $\mathcal{U}$, and $\vartheta(\mathcal{U}) y = x \cdot y$ if $y \in Y$ satisfies $s(x) = r(y)$. □

5. The universal action

In this section, we build the universal action. We first treat the case $\mathcal{R} = \varnothing$. Then we reduce the general case to this one. A crucial ingredient for the construction is the iterated composition of $\mathcal{X}$ with itself. Let $\mathcal{X}_0 := \mathcal{G}$ be the identity groupoid correspondence and let $\mathcal{X}_n := \mathcal{X}_{n-1} \circ \mathcal{X}$; in particular, $\mathcal{X}_1 = \mathcal{X}$. These are all groupoid correspondences (see [Antunes et al. 2022]). So their right $\mathcal{G}$-action is free and proper, so that the orbit spaces $\mathcal{X}_n/\mathcal{G}$ are locally compact Hausdorff. The left $\mathcal{G}$-action on $\mathcal{X}_n$ induces a $\mathcal{G}$-action on $\mathcal{X}_n/\mathcal{G}$. There are canonical $\mathcal{G}$-equivariant maps

$$\pi_n^m : \mathcal{X}_n/\mathcal{G} \to \mathcal{X}_m/\mathcal{G}, \quad [x_1, \dots, x_n] \mapsto [x_1, \dots, x_m],$$

for $m \le n$, which satisfy $\pi^k_m \circ \pi^m_n = \pi^k_n$ for $k \le m \le n$. Unlike in [Meyer 2022], the maps π^m_n are not proper, so that it is not useful to form their projective limit: this space need not be locally compact.

Let Y carry an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$. The multiplication map μ induces iterated multiplication maps

$$\mu_n : \mathcal{X}_n \circ Y \to Y, \quad \mu_n([x_1, \dots, x_n, y]) := x_1 \cdot \mu_{n-1}([x_2, \dots, x_n, y])$$

for $n \ge 1$. This is just μ for $n = 1$. We let $\mu_0 : \mathcal{X}_0 \circ Y = \mathcal{G} \circ Y \to Y$ be the $\mathcal{G}$-action on Y. The maps μ_n for $n \in \mathbb{N}$ are homeomorphisms onto open subsets in Y.

Lemma 5.1. *For $n \in \mathbb{N}$ and $h \in C_0(\mathcal{X}_n/\mathcal{G})$, define a function $\pi^*_n h : Y \to \mathbb{C}$ by $(\pi^*_n h)(y) := 0$ for $y \notin \mathcal{X}_n \cdot Y$, and $(\pi^*_n h)(\omega \cdot y) := h([\omega])$ for all $\omega \in \mathcal{X}_n$, $y \in Y$ with $s(\omega) = r(y)$. These functions are continuous.*

Proof. For $n = 0$, the statement is trivial. For $n = 1$, the statement is equivalent to (3.1.4) by Lemma 3.8. So it remains to prove that the maps μ_n inherit this extra property from μ. We show this by induction on n, using the equivalent characterisation in (3.8.1). So take a net (x_n, ω_n, y_n) in $\mathcal{X} \times_{s,r} \mathcal{X}_{n-1} \times_{s,r} Y$ such that the net $[x_n, \omega_n]$ converges in $\mathcal{X}_n$ and $x_n \cdot \omega_n \cdot y_n$ converges in Y. Then there are arrows $g'_n \in \mathcal{G}$ such that $(x_n \cdot g, g^{-1} \cdot \omega_n)$ converges in $\mathcal{X} \times_{s,r} \mathcal{X}_{n-1}$. We may change our net to arrange, without loss of generality, that already (x_n, ω_n) converges in $\mathcal{X} \times_{s,r} \mathcal{X}_{n-1}$. In particular, ω_n converges in $\mathcal{X}_{n-1}$. Since $x_n \cdot \omega_n \cdot y_n$ converges in Y, (3.8.1) implies that $\omega_n \cdot y_n$ converges in Y. Then the induction assumption about μ_{n-1} implies that y_n converges. □

So an action on Y induces *-homomorphisms from $C_0(\mathcal{X}_n/\mathcal{G})$ to $C_b(Y)$ for all $n \in \mathbb{N}$. We now define a C^*-algebra that allows us to combine these maps for different n.

Definition 5.2. For $0 \le m \le n$, let $A_{[m,n]} := \bigoplus_{k=m}^n C_0(\mathcal{X}_k/\mathcal{G})$ with the pointwise *-operation $(f_k)^* := (f_k^*)$ and the multiplication where $(f_k) \cdot (h_l)$ is the family with j-th entry

$$\sum_{k=0}^{j} (\pi^k_j)^*(f_k) \cdot h_j + f_j \cdot (\pi^k_j)^*(h_k).$$

Lemma 5.3. *There is a unique norm that makes $A_{[m,n]}$ into a commutative C^*-algebra. If $m \le k \le n$, then $A_{[m,k]} \subseteq A_{[m,n]}$ is a C^*-subalgebra and $A_{[k,n]} \subseteq A_{[m,n]}$ is a closed ideal, with quotient isomorphic to $A_{[m,k-1]}$.*

Proof. Routine computations show that $A_{[m,n]}$ is a commutative *-algebra, that $A_{[m,k]} \subseteq A_{[m,n]}$ is a *-subalgebra, and that $A_{[k,n]}$ is a *-ideal in it such that $A_{[m,n]}/A_{[k,n]} \cong A_{[m,k-1]}$. We prove by induction on $n - m$ that $A_{[m,n]}$ is a C^*-algebra in a unique C^*-norm. If $n - m = 0$, then $A_{[m,n]} = C_0(\mathcal{X}_n/\mathcal{G})$ is indeed a C^*-algebra and the assertion follows. If the assertion is known for some $n - m$, then in the extension of *-algebras

$$C_0(\mathcal{X}_n/\mathcal{G}) \rightarrowtail A_{[m,n]} \twoheadrightarrow A_{[m,n-1]}, \tag{5.4}$$

both the kernel and quotient are known to be C^*-algebras in a unique C^*-norm. The inclusion $A_{[m,n-1]} \to A_{[m,n]}$ induces a map from $A_{[m,n-1]}$ to the multiplier algebra of $C_0(\mathcal{X}_n/\mathcal{G})$. Then the obvious map provides an injective *-homomorphism $A_{[m,n]} \to \mathcal{M}(C_0(\mathcal{X}_n/\mathcal{G})) \times A_{[m,n-1]}$. This induces a C^*-norm on $A_{[m,n]}$.

Since it generates the product topology on our direct sum, $A_{[m,n]}$ is complete in this norm, and so it carries only one C^*-norm. □

Let $\Omega_{[m,n]}$ be the spectrum of $A_{[m,n]}$. To understand its topology, we recall a general topological construction of Whyburn:

Lemma 5.5. *Let X and Y be locally compact Hausdorff spaces and let $f : X \to Y$ be a continuous map. There is a unique locally compact Hausdorff topology on $Z := X \sqcup Y$ with the following properties*:

- *the inclusion $X \to Z$ is a homeomorphism onto an open subset*;
- *the inclusion $Y \to Z$ is a homeomorphism onto a closed subset*;
- *the map $(f, \mathrm{id}_Y) : X \sqcup Y \to Y$ is continuous and proper.*

A subset $V \subseteq Z$ is open if and only if $V \cap X$ is open in X, $V \cap Y$ is open in Y, and for every compact subset $K \subseteq V \cap Y$, $f^{-1}(K) \setminus (V \cap X) \subseteq X$ is compact.

Proof. It is shown by Whyburn [1953] that the topology in the statement has the three properties in the lemma. Now give Z any topology with these properties. Being locally compact and Hausdorff, Z is determined by the commutative C^*-algebra $C_0(Z)$. The latter contains $C_0(X)$ as an ideal because X is homeomorphic to an open subset. Restriction to Y identifies $C_0(Z)/C_0(X)$ with $C_0(Y)$ because Y is homeomorphic to $Z \setminus X$. Since the map (f, id_Y) is proper, it induces a *-homomorphism $\sigma : C_0(Y) \to C_0(Z)$. That map is a section for the quotient map. Therefore there is a *-isomorphism

$$C_0(Z) \xrightarrow{\sim} \{(\xi, \eta) \in C_b(X) \oplus C_0(Y) : \xi - \eta \circ f \in C_0(X)\}, \quad \zeta \mapsto (\zeta|_X, \zeta|_Y).$$

Since its target only depends on f, the topology on its spectrum Z is unique. □

If Y is a one-point space, then the map f is constant and the space Z in Lemma 5.5 is the one-point compactification of X. In general, we think of X as a bundle of spaces over Y with fibres $f^{-1}(y)$ for $y \in Y$. When we apply the one-point compactification to each fibre, we add one point to each fibre, getting $X \sqcup Y$ as a set. The compactness of the fibres translates into the projection $Z \to Y$ being proper. Thus the lemma describes a unique topology that makes Z into a fibrewise one-point compactification. This is why we call Z the *fibrewise one-point compactification for f*. Whyburn [1953] and Antunes [2023] call it the unified space instead.

Lemma 5.6. *The fibrewise one-point compactification construction is compatible with pullbacks. More precisely*, *take a diagram*

$$\begin{array}{ccccc} & & & & A \\ & & & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y & \xrightarrow{\pi} & Y' \end{array}$$

Let Z be the fibrewise one-point compactification of f and let $\tilde{f} : Z \to Y$ be the canonical map. Then the pullback of $\pi \circ \tilde{f}$ and g is homeomorphic to the fibrewise one-point compactification of the pullback map $\mathrm{id}_A \times_{Y'} f : A \times_{Y'} X \to A \times_{Y'} Y$.

Proof. Pullbacks of proper maps are again proper (see [Antunes et al. 2022, Lemma 5.1]). Therefore the topology of the pullback of $\mathrm{id}_A \times_{Y'} f$ is a locally compact topology with the three properties required in Lemma 5.5. □

Now we show how the spectra $\Omega_{[m,n]}$ may be built through fibrewise one-point compactifications. The inclusion of the ideal $C_0(\mathcal{X}_n/\mathcal{G})$ in $A_{[m,n]}$ identifies $\mathcal{X}_n/\mathcal{G}$ with an open subset of $\Omega_{[m,n]}$. The complement is naturally homeomorphic to $\Omega_{[m,n-1]}$. The formula for the multiplication in $A_{[m,n]}$ shows that the inclusion *-homomorphism $A_{[m,n-1]} \hookrightarrow A_{[m,n]}$ is nondegenerate. Therefore it is induced by a continuous, proper map

$$\pi_{[m,n]}^{[m,n-1]} : \Omega_{[m,n]} \twoheadrightarrow \Omega_{[m,n-1]}. \tag{5.7}$$

This is a retraction onto the subset $\Omega_{[m,n-1]}$. Comparing with the proof of Lemma 5.5, we see that $\Omega_{[m,n]}$ is the fibrewise one-point compactification for the continuous map

$$\mathcal{X}_n/\mathcal{G} \hookrightarrow \Omega_{[m,n]} \xrightarrow{\pi_{[m,n]}^{[m,n-1]}} \Omega_{[m,n-1]}. \tag{5.8}$$

Since $\Omega_{[m,m]} = \mathcal{X}_m/\mathcal{G}$, we may build the spaces $\Omega_{[m,n]}$ from the spaces $\mathcal{X}_k/\mathcal{G}$ by repeated fibrewise one-point compactification. As a result, there is a (discontinuous) bijection between $\Omega_{[m,n]}$ and $\bigsqcup_{k=m}^{n} \mathcal{X}_k/\mathcal{G}$. Here $x \in \mathcal{X}_k/\mathcal{G}$ corresponds to the character on $A_{[m,n]}$ that vanishes on $C_0(\mathcal{X}_m/\mathcal{G})$ for $m > k$ and is evaluation at $\pi_k^m(x)$ for $m \le k$.

Definition 5.9. Let $A_{[m,\infty)}$ be the colimit C^*-algebra of the inductive system of C^*-algebras $A_{[m,n]}$ for the obvious inclusion maps $A_{[m,n-1]} \to A_{[m,n]}$. Let $\Omega_{[m,\infty)}$ be the spectrum of $A_{[m,\infty)}$.

The spectrum of the inductive limit of commutative C^*-algebras $\varinjlim A_{[m,n]}$ is the projective limit of the spectra of $A_{[m,n]}$. So $\Omega_{[m,\infty)}$ is the projective limit of the projective system of maps in (5.7). Lemma 5.3 implies that $A_{[m,\infty)}$ for $m \in \mathbb{N}$ is isomorphic to an ideal in $A_{[0,\infty)}$ and that these ideals form a decreasing chain, with

$$A_{[m,\infty)}/A_{[m+1,\infty)} \cong C_0(\mathcal{X}_m/\mathcal{G}).$$

Lemma 5.10. *Let Y be a topological space with a $(\mathcal{G}, \mathcal{X}, \varnothing)$-action. The maps $\pi_m^* : C_0(\mathcal{X}_m/\mathcal{G}) \to C_b(Y)$ for $m \in \mathbb{N}$ constructed in Lemma 5.1 induce a *-homomorphism $C_0(\Omega_{[0,\infty)}) \to C_b(Y)$, which is nondegenerate in the sense that it comes from a continuous map $\varrho : Y \to \Omega_{[0,\infty)}$.*

Proof. The *-homomorphism $\pi_m^* : C_0(\mathcal{X}_m/\mathcal{G}) \to C_b(Y)$ was built by pulling back functions along the projection $\pi_m : \mathcal{X}_m \cdot Y \to \mathcal{X}_m/\mathcal{G}$ and then extending by 0. Let $0 \le m \le n$. Using that $\pi_n^m \circ \pi_n = \pi_m$ on $\mathcal{X}_n \cdot Y \subseteq \mathcal{X}_m \cdot Y$, it follows that the resulting map $A_{[0,n]} \to C_b(Y)$, $(f_k) \mapsto \sum_{k=0}^{n} \pi_k^*(f_k)$, is a *-homomorphism. These maps induce a *-homomorphism $\varrho^* : C_0(\Omega_{[0,\infty)}) = A_{[0,\infty)} \to C_b(Y)$. Each $y \in Y$ defines a nonzero character ev_y on $C_b(Y)$. The character $\mathrm{ev}_y \circ \varrho^*$ on $C_0(\Omega_{[0,\infty)})$ is also nonzero because it is even nonzero on $C_0(\mathcal{X}_0/\mathcal{G}) = C_0(\mathcal{G}^0) \subseteq C_0(\Omega_{[0,\infty)})$. Then it must be the evaluation character at some point $\varrho(y) \in \Omega_{[0,\infty)}$. The map $\varrho : Y \to \Omega_{[0,\infty)}$ defined in this way is continuous because its composite with all C_0-functions on $\Omega_{[0,\infty)}$ is continuous. □

Each space $\mathcal{X}_n/\mathcal{G}$ carries a $\mathcal{G}$-action and the maps $\pi_n^m : \mathcal{X}_n/\mathcal{G} \to \mathcal{X}_m/\mathcal{G}$ are $\mathcal{G}$-equivariant. Recall that $\Omega_{[m,n]} = \bigsqcup_{k=m}^n \mathcal{X}_k/\mathcal{G}$ as a set. This carries an obvious $\mathcal{G}$-action. This is inherited by the projective limit sets $\Omega_{[m,\infty)}$ for $m \in \mathbb{N}$. As a set,

$$\Omega_{[m,n]} \cong \bigsqcup_{k=m}^{n} \mathcal{X}_k/\mathcal{G} \cong \bigsqcup_{k=0}^{n-m} \mathcal{X}_m \circ (\mathcal{X}_k/\mathcal{G}) \cong \mathcal{X}_m \circ \bigsqcup_{k=0}^{n-m} (\mathcal{X}_k/\mathcal{G}) \cong \mathcal{X}_m \circ \Omega_{[0,n-m]}. \tag{5.11}$$

Passing to the projective limit sets, these bijections induce a canonical bijection

$$\Omega_{[m,\infty)} \cong \mathcal{X}_m \circ \Omega_{[0,\infty]}. \tag{5.12}$$

Lemma 5.13. *The canonical $\mathcal{G}$-action on $\Omega_{[m,n]}$ is continuous.*

Proof. The anchor map $\Omega_{[m,n]} \to \mathcal{G}^0$ factors through the continuous maps $\Omega_{[m,n]} \to \Omega_{[m,m]} = \mathcal{X}_m/\mathcal{G} \to \mathcal{G}^0$, so that it is continuous. It remains to prove the continuity of a map $\mathcal{G} \times_{s,\mathcal{G}^0,r} \Omega_{[m,n]} \to \Omega_{[m,n]}$. This is clear for $n = m$, and we prove this by induction over $n - m$. We have seen above that $\Omega_{[m,n]}$ is the fibrewise one-point compactification of a certain continuous $\mathcal{G}$-equivariant map $\mathcal{X}_n/\mathcal{G} \to \Omega_{[m,n-1]}$. By Lemma 5.6, $\mathcal{G} \times_{s,\mathcal{G}^0,r} \Omega_{[m,n]}$ is homeomorphic to the fibrewise one-point compactification of the induced map $\mathcal{G} \times_{s,\mathcal{G}^0,r} \mathcal{X}_n/\mathcal{G} \to \mathcal{G} \times_{s,\mathcal{G}^0,r} \Omega_{[m,n-1]}$. Since the fibrewise one-point compactification is natural for commuting squares of maps, this implies the desired continuity. □

Lemma 5.14. *The bijections $\Omega_{[m,n]} \cong \mathcal{X}_m \circ \Omega_{[0,n-m]}$ in* (5.11) *are homeomorphisms.*

Proof. Since $\mathcal{X}_m = \mathcal{X} \circ \mathcal{X}_{m-1}$, an induction on m shows that it suffices to prove the claim for $m = 0$ and $m = 1$, and the case $m = 0$ follows from Lemma 5.13. So let $m = 1$. We argue by induction on $n - m$. The case $n - m = 0$ reduces to the true statement $\mathcal{X} \circ \mathcal{X}_m/\mathcal{G} \cong \mathcal{X}_{m+1}/\mathcal{G}$. Assume the statement for all shorter intervals than $[m, n]$, and in particular for $[m, n-1]$. The split exact sequence of C^*-algebras in (5.4) and the proof of Lemma 5.5 show that $\Omega_{[m,n]}$ is the fibrewise one-point compactification of the map $\mathcal{X}_n/\mathcal{G} \to \Omega_{[m,n-1]}$. By the induction assumption, the canonical maps are homeomorphisms $\mathcal{X}_n \cong \mathcal{X} \circ \mathcal{X}_{n-1}/\mathcal{G}$ and $\Omega_{[m,n-1]} \cong \mathcal{X} \circ \Omega_{[m-1,n-2]}$. Now $\mathcal{X} \circ \Omega_{[m-1,n-1]}$ is another locally compact Hausdorff space with the properties in Lemma 5.5 for the canonical map $\mathcal{X}_n/\mathcal{G} \to \Omega_{[m,n-1]}$. Since the topology in Lemma 5.5 is unique, it follows that the canonical map in (5.11) is an isomorphism, completing the induction step. □

Proposition 5.15. *Let (Y_n) be a projective system of $\mathcal{G}$-spaces. Then the induced action on $\varprojlim Y_n$ is continuous and the canonical map $\mathcal{X} \circ \varprojlim Y_n \to \varprojlim \mathcal{X} \circ Y_n$ is a homeomorphism.*

Proof. The canonical map exists and is continuous by the universal property of the limit and the naturality of $Y \mapsto \mathcal{X} \circ Y$. It is also easy to see that it is bijective. It is straightforward to check that the induced $\mathcal{G}$-action on the projective limit remains continuous. If $\mathcal{U} \in \mathcal{B}(\mathcal{X})$, then Lemma 4.1 provides a homeomorphism from $r^{-1}(s(\mathcal{U})) \subseteq Y$ to an open subset of $\mathcal{X} \circ Y$ for any $\mathcal{G}$-space Y. Applying this to the induced $\mathcal{G}$-action on $\varprojlim Y_n$ gives a homeomorphism from the preimage of $s(\mathcal{U})$ in $\varprojlim Y_n$ onto an open subset of $\mathcal{X} \circ \varprojlim Y_n$. Applying this to each Y_n for $n \in \mathbb{N}$ and using the naturality of the projective limit, we get that this map is also a homeomorphism onto an open subset of $\varprojlim \mathcal{X} \circ Y_n$. Therefore the canonical map $\mathcal{X} \circ \varprojlim Y_n \to \varprojlim \mathcal{X} \circ Y_n$ is a homeomorphism on certain open subsets of the two spaces. These open subsets cover the whole sets because the slices $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ cover $\mathcal{X}$. Therefore the map is a homeomorphism everywhere. □

Lemma 5.16. *The bijections in* (5.12) *are homeomorphisms.*

Proof. Lemma 5.13 allows us to apply Proposition 5.15, and then the claim follows from Lemma 5.14. □

Proposition 5.17. *The $\mathcal{G}$-action on $\Omega_{[0,\infty)} \cong \varprojlim \Omega_{[0,n]}$ and the homeomorphism*

$$\mathcal{X} \circ \Omega_{[0,\infty)} \xrightarrow{\sim} \Omega_{[1,\infty)} \subseteq \Omega_{[0,\infty)}$$

are an action of $(\mathcal{G}, \mathcal{X}, \emptyset)$ on $\Omega_{[0,\infty)}$. Its anchor map $\Omega_{[0,\infty)} \to \mathcal{G}^0$ is proper and surjective. This action of $(\mathcal{G}, \mathcal{X}, \emptyset)$ is universal.

Proof. Recall that $\Omega_{[1,\infty)}$ is an open subset of $\Omega_{[0,\infty)}$, being the spectrum of the ideal $A_{[1,\infty)}$ in $A_{[0,\infty)}$. The conditions (3.1.1) and (3.1.2) are easily seen to be satisfied, and (3.1.3) is empty for $\mathcal{R} = \emptyset$. The anchor map is proper (and surjective) because it is the map on spectra induced by the inclusion $C_0(\mathcal{G}^0) = A_{[0,0]} \hookrightarrow A_{[0,\infty)}$. This implies (3.1.4) by Lemma 3.7.

To prove the universality, take another action of $(\mathcal{G}, \mathcal{X}, \emptyset)$ on a locally compact space Y. Lemma 5.10 provides a continuous map $\varrho : Y \to \Omega_{[0,\infty)}$. The canonical $\mathcal{G}$-action on $\Omega_{[0,\infty)}$ makes this map $\mathcal{G}$-equivariant. In addition, $y \in Y$ belongs to $\mathcal{X} \cdot Y$ if and only if there is $h \in C_0(\mathcal{X}/\mathcal{G})$ such that $\pi_1^*(h)(y) \neq 0$. This implies that $\varrho^{-1}(\mathcal{X} \cdot \Omega_{[0,\infty)}) = \varrho^{-1}(\Omega_{[1,\infty)}) = \mathcal{X} \cdot Y$. Our canonical constructions ensure that $\varrho(x \cdot y) = x \cdot \varrho(y)$ for all $x \in \mathcal{X}$ and $y \in Y$ with $s(x) = r(y)$. So ϱ is indeed $(\mathcal{G}, \mathcal{X})$-equivariant.

Now let $\varphi : Y \to \Omega_{[0,\infty)}$ be any $(\mathcal{G}, \mathcal{X})$-equivariant map. We claim that $\varphi = \varrho$. We prove that the projections of φ and ϱ to $\Omega_{[0,n]}$ are the same for all $n \in \mathbb{N}$. This proves the claim. The claim for $n = 0$ concerns the projections to $\Omega_{[0,0]} = \mathcal{G}^0$. These are the anchor maps, and $r_{\Omega_{[0,\infty)}} \circ \varphi = r_Y = r_{\Omega_{[0,\infty)}} \circ \varrho$ implies the claim. Assume the claim has been shown for $n-1$. We are going to prove it for n. Let $y \in \mathcal{X} \cdot Y$ and write $y = x \cdot y'$ for $x \in \mathcal{X}$ and $y' \in Y$. Then $\varphi(y) = x \cdot \varphi(y')$ and $\varrho(y) = x \cdot \varrho(y')$. Now if $\omega \in \Omega_{[0,\infty)}$, then the image of $x \cdot \omega$ in $\Omega_{[0,n]}$ depends only on the image of ω in $\Omega_{[0,n-1]}$. So the induction assumption implies $\varphi(y) = \varrho(y)$ for all $y \in \mathcal{X} \cdot Y$. Since $\varphi^{-1}(\mathcal{X} \cdot \Omega_{[0,\infty)}) = \mathcal{X} \cdot Y$ and $\varrho^{-1}(\mathcal{X} \cdot \Omega_{[0,\infty)}) = \mathcal{X} \cdot Y$, both maps must send any $y \in Y \setminus \mathcal{X} \cdot Y$ to a point in $\Omega_{[0,\infty)} \setminus \Omega_{[1,\infty)} = \mathcal{G}^0$. Since both maps are compatible with the anchor map to $\mathcal{G}^0$, $\varphi(y) = r(y) = \varrho(y)$ holds for all these y. This completes the induction step and shows that ϱ is the unique equivariant map $Y \to \Omega_{[0,\infty)}$. □

Proposition 5.18. *There is a bijection between $\Omega_{[0,\infty)}$ and*

$$(\varprojlim \mathcal{X}_m/\mathcal{G}) \sqcup \bigsqcup_{m=0}^{\infty} \mathcal{X}_m/\mathcal{G},$$

such that the projection from $\Omega_{[0,\infty)}$ to $\Omega_{[0,n]} \cong \bigsqcup_{m=0}^{n} \mathcal{X}_m/\mathcal{G}$ becomes the identity map on the components $\mathcal{X}_m/\mathcal{G}$ for $m \leq n$, π_m^n on $\mathcal{X}_m/\mathcal{G}$ for $m > n$, and the canonical map to $\mathcal{X}_n/\mathcal{G}$ on the projective limit.

Proof. Identify $\Omega_{[0,n]}$ with $\bigsqcup_{k=0}^{n} \mathcal{X}_k/\mathcal{G}$ as a set. The map $\pi_{[0,m]}^{[0,n]} : \Omega_{[0,m]} \to \Omega_{[0,n]}$ for $m > n$ is the identity map on the components $\mathcal{X}_k/\mathcal{G}$ for $k \le n$ and restricts to π_k^n for $k > n$. An element of $\Omega_{[0,\infty)} = \varprojlim \Omega_{[0,n]}$ is a family of elements $x_n \in \Omega_{[0,n]}$ with $\pi_{[0,m]}^{[0,n]}(x_m) = x_n$ for all $m \ge n$. The preimage of $\bigsqcup_{k=0}^{n-1} \mathcal{X}_k/\mathcal{G} \subseteq \Omega_{[0,n]}$ is the same subset of $\Omega_{[0,m]}$, and $\pi_{[0,m]}^{[0,n]}$ restricts to the identity map on this subset. Therefore, if there is $n \in \mathbb{N}$ with $x_n \in \mathcal{X}_k/\mathcal{G}$ for some $k < n$, then x_m must be the same element of $\mathcal{X}_k/\mathcal{G}$ for all $m \ge n$. This gives the subsets $\mathcal{X}_k/\mathcal{G}$ in $\Omega_{[0,\infty)}$. If this does not happen, then $x_n \in \mathcal{X}_n/\mathcal{G}$ for all $n \in \mathbb{N}$. Then $x_n = \pi_m^n(x_m)$ for $m \ge n$. So these elements of $\Omega_{[0,\infty)}$ correspond to elements of the projective limit $\varprojlim \mathcal{X}_n/\mathcal{G}$. □

From now on, let $\mathcal{R} \subseteq \mathcal{G}^0$ be an open $\mathcal{G}$-invariant subset such that $r_* : \mathcal{X}/\mathcal{G} \to \mathcal{G}^0$ restricts to a proper map $r_*^{-1}(\mathcal{R}) \to \mathcal{R}$. We are going to build a universal action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$. For $n \in \mathbb{N}$, let

$$(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}} := \{[\omega] \in \mathcal{X}_n/\mathcal{G} : s(\omega) \in \mathcal{R}\};$$

since $\mathcal{R}$ is invariant, $s(\omega) \in \mathcal{R}$ implies $s(\omega \cdot g) \in \mathcal{R}$ for all $g \in \mathcal{G}$ with $s(\omega) = r(g)$.

Lemma 5.19. *Describe the underlying set of $\Omega_{[0,\infty)}$ as in Proposition 5.18. The subsets $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}} \subseteq \bigsqcup_{k\in\mathbb{N}} \mathcal{X}_k/\mathcal{G} \subseteq \Omega_{[0,\infty)}$ are open for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Since pullbacks of proper maps are again proper, the restriction of the projection $\mathcal{X}_n \times_{s,r} \mathcal{X} \to \mathcal{X}_n \times_{s,r} \mathcal{G}^0 \cong \mathcal{X}_n$ to the preimage of $\mathcal{X}_n \times_{s,r} \mathcal{R}$ is again proper. This implies that the induced map $\mathcal{X}_{n+1}/\mathcal{G} \to \mathcal{X}_n/\mathcal{G}$ restricts to a proper map on the preimage of $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$. The description of the topology of the fibrewise one-point compactification in Lemma 5.5 shows that $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ is open in the fibrewise one-point compactification $\Omega_{[0,n+1]}$ for the map $\mathcal{X}_{n+1}/\mathcal{G} \to \mathcal{X}_n/\mathcal{G} \subseteq \Omega_{[0,n]}$. The image of $\mathcal{X}_k/\mathcal{G} \to \Omega_{[0,k-1]}$ is contained in $\mathcal{X}_{k-1}/\mathcal{G}$ and thus disjoint from $\mathcal{X}_n/\mathcal{G}$ for $k > n+1$. Therefore the image of $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ in $\Omega_{[0,k]}$ for $k \ge n+1$ is simply the preimage of $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ in $\Omega_{[0,n+1]}$ under the canonical projection $\pi_{[0,k]}^{[0,n+1]} : \Omega_{[0,k]} \to \Omega_{[0,n+1]}$. This remains true in the projective limit $\Omega_{[0,\infty)}$. As the preimage of an open subset under a continuous map, $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ is open in $\Omega_{[0,\infty)}$. □

Lemma 5.20. *The complement*

$$\Omega(\mathcal{R}) := \Omega_{[0,\infty)} \setminus \bigsqcup_{n=0}^{\infty} (\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$$

is closed in $\Omega_{[0,\infty)}$, and the restriction of the anchor map $r : \Omega(\mathcal{R}) \to \mathcal{G}^0$ is proper. There is a unique action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ on $\Omega(\mathcal{R})$ for which the inclusion into $\Omega_{[0,\infty)}$ becomes equivariant.

Proof. Since each subset $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ is open in $\Omega_{[0,\infty)}$ by Lemma 5.19, their union remains open and so $\Omega(\mathcal{R})$ is closed. Since $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ is $\mathcal{G}$-invariant in $\mathcal{X}_n/\mathcal{G}$ for all $n \in \mathbb{N}$, $\Omega(\mathcal{R})$ is $\mathcal{G}$-invariant in $\Omega_{[0,\infty)}$. So the $\mathcal{G}$-action restricts to it. The image of $\mathcal{X} \circ (\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ under the canonical homeomorphism $\mathcal{X} \circ \mathcal{X}_n/\mathcal{G} \cong \mathcal{X}_{n+1}/\mathcal{G}$ is equal to $(\mathcal{X}_{n+1}/\mathcal{G})_{\mathcal{R}}$. Thus the homeomorphism $\mathcal{X} \circ \Omega_{[0,\infty)} \to \Omega_{[1,\infty)}$ maps $\mathcal{X} \circ \Omega(\mathcal{R})$ onto the complement of $\bigsqcup_{n=0}^{\infty} (\mathcal{X}_{n+1}/\mathcal{G})_{\mathcal{R}}$. This implies that the map $\mathcal{X} \circ \Omega_{[0,\infty)} \to \Omega_{[0,\infty)}$ restricts to a map from $\mathcal{X} \circ \Omega(\mathcal{R})$ onto $\Omega(\mathcal{R}) \cap (\mathcal{X} \cdot \Omega_{[0,\infty)})$. This restriction remains a homeomorphism onto its image, and it inherits the technical property (3.1.4) because the anchor map $\Omega(\mathcal{R}) \to \mathcal{G}^0$ is proper. The preimage of $\mathcal{R}$ under the anchor map $\Omega(\mathcal{R}) \to \mathcal{G}^0$ is contained in $\mathcal{X} \cdot \Omega_{[0,\infty)} \cap \Omega(\mathcal{R})$ because we removed $\mathcal{R}$ from $\mathcal{G}^0$. We have seen that this is equal to $\mathcal{X} \cdot \Omega_{[0,\infty)} \cap \Omega(\mathcal{R}) = \mathcal{X} \cdot \Omega(\mathcal{R})$. So our action satisfies (3.1.3). □

Theorem 5.21. *The $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action on $\Omega(\mathcal{R})$ is the universal one.*

Proof. Let Y carry an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$. This is also an action of $(\mathcal{G}, \mathcal{X}, \varnothing)$, and the conditions on equivariant maps do not involve $\mathcal{R}$. So Proposition 5.17 provides a unique equivariant map $\varrho : Y \to \Omega_{\varnothing}$. It remains to check that the image of $\varrho : Y \to \Omega_{[0,\infty)}$ is always disjoint from $\bigsqcup_{n=0}^{\infty}(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}} \subseteq \Omega_{[0,\infty)}$.

The map ϱ sends all points in $\mathcal{X} \cdot Y$ to $\mathcal{X} \cdot \Omega_{[0,\infty)} = \Omega_{[1,\infty)}$. So only points in $Y \setminus \mathcal{X} \cdot Y$ are mapped to $\mathcal{G}^0$, and this is done by the anchor map $r : Y \to \mathcal{G}^0$. By assumption, $r^{-1}(\mathcal{R}) \subseteq \mathcal{X} \cdot Y$. So no point in Y can be mapped to $\mathcal{R} \subseteq \mathcal{G}^0$. Now let $\omega \in \mathcal{X}_n$ and $y \in Y$ satisfy $s(\omega) = r(y)$. Then $\omega \cdot y \in \mathcal{X}_n \cdot Y$ is only mapped to $\mathcal{X}_n/\mathcal{G} \subseteq \Omega_{[0,\infty)}$ if it does not belong to $\mathcal{X}_{n+1} \cdot Y = \mathcal{X}_n \cdot (\mathcal{X} \cdot Y)$. Hence $y \notin \mathcal{X} \cdot Y$, so that $s(\omega) = r(y) \notin \mathcal{R}$. Thus $\varrho(\omega \cdot y) \notin (\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$. ☐

Example 5.22. Let $\mathcal{G} = V$ be just a discrete set with only identity arrows. Then a correspondence $\mathcal{X} : \mathcal{G} \leftarrow \mathcal{G}$ is a discrete set E with two maps $r, s : E \rightrightarrows V$. Therefore

$$\mathcal{X}_n/\mathcal{G} = \mathcal{X}_n = \{(e_1, \dots, e_n) \in E^n : s(e_j) = r(e_{j+1}) \text{ for } j = 1, \dots, n-1\}.$$

So we may identify $\mathcal{X}_m/\mathcal{G}$ with the set of paths of length m and $\varprojlim \mathcal{X}_m/\mathcal{G}$ with the set of infinite paths in the directed graph described by $r, s : E \rightrightarrows V$. The space $\Omega_{[0,\infty)}$ comprises these finite and infinite paths and carries a locally compact Hausdorff topology in which the final vertex map from paths to V is a proper continuous map. In particular, if V is finite, then $\Omega_{[0,\infty)}$ is compact.

Let $\mathcal{R} \subseteq V$ be the set of regular vertices, that is, those vertices $v \in V$ for which $r^{-1}(v)$ is finite and nonempty. Lemma 5.20 defines $\Omega(\mathcal{R}) \subset \Omega_{[0,\infty)}$ as the subset consisting of the infinite paths and those finite paths that start in an irregular vertex $v \in V \setminus \mathcal{R}$. Paterson [2002] has already described the graph C^*-algebra of a possibly irregular graph as a groupoid C^*-algebra. We remark without proof that the space $\Omega(\mathcal{R})$ is identical to the object space of Paterson's groupoid.

The universal action $\Omega(\mathcal{R})$ is a key ingredient in our construction because it is the object space of the groupoid model by Proposition 3.12. In this context, it is useful to study an extra property of it.

Recall that a groupoid is *ample* if and only if its object space is totally disconnected, that is, every point has a compact open neighbourhood. The following proposition implies that the groupoid model is ample if the groupoid $\mathcal{G}$ is:

Proposition 5.23. *Assume that $\mathcal{G}^0$ is totally disconnected. Then so is $\Omega(\mathcal{R})$.*

Proof. Since the orbit space projection $\mathcal{X}_n \to \mathcal{X}_n/\mathcal{G}$ and the source map $\mathcal{X}_n \to \mathcal{G}^0$ are local homeomorphisms, compact open neighbourhoods in $\mathcal{G}^0$ provide compact open neighbourhoods in $\mathcal{X}_n$ and then in $\mathcal{X}_n/\mathcal{G}$. The explicit description of the topology on the fibrewise one-point compactification of $f : X \to Y$ in Lemma 5.5 shows that it is totally disconnected once X and Y are so. Thus the spaces $\Omega_{[0,n]}$ for $n \in \mathbb{N}$ are totally disconnected. Finally, Tychonov's theorem implies that the preimage of a compact open subset in $\Omega_{[0,n]}$ in the projective limit $\Omega_{[0,\infty)}$ is again compact open. It follows that $\Omega_{[0,\infty)}$ is totally disconnected. This is inherited by the closed subset $\Omega(\mathcal{R})$. ☐

6. Construction of the groupoid model

Definition 6.1. Let $\mathcal{I} = \mathcal{I}(\mathcal{G}, \mathcal{X})$ be the inverse semigroup that is generated by the set $\mathcal{B} = \mathcal{B}(\mathcal{G}) \sqcup \mathcal{B}(\mathcal{X})$ and the relations in (4.4.1) and (4.4.2), that is, $\Theta(\mathcal{U}\mathcal{V}) = \Theta(\mathcal{U})\Theta(\mathcal{V})$ if $\mathcal{U}, \mathcal{V} \in \mathcal{B}$ and $\mathcal{U} \in \mathcal{B}(\mathcal{X})$ or $\mathcal{V} \in \mathcal{B}(\mathcal{X})$, and $\Theta(\mathcal{U}_1)^*\Theta(\mathcal{U}_2) = \Theta(\langle \mathcal{U}_1 \mid \mathcal{U}_2 \rangle)$ for all $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{B}(\mathcal{X})$; here we write $\Theta(\mathcal{U})$ for $\mathcal{U}$ viewed as a generator of $\mathcal{I}(\mathcal{G}, \mathcal{X})$.

To construct $\mathcal{I}$, we first let $\mathrm{Free}(\mathcal{B})$ be the free inverse semigroup on the set $\mathcal{B}$ (see [Higgins 1992, Theorem 1.1.10]). Then we let $\mathcal{I}$ be the quotient of $\mathrm{Free}(\mathcal{B})$ by the smallest congruence relation that contains the relations in (4.4.1) and (4.4.2). This is an inverse semigroup by [Higgins 1992, Corollary 1.1.8]. By definition, there is a natural bijection between inverse semigroup homomorphisms $\mathcal{I} \to S$ for another inverse semigroup S and maps $\vartheta : \mathcal{B} \to S$ that satisfy (4.4.1) and (4.4.2).

Remark 6.2. The element $\Theta(\mathcal{G}^0)$ is a unit element because of (4.4.1). The relations defining $\mathcal{I}$ imply that $\mathcal{B}(\mathcal{G})$ is an inverse subsemigroup of $\mathcal{I}$.

Lemma 6.3. *Any element in $\mathcal{I}$ is of the form $a_1 \cdots a_n \cdot b_1^* \cdots b_m^*$ or $a_1 \cdots a_n$ or $b_1^* \cdots b_m^*$ with $a_i, b_j \in \mathcal{B}(\mathcal{X})$, or just $a \in \mathcal{B}(\mathcal{G})$.*

Proof. Any element of $\mathcal{I}$ may be written as a product of generators a and a^* with $a \in \mathcal{B}$. Since $\mathcal{B}(\mathcal{G})$ is an inverse subsemigroup, we do not need the letters a^* for $a \in \mathcal{B}(\mathcal{G})$. The defining relations allow us to shorten any word in the generators that contains an expression a^*b with $a, b \in \mathcal{B}$. Therefore every element in $\mathcal{I}$ may be rewritten as $a_1 \cdots a_n \cdot b_1^* \cdots b_m^*$ with $a_i, b_j \in \mathcal{B}$. If $n \geq 1$, then we may further shorten the word if some a_i belongs to $\mathcal{B}(\mathcal{G})$, and similarly for $m \geq 1$ and for $b_j \in \mathcal{B}(\mathcal{G})$. This reduces every word in the generators to the form claimed in the lemma. □

Definition 6.4. Let S be an inverse semigroup and let Y and Z be topological spaces equipped with actions $\vartheta_Y : S \to I(Y)$ and $\vartheta_Z : S \to I(Z)$ of S by partial homeomorphisms. A continuous map $f : Y \to Z$ is *S-equivariant* if $f \circ \vartheta_Y(t) = \vartheta_Z(t) \circ f$ as partial maps for all $t \in S$.

In particular, the equality of $f \circ \vartheta_Y(t) = \vartheta_Z(t) \circ f$ says that both maps have the same domain, that is, $f^{-1}(\mathrm{Dom}\,\vartheta_Z(t)) = \mathrm{Dom}\,\vartheta_Y(t)$.

Proposition 6.5. *An action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ on a topological space Y is equivalent to an inverse semigroup homomorphism $\mathcal{I} \to I(Y)$ for which there exists an $\mathcal{I}$-equivariant continuous map $Y \to \Omega(\mathcal{R})$.*

Proof. Lemma 4.4 shows that the maps $\vartheta(\mathcal{U})$ defined by an action of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ satisfy the relations that are needed for ϑ to induce a homomorphism $\mathcal{I} \to I(Y)$. In addition, Lemma 4.5 implies easily that a map is $(\mathcal{G}, \mathcal{X})$-equivariant if and only if it is equivariant for the induced actions of $\mathcal{I}$. In particular, the continuous map $Y \to \Omega(\mathcal{R})$ is $\mathcal{I}$-equivariant. Conversely, assume that $\vartheta : \mathcal{I} \to I(Y)$ is a homomorphism and that there is an $\mathcal{I}$-equivariant map $\varrho : Y \to \Omega(\mathcal{R})$. Lemma 4.4 applied to the $(\mathcal{G}, \mathcal{X}, \mathcal{R})$-action on $\Omega(\mathcal{R})$ shows that the action $\vartheta_{\Omega(\mathcal{R})}$ on $\Omega(\mathcal{R})$ satisfies all the conditions in Lemma 4.4. The equivariance of ϱ requires, among others, that ϱ^{-1} maps the domain of $\vartheta_{\Omega(\mathcal{R})}(t)$ for $t \in \mathcal{I}$ to the domain of $\vartheta_Y(t)$. In particular, for the empty slice and the unit slice in $\mathcal{G}$, this gives (4.4.3) for ϑ_Y. Since taking preimages commutes with unions and the closure of a preimage is contained in the preimage of the closure, the action ϑ_Y also inherits the properties in (4.4.5) and (4.4.6) from $\vartheta_{\Omega(\mathcal{R})}$. □

We form the transformation groupoid $\Omega \rtimes S$ for an action of an inverse semigroup S on a space Ω as in [Exel 2008], where it is called a groupoid of germs. The following proposition describes the transformation groupoid through a universal property:

Proposition 6.6. *Let S be an inverse semigroup and let X be a space with an S-action ϑ_X by partial homeomorphisms. Let Y be a space. There is a natural bijection between actions of the transformation groupoid $X \rtimes S$ on Y and pairs (ϑ_Y, f) consisting of an action ϑ_Y of S on Y and an S-equivariant map $f : Y \to X$.*

Proof. In this proof, we identify an idempotent partial homeomorphism of a space with its domain, which is an open subset. Assume that $X \rtimes S$ acts on Y. Let $f : Y \to X$ be the anchor map of the action. Any $t \in S$ gives a slice Θ_t of $X \rtimes S$. We define a partial map $\vartheta_{Y,t}$ of Y with domain $f^{-1}(\vartheta_X(t^*t))$ by $\vartheta_{Y,t}(y) := \gamma \cdot y$ for the unique $\gamma \in \Theta_t$ with $s(\gamma) = f(y)$ in X. This is a partial homeomorphism because ϑ_{Y,t^*} is a partial inverse for it. The partial homeomorphisms $\vartheta_{Y,t}$ for $t \in S$ define an action ϑ_Y of S on Y such that f is S-equivariant. Conversely, let (ϑ_Y, f) be given. We are going to define an action of $X \rtimes S$ with anchor map f. Let $\gamma \in X \rtimes S$ and $y \in Y$ satisfy $s(\gamma) = f(y)$. There is $t \in S$ with $\gamma \in \Theta_t$. Since $s(\Theta_t)$ is the open subset in X corresponding to the idempotent element $t^*t \in S$, $f(y) = s(\gamma) \in \vartheta_X(t^*t)$. Then $y \in \vartheta_Y(t^*t)$ because f is S-equivariant. So $\gamma \cdot y := \vartheta_{Y,t}(y)$ is defined. If $\gamma \in \Theta_t \cap \Theta_u$, then there is an idempotent element $e \in S$ with $te = ue$ and $s(\gamma) \in \vartheta_X(e)$. Then $y \in \vartheta_Y(e)$ as well, and hence $\vartheta_{Y,t}(y) = \vartheta_{Y,te}(y) = \vartheta_{Y,ue}(y) = \vartheta_{Y,u}(y)$. Thus $\gamma \cdot y$ does not depend on the choice of $t \in S$ with $y \in \Theta_t$. The multiplication map $(X \rtimes S) \times_{s,X,f} Y \to Y$ is continuous because this holds on each $\Theta_t \times_{s,X,f} Y$. Routine computations show that the multiplication satisfies $\gamma_1 \cdot (\gamma_2 \cdot y) = (\gamma_1 \cdot \gamma_2) \cdot y$ for $(\gamma_1, \gamma_2, y) \in (X \rtimes S) \times_{s,X,r} (X \rtimes S) \times_{s,X,f} Y$ and $1_{f(y)} \cdot y = y$ for all $y \in Y$. Thus the pair (ϑ_Y, f) gives rise to an action of $X \rtimes S$. The two constructions above are inverse to each other, so that we get the desired bijection. Both are natural; that is, a map $\varphi : Y \to Y'$ is equivariant with respect to two actions of $X \rtimes S$ if and only if it is S-equivariant and satisfies $f' \circ \varphi = f$. □

Theorem 6.7. *The transformation groupoid $\Omega(\mathcal{R}) \rtimes \mathcal{I}(\mathcal{G}, \mathcal{X})$ is a groupoid model for $(\mathcal{G}, \mathcal{X}, \mathcal{R})$.*

Proof. This follows by combining the universal property of the transformation groupoid in Proposition 6.6 with the description of actions of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ on a space Y in Proposition 6.5. □

Proposition 6.8. *The restriction of $\mathcal{M} := \Omega_{[0,\infty)} \rtimes \mathcal{I}(\mathcal{G}, \mathcal{X})$ to the open subset $\mathcal{R} \subseteq \mathcal{G}^0 \subseteq \Omega_{[0,\infty)}$ is isomorphic to $\mathcal{G}_{\mathcal{R}}$, and the orbit $\mathcal{M} \cdot \mathcal{R}$ is $\bigsqcup_{n\in\mathbb{N}} (\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$. The restriction of $\mathcal{M}$ to $\bigsqcup_{n\in\mathbb{N}} (\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ is Morita equivalent to $\mathcal{G}_{\mathcal{R}}$.*

Proof. If $a \in \mathcal{B}(\mathcal{X})$, then ϑ_a maps $\Omega_{[0,\infty)}$ into $\mathcal{X} \cdot \Omega_{[0,\infty)} = \Omega_{[1,\infty)}$, which is disjoint from $\mathcal{R} \subseteq \mathcal{G}^0 \subseteq \Omega_{[0,\infty)}$. Therefore any element of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ of the form sa^* with $s \in \mathcal{I}(\mathcal{G}, \mathcal{X})$ vanishes on $\mathcal{R}$. By Lemma 6.3, the only remaining elements of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ are of the form $a_1 \cdots a_n$ for $a_1, \dots, a_n \in \mathcal{B}(\mathcal{X})$ or $a \in \mathcal{B}(\mathcal{G})$. An element of the first type maps $\mathcal{G}^0$ into $\mathcal{X}_n/\mathcal{G} \subseteq \Omega_{[n,\infty)}$, so that its codomain is disjoint from $\mathcal{R}$. Therefore an arrow in $\Omega_{[0,\infty)} \rtimes \mathcal{I}(\mathcal{G}, \mathcal{X})$ with range and source in $\mathcal{R}$ must belong to Θ_a for some $a \in \mathcal{B}(\mathcal{G})$. The germ relation for these arrows is also the same as for $\mathcal{R} \rtimes \mathcal{B}(\mathcal{G}) = \mathcal{G}_{\mathcal{R}}$. Thus the restriction of $\mathcal{M}$ to $\mathcal{R}$ is isomorphic to $\mathcal{G}_{\mathcal{R}}$. Let $x_{n+1} \in \mathcal{R}$ belong to the domain of a slice of the form $a_1 \cdots a_n$ with $a_1, \dots, a_n \in \mathcal{B}(\mathcal{X})$. Let $x_1, \dots, x_n \in \mathcal{X}$ be such that $x_j \in a_j$ and $s(x_j) = r(x_{j+1})$ for $j = 1, \dots, n+1$. An induction on n shows that $\vartheta_{a_1 \cdots a_n}(x_{n+1}) = [x_1, \dots, x_n] \in \mathcal{X}_n/\mathcal{G} \subseteq \Omega_{[0,\infty)}$. The latter belongs to $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$. Conversely, any element $[x_1, \dots, x_n]$ of $(\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$ is of this form by choosing $x_{n+1} := s(x_n) \in \mathcal{R}$ and slices $a_j \in \mathcal{B}(\mathcal{G})$ with $x_j \in a_j$. Thus $\mathcal{M} \cdot \mathcal{R} = \bigsqcup_{n \in \mathbb{N}} (\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}$. Since $\mathcal{R}$ is open and the multiplication map in an étale groupoid is open, $\mathcal{M} \cdot \mathcal{R} \subseteq \Omega_{[0,\infty)}$ is open. The subset

$$\mathcal{M}_{\mathcal{R}} := \{\gamma \in \mathcal{M} : s(\gamma) \in \mathcal{R}\} \subseteq \mathcal{M}$$

is also open, and the left and right multiplication actions of $\mathcal{M}$ on its arrow space restrict to actions of $\mathcal{M}|_{\mathcal{M}\cdot\mathcal{R}}$ and $\mathcal{M}|_{\mathcal{R}}$ on $\mathcal{M}_{\mathcal{R}}$ on the left and right, respectively. These actions remain free and proper, and the range and source maps induce homeomorphisms $\mathcal{M}_{\mathcal{R}}/\mathcal{M}|_{\mathcal{R}} \cong \mathcal{M} \cdot \mathcal{R}$, $\mathcal{M}|_{\mathcal{M}\cdot\mathcal{R}} \backslash \mathcal{M}_{\mathcal{R}} \cong \mathcal{R}$. Thus $\mathcal{M}_{\mathcal{R}}$ is a Morita equivalence between the groupoids $\mathcal{M}|_{\mathcal{M}\cdot\mathcal{R}}$ and $\mathcal{M}|_{\mathcal{R}}$. □

7. A universal property for transformation groupoid C^*-algebras

Cuntz–Pimsner algebras are defined by a universal property that specifies their *-homomorphisms to arbitrary C^*-algebras. We would like a similar universal property for the C^*-algebra of our groupoid model. In this section, we formulate the relevant universal property, which applies to all transformation groupoids of inverse semigroup actions. Similar results exist in the literature, but with extra assumptions, such as assuming the groupoid to be Hausdorff ([Buss et al. 2018, Theorem 7.6]) or second countable ([Exel 2008, Proposition 9.8]).

Let Y be a locally compact space and let S be an inverse semigroup. Let $\vartheta : S \to I(Y)$ be an inverse semigroup homomorphism. Let $Y \rtimes S$ be the resulting transformation groupoid. Recall that this is an étale groupoid with unit space Y. For each $a \in S$, let $\Theta_a \subseteq Y \rtimes S$ be the set of all arrows $a : y \to \vartheta_a(y)$ for $y \in D_{a^*a}$. These subsets of the arrow space are slices that cover $Y \rtimes S$, and $\Theta_a \cap \Theta_b$ is the union of all Θ_c for $c \leq a, b$. Extend functions in $C_c(\Theta_a)$ by zero to the arrow space $Y \rtimes S$ and take linear combinations. The resulting space of functions is a *-algebra $\mathfrak{S}(Y \rtimes S)$ for the usual convolution and involution. The groupoid C^*-algebra $C^*(Y \rtimes S)$ is defined as the completion of $\mathfrak{S}(Y \rtimes S)$ in the maximal C^*-seminorm.

We are going to describe nondegenerate *-homomorphisms $C^*(Y \rtimes S) \to \mathbb{K}(\mathcal{E})$ for a Hilbert module $\mathcal{E}$ over some C^*-algebra D. This differs slightly from [Buss et al. 2018] in that we only allow representations by compact operators. This makes it easier to prove that certain operators are adjointable. Our definition of a covariant representation differs from the one in [Buss et al. 2018] because we want to use generators and relations for an inverse semigroup and thus want the action of the inverse semigroup to be an action by partial maps. On a Hilbert module, the right partial maps are the following:

Definition 7.1. A *partial unitary* on $\mathcal{E}$ is a unitary operator between two Hilbert submodules of $\mathcal{E}$.

The set of partial unitaries is a unital inverse subsemigroup of the inverse semigroup of partial bijections on $\mathcal{E}$, that is, the identity map is a partial unitary, the partial inverse of a partial unitary is a partial unitary, and the composite of two partial unitaries (as partial maps) is again a partial unitary.

Since $\mathcal{E} = D$ is possible, our theory contains nondegenerate *-homomorphisms to any C^*-algebra D as a special case. Conversely, since $\mathbb{K}(\mathcal{E})$ is also a C^*-algebra, what we do is equivalent to describing nondegenerate *-homomorphisms to all C^*-algebras. If $\mathcal{E} = D$ for a C^*-algebra, a partial unitary is a map between two closed right ideals in D that becomes unitary when these are viewed as Hilbert modules. This idea becomes much more natural in the setting of Hilbert modules.

Before we define covariant representations, we prove a useful lemma about the domains of the partial unitaries that occur in our covariant representations.

Lemma 7.2. *Let $\mathcal{E}$ be a Hilbert module over some C^*-algebra and let $\varphi : C_0(Y) \to \mathbb{B}(\mathcal{E})$ be a representation. Let $W \subseteq Y$ be an open subset and let (f_n) be a bounded approximate unit in $C_0(W)$. Then a vector $\xi \in \mathcal{E}$ is of the form $\varphi(g)\eta$ for some $g \in C_0(W)$, $\eta \in \mathcal{E}$ if and only if the net $\varphi(f_n)\xi$ converges towards ξ. If W_1, $W_2 \subseteq Y$ are open, then*

$$C_0(W_1)\mathcal{E} \cap C_0(W_2)\mathcal{E} = C_0(W_1 \cap W_2)\mathcal{E}.$$

Proof. The approximate unit property implies immediately that $\varphi(f_n)\varphi(g)\eta$ converges towards $\varphi(g)\eta$ for g and η as above. For the converse, we use the Cohen–Hewitt factorisation lemma. It shows that the subset of $\varphi(g)\eta$ is a closed linear subspace. Since it contains $\varphi(f_n)\xi$, it also contains the limit of $\varphi(f_n)\xi$, if that limit exists.

For the second statement, the inclusion $C_0(W_1)\mathcal{E} \cap C_0(W_2)\mathcal{E} \supseteq C_0(W_1 \cap W_2)\mathcal{E}$ is obvious. For the converse, let $(f_{j,n})$ be bounded approximate units in $C_0(W_j)$ for $j = 1, 2$, respectively. Then $(f_{1,n} f_{2,n})$ is an approximate unit in $C_0(W_1 \cap W_2)\mathcal{E}$. If $\xi \in C_0(W_1)\mathcal{E} \cap C_0(W_2)\mathcal{E}$, then $\varphi(f_{2,n})\xi$ converges towards ξ, and then $\varphi(f_{1,n})\varphi(f_{2,n})\xi$ converges towards ξ as well because $f_{1,n}$ is bounded. This implies $\xi \in C_0(W_1 \cap W_2)\mathcal{E}$ by the first statement. □

Definition 7.3 (compare [Buss et al. 2018]). Let Y be a locally compact Hausdorff space with a unital action of a unital inverse semigroup S by partial homeomorphisms $\vartheta_a : D_{a^*a} \xrightarrow{\sim} D_{aa^*}$ between open subsets $D_e \subseteq Y$ for idempotent $e \in S$. A *compact covariant representation* of this system on a Hilbert D-module $\mathcal{E}$ consists of a nondegenerate representation $\varphi : C_0(Y) \to \mathbb{K}(\mathcal{E})$ and a family of unitaries $T_a : \mathcal{E}_{a^*a} \xrightarrow{\sim} \mathcal{E}_{aa^*}$ for $a \in S$, where $\mathcal{E}_e := \varphi(C_0(D_e))\mathcal{E}$, that satisfy the following conditions:

(7.3.1) $T_1 = \mathrm{id}_{\mathcal{E}}$ and $T_a T_b = T_{ab}$ for all $a, b \in S$, where $T_a T_b$ denotes the composition of partial maps on $\mathcal{E}$;

(7.3.2) $T_a^* \varphi(f) T_a = \varphi(f \circ \vartheta_a)$ as elements of $\mathbb{K}(\mathcal{E}_{a^*a})$, for all $f \in C_0(D_{aa^*})$.

Theorem 7.4. *There is a natural bijection between compact covariant representations and nondegenerate *-homomorphisms $C^*(Y \rtimes S) \to \mathbb{K}(\mathcal{E})$. Naturality means two things*:

(7.4.1) *A bounded, possibly nonadjointable operator $\xi : \mathcal{E}_1 \to \mathcal{E}_2$ intertwines two nondegenerate *-homomorphisms $\psi_j : C^*(Y \rtimes S) \to \mathbb{K}(\mathcal{E}_j)$ for $j = 1, 2$ if and only if it intertwines the* *corresponding covariant representations* (φ_j, T_j), *that is, it intertwines the representations* φ_1 *and* φ_2, *and* $\xi T_{1,a} = T_{2,a}\xi$ *holds as maps* $\mathcal{E}_{1,a^*a} \to \mathcal{E}_{2,aa^*}$ *for all* $a \in A$.

(7.4.2) *Let* $\mathcal{F}$ *be a Hilbert* D'*-module for another* C^**-algebra with a nondegenerate left action of* D *by compact operators. Then a nondegenerate* **-homomorphism* $C^*(Y \rtimes S) \to \mathbb{K}(\mathcal{E})$ *induces a nondegenerate* **-homomorphism* $C^*(Y \rtimes S) \to \mathbb{K}(\mathcal{E} \otimes_D \mathcal{F})$, *and a compact covariant representation* (φ, T) *induces a compact covariant representation* $(\varphi \otimes_D \mathrm{id}_{\mathcal{F}}, T_a \otimes_D \mathrm{id}_{\mathcal{F}})$ *on* $\mathcal{E} \otimes_D \mathcal{F}$. *The bijection between representations is compatible with these two constructions.*

The first naturality property differs slightly from the one in [Buss et al. 2018, Theorem 3.23] in that we allow nonadjointable bounded operators instead of nonadjointable isometries. Michelle Göbel [2025] needed this extra generality in her recent work about induction theorems for C^*-hulls, and it is no extra effort to prove the stronger property.

Proof. First let T_a and φ be a covariant representation. Let $a \in S$ and $f \in C_c(\Theta_a)$. The source map $s|_{\Theta_a}$ is a homeomorphism onto $\Theta_a^*\Theta_a = \Theta_{a^*a} = D_{a^*a}$. So we may transfer f to a function $f^s \in C_c(s(\Theta_a)) = C_c(D_{a^*a})$ by composing with the inverse of the homeomorphism $s|_{\Theta_a} : \Theta_a \xrightarrow{\sim} s(\Theta_a)$. Define $f^r \in C_c(D_{aa^*})$ similarly. We claim that $\varphi(f^s)$ belongs to the image of $\mathbb{K}(\mathcal{E}_{a^*a})$ in $\mathbb{K}(\mathcal{E})$, which is a hereditary C^*-subalgebra. To see this, we use that products of functions $f_1 \cdot f_2 \cdot f_3$ with $f_j \in C_c(D_{a^*a})$ are dense in $C_0(D_{a^*a})$ and that $\varphi(f_1)|\xi\rangle\langle\eta|\varphi(f_3) = |\varphi(f_1)\xi\rangle\langle\varphi(f_3^*)\eta| \in \mathbb{K}(\mathcal{E}_{a^*a})$ for all $\xi, \eta \in \mathcal{E}$. Since T_a is an isometry from $\mathcal{E}_{a^*a}$ to $\mathcal{E}_{aa^*}$,

$$\psi_a(f) := T_a\varphi(f)$$

is a compact operator from $\mathcal{E}_{a^*a}$ to $\mathcal{E}_{aa^*}$ for all $a \in S$ and $f \in C_c(\Theta_a)$. The space of such compact operators is contained in $\mathbb{K}(\mathcal{E})$.

We claim that the linear maps $\psi_a : C_c(\Theta_a) \to \mathbb{K}(\mathcal{E})$ defined above piece together to a representation of $\mathfrak{S}(Y \rtimes S)$. First we check that it is compatible with the involution and convolution for functions that live on single slices. This requires a slightly more general formula. Any $f \in C_c(\Theta_a)$ is a pointwise product $f = f_1 f_2$ of two functions in $C_c(\Theta_a)$; for instance, take $f_1 = \sqrt{|f|}$ and $f_2(g) = f(g)/f_1(g)$ if $f(g) = 0$ and 0 otherwise. Then $T_a^*\varphi(f_1^r)T_a = \varphi(f_1^r \circ \vartheta_a) = \varphi(f_1^s)$ as operators $\mathcal{E}_{a^*a} \to \mathcal{E}_{a^*a}$. Since $\varphi(f_2^s) \in \mathbb{K}(\mathcal{E}_{a^*a}, \mathcal{E}_{a^*a})$, we compute

$$\psi_a(f_1 f_2) = T_a\varphi(f_1^s f_2^s) = T_a\varphi(f_1^s)\varphi(f_2^s) = T_a T_a^*\varphi(f_1^r)T_a\varphi(f_2^s) = \varphi(f_1^r)T_a\varphi(f_2^s),$$

where the last step uses that $\varphi(f_1^r)T_a\varphi(f_2^s)$ is a compact operator to $\mathcal{E}_{aa^*}$ and that $T_a T_a^*$ is the identity map on that Hilbert submodule.

Since $a \mapsto T_a$ is an inverse semigroup homomorphism, $T_{a^*} = T_a^*$. Let $f^\dagger(g) := \overline{f(g)}$ denote the pointwise complex conjugate. This is the adjoint in $C_0(Y \rtimes S)$, but not in $\mathfrak{S}(Y \rtimes S)$. We compute

$$\psi_a(f_1 f_2)^* = \varphi(f_2^s)^* T_a^* \varphi(f_1^s)^* = \varphi((f_2^s)^\dagger)T_{a^*}\varphi((f_1^r)^\dagger) = \psi_{a^*}((f_1 f_2)^*)$$

because $(f_1 f_2)^* \in C_c(\Theta_{a^*})$ is given by $(f_1 f_2)^*(g) = (f_1 f_2)^\dagger(g^{-1})$ and the inverse flips range and source maps. Thus $\psi_a(f)^* = \psi_{a^*}(f^*)$ for all $f \in C_c(\Theta_a)$, $a \in S$.

Next, let $a, b \in S$ and $f_a \in C_c(\Theta_a)$, $f_b \in C_c(\Theta_b)$. Write them as pointwise products $f_a = f_{a,1} f_{a,2}$ and $f_b = f_{b,1} f_{b,2}$. Recall that $f_a * f_b \in C_c(\Theta_{ab})$ with $(f_a * f_b)(g \cdot h) = f_a(g) f_b(h)$ for all $g \in \Theta_a$, $h \in \Theta_b$ with $s(g) = r(h)$. Here $r(g \cdot h) = r(g)$ and $s(g \cdot h) = s(h) = \vartheta_b^*(s(g))$. Then

$$\begin{aligned}\psi_a(f_a)\psi_b(f_b) &= \varphi(f_{a,1}^r) T_a \varphi(f_{a,2}^s)\varphi(f_{b,1}^r) T_b \varphi(f_{b,2}^s)\\ &= \varphi(f_{a,1}^r) T_a \varphi(f_{a,2}^s \cdot f_{b,1}^r) T_b \varphi(f_{b,2}^s) = \varphi(f_{a,1}^r) T_a T_b \varphi\big(((f_{a,2}^s \cdot f_{b,1}^r) \circ \vartheta_b) \cdot f_{b,2}^s\big)\\ &= \varphi(f_{a,1}^r) T_{ab} \varphi\big(((f_{a,2}^s \cdot f_{b,1}^r) \circ \vartheta_b) \cdot f_{b,2}^s\big) = \psi_{ab}(f_a * f_b),\end{aligned}$$

because the functions $f_{k,j}$ provide a suitable factorisation of $f_a * f_b$. Thus the family of maps ψ_a is multiplicative.

Next, we claim that ψ_a and ψ_b coincide on $C_c(\Theta_a \cap \Theta_b)$ for all $a, b \in S$. To prove this, we use that $\Theta_a \cap \Theta_b = \bigcup \Theta_c$ for $c \in S$ with $c \leq a$ and $c \leq b$. Thus each such c is of the form $c = a \cdot e = b \cdot e$ with $e = c^* c$ idempotent. Thus $T_b T_e = T_c = T_a T_e$. Any idempotent partial unitary on $\mathcal{E}$ is the identity map on a Hilbert submodule. So the partial unitaries T_a and T_b restrict to the same map on the image $\mathcal{E}_e$ of the idempotent e. Then they also restrict to the same map on the sum of $\mathcal{E}_{c^*c}$ over all $c \leq a, b$. This sum is dense in $C_c(s(\Theta_a \cup \Theta_b))\mathcal{E}$. Therefore, if $f \in C_c(\Theta_a \cap \Theta_b)$, then T_a and T_b restrict to the same map on $\varphi(f^s)\xi$ for all $\xi \in \mathcal{E}$. It follows that $\psi_a(f) = T_a \varphi(f^s)$ and $\psi_b(f) = T_b \varphi(f^s)$ are equal in $\mathbb{K}(\mathcal{E})$, as desired.

Putting the maps $\psi_a : C_c(\Theta_a) \to \mathbb{K}(\mathcal{E})$ together gives a map $\bigoplus_{a \in S} C_c(\Theta_a) \to \mathbb{K}(\mathcal{E})$. Since ψ_a and ψ_b coincide on $C_c(\Theta_a \cap \Theta_b)$ for all $a, b \in S$, [Buss and Meyer 2017, Proposition B.2] implies that this map descends to a well-defined map $\psi : \mathfrak{S}(Y \rtimes S) \to \mathbb{K}(\mathcal{E})$. The computations above show that this is a *-homomorphism. Its restriction to the unit slice Θ_1 is φ because $T_1 = \mathrm{id}_{\mathcal{E}}$. Therefore ψ is nondegenerate. It extends to a nondegenerate representation on the C^*-completion $C^*(Y \rtimes S)$ of $\mathfrak{S}(Y \rtimes S)$.

Conversely, let $\psi : C^*(Y \rtimes S) \to \mathbb{K}(\mathcal{E})$ be a nondegenerate *-homomorphism. Restricting to the unit slice gives a nondegenerate *-homomorphism $\varphi : C_0(Y) \to \mathbb{K}(\mathcal{E})$. Fix $a \in S$. Let $f_1, f_2 \in C_c(\Theta_a)$ and $\xi_1, \xi_2 \in \mathcal{E}$. We compute

$$\langle \psi(f_1)\xi_1 \mid \psi(f_2)\xi_2 \rangle = \langle \xi_1 \mid \varphi(f_1^* * f_2)\xi_2 \rangle = \langle \xi_1 \mid \varphi((f_1^s)^\dagger \cdot f_2^s)\xi_2 \rangle = \langle \varphi(f_1^s)\xi_1 \mid \varphi(f_2^s)\xi_2 \rangle.$$

Thus the map $\varphi(f^s)\xi \mapsto \psi(f)\xi$ defines an isometric map from $\varphi(C_c(D_{a^*a}))\mathcal{E}$ to $\psi(C_c(\Theta_a))\mathcal{E}$. Extending this to the closed linear spans gives a partial unitary T_a on $\mathcal{E}$, whose domain is $\mathcal{E}_{a^*a} := \varphi(C_0(D_{a^*a}))\mathcal{E}$ and whose codomain is the closure of $\psi(C_c(\Theta_a))\mathcal{E}$. We claim that the latter is equal to $\mathcal{E}_{aa^*}$. To prove this, we use the chain of inclusions

$$\psi(C_c(\Theta_a))\mathcal{E} \supseteq \psi(C_c(\Theta_a))\psi(C_c(\Theta_a))^*\mathcal{E} \supseteq \psi(C_c(\Theta_a))\psi(C_c(\Theta_a))^*\psi(C_c(\Theta_a))\mathcal{E}.$$

Since $\Theta_a \Theta_a^* = D_{aa^*}$, the closed linear span of $\psi(f_1)\psi(f_2)^*\xi$ with $f_1, f_2 \in C_c(\Theta_a)$ and $\xi \in \mathcal{E}$ is $\mathcal{E}_{aa^*} := \varphi(C_0(D_{aa^*}))\mathcal{E}$. Since any element of $C_c(\Theta_a)$ is a product $f_1 * f_2^* * f_3$ for $f_j \in C_c(\Theta_a)$, the sets $\psi(C_c(\Theta_a))\mathcal{E}$ and $\psi(C_c(\Theta_a))\psi(C_c(\Theta_a))^*\psi(C_c(\Theta_a))\mathcal{E}$ are equal. Since $\mathcal{E}_{aa^*}$ is sandwiched between them, this proves the claim. We have now attached partial unitaries $T_a : \mathcal{E}_{a^*a} \to \mathcal{E}_{aa^*}$ to ψ. By construction, these satisfy $\psi(f) = T_a \varphi(f^s)$ for all $f \in C_c(\Theta_a)$.

If $f_1, f_2 \in C_c(\Theta_a)$, then $\varphi(f_1^r)\psi(f_2) = \psi(f_1 f_2)$, where $f_1 f_2 \in C_c(\Theta_a)$ means the pointwise product. This gives the more symmetric formula

$$\psi(f_1 f_2) = \varphi(f_1^r) T_a \varphi(f_2^s)$$

for all $f_1, f_2 \in C_c(\Theta_a)$. So $\psi(f) \in \mathbb{K}(\mathcal{E}_{a^*a}, \mathcal{E}_{aa^*}) \subseteq \mathbb{K}(\mathcal{E})$ for all $f \in C_c(\Theta_a)$.

Since $\psi(f_1 f_2) = T_a\varphi(f_1^s)\varphi(f_2^s)$ as well, we get $T_a\varphi(f_1^s)\varphi(f_2^s) = \varphi(f_1^r)T_a\varphi(f_2^s)$. This implies $T_a^*T_a\varphi(f_1^s)\varphi(f_2^s) = T_a^*\varphi(f_1^r)T_a\varphi(f_2^s)$. Since $\varphi(f_1^s)$ takes values in $\mathcal{E}_{a^*a}$ where $T_a^*T_a$ is the identity, we may leave out $T_a^*T_a$. Now we interpret $\varphi(f_1^s)$ and $T_a^*\varphi(f_1^r)T_a$ as compact operators on $\mathcal{E}_{a^*a}$. Since $\mathcal{E}_{a^*a}$ is the span of $\varphi(f_2^s)\xi$ for $f_2 \in C_c(\Theta_a)$, $\xi \in \mathcal{E}$, the equality we have proven implies (7.3.2).

Next we prove (7.3.1) for $a, b \in S$. The operator T_{ab} has the domain $\mathcal{E}_{b^*a^*ab}$. This is contained in the domain $\mathcal{E}_{b^*b}$ of T_b, so that T_b is defined there. The image of T_b is $\mathcal{E}_{bb^*}$. Lemma 7.2 implies

$$\mathcal{E}_{bb^*} \cap \mathcal{E}_{a^*a} = \varphi(C_0(D_{bb^*}))\mathcal{E} \cap \varphi(C_0(D_{a^*a}))\mathcal{E} = \varphi(C_0(D_{bb^*} \cap D_{a^*a}))\mathcal{E}.$$

The property (7.3.2) that we have already proven shows that the T_b-preimage of $\varphi(C_0(D_{bb^*} \cap D_{a^*a}))\mathcal{E}$ is $C_0(D_{b^*a^*ab})\mathcal{E}$ because the partial homeomorphism ϑ_b associated to b restricts to a homeomorphism

$$D_{b^*a^*ab} = D_{b^*b} \cap D_{b^*a^*ab} \xrightarrow{\sim} D_{bb^*} \cap D_{a^*a}.$$

Thus T_aT_b has the same domain as T_{ab}.

Let $f_1 \in C_c(\Theta_a)$, $f_2 \in C_c(\Theta_b)$, and $f_3 \in C_c(D_{b^*a^*ab})$. Then $f_1 * f_2 \in C_c(\Theta_{ab})$ and

$$T_{ab}\varphi((f_1 * f_2)^s)\varphi(f_3) = \psi(f_1 * f_2)\varphi(f_3) = \psi(f_1)\psi(f_2)\varphi(f_3) = T_a\varphi(f_1^s)T_b\varphi(f_2^s)\varphi(f_3).$$

Here we use that the various functions belong to the domain where $\psi_x(f) = T_x\varphi(f^s)$ holds for $x \in \{a, b, ab\}$. The right-hand side further simplifies to $T_aT_b\varphi(f_1^s \circ \vartheta_b^*)\varphi(f_2^s)\varphi(f_3)$ because the domain of f_3 is small enough. Now we may let f_1^s and f_2^s run through approximate units in $C_0(D_{a^*a})$ and $C_0(D_{bb^*})$, respectively. Then $(f_1 * f_2)^s$ and $(f_1^s \circ \vartheta_b^*) \cdot f_2^s$ run through approximate units in $C_0(D_{b^*a^*ab})$. Therefore we get $T_aT_b\varphi(f_3) = T_{ab}\varphi(f_3)$ for all $f_3 \in C_c(D_{b^*a^*ab})$. Since both T_aT_b and T_{ab} are defined only on $\varphi(C_c(D_{b^*a^*ab}))\mathcal{E}$, this implies $T_aT_b = T_{ab}$. Thus any nondegenerate representation $\psi : C_c(Y \rtimes S) \to \mathbb{K}(\mathcal{E})$ gives a covariant pair (φ, T).

When we start with a nondegenerate representation ψ, make a covariant pair (φ, T), and turn this into a nondegenerate representation again, we get back ψ because $\psi(f) = T_a\varphi(f^s)$ for all $f \in C_c(\Theta_a)$, $a \in S$, and these functions generate $\mathfrak{S}(Y \rtimes S)$. The converse is also true because the representation ψ built from a covariant pair (φ, T) restricts to φ on the unit slice and the property $\psi(f) = T_a\varphi(f^s)$ for all $f \in C_c(\Theta_a)$, $a \in S$ determines T_a as a partial unitary $\mathcal{E}_{a^*a} \to \mathcal{E}_{aa^*}$. So the two constructions are inverse to each other.

Routine computations show that our constructions in both directions have the two naturality properties stated in the theorem. □

8. Comparison to the relative Cuntz–Pimsner algebra

In this section, we are going to identify the groupoid C^*-algebra of the groupoid model of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$ with the Cuntz–Pimsner algebra of $C^*(\mathcal{X})$ relative to $C^*(\mathcal{G}_\mathcal{R})$. Since the groupoid model is unique up to isomorphism by Proposition 3.13, we may work with the specific groupoid model built in Theorem 6.7. This is the transformation groupoid

$$\mathcal{M} := \Omega(\mathcal{R}) \rtimes \mathcal{I}(\mathcal{G}, \mathcal{X})$$

for the inverse semigroup $\mathcal{I}(\mathcal{G}, \mathcal{X})$ acting on the space $\Omega(\mathcal{R})$ of Lemma 5.20. Let $\vartheta_a : D_{a^*a} \to D_{aa^*}$ for $a \in \mathcal{I}(\mathcal{G}, \mathcal{X})$ denote the partial homeomorphisms by which $\mathcal{I}(\mathcal{G}, \mathcal{X})$ acts on $\Omega(\mathcal{R})$.

Proposition 8.1. *Let $\mathcal{E}$ be a Hilbert module over a C^*-algebra D. There is a bijection between nondegenerate *-homomorphisms $\psi : C^*(\mathcal{M}) \to \mathbb{K}(\mathcal{E})$ and pairs (φ, T) where $\varphi : C_0(\Omega(\mathcal{R})) \to \mathbb{K}(\mathcal{E})$ is a nondegenerate *-homomorphism and T_a for $a \in \mathcal{B} = \mathcal{B}(\mathcal{G}) \sqcup \mathcal{B}(\mathcal{X})$ are partial unitaries, for which the following hold*:

(8.1.1) $T_a T_b = T_{ab}$ *if $a, b \in \mathcal{B}$ and $a \in \mathcal{B}(\mathcal{G})$ or $b \in \mathcal{B}(\mathcal{G})$*;

(8.1.2) *the domain of T_a for $a \in \mathcal{B}$ is $\mathcal{E}_{r^{-1}(s(a))}$, where $s(a) \subseteq \mathcal{G}^0$ and we define $\mathcal{E}_X := \varphi(C_0(X))\mathcal{E}$ for an open subset $X \subseteq \Omega(\mathcal{R})$*;

(8.1.3) *the codomain of T_a is $\mathcal{E}_{\pi^{-1}(p(a))}$ for $a \in \mathcal{B}(\mathcal{X})$, where $p : \mathcal{X} \to \mathcal{X}/\mathcal{G} \subseteq \Omega_{[0,1]}$ is the orbit space projection and $\pi : \Omega(\mathcal{R}) \subseteq \Omega_{[0,\infty)} \to \Omega_{[0,1]}$ is the canonical projection*;

(8.1.4) $T_a^* \varphi(f_1) T_a \varphi(f_2) = \varphi((f_1 \circ \vartheta_a) f_2)$ *if $a \in \mathcal{B}$, $f_1 \in C_0(D_{aa^*})$, and $f_2 \in C_0(D_{a^*a})$*;

(8.1.5) $T_a^* T_b = T_{\langle a | b \rangle}$ *for all $a, b \in \mathcal{B}(\mathcal{X})$.*

This bijection has the two naturality properties in Theorem 7.4. Condition (8.1.5) *is redundant*, *that is*, *it follows from the other conditions.*

Proof. We first start with a nondegenerate *-homomorphism $C^*(\mathcal{M}) \to \mathbb{K}(\mathcal{E})$ and construct a pair (φ, T) as in the proposition. Recall that $\mathcal{M} = \Omega(\mathcal{R}) \rtimes \mathcal{I}(\mathcal{G}, \mathcal{X})$. Theorem 7.4 provides a natural bijection between nondegenerate *-homomorphisms $C^*(\mathcal{M}) \to \mathbb{K}(\mathcal{E})$ and covariant representations (φ, T) of the action of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ on $C_0(\Omega(\mathcal{R}))$. The inverse semigroup $\mathcal{I}(\mathcal{G}, \mathcal{X})$ is the universal inverse semigroup generated by the set $\mathcal{B}$ subject to the relations in (8.1.1) and (8.1.5). In particular, a representation T of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ provides partial unitaries T_a for all $a \in \mathcal{B}$ that satisfy the relations in (8.1.1) and (8.1.5). In addition, the domain and codomain of T_a in a covariant representation are $\mathcal{E}_{a^*a}$ and $\mathcal{E}_{aa^*}$, respectively. These submodules involve the domains and codomains of the partial homeomorphisms induced by the action on $\Omega(\mathcal{R})$, and Lemma 4.1 allows us to compute these. Putting things together, it follows that the domain of T_a for $a \in \mathcal{B}$ is $\mathcal{E}_{a^*a} = \mathcal{E}_{r^{-1}(s(a))}$ as required in (8.1.2) and the codomain of T_a for $a \in \mathcal{B}(\mathcal{X})$ is $\mathcal{E}_{aa^*} = \mathcal{E}_{\pi^{-1}(p(a))}$ as required in (8.1.3). Thus a nondegenerate *-homomorphism $C^*(\mathcal{M}) \to \mathbb{K}(\mathcal{E})$ gives rise to a pair (φ, T) satisfying (8.1.1)–(8.1.5).

Conversely, assume that such a pair (φ, T) is given. The conditions (8.1.1) and (8.1.5) ensure that T extends to a homomorphism $\overline{T}$ from $\mathcal{I}(\mathcal{G}, \mathcal{X})$ to the inverse semigroup of partial unitaries on $\mathcal{E}$. We claim that $(\varphi, \overline{T})$ is a covariant representation of the action of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ on $\Omega(\mathcal{R})$. Thus (φ, T) induces a nondegenerate *-homomorphism $C^*(\mathcal{M}) \to \mathbb{K}(\mathcal{E})$. The property $T_a T_b = T_{ab}$ for $a, b \in \mathcal{I}(\mathcal{G}, \mathcal{X})$ in (7.3.1) is already built in. The unit element of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ is the unit slice in $\mathcal{G}$. This is idempotent and (8.1.2) says that its domain is all of $\mathcal{E}$. So $T_1 = \mathrm{id}_{\mathcal{E}}$, as required in (8.1.2).

We call $a \in \mathcal{I}(\mathcal{G}, \mathcal{X})$ *good* if the domain and codomain of T_a are $\mathcal{E}_{a^*a}$ and $\mathcal{E}_{aa^*}$, respectively, and $T_a^*\varphi(f)T_a = \varphi(f \circ \vartheta_a)$ in $\mathbb{K}(\mathcal{E}_{a^*a})$ for all $f \in C_0(D_{aa^*})$. The pair $(\varphi, \overline{T})$ is a covariant representation if and only if all elements of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ are good. We are going to prove this by showing that the generators $a \in \mathcal{B}$ are good and that being good is hereditary for products and adjoints in $\mathcal{I}(\mathcal{G}, \mathcal{X})$. That is, the good elements form an inverse subsemigroup of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ that contains all the generators, and this forces it to be all of $\mathcal{I}(\mathcal{G}, \mathcal{X})$.

We have already discussed that the domain of T_a for $a \in \mathcal{B}$ is $\mathcal{E}_{a^*a}$ if and only if (8.1.2) is satisfied. Since $a^* \in \mathcal{B}$ for $a \in \mathcal{B}(\mathcal{G})$, this also gives the correct codomain for $a \in \mathcal{B}(\mathcal{G})$. If $a \in \mathcal{B}(\mathcal{X})$, then (8.1.3) ensures that the codomain of T_a is $\mathcal{E}_{aa^*}$. The covariance condition (7.3.2) identifies $T_a^*\varphi(f_1)T_a$ and $\varphi(f_1 \circ \vartheta_a)$ in $\mathbb{K}(\mathcal{E}_{a^*a})$ for all $f_1 \in C_0(D_{aa^*})$. This is equivalent to $T_a^*\varphi(f_1)T_a\varphi(f_2) = \varphi(f_1 \circ \vartheta_a)\varphi(f_2)$ as elements of $\mathbb{K}(\mathcal{E})$ for all $f_1 \in C_0(D_{aa^*})$ and $f_2 \in C_0(D_{a^*a})$ by the definition of $\mathcal{E}_{a^*a}$; this is what is required in (8.1.4) for $a \in \mathcal{B}$. This finishes the proof that all elements of $\mathcal{B}$ are good.

It is easy to see that a^* is good if a is. To show that all elements of $\mathcal{I}(\mathcal{G}, \mathcal{X})$ are good, it remains to prove that ab is good if a and b are good. So let a and b be good. The intersection of the image of T_b with the domain of T_a is

$$\mathcal{E}_{bb^*} \cap \mathcal{E}_{a^*a} = C_0(D_{bb^*})\mathcal{E} \cap C_0(D_{a^*a})\mathcal{E} = C_0(D_{bb^*} \cap D_{a^*a})\mathcal{E}$$

by Lemma 7.2. If $f \in C_0(D_{bb^*} \cap D_{a^*a})$, then $T_b^*\varphi(f)T_b = \varphi(f \circ \vartheta_b)$ as operators on $\mathcal{E}_{b^*b}$ because b is good. So the domain of $T_{ab} = T_a T_b$ is $C_0(\vartheta_b^{-1}(D_{bb^*} \cap D_{a^*a}))\mathcal{E} = C_0(D_{(ab)^*ab})\mathcal{E} = \mathcal{E}_{(ab)^*ab}$, as desired. The codomain is

$$T_a(C_0(D_{bb^*} \cap D_{a^*a})\mathcal{E}) = T_a\varphi(C_0(D_{bb^*}))\mathcal{E}_{a^*a} = \varphi\big(C_0(\vartheta_a(D_{bb^*}))\big)\mathcal{E}_{aa^*} = \mathcal{E}_{abb^*a^*},$$

as needed. Finally, the equalities

$$T_{ab}^*\varphi(f)T_{ab} = T_b^*T_a^*\varphi(f)T_aT_b = T_b^*\varphi(f \circ \vartheta_a)T_b = \varphi(f \circ \vartheta_a \circ \vartheta_b) = \varphi(f \circ \vartheta_{ab})$$

hold as operators on $\mathcal{E}_{(ab)^*ab}$ for all $f \in C_0(D_{ab(ab)^*})$. So ab is good, as desired.

Finally, we show that (8.1.5) is redundant. Let $a, b \in \mathcal{B}(\mathcal{X})$. First we claim that (8.1.5) follows if

$$T_a T_a^* T_b = T_a T_{\langle a|b\rangle}. \tag{8.2}$$

The codomain of $T_{\langle a|b\rangle}$ is the same as the domain of $T_{\langle a|b\rangle}^* = T_{\langle b|a\rangle}$, which is $\mathcal{E}_{r^{-1}(s(\langle b|a\rangle))}$ by (8.1.2). This is contained in $\mathcal{E}_{r^{-1}(s(a))}$, which is the domain of T_a by (8.1.2). This is where $T_a^*T_a$ is the identity map. So $T_a^*T_aT_{\langle a|b\rangle} = T_{\langle a|b\rangle}$. In addition, $T_a^*T_aT_a^* = T_a^*$. Therefore (8.2) implies

$$T_a^*T_b = T_a^*T_aT_a^*T_b = T_a^*T_aT_{\langle a|b\rangle} = T_{\langle a|b\rangle},$$

which is (8.1.5). So it suffices to prove (8.2). Now $T_a T_{\langle a|b\rangle} = T_{a\langle a|b\rangle}$ by (8.1.1). Since $x\langle x \mid y\rangle = y$ for all $x, y \in \mathcal{X}$ with $p(x) = p(y)$, the slice $a\langle a \mid b\rangle$ is contained in b. Namely, it consists of all $y \in b$ with $p(y) \in p(a)$. Let $W = s(a\langle a \mid b\rangle)$. This is an open subset of $\mathcal{G}^0$ and thus an idempotent element of $\mathcal{B}(\mathcal{G})$. The computation above implies $a\langle a \mid b\rangle = bW$. So $T_a T_{\langle a|b\rangle} = T_b T_W$. By (8.1.1), T_W is the identity map on its domain, which is $\mathcal{E}_{r^{-1}(W)}$ by (8.1.2). So $T_b T_W$ is the restriction of T_b to $\mathcal{E}_{r^{-1}(W)}$. Next $T_a T_a^*$ is the identity map on the codomain of T_a, which is $\mathcal{E}_{\pi^{-1}(p(a))}$ by (8.1.3). Let (f_n) be an approximate unit in $C_0(p(a)) \subseteq C_0(\mathcal{X}/\mathcal{G}) \subseteq C_0(\Omega_{[0,1]})$. Then $(f_n \circ \pi)$ is an approximate unit in $C_0(\pi^{-1}(p(a)))$. Lemma 7.2 implies that $T_a T_a^* T_b$ is defined at $\xi \in \mathcal{E}$ if and only if $\xi \in \mathcal{E}_{s(b)}$ and $\lim \varphi(f_n \circ \pi) T_b(\xi) = T_b(\xi)$. If $\xi \in \mathcal{E}_{s(b)}$, then $\xi = \varphi(f_b)\xi'$ for some $f_b \in C_0(s(b))$ and $\xi \in \mathcal{E}$ by the Cohen–Hewitt factorisation theorem. So (8.1.4) implies

$$\varphi(f_n \circ \pi) T_b(\xi) = T_b \varphi(f_n \circ \pi \circ \vartheta_b) \varphi(f_b)\xi' = T_b \varphi(f_n \circ \pi \circ \vartheta_b)\xi.$$

If $\omega \in \Omega(\mathcal{R})$, then $\pi(\vartheta_b(\omega)) \in \Omega_{[0,1]}$ is defined if and only if $r(\omega) \in \mathcal{G}^0$ is in $s(b)$, and then it is $[x] \in \mathcal{X}/\mathcal{G} \subseteq \Omega_{[0,1]}$ for the unique $x \in b$ with $s(x) = r(\omega)$. It follows that $f_n \circ \pi \circ \vartheta_b(\omega)$ converges to 1 if $r(\omega) \in W$ and vanishes otherwise, with $W \subseteq \mathcal{G}^0$ as above. So we may write $f_n \circ \pi \circ \vartheta_b(\omega) = f_n' \circ r(\omega)$ with an approximate unit f_n' for the ideal $C_0(W)$ in $C_0(\mathcal{G}^0)$. Using Lemma 7.2 once again, we conclude that the domain of $T_a T_a^* T_b$ is also equal to $\mathcal{E}_{r^{-1}(W)}$. Since $T_a T_a^*$ is idempotent, $T_a T_a^* T_b$ is the restriction of T_b to this domain. This is equal to $T_a T_{\langle a|b\rangle} = T_{a\langle a|b\rangle} = T_b T_W$. So (8.2), and thus (8.1.5), follow from the other conditions in the proposition. □

Remark 8.3. The last sentence in Proposition 8.1 implies that the transformation groupoid $\Omega(\mathcal{R}) \rtimes \mathcal{I}(\mathcal{G}, \mathcal{X})$ does not change if we drop the relation $\Theta(\mathcal{U}_1)^*\Theta(\mathcal{U}_2) = \Theta(\langle \mathcal{U}_1 \mid \mathcal{U}_2\rangle)$ for $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{B}(\mathcal{X})$ in the definition of $\mathcal{I}(\mathcal{G}, \mathcal{X})$.

The next proposition describes nondegenerate representations of the Cuntz–Pimsner algebra $\mathcal{O} := \mathcal{O}_{C^*(\mathcal{X}), C^*(\mathcal{G}_\mathcal{R})}$ in $\mathbb{K}(\mathcal{E})$ in a somewhat similar fashion. It uses the partial map ϱ_a on $\mathcal{G}^0$ induced by a slice $a \in \mathcal{B}(\mathcal{X})$. This is defined on the open subset $s(a) \subseteq \mathcal{G}^0$ and maps $s(x)$ for $x \in a$ to $r(x) \in r(a) \subseteq \mathcal{G}^0$. Since r is not a local homeomorphism, this map $\varrho_a : s(a) \to r(a)$ is not a homeomorphism. Instead, it is the composite of the homeomorphism $\vartheta_a : s(a) \to p(a) \subseteq \mathcal{X}/\mathcal{G}$ with $r_* : \mathcal{X}/\mathcal{G} \to \mathcal{G}^0$. If $f \in C_0(\mathcal{G}^0)$, then $f \circ \varrho_a \in C_b(s(a))$, and it need not belong to $C_0(s(a))$ unless the correspondence $\mathcal{X}$ is proper. If $f_1 \in C_0(\mathcal{G}^0)$ and $f_2 \in C_0(s(a))$, then $(f_1 \circ \varrho_a) \cdot f_2 \in C_0(s(a)) \subseteq C_0(\mathcal{G}^0)$, although $f_1 \circ \varrho_a$ is only defined on $s(a)$.

Proposition 8.4. *Let $\mathcal{E}$ be a Hilbert module over a C^*-algebra D. There is a natural bijection between nondegenerate *-homomorphisms $\psi : \mathcal{O}_{C^*(\mathcal{X}), C^*(\mathcal{G}_\mathcal{R})} \to \mathbb{K}(\mathcal{E})$ and pairs (φ_0, T), where $\varphi_0 : C_0(\mathcal{G}^0) \to \mathbb{K}(\mathcal{E})$ is a nondegenerate *-homomorphism and T_a for $a \in \mathcal{B} = \mathcal{B}(\mathcal{G}) \sqcup \mathcal{B}(\mathcal{X})$ are partial unitaries, such that the following hold*:

(8.4.1) $T_a T_b = T_{ab}$ *if $a, b \in \mathcal{B}$ and $a \in \mathcal{B}(\mathcal{G})$ or $b \in \mathcal{B}(\mathcal{G})$.*

(8.4.2) *The domain of T_a for $a \in \mathcal{B}$ is $\mathcal{E}_{s(a)}$, where $s(a) \subseteq \mathcal{G}^0$ and we define $\mathcal{E}_W := \varphi_0(C_0(W))\mathcal{E}$ for an open subset $W \subseteq \mathcal{G}^0$.*

(8.4.3) *$\mathcal{E}_\mathcal{R}$ is contained in the closed linear span of the images of T_a for $a \in \mathcal{B}(\mathcal{X})$.*

(8.4.4) $T_a^*\varphi_0(f_1)T_a\varphi_0(f_2) = \varphi_0((f_1 \circ \varrho_a)\cdot f_2)$ *if* $a \in \mathcal{B}$, $f_1 \in C_0(\mathcal{G}^0)$, *and* $f_2 \in C_0(s(a))$.

(8.4.5) *Let* $a, b \in \mathcal{B}(\mathcal{X})$, $f_j \in C_0(j)$, *and* $\xi_j \in \mathcal{E}$ *for* $j = a, b$. *Then*

$$\langle T_a\varphi_0(f_a^s)\xi_a \mid T_b\varphi_0(f_b^s)\xi_b\rangle = \langle \xi_a \mid T_{\langle a|b\rangle}\varphi_0((f_a^s \circ \vartheta_{\langle a|b\rangle})^\dagger f_b^s)\xi_b\rangle.$$

Here $\dagger$ *denotes pointwise complex conjugation, and* $f_j^s \in C_c(s(j))$ *is* $f_j \circ (s|_j)^{-1}$ *for the homeomorphism* $(s|_j)^{-1} : s(j) \xrightarrow{\sim} j$ *for* $j = a, b$.

This bijection has the two naturality properties in Theorem 7.4.

Proof. The universal property says that a *-homomorphism $\mathcal{O} \to \mathbb{K}(\mathcal{E})$ is equivalent to a Toeplitz representation $C^*(\mathcal{X}) \to \mathbb{K}(\mathcal{E})$ that is Cuntz–Pimsner covariant on the ideal $C^*(\mathcal{G}_\mathcal{R})$. Here a Toeplitz representation is a pair of maps $\psi : C^*(\mathcal{G}) \to \mathbb{K}(\mathcal{E})$ and $L : C^*(\mathcal{X}) \to \mathbb{K}(\mathcal{E})$, where ψ is a *-homomorphism and L is a linear map that satisfies $\psi(f)L(\xi) = L(f * \xi)$, $L(\xi)\psi(f) = L(\xi * f)$, and $L(\xi_1)^*L(\xi_2) = \psi(\langle \xi_1 \mid \xi_2\rangle)$ for $f \in C^*(\mathcal{G})$ and $\xi, \xi_1, \xi_2 \in C^*(\mathcal{X})$. By [Meyer and Sehnem 2019, Proposition 2.15], this Toeplitz representation is Cuntz–Pimsner covariant on the ideal $C^*(\mathcal{G}_\mathcal{R}) \subseteq C^*(\mathcal{G})$ if and only if $\psi(C^*(\mathcal{G}_\mathcal{R}))\mathcal{E} \subseteq L(C^*(\mathcal{X}))\mathcal{E}$. Since $C_0(\mathcal{R})$ is a nondegenerate C^*-subalgebra of $C^*(\mathcal{G}_\mathcal{R})$, this is equivalent to $\psi(C_0(\mathcal{R}))\mathcal{E} \subseteq L(C^*(\mathcal{X}))\mathcal{E}$. In addition, the *-homomorphism $\mathcal{O} \to \mathbb{K}(\mathcal{E})$ is nondegenerate if and only if ψ is nondegenerate.

We may write $\mathcal{G}$ as the inverse semigroup transformation groupoid $\mathcal{G}^0 \rtimes \mathcal{B}(\mathcal{G})$. By Theorem 7.4, a nondegenerate *-homomorphism $\psi : C^*(\mathcal{G}) \to \mathbb{K}(\mathcal{E})$ is equivalent to a covariant representation (φ_0, T) of $\mathcal{G}^0 \rtimes \mathcal{B}(\mathcal{G})$ as in Definition 7.3. This consists of a nondegenerate *-homomorphism $\varphi_0 : C_0(\mathcal{G}^0) \to \mathbb{K}(\mathcal{E})$ and partial unitaries $T_a : \mathcal{E}_{s(a)} \to \mathcal{E}_{r(a)}$ for $a \in \mathcal{B}(\mathcal{G})$. The conditions in Definition 7.3 are equivalent to the conditions in (8.4.1), (8.4.2), and (8.4.4) for $a \in \mathcal{B}(\mathcal{G})$; compare with the proof of Theorem 7.4 for why we may add the factor $\varphi_0(f_2)$ in (8.4.4).

Next we relate the map $L : C^*(\mathcal{X}) \to \mathbb{K}(\mathcal{E})$ in a Toeplitz representation to a family of partial unitaries T_a for $a \in \mathcal{B}(\mathcal{X})$. By the construction in [Antunes et al. 2022], $C^*(\mathcal{X})$ is defined as the Hilbert module completion of the space of functions $\mathfrak{S}(\mathcal{X})$, which is the closed linear span of functions in $C_c(a)$ for slices $a \subseteq \mathcal{X}$, extended by zero outside a. So L is determined uniquely by its restrictions to $C_c(a) \subseteq C^*(\mathcal{X})$ for all $a \in \mathcal{B}(\mathcal{X})$. Given φ_0 and T_a as in the statement, we define a map

$$L : \bigoplus_{a\in\mathcal{B}(\mathcal{X})} C_c(a) \to \mathbb{K}(\mathcal{E}), \quad L((f_a)_{a\in\mathcal{B}(\mathcal{X})}) := \sum T_a\varphi_0(f_a^s).$$

Condition (8.4.5) is equivalent to $\langle L(f_a)\xi_a \mid L(f_b)\xi_b\rangle = \langle \xi_a \mid \psi(\langle f_a \mid f_b\rangle)\xi_b\rangle$ for $f_a \in C_c(a)$ and $f_b \in C_c(b)$, for $a, b \in \mathcal{B}(\mathcal{X})$ and $\xi_a, \xi_b \in \mathcal{E}$ because $\varphi_0(f_a^s)\xi_a \in \mathcal{E}_{s(a)}$, $\varphi_0(f_b^s)\xi_b \in \mathcal{E}_{s(b)}$, and $\psi(\langle f_a \mid f_b\rangle) = T_{\langle a|b\rangle}\varphi_0((f_a^* * f_b)^s) = T_{\langle a|b\rangle}\varphi_0((f_a^s \circ \vartheta_{\langle a|b\rangle})^\dagger f_b^s)$. So (8.4.5) is equivalent to $L(f_a)^*L(f_b) = \psi(\langle f_a \mid f_b\rangle)$. By linearity, $L(f_1)^*L(f_2) = \psi(\langle f_1 \mid f_2\rangle)$ for all $f_1, f_2 \in \bigoplus C_c(a)$ with the scalar product that defines $C^*(\mathcal{X})$. So L factors through an isometric map $C^*(\mathcal{X}) \to \mathbb{K}(\mathcal{E})$, which we also denote by L.

Let $a \in \mathcal{B}(\mathcal{G})$, $b \in \mathcal{B}(\mathcal{X})$, $f_a \in C_c(a)$, and $f_b \in C_c(b)$. The product $f_a * f_b$ in $\mathfrak{S}(\mathcal{X})$ is defined in [Antunes et al. 2022, (7.3)]. It is supported on the slice $ab \subseteq \mathcal{X}$ and has the value $f_a(g_a)f_b(x_b)$ at $g_a x_b$ if

$g_a \in a$, $x_b \in b$ are such that $s(g_a) = r(x_b)$. Thus $(f_a * f_b)^s = (f_a^s \circ \varrho_b) f_b^s$. Then

$$\psi(f_a)L(f_b) = T_a\varphi_0(f_a^s)T_b\varphi_0(f_b^s) = T_aT_b\varphi_0((f_a^s \circ \varrho_b) f_b^s) = T_{ab}\varphi_0((f_a * f_b)^s) = L(f_a * f_b)$$

by (8.4.4) and (8.4.1). This implies that L is a left $C^*(\mathcal{G})$-module map. A similar computation shows that it is a right $C^*(\mathcal{G})$-module map. So (ψ, L) is a Toeplitz representation. We have argued above that (ψ, L) is Cuntz–Pimsner covariant on $C^*(\mathcal{G}_{\mathcal{R}})$ if and only if $\mathcal{E}_{\mathcal{R}} \subseteq L(C^*(\mathcal{X}))\mathcal{E}$. The latter is equivalent to (8.4.3). Summing up, we get a *-homomorphism $\mathcal{O} \to \mathbb{K}(\mathcal{E})$ from the data and conditions in the proposition.

Conversely, let (ψ, L) be a Toeplitz representation. Let $a \in \mathcal{B}(\mathcal{X})$ and $f_1, f_2 \in C_c(a) \subseteq C^*(\mathcal{X})$. If $x \in \mathcal{X}$ and $g \in \mathcal{G}$ are such that $x \in a$ and $xg \in a$, then g is a unit because the orbit space projection is injective on a and $\mathcal{G}$ acts freely on $\mathcal{X}$. In addition $g = s(x)$, because it is composable with x. Therefore the Hilbert module scalar product on $\mathfrak{S}(\mathcal{X})$ in [Antunes et al. 2022, (7.2)] specialises to $\langle f_1 \mid f_2\rangle(s(x)) = \overline{f_1(x)}\, f_2(x)$ for $x \in a$ and $\langle f_1 \mid f_2\rangle(g) = 0$ for $g \in \mathcal{G} \setminus s(a)$. More briefly, $\langle f_1 \mid f_2\rangle = (f_1^s)^\dagger f_2^s$. So $L(f_1)^*L(f_2) = \psi(\langle f_1 \mid f_2\rangle)$ implies that there is a unique isometric map $T_a : \mathcal{E}_a \to \mathcal{E}$ with $T_a(\varphi_0(f^s)\xi) = L(f)\xi$ for all $f \in C_c(a)$, $\xi \in \mathcal{E}$. By construction, T_a has the domain required in (8.4.2).

Let $a \in \mathcal{B}(\mathcal{G})$, $b \in \mathcal{B}(\mathcal{X})$, and $f_j \in C_c(j)$ for $j = a, b$. Then $f_a * f_b$ is supported on the slice ab and $(f_a * f_b)^s = (f_a^s \circ \varrho_b) f_b^s$. So

$$T_a\varphi_0(f_a^s)T_b\varphi_0(f_b^s) = \psi(f_a)L(f_b) = L(f_a * f_b) = T_{ab}\varphi_0((f_a * f_b)^s) = T_{ab}\varphi_0(f_a^s \circ \varrho_b)\varphi_0(f_b^s). \quad (8.5)$$

If a is the unit slice $\mathcal{G}^0$, then $T_{\mathcal{G}^0}$ is the identity map on all of $\mathcal{E}$ by the construction above. Then (8.5) specialises to (8.4.4) with b instead of a. Letting a be general again, (8.4.4) identifies the left-hand side in (8.5) with $T_aT_b\varphi_0(f_a^s \circ \varrho_b)\varphi_0(f_b^s) = T_aT_b\varphi_0((f_a * f_b)^s)$. Any function $f \in C_c(s(ab))$ may be written as $(f_a * f_b)^s$ for f_a and f_b as above. So (8.5) says that $T_{ab}\varphi_0(f) = T_aT_b\varphi_0(f)$ for all $f \in C_c(s(ab))$. Therefore T_aT_b is defined on $\mathcal{E}_{s(ab)}$, and equal to T_{ab} there.

Next we show that the domain of T_aT_b is contained in the domain of T_{ab}. By Lemma 7.2, T_j is defined at $\xi \in \mathcal{E}$ if and only if $\lim \varphi_0(f_j^s)\xi = \xi$ in norm for an approximate unit (f_j) in $C_c(j)$ for $j \in \{a, b, ab\}$. We may choose the functions in (8.5) so that f_a^s and f_b^s converge to 1 uniformly on compact subsets of $s(j) \subseteq \mathcal{G}^0$ for $j = a, b$, respectively. Let $\xi \in \mathcal{E}$ belong to the domain of T_aT_b. Then $\lim \varphi_0(f_b^s)\xi = \xi$ and $\lim \varphi_0(f_a^s)T_b(\xi) = T_b(\xi)$ together imply that $\lim \varphi_0(f_a^s)T_b\varphi_0(f_b^s)\xi = T_b(\xi)$. Using (8.4.4), this implies $\lim T_b\varphi_0((f_a * f_b)^s)\xi = T_b(\xi)$. Since T_b is a partial unitary, this implies $\lim \varphi_0((f_a * f_b)^s)\xi = \xi$, and this says that ξ belongs to the domain of T_{ab}. This finishes the proof that the partial unitaries T_a satisfy (8.4.1) for $a \in \mathcal{B}(\mathcal{G})$, $b \in \mathcal{B}(\mathcal{X})$. Similar computations establish (8.4.1) for $a, b \in \mathcal{B}(\mathcal{G})$ and for $a \in \mathcal{B}(\mathcal{X})$ and $b \in \mathcal{B}(\mathcal{G})$. That is, (8.4.1) holds.

Let $a, b \in \mathcal{B}(\mathcal{X})$ and $f_j \in C_c(j)$ for $j = a, b$. Then $L(f_a)^*L(f_b) = \psi(\langle f_a \mid f_b\rangle)$ for $f_a \in C_c(a)$, for $j = a, b$. Since $L(f_j) = T_j\varphi_0(f_j^s)$ for $j = a, b$ and $\psi(\langle f_a \mid f_b\rangle) = T_{\langle a|b\rangle}\varphi_0((f_a^s \circ \vartheta_{\langle a|b\rangle})^\dagger f_b^s)$, this implies (8.4.5).

The two constructions outlined above are inverse to each other and so give a bijection between nondegenerate representations $\mathcal{O} \to \mathbb{K}(\mathcal{E})$ and the families (φ_0, T_a) with the properties in the statement of the proposition. □

Comparing the descriptions of representations of $C^*(\mathcal{M})$ and $\mathcal{O}$, the main difference is that in Proposition 8.1 we require a representation of $C_0(\Omega(R))$, whereas we only require a representation of $C_0(\mathcal{G}^0)$ in Proposition 8.4.

Lemma 8.6. *A nondegenerate* **-homomorphism* $C^*(\mathcal{M}) \to \mathbb{K}(\mathcal{E})$ *induces a nondegenerate* **-homomorphism* $\mathcal{O} \to \mathbb{K}(\mathcal{E})$.

Proof. By composing with the proper map $r : \Omega(\mathcal{R}) \to \mathcal{G}^0$, a *-homomorphism on $C_0(\Omega(\mathcal{R}))$ induces one on $C_0(\mathcal{G}^0)$. So the data in Proposition 8.1 that is equivalent to a nondegenerate *-homomorphism $C^*(\mathcal{M}) \to \mathbb{K}(\mathcal{E})$ produces the data in Proposition 8.4 that is equivalent to a nondegenerate *-homomorphism $\mathcal{O} \to \mathbb{K}(\mathcal{E})$. The conditions (8.1.1) and (8.4.1) are identical, and (8.1.2) is equivalent to (8.4.2). In $\Omega(\mathcal{R})$, $r^{-1}(\mathcal{R}) \subseteq \mathcal{X} \cdot \Omega(\mathcal{R})$. This is covered by the subsets $\pi^{-1}(p(a))$ for $a \in \mathcal{B}(\mathcal{X})$. So (8.1.3) implies (8.4.3). Computations as in the proof of Proposition 8.1 show that $(f \circ r \circ \vartheta_a)(\omega) = f \circ \varrho_a \circ r(\omega)$ for $f \in C_0(\mathcal{G}^0)$, $a \in \mathcal{B}(\mathcal{X})$, and $\omega \in \Omega(\mathcal{R})$ with $r(\omega) \in s(a)$. Therefore (8.1.4) for functions on $\Omega(\mathcal{R})$ of the form $f \circ r$ implies (8.4.4). In the situation of (8.4.5), let (h_n) be a positive approximate unit in $C_0(p(a))$. Then $T_a\varphi_0(f_a^s) = \lim \varphi(h_n \circ \pi)T_a\varphi_0(f_a^s)$ by Lemma 7.2 and (8.1.3). A computation in the proof of Proposition 8.1 shows that $h_n \circ \pi \circ \vartheta_b$ is of the form $h_n' \circ r$ for an approximate unit (h_n') in $C_0(s(\langle a \mid b\rangle))$. Now

$$\langle T_a\varphi_0(f_a^s)\xi_a \mid T_b\varphi_0(f_b^s)\xi_b\rangle = \lim \langle \varphi(h_n \circ \pi)T_a\varphi_0(f_a^s)\xi_a \mid T_b\varphi_0(f_b^s)\xi_b\rangle = \lim \langle T_a\varphi_0(f_a^s)\xi_a \mid T_b\varphi_0(h_n' f_b^s)\xi_b\rangle.$$

Therefore it suffices to prove the equation in (8.4.5) when f_b^s is supported in $s(\langle a \mid b\rangle) \subseteq s(b)$. We assume this. Then $T_b(f_b^s)$ belongs to $\mathcal{E}_{\pi^{-1}(p(a)\cap p(b))}$, which is contained in the codomain of T_a^*. So

$$\begin{aligned}\langle T_a\varphi_0(f_a^s)\xi_a \mid T_b\varphi_0(f_b^s)\xi_b\rangle &= \langle \varphi_0(f_a^s)\xi_a \mid T_a^* T_b\varphi_0(f_b^s)\xi_b\rangle \\ &= \langle \varphi_0(f_a^s)\xi_a \mid T_{\langle a|b\rangle}\varphi_0(f_b^s)\xi_b\rangle = \langle \xi_a \mid \varphi_0((f_a^s)^\dagger)T_{\langle a|b\rangle}\varphi_0(f_b^s)\xi_b\rangle.\end{aligned}$$

This together with (8.4.4) implies (8.4.5). So all the conditions in Proposition 8.4 hold. ☐

Applied to the identity representation on $\mathcal{E} = C^*(\mathcal{M})$, the lemma implies that there is a nondegenerate *-homomorphism $\mathcal{O} \to C^*(\mathcal{M})$. In fact, it is not hard to describe the Toeplitz representation $C^*(\mathcal{G}) \to C^*(\mathcal{M})$ and $C^*(\mathcal{X}) \to C^*(\mathcal{M})$ directly. The more difficult point is that this map is an isomorphism or, equivalently, both C^*-algebras have the same nondegenerate compact representations. This means that, in the presence of the operators T_a as in Proposition 8.4, we may always extend the homomorphism $\varphi_0 : C_0(\mathcal{G}^0) \to \mathbb{K}(\mathcal{E})$ to all of $A_{[0,\infty)} \cong C_0(\Omega_{[0,\infty)})$.

Lemma 8.7. *The pointwise multiplication action of* $C_0(\mathcal{X}_n/\mathcal{G})$ *on* $\mathfrak{S}(\mathcal{X})$ *given by* $(M_f h)(x) := f([x])h(x)$ *for* $f \in C_0(\mathcal{X}_n/\mathcal{G})$, $h \in \mathfrak{S}(\mathcal{X})$, *and* $x \in \mathcal{X}_n$ *extends to a compact operator on* $C^*(\mathcal{X}_n)$, *and this defines a nondegenerate* **-homomorphism* $C_0(\mathcal{X}_n/\mathcal{G}) \to \mathbb{K}(C^*(\mathcal{X}_n))$.

Proof. The orbit space projection $\mathcal{X}_n \to \mathcal{X}_n/\mathcal{G}$ may be viewed as an action of the space $\mathcal{X}_n/\mathcal{G}$ on $\mathcal{X}_n$. It manifestly commutes with the right $\mathcal{G}$-action, and so it turns $\mathcal{X}_n$ into a groupoid correspondence $\mathcal{X}_n/\mathcal{G} \leftarrow \mathcal{G}$. Since the anchor map of the left action induces the identity homeomorphism $\mathcal{X}_n/\mathcal{G} \xrightarrow{\sim} \mathcal{X}_n/\mathcal{G}$, this groupoid correspondence is also proper. So the left action of $C^*(\mathcal{X}_n/\mathcal{G}) \cong C_0(\mathcal{X}_n/\mathcal{G})$ on $C^*(\mathcal{X})$ is by compact operators by [Antunes et al. 2022, Theorem 7.14]. This left action comes from the left $\mathfrak{S}(\mathcal{X}_n/\mathcal{G})$-module structure on $\mathfrak{S}(\mathcal{X}_n)$ that is defined in [Antunes et al. 2022, (7.1)]. In the case at hand, this simplifies to pointwise multiplication. □

Proposition 8.8. *Let $\mathcal{R} = \varnothing$, so that $\mathcal{O}$ is the Toeplitz C^*-algebra of $C^*(\mathcal{X})$. There is a canonical *-homomorphism $C_0(\Omega_{[0,\infty)}) = A_{[0,\infty)} \to \mathcal{O}$, which together with the partial unitaries T_a for $a \in \mathcal{B}$ associated to the identity representation of $\mathcal{O}$ on itself produces a nondegenerate *-homomorphism $C^*(\mathcal{M}) \to \mathcal{O}$.*

Proof. Let $D = C^*(\mathcal{G})$ and let $\mathcal{E} = C^*(\mathcal{X})$. Let $\mathcal{E}^{\otimes n}$ be the n-fold tensor product of $\mathcal{E}$ with itself over D. The multiplicativity of $\mathcal{X} \mapsto C^*(\mathcal{X})$ in [Antunes et al. 2022, Proposition 7.12] implies that

$$C^*(\mathcal{X}_n) \otimes_{C^*(\mathcal{G})} C^*(\mathcal{X}) \cong C^*(\mathcal{X}_{n+1})$$

for all $n \in \mathbb{N}$. So there are natural isomorphisms $\mathcal{E}^{\otimes n} \cong C^*(\mathcal{X}_n)$. Recall from [Pimsner 1997] that the Toeplitz C^*-algebra of $\mathcal{E}$ contains a copy of $\mathbb{K}(\mathcal{E}^{\otimes n})$ for all $n \in \mathbb{N}$ and that the product of $T_n \in \mathbb{K}(\mathcal{E}^{\otimes n})$ and $T_m \in \mathbb{K}(\mathcal{E}^{\otimes m})$ for $m \le n$ is the product of T_n with the adjointable operator $T_m \otimes \mathrm{id}_{\mathcal{E}^{\otimes n-m}}$.

Lemma 8.7 provides nondegenerate *-homomorphisms

$$C_0(\mathcal{X}_n/\mathcal{G}) \to \mathbb{K}(\mathcal{X}_n) \cong \mathbb{K}(\mathcal{E}^{\otimes n}) \to \mathcal{O}.$$

If $m \le n$ and $f_m \in C_0(\mathcal{X}_m/\mathcal{G})$, then the operator on $\mathcal{E}^{\otimes n}$ induced by pointwise multiplication by f_m is pointwise multiplication by the function $f_m \circ \pi_n^m$ on $\mathcal{X}_n/\mathcal{G}$. Therefore the product of $f_n \in C_0(\mathcal{X}_n/\mathcal{G})$ and $f_m \in C_0(\mathcal{X}_m/\mathcal{G})$ in $\mathcal{O}$ is the product of f_n and $f_m \circ \pi_n^m$ as functions on $\mathcal{X}_n/\mathcal{G}$. The same product is used to define the commutative C^*-algebras $A_{[0,n]}$ in Definition 5.2. Therefore we get *-homomorphisms $A_{[0,n]} \to \mathcal{O}$. These are compatible with the canonical maps $A_{[0,n]} \to A_{[0,m]}$ for $n \le m$. So they combine to a *-homomorphism $\varphi : C_0(\Omega_{[0,\infty)}) = A_{[0,\infty)} \to \mathcal{O}$.

Now use Proposition 8.4 to associate to the identity map on $\mathcal{O}$ a nondegenerate *-homomorphism $\varphi_0 : C_0(\mathcal{G}^0) \to \mathcal{O}$ and a family of partial unitaries T_a on $\mathcal{O}$, viewed as a Hilbert module over itself. The Toeplitz representation (ψ, L) that corresponds to the identity map on $\mathcal{O}$ is such that ψ and L are the canonical inclusions of $C^*(\mathcal{G}) = D$ and $C^*(\mathcal{X}) = \mathcal{E}$ into the Toeplitz C^*-algebra $\mathcal{O}$ of $\mathcal{E}$, respectively. The *-homomorphism φ_0 is the restriction of the canonical map $C^*(\mathcal{G}) \to \mathcal{O}$ to $C_0(\mathcal{G}^0)$. This agrees with the restriction of φ to $C_0(\mathcal{G}^0) \subseteq C_0(\Omega_{[0,\infty)})$. The partial unitary T_a is defined by $T_a(\psi(f^s)\xi) = L(f)\xi$ for all $f \in C_c(a) \subseteq C^*(\mathcal{X})$ and $\xi \in \mathcal{O}$.

We claim that φ and the partial unitaries T_a satisfy the conditions in Proposition 8.1 for $\mathcal{R} = \varnothing$ with $\Omega(\varnothing) = \Omega_{[0,\infty)}$. The conditions (8.4.1) and (8.4.2) imply (8.1.1) and (8.1.2) because φ_0 is the restriction of φ to $C_0(\mathcal{G}^0) \subseteq C_0(\Omega_{[0,\infty)})$.

We now check (8.1.3). The codomain of T_a for $a \in \mathcal{B}(\mathcal{X})$ is the closed right ideal in $\mathcal{O}$ generated by $C_c(a) \subseteq C^*(\mathcal{X}) \subseteq \mathcal{O}$. Let $f_1, f_2 \in C_c(a)$ and $f_3 \in \mathfrak{S}(\mathcal{X})$. Then

$$\begin{aligned} |f_1\rangle\langle f_2| f_3(x) &= (f_1 * \langle f_2 \mid f_3\rangle)(x) \\ &= \sum_{g \in \mathcal{G}^{s(x)}} f_1(xg)\langle f_2 \mid f_3\rangle(g^{-1}) = \sum_{g \in \mathcal{G}^{s(x)}} \sum_{\{y \in \mathcal{X} : s(y) = s(g)\}} f_1(xg)\overline{f_2(y)} f_3(yg^{-1}). \end{aligned}$$

If g and y occur in a nonzero summand, then $s(y) = s(xg)$ and $xg, y \in a$. This forces $y = xg$ because a is a slice. So

$$|f_1\rangle\langle f_2| f_3(x) = \sum_{g\in\mathcal{G}^{s(x)}} f_1(xg)\overline{f_2(xg)} f_3(x).$$

In other words, $|f_1\rangle\langle f_2|$ is the operator of pointwise multiplication by the function $\mathcal{X}/\mathcal{G} \to \mathbb{C}$, $[x] \mapsto \sum_{g\in\mathcal{G}^{s(x)}} f_1(xg)\overline{f_2(xg)}$. If $f_3 \in C_c(a)$ as well, then $|f_1\rangle\langle f_2| f_3 \in C_c(a)$ is the pointwise product $f_1 f_2^\dagger f_3$. Thus any function in $C_c(a)$ may be written as $|f_1\rangle\langle f_2| f_3$ for $f_1, f_2, f_3 \in C_c(a)$. This implies that the closed right ideal in $\mathcal{O}$ generated by $C_c(a)$ is equal to the closed right ideal generated by $|f_1\rangle\langle f_2| \in \mathbb{K}(C^*(\mathcal{X})) \subseteq \mathcal{O}$ for $f_1, f_2 \in C_c(a)$. The proof of Proposition 8.8 shows that this is the closed right ideal in $\mathcal{O}$ generated by $C_c(p(a)) \subseteq C_0(\mathcal{X}/\mathcal{G}) \subseteq C_0(\Omega_{[0,\infty)})$. Then (8.1.3) follows.

Next we check (8.1.4) for $a \in \mathcal{B}$. Since the sum of $C_0(\mathcal{X}_n/\mathcal{G})$ is dense in $A_{[0,\infty)}$, we may assume without loss of generality that f_1 belongs to the image of $C_0(\mathcal{X}_n/\mathcal{G})$ for some $n \in \mathbb{N}$; this image is $\pi_n^*(C_0(\mathcal{X}_n/\mathcal{G}))$ for the canonical map $\pi_n : \Omega_{[0,\infty)} \to \Omega_{[0,n]}$, where we tacitly embed $\mathcal{X}_n/\mathcal{G}$ into $\Omega_{[0,n]}$. Then

$$\varphi(f_1) \in \mathbb{K}(\mathcal{E}^{\otimes n}) \subseteq \mathcal{O}.$$

The domain D_{a^*a} is $r^{-1}(s(a)) \subseteq \Omega_{[0,\infty)}$. Therefore $C_0(D_{a^*a}) = r^*(C_0(s(a))) \cdot C_0(\Omega_{[0,\infty)})$, and we may assume without loss of generality that $f_2 = r^*(f_3) \cdot f_4$ for some $f_3 \in C_0(s(a))$ and $f_4 \in C_0(\Omega_{[0,\infty)})$. We are going to prove that

$$T_a^* \varphi(f_1) T_a \varphi_0(f_3^s) = \varphi(f_1 \circ \vartheta_a)\varphi_0(f_3^s)$$

for $f_1 \in \pi_n^* C_0(\mathcal{X}_n/\mathcal{G})$ and $f_3 \in C_0(a)$, so that $f_3^s \in C_0(s(a))$. This implies (8.1.4). We distinguish the cases $a \in \mathcal{B}(\mathcal{G})$ and $a \in \mathcal{B}(\mathcal{X})$.

Let $a \in \mathcal{B}(\mathcal{G})$. Then $T_a\varphi_0(f_3^s) = \psi(f_3) \in C_c(a) \subseteq C^*(\mathcal{G})$. When we multiply with $\varphi(f_1) \in C_0(\mathcal{X}_n/\mathcal{G}) \subseteq \mathbb{K}(\mathcal{E}^{\otimes n})$ in the Toeplitz C^*-algebra, we compose $\varphi(f_1)$ with the operator $\psi(f_3) \otimes \mathrm{id}_{\mathcal{E}^{\otimes n-1}} \in \mathbb{K}(\mathcal{E}^{\otimes n})$. The latter maps a function $h \in \mathfrak{S}(\mathcal{X}_n) \subseteq C^*(\mathcal{X}_n) \cong \mathcal{E}^{\otimes n}$ to the function

$$[x_1, \dots, x_n] \mapsto \begin{cases} f_3(g)h[g^{-1} \cdot x_1, x_2, \dots, x_n] & \text{if } r(x_1) \in r(a), \\ 0 & \text{otherwise.} \end{cases}$$

Here $x_1, \dots, x_n \in \mathcal{X}$ satisfy $s(x_j) = r(x_{j+1})$ for $j = 1, \dots, n-1$, $[x_1, \dots, x_n]$ denotes the equivalence class of $(x_1, \dots, x_n)$ in $\mathcal{X}_n$, and $g \in a$ is the unique element with $r(a) = r(x_1)$. Then $\varphi(f_1)$ multiplies pointwise with the function $f_1 \in C_0(\mathcal{X}_n/\mathcal{G})$, viewed as a $\mathcal{G}$-invariant function on $\mathcal{X}_n$. Finally, T_a^* gives the function

$$[x_1, \dots, x_n] \mapsto \begin{cases} f_1[g \cdot x_1, x_2, \dots, x_n] f_3(g) h[g^{-1} g x_1, x_2, \dots, x_n] & \text{if } r(x_1) \in s(a), \\ 0 & \text{otherwise.} \end{cases}$$

Here $g \in a$ is the unique element with $s(g) = r(x_1)$, and then it is also the unique element of a with $r(g) = r(g \cdot x_1)$. The function $f_1 \circ \vartheta_a$ is defined on the set of $[x_1, \dots, x_n] \in \mathcal{X}_n$ with $r[x_1, \dots, x_n] = r(x_1) \in s(a)$, and there it has the value $f_1[g \cdot x_1, \dots, x_n]$ for the unique $g \in a$ with $s(g) = r(x_1)$. So the composite operator $T_a^*\varphi(f_1)T_a\varphi_0(f_3^s)$ first multiplies pointwise with $f_3^s(r[x_1, \dots, x_n])$ and then with $f_1 \circ \vartheta_a$. That is, it is equal to $\varphi(f_1 \circ \vartheta_a)\varphi_0(f_3^s)$, as desired.

Next let $a \in \mathcal{B}(\mathcal{X})$. The computation is similar. Now $T_a\varphi_0(f_3^s) = L(f_3) \in C^*(\mathcal{X})$ maps $\mathcal{E}^{\otimes n-1} \to \mathcal{E}^{\otimes n}$ in the Toeplitz C^*-algebra. So the composite with $\varphi(f_1)$ is now the operator in $\mathbb{K}(\mathcal{E}^{\otimes n-1}, \mathcal{E}^{\otimes n})$ that maps $h \in \mathfrak{S}(\mathcal{X}_{n-1}) \subseteq C^*(\mathcal{X}_{n-1}) \cong \mathcal{E}^{\otimes n-1}$ to the function

$$\mathcal{X}_n \to \mathbb{C}, \quad [x_1, \dots, x_n] \mapsto \begin{cases} f_3(x_1)h[x_2, x_3, \dots, x_n] & \text{if } p(x_1) \in p(a), \\ 0 & \text{otherwise.} \end{cases}$$

Here the representative $[x_1, x_2, \dots, x_n]$ is chosen with $x_1 \in a$, which is possible if and only if $p(x_1) \in p(a)$. Multiplying pointwise with f_1 and applying T_a^* gives the operator in $\mathbb{K}(\mathcal{E}^{\otimes n-1})$ that maps $h \in \mathfrak{S}(\mathcal{X}_{n-1}) \subseteq C^*(\mathcal{X}_{n-1}) \cong \mathcal{E}^{\otimes n-1}$ to the function on $\mathcal{X}_{n-1}$ given by

$$[x_2, \dots, x_n] \mapsto \begin{cases} f_1[x_1, \dots, x_n] f_3(x_1) h[x_2, x_3, \dots, x_n] & \text{if } r(x_2) \in s(a), \\ 0 & \text{otherwise.} \end{cases}$$

Here $x_1 \in a$ is the unique element with $s(x_1) = r(x_2)$. The function $f_1 \circ \vartheta_a$ is defined at $[x_2, \dots, x_n]$ if and only if $r(x_2) \in s(a)$ and then takes the value $f_1[x_1, \dots, x_n]$ for x_1 as above. So the composite operator $T_a^*\varphi(f_1)T_a\varphi_0(f_3^s)$ is equal to the image of $\varphi(f_1 \circ \vartheta_a)\varphi_0(f_3^s)$ in $\mathbb{K}(\mathcal{E}^{\otimes n-1})$, as desired. This finishes the proof of (8.1.4). Since (8.1.5) is redundant, the representation φ and the operators T_a have all the conditions required in Proposition 8.1. □

Finally, we may state and prove our main theorem:

Theorem 8.9. *Let $\mathcal{G}$ be an étale locally compact groupoid and let $\mathcal{X} : \mathcal{G} \leftarrow \mathcal{G}$ be an étale locally compact groupoid correspondence. Let $\mathcal{R} \subseteq \mathcal{G}^0$ be an open subset such that $\mathcal{X}$ is proper on $\mathcal{R}$. Let $\mathcal{M} := \Omega(\mathcal{R}) \rtimes \mathcal{I}(\mathcal{G}, \mathcal{X})$ be the groupoid model of $(\mathcal{G}, \mathcal{X}, \mathcal{R})$. Then $C^*(\mathcal{M})$ is isomorphic to the Cuntz–Pimsner algebra of the C^*-correspondence $C^*(\mathcal{X}) : C^*(\mathcal{G}) \leftarrow C^*(\mathcal{G})$ relative to the ideal $C^*(\mathcal{G}_{\mathcal{R}})$, which acts on $C^*(\mathcal{X})$ by compact operators.*

Proof. We abbreviate $\mathcal{I} = \mathcal{I}(\mathcal{G}, \mathcal{X})$ as above. Let $\mathcal{T}$ denote the Toeplitz C^*-algebra of $C^*(\mathcal{X})$, which is the Cuntz–Pimsner algebra relative to the zero ideal, which corresponds to $\mathcal{R} = \emptyset$. Then Lemma 8.6 and Proposition 8.8 produce canonical nondegenerate *-homomorphisms $\mathcal{T} \to C^*(\Omega_{[0,\infty)} \rtimes \mathcal{I})$ and $C^*(\Omega_{[0,\infty)} \rtimes \mathcal{I}) \to \mathcal{T}$. On the level of the universal properties in Propositions 8.1 and 8.4, these nondegenerate *-homomorphisms map the representation of $C^*(\Omega_{[0,\infty)} \rtimes \mathcal{I})$ corresponding to (φ, T_a) to the representation of $\mathcal{T}$ corresponding to (φ_0, T_a) with the same partial unitaries T_a and $\varphi_0 = \varphi \circ r^*$ for $r : \Omega_{[0,\infty)} \to \mathcal{G}^0$. The proof of Proposition 8.8 shows that φ_0 may be extended in a unique way to φ so as to satisfy the conditions in Proposition 8.1. In particular, we get a bijection on the level of the data in Propositions 8.1 and 8.4. So the two maps above are inverse to each other.

Now let $\mathcal{R}$ be arbitrary. Lemma 8.6 shows that the map $\mathcal{T} \to C^*(\Omega_{[0,\infty)} \rtimes \mathcal{I})$ descends to a nondegenerate *-homomorphism $\mathcal{O} \to C^*(\Omega(\mathcal{R}) \rtimes \mathcal{I})$. It remains to prove that the nondegenerate *-homomorphism $C^*(\Omega_{[0,\infty)} \rtimes \mathcal{I}) \to \mathcal{T}$ descends to $C^*(\Omega(\mathcal{R}) \rtimes \mathcal{I}) \to \mathcal{O}$. The embeddings of $C_0(\mathcal{G}^0)$ and $C_0(\mathcal{X}/\mathcal{G})$ into $\mathcal{O}$ combine to a canonical map $C_0(\Omega_{[0,1]}) \to \mathcal{O}$. The Cuntz–Pimsner covariance condition implies that if $f \in C_0(\mathcal{R})$, then $f \in C_0(\mathcal{G}^0)$ and $f \circ r_* \in C_0(\mathcal{X}/\mathcal{G})$ have the same image in $\mathcal{O}$. This says that the *-homomorphism $C_0(\Omega_{[0,1]}) \to \mathcal{O}$ vanishes on the ideal $C_0(\mathcal{R})$. The covariance (8.1.4) implies that the kernel of the *-homomorphism $C_0(\Omega_{[0,\infty)}) \to \mathcal{T} \to \mathcal{O}$ is invariant under the action of $\mathcal{I}$. By Proposition 6.8, the $\mathcal{I}$-orbit of $\mathcal{R} \subseteq \mathcal{G}^0$ is equal to the complement of $\Omega(\mathcal{R})$ in $\Omega_{[0,\infty)}$. This gives the desired factorisation of the *-homomorphism $C_0(\Omega_{[0,\infty)}) \to \mathcal{T} \to \mathcal{O}$ through $C_0(\Omega(\mathcal{R}))$. □

Remark 8.10. Since the construction of full groupoid C^*-algebras is exact, the closed invariant subset $\Omega(\mathcal{R}) \subseteq \Omega_{[0,\infty)}$ gives rise to an extension of groupoid C^*-algebras

$$C_0\biggl(\bigsqcup_{n\in\mathbb{N}} (\mathcal{X}_n/\mathcal{G})_{\mathcal{R}}\biggr) \rtimes \mathcal{I}(\mathcal{X},\mathcal{G}) \rightarrowtail C_0(\Omega_{[0,\infty)}) \rtimes \mathcal{I}(\mathcal{X},\mathcal{G}) \twoheadrightarrow C_0(\Omega_{\mathcal{R}}) \rtimes \mathcal{I}(\mathcal{X},\mathcal{G}).$$

So the kernel in this extension is the kernel of the canonical quotient map from the Toeplitz C^*-algebra to the relative Cuntz–Pimsner algebra. It is well known that this kernel is isomorphic to the C^*-algebra of compact operators on $\mathcal{F} \cdot C^*(\mathcal{G}_{\mathcal{R}})$, where $\mathcal{F}$ denotes the Fock module associated to $C^*(\mathcal{X})$. This is Morita–Rieffel equivalent to the ideal $C^*(\mathcal{G}_{\mathcal{R}})$. The Morita–Rieffel equivalence of this ideal and $C^*(\mathcal{G}_{\mathcal{R}})$ also follows from Theorem 2.3 and Proposition 6.8.

9. Some more examples

We have discussed throughout the text how graph C^*-algebras fit into our theory. Namely, this is the special case where the groupoid $\mathcal{G}$ is just a discrete set V and $\mathcal{R}$ is the set of regular vertices in the graph; a groupoid correspondence on V is the same as a directed graph. It is well known that the Cuntz–Pimsner algebra of the associated C^*-correspondence relative to the ideal $C_0(\mathcal{R})$ is the graph C^*-algebra. Finally, we consider the case of groupoid correspondences of discrete groupoids in complete generality. These groupoid correspondences are roughly the same as the self-similar groupoid actions of [Laca et al. 2018], except that we do not require the graph in question to be finite.

So let $\mathcal{G}$ be a groupoid with the discrete topology. Then $\mathcal{X}$ also carries the discrete topology because $s\colon \mathcal{X} \to \mathcal{G}^0$ is a local homeomorphism. Being a groupoid correspondence means that the set $\mathcal{X}$ carries commuting actions of $\mathcal{G}$ on the left and right, where the right action is free. Then we may choose a fundamental domain $F \subseteq \mathcal{X}$ for the right $\mathcal{G}$-action so that the following map is a bijection:

$$F \times_{s,\mathcal{G}^0,r} \mathcal{G} \xrightarrow{\sim} \mathcal{X}, \quad (x, g) \mapsto x \cdot g.$$

Since the left action commutes with the right action, it must be of the form

$$h \cdot (x, g) = (h \circ x, h|_x \cdot g) \tag{9.1}$$

for $h \in \mathcal{G}$, $x \in F$, and $g \in \mathcal{G}$ with $s(h) = r(x)$ and $s(x) = r(g)$; here $\circ$ is a $\mathcal{G}$-action on F and $h|_x \in \mathcal{G}$ is defined for $h \in \mathcal{G}$ and $x \in F$ with $s(h) = r(x)$, and it satisfies $s(h|_x) = s(x) = r(g)$ and $(hg)|_x = h|_{g\circ x} \cdot g|_x$ for composable h, g, and x. We refer to [Antunes et al. 2022, Proposition 4.3] for more details in the special case where $\mathcal{G} = V \rtimes \Gamma$ is the transformation groupoid of a group action on a discrete set, or to [Laca et al. 2018].

The maps $r, s\colon \mathcal{X} \rightrightarrows \mathcal{G}^0$ restrict to the fundamental domain and allow us to view it as a directed graph. This point of view is taken in [Laca et al. 2018]. The groupoid $\mathcal{G}$ acts both on the set $\mathcal{G}^0$ of vertices of the graph and on the edges by the action $\circ$. However, (9.1) requires the relation

$$s(h \circ x) = r(h|_x \cdot g) = h|_x \cdot r(g) = h|_x \cdot s(x)$$

instead of $s(h \circ x) = h \cdot s(x)$. The relation is also used in [Laca et al. 2018], but it is rather unnatural if one thinks of an action on a graph. In fact, the original article [Exel and Pardo 2017] assumes both $s(h \circ x) = h \cdot s(x)$ and $h|_x \cdot s(x) = h \cdot s(x)$.

Let $\mathcal{R} \subseteq \mathcal{G}^0$. Then $\mathcal{X}$ is proper on $\mathcal{R}$ if and only if $r^{-1}(v) \cap F \subseteq F$ is finite for all $v \in \mathcal{R}$; equivalently, $r^{-1}(v)/\mathcal{G}$ is finite in $\mathcal{X}/\mathcal{G}$. We call $\mathcal{X}$ *regular on* $\mathcal{R}$ if, in addition, $r^{-1}(v)$ is nonempty for all $v \in \mathcal{R}$. It is reasonable to restrict attention to this case, although our theory also works without this assumption.

Since $\mathcal{G}$ and $\mathcal{X}$ are discrete, the sets of δ-functions δ_x for $x \in \mathcal{G}$ or $x \in \mathcal{X}$ are bases of $\mathfrak{S}(\mathcal{G})$ and $\mathfrak{S}(\mathcal{X})$, respectively, and the *-algebra structure on $\mathfrak{S}(\mathcal{G})$ and the $\mathfrak{S}(\mathcal{G})$-bimodule structure and inner product on $\mathfrak{S}(\mathcal{X})$ are uniquely determined by their values on the δ-functions. These are given by $\delta_g * \delta_h := \delta_{gh}$ if $s(g) = r(h)$ and 0 otherwise, $\delta_g^* = \delta_{g^{-1}}$ for $g \in \mathcal{G}$, and $\langle \delta_x \mid \delta_y \rangle = \delta_h$ if $s(x) = r(h)$ and $x \cdot h = y$ and 0 if there is no $h \in \mathcal{G}$ with $s(x) = r(h)$ and $x \cdot h = y$; the relation for $\delta_g * \delta_h$ also applies if one of $\{g, h\}$ belongs to $\mathcal{X}$ and then describes the $\mathfrak{S}(\mathcal{G})$-bimodule structure on $\mathfrak{S}(\mathcal{X})$. This implies the following:

Theorem 9.2. *Let $\mathcal{G}$ be a discrete groupoid and let $\mathcal{X} : \mathcal{G} \leftarrow \mathcal{G}$ be a groupoid correspondence that is proper on a subset $\mathcal{R} \subseteq \mathcal{G}^0$. Let D be a C^*-algebra. A Toeplitz representation $C^*(\mathcal{X}) \to D$ is equivalent to elements $T_x \in D$ for $x \in \mathcal{G} \sqcup \mathcal{X}$ that satisfy the following relations*:

- *let $x, y \in \mathcal{G} \sqcup \mathcal{X}$ and $x \in \mathcal{G}$ or $y \in \mathcal{G}$; then $T_x * T_y := T_{xy}$ if $s(x) = r(y)$ and $T_x * T_y := 0$ if $s(x) \neq r(y)$;*
- *$T_g^* = T_{g^{-1}}$ for $g \in \mathcal{G}$;*
- *if $x, y \in \mathcal{X}$ and there is $h \in \mathcal{G}$ with $s(x) = r(h)$ and $x \cdot h = y$, then $T_x^* T_y = T_h$, and $T_x^* T_y = 0$ if no such h exists.*

This representation is Cuntz–Pimsner covariant on $C^*(\mathcal{G}_\mathcal{R})$ if and only if

$$T_v = \sum_{x \in r_\mathcal{X}^{-1}(v) \cap F} T_x T_x^* \tag{9.3}$$

holds for all $v \in \mathcal{R}$. As a consequence, the Cuntz–Pimsner algebra of $C^*(\mathcal{X})$ is the universal C^*-algebra with the generators T_x for $x \in \mathcal{G}^0 \sqcup \mathcal{X}$ and the relations above.

If v does not belong to the range of $r : \mathcal{X} \to Y$, then the relation (9.3) says that $T_v = 0$. This is why it is reasonable to restrict attention to the case when $\mathcal{X}$ is regular on $\mathcal{R}$.

Remark 9.4. Recall that $F \subseteq \mathcal{X}$ is a fundamental domain. The products $T_x T_g = T_{xg}$ for all $x \in F$ and $g \in \mathcal{G}$ with $s(x) = r(g)$ exhaust the generators for elements of $\mathcal{X}$. So the elements T_x for $x \in \mathcal{G}^0 \sqcup F$ suffice to generate the Toeplitz and Cuntz–Pimsner algebras of $C^*(\mathcal{X})$. When we use only these generators, there is no longer any relation for $T_x T_g$ for $x \in F$ and $g \in \mathcal{G}$, and the product relation for elements of $\mathcal{G}$ and $\mathcal{X}$ becomes $T_g T_x = T_{g \circ x} T_{g|_x}$ for $g \in \mathcal{G}$ and $x \in F$ with $s(g) = r(x)$, with the notation in (9.1). If $x, y \in F$, then $T_x^* T_y$ is only nonzero if $x = y$ because $p|_F$ is injective. So $T_x^* T_y = \delta_{x,y} T_{s(x)}$ for $x, y \in F$, and this relation implies, and therefore completely replaces, the more general relation $T_x^* T_y = T_{\langle x | y \rangle}$ for $x, y \in \mathcal{X}$.

Now let $\mathcal{R}$ be the set of regular vertices for the graph $r, s : F \rightrightarrows \mathcal{G}^0$ in the usual sense. Then the relations for the generators T_x for $x \in \mathcal{G}^0 \sqcup F$ in the Cuntz–Pimsner algebra $\mathcal{O}$ of $C^*(\mathcal{X})$ relative to $C^*(\mathcal{G}_{\mathcal{R}})$ are exactly the defining relations of a Cuntz–Krieger family for the graph $r, s : F \rightrightarrows \mathcal{G}^0$. Thus these generators generate a *-homomorphism from the graph C^*-algebra of this graph to $\mathcal{O}$. The generators T_x for $x \in \mathcal{G} \supseteq \mathcal{G}^0$ satisfy the same relations as in the groupoid C^*-algebra $C^*(\mathcal{G})$. Finally, there are some extra commutation relations for $T_x T_y$, for $x \in \mathcal{G}$ and $y \in F$.

If $\mathcal{R} \subseteq \mathcal{G}^0$ is a different subset, then the statements above remain true if we replace the graph C^*-algebra by the relative graph C^*-algebra, where the relation $p_v = \sum_{x \in r^{-1}(v) \cap F} T_x T_x^*$ is imposed for $v \in \mathcal{R}$.

Our main theorem, Theorem 8.9, says that $\mathcal{O}$ is the groupoid C^*-algebra of the groupoid $\mathcal{M} = \Omega(\mathcal{R}) \rtimes \mathcal{I}(\mathcal{G}, \mathcal{X})$. We have also described $\mathcal{M}$ through a universal property, which specifies how it acts on locally compact Hausdorff spaces. Such an action on a space Y consists of an action of $\mathcal{G}$ and a map $\mathcal{X} \times_{s,\mathcal{G}^0,r} Y \to Y$ with certain properties. Since $\mathcal{G}^0$ is discrete, the anchor map $r : Y \to \mathcal{G}^0$ is equivalent to a disjoint union decomposition $Y = \bigsqcup_{x \in \mathcal{G}^0} Y_x$, namely, $Y_x := r_Y^{-1}(x) \subseteq Y$. Then $g \in \mathcal{G}$ acts on Y by a homeomorphism $\vartheta_g : Y_{s(g)} \to Y_{r(g)}$, $y \mapsto g \cdot y$. If $x \in \mathcal{X}$, then left multiplication by x must be a homeomorphism ϑ_x from $Y_{s(x)}$ onto a clopen subset of $Y_{r(x)}$. In fact, each singleton in $\mathcal{G}$ and $\mathcal{X}$ is a slice, and $\vartheta_x = \vartheta(\{x\})$ is the partial homeomorphism associated to a singleton slice. The product of two singleton slices $\{x\}$ and $\{y\}$ is either the singleton $\{x \cdot y\}$ if $s(x) = r(y)$ or empty otherwise, and the empty slice acts by the empty map on Y. Similarly, $\langle \{x\} \mid \{y\} \rangle = \{\langle x \mid y \rangle\}$ if $x, y \in \mathcal{X}$ satisfy $p(x) = p(y)$, so that $\langle x \mid y \rangle$ is defined and $\langle \{x\} \mid \{y\} \rangle$ is empty otherwise. With these clarifications, our partial homeomorphisms $\vartheta_x = \vartheta(\{x\})$ for $x \in \mathcal{G} \sqcup \mathcal{X}$ now satisfy the relations (4.4.1) and (4.4.2). In addition, it is clear that the graph of $\vartheta(\mathcal{U})$ for a slice $\mathcal{U}$ is simply the disjoint union of the graphs of ϑ_x for all $x \in \mathcal{U}$. Thus the partial homeomorphisms ϑ_x for $x \in \mathcal{G} \sqcup \mathcal{X}$ determine the action of all slices on Y. In fact, we may replace $\mathcal{I}$ in our description of $\mathcal{M}$ by the inverse semigroup with zero element that is generated by the singleton slice generators $\Theta(x)$ for $x \in \mathcal{G} \sqcup \mathcal{X}$ with the analogues of the relations (4.4.1) and (4.4.2), where the right-hand side is understood to be the zero element if the relevant slice is empty.

For $n \in \mathbb{N}$, let

$$F_n := \{(x_1, \dots, x_n) \in F^n : s(x_j) = r(x_{j+1}) \text{ for } j = 1, \dots, n-1\}.$$

This is in bijection with a fundamental domain for the composite groupoid correspondence $\mathcal{X}_n = \mathcal{X}^{\circ n}$. That is, $F_n \cong \mathcal{X}_n / \mathcal{G}$. So $\Omega_{[0,n]} = \bigsqcup_{k=0}^{n} F_k$ as a set. If F and $\mathcal{G}^0$ are finite, then it follows that all the sets F_k are finite and so the topology on $\Omega_{[0,n]}$ is still discrete. Then $\Omega_{[0,\infty)}$ carries the projective limit topology of these finite compact spaces. In general, $\Omega_{[0,n]}$ carries the locally compact Hausdorff topology induced by the commutative C^*-algebra $A_{[0,n]}$ in Definition 5.2. We also get this topology by iterating the fibrewise one-point compactification construction several times. We refrain from giving more details. The space $\Omega(\mathcal{R})$ is also the object space of the groupoid that underlies the graph C^*-algebra of $r, s : F \rightrightarrows \mathcal{G}^0$ inside $\mathcal{O}$.

Remark 9.5. The definition of a self-similar groupoid action in [Laca et al. 2018] imposes some extra assumptions. Namely, the vertex and edge sets of the graph, $\mathcal{G}^0$ and F, are assumed to be finite, and the induced action of $\mathcal{G}$ on the path space of the graph F is assumed to be faithful. In this case, the C^*-algebra of the self-similar groupoid action that is studied in [Laca et al. 2018] is the absolute Cuntz–Pimsner algebra of $C^*(\mathcal{X})$, that is, for $\mathcal{R} = \mathcal{G}^0$ (compare [Laca et al. 2018, Proposition 4.7]).

Summing up, our construction for a discrete groupoid $\mathcal{G}$ gives the C^*-algebras of self-similar groupoid actions of [Laca et al. 2018], except that we drop some assumptions, such as the graph being finite or the action on the path space being faithful. Thus we propose that our construction for a general étale locally compact groupoid is a good definition of a self-similar action of an étale locally compact groupoid. Of course, topological graphs are a special case of our theory. To get Katsura's topological graph C^*-algebras [2004], we must let $\mathcal{R}$ be the largest subset of the vertex set where the range map is surjective and proper. For a more general groupoid correspondence $\mathcal{X}$, it is not clear whether Katsura's ideal for the corresponding C^*-correspondence $C^*(\mathcal{X})$ is of the form $C^*(\mathcal{G}_{\mathcal{X}})$ for an open invariant subset $\mathcal{X} \subseteq \mathcal{G}^0$. We seem to need this to be the case in order for the resulting Cuntz–Pimsner algebra to be a groupoid C^*-algebra in a natural way.

Finally, we specialise to the case of a groupoid correspondence $\mathcal{X} : \mathcal{G} \leftarrow \mathcal{G}$ for a discrete group $\mathcal{G}$. By [Antunes et al. 2022, Example 4.2], this is the same as a discrete set $\mathcal{X}$ with commuting actions of $\mathcal{G}$ on the left and right, such that the right action is free and proper. The only two options for $\mathcal{R} \subseteq \mathcal{G}^0$ are the empty set and the one-point set $\mathcal{G}^0$ itself. If $\mathcal{R} = \mathcal{G}^0$, then we need $\mathcal{X}$ to be proper, which amounts to $\mathcal{X}/\mathcal{G}$ being finite. If we add the condition that the $\mathcal{G}$-action on the set $\varprojlim \mathcal{X}_n/\mathcal{G}$ should be free, then this is the same as a self-similar group. If $\mathcal{R} = \varnothing$, then any groupoid correspondence $\mathcal{X}$ is allowed, that is, we may also allow $\mathcal{X}/\mathcal{G}$ to be infinite. This goes beyond the scope of the theory of self-similar groups.

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Ancient Ricci flows of bounded girth

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For each $n \geq 3$, we construct a "pancake-like" ($O(2)\times O(n-1)$-invariant with positive curvature operator and bounded "girth") ancient Ricci flow, which generalises the two-dimensional "ancient sausage" solution. We also establish explicit asymptotics backwards in time. This solution is new even in dimension three and opens new avenues into the analysis of ancient solutions. Indeed, the symmetry ansatz does *not* reduce the problem to the analysis of a system of ODEs; instead we obtain a coupled system of parabolic PDEs with one space variable, which we use to establish the invariance under Ricci flow of certain inequalities on the curvature and its spatial derivatives (which does not follow from Hamilton's tensor maximum principle) and exploit the evolution of volume, Myers' theorem and the differential Harnack inequality.

1. Introduction

A solution to the Ricci flow equation $(\partial/\partial t)g(t) = -2\,\mathrm{Ric}_{g(t)}$ is called *ancient* if it is defined for all $t \in (-\infty, T)$. Ancient solutions to geometric flows have aroused a great deal of interest in recent years, as they are natural parabolic counterparts of complete solutions to the corresponding elliptic equation; see [Daskalopoulos 2020] for a recent survey. We are particularly interested in the Liouville-type rigidity phenomena they may exhibit, in the spirit of Appell's theorem [Hirschman 1952] stating that positive ancient solutions to the heat equation on $\mathbb{R}^n$ with subexponential growth must be constant.

With Appell's theorem in mind, it is natural to consider ancient Ricci flows with positive curvature operator. In fact, this is not actually a restriction, at least in low dimensions, since Ricci flow tends to force curvature towards the positive [Brendle 2019; Chen et al. 2013; Hamilton 1999; Ivey 1993]. In the compact case, positivity of the curvature operator forces the underlying manifold to be S^n, up to a

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quotient [Böhm and Wilking 2008]. Ancient Ricci flows on S^2 are completely classified [Daskalopoulos et al. 2012], the only examples being the shrinking round spheres and the "ancient sausage" solution.[1] Remarkably, in a recent major achievement, positively curved ancient Ricci flows on S^3 have also been classified, under an additional noncollapsing assumption [Brendle et al. 2021]; the only examples are the shrinking round spheres and Perelman's "football" solution [2003b]. This classification has been extended to higher dimensions $n \geq 4$, under suitable curvature positivity assumptions [Brendle et al. 2023]. We note here that the noncollapsing assumption, however natural from the singularity formation viewpoint (due to Perelman's "no local collapsing" theorem [2002]), is not as natural in the compact case. (Indeed, any compact blowup limit of a finite-time Ricci flow singularity must be a shrinking soliton [Zhang 2007].) Nonetheless, given the difficulty of the problem, these results are a remarkable achievement and provide spectacular progress towards a general classification of ancient Ricci flows in low dimensions.

Our main result is the construction of an interesting new ancient Ricci flow on S^n with positive curvature operator in each dimension $n \geq 3$.

Theorem 1.1. *For each $n \geq 3$, there exists an $O(2)\times O(n-1)$-invariant ancient Ricci flow on S^n with positive curvature operator and bounded* ***girth***. *Pointed limits of this solution are asymptotic*, *as $t \to -\infty$, to either a flat cylinder $S^1 \times \mathbb{R}^{n-1}$ or a cigar plane* $(\mathbb{R}^2, g_{\text{cigar}}) \times \mathbb{R}^{n-2}$.

Roughly speaking, the *girth* is the length of a shortest closed geodesic; see Section 2.1 for the precise definition. The bounded girth condition implies that the solution is collapsed. The group $O(2)\times O(n-1) \subset O(n+1)$ (standard block embedding) acts on $\mathbb{R}^{n+1} \supset S^n$ by orthogonal transformations, with generic orbits of codimension one when restricted to S^n. We call the fixed-point sets of the $O(2)$- and $O(n-1)$-actions the *tip* and the *waist*, respectively. When the sequence of marked points stays within bounded distance from the *waist*, the asymptotic limit is the flat cylinder, whereas if they stay close to the *tip*, the limit is a cigar plane. We note that the tip and waist regions are both of the same scale: the asymptotic cylinder at the waist and the asymptotic cylinder of the asymptotic cigar both have the same radius.

Geometrically, our terminology for the fixed-point sets is justified by considering, for each time slice $(S^n, g(t))$, a totally geodesic S^2 intersecting each $O(n-1)$-orbit orthogonally. Such a surface, which we refer to as the *section*, exists in fact for any $O(2)\times O(n-1)$-invariant metric on S^n (indeed, the $O(n-1)$-action is polar). In the case of our solution, it is a highly elongated, rotationally symmetric 2-sphere, whose equator (the waist) realises the girth, which is uniformly bounded as $t \to -\infty$, and whose poles are the so-called tips. Our solution may be thought of as "rotating" these elongated 2-sphere while keeping their waist fixed; if instead of 2-spheres we had highly elongated ovals, our solution in dimension three would resemble a pancake, with bounded thickness but diverging radius as $t \to -\infty$.

The proof of Theorem 1.1 is in part inspired by the construction of ancient mean curvature flows in [Bourni et al. 2021]; cf. [Bourni et al. 2020]. We consider a sequence of old-but-not-ancient $O(2)\times O(n-1)$-invariant Ricci flow solutions on S^n whose initial data has a section S^2 which is a time-slice of the ancient sausage solution. The estimates required to extract a limit, to estimate the existence

[1]Discovered independently by Fateev, Onofri and Zamolodchikov [Fateev et al. 1993], King [1993] and Rosenau [1995].

time and to analyse the asymptotic behaviour, rely crucially on a rather precise space-time control of the four different eigenvalues of the curvature operator; see Propositions 3.2 and 3.3 (we note here that the second fundamental form in the mean curvature flow solution [Bourni et al. 2021] has only two different eigenvalues). Unfortunately, Hamilton's tensor maximum principle is not applicable in this situation, and we must rely on a subtle analysis of the PDE system satisfied by the curvature operator under this symmetry assumption, which has unbounded coefficients due to the presence of lower-dimensional orbits.

There do not appear to be many other known examples of collapsed ancient Ricci flows with positive curvature operator. In dimension two, a classification of all complete ancient Ricci flows is possible [Chow et al. 2006; Daskalopoulos et al. 2012; Daskalopoulos and Sesum 2006; Hamilton 1988; Chu 2007]; all of them are collapsed with positive curvature operator, apart from those with constant sectional curvature. On S^3, we know that any ancient solution on S^3 has positive curvature (nonnegativity follows from [Chen 2009], and positivity by the strong maximum principle), and yet the classification result of [Brendle et al. 2011] implies that this positive curvature cannot be pinched for nonround solutions. To our knowledge, an exhaustive list of known collapsed examples on S^3 consists of Fateev's "ancient hypersausage" [1995; 1996], the homogeneous "ancient Hopf fibrations" of Bakas, Kong and Ni [Bakas et al. 2012] (see also [Buzano 2014; Cao and Saloff-Coste 2009; Isenberg and Jackson 1992; Lauret 2013]), and a family interpolating between them, which we shall call the "twisted ancient hypersausages" [Bakas et al. 2012, §3]. In the noncompact setting, Y. Lai [2025; 2024] was able to construct, in each dimension $n \geq 3$, a family of "flying wing" steady Ricci solitons on $\mathbb{R}^n$, which are $O(n-1)$-invariant and interpolate between a cigar-hyperplane and Bryant's soliton; our construction is in some ways analogous to Lai's: in both cases we are solving a geometric partial differential equation on a noncompact domain (hers elliptic and ours parabolic) with a symmetry assumption which leaves two degrees of freedom (subverting any attempt at reduction to a system of ODEs), and in neither case are the powerful tools related to noncollapsing available.

On the other hand, much is known about ancient Ricci flows that are *noncollapsed*, at least in the presence of various curvature nonnegativity assumptions. For example, noncollapsed ancient Ricci flows with uniform PIC and weak PIC2 have been classified [Angenent et al. 2022; Brendle 2020; Brendle et al. 2021]; they consist of the shrinking spheres, the Bryant solitons and Perelman's "football" solutions. If we maintain weak PIC2 and noncollapsing, and replace uniform PIC with type-I curvature growth, the ancient Ricci flow is a symmetric space [Lynch and Abrego 2024]. On the other hand, if we drop the type-I curvature growth condition, but keep nonnegative curvature operator and noncollapsing, there appear to be many more: analogous solutions to Perelman's football [2003b] in higher dimensions with $O(k)\times O(n+1-k)$-symmetry (see [Chow et al. 2010; 2008] for $k = 1$, and [Buttsworth 2022] for the case $k = 2$ and $n = 4$), and also Haslhofer's new solutions [2024].

We also mention some results on homogeneous ancient Ricci flows, i.e., those ancient solutions which admit a transitive and isometric group action. Homogeneous ancient solutions on spheres of any dimension have been recently classified by Sbiti [2022], and it appears that only the round sphere has positive curvature operator. Further examples on homogeneous torus bundles were constructed by Lu and Wang [2017], Buzano [2014] and Böhm, Lafuente and Simon [Böhm et al. 2019]. In fact, a compact homogeneous space admits a collapsing ancient homogeneous Ricci flow (of not necessarily positive curvature) if and only if it is the total space of a homogeneous torus bundle; all known homogeneous examples are invariant under a corresponding torus action (this is known to be necessary under certain assumptions [Krishnan et al. 2026]) and, after appropriately rescaling, collapse the torus fibres as time tends to minus infinity and Gromov–Hausdorff converge to an Einstein metric on the base [Cao and Saloff-Coste 2009; Pediconi and Sbiti 2022].

Finally, despite the fact that there are clearly many ancient Ricci flows — as indicated in the previous paragraphs — we conjecture that our example is unique amongst $O(n-1)$-invariant ancient Ricci flows on S^n for $n \geq 4$, with positive curvature operator and bounded girth.

Conjecture 1.2. *Let $n \geq 4$. Then, up to isometry, parabolic rescalings and time-translation, the Ricci flow solution on S^n constructed in Theorem 1.1 is the unique $O(n-1)$-invariant ancient solution with positive curvature operator and bounded girth.*

Remark 1. We do not assume the full $O(2)\times O(n-1)$-symmetry in this conjecture. Also, the conjecture fails in dimension three due to the existence of the (torus fibred) ancient hypersausage. (The $n=3$ case is special is because $O(2)$ is abelian.)

2. Preliminaries

We begin by developing a precise notion of "girth" suitable for Theorem 1.1 and Conjecture 1.2. We then describe the class of Riemannian metrics on S^n that we work with, and collect some basic facts about how these metrics behave under the Ricci flow.

2.1. ***The girth of a Riemannian sphere.*** Given a smooth manifold M, let ΛM be the space of piecewise smooth maps $S^1 \to M$, and let $\Lambda^0 M \subset \Lambda M$ denote the constant maps. Recall that the energy and length functionals $E, \ell : \Lambda M \to \mathbb{R}$ are defined by

$$E(c) := \frac{1}{2}\int_{S^1} |c'|^2 \, d\theta, \quad \text{length}(c) := \int_{S^1} |c'| \, d\theta.$$

Observe that $\text{length}(c) \leq \sqrt{2E(c)}$, with equality when the parametrisation is proportional to arclength. Consider the homotopy class Σ_1 of continuous maps

$$F : B^{n-1} \to \Lambda M, \quad F(\partial B^{n-1}) \subset \Lambda^0 M,$$

which have degree one, in a sense to be made precise below. Elements of Σ_1 are called 1-*sweepouts*.

Definition 2.1. The 1-*width* and *girth* of (S^n, g) are defined by

$$\text{width}_1(S^n, g) := \inf_{F\in\Sigma_1} \max_{p\in B^{n-1}} E(F(p)), \quad \text{girth}(S^n, g) := \sqrt{2\,\text{width}_1(S^n, g)}.$$

It follows from [Klingenberg 1978, Appendix 1] that these invariants are positive and in fact realised by a closed geodesic in (S^n, g). Indeed, his Lemma A.1.4 provides a homotopy which decreases the maximum energy of a sweepout F if there are no closed geodesics of that energy level.

Following [Klingenberg 1978], we now explain how a continuous map

$$F : B^{n-1} \to \Lambda M, \quad F(\partial B^{n-1}) \subset \Lambda^0 M$$

induces a continuous map $f : S^n \to S^n$, which we use to define the degree $\deg F := \deg f$ of the sweepout F. Using coordinates $(x_0, \ldots, x_n)$ for $\mathbb{R}^{n+1}$, we may view B^{n-1} as a half-equator

$$B^{n-1} \simeq \{x_0 \geq 0, x_1 = 0\} \subset S^n \subset \mathbb{R}^{n+1}.$$

For each $p \in B^{n-1}$ there is a circle $\theta \mapsto a_p(\theta)$ in S^n for $\theta \in S^1$ which starts at $a_p(1) = p$ in the direction of $x_1 \geq 0$, and along which the coordinates $x_2, \ldots, x_n$ remain constant. Notice that for $p \in \partial B^{n-1} \simeq \{x_0 = x_1 = 0\}$, $a_p(\theta) \equiv p$ is constant. Thus any $q \in S^n$ admits a representation $q = a_p(\theta)$, for a unique $p \in B^{n-1}$, and $\theta \in S^1$ which is unique when $p \notin \partial B^{n-1}$. We then simply set

$$f(a_p(\theta)) := F(p)(\theta).$$

Importantly, it is easy to see that a homotopy $[0, 1] \times (B^{n-1}, \partial B^{n-1}) \to (\Lambda M, \Lambda^0 M)$ of F induces a homotopy $[0, 1] \times S^n \to S^n$ of f.

Remark 2. The analogous notion of 2-width is known to interact well with the Ricci flow. Indeed, Colding and Minicozzi [2008] obtained a differential inequality for the evolution of the 2-width of any homotopy 3-sphere under Ricci flow-with-surgery, which they were able to exploit to provide a different proof that the flow-with-surgery terminates in finite time in this setting; see also Hamilton [1999] and Perelman [2003a].

Remark 3. The above notion of width originates in work of Birkhoff [1917]. A number of related (but not necessarily equivalent) notions of width have also been developed, including the Almgren–Pitts width, which is based on Almgren's isomorphism theorem and was exploited in the celebrated Almgren–Pitts min–max theory [Almgren 1962; Pitts 1981] (see also [Marques and Neves 2016] for a survey), and Gromov's [1983] definition in terms of level sets of proper functions $M^n \to \mathbb{R}^{n-1}$. The later was invoked in the study of Ricci flows on surfaces by Daskalopoulos and Hamilton [2004] and Daskalopoulos and Sesum [2006]; see also Chu [2007]. For our purposes here, these notions of 1-width can be used in place of the 1-girth defined above. We have chosen to use the latter, however, as it easier to work with (e.g., it is always attained by the length of some smooth closed geodesic).

2.2. *$O(n-1)$-invariant Riemannian metrics.* Consider the standard action of $O(n-1)$ on $S^n \subset \mathbb{R}^{n+1}$ by rotating the last $n-1$ coordinates, and let g be any $O(n-1)$-invariant Riemannian metric on S^n, i.e., one for which $O(n-1)$ acts by isometries. The image $\Sigma \subset S^n$ of the embedding $\gamma : S^2 \to S^n$,

$$\gamma(x_1, x_2, x_3) = (x_1, x_2, 0, \ldots, 0, x_3),$$

is a totally geodesic submanifold of (S^n, g), as it coincides with the fixed-point set of the closed subgroup $O(n-2) \subset O(n-1)$ rotating the $n-2$ coordinates $x_3, \ldots, x_n$, which is nontrivial for $n \geq 3$. Similarly, the image $\gamma(E)$ of the equator $E = \{(x_1, x_2, 0) \in S^2\}$ is a closed geodesic, being the fixed-point set of the full group $O(n-1)$. Given any $p \in S^n \setminus \gamma(E)$, its $O(n-1)$-orbit is an $(n-2)$-dimensional sphere, intersecting Σ

exactly twice. We thus define $\Sigma_+ := \gamma(S^2_+)$ where $S^2_+ = \{x_3 > 0\}$ denotes the northern hemisphere. It follows that Σ_+ intersects each $O(n-1)$-orbit exactly once, and $\partial\Sigma_+ = \gamma(E)$ is a closed geodesic.

For each point $x \in S^2 \setminus E$ we write

$$T_{\gamma(x)}S^n = T_{\gamma(x)}\gamma(S^2) \oplus K, \tag{1}$$

where K is the subspace of $\mathbb{R}^{n+1}$ consisting of points x where the first, second and last entries are zero (recall that $n+1 \geq 4$, and thus $K \neq 0$). The Euclidean inner product on $\mathbb{R}^{n+1}$ induces an $O(n-2)$-invariant inner product Q on K.

Lemma 2.2. *Let g be an $O(n-1)$-invariant Riemannian metric on S^n for $n \geq 3$. Along Σ_+, g can be written as a warped product*

$$g = g^\top + \varphi^2 g_{S^{n-2}}, \tag{2}$$

where $g^\top = g|_{T\Sigma}$, $g_{S^{n-2}}$ is the round metric on S^{n-2}, and $\varphi : \Sigma^+ \to \mathbb{R}_{>0}$ is smooth.

Proof. For nonequator points $p \in \Sigma_+$, the action of the isotropy subgroup $O(n-2)$ on $T_pS^n = T_p\Sigma \oplus K$ is trivial on $T_p\Sigma$ and contains no trivial subspaces in K (even if $n = 3$, where the action is simply a reflection). It follows from Schur's lemma that g makes (1) orthogonal, and that restricted to K it must be a multiple of Q. □

Remark 4. The converse of this lemma is not true. In order for the expression in (2) to define a Riemannian metric g which extends smoothly to all of S^n it is necessary that

$$\bar{\varphi}(x_1, x_2, x_3) := \begin{cases} \varphi(\gamma(x_1, x_2, x_3)), & x_3 > 0, \\ 0, & x_3 = 0, \\ -\varphi(\gamma(x_1, x_2, x_3)), & x_3 < 0, \end{cases}$$

is a smooth function on S^2.

2.3. *$O(2)\times O(n-1)$-invariant Riemannian metrics.* The natural action of $O(2) \times O(n-1)$ on $S^n \subset \mathbb{R}^2 \oplus \mathbb{R}^n$ is of cohomogeneity one. In the coordinates described below, the quotient map of this action is given by $r : S^n \to \left[0, \tfrac{1}{2}\pi\right]$. The principal orbits $r^{-1}(r_0)$, $r_0 \in \left(0, \tfrac{1}{2}\pi\right)$, are diffeomorphic to $S^1 \times S^{n-2}$. The singular orbit $r^{-1}(0) \simeq S^1$ will be called the *waist*, and the singular orbit $r^{-1}\left(\tfrac{1}{2}\pi\right) \simeq S^{n-2}$ will be referred to as the *tip*. This terminology is justified when considering the 2-dimensional picture obtained by quotienting out the $O(n-1)$-action.

To describe $O(2)\times O(n-1)$-invariant Riemannian metrics on S^n, we define

$$\gamma : \left[0, \tfrac{1}{2}\pi\right] \to S^n, \quad \gamma(r) = (\cos(r), 0, \dots, 0, \sin(r)).$$

This curve starts at the waist, ends at the tip, and intersects all the orbits exactly once. Moreover, the reflection about the $\langle e_1, e_{n+1}\rangle$-plane in $\mathbb{R}^{n+1}$ is an element in $O(2) \times O(n-1)$ which pointwise fixes the curve, thus γ is a geodesic for any invariant metric. The principal domain for the action (i.e., the union of all orbits through $\gamma(r)$ for $r \in \left(0, \tfrac{1}{2}\pi\right)$) is diffeomorphic to $\left(0, \tfrac{1}{2}\pi\right) \times S^1 \times S^{n-2}$. We denote by θ the standard coordinate on S^1 and by $g_{S^{n-2}}$ the round metric on S^{n-2}. Also, given a smooth function

$f : \left[0, \tfrac{1}{2}\pi\right] \to \mathbb{R}$, we call it *even* (resp. *odd*) *at* 0, if its even (resp. odd) extension to $\left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right)$ is smooth. Analogous terminology will be used at $\tfrac{1}{2}\pi$. We then have:

Proposition 2.3. *For any $O(2)\times O(n-1)$-invariant Riemannian metric g on S^n, for $n \geq 3$, there exist smooth functions $\chi, \psi, \varphi : \left[0, \tfrac{1}{2}\pi\right] \to \mathbb{R}$, positive on $\left(0, \tfrac{1}{2}\pi\right)$, such that*

$$g_{\gamma(r)} = \chi(r)^2\, dr^2 + \psi(r)^2\, d\theta^2 + \varphi(r)^2 g_{S^{n-2}} \quad \textit{for all } r \in \left(0, \tfrac{1}{2}\pi\right), \tag{3}$$

while at $r = 0, \tfrac{1}{2}\pi$, the following "smoothness conditions" are satisfied:

(1) χ *is even and positive at* 0 *and* $\tfrac{1}{2}\pi$;

(2) ψ *is even and positive at* 0 *and odd at* $\tfrac{1}{2}\pi$, *and* $\psi'\left(\tfrac{1}{2}\pi\right) = -\chi\left(\tfrac{1}{2}\pi\right)$;

(3) φ *is even and positive at* $\tfrac{1}{2}\pi$ *and odd at* 0, *and* $\varphi'(0) = \chi(0)$.

Conversely, any smooth functions χ, ψ and φ satisfying (1)–(3) *give rise to a metric via* (3) *which extends to a smooth, $O(2)\times O(n-1)$-invariant Riemannian metric on S^n.*

Proof. By considering the reflections through the subspaces $\langle e_2\rangle, \langle e_3, \dots, e_n\rangle \subset \mathbb{R}^{n+1}$ it follows that any invariant metric makes $\left(0, \tfrac{1}{2}\pi\right)$, S^1 and S^{n-2} orthogonal. Since its restriction to S^{n-2} is $O(n-1)$-invariant, said restriction must be a multiple of $g_{S^{n-2}}$ and hence (3) follows. Regarding the smoothness conditions, these are well known and we refer the reader to [Petersen 2016, Chapter 4] for the details. ☐

Note that since these metrics are $O(n-1)$-invariant, all of the material of Section 2.2 still applies. In particular, we denote by $g^\top$ the Riemannian metric on the totally geodesic submanifold $\Sigma \simeq S^2 \subset S^n$, which on the principal part is given by

$$g^\top_{\gamma(r)} = \chi(r)^2\, dr^2 + \psi(r)^2\, d\theta^2, \quad r \in \left(0, \tfrac{1}{2}\pi\right).$$

Note that $g^\top$ is now $O(2)$-invariant, instead of merely $O(1)$-invariant.

2.4. *$O(2)\times O(n-1)$-invariant Ricci flow on S^n.*

Consider now a *time-dependent* Riemannian metric g on S^n which is evolving by Ricci flow. If the initial metric is $O(2)\times O(n-1)$-invariant, the uniqueness and diffeomorphism-invariance properties of the Ricci flow imply that the Riemannian metric will continue to be $O(2)\times O(n-1)$-invariant for all future times. As a consequence, on the principal domain the Riemannian metric g must have the form of (3) for some functions χ, ψ and φ which are now time-dependent. Then, using the formula for Ricci curvature (33), the Ricci flow equation gives rise to a system of PDEs for the functions $\chi(r,t)$, $\psi(r,t)$ and $\varphi(r,t)$. Introducing now the (time-dependent) arc-length parameter $s(r,t) = \int_0^r \chi(v,t)\, dv$, the Ricci flow equation becomes

$$\chi_t = \chi\left(\frac{\psi_{ss}}{\psi} + (n-2)\frac{\varphi_{ss}}{\varphi}\right), \tag{4a}$$

$$\psi_t = \psi_{ss} + (n-2)\frac{\psi_s\varphi_s}{\varphi}, \tag{4b}$$

$$\varphi_t = \varphi_{ss} + \frac{\varphi_s\psi_s}{\psi} - (n-3)\frac{1-\varphi_s^2}{\varphi}. \tag{4c}$$

Note that the commutator of ∂_t and ∂_s is given by

$$[\partial_t, \partial_s] = -\left(\frac{\psi_{ss}}{\psi} + (n-2)\frac{\varphi_{ss}}{\varphi}\right)\partial_s. \tag{5}$$

2.5. ***Evolution of curvature under $O(2)\times O(n-1)$-invariant Ricci flow.*** As described in Appendix A, the Riemann curvature operator $\mathrm{Rm} : TS^n \wedge TS^n \to TS^n \wedge TS^n$ of the metric (3) has at most four distinct eigenvalues, and these coincide with the following sectional curvatures:

- $K^\top = -\psi_{ss}/\psi$, the Gauss curvature of $(S^2, g^\top)$ (multiplicity one);
- $K_1^\perp = -\varphi_{ss}/\varphi$, the sectional curvature of planes spanned by ∂_r and S^{n-2} (multiplicity $n-2$);
- $K_2^\perp = -\psi_s\varphi_s/(\psi\varphi)$, the sectional curvature of planes between ∂_θ and S^{n-2} (multiplicity $n-2$);
- $L = (1-\varphi_s^2)/\varphi^2$, the sectional curvature in the S^{n-2} direction (only appears if $n \geq 4$; multiplicity $\binom{n-2}{2}$).

Notice that even though these four quantities are only defined along principal orbits, the smoothness conditions from Proposition 2.3 ensure that they can be extended to smooth invariant scalar functions on S^n.

It is well-known that, with respect to a moving orthonormal frame, the curvature operator evolves under the Ricci flow according to

$$\frac{\partial\,\mathrm{Rm}}{\partial t} = \Delta\mathrm{Rm} + 2(\mathrm{Rm}^2 + \mathrm{Rm}^\sharp),$$

where Δ denotes the tensor Laplacian and the operator $\cdot^\sharp$ denotes the Lie algebra square, both taken with respect to the time-varying metric $g(t)$, and $\partial/\partial t$ is interpreted according to the Uhlenbeck trick; see [Chow et al. 2006, §2.7.2] or [Andrews and Hopper 2011, §6]. Using the expressions for all of these terms in Appendix A, we obtain the evolution equations

$$(\partial_t-\Delta)K^\top = 2((K^\top)^2+(n-2)K_1^\perp K_2^\perp)-2(n-2)\left(\frac{\varphi_s}{\varphi}\right)^2(K^\top-K_2^\perp), \tag{6a}$$

$$(\partial_t-\Delta)K_1^\perp = 2((K_1^\perp)^2+K^\top K_2^\perp+(n-3)K_1^\perp L)+2\left(\frac{\psi_s}{\psi}\right)^2(K_2^\perp-K_1^\perp)-2(n-3)\left(\frac{\varphi_s}{\varphi}\right)^2(K_1^\perp-L), \tag{6b}$$

$$(\partial_t-\Delta)K_2^\perp = 2((K_2^\perp)^2+K^\top K_1^\perp+(n-3)K_2^\perp L)-2\left(\frac{\psi_s}{\psi}\right)^2(K_2^\perp-K_1^\perp)+2\left(\frac{\varphi_s}{\varphi}\right)^2(K^\top-K_2^\perp), \tag{6c}$$

$$(\partial_t-\Delta)L = 2((K_1^\perp)^2+(K_2^\perp)^2+(n-3)L^2)+4\left(\frac{\varphi_s}{\varphi}\right)^2(K_1^\perp-L), \tag{6d}$$

where Δ is the Laplace–Beltrami operator acting on functions. By differentiating these expressions in s, and using (5), we also obtain

$$\begin{aligned}(\partial_t-\Delta)K_s^\top = {}&\left(4K^\top-\left(\frac{\psi_s}{\psi}\right)^2-3(n-2)\left(\frac{\varphi_s}{\varphi}\right)^2\right)K_s^\top+2(n-2)K_2^\perp(K_1^\perp)_s+2(n-2)\left(K_1^\perp+\left(\frac{\varphi_s}{\varphi}\right)^2\right)(K_2^\perp)_s\\ &+4(n-2)\frac{\varphi_s}{\varphi}\left(\left(\frac{\varphi_s}{\varphi}\right)^2+K_1^\perp\right)(K^\top-K_2^\perp)\end{aligned} \tag{7a}$$

and

$$(\partial_t-\Delta)(K_1^\perp)_s=\left[4K_1^\perp+(n-3)L-3\left(\frac{\psi_s}{\psi}\right)^2-(3n-8)\left(\frac{\varphi_s}{\varphi}\right)^2\right](K_1^\perp)_s+2K_2^\perp K_s^\top$$
$$+2\left[K^\top+\left(\frac{\psi_s}{\psi}\right)^2\right](K_2^\perp)_s+\left[(n-3)K_1^\perp+2(n-3)\left(\frac{\varphi_s}{\varphi}\right)^2\right]L_s$$
$$-4\frac{\psi_s}{\psi}\left[K^\top+\left(\frac{\psi_s}{\psi}\right)^2\right](K_2^\perp-K_1^\perp)+2\frac{\varphi_s}{\varphi}\left[K_1^\perp+\left(\frac{\varphi_s}{\varphi}\right)^2\right](K_1^\perp-L). \tag{7b}$$

3. Existence

We will construct an $O(2)\times O(n-1)$-invariant ancient Ricci flow by taking a limit of a family of "very old" Ricci flow solutions which start at symmetric extensions to S^n by "very early" time slices of the $O(2)$-invariant ancient sausage Ricci flow on S^2 (see Section B.2).

3.1. ***The initial metrics.*** For each $\tau<0$, consider the functions

$$\chi_\tau(r)=\sqrt{\frac{\tanh(-2\tau)}{1-\sin^2(r)\tanh^2(-2\tau)}},\quad \psi_\tau(r)=\chi_\tau(r)\cos(r),\quad \varphi_\tau(r)=\frac{\operatorname{arctanh}(\tanh(-2\tau)\sin(r))}{\sqrt{\tanh(-2\tau)}}. \tag{8}$$

It is straightforward to verify that these functions satisfy the boundary smoothness conditions of Proposition 2.3, so the expression

$$\underline{g}_\tau:=\chi_\tau^2\,dr^2+\psi_\tau^2\,d\theta^2+\varphi_\tau^2 g_{S^{n-2}}. \tag{9}$$

defines a smooth $O(2)\times O(n-1)$-invariant Riemannian metric on S^n.

Notice that

$$\chi_\tau^2\,dr^2+\psi_\tau^2\,d\theta^2=g_{\text{sausage}}(\tau)$$

is a time slice of the ancient sausage solution on S^2 (see Section B.2), whereas φ_τ has been chosen so that $\varphi_\tau'(r)/(\psi_\tau(r)\chi_\tau(r))=C$ is constant in r (the value of C is determined by χ_τ, ψ_τ and the smoothness conditions from Proposition 2.3). This in particular yields that the sectional curvatures of $\underline{g}_\tau$ satisfy $K_1^\perp=K_2^\perp$. In fact, even more is true:

Lemma 3.1. *The sectional curvatures of* $(S^n,\underline{g}_\tau)$ *satisfy*

$$K^\top\geq K_2^\perp\geq K_1^\perp\geq L>0, \tag{10a}$$

and

$$K_s^\top,\,(K_1^\perp)_s,\,(K_2^\perp)_s,\,L_s\geq 0. \tag{10b}$$

Proof. We compute

$$\frac{\psi_\tau'(r)}{\chi_\tau(r)}=\frac{\sin(r)}{\sin^2(r)\sinh^2(-2\tau)-\cosh^2(-2\tau)},\quad \frac{\varphi_\tau'(r)}{\chi_\tau(r)}=\frac{\psi_\tau(r)}{\sqrt{\tanh(-2\tau)}}.$$

Therefore,

$$K^\top = -\frac{\psi_{ss}}{\psi} = -\frac{1}{\psi_\tau \chi_\tau}\left(\frac{\psi_\tau'}{\chi_\tau}\right)' = \frac{1+\sin^2(r)\tanh^2(-2\tau)}{(\sinh(-2\tau)\cosh(-2\tau))(1-\sin^2(r)\tanh^2(-2\tau))}.$$

In particular, $K^\top$ is positive and monotone increasing in r (and therefore in s as well).

Next, we observe that since $\varphi_\tau'/(\chi_\tau\psi_\tau) = (\varphi_\tau)_s/\psi_\tau$ is constant in r and s, we can conclude that $(\varphi_\tau)_{ss}/\varphi_\tau = (\varphi_\tau)_s(\psi_\tau)_s/(\varphi_\tau\psi_\tau)$, so $K_1^\perp = K_2^\perp$ uniformly. Observe that

$$K_2^\perp = \frac{\sin(r)}{\cosh^2(-2\tau)\operatorname{arctanh}(\tanh(-2\tau)\sin(r))(1-\sin^2(r)\tanh^2(-2\tau))},$$

so that

$$\begin{aligned}\sinh(-2\tau)\cosh(-2\tau)&(1-\sin^2(r)\tanh^2(-2\tau))(K^\top - K_2^\perp)\\ &= 1+\sin^2(r)\tanh^2(-2\tau) - \frac{\sin(r)\tanh(-2\tau)}{\operatorname{arctanh}(\tanh(-2\tau)\sin(r))}.\end{aligned}$$

The nonnegativity of the function $1-x/\operatorname{arctanh}(x)$ implies that the right-hand side is nonnegative. Finally,

$$\operatorname{arctanh}^2(\tanh(-2\tau)\sin(r))L = \tanh(-2\tau) - \frac{\cos^2(r)\tanh(-2\tau)}{1-\sin^2(r)\tanh^2(-2\tau)},$$

so that L is strictly positive on $\left[0, \frac{1}{2}\pi\right]$ (in the sense of limits for $r = 0, \frac{1}{2}\pi$), and

$$\begin{aligned}(1-\sin^2(r)\tanh^2(-2\tau))&(K_2^\perp - L)\operatorname{arctanh}(\tanh(-2\tau)\sin(r))\\ &= \frac{\sin(r)}{\cosh^2(-2\tau)} - \frac{\tanh(-2\tau)(1-\sin^2(r)\tanh^2(-2\tau)) - \cos^2(r)\tanh(-2\tau)}{\operatorname{arctanh}(\tanh(-2\tau)\sin(r))}\\ &= \frac{\sin(r)\operatorname{arctanh}(\tanh(-2\tau)\sin(r)) + \cosh(-2\tau)\sinh(-2\tau)\sin^2(r)(\tanh^2(-2\tau)-1)}{\operatorname{arctanh}(\tanh(-2\tau)\sin(r))\cosh^2(-2\tau)},\end{aligned}$$

which is also nonnegative, due to the estimate $\operatorname{arctanh}(x) \geq x$ for $x \geq 0$.

We have established $K^\top \geq K_2^\perp = K_1^\perp \geq L > 0$ and $(K^\top)_s \geq 0$. The other required derivative estimates follow from (34) in Section A.4. □

Since $\underline{g}_\tau$ has positive curvature operator by Lemma 3.1 and Section A.3, the work of Böhm and Wilking [2008] implies that the maximal Ricci flow starting at $\underline{g}_\tau$ continues to have positive curvature operator, and develops a singularity in finite time modelled on the round sphere. By shifting time so that the singularity occurs at $t = 0$, we thereby obtain a τ-indexed family of Ricci flows

$$(S^n, g_\tau(t))_{t\in[\alpha_\tau,0)}, \quad g_\tau(\alpha_\tau) = \underline{g}_\tau,$$

with $1/(-2(n-1)t)g_\tau(t)$ smoothly converging to the round metric as $t \to 0$.

3.2. *Sturmian properties of the sectional curvatures.* In this section, we establish two remarkable properties of $O(2)\times O(n-1)$-invariant Ricci flows.

Proposition 3.2. *The curvature conditions* (10a) *are preserved under the Ricci flow of* $O(2)\times O(n-1)$*-invariant metrics on* S^n.

Proof. Given any $O(2)\times O(n-1)$-invariant Ricci flow on S^n, defined on the time interval $[\alpha, 0)$, Hamilton [1986] showed that positivity of the curvature operator is preserved under Ricci flow, so that in particular the inequality $\min\{K^\top, K_1^\perp, K_2^\perp, L\} > 0$ is preserved. By (6) we have

$$\begin{aligned}\frac{\partial}{\partial t}(K^\top - K_2^\perp) = \Delta(K^\top - K_2^\perp) - 2(n-1)\left(\frac{\varphi_s}{\varphi}\right)^2 (K^\top - K_2^\perp)\\ +2\left(\frac{\psi_s}{\psi}\right)^2 (K_2^\perp - K_1^\perp) + 2(K^\top - K_2^\perp)(K^\top + K_2^\perp - K_1^\perp)\\ +2(n-3)K_2^\perp(K_1^\perp - L),\end{aligned} \tag{11a}$$

$$\begin{aligned}\frac{\partial}{\partial t}(K_2^\perp - K_1^\perp) = \Delta(K_2^\perp - K_1^\perp) - 4\left(\frac{\psi_s}{\psi}\right)^2 (K_2^\perp - K_1^\perp) + 2\left(\frac{\varphi_s}{\varphi}\right)^2 (K^\top - K_2^\perp) + 2(n-3)\left(\frac{\varphi_s}{\varphi}\right)^2 (K_1^\perp - L)\\ +2(K_2^\perp - K_1^\perp)(K_2^\perp + K_1^\perp - K^\top + (n-3)L),\end{aligned} \tag{11b}$$

$$\begin{aligned}\frac{\partial}{\partial t}(K_1^\perp - L) = \Delta(K_1^\perp - L) + 2\left(\frac{\psi_s}{\psi}\right)^2 (K_2^\perp - K_1^\perp)\\ -2(n-1)\left(\frac{\varphi_s}{\varphi}\right)^2 (K_1^\perp - L) + 2K_2^\perp(K^\top - K_2^\perp) + 2(n-3)L(K_1^\perp - L).\end{aligned} \tag{11c}$$

We work in the nongeometric, time-independent coordinate $r \in \left[0, \frac{1}{2}\pi\right]$. The smoothness conditions on χ, ψ and φ (Proposition 2.3) imply that, close to $r = 0$,

$$\chi(r,t) = \chi(0,t) + r^2\chi_{\text{waist}}(r,t), \quad \psi(r,t) = \psi(0,t) + r^2\psi_{\text{waist}}(r,t), \quad \varphi(r,t) = r\chi(0,t) + r^3\varphi_{\text{waist}}(r,t),$$

where $\chi(0,t)$ and $\psi(0,t)$ are both positive, and the functions χ_{waist}, ψ_{waist}, and φ_{waist} are all smooth and even. Similarly, close to $\frac{1}{2}\pi$, we have

$$\begin{aligned}\chi\left(\tfrac{1}{2}\pi - r, t\right) &= \chi\left(\tfrac{1}{2}\pi, t\right) + r^2\chi_{\text{tip}}(r,t),\\ \psi\left(\tfrac{1}{2}\pi - r, t\right) &= r\chi\left(\tfrac{1}{2}\pi, t\right) + r^3\psi_{\text{tip}}(r,t),\\ \varphi\left(\tfrac{1}{2}\pi - r, t\right) &= \varphi\left(\tfrac{1}{2}\pi, t\right) + r^2\varphi_{\text{tip}}(r,t),\end{aligned}$$

where $\chi\left(\frac{1}{2}\pi, t\right)$ and $\varphi\left(\frac{1}{2}\pi, t\right)$ are positive, and χ_{tip}, ψ_{tip} and φ_{tip} are smooth and even with respect to $r = 0$. As a result, from (32) and $\partial_s = \chi^{-1}\partial_r$ we deduce that the curvatures near the waist satisfy

$$\begin{aligned}K^\top(r,t) &= -\frac{2\psi_{\text{waist}}(0,t)}{\chi(0,t)^2\psi(0,t)} + r^2 K^\top_{\text{waist}}(r,t)\\ K_2^\perp(r,t) &= -\frac{2\psi_{\text{waist}}(0,t)}{\chi(0,t)^2\psi(0,t)} + r^2 K^\perp_{2,\text{waist}}(r,t),\\ K_1^\perp(r,t) &= -\frac{6\varphi_{\text{waist}}(0,t)}{\chi^3(0,t)} + \frac{2\chi_{\text{waist}}(0,t)}{\chi^3(0,t)} + r^2 K^\perp_{1,\text{waist}}(r,t),\\ L(r,t) &= -\frac{6\varphi_{\text{waist}}(0,t)}{\chi^3(0,t)} + \frac{2\chi_{\text{waist}}(0,t)}{\chi^3(0,t)} + r^2 L_{\text{waist}}(r,t),\end{aligned} \tag{12}$$

where all of the waist functions are smooth and even in r. At the other end, we have

$$
\begin{aligned}
K^{\top}\big(\tfrac{1}{2}\pi-r\big) &= -\frac{6\psi_{\mathrm{tip}}(0,t)}{\chi^3\big(\tfrac{1}{2}\pi,t\big)} + \frac{2\chi_{\mathrm{tip}}(0,t)}{\chi^3\big(\tfrac{1}{2}\pi,t\big)} + r^2 K^{\top}_{\mathrm{tip}}(r,t),\\
K_2^{\perp}\big(\tfrac{1}{2}\pi-r\big) &= -\frac{2\varphi_{\mathrm{tip}}(0,t)}{\chi\big(\tfrac{1}{2}\pi,t\big)^2\varphi(0,t)} + r^2 K^{\perp}_{2,\mathrm{tip}}(r,t),\\
K_1^{\perp}\big(\tfrac{1}{2}\pi-r\big) &= -\frac{2\varphi_{\mathrm{tip}}(0,t)}{\chi\big(\tfrac{1}{2}\pi,t\big)^2\varphi(0,t)} + r^2 K^{\perp}_{1,\mathrm{tip}}(r,t),\\
L\big(\tfrac{1}{2}\pi-r\big) &= \frac{1}{\varphi\big(\tfrac{1}{2}\pi,t\big)^2} + r^2 L_{\mathrm{tip}}(r,t),
\end{aligned}
\tag{13}
$$

and all of the tip functions are smooth and even around $r=0$.

Define now a weight function $w:\big(0,\tfrac{1}{2}\pi\big)\to\mathbb{R}^+$ to be a smooth time-independent function such that $w(r)=1/r^2$ for $r<\tfrac{1}{6}\pi$, and $w(r)=1$ for $r>\tfrac{1}{3}\pi$. Consider the weighted quantities

$$
\begin{aligned}
X(r,t) &= w(r)(K^{\top}(r,t)-K_2^{\perp}(r,t)),\\
Y(r,t) &= w\big(\tfrac{1}{2}\pi-r\big)(K_2^{\perp}(r,t)-K_1^{\perp}(r,t)),\\
Z(r,t) &= w(r)(K_1^{\perp}(r,t)-L(r,t)).
\end{aligned}
\tag{14}
$$

The functions X, Y and Z of (14) are smooth on $\big[0,\tfrac{1}{2}\pi\big]$, and satisfy Neumann conditions at the boundary by (12) and (13). We now write evolution equations for these quantities. We have $\partial_s=\partial_r/\chi$, so

$$
\Delta u = u_{ss} + \left(\frac{\psi_s}{\psi}+(n-2)\frac{\varphi_s}{\varphi}\right)u_s = \frac{u_{rr}}{\chi^2} + \left(\frac{\psi_r}{\psi\chi^2}+(n-2)\frac{\varphi_r}{\varphi\chi^2}-\frac{\chi_r}{\chi^3}\right)u_r,
$$

implying

$$
\begin{aligned}
\Delta\left(\frac{u}{r^2}\right) &= \frac{1}{\chi^2}\left(\frac{u}{r^2}\right)_{rr} + \left(\frac{\psi_r}{\psi\chi^2}+(n-2)\frac{\varphi_r}{\varphi\chi^2}-\frac{\chi_r}{\chi^3}\right)\left(\frac{u}{r}\right)_r\\
&= \frac{1}{\chi^2}\left(\frac{u_{rr}}{r^2}-\frac{4u_r}{r^3}+\frac{6u}{r^4}\right) + \left(\frac{\psi_r}{\psi\chi^2}+(n-2)\frac{\varphi_r}{\varphi\chi^2}-\frac{\chi_r}{\chi^3}\right)\left(\frac{u_r}{r^2}-\frac{2u}{r^3}\right)\\
&= \frac{\Delta u}{r^2} - \frac{4u_r}{\chi^2 r^3} + u\left(\frac{6}{\chi^2 r^4}-\frac{2}{r^3}\left(\frac{\psi_r}{\psi\chi^2}+(n-2)\frac{\varphi_r}{\varphi\chi^2}-\frac{\chi_r}{\chi^3}\right)\right)\\
&= \frac{\Delta u}{r^2} - \frac{4}{\chi^2 r}\left(\frac{u}{r^2}\right)_r - \frac{u}{r^2}\left(\frac{2}{\chi^2 r^2}+\frac{2}{r}\left(\frac{\psi_r}{\psi\chi^2}+(n-2)\frac{\varphi_r}{\varphi\chi^2}-\frac{\chi_r}{\chi^3}\right)\right).
\end{aligned}
$$

We use this, together with (11), to write

$$
\begin{aligned}
\frac{\partial X}{\partial t} &= \frac{X_{rr}}{\chi^2} + D_1 X_r + E_{11}X + E_{12}Y + E_{13}Z,\\
\frac{\partial Y}{\partial t} &= \frac{Y_{rr}}{\chi^2} + D_2 Y_r + E_{21}X + E_{22}Y + E_{23}Z,\\
\frac{\partial Z}{\partial t} &= \frac{Z_{rr}}{\chi^2} + D_3 Z_r + E_{31}X + E_{32}Y + E_{33}Z,
\end{aligned}
$$

for $r \in \left[0, \frac{1}{2}\pi\right]$. Note that, for $r \in \left(0, \frac{1}{6}\pi\right)$, say, we get

$$
\begin{aligned}
D_1 &= D_3 = D_2 + \frac{4}{\chi^2 r} = \frac{4}{\chi^2 r} + \left(\frac{\psi_r}{\psi\chi^2} + (n-2)\frac{\varphi_r}{\varphi\chi^2} - \frac{\chi_r}{\chi^3}\right),\\
E_{11} &= \frac{2}{\chi^2 r^2} + \frac{2}{r}\left(\frac{\psi_r}{\psi\chi^2} + (n-2)\frac{\varphi_r}{\varphi\chi^2} - \frac{\chi_r}{\chi^3}\right) - 2(n-1)\left(\frac{\varphi_r}{\chi\varphi}\right)^2 + 2(K^\top + K_2^\perp - K_1^\perp),\\
E_{12} &= \frac{2}{r^2}\left(\frac{\psi_r}{\chi\psi}\right)^2, \quad E_{13} = 2(n-3)K_2^\perp, \quad E_{21} = 2r^2\left(\frac{\varphi_r}{\chi\varphi}\right)^2,\\
E_{22} &= -4\left(\frac{\psi_r}{\chi\psi}\right)^2 + 2(K_2^\perp + K_1^\perp - K^\top + (n-3)L),\\
E_{23} &= 2(n-3)r^2\left(\frac{\varphi_r}{\chi\varphi}\right)^2, \quad E_{31} = 2K_2^\perp, \quad E_{32} = \frac{2}{r^2}\left(\frac{\psi_r}{\chi\psi}\right)^2,\\
E_{33} &= \frac{2}{\chi^2 r^2} + \frac{2}{r}\left(\frac{\psi_r}{\psi\chi^2} + (n-2)\frac{\varphi_r}{\varphi\chi^2} - \frac{\chi_r}{\chi^3}\right) - 2(n-1)\left(\frac{\varphi_r}{\chi\varphi}\right)^2 + 2(n-3)L.
\end{aligned}
$$

The key information we need from this list is that for any cutoff time $\sigma \in (\alpha, 0)$, the coefficients satisfy the following for $r \in [0, \frac{\pi}{6}]$ and $t \in [\alpha, \sigma]$:

- $D_1 - (n+2)/\chi^2 r = D_3 - (n+2)/\chi^2 r = D_2 - (n-2)/\chi^2 r$ is a smooth odd function around $r = 0$,
- E_{ij} are bounded above for all $i, j = 1, \dots, 3$, and are nonnegative when $i \neq j$.

The computation for the tip is similar, except we first perform the change $r \mapsto r - \frac{\pi}{2}$ to simplify the outputs:

$$
\begin{aligned}
D_1 &= D_3 = D_2 - \frac{4}{\chi^2 r} = \frac{\psi_r}{\psi\chi^2} + (n-2)\frac{\varphi_r}{\varphi\chi^2} - \frac{\chi_r}{\chi^3},\\
E_{11} &= -2(n-1)\left(\frac{\varphi_r}{\chi\varphi}\right)^2 + 2(K^\top + K_2^\perp - K_1^\perp),\\
E_{12} &= 2r^2\left(\frac{\psi_r}{\chi\psi}\right)^2, \quad E_{13} = 2(n-3)K_2^\perp, \quad E_{21} = \frac{2}{r^2}\left(\frac{\varphi_r}{\chi\varphi}\right)^2,\\
E_{22} &= \frac{2}{\chi^2 r^2} + \frac{2}{r}\left(\frac{\psi_r}{\psi\chi^2} + (n-2)\frac{\varphi_r}{\varphi\chi^2} - \frac{\chi_r}{\chi^3}\right) - 4\left(\frac{\psi_r}{\psi\chi}\right)^2 + 2(K_2^\perp + K_1^\perp - K^\top + (n-3)L),\\
E_{23} &= \frac{2}{r^2}(n-3)\left(\frac{\varphi_r}{\chi\varphi}\right)^2, \quad E_{31} = 2K_2^\perp, \quad E_{32} = 2r^2\left(\frac{\psi_r}{\chi\psi}\right)^2, \quad E_{33} = -2(n-1)\left(\frac{\varphi_r}{\chi\varphi}\right)^2 + 2(n-3)L.
\end{aligned}
$$

As for the waist region, here the coefficients E_{ij} are all bounded above, nonnegative if $i \neq j$, and $D_1 - 1/(\chi^2 r) = D_3 - 1/(\chi^2 r) = D_2 - 5/(\chi^2 r)$ is smooth and odd around $r = 0$ (corresponding to the tip due to our change of variables).

Now, for each $\varepsilon > 0$ and $\sigma < 0$, define

$$
u_{\varepsilon,\sigma} = \min\{X, Y, Z\} + \varepsilon e^{(C+1)(t-\alpha)}, \quad C := 3 \max_{[0,\pi/6]\times[\alpha,\sigma]} \{E_{ij}(r,t) : 1 \le i, j \le 3\}.
$$

We claim that $u_{\varepsilon,\sigma}(r, t) \geq 0$ holds for all $(r, t) \in \left[0, \tfrac{1}{2}\pi\right] \times [\alpha, \sigma]$, provided that $\min\{X, Y, Z\} \geq 0$ holds at $t = \alpha$. Since $\varepsilon > 0$ is arbitrary, the proposition will follow from this. The assumption implies that $u_{\varepsilon,\sigma}(r, \alpha) > 0$ for all r. Arguing by contradiction, we let $t_0 > \alpha$ be the first time where $u_{\varepsilon,\sigma}(x_0, t_0) = 0$ for some $x_0 \in \left[0, \tfrac{1}{2}\pi\right]$. At least one of the following holds:

$$\begin{aligned}
u(x_0, t_0) &= X(x_0, t_0) + \varepsilon e^{(C+1)(t_0-\alpha)} = 0 \quad \text{and} \quad \min\{Y(x_0, t_0), Z(x_0, t_0)\} \geq X(x_0, t_0),\\
u(x_0, t_0) &= Y(x_0, t_0) + \varepsilon e^{(C+1)(t_0-\alpha)} = 0 \quad \text{and} \quad \min\{X(x_0, t_0), Z(x_0, t_0)\} \geq Y(x_0, t_0),\\
u(x_0, t_0) &= Z(x_0, t_0) + \varepsilon e^{(C+1)(t_0-\alpha)} = 0 \quad \text{and} \quad \min\{X(x_0, t_0), Y(x_0, t_0)\} \geq Z(x_0, t_0).
\end{aligned}$$

Suppose we are in the first case, so that $\widetilde{X}(x, t) := X(x, t) + \varepsilon e^{(C+1)(t-\alpha)} \geq 0$ for all $t \leq t_0$ and $x \in \left[0, \tfrac{1}{2}\pi\right]$, but $\widetilde{X}(x_0, t_0) = 0$. Then, since X is smooth on $\left[0, \tfrac{1}{2}\pi\right]$ with Neumann boundary conditions, we have

$$\widetilde{X}_t(x_0, t_0) \leq 0, \qquad \widetilde{X}_r(x_0, t_0) = 0, \qquad \widetilde{X}_{rr}(x_0, t_0) \geq 0. \tag{15}$$

Consider now the quantities

$$\begin{aligned}
Q(r, t) &= X_t(r, t) - \frac{X_{rr}(r, t)}{\chi^2(r, t)} - D_1(r, t) X_r(r, t),\\
\widetilde{Q}(r, t) &= \widetilde{X}_t(r, t) - \frac{\widetilde{X}_{rr}(r, t)}{\chi^2(r, t)} - D_1(r, t) \widetilde{X}_r(r, t).
\end{aligned}$$

The fact that X is smooth on the boundary with Neumann conditions, combined with the D_1 estimates at said boundary, imply that Q and $\widetilde{Q}$ are continuously extendable to the boundary. We furthermore claim that $\widetilde{Q}(x_0, t_0) \leq 0$. Indeed, if $x_0 \in \left(0, \tfrac{1}{2}\pi\right)$, then D_1 is smooth at x_0 and the claim follows immediately from (15). On the other hand, if $x_0 = 0$, (15) still holds, but we must study the sign of the first order term $D_1 \widetilde{X}$. The D_1 expression implies that there is a function $D_{1,\text{waist}}$ which is smooth at $r = 0$, and such that

$$D_1 = D_{1,\text{waist}} + \frac{n+2}{\chi^2 r}$$

for $r \in \left(0, \tfrac{1}{6}\pi\right)$. L'Hôpital's rule and (15) then yield

$$\lim_{r\to 0^+} D_1 \widetilde{X}_r = \lim_{r\to 0^+} D_{1,\text{waist}} X_r + \frac{n+2}{\chi^2}\frac{X_r}{r} = \frac{n+2}{\chi^2(0, t_0)} \widetilde{X}_{rr}(0, t_0) \geq 0.$$

The case $x_0 = \tfrac{1}{2}\pi$ is analogous.

Therefore,

$$\begin{aligned}
0 \geq \widetilde{Q}(x_0, t_0) &= (C+1)\varepsilon e^{(C+1)(t_0-\alpha)} + Q(x_0, t_0)\\
&= (C+1)\varepsilon e^{(C+1)(t_0-\alpha)} + \lim_{r\to x_0}\left(\frac{\partial X}{\partial t}(r, t_0) - X_{rr}(r, t_0) - D_0(r, t_0) X_r(r, t_0)\right)\\
&= (C+1)\varepsilon e^{(C+1)(t_0-\alpha)} + D_1(x_0, t_0) X(x_0, t_0) + D_2(x_0, t_0) Y(x_0, t_0) + D_3(x_0, t_0) Z(x_0, t_0)\\
&\geq (C+1)\varepsilon e^{(C+1)(t_0-\alpha)} + (D_1(x_0, t_0) + D_2(x_0, t_0) + D_3(x_0, t_0)) X(x_0, t_0)\\
&= (C+1)\varepsilon e^{(C+1)(t_0-\alpha)} - (D_1(x_0, t_0) + D_2(x_0, t_0) + D_3(x_0, t_0))\varepsilon e^{(C+1)(t_0-\alpha)} > 0,
\end{aligned}$$

a contradiction. The computations for the second and third cases are very similar. □

In fact, it turns out that all of (10) is preserved by the Ricci flow.

Proposition 3.3. *The curvature conditions* (10) *are preserved under Ricci flow of* $O(2)\times O(n-1)$*-invariant metrics on* S^n.

Proof. Choose any $O(2)\times O(n-1)$-invariant Ricci flow on S^n, defined on the time interval $[\alpha, 0)$. By Proposition 3.2, we know that

$$K^{\top} \geq K_2^{\perp} \geq K_1^{\perp} \geq L > 0 \tag{16}$$

is preserved, so it suffices to verify the curvature monotonicity conditions

$$K_s^{\top}, (K_1^{\perp})_s, (K_2^{\perp})_s, L_s \geq 0 \tag{17}$$

are preserved, under the assumption that our Ricci flow satisfies (16). Observe that (34) implies immediately that $(K_2^{\perp})_s \geq 0$ and $L_s \geq 0$. The other inequalities then follow from the maximum principle, much as for the curvature differences, applied to (7). Indeed, we can write these evolution equations as

$$(\partial_t - \Delta)K_s^{\top} = A_1 K_s^{\top} + A_2 (K_1^{\perp})_s + A_3, \quad (\partial_t - \Delta)(K_1^{\perp})_s = B_1 K_s^{\top} + B_2 (K_1^{\perp})_s + B_3,$$

where A_3 and B_3 are nonnegative, A_1 and B_2 are bounded from above, and A_2 and B_1 are nonnegative and bounded from above. Consider, for any $\varepsilon > 0$ and $\sigma \in (\alpha, 0)$, the function

$$u_{\varepsilon,\sigma}(s, t) = \min\{K_s^{\top}(s, t), (K_1^{\perp})_s(s, t)\} + \varepsilon e^{(C+1)(t-\alpha)},$$

where C_σ is twice the supremum of $\max\{A_1, A_2, B_1, B_2\}$ on the time interval $[\alpha, \sigma]$. We claim that $u_{\varepsilon,\sigma}$ is nonnegative in $[\alpha, \sigma]$. Indeed, since the sectional curvatures are smooth functions satisfying Neumann conditions at the boundary, it follows that, $u_{\varepsilon,\sigma} > 0$ at the singular orbits, so if this quantity even became zero, say at a point (t_0, s_0), then $t_0 > \alpha$ (since $K_s^{\top}$ and $(K_1^{\perp})_s$ are both nonnegative initially), and for s_0 corresponding to a principal orbit. Then if $(K_1^{\perp})_s(s_0, t_0) \geq K_s^{\top}(s_0, t_0) = -\varepsilon e^{(C+1)(t-\alpha)}$, then

$$\begin{aligned}
0 &\geq (\partial_t - \Delta)(\varepsilon e^{(C+1)(t-\alpha)} + K_s^{\top})|_{(s_0,t_0)} \\
&= (C+1)\varepsilon e^{(C+1)(t_0-\alpha)} + A_1(s_0, t_0)K_s^{\top}(s_0, t_0) + A_2(s_0, t_0)(K_1^{\perp})_s(s_0, t_0) + A_3(s_0, t_0) \\
&\geq (C+1)\varepsilon e^{(C+1)(t_0-\alpha)} + (A_1(s_0, t_0) + A_2(s_0, t_0))K_s^{\top}(s_0, t_0) \\
&= (C+1 - A_1(s_0, t_0) - A_2(s_0, t_0))\varepsilon e^{(C+1)(t_0-\alpha)} > 0,
\end{aligned}$$

which is absurd. The case that $K_s^{\top}(s_0, t_0) \geq (K_1^{\perp})_s(s_0, t_0) = -\varepsilon e^{(C+1)(t-\alpha)}$ is treated identically. □

Thus our Ricci flows $(S^n, g_\tau(t))_{t\in[\alpha_\tau,0)}$ satisfy (10) at all times.

3.3. *Crude estimates for key geometric quantities.* Define, along the Ricci flows $(S^n, g_\tau(t))_{t\in[\alpha_\tau,0)}$,

$$\begin{aligned}
\ell_\tau(t) &:= \int_0^{\pi/2} \chi_\tau(r, t)\, dr, & h_\tau(t) &:= \psi_\tau(0, t) = \max_{r\in[0,\pi/2]} \psi_\tau(r, t), \\
A_\tau(t) &:= 4\pi \int_0^{\ell_\tau(t)} \psi(s, t)\, ds, & d_\tau(t) &:= \varphi_\tau\big(\tfrac{1}{2}\pi, t\big) = \max_{r\in[0,\pi/2]} \varphi_\tau(r, t).
\end{aligned} \tag{18}$$

Note that $A_\tau(t) = \mathrm{area}(\Sigma^2, g_\tau^\top(t))$ is the area of the totally geodesic 2-sphere $\Sigma \subset S^n$ with respect to the metric $g_\tau^\top(t)$ induced by $g_\tau(t)$, $h_\tau(t)$ is the "radius" of the S^1 singular orbit, $d_\tau(t)$ is the "radius" of the S^{n-2} singular orbit, and $\ell_\tau(t)$ is the length of a minimising geodesic connecting these two singular orbits.

We shall derive a series of crude estimates for these quantities.

Lemma 3.4. *Along the Ricci flows $(S^n, g_\tau(t))_{t\in[\alpha_\tau,0)}$ we have*

$$-8\pi t \le A_\tau(t) \le -8\pi(n-1)t \quad \textit{and} \quad \frac{-\tau}{n-1} \le -\alpha_\tau \le -\tau.$$

Proof. We fix τ and drop the τ subscripts throughout the proof. We directly compute, using the Ricci flow equation,

$$A' = \frac{1}{2}\int_{\Sigma^2} \mathrm{tr}_{g^\top}(\partial_t g^\top)\, d\mu_{g^\top} = -\int_{\Sigma^2} (2K^\top + (n-2)(K_1^\perp + K_2^\perp))\, d\mu_{g^\top}.$$

Owing to Proposition 3.3, (10) and the Gauss–Bonnet theorem, we obtain

$$-8\pi(n-1) \le A' \le -8\pi.$$

Integrating this estimate and using $\lim_{t\to 0} A(t) = 0$ gives the inequalities for $A(t)$. The estimate for α then follows immediately from $A(\alpha) = -8\pi\tau$ (35). □

Lemma 3.5. *Along the Ricci flows $(S^n, g_\tau(t))_{t\in[\alpha_\tau,-(n-1))}$ we have*

$$1 - \frac{n-1}{-t} \le h_\tau(t) \le 1 \quad \textit{and} \quad -2t \le \ell_\tau(t) \le \frac{-4(n-1)t}{1-(n-1)/(-t)}.$$

Proof. We fix τ and drop τ subscripts throughout the proof. Since $K^\top > 0$, $\psi(s,t)$ is a concave function of s, with $\psi_s(0,t) = 0$ by Proposition 2.3. Thus for $s \in (0, \ell(t))$ we have

$$h(t) = \psi(0,t) \ge \psi(s,t) \ge h(t)(1 - s/\ell(t)). \tag{19}$$

Curvature positivity also implies that $\psi(s,t)$ is nonincreasing in t; see (4b). Hence, $h(t) \le h(\alpha) = 1$.

The formula for the area in (18) together with (19) immediately yield

$$2\pi h(t)\ell(t) \le A(t) \le 4\pi h(t)\ell(t). \tag{20}$$

Using $h(t) \le 1$ and Lemma 3.4 we get $\ell(t) \ge -2t$.

Noticing that $2\ell(t)$ is the length of a minimising geodesic in $(\Sigma^2, g^\top(t))$, the contrapositive of Myers' theorem and $(K^\top)_s \ge 0$ (Proposition 3.3) imply

$$K^\top(0) = \min_{\Sigma^2} K^\top \le \frac{\pi^2}{4\ell(t)^2}. \tag{21}$$

By using l'Hôpital's rule at $r = 0$ in (4b) we then obtain

$$h'(t) = -(n-1)h(t)K^\top|_{r=0} \ge -\frac{\pi^2(n-1)}{4}\frac{h(t)}{\ell(t)^2} \ge -\frac{\pi^2(n-1)}{16t^2} \ge -\frac{n-1}{t^2}. \tag{22}$$

Integrating gives our lower bound for $h(t)$.

Finally, (20), Lemma 3.4 and the lower bound for $h(t)$ give

$$\ell(t) \le \frac{A(t)}{2\pi h(t)} \le \frac{-4(n-1)t}{1-(n-1)(\alpha^{-1}-t^{-1})}.$$

(Notice that, since $t < -(n-1)$, the denominator in the right-hand side is positive.) □

Lemma 3.6. *Along the Ricci flows* $(S^n, g_\tau(t))_{t\in[\alpha_\tau,0)}$ *we have*

$$d_\tau \ge \delta \log(-t) - C \quad \textit{for } t \le -2(n-1),$$

where $\delta \doteqdot 1/(4(n-1))$ *and* $C \doteqdot \log(2(n-1))/(4(n-1))$.

Proof. We again fix τ and drop the τ subscripts throughout the proof. Estimating $\varphi \le d$ and $K_1^\perp \le K_1^\perp|_{r=\pi/2}$, we find that

$$1 = -\varphi_s|_{s=\ell(t)} = \int_0^{\ell(t)} (-\varphi_{ss})\, ds = \int_0^{\ell(t)} \varphi K_1^\perp \, ds \le d\ell K_1^\perp|_{r=\pi/2}.$$

Since $K_2^\perp = K_1^\perp$ at the tip, $s=\ell$, (4c) and l'Hôpital's rule then yield

$$d' \le -\frac{2}{\ell}.$$

Applying the estimate for ℓ from Lemma 3.5 and integrating from time t to time $-2(n-1)$ yields the claim. □

Finally, we obtain a uniform upper bound for the scalar curvature, and hence the entire curvature operator, through Hamilton's Harnack inequality [Hamilton 1993].

Lemma 3.7. *There exists a constant* $C(n) > 0$ *such that for any sufficiently large* τ, *and for each* $t \in \left(\frac{1}{10}\alpha_\tau, -2(n-1)\right)$, *the maximal scalar curvature* $R^\tau_{\max}$ *of the Ricci flow* $(S^n, g_\tau(t))_{t\in[\alpha_\tau,0)}$ *is bounded by*

$$R^\tau_{\max}(t) \le \frac{Ce^{-Ct}}{t^2}.$$

Proof. Once again, we drop the τ superscript. The Harnack inequality implies that

$$R_{\max}(t) \le \frac{\sigma-\alpha}{t-\alpha} e^{\ell^2(t)/(2(\sigma-t))} R_{\min}(\sigma)$$

for $\alpha < t < \sigma < 0$. Setting $\sigma = \frac{1}{2}t$ gives

$$R_{\max}(t) \le \frac{\frac{1}{2}t-\alpha}{t-\alpha} e^{\ell^2(t)/(-t)} R_{\min}\left(\tfrac{1}{2}t\right) \le 10 e^{\ell^2(t)/(-t)} R_{\min}\left(\tfrac{1}{2}t\right),$$

provided $t \in \left(\frac{1}{10}\alpha, 0\right)$. Since $\ell\left(\frac{1}{2}t\right) \ge -2\pi t$ (Lemma 3.5), we can use (21) to conclude that

$$R_{\min}\left(\tfrac{1}{2}t\right) \le n(n-1)K^\top_{\min} \le \frac{n(n-1)}{16t^2}.$$

Estimating

$$\ell(t) \le -8(n-1)t$$

for $t < -2(n-1)$ via Lemma 3.5 yields the claim. □

Note that, for $t \geq -2(n-1)$, we may estimate $\ell(t) \leq \ell(-2(n-1)) \leq 16(n-1)^2$, which yields the extremely bad (but uniform in τ) estimate

$$R^{\tau}_{\max} \leq \frac{Ce^{C/(-t)}}{t^2}, \tag{23}$$

where $C = C(n)$.

3.4. ***Taking the limit.*** We are now in a position to construct the desired $O(2)\times O(n-1)$-invariant ancient Ricci flow on S^n.

Proof of Theorem 1.1. Choose an arbitrary time interval I compactly contained in $(-\infty, 0)$. Since $\alpha_\tau \to -\infty$ as $\tau \to -\infty$, the estimates of Lemma 3.7 and (23) yield a uniform-in-τ bound for the Riemann curvature operator Rm of the Riemannian metrics $\{g_\tau(t)\}_{t\in I}$. Shi's estimates [1989] then imply uniform-in-τ estimates for all spatial derivatives of Rm (and hence also of the sectional curvatures) on I.

Define now the functions $A_\tau, B_\tau : \mathbb{R} \times I \to \mathbb{R}$ (for sufficiently large τ) with

$$A_\tau(s,t) = \begin{cases} \psi_\tau(s,t) & \text{if } 0 \leq s \leq \ell_\tau(t), \\ -\psi_\tau(2\ell_\tau(t)-s,t) & \text{if } \ell_\tau(t) \leq s \leq 2\ell_\tau(t), \\ -\psi_\tau(s-2\ell_\tau(t)) & \text{if } 2\ell_\tau(t) \leq s \leq 3\ell_\tau(t), \\ \psi_\tau(4\ell_\tau(t)-s,t) & \text{if } 3\ell_\tau(t) \leq s \leq 4\ell_\tau(t), \end{cases}$$

$$B_\tau(s,t) = \begin{cases} \varphi_\tau(s,t) & \text{if } 0 \leq s \leq \ell_\tau(t), \\ \varphi_\tau(2\ell_\tau(t)-s,t) & \text{if } \ell_\tau(t) \leq s \leq 2\ell_\tau(t), \\ -\varphi_\tau(s-2\ell_\tau(t)) & \text{if } 2\ell_\tau(t) \leq s \leq 3\ell_\tau(t), \\ -\varphi_\tau(4\ell_\tau(t)-s,t) & \text{if } 3\ell_\tau(t) \leq s \leq 4\ell_\tau(t), \end{cases}$$

and then extended to all of $\mathbb{R}$ by insisting that $A_\tau(\,\cdot\,,t)$ and $B_\tau(\,\cdot\,,t)$ are both $4\ell_\tau(t)$-periodic. The functions are smooth because of the parity conditions of Proposition 2.3. We claim that all derivatives of A_τ and B_τ are uniformly bounded on $I \times \mathbb{R}$, independently of τ. Indeed, Lemma 3.5 gives us uniform bounds on $\ell_\tau(t)$, and Lemma 3.7 gives us uniform bounds on A_{ss}/A and B_{ss}/B; combining with the conditions $A_\tau(\ell_\tau(t),t) = 0$, $(A_\tau)_s(\ell_\tau(t),t) = -1$, $B_\tau(0,t) = 0$ and $(B_\tau)_s(0,t) = 1$ gives uniform bounds on A_τ and B_τ and their first spatial derivatives. Uniform bounds on all spatial derivatives then follow from the uniform bounds on all spatial derivatives of curvature. Bounds on all mixed space/time derivatives then follows from the evolution equations (4) and the Lie bracket equation (5).

The Arzelà–Ascoli theorem now implies that, for any sequence of times $\tau_j \to -\infty$ and any $k \in \mathbb{N}$, there exist functions $A_I, B_I : \mathbb{R} \times I \to \mathbb{R}$ to which A_{τ_j} and B_{τ_j} converge in C^k as $j \to \infty$. A standard diagonal argument can be used to extend these limiting functions to all of the time interval $(-\infty, 0)$. The lower bounds for $d_\tau(t)$ and $h_\tau(t)$, upper and lower bounds for $\ell_\tau(t)$ and positive curvature ensures that these functions can be used to construct an $O(2)\times O(n-1)$-invariant time-varying metric which evolves by Ricci flow and has the same curvature properties (10). Note that the time interval $(-\infty, 0)$ on which our solution has been constructed is maximal due to the upper bound for the area A in Lemma 3.4. □

4. Asymptotics

We now examine the backwards asymptotics of our constructed ancient Ricci flow $(S^n, g(t))_{t\in(-\infty,0)}$. First note that since $g(t)$ is $O(2)\times O(n-1)$-invariant, it has the form (3) for some functions χ, ψ and φ. We can again define the geometric quantities (18), and by Lemmas 3.4, 3.5 and 3.6, our ancient solution satisfies

$$\begin{aligned} -8\pi t \le A(t) \le -8\pi(n-1)t, \qquad & 1-\frac{n-1}{-t} \le h(t) \le 1, \\ -2t \le \ell(t) \le -8(n-1)t, \qquad & \frac{1}{4(n-1)}\log\Big(\frac{-t}{2(n-1)}\Big) \le d(t) \end{aligned} \tag{24}$$

for all $t < -2(n-1)$. We start with asymptotics near the waist:

Lemma 4.1. *Let $p_{\text{waist}} \in S^n$ be a fixed reference point on the "waist". Then the time-translated pointed Ricci flows $(S^n, g(t-\tau), p_{\text{waist}})_{t\in(-\infty,\tau)}$ converge smoothly as $\tau \to \infty$ to the stationary flat solution $S^1(2\pi)\times\mathbb{R}^{n-1}$.*

Proof. By the trace Harnack inequality for ancient Ricci flows [Hamilton 1993], $\partial_t R \ge 0$; thus R is uniformly bounded as $\tau \to \infty$. By curvature positivity this implies that the full curvature tensor remains uniformly bounded. The length of the S^1-orbit converges to 2π, for $h(t) \to 1$ as $t \to -\infty$. Hence for any sequence $\tau_k \to \infty$ there is a subsequence of time-translated Ricci flows which converge to some ancient Ricci flow. The Riemann curvature operator at the S^1 singular orbit vanishes as $t \to -\infty$ because of (21) and the $\ell(t)$ estimate in (24), so by the strong maximum principle this limiting Ricci flow must be flat. Since $\lim_{t\to-\infty}\ell(t) = \infty$ and $\lim_{t\to-\infty} h(t) = 1$, the resulting Ricci flow must be the flat product metric on $S^1\times\mathbb{R}^{n-1}$, and S^1 has length 2π. Uniqueness of the resulting Ricci flow implies convergence as $\tau\to\infty$, and not just along subsequences. □

Next we study the asymptotics in the vicinity of the "tip" (i.e., the S^{n-2} singular orbit). To that end, let $(\mathbb{R}^2, g_{\text{cigar},\lambda})$ denote Hamilton's cigar Ricci flow of scale λ (with girth equal to $2\pi\lambda$); see Appendix B.

Lemma 4.2. *Let $p_{\text{tip}} \in S^n$ be a fixed point in the "tip" region. There exists $\lambda \in (0, 1]$ such that the time-translated pointed Ricci flows $(S^n, g(t-\tau), p_{\text{tip}})_{t\in(-\infty,\tau)}$ converge smoothly as $\tau \to \infty$ to the product Ricci flow $(\mathbb{R}^2, g_{\text{cigar},\lambda}) \times (\mathbb{R}^{n-2}, g_{\text{flat}})$,*

Proof. The Harnack inequality and curvature positivity again imply that the pointed Ricci flows $(g(t-\tau_k), p_{\text{tip}})$ converge subsequentially to another Ricci flow. The estimate for $d(t)$ in (24) implies that the geodesics in S^{n-2} converge to lines, and so the Ricci flow has the direct product form $g^\top + g_{\text{flat}}$ on $M \oplus \mathbb{R}^{n-2}$ for some noncompact two-dimensional manifold M. Being rotationally invariant, the ancient Ricci flow $g^\top$ must be a cigar soliton of a certain scale λ, a priori depending on the sequence. The estimate for $h(t)$ in (24) implies that the girth $2\pi\lambda$ of this cigar soliton must be no greater than 2π, so $\lambda \le 1$.

Now, since $\partial_t R(p_{\text{tip}}, t) \ge 0$, the limit $\lim_{t\to-\infty} R(p_{\text{tip}}, t)$ exists. Therefore the scalar curvature at the tip, and hence also the scale λ, of our limiting cigar soliton is independent of the sequence $(\tau_k)_k$. The uniqueness of the limiting ancient flow then implies that we obtain full convergence, not just along subsequences. □

Ultimately, we would like to show that the asymptotic cigar has the right scale $\lambda = 1$, making it compatible with the asymptotic limit in the waist region.

Proposition 4.3. *The scale of the cigar factor in the asymptotic limit on the tip region* (*Lemma 4.2*) *is* $\lambda = 1$.

To prove this, we examine first how the scale λ affects the geometry of the ancient solution.

Lemma 4.4. *For each* $\bar{\lambda} > \lambda$, *there exists* $t_0 = t_0(\lambda, \bar{\lambda})$ *such that, for all* $t < t_0$, *we have the length estimate*

$$\ell(t) \geq -2\bar{\lambda}^{-1} t - C, \tag{25}$$

where $C := -2\bar{\lambda}^{-1} t_0 - \ell(t_0)$.

Proof. In order to estimate the time derivative of ℓ, we introduce the time-independent parameters $\delta = \delta(\lambda, \bar{\lambda}) > 0$, $\varepsilon = \varepsilon(\delta, \lambda, \bar{\lambda}) > 0$ and $t_0 = t_0(\delta, \varepsilon) < 0$. Given ε and δ, we choose t_0 small enough that

$$\mathrm{Rc}_1(s, t) \geq K_{\mathrm{cigar},\lambda}(\ell - s) - \varepsilon$$

for all $s \in [\ell - \delta, \ell]$ and all $t < t_0$; this is possible because of the backwards time convergence established in Lemma 4.2. We compute

$$\begin{aligned} -\frac{d\ell}{dt} &= \int_0^\ell \mathrm{Rc}_1(s, t)\, ds \geq \int_{\ell-\delta}^{\ell} \mathrm{Rc}_1(s, t)\, ds \geq \int_0^\delta K_{\mathrm{cigar},\lambda}(u)\, du - \varepsilon\delta = \frac{2}{\lambda} \tanh\Big(\frac{\delta}{\lambda}\Big) - \varepsilon\delta \\ &\geq \frac{2}{\lambda}(1 - 2e^{-2\delta/\lambda}) - \varepsilon\delta \geq \frac{2}{\bar{\lambda}}, \end{aligned}$$

where u denotes the arc-length parameter of the cigar geodesic (scale λ), and in the last step we chose first δ and then ε so that

$$\frac{4}{\lambda} e^{-2\delta/\lambda} = \frac{1}{\lambda} - \frac{1}{\bar{\lambda}} = \varepsilon\delta.$$

The lemma follows by integrating the above estimate on $[t, t_0]$. □

On the other hand, we obtain the following upper bound for the area.

Lemma 4.5. *For each small* $\varepsilon > 0$, *there exist* $t_0 < 0$ *and* $C(\varepsilon) > 0$ *such that*

$$\tfrac{1}{2} A(t) \leq -4\pi(1 + \varepsilon)t + C(\varepsilon) \quad \textit{for all } t \leq t_0. \tag{26}$$

Proof. Observe that

$$\tfrac{1}{2} A(t) = \int_t^0 -\tfrac{1}{2} A'(\tau)\, d\tau = \int_t^0 \int_{\Sigma^2} (K^\top + (n-2)K^\perp)\, d\mu_{g^\top}\, d\tau = -4\pi t + (n-2) \int_t^0 \int_{\Sigma^2} K^\perp\, d\mu_{g^\top}\, d\tau.$$

To deal with the second term, first observe that by (10) we have

$$\int_{-1}^0 \int_{\Sigma^2} K^\perp\, d\mu_{g^\top}\, d\tau \leq \int_{-1}^0 \int_{\Sigma^2} K^\top\, d\mu_{g^\top}\, d\tau = 4\pi.$$

Regarding the interval $(t, -1)$, we use the estimate

$$\int_t^{-1} K^\perp\big(\tfrac{1}{2}\pi, \tau\big)\, d\tau \leq O(\log(-t)) \quad \text{as } t \to -\infty, \tag{27}$$

which follows from the estimates

$$-\frac{d}{dt}\log d(t) \geq 2K^{\perp}|_{r=\pi/2} \quad \text{and} \quad d \leq O(-t).$$

This is enough to proceed analogously to [Bourni et al. 2021] to obtain (26). To that end, we introduce the coordinate $\rho = \rho(s,t) := \arcsin(-\psi_s(s)) \in \left(0, \tfrac{1}{2}\pi\right)$ — recall that ψ_s takes values on $(-1,0)$. For a fixed $\rho_0 > 0$ we also consider $s_0 = s_0(\rho_0, t)$ defined so that $\rho(s_0,t) = \rho_0$, i.e.,

$$\sin\rho_0 = -\psi_s(s_0).$$

This yields a "partial Gauss–Bonnet theorem" on $\Sigma^2_{<s_0} := \{s < s_0\} \subset \Sigma^2$:

$$\int_{\Sigma^2_{<s_0}} K^{\top}\,d\mu_{g^{\top}} = 2\int_0^{s_0} 2\pi\psi K^{\top}\,ds = -4\pi\int_0^{s_0}\psi_{ss}\,ds = 4\pi\,\sin(\rho_0). \tag{28}$$

Observe also that, since ρ_0 is fixed, $\ell - s_0$ stays uniformly bounded independent of t. Indeed, using that $-\psi_s$ is nondecreasing we have

$$\sin(\rho_0)(\ell - s_0) \leq \int_{s_0}^{\ell} -\psi_s(s)\,ds = \psi(s_0) \leq 1.$$

This implies the uniform-in-time area estimate

$$\int_{\Sigma^2_{\geq s_0}} d\mu_{g^{\top}} = 2\int_{s_0}^{\ell} 2\pi\psi\,ds \leq 4\pi(\ell - s_0) \leq \frac{4\pi}{\sin(\rho_0)}. \tag{29}$$

Thus for each $t < 0$ we split $\Sigma^2 = \Sigma^2_{<s_0} \cup \Sigma^2_{\geq s_0}$ and use (10), (28) and (29) to obtain

$$\int_{\Sigma^2} K^{\perp}(t)\,d\mu_{g^{\top}} \leq \int_{\Sigma^2_{<s_0}} K^{\top}\,d\mu_{g^{\top}} + K^{\perp}\left(\tfrac{1}{2}\pi, t\right)\int_{\Sigma^2_{\geq s_0}} d\mu_{g^{\top}} \leq 4\pi\,\sin(\rho_0) + K^{\perp}\left(\tfrac{1}{2}\pi, t\right)\frac{4\pi}{\sin(\rho_0)}. \tag{30}$$

Finally, integrating (30) and using (27) we immediately get (26) after choosing $\rho_0 > 0$ small enough. ☐

Proof of Proposition 4.3. Suppose, to the contrary, that $\lambda < 1$. For each $t < 0$ and $d_0 \in (0, \ell(t))$, curvature positivity implies

$$\psi(s,t) \geq \frac{(\ell(t) - d_0 - s)\psi(0,t) + s\psi(\ell(t) - d_0, t)}{\ell(t) - d_0}$$

for all $s \in (0, \ell(t) - d_0)$. Therefore we can estimate the $g^{\top}$ area of S^2 with

$$\tfrac{1}{2}A(t) = 2\pi\int_0^{\ell(t)}\psi(s,t)\,ds \geq 2\pi\int_0^{\ell(t)-d_0}\psi(s,t)\,ds \geq \pi(\ell(t) - d_0)\big(\psi(\ell(t) - d_0, t) + \psi(0,t)\big).$$

Then by Lemmas 4.4 and 4.5, for each $\varepsilon > 0$ and $\bar{\lambda} \in (\lambda, 1)$, there are t_0, C_1 and C_2 such that for all $t \leq t_0$,

$$\begin{aligned}\left(2\bar{\lambda}^{-1} + \frac{C_1 - d_0}{-t}\right)\big(\psi(\ell(t) - d_0, t) + \psi(0,t)\big) &\leq \frac{(\ell(t) - d_0)}{-t}\big(\psi(\ell(t) - d_0, t) + \psi(0,t)\big) \\ &\leq \frac{A(t)}{-2\pi t} \leq 4(1+\varepsilon) + \frac{C_2}{-t}.\end{aligned}$$

But this is impossible, because $\lim_{d_0\to\infty}\big(\lim_{t\to-\infty}\psi(\ell(t) - d_0, t) + \psi(0,t)\big) = \lambda + 1$. ☐

Finally, we treat asymptotics of points that have neither bounded distance to the S^1 singular orbit, nor to the S^{n-2} singular orbit.

Lemma 4.6. *For any sequence of times* $\{t_k\}_{k=1}^{\infty}$ *with* $\lim_{k\to\infty} t_k = -\infty$, *and any sequence of points* $\{p_k\}_{k=1}^{\infty}$ *with unbounded* $g(t_k)$ *distance to the* S^{n-2} *and* S^1*-singular orbits, the time-translated pointed Ricci flows* $(S^n, g(t-t_k), p_k)_{t\in(-\infty,-t_k)}$ *converge to the stationary flat cylinder* $S^1 \times \mathbb{R}^{n-1}$ *(of unit radius).*

Proof. Consider the arc-length parametrisation of the functions ψ_k and φ_k, recentred so that $s = 0$ corresponds to p_k. Since the distance to both singular orbits is increasing without bound, $(\varphi_k)_s \geq 0$, $(\varphi_k)_{ss} \leq 0$ and φ_k is also increasing without bound at the S^{n-2} singular orbit, we conclude that $1/\varphi_k$ converges to 0 uniformly on compact subsets (of the s variable). Similarly, since $(\psi_k)_s \leq 0$, ψ_k converges to 1 at the singular orbits, and the girth of the cigar solitons forming near the S^{n-2} orbit is 2π, we conclude that ψ_k must converge to 1 uniformly on compact subsets. The convergence of the ψ_k and φ_k functions give us the required geometric convergence. □

Appendix A. $O(2)\times O(n-1)$-invariant Riemannian metrics on S^n

The purpose of this appendix is to collect a number of important, but computationally intensive observations regarding the geometry of the Riemannian metrics on S^n which have the form (3).

A.1. ***Orthonormal frames for doubly warped product metrics.*** To begin, observe that there is an $O(2)\times O(n-1)$-invariant diffeomorphism from the principal part of S^n and $\left(0, \tfrac{1}{2}\pi\right) \times S^1 \times S^{n-2}$, on which the Riemannian metric takes the form

$$\chi(r)^2\, dr^2 + \psi(r)^2\, d\theta^2 + \varphi(r)^2 g_{S^{n-2}},$$

where θ is the usual parameter for S^1, and $g_{S^{n-2}}$ is the round metric of radius 1 on S^{n-2}. This doubly warped product structure allows us to compute many of the important geometric terms relevant to this paper.

We start by defining $\tilde{e}_1 = \partial_r$, the vector field evolving the natural coordinate on $\left(0, \tfrac{1}{2}\pi\right)$, $\tilde{e}_2 = \partial_\theta$, the vector field evolving the natural coordinate on S^1, and a frame of vector fields $\{\tilde{e}_k\}_{k=3}^{n}$ which are orthonormal with respect to the round radius-1 metric $g_{S^{n-2}}$ on the S^{n-2} component, locally defined around a point $p \in S^{n-2}$. Note that $[\tilde{e}_i, \tilde{e}_j] = 0$, unless i and j are both in $\{3, \ldots, n\}$. Then an orthonormal frame of vector fields for g around the point (r, θ, p) is given by

$$e_1 = \frac{\tilde{e}_1}{\chi(r)}, \qquad e_2 = \frac{\tilde{e}_2}{\psi(r)}, \qquad e_i = \frac{\tilde{e}_i}{\varphi(r)} \quad \text{for all } i = 3, \ldots, n.$$

Observe that for $i, j \in \{3, \ldots, n\}$,

$$[e_1, e_2] = -\frac{\psi'}{\chi\psi} e_2, \quad [e_1, e_j] = -\frac{\varphi'}{\chi\varphi} e_j, \quad [e_2, e_j] = 0, \quad [e_i, e_j] = \frac{1}{\varphi^2}[\tilde{e}_i, \tilde{e}_j].$$

It is convenient to introduce the vector field $\partial_s = \partial_r/\chi$, so that the previous Lie bracket relations become

$$[e_1, e_2] = -\frac{\psi_s}{\psi} e_2, \quad [e_1, e_j] = -\frac{\varphi_s}{\varphi} e_j, \quad [e_2, e_j] = 0, \quad [e_i, e_j] = \frac{1}{\varphi^2}[\tilde{e}_i, \tilde{e}_j].$$

We also define ω_i to be the dual one-form associated to e_i.

A.2. ***The Levi-Civita connection.*** Use of the Koszul formula

$$g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) + g([e_k, e_j], e_i),$$

allows for the computation of the (1, 1)-tensor field ∇e_i for each $i \in \{1, \dots, n\}$:

$$\begin{aligned} \nabla e_1 &= \frac{\psi_s}{\psi}\omega_2 \otimes e_2 + \sum_{j=3}^{n} \frac{\varphi_s}{\varphi}\omega_j \otimes e_j, \qquad \nabla e_2 = -\frac{\psi_s}{\psi}\omega_2 \otimes e_1, \\ \nabla e_i &= -\frac{\varphi_s}{\varphi}\omega_i \otimes e_1 + \sum_{k,l=3}^{n} \frac{\widetilde{\Gamma}_{\mathrm{kil}}}{\varphi}\omega_k \otimes e_l \quad \text{for } i = 3, \dots, n. \end{aligned} \tag{31}$$

Here $\widetilde{\Gamma}_{ijk}$ are the Christoffel symbols of $(S^{n-2}, g_{S^{n-2}})$ in the basis $\{\tilde{e}_i\}_{i=3}^n$. Similarly, we can compute the covariant derivatives of ω_i using the rule $\nabla_X(\omega_i)(Y) = X(\omega_i(Y)) - \omega_i(\nabla_X Y)$:

$$\begin{aligned} \nabla \omega_1 &= \frac{\psi_s}{\psi}\omega_2 \otimes \omega_2 + \sum_{j=3}^{n} \frac{\varphi_s}{\varphi}\omega_j \otimes \omega_j, \qquad \nabla \omega_2 = -\frac{\psi_s}{\psi}\omega_2 \otimes \omega_1, \\ \nabla \omega_i &= -\frac{\varphi_s}{\varphi}\omega_i \otimes \omega_i + \sum_{k,l=3}^{n} \frac{\widetilde{\Gamma}_{\mathrm{kil}}}{\varphi}\omega_k \otimes \omega_l \quad \text{for } i = 3, \dots, n, \end{aligned}$$

A.3. ***Curvature.*** Using the expressions in (31) we compute that for each point x in a neighbourhood of $(r, \theta, p) \in \left(0, \tfrac{1}{2}\pi\right) \times S^1 \times S^{n-2}$, the Riemann curvature operator $\mathrm{Rm} : T_x S^n \wedge T_x S^n \to T_x S^n \wedge T_x S^n$ is diagonal in the basis $\{e_i \wedge e_j\}_{1 \le i < j \le n}$, and is given by

$$\begin{aligned} K^{\top} &:= \mathrm{Rm}_{1212} = -\frac{\psi_{ss}}{\psi} \qquad K_1^{\perp} := \mathrm{Rm}_{1i1i} = -\frac{\varphi_{ss}}{\varphi}, \\ K_2^{\perp} &:= \mathrm{Rm}_{2i2i} = -\frac{\psi_s \varphi_s}{\psi\varphi}, \qquad L := \mathrm{Rm}_{ijij} = \frac{1 - \varphi_s^2}{\varphi^2}, \end{aligned} \tag{32}$$

for all $i, j \in \{3, \dots, n\}$ with $i \ne j$. Then the Ricci curvature is diagonal in the basis $\{e_i\}_{i=1}^n$ and is given by

$$\begin{aligned} \mathrm{Rc}_{11} &= -\frac{\psi_{ss}}{\psi} - (n-2)\frac{\varphi_{ss}}{\varphi}, \qquad \mathrm{Rc}_{22} = -\frac{\psi_{ss}}{\psi} - (n-2) - \frac{\psi_s \varphi_s}{\psi\varphi}, \\ \mathrm{Rc}_{ii} &= -\frac{\varphi_{ss}}{\varphi} - \frac{\psi_s \varphi_s}{\psi\varphi} + (n-3)\frac{1 - \varphi_s^2}{\varphi^2}, \quad \text{for each } i = 3, \dots, n. \end{aligned} \tag{33}$$

A.4. ***Derivatives of curvature.*** We produce some useful formulas which relate derivatives of the sectional curvatures in terms of the warping functions χ, ψ and φ, as well as the sectional curvatures themselves. First, observe that

$$(K_2^{\perp})_s = \frac{\varphi_s}{\varphi}(K^{\top} - K_2^{\perp}) - \frac{\psi_s}{\psi}(K_2^{\perp} - K_1^{\perp}), \quad L_s = \frac{2\varphi_s}{\varphi}(K_1^{\perp} - L). \tag{34}$$

The tensor Laplacian $\Delta\,\mathrm{Rm}$ can be computed through use of the rule for covariant derivatives of a (2, 2)-tensor field T:

$$\begin{aligned}(\nabla_{e_k}T)(e_a, e_b, \omega_c, \omega_d) - e_k(T(e_a, e_b, \omega_c, \omega_d))\\ = -T(\nabla_{e_k}e_a, e_b, \omega_c, \omega_d) - T(e_a, \nabla_{e_k}e_b, \omega_c, \omega_d) - T(e_a, e_b, \nabla_{e_k}\omega_c, \omega_d) - T(e_a, e_b, \omega_c, \nabla_{e_k}\omega_d).\end{aligned}$$

We conclude that $\Delta\,\mathrm{Rm} = \sum_{k=1}^{n} \nabla_{e_k}\nabla_{e_k}\mathrm{Rm} - \nabla_{\nabla_{e_k}e_k}\mathrm{Rm}$ is diagonal, and that

$$\begin{aligned}(\Delta\,\mathrm{Rm})(e_i, e_j, \omega_i, \omega_j) - \Delta(\mathrm{Rm}(e_i, e_j, \omega_i, \omega_j))\\ = 2\sum_{k=1}^{n}\mathrm{Rm}(\nabla_{e_k}\nabla_{e_k}e_i, e_j, \omega_i, \omega_j) + \mathrm{Rm}(e_i, \nabla_{e_k}\nabla_{e_k}e_j, \omega_i, \omega_j)\\ + 2\sum_{k=1}^{n}\mathrm{Rm}(\nabla_{e_k}e_i, e_j, \nabla_{e_k}\omega_i, \omega_j) + \mathrm{Rm}(e_i, \nabla_{e_k}e_j, \omega_i, \nabla_{e_k}\omega_j)\\ + 4\sum_{k=1}^{n}\mathrm{Rm}(\nabla_{e_k}e_i, \nabla_{e_k}e_j, \omega_i, \omega_j) + \mathrm{Rm}(\nabla_{e_k}e_i, e_j, \omega_i, \nabla_{e_k}\omega_j),\end{aligned}$$

with the scalar Laplacian given by $\Delta f = f_{ss} + (\psi_s/\psi + (n-2)(\varphi_s/\varphi)) f_s$. Note that the terms of the form $e_k(\mathrm{Rm}(\nabla_{e_k}e_i, e_j, \omega_i, \omega_j))$ all vanish for $k = 1$ because the frame is parallel along the ∂_r direction. They also vanish for $k \geq 2$ because the term in brackets is constant in the direction of the principal orbits.

Completing the computation gives

$$\begin{aligned}(\Delta\,\mathrm{Rm})_{1212} &= \Delta K^{\top} - 2(n-2)\left(\frac{\varphi_s}{\varphi}\right)^2 (K^{\top} - K_2^{\perp}),\\ (\Delta\,\mathrm{Rm})_{1i1i} &= \Delta K_1^{\perp} + 2\left(\frac{\psi_s}{\psi}\right)^2 (K_2^{\perp} - K_1^{\perp}) - 2(n-3)\left(\frac{\varphi_s}{\varphi}\right)^2 (K_1^{\perp} - L),\\ (\Delta\,\mathrm{Rm})_{2i2i} &= \Delta K_2^{\perp} - 2\left(\frac{\psi_s}{\psi}\right)^2 (K_2^{\perp} - K_1^{\perp}) + 2\left(\frac{\varphi_s}{\varphi}\right)^2 (K^{\top} - K_2^{\perp}),\\ (\Delta\,\mathrm{Rm})_{ijij} &= \Delta L + 4\left(\frac{\varphi_s}{\varphi}\right)^2 (K_1^{\perp} - L).\end{aligned}$$

A.5. ***Square and Lie algebra square of curvature.*** Take a principal point $p \in S^n$ and the orthonormal frame $\{e_i\}_{i=1}^n$ for $V = T_pS^n$ described in Section A.1. This orthonormal frame allows us to identify $V \wedge V$ with $\mathfrak{so}(n)$ via $e_i \wedge e_j \mapsto e_{ij} - e_{ji}$. The set $\{e_i \wedge e_j\}_{1\leq i<j\leq n}$ then forms an orthonormal basis for $\mathfrak{so}(n)$ equipped with the inner product $\langle A, B\rangle_{\mathfrak{so}(n)} = -\frac{1}{2}\operatorname{tr}(AB)$. This identification allows us to interpret the Riemann curvature operator Rm at the point p as a self-adjoint linear operator on $\mathfrak{so}(n)$, and the formula for its Lie algebra square is

$$\langle \mathrm{Rm}^{\sharp}(h), h\rangle = \frac{1}{2}\sum_{\alpha,\beta=1}^{N}\langle[\mathrm{Rm}(b_\alpha), \mathrm{Rm}(b_\beta)], h\rangle \cdot \langle[b_\alpha, b_\beta], h\rangle,$$

where $\{b_\alpha\}_{\alpha=1}^N$ is an orthonormal basis for $\mathfrak{so}(n)$. Since Rm is diagonal in the $\{e_i \wedge e_j\}_{1\le i<j\le n}$ basis, we find immediately that $\mathrm{Rm}^\sharp$ is also diagonal, and we compute

$$
\begin{aligned}
(\mathrm{Rm}^2+\mathrm{Rm}^\sharp)_{1212} &= (K^\top)^2 + (n-2)K_1^\perp K_2^\perp,\\
(\mathrm{Rm}^2+\mathrm{Rm}^\sharp)_{1212} &= (K_1^\perp)^2 + K^\top K_2^\perp + (n-3)K_1^\perp L,\\
(\mathrm{Rm}^2+\mathrm{Rm}^\sharp)_{2i2i} &= (K_2^\perp)^2 + K^\top K_1^\perp + (n-3)K_2^\perp L,\\
(\mathrm{Rm}^2+\mathrm{Rm}^\sharp)_{ijij} &= (K_1^\perp)^2 + (K_2^\perp)^2 + (n-3)L^2,
\end{aligned}
$$

for all $i, j \in \{3, \dots, n\}$ with $i \neq j$.

Appendix B. Cohomogeneity-one ancient Ricci flows in low dimensions

The purpose of this appendix is to give detailed accounts of a number of low-dimensional ancient Ricci flows that are highly relevant to our construction.

B.1. ***Hamilton's cigar.*** The cigar solution of the Ricci flow is a time-varying $O(2)$-invariant metric $g_{\mathrm{cigar}}(t)$ on $\mathbb{R}^2$ given in polar coordinates by

$$g_{\mathrm{cigar}}(t) = \frac{e^{2t}(\cosh^2(r)\,dr\otimes dr + \sinh^2(r)\,d\theta\otimes d\theta)}{1+e^{2t}\sinh^2(r)}.$$

By introducing the arc-length parameter $s(r,t) = \operatorname{arcsinh}(\sinh(r)e^t)$, we can construct a time-varying $O(2)$-invariant diffeomorphism of $\mathbb{R}^2$ so that the pull-back metric is

$$g_{\mathrm{cigar}} = ds\otimes ds + \tanh^2(s)\,d\theta\otimes d\theta;$$

since there is no explicit t-dependence, we can conclude that g_{cigar} is a steady Ricci soliton. The scalar curvature is $4\,\mathrm{sech}^2(s)$; this is four at the tip, and vanishes in the limit as $s\to\infty$. In particular, the cigar soliton is not κ-noncollapsed on all scales, since the limit of the length of the S^1 principal orbits under the $O(2)$ action is 2π. We refer to g_{cigar} as the cigar soliton *of scale* 1. The cigar soliton *of scale* λ is given in arclength parametrisation by

$$g_{\mathrm{cigar},\lambda} = ds\otimes ds + \lambda^2\tanh^2(\lambda^{-1}s)\,d\theta\otimes d\theta.$$

B.2. ***The ancient sausage solution.*** The ancient sausage solution is an $O(2)$-invariant ancient Ricci flow on S^2 given by

$$g_{\mathrm{sausage}}(t) = \chi(r,t)^2(dr\otimes dr + \cos^2(r)\,d\theta\otimes d\theta),$$

where $\chi(r,t) = \sqrt{\tanh(-2t)/(1-\sin^2(r)\tanh^2(-2t))}$. Here $r=0$ corresponds to the equator, and $r=\pm\frac{1}{2}\pi$ corresponds to the singular orbits of the $O(2)$ action.

Observe that this Ricci flow has positive sectional curvature (see the computation in the proof of (10)), so near the singularity at $t=0$, the normalised Ricci flow converges smoothly to the round sphere. Also, since by Gauss–Bonnet the area of a Ricci flow on S^2 decays linearly at a speed of -8π,

$$\operatorname{area}(S^2, g_{\mathrm{sausage}}(t)) = -8\pi t. \tag{35}$$

For the backwards asymptotics we introduce the coordinate $R = \cosh(-2t)/\tan(r)$ so that the metric becomes

$$g_{\text{sausage}}(t) = \frac{\tanh(-2t)}{1+R^2}\left(\frac{\cosh(-2t)^2}{\cosh(-2t)^2+R^2}\,dR\otimes dR + R^2\,d\theta\otimes d\theta\right);$$

this converges to the metric $(dR\otimes dR + R^2\,d\theta\otimes d\theta)/(1+R^2)$, which we recognise as the cigar soliton with $R = \sinh(\rho)$. The supremum length of the S^1 fibres is 2π.

Since the ancient sausage solution has positive curvature, and the length of the S^1 orbit at the equator converges to 2π, the other asymptotics must converge to the cylinder $S^1\times\mathbb{R}$ of waist 2π.

B.3. ***The $O(2)\times O(2)$-invariant hypersausage solution on S^3.*** In order to put our solutions into some context, we recall that Fateev's ancient "hypersausage" solution is an $O(2)\times O(2)$-invariant Ricci flow on S^3 which is given, in the principal domain $\left(0, \tfrac{1}{2}\pi\right)\times S^1\times S^1$, by

$$g_{\text{hypersausage}} = \chi^2\,dr^2 + \psi^2\,d\theta^2 + \varphi^2\,d\omega^2,$$

where

$$\chi^2(r,t) := \frac{\cosh(-2t)\sinh(-2t)}{2(\cos^2 r+\sin^2 r\cosh(-2t))(\sin^2 r+\cos^2 r\cosh(-2t))},$$

$$\psi^2(r,t) := \frac{\cos^2 r\sinh(-2t)}{2(\sin^2 r+\cos^2 r\cosh(-2t))},\quad \varphi^2(r,t) := \frac{\sin^2 r\sinh(-2t)}{2(\cos^2 r+\sin^2 r\cosh(-2t))}.$$

Note that, as $t\to-\infty$, both $\psi(0,t)$ and $\varphi\left(\tfrac{1}{2}\pi, 0\right)$ are bounded. Since only one of these quantities is bounded for the ancient Ricci flow on S^3 of Theorem 1.1, the two solutions are not isometric.

Remark 5. This solution corresponds to "$k = 0$" in the family of "twisted hypersausage" solutions on S^3 constructed by Bakas, Kong and Ni [Bakas et al. 2012] (following Fateev [1995; 1996]). But the other metrics in the family are not $O(2)\times O(2)$-invariant.

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Bundle your figure files into a single archive (using zip, tar, rar or other format of your choice) and upload on the link you been provided at acceptance time. Each figure should be captioned and numbered so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text (“the curve looks like this:”). It is acceptable to submit a manuscript with all figures at the end, if their placement is specified in the text by means of comments such as “Place Figure 1 here”. The same considerations apply to tables.

White Space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal’s preferred fonts and layout.

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